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Ullery, William Davis

THE ISOMORPHISM PROBLEM FOR COMMUTATIVE GROUP ALGEBRAS

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THE ISOMORPHISM PROBLEM FOR
COMMUTATIVE GROUP ALGEBRAS

by
William Davis Ullery

A Dissertation Submitted to the Faculty of the
DEPARTMENT OF MATHEMATICS
In Partial Fulfillment of the Requirements
For the Degree of
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In the Graduate College
THE UNIVERSITY OF ARIZONA

1 9 8 3
As members of the Final Examination Committee, we certify that we have read the dissertation prepared by William Davis Ullery entitled "The Isomorphism Problem for Commutative Group Algebras" and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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SIGNED: William Davis Ullery
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W.D.U.

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Let $R$ be a commutative ring with identity and let $G$ and $H$ be abelian groups with the group algebras $RG$ and $RH$ isomorphic as $R$-algebras. In this dissertation we investigate the relationships between $G$ and $H$.

Let $\text{inv}(R)$ denote the set of rational prime numbers that are units in $R$ and let $G_R$ (respectively, $H_R$) be the direct sum of the $p$-components of $G$ (respectively, $H$) with $p \in \text{inv}(R)$. It is known that if $G_R$ or $H_R$ is non-trivial then it is not necessarily true that $G$ and $H$ are isomorphic. However, if $R$ is an integral domain of characteristic 0 or a finitely generated indecomposable ring of characteristic 0 then $G/G_R \cong H/H_R$.

This leads us to make the following definition: We say that $R$ satisfies the Isomorphism Theorem if whenever $RG \cong RH$ as $R$-algebras for abelian groups $G$ and $H$ then $G/G_R \cong H/H_R$. Thus integral domains of characteristic 0 and finitely generated indecomposable rings of characteristic 0 satisfy the Isomorphism Theorem. Our first major result shows that indecomposable rings of characteristic 0 (no restrictions on generation) satisfy the Isomorphism Theorem.
It has been conjectured that all rings $R$ satisfy the Isomorphism Theorem. However, we show that the conjecture may fail if nontrivial idempotents are present in $R$. This leads us to consider necessary and sufficient conditions for rings to satisfy the Isomorphism Theorem.

We say that $R$ is an ND-ring if whenever $R$ is written as a finite product of rings then one of the factors, say $R_i$, satisfies $\text{inv}(R_i) = \text{inv}(R)$. We show that every ring satisfying the Isomorphism Theorem is an ND-ring. Moreover, if $R$ is an ND-ring and if $\text{inv}(R)$ is not the complement of a single prime we show that $R$ must satisfy the Isomorphism Theorem. This result together with some other fragmentary evidence leads us to conjecture that $R$ satisfies the Isomorphism Theorem if and only if $R$ is an ND-ring.

Finally we obtain several equivalent formulations of our conjecture. Among them is the result that every ND-ring satisfies the Isomorphism Theorem if and only if every field of prime characteristic satisfies the Isomorphism Theorem.
CHAPTER 1

INTRODUCTION

Throughout this paper "ring" will always mean "commutative ring with identity" and $R$ will always denote a ring with identity element $1_R$ or simply 1. Unless explicitly stated to the contrary, all groups will be written multiplicatively with identity element $e$. No confusion should result from using the same letter to denote the identity of several groups simultaneously.

Let $G$ be a group. The group algebra $RG$ is the additive free $R$-module generated by the elements of $G$ with a multiplication induced in the obvious fashion by the multiplications in $R$ and $G$. Thus $RG$ may be regarded as the $R$-algebra consisting of all formal sums $\alpha = \sum \{a(g)g|g \in G\}
= \sum a(g)g$ where $a(g) \in R$ for all $g \in G$ and $a(g) = 0$ for almost all $g \in G$. We shall also use the notation $\alpha = r_1 g_1 + \ldots + r_n g_n$ ($r_i \in R, g_i \in G, 1 \leq i \leq n$) to denote elements of $RG$.

The problem we intend to investigate will be referred to as the Isomorphism Problem (for group algebras). The Isomorphism Problem can be loosely stated as follows: If $G$ is a group, what can be said about $G$ if $R$ and $RG$ are known? More precisely, if $G$ and $H$ are groups with $\cong$ as
R-algebras, how are $G$ and $H$ related? For example, are $G$ and $H$ necessarily isomorphic? For general accounts of the Isomorphism Problem from various perspectives see Passman [1] and Sehgal [1].

In this paper we will only consider the Isomorphism Problem for commutative group algebras and, unless explicitly stated to the contrary, all group algebras considered will be commutative. In this context the Isomorphism Problem becomes: If $G$ and $H$ are abelian groups with $RG \cong RH$ as $R$-algebras, how are $G$ and $H$ related? For example, it is well known that if $R = \mathbb{Z}$, the ring of rational integers, then $RG \cong RH$ as $R$-algebras implies that $G \cong H$ for all abelian groups $G$ and $H$. We will give an elementary proof of this fact inspired by Dennis [1] later in the chapter.

If $X$ is any abelian group we shall denote its torsion subgroup by $X_0$.

The Isomorphism Problem for commutative group algebras becomes more difficult if $R$ is even "reasonably" arbitrary. Let $G$ be an abelian group. Being able to distill a complete system of invariants for $G$ from $RG$ seems to depend a great deal on a certain relationship between the arithmetic of $R$ and the structure of $G_0$ (see May [1, 3, 4]). To discuss this relationship more precisely we need some notation.

For any $R$ there is a unique ring-homomorphism $\mathbb{Z} \to R$ carrying 1 to $1_R$. The nonnegative integer $n$ that
generates the kernel of this homomorphism (as an ideal of \( \mathbb{Z} \))

is known as the characteristic of \( R \) and we write \( \text{char}(R) = n \). If the image of an integer \( m \) is a unit in \( R \) we say that \( m \) inverts in \( R \). We shall frequently abuse notation and write \( \frac{1}{m} \in R \) to mean that \( m \) inverts in \( R \). The set of all prime numbers in \( \mathbb{Z} \) that invert in \( R \) is denoted by \( \text{inv}(R) \).

We note that if \( G \) is a group then \( R \) can (and will) be considered as an \( R \)-subalgebra of \( RG \) by means of the identification \( r \rightarrow re \) for all \( r \in R \). In a similar fashion we may also regard subrings of \( R \) as subrings of \( RG \).

Let \( G \) be an abelian group. It is a well known elementary fact that the torsion subgroup \( G_0 \) decomposes uniquely as the direct sum of its primary components. If \( p \) is a prime number then the \( p \)-primary component of \( G_0 \) is denoted by \( G_p \). We call the set of all prime numbers \( p \) such that \( G_p \) is nontrivial the support of \( G \) and denote it by \( \text{supp}(G) \).

The symbol \( \oplus \) will always be used to denote the internal direct sum of subgroups of an abelian group. Thus if \( G \) is an abelian group, \( G_0 = \bigoplus \{ G_p \mid p \in \text{supp}(G) \} = \bigoplus_p G_p \), the last sum being taken over all prime numbers \( p \). The subgroup of \( G_0 \) given by \( \bigoplus \{ G_p \mid p \in \text{inv}(R) \} \) is denoted by \( G_R \).

We finally note that any subgroup \( H \) of \( G \) can (and will) be considered as a subgroup of \( RG^* \), the full group of units
of $RG$, via the identification $g \leftrightarrow lg$ for all $g \in H$.

We are now ready to discuss the dependence of the structure of $G$ on $RG$ and a certain relationship between the arithmetic of $R$ and the structure of $G_0$. Suppose $G$ and $H$ are abelian groups with $RG \cong RH$ as $R$-algebras. May [4] gives an explicit example to show that $H$ and $G$ are not necessarily isomorphic if $G_R$ or $H_R$ is nontrivial. On the other hand, in May [3] and [4] it is shown that if $R$ is either an integral domain of characteristic 0 or a finitely generated indecomposable ring of characteristic 0 then $RG \cong RH$ implies that $G/G_R \cong H/H_R$. In particular, if $G_R$ and $H_R$ are trivial (which is the case if $R = \mathbb{Z}$) then $G \cong H$.

The above discussion motivates the following definition: A ring $R$ is said to satisfy the Isomorphism Theorem if whenever $G$ and $H$ are abelian groups with $RG \cong RH$ as $R$-algebras then $G/G_R \cong H/H_R$. Thus the results in May [3] and [4] supply us with two classes of rings satisfying the Isomorphism Theorem; namely, the integral domains of characteristic 0 and the finitely generated indecomposable rings of characteristic 0. In Chapter 3 we unite these two classes of examples by proving that indecomposable rings of characteristic 0 (no restriction on generation) satisfy the Isomorphism Theorem.

In May [4] it is conjectured that every ring $R$ satisfies the Isomorphism Theorem. In Chapter 4 we show, by
example, that the conjecture may not hold if nontrivial idempotents are present. In fact we show that if \( p_1, \ldots, p_n \) are distinct prime numbers then \( R = \mathbb{Z}[1/p_1] \times \ldots \times \mathbb{Z}[1/p_n] \) does not satisfy the Isomorphism Theorem if \( n \geq 2 \). Moreover, we provide a necessary and sufficient condition for a finite product of indecomposable rings of characteristic 0 to satisfy the Isomorphism Theorem. In an appendix we examine further the algebraic invariants of group algebras over rings of this type.

In Chapter 5 we continue our investigation of rings which satisfy the Isomorphism Theorem. We show that every ring \( R \) satisfying the Isomorphism Theorem necessarily satisfies the following property: Whenever \( R \) is expressed as a product of rings, say \( R = R_1 \times \ldots \times R_n \) (with each \( R_i \) not necessarily indecomposable), then there exists an \( i, 1 \leq i \leq n \), with \( \text{inv}(R_i) = \text{inv}(R) \). Rings satisfying this property are referred to as ND-rings (with "ND" standing for "nicely decomposing"). Thus every ring satisfying the Isomorphism Theorem is an ND-ring. We then proceed to show that a very wide class of ND-rings satisfies the Isomorphism Theorem. This leads us to conjecture that the rings which satisfy the Isomorphism Theorem are precisely the ND-rings.

In Chapter 6 we obtain several equivalent formulations of the above conjecture. In particular, we show that resolving our conjecture is equivalent to showing that every
field of prime characteristic satisfies the Isomorphism Theorem. Unfortunately, very little is known at this time about the latter problem.

We now provide the promised proof that \( \mathbb{Z} \) satisfies the Isomorphism Theorem. We include a proof of this basic result because its use of the unit group \( \mathbb{Z}G^* \) is similar to our later use of \( RG^* \) in more general settings. The basic idea is to show that \( G \) is a direct summand of \( RG^* \) in a nice enough way so that \( G \) is determined up to isomorphism. The functions \( \delta \) and \( \delta' \) defined below and their basic properties are due to R. K. Dennis (see Dennis [1]).

**Theorem 1.1.** \( \mathbb{Z} \) satisfies the Isomorphism Theorem.

**Proof.** Let \( G \) and \( H \) be abelian groups with \( \mathbb{Z}G \cong \mathbb{Z}H \) as \( \mathbb{Z} \)-algebras. Since \( \text{inv}(\mathbb{Z}) = \phi \) we need to show that \( G \cong H \). Set \( R = \mathbb{Z} \).

Viewing \( RG \) as an additive abelian group we obtain a group-homomorphism \( \delta:RG \rightarrow G \) given by \( \delta(\alpha) = \bigoplus\{a(g)g \in G\} \) for all \( \alpha = \bigoplus\{a(g)g \in RG\} \). Hence \( \delta(\alpha + \beta) = \delta(\alpha)\delta(\beta) \) for all \( \alpha, \beta \in RG \). Let \( \text{aug}:RG \rightarrow R \) be the \( R \)-algebra homomorphism given by \( \text{aug}(\alpha) = \bigoplus\{a(g)g \in G\} \) for all \( \alpha \in RG \). (We note that \( \text{aug} \) is the augmentation mapping introduced in Chapter 2.) A routine computation reveals that \( \delta(\alpha\beta) = \delta(\alpha)\text{aug}(\beta)\delta(\beta)\text{aug}(\alpha) \) for all \( \alpha, \beta \in RG \). In a
similar fashion we obtain maps \( \delta' : RH \to H \) and \( \text{aug}' : RH \to R \) with the corresponding properties.

Let \( f:RG \to RH \) be any \( R \)-algebra isomorphism. Define a function \( \phi : RG \to RG \) by setting \( \phi(g) = \text{aug}'(f(g^{-1})) \cdot g \) and then extending linearly to all of \( RG \). It is easily seen that \( \phi \) is an \( R \)-algebra automorphism of \( RG \) and that \( \text{aug}(a) = \text{aug}'((f \circ \phi)(a)) \). Thus we can replace \( f \) by \( f \circ \phi \) and assume that \( \text{aug}(a) = \text{aug}'(f(a)) \) for all \( a \in RG \).

Set \( RG^* = \{ a \in RG^* | \text{aug}(a) = 1 \} \) and \( RH^* = \{ a' \in RH^* | \text{aug}'(a') = 1 \} \). Then, \( f(RG^*) = RH^* \). Note that \( \delta_1 = \delta | RG^* \) is a group-homomorphism \( \delta_1 : RG^* \to G \) which splits the inclusion map \( G \to RG^* \). Thus \( RG^* = G \oplus \ker(\delta_1) \). Likewise if we set \( \delta'_1 = \delta' | RH^* \) then \( RH^* = H \oplus \ker(\delta'_1) \).

Since \( f(\ker(\delta_1)) = \ker(\delta'_1) \) we conclude that \( G \cong RG^*/\ker(\delta_1) \cong f(RG^*)/f(\ker(\delta_1)) = RH^*/\ker(\delta'_1) \cong H \).

Q.E.D.

We note that the above proof works because every abelian group happens to be a \( \mathbb{Z} \)-module. Thus it seems unlikely that the proof can be generalized to more arbitrary \( R \).
CHAPTER 2

HOMOMORPHISMS, THE INTEGRAL SUBALGEBRA AND UNITS

This chapter is mainly expository. Here we will lay the foundation for our study of the Isomorphism Problem which begins in Chapter 3.

The first section consists mainly of some well known facts concerning homomorphisms of group algebras. Most of these results are easily found in the literature and references will be given as appropriate.

In the last two sections we present some material from May [3] and May [4]. May's work concerns algebraic elements and units of group algebras and generalizes Higman's study of units in Higman [1]. In some cases we will be able to simplify the original proofs of results given in May [3]. For the sake of brevity we will not always state results in their greatest known generality. Rather, we will specialize certain results, tailoring them for our later applications.

§1. Homomorphisms of Group Algebras

As usual $\mathbb{R}$ denotes a commutative ring with identity $1_{\mathbb{R}}$ or simply 1. Throughout this section $G$ and $G'$ are abelian groups (written multiplicatively with identity elements denoted by $e$).
Our convention will be that homomorphisms of rings and algebras will carry the identity to the identity. In particular, if \( R \) is a subring of the ring \( S \) we will require that the inclusion map \( R \to S \) be a ring-homomorphism so that \( 1_R = 1_S \).

Let \( S \) be a ring and suppose \( \phi: R \to S \) is a ring-homomorphism. Then \( S \) has an \( R \)-algebra structure given by \( r \cdot s = \phi(r)s \ (r \in R, \ s \in S) \). Thus \( S G' \) is an \( R \)-algebra. If \( \psi: G \to G' \) is a group-homomorphism we have an induced \( R \)-algebra homomorphism \( RG \to SG' \) given by \( \alpha \mapsto \sum (\phi(\alpha(g))) \cdot \psi(g) \mid g \in G \) for all \( \alpha = \sum \alpha(g)g \in RG \). If \( I \) is an ideal of \( R \) and if \( H \) is a subgroup of \( G \), the \( R \)-algebra homomorphisms \( RG \to (R/I)G, \ RG \to R(G/H) \) and \( RG \to (R/I)(G/H) \) induced by the respective quotient and inclusion maps will always be referred to as natural maps.

Suppose \( \{ \phi_i: R \to S_i \} \) is a collection of ring-homomorphisms and \( \{ \psi_i: G \to G_i \} \) is a collection of group-homomorphisms. In this situation we have an induced \( R \)-algebra homomorphism \( RG \to \prod_i S_i G_i \) into the product. This homomorphism is described explicitly as follows: If \( \alpha = \sum \alpha(g)g \in RG \) then \( \alpha \mapsto x \in \prod_i S_i G_i \) where \( x \) has \( i \)-th component \( x_i = \sum (\phi_i(\alpha(g))) \cdot \psi_i(g) \mid g \in G \). Such an algebra-homomorphism will also be called "natural" if all \( \phi_i \) and \( \psi_i \) are inclusion or quotient maps.
The group-homomorphism $G \to \{e\}$ induces an $R$-algebra homomorphism $RG \to R$ called the **augmentation map**. The augmentation map will be denoted by $\text{aug}$ or by $\text{aug}_R$ if the dependence on $R$ is to be emphasized. An $R$-algebra homomorphism $f: RG \to RG'$ is called **augmentation preserving** if $\text{aug} = \text{aug} \circ f$. An augmentation preserving $R$-algebra isomorphism is called **normalized**.

A reference for our first result is Losey [1].

**Lemma 2.1.1.** If $RG \cong RG'$ as $R$-algebras then there exists a normalized $R$-algebra isomorphism $f: RG \to RG'$.

**Proof.** Let $\phi: RG \to RG'$ be an $R$-algebra isomorphism. Define $\psi: RG \to RG$ by setting $\psi(g) = \text{aug}(\phi(g^{-1})) \cdot g$ for all $g \in G$ and then extending linearly to all of $RG$. It is now easily checked that $\psi$ is an $R$-algebra automorphism of $RG$ and that $f = \phi \circ \psi: RG \to RG'$ is normalized. Q.E.D.

Let $H$ be a subgroup of $G$. We will often be concerned with the kernel of the natural map $RG \to R(G/H)$. The following result gives us a characterization of this kernel. References in the literature for this result are Connell [1] or Gruenberg [1].

**Lemma 2.1.2.** Let $H$ be a subgroup of $G$ generated by a set $S \subseteq H$. Then the kernel of the natural map $\phi: RG \to R(G/H)$ is generated as an ideal of $RG$ by $\{s - 1|s \in S\}$. 
Proof. If \( T \) is any subset of \( RG \) let \( \langle T \rangle \) denote the ideal of \( RG \) generated by \( T \). Set \( I = \langle \{ h - 1 | h \in H \} \rangle \).

If \( s_1, s_2 \in S \) then the identities \( s_1s_2 - 1 = (s_1 - 1)(s_2 - 1) + (s_1 - 1) + (s_2 - 1) \) and \( s_1^{-1} - 1 = -s_1^{-1}(s_1 - 1) \) show that \( I = \langle \{ s - 1 | s \in S \} \rangle \). Since clearly \( I \subseteq \ker(\phi) \) it suffices to show that \( \ker(\phi) \subseteq I \).

Let \( \{ g_i | i \in \Lambda \} \) be a complete set of coset representatives for \( H \) in \( G \) indexed by a set \( \Lambda \). Suppose \( \alpha \in RG \) and write \( \alpha = \sum_{i \in \Lambda} (\sum_{h \in H} r_{i,h}) g_i \) where \( r_{i,h} \in R \)

for all \( i \in \Lambda \) and \( h \in H \). If \( \alpha \in \ker(\phi) \) then \( 0 = \phi(\alpha) = \sum_{i \in \Lambda} (\sum_{h \in H} r_{i,h}) g_i \) and so \( r_{i,h} = \sum_{h \notin e} (-r_{i,h}) \) for every \( i \in \Lambda \). Consequently \( \alpha = \sum_{i \in \Lambda} (\sum_{h \notin e} r_{i,h}) g_i H \) showing \( \ker(\phi) \subseteq I \) as required.

Q.E.D.

Since \( \text{aug} \) is equivalent to the natural map induced by \( G \to G/G \) we have the following.

Corollary. Suppose \( G \) is generated by a set \( S \).

Then the kernel of the augmentation map \( \text{aug}:RG \to R \) is generated by \( \{ s - 1 | s \in S \} \).

The next result shows that if \( R \) is an indecomposable ring with \( \text{char}(R) = 0 \) and \( p \not\in \text{inv}(R) \) then \( R \) has a homomorphic image \( R/P \) which is an integral domain enjoying the same properties as \( R \). This will be applied repeatedly via the natural map \( RG \to (R/P)G \) to transfer questions.
concerning group algebras over indecomposable rings of characteristic 0 to questions concerning group algebras over integral domains of characteristic 0.

**Lemma 2.1.3 (May [3]).** Let R be an indecomposable ring with $\text{char}(R) = 0$ and $p \not\in \text{inv}(R)$. Then there exists a minimal prime ideal $P$ of $R$ such that $\text{char}(R/P) = 0$ and $p \not\in \text{inv}(R/P)$.

**Proof.** Let $S$ be the subset of $R$ consisting of all elements of the form $p^n(px + 1)$, $n \geq 0$ and $x \in R$. Note that $S$ is closed under multiplication and $1 \in S$. We claim that $0 \not\in S$.

Suppose to the contrary that $0 \in S$. Then, by replacing $x$ by $-x$ if necessary, we may assume that $p^n(px - 1) = 0$ for some $n \geq 0$ and $x \in R$. Then $p^{n+1}x = p^n$ and so $n > 0$ since $p \not\in \text{inv}(R)$. Thus $p^n = p(p^n x)$ = $p^{n+2}x^2$ and by induction $p^n = p^{2n}x^n$. Therefore $(p^n x^n)^2 = (p^{2n}x^n)x^n = p^n x^n$ and so $p^n x^n$ is idempotent. Hence $p^n x^n = 1$ or $p^n x^n = 0$.

Since $p \not\in \text{inv}(R)$ and $n > 0$, $p^n x^n \neq 1$. If $p^n x^n = 0$ then $p^n = p^{2n} x^n = p^n(p^n x^n) = 0$, contradicting $\text{char}(R) = 0$. Therefore $0 \not\in S$ as claimed.

Now, since $S$ is closed under multiplication, $1 \in S$ and $0 \not\in S$, there exists a prime ideal $P$ of $R$ with $P \cap S = \phi$. Moreover, since every prime ideal contains a minimal
prime ideal, $P$ may be chosen to be a minimal prime of $R$.

Finally, $P \cap S = \emptyset$ implies that $p \not\in \text{inv}(R/P)$ and
$\text{char}(R/P) \neq p$. Therefore, $\text{char}(R/P) = 0$. Q.E.D.

The following generalization of Lemma 2.1.3 is also needed.

**Corollary.** Suppose $\{R_j | j \in J\}$ is a collection of
indecomposable rings of characteristic 0 indexed by a set
$J$. If $R$ is the product $\prod_j R_j$ and if $p \not\in \text{inv}(R)$
then there exists a minimal prime ideal $P$ of $R$ such that
$\text{char}(R/P) = 0$ and $p \not\in \text{inv}(R/P)$.

**Proof.** Since $p \not\in \text{inv}(R)$ there exists $i \in J$ with
$p \not\in \text{inv}(R_i)$. By Lemma 2.1.3 there exists a minimal prime
ideal $P_i$ of $R_i$ such that $p \not\in \text{inv}(R_i/P_i)$ and
$\text{char}(R_i/P_i) = 0$. Let $P$ be the set of all $r \in R$ such that the $i$-th
component of $r$ lies in $P_i$. Then $P$ is a minimal prime
ideal of $R$ with $R/P \cong R_i/P_i$. Q.E.D.

Next we discuss two important identifications which
will be used frequently in the sequel. First suppose $G$ is
an abelian group with $G = H \oplus K$ for subgroups $H$ and $K$.
Let $S$ be the commutative $R$-algebra $RH$. In this situation
we obtain an $S$-algebra isomorphism $SK \rightarrow RG$ given as follows:
Suppose $sk \in SK$ with $s \in S$ and $k \in K$. Write $s = r_1h_1$
$+ \ldots + r_nh_n$ with $r_i \in R$, $h_i \in H$, $1 \leq i \leq n$. Define
sk \mapsto r_1(h_1k) + ... + r_n(h_nk) \text{ and then extend linearly to all of } SK \text{ to obtain the desired isomorphism. Thus we may view the group algebra } \text{RG as the group algebra } \text{SK by using the inverse of this isomorphism.}

Finally suppose S is a commutative R-algebra and let G be an abelian group. In this situation there is a natural S-algebra isomorphism from \( S \otimes_R \text{RG} \) to the group algebra \( S \text{G} \) defined as follows: Given an elementary tensor \( s \otimes \alpha \in S \otimes_R \text{RG} \) write \( \alpha = \sum \{ (\alpha(g)g | g \in G \} \in \text{RG} \) and define \( s \otimes \alpha \mapsto \sum \{ (\alpha(g) \cdot s)g | g \in G \} \in S \text{G} \). By extending linearly to all of \( S \otimes_R \text{RG} \) it is easily checked that a well-defined isomorphism is obtained.

§2. The Integral Subalgebra

Throughout this section R is a ring and G is an abelian group. Viewing R as a subalgebra of RG let \( A(\text{RG}) \) be the subset of \( \text{RG} \) consisting of all elements of \( \text{RG} \) integral over R. Thus \( A(\text{RG}) \) is the set containing all \( \alpha \in \text{RG} \) with \( f(\alpha) = 0 \) for some monic polynomial \( f \in R[X] \), the polynomial ring over R. Then in fact \( A(\text{RG}) \) is an R-subalgebra of \( \text{RG} \) with \( R \subseteq A(\text{RG}) \subseteq \text{RG} \). We call \( A(\text{RG}) \) the integral subalgebra of \( \text{RG} \). (See May [3] for a reference.)

If S is any ring let \( N(S) \) be the nilradical of S; that is, \( N(S) \) is the set of all nilpotent elements of
S. \( N(S) \) is the ideal of \( S \) formed by intersecting all the minimal prime ideals of \( S \). If \( N(S) = 0 \) then \( S \) is called reduced. (See Gilmer [1], Kaplansky [2] and Zariski and Samuel [1] for references.)

Viewing \( RG_0 \) as a subalgebra of \( RG \) it is easy to see that \( RG_0 + N(RG) \subseteq A(RG) \). A result in May [3] shows that this is actually an equality. Before proving this we need a definition and a Lemma.

If \( \alpha = \sum \{ \alpha(g)g \mid g \in G \} \in RG \) then the support of \( \alpha \), written \( \text{supp}(\alpha) \), is the set of all \( g \in G \) with \( \alpha(g) \neq 0 \). If \( \alpha \in RG^* \), the group of units of \( RG \), and \( |\text{supp}(\alpha)| = 1 \) then \( \alpha \) is called a trivial unit. Thus a trivial unit of \( RG \) is of the form \( rg \) with \( r \in R^* \) and \( g \in G \).

Part (1) of the Lemma below is essentially contained in the proof of Proposition 2 in May [3]. We isolate this result as a part of our Lemma because it will prove useful later. Parts (2) and (3) constitute Lemma 1 in May [3].

**Lemma 2.2.1.** (May [3]). Suppose \( F \) is a torsion free abelian group. Then:

1. \( \alpha = \sum \alpha(g)g \in N(RF) \) implies that \( \alpha(g) \in N(R) \) for all \( g \in F \). Thus \( N(RF) = N(R)F \) where \( N(R)F \) is the ideal of \( RF \) consisting of all elements with coefficients in \( N(R) \).
(2) If \( R \) is reduced, \( A(RF) = R \). In particular, \( N(RF) = 0 \).

(3) If \( R \) is an integral domain then every unit of \( RF \) is a trivial unit.

Proof. Any torsion free abelian group can be made into an ordered group (see Gilmer [1] for a reference). So, we may assume that \( F \) is ordered.

(1) Clearly \( N(R)F \leq N(RF) \). Suppose \( \alpha = r_1g_1 + \ldots + r_kg_k \in N(RF) \) with \( g_1 < g_2 < \ldots < g_k \) and say \( \alpha^n = 0 \).

Then \( 0 = \beta + r_k^n g_k^n \) (\( \beta \in RF \)) where every element of \( \text{supp}(\beta) \) is strictly less than \( g_k^n \). Thus \( r_k^n = 0 \) and so \( r_k \in N(R) \). Since \( \alpha - r_k g_k \) is nilpotent the result follows by induction on \( k \).

(2) Suppose \( \alpha \in A(RF) \) and choose \( f \in R[X] \) monic of degree \( n \) with \( f(\alpha) = 0 \). We may assume that \( \alpha \neq 0 \) so that \( \text{supp}(\alpha) \) is nonempty. Note that \( n \geq 1 \).

Let \( g \) be the greatest element in \( \text{supp}(\alpha) \). If \( g > e \) then \( g^n \in \text{supp}(f(\alpha)) \) since \( R \) is reduced. But this contradicts \( f(\alpha) = 0 \) and so \( g \leq e \). If \( h \) is the least element of \( \text{supp}(\alpha) \) then a similar argument shows that \( h > e \). Therefore \( \text{supp}(\alpha) = \{ e \} \) and \( \alpha \in R \) as required.

(3) Suppose \( \alpha \in RF^* \) where \( R \) is an integral domain. Let \( g_1 \) (respectively, \( g_2 \)) be the greatest element in \( \text{supp}(\alpha) \) (respectively, \( \text{supp}(\alpha^{-1}) \)). Let \( h_1 \) (respectively, \( h_2 \)) be the least element in \( \text{supp}(\alpha) \) (respectively, \( \text{supp}(\alpha^{-1}) \)). Then \( g_1 g_2 \) and \( h_1 h_2 \) are
both in \( \text{supp}(a \alpha^{-1}) = \{e\} \). Therefore \( g_1 g_2 = h_1 h_2 \) and so \( g_1 = h_1 \). Thus \( \alpha \) is a trivial unit. \( \text{Q.E.D.} \)

We are now ready to prove the main result of this section. This is Proposition 1 in May [3].

**Proposition 2.2.2.** (May [3]). \( A(RG) = RG_0 + N(RG) \).

**Proof.** As observed above, \( RG_0 + N(RG) \subseteq A(RG) \). So, suppose \( \alpha \in A(RG) \). Then \( \alpha \in RG_1 \) where \( G_1 \) is a finitely generated subgroup of \( G \). Write \( G_1 = H \oplus F \) where \( H \) is finite and \( F \) is free. Put \( S = RH \) and view \( RG_1 \) as \( SF \). We need to show that \( \alpha \in S + N(SF) \).

Let \( \phi: SF \to (S/N(S))F \) be the natural map. Then \( \phi(\alpha) \) is integral over \( S/N(S) \) and so \( \phi(\alpha) \in S/N(S) \) by Lemma 2.2.1(2). Say \( \phi(\alpha) = s + N(S) \), \( s \in S \). Then \( \alpha - s \in \ker(\phi) = N(S)F \subseteq N(SF) \) and so \( \alpha \in S + N(SF) \) as desired. \( \text{Q.E.D.} \)

**Corollary.** (May [3]). If \( R \) is an integral domain with \( \text{char}(R) = 0 \) and if \( G \) is an abelian group then \( N(RG) = 0 \). Thus \( A(RG) = RG_0 \).

**Proof.** Suppose \( \alpha \in N(RG) \). Then \( \alpha \in RG_1 \) where \( G_1 \) is a finitely generated subgroup of \( G \). Write \( G_1 = H \oplus F \) where \( H \) is finite and \( F \) is free. Choose an algebraically closed field \( k \) containing \( R \). Then \( \text{char}(k) = 0 \) and it follows from Maschke's Theorem that
kH \cong k \times \ldots \times k \text{ with } |H| \text{ factors (see Pierce [1] for a reference). Thus } kH \text{ is reduced and so } (kH)F \cong kG_1 \text{ is reduced by Lemma 2.2.1(2). Therefore } RG_1 \text{ is reduced and } \alpha = 0. \quad \text{Q.E.D.}

§3. Units of Group Algebras

In this section we will analyze the structure of $RG^*$, the group of units of $RG$, when $G$ is an abelian group. Looking forward to our applications in Chapter 3, we will be especially interested in the case where $R$ is indecomposable. All of these results can be found in the papers May [3] and May [4]. However, in some cases we are able to simplify the original proofs given by May.

Let $R$ be any ring and suppose $G$ is an abelian group. Let $U(RG) = \{ \alpha \in RG^* | \text{aug}(\alpha) = 1 \}$. Thus $U(RG)$ is a subgroup of the full group of units $RG^*$. Our first Lemma shows that to study $G$ as a subgroup of $RG^*$ it suffices to study $U(RG)$. A reference for this elementary fact is May [3].

Lemma 2.3.1. Let $G$ be an abelian group. Then $RG^* = R^* \oplus U(RG)$ where $R^*$ is the group of units of $R$.

Proof. Clearly $R^* \cap U(RG) = \{1\}$ and $R^* \cdot U(RG) \subseteq RG^*$. If $\alpha \in RG^*$ then $\text{aug}(\alpha) \in R^*$ and

$\alpha = (\text{aug}(\alpha))(\text{aug}(\alpha)^{-1}\alpha) \in R^* \cdot U(RG). \quad \text{Q.E.D.}$
If $G$ is an abelian group let $V(RG) = \{\alpha \in A(RG)^* \mid \text{aug}(\alpha) = 1\}$. Then $V(RG)$ is a subgroup of $U(RG)$ with $G \cap V(RG) = G_0$. Our first goal is to prove a result proved by May in [3]; namely, if $R$ is an integral domain with $\text{char}(R) = 0$ and if $G_R$ is trivial then $V(RG)_0 = G_0$. The proof we give simplifies the original proof given by May. We first need the following standard Lemma (for a reference see Sehgal [1], Proposition I.1.1).

**Lemma 2.3.2.** Let $K$ be a field with $\text{char}(K) = 0$ and let $\mathbb{Q}$ be the field of rational numbers regarded as a subfield of $K$. Suppose $G$ is a finite group (not necessarily abelian). If $\alpha = \sum_{g \in G} \alpha(g)g \in KG$ is a nontrivial idempotent then $\alpha(e) \in \mathbb{Q}$ and $0 < \alpha(e) < 1$.

**Proof.** Regard $KG$ as a $K$-space with basis consisting of the elements of $G$. For $x \in KG$ let $R_x: KG \to KG$ be the linear transformation given by $R_x(y) = xy$ for all $y \in KG$. Then $R_\alpha = \sum_{g \in G} \alpha(g)R_g$ and the matrix representing $R_g$ has all zeros along its diagonal unless $g = e$. Thus $\text{tr}(R_\alpha) = \alpha(e) \cdot |G|$ where tr denotes the trace of linear transformations. Since the eigenvalues of $R_\alpha$ are all 0 or 1, $\text{tr}(R_\alpha)$ is a nonnegative integer no bigger than $|G|$. Moreover, $\text{tr}(R_\alpha) = 0$ (respectively, $|G|$) if and only if $\alpha = 0$ (respectively, 1). Therefore $\alpha(e) = \text{tr}(R_\alpha)/|G| \in \mathbb{Q}$ and $0 < \text{tr}(R_\alpha)/|G| < 1$. Q.E.D.
Corollary (Sehgal [1]). Let $K$ be a field with $\text{char}(K) = 0$ and let $G$ be a finite group (not necessarily abelian). Suppose $\alpha$ is a nontrivial idempotent of $KG$ and write $\alpha = \sum \alpha(g)g$ where $\alpha(e) = r/s$ with $r, s \in \mathbb{Z}$ and relatively prime. If $p$ is a prime number dividing $s$ then the $p$-Sylow subgroup of $G$ is nontrivial.

Proof. From the proof of the Lemma we see that $r/s = \text{tr}(R)/|G|$ and so $s$ divides $r \cdot |G|$. Therefore $s$ divides $|G|$ and the result follows from Sylow's Theorem.

Q.E.D.

We now come to the first main result of this section. The original proof in May [3] involved a nontrivial use of some machinery from algebraic number theory; e.g., ideal theory and the finiteness of class number. The proof we give here is, with only slight modifications, the proof given by Sehgal (see Sehgal [1], Theorem II.1.1). This proof not only shortens the original proof of May, but the most sophisticated ideas employed involve only an application of Maschke's Theorem and some basic facts about the norm from $\mathbb{Q}(\rho)$ to $\mathbb{Q}$, $\rho$ a primitive root of unity.

Theorem 2.3.3. (May [3]). Let $R$ be an integral domain with $\text{char}(R) = 0$ and suppose $G$ is an abelian group with $G_R$ trivial. Then every unit of finite order in $RG$ is a trivial unit. Hence, $V(RG)_0 = G_0$. 
Proof. By the Corollary to Proposition 2.2.2 every unit of finite order must lie in $RG_0$ so we may assume that $G$ is finite. Let $F$ be an algebraic closure of the quotient field of $R$. By Maschke's Theorem, $FG = F\beta_1 \oplus \ldots \oplus F\beta_t$ where $\{\beta_1, \ldots, \beta_t\}$ is an orthogonal set of idempotents of $FG$ with $t = |G|$ and $\sum \beta_i = 1$. (Here $\oplus$ denotes the internal direct product of ideals.) Since the Theorem is trivial if $|G| = 1$ we may assume that $t > 1$ and that each $\beta_i$ is nontrivial.

Suppose $\alpha = \sum g \in RG$ is a unit of finite order. Say $\alpha^n = 1$. We may assume $\alpha(e) \neq 0$. (Otherwise replace $\alpha$ by $\beta = g^{-1}\alpha$ where $\alpha(g) \neq 0$ and observe that $\beta$ is a unit of finite order if and only if $\alpha$ is. Moreover, $\beta$ is a trivial unit if and only if $\alpha$ is.) We want to show that $\alpha = \alpha(e)$.

Write $\alpha = \rho_1\beta_1 + \ldots + \rho_t\beta_t$ where $\rho_i \in F$. Then $\rho_i^n = 1$ and $\alpha(e) = \rho_1\beta_1(e) + \ldots + \rho_t\beta_t(e)$. By Lemma 2.3.2 each $\beta_i(e) \in \mathcal{O}$ and so $\alpha(e) \in K = \mathcal{O}(\rho)$, $\rho$ a primitive $n$-th root of unity. Since every automorphism of $K$ is induced by $\rho \mapsto \rho^k$ for a suitable positive integer $k$, every conjugate of $\alpha(e)$ is of the form $\rho_1^{k}\beta_1(e) + \ldots + \rho_t^{k}\beta_t(e) = (\alpha^k)(e) \in R$. Thus if $N$ denotes the norm from $K$ down to $\mathcal{O}$ then $N(\alpha(e)) \in R \cap \mathcal{O}$.

For each $i$ we can apply Lemma 2.3.2 and write $\beta_i(e) = r_i/s_i$, $r_i$ and $s_i$ relatively prime positive
integers with $\sum (r_i/s_i) = 1$. Then,

$$\alpha(e) = \sum (r_i/s_i) \rho_i \quad (*)$$

and

$$s \cdot \alpha(e) = \prod r'_i \rho_i \quad (**)$$

where $s = \prod s_i$ and each $r'_i$ is a positive integer with $\sum r'_i = s$.

Applying $N$ to both sides of (**) and setting

$k = [K:\emptyset]$ we see that $s^k N(\alpha(e))$ is a product of elements of the form $\prod r'_i \rho_i^k$, $k$ a positive integer. Thus $s^k N(\alpha(e))$ is an algebraic integer (i.e., algebraic over $\mathbb{Z}$) and rational. Therefore $s^k N(\alpha(e)) \in \mathbb{Z}$ and $N(\alpha(e)) \in \mathbb{R}$. Since a prime number dividing $s^k$ divides some $s_i$ we apply the Corollary to Lemma 2.3.2 and the hypothesis that $G_R$ is trivial to conclude that no prime divisor of $s^k$ inverts in $R$. Consequently $s^k$ does not invert in $R$ and so $N(\alpha(e)) \in \mathbb{Z}$.

Considering $\alpha(e) \in K = \emptyset(\rho)$ as an element of $\mathfrak{c}$ and taking absolute values of both sides of (*) we see that $|\alpha(e)| \leq \sum (r_i/s_i) = 1$. Likewise, every conjugate of $\alpha(e)$ has absolute value less than or equal to 1. Thus

$0 < |N(\alpha(e))| \leq 1$ and so $|N(\alpha(e))| = 1$ since $N(\alpha(e)) \in \mathbb{Z}$. 
We conclude that $|\alpha(e)| = 1$. Thus, $1 = |\alpha(e)|$
$= \sum (r_i/s_i) \rho_i \leq \sum (r_i/s_i) \rho_i = 1$; so, equality holds for
the triangle inequality. We must then have $\rho_1 = \ldots = \rho_t$
and consequently $\alpha(e) = \rho_1 = \sum \rho_i \beta_i = \alpha$ as desired. Q.E.D.

Our next task is to show that if $R$ is an integral
domain with $\text{char}(R) = 0$ and if $G$ is an abelian group with
$G_R$ trivial then $G_0$ is a direct summand of $V(RG)$. By
Theorem 2.3.3 this is the same as showing that $V(RG)$ is a
split group. To show this we need the following splitting
criterion of May [4]: An abelian group $V$ splits if and
only if there exists an ascending sequence $V_1 \subseteq \ldots \subseteq V_n$
$\subseteq \ldots$ of subgroups of $V$ such that
(1) $\bigcup \{V_n | n \geq 1 \} = V$;
(2) $(V_n)_0$ is a bounded group for every $n$;
(3) $(V/V_n)_0 = (V_0/V_n)/V_n$ for every $n$.
For a streamlined proof of this result see Fuchs [2], page
188.

Theorem 2.3.4. (May [4]). Let $R$ be an integral
domain with $\text{char}(R) = 0$ and suppose $G$ is an abelian group
with $G_R$ trivial. Then, $G_0$ is a direct summand of $V(RG)$.

Proof. Let $V = V(RG)$ and choose a sequence
of subgroups $G_1 \subseteq \ldots \subseteq G_n \subseteq \ldots$ such that
$G_0 = \bigcup \{G_n | n \geq 1 \}$ and $G_n$ is bounded for every $n$. For
each n let $\phi_n : R(G_0/G_n)$ be the natural map.

By the Corollary to Proposition 2.2, $V \subseteq R(G_0)$ so each $\phi_n$ restricts to a group-homomorphism $V \rightarrow V(R(G_0/G_n))$ with kernel $V_n$. Then $V_1 \subseteq \cdots \subseteq V_n \subseteq \cdots$ is a sequence of subgroups of $V$ with $V = \bigcup \{ V_n | n \geq 1 \}$. Moreover, using Theorem 2.3.3 it is not too hard to see that $(V_n)_0$ is bounded and $(V/V_n)_0 = (V_0/V_n)/V_n$ for all $n$. By May's splitting criterion $V_0 = G_0$ is a direct summand of $V$.

Q.E.D.

In Chapter 3 we will generalize Theorem 2.3.4 to indecomposable rings of characteristic 0.

Since both $G$ and $V(RG)$ are subgroups of $U(RG)$ one might expect that the structure of $U(RG)$ is somewhat determined by $G$ and $V(RG)$. This is indeed the case if $R$ is indecomposable and $G$ is an abelian group with $G_R$ trivial. Our next goal will be to prove Theorem 2.3.10 which is due to May: If $R$ is indecomposable and if $G$ is an abelian group with $G_R$ trivial then $U(RG)$ is the pushout of $G$ and $V(RG)$ by $G_0$ (i.e., $U(RG) = G \cdot V(RG)$ and $G \cap V(RG) = G_0$).

If $R$ is indecomposable it will be convenient to have some information about the idempotents of $RG$. In Coleman [1] it was shown that if $R$ is an integral domain and if $G$ is a finite group (not necessarily abelian) then $RG$
contains nontrivial idempotents if and only if a prime in \( \text{inv}(R) \) divides \(|G|\). In May [3] this result was generalized for abelian groups. May showed that if \( G \) is an abelian group and if \( R \) is indecomposable then \( RG \) is indecomposable if and only if \( G_R \) is trivial (May [3], Proposition 3). We will not need the full force of this last result but instead we will use the weaker result proved in our Proposition 2.3.7: If \( R \) is indecomposable and if \( G \) is an abelian group with \( G_R \) trivial then \( RG_0 \) is indecomposable.

If \( S \) is any ring let \( S^n \) denote the direct product of \( n \) copies of \( S \). Let \( G \) be a finite abelian group with \(|G| = n \) and let \( R \) be an integral domain with \( 1/n \in R \) and \( \rho_n \in R \) where \( \rho_n \) is a primitive \( n \)-th root of unity. In May [3] it is observed that \( RG \) and \( R^n \) are isomorphic as \( R \)-algebras. This generalizes a result proved in Higman [1]. An explicit description of this isomorphism is known but this description will not be needed and is omitted.

We will need the following technical Lemma whose proof is essentially contained in the proofs of Lemmas 3 and 7 in May [3].

**Lemma 2.3.5.** Suppose \( R \) is an integral domain, \( M \) a prime ideal of \( R \) and \( G \) a finite abelian group. Let \( \psi: RG \to (R/M)G \) be the natural map. Then:

1. \( \psi \) is injective when restricted to the set of idempotents of \( RG \).
(2) If $p \in \text{inv}(R)$ then $\psi$ is injective when restricted to $(RG^*)_p$ the $p$-primary component of $RG^*$.

Proof. Write $G = G_1 \oplus G_2$ where $\text{supp}(G_1) \subseteq \text{inv}(R)$ and $\text{supp}(G_2) \cap \text{inv}(R) = \emptyset$. Set $n = |G_1|$ and $R_1 = R[\zeta_n]$ where $\zeta_n$ is a primitive $n$-th root of unity over $R$. Since $R_1$ is integral over $R$, $\text{inv}(R_1) = \text{inv}(R)$. Moreover, if $M_1$ is a prime ideal of $R_1$ lying over $M$ then $\psi$ extends to the natural map $\psi_1: R_1 G \to (R_1/M_1) G$. So, if $\psi_1$ is injective on the set of idempotents of $R_1 G$ or injective on $(R_1 G^*)_p$ then the corresponding result holds for $\psi$. Thus, it does no harm to assume that $\zeta_n \in R$. Note further that $1/n \in R$.

From the discussion preceding the Lemma we see that $R G_1 \cong R^n$ as $R$-algebras and so $R G \cong (R G_1) G_2 \cong (R^n)(G_2) \cong (R G_2)^n$. Moreover, if we regard $\psi$ as defined on $(R G_2)^n$ then $\psi$ is equivalent to the natural map $(R G_2)^n \to ((R/M) G_2)^n$ induced by $R G_2 \to (R/M) G_2$. We now consider the two assertions of the Lemma separately:

(1) If $\alpha = (\alpha_1, \ldots, \alpha_n) \in (R G_2)^n$ is idempotent then each $\alpha_i$ is idempotent and Coleman's Theorem (Coleman [1]) implies that each $\alpha_i$ is 0 or 1. Thus (1) follows.

(2) If $\alpha = (\alpha_1, \ldots, \alpha_n) \in ((R G_2)^n)_p^*$ has order $p^r$ then $\alpha_i^{p^r} = 1$, $1 \leq i \leq n$. We claim that $\alpha_i \in R$ for all $i$. 
If \( \text{char}(R) = 0 \) then each \( \alpha_i \) is a trivial unit by Theorem 2.3.3. Therefore since \( p \) does not divide \( |G_2| \), each \( \alpha_i \) must lie in \( R \). If \( R \) has prime characteristic \( q \) then \( G_2 \) must be a \( q \)-group, say \( |G_2| = q^t \), and \( q \neq p \).

Also, it is easily seen that \( \alpha_i^{q^t} \in R \) for all \( i \). Select integers \( a \) and \( b \) such that \( aq^t + bp^r = 1 \). Then

\[
\alpha_i = (\alpha_i^{q^t})^a(\alpha_i^{p^r})^b = (\alpha_i^{q^t})^a \in R. \quad \text{Thus } \alpha_i \in R \text{ for all } i \]
as claimed.

Consequently, \( (RG_2)_p^* = R^*_p \) and so \( ((RG_2)^*)^n = ((RG_2)_p^*)^n \)

\( = (R^*_p)^n \). Thus \( \psi \) will be injective on \( ((RG_2)_p^*)^n \) if and only if the natural map \( \phi: R + R/M \) is injective on \( R^*_p \).

Let \( \rho_p \) be a primitive \( p \)-th root of unity over \( R \). If \( \rho_p \not\in R \) then \( R^*_p \) is trivial and so \( \phi|_{R^*_p} \) is injective.

So, we may assume that \( \rho_p \in R \).

If \( R \) has prime characteristic \( q \) then \( q \neq p \) since \( p \in \text{ inv}(R) \). Thus in all cases the polynomial \( x^{p^2} - 1 \)

has \( p \) distinct roots over \( R \). Thus \( \rho_p \neq 1 \) and \( \rho_j \),

\( 1 \leq j \leq p - 1 \), are the roots of \( f(X) = x^{p^2} - 1 + \ldots + X + 1 \)

\( \in R[X] \). Hence \( f(X) = \prod_j (X - \rho_j^p) \). Therefore \( p = f(1) \)

\( = \prod_j (1 - \rho_j^p) \) and so \( 1 - \rho_p \) divides \( p \) in \( R \). Since \( p \in \text{ inv}(R) \), \( 1 - \rho_p \in R^* \). In particular, \( 1 - \rho_p \not\in M \). This implies that \( \phi|_{R^*_p} \) is injective as required.

Q.E.D.

We will find the next Lemma useful in dealing with group algebras over indecomposable rings.
Lemma 2.3.6. (May [3]). Suppose \( R \) is indecomposable with finitely many minimal prime ideals. Suppose the set of minimal primes is partitioned into two nonempty disjoint sets \( T_1 \) and \( T_2 \). Then there exist a maximal ideal \( M \) of \( R \) and minimal primes \( P \in T_1, Q \in T_2 \) such that both \( P \) and \( Q \) are contained in \( M \).

Proof. Let \( I \) (respectively, \( J \)) be the intersection of the ideals in \( T_1 \) (respectively, \( T_2 \)). Then \( I \cap J = N(R) \). Note that \( I + J \neq R \) for if \( I + J = R \) then the indecomposable ring \( R/N(R) \) would decompose as \( I/N(R) + J/N(R) \). Thus there exists a maximal ideal \( M \) of \( R \) with \( I + J \subseteq M \). Since \( M \) is prime and since \( I \) is the intersection of the ideals in \( T_1 \), some \( P \in T_1 \) is contained in \( M \). Likewise, \( Q \subseteq M \) for some \( Q \in T_2 \). Q.E.D.

We will use the terminology of May [3] and call Lemma 2.3.6 the partition principle.

We are now ready to prove what we need for handling the idempotents of \( RG, R \) indecomposable. This is a special case of the Corollary to Proposition 3 in May [3].

Proposition 2.3.7. Suppose \( R \) is indecomposable and \( G \) is an abelian group with \( G_R \) trivial. Then \( RG_0 \) is indecomposable.
Proof. Suppose \( \alpha \in R G_0 \) is idempotent. Without loss of generality we may assume that \( G_0 \) is finite and that \( R \) is finitely generated. Since \( R \) is Noetherian it has only finitely many minimal prime ideals (see Kaplansky [2]). Say \( S = \{P_1, \ldots, P_m\} \) is the set of minimal prime ideals of \( R \). We will show that \( \alpha = 0 \) or \( 1 \) by induction on \( |\text{supp} (G)| \). Of course the result is trivial if \( |\text{supp} (G)| = 0 \).

Suppose first that \( |\text{supp} (G)| = 1 \), say \( \text{supp} (G) = \{p\} \). Since \( p \not\in \text{inv}(R) \), there exists at least one minimal prime ideal, say \( P_1 \), with \( p \not\in \text{inv}(R/P_1) \). Let \( \tilde{\alpha} : R G_0 \rightarrow (R/P_1)G_0 \times \cdots \times (R/P_m)G_0 \) be the natural map. Then \( \tilde{\alpha}(\alpha) = (\alpha_1, \ldots, \alpha_m) \) with \( \alpha_i \in (R/P_i)G_0 \) idempotent. Since \( \text{supp} (G) \cap \text{inv}(R/P_1) = \emptyset \), Coleman's Theorem (Coleman [1]) implies that \( \alpha_1 = 0 + P_1 \) or \( \alpha_1 = 1 + P_1 \). In particular, \( \alpha_1 \in R/P_1 \).

Let \( T_1 = \{P_i \in S | \alpha_i \in R/P_i \} \) and \( T_2 = S - T_1 \). We claim that \( T_2 \) is empty. If not, the partition principle gives minimal prime ideals \( P_r \in T_1 \) and \( P_S \in T_2 \) and a maximal ideal \( M \) of \( R \) with \( P_r, P_S \subseteq M \). Let \( \psi_j : (R/P_j)G_0 \rightarrow (R/M)G_0 \) (\( j = r, s \)) be the natural maps. Then \( \psi_s(\alpha_s) = \psi_r(\alpha_r) \in R/M \), a field. By Lemma 2.3.5(1), \( \psi_s \) is injective on the set of idempotents of \( (R/P_s)G_0 \). Thus \( \alpha_s = 0 + P_s \) or \( 1 + P_s \). In particular, \( \alpha_s \in R/P_s \), a
contradiction. So $T_2$ is empty as claimed. Therefore, for all $i$, either $a_i = 0 + P_i$ or $a_i = 1 + P_i$.

Suppose $a_i = 0 + P_i$ and $a_j = 1 + P_j$ for some $i$ and $j$. By the partition principle there exist minimal prime ideals $P_k$ and $P_\ell$ and a maximal ideal $M$ such that $a_k = 0 + P_k$, $a_\ell = 1 + P_\ell$, and $P_k, P_\ell \subseteq M$. But the images of $a_k$ and $a_\ell$ in $(R/M)G_0$ are the same so $0 + M = 1 + M$, a contradiction. Therefore either $a_i = 0 + P_i$ for all $i$ or $a_i = 1 + P_i$ for all $i$.

If $a_i = 0 + P_i$ for all $i$ then all coefficients of $a$ lie in $P_1 \cap \ldots \cap P_m = N(R)$. Thus $a$ is a nilpotent idempotent and so $a = 0$. If $a_i = 1 + P_i$ for all $i$ then $a = 1 + \beta$ where $\beta \in N(R)G_0 \subseteq N(RG_0)$ and so $a$ is an idempotent unit. Thus $a = 1$. This completes the proof of the Proposition if $|\text{supp}(G)| = 1$.

Now suppose $|\text{supp}(G)| > 1$ and select $q \in \text{supp}(G)$. Set $H = \bigoplus_{p \neq q} G_p$ the sum being taken over all primes $p$ different from $q$. By induction, $RH$ is indecomposable and since $RH$ is integral over $R$, $\text{inv}(RH) = \text{inv}(R)$. Therefore the case $|\text{supp}(G)| = 1$ implies that $(RH)G_q \cong RG_0$ is indecomposable.

Q.E.D.

Our next two Lemmas lead us to the useful pushout result of Theorem 2.3.10. Our Lemmas 2.3.8 and 2.3.9 are respectively Lemma 4 and Proposition 4 in May [3]. However,
our proof of Lemma 2.3.9 is shorter than the original proof for it requires only our Proposition 2.3.7 rather than the more general Proposition 3 in May [3] whose proof is longer and more difficult.

**Lemma 2.3.8.** (May [3]). Suppose $R$ is indecomposable and $F$ is a torsion free abelian group. Then $U(RF) = F \oplus (1 + N(RF)^0)$ where $N(RF)^0 = N(RF) \cap \ker(aug_R)$.

**Proof.** Clearly $1 + N(RF)^0$ is a subgroup of $U(RF)$ and $F \cap (1 + N(RF)^0) = \{1\}$. So we need only suppose that $\alpha \in U(RF)$ and show that $\alpha \in F \cdot (1 + N(RF)^0)$.

We may assume that $R$ is finitely generated with finitely many minimal prime ideals $P_1, \ldots, P_m$. Let $\phi: RF \to (R/P_1)F \times \cdots \times (R/P_m)F$ be the natural map and write $\phi(\alpha) = (\alpha_1, \ldots, \alpha_m)$. Lemma 2.2.1(3) implies that $\alpha_i = f_i \in F$, $1 \leq i \leq m$.

We claim that $f_i = f_j$ for all $i$ and $j$. If not we apply the partition principle to obtain a contradiction. So, $f_i = f \in F$ for all $i$.

Note that $\phi(\alpha - f) = 0$ and so each coefficient of $\beta = \alpha - f$ is in $P_1 \cap \cdots \cap P_m = N(R)$. Thus $\beta \in N(RF)^0$ and $\alpha = f(1 + f^{-1}\beta) \in F \cdot (1 + N(RF)^0)$. Q.E.D.

**Lemma 2.3.9.** (May [3]). Suppose $R$ is indecomposable and $G$ is an abelian group with $G_R$ trivial. Let
\( \phi: RG \to R(G/G_0) \) be the natural map. If \( \alpha \in U(RG) \) and
\( \phi(\alpha) = 1 \) then \( \alpha \in V(RG) \).

**Proof.** We may assume that \( G \) is finitely generated and write \( G = G_0 \oplus F \) where \( G_0 \) is finite and \( F \) is free. By Proposition 2.3.7, \( S = RG_0 \) is indecomposable. Thus Lemmas 2.3.1 and 2.3.8 imply that \( \alpha \) can be written uniquely as \( \alpha = sf(l + \beta) \) where \( s \in S^* \), \( f \in F \), \( \beta \in N(SF)^0 \).

Since \( R(G/G_0) \) is naturally isomorphic to \( RF \) we may regard \( \phi \) as the map \( SF = (RG_0)F + RF \subseteq SF \) induced by \( \text{aug}_R:S \to R \subseteq S \). Hence \( 1 = \phi(\alpha) = \text{aug}_R(s) \cdot f(l + \phi(\beta)) \).

This last expression must be the unique factorization of \( 1 \) in \( S^* \oplus F \oplus (1 + N(SF)^0) \). In particular, \( f = e \). Thus \( \alpha = s(l + \beta) \) is integral over \( S \) and hence over \( R \). Q.E.D.

We can now prove the desired pushout result.

**Theorem 2.3.10.** (May [3].) Suppose \( R \) is indecomposable and \( G \) is an abelian group with \( G_R \) trivial. Then \( U(RG) \) is the pushout of \( G \) and \( V(RG) \) by \( G_0 \).

**Proof.** Clearly \( G \cap V(RG) = G_0 \) and \( G \cdot V(RG) \subseteq U(RG) \). So, we need only suppose that \( \alpha \in U(RG) \) and show that \( \alpha \in G \cdot V(RG) \).

Let \( \phi: RG \to R(G/G_0) \) be the natural map. Then \( \phi(\alpha) \in U(R(G/G_0)) \) and \( U(R(G/G_0)) = (G/G_0) \oplus (1 + N(R(G/G_0))^0) \)
by Lemma 2.3.8. Write \( \phi(\alpha) = (gG_0)(1 + \beta) \) where \( g \in G \) and \( \beta \in N(R(G/G_0)) \). By Lemma 2.2.1(1) all coefficients of \( \beta \) are in \( N(R) \) so there exists \( \beta \in N(RG)_0 \) with \( \phi(\beta) = \beta \). Thus \( \alpha g^{-1}(1 + \beta)^{-1} \in U(RG) \) and \( \phi(\alpha g^{-1}(1 + \beta)^{-1}) = 1 \). By Lemma 2.3.9, \( \alpha g^{-1}(1 + \beta)^{-1} \in V(RG) \). Since \( 1 + \beta \) is also in \( V(RG) \) we conclude that \( \alpha \in G \cdot V(RG) \) as required.

Q.E.D.

By combining Theorems 2.3.4 and 2.3.10 we may conclude that \( G \) is a direct summand of \( U(RG) \) if \( G_R \) is trivial and if \( R \) is an integral domain of characteristic 0. In Chapter 3 we will generalize this to indecomposable rings of characteristic 0. This will be the key to showing that such rings satisfy the Isomorphism Theorem.

Let \( R \) be an indecomposable ring of characteristic 0 and suppose \( G \) is an abelian group. Our final goal in this chapter is to give a characterization of the kernel of the natural map \( RG \to R(G/G_R) \) in terms of algebraic invariants of \( RG \). We will see that this kernel is determined by the structure of \( V(RG)_0 \). Our characterization is a corollary of a result in May [3]: If \( G_R \) is trivial and \( p \in \text{inv}(R) \) then \( V(RG)_p \) is trivial. May proved this result with no restriction on \( \text{char}(R) \); however, the result is easier to prove if \( \text{char}(R) = 0 \) and this weaker formulation is all that we need. We first prove a special case:
Lemma 2.3.11. Let \( R \) be an indecomposable ring with \( \text{char}(R) = 0 \) and suppose \( G \) is a finite abelian group with \( G_R \) trivial. If \( p \in \text{inv}(R) \) then \( V(RG)_p \) is a trivial group.

Proof. Suppose \( \alpha \in V(RG)_p \) with \( \alpha^p = 1 \). It suffices to show that \( \alpha = 1 \). We may assume that \( R \) is finitely generated with minimal prime ideals \( P_1, \ldots, P_m \). Let \( \phi: RG \to (R/P_1)G \times \cdots \times (R/P_m)G \) be the natural map and set \( \phi(\alpha) = (\alpha_1, \ldots, \alpha_m) \). We now argue by induction on \( |\text{supp}(G)| \). Of course if \( |\text{supp}(G)| = 0 \) the result is trivial.

If \( |\text{supp}(G)| = 1 \) then Lemma 2.1.3 implies that there exists a minimal prime ideal \( P \) of \( R \) with \( \text{char}(R/P) = 0 \) and \( \text{supp}(G) \cap \text{inv}(R/P) = \emptyset \). Without loss of generality we may assume that \( P = P_1 \). Since \( \alpha_1 \in (R/P_1)G \) is a unit of finite order, it must be a trivial unit by Theorem 2.3.3. Since \( \alpha_1 \) has order dividing \( p \) and augmentation \( 1 + P_1 \) we conclude that \( \alpha_1 = (1 + P_1)e \) since \( G_p \) is trivial.

We claim that \( \alpha_i = (1 + P_i)e \) for all \( i, 1 \leq i \leq m \). If not, the partition principle supplies minimal prime ideals \( P_r \) and \( P_s \) and a maximal ideal \( M \) containing both \( P_r \) and \( P_s \) with \( \alpha_r = (1 + P_r)e \) and \( \alpha_s \neq (1 + P_s)e \). Let \( \psi_j: (R/P_j)G \to (R/M)G \) (\( j = r, s \)) be the natural maps. Then \( \psi_s(\alpha_s) = \psi_r(\alpha_r) = (1 + M)e \). Lemma 2.3.5(2) now implies that
\[ \alpha_i = (1 + P_i)e, \] a contradiction. Therefore \( \alpha_i = (1 + P_i)e \)
for all \( i \) as claimed.

Consequently \( \alpha = 1 + \beta \) (\( \beta \in RG \)) where the coefficients of \( \beta \) lie in \( P_1 \cap \ldots \cap P_m = N(R) \). Thus \( \beta \) is nilpotent and \( 1 = \alpha^p = (1 + \beta)^p = 1 + p\beta + \ldots + \beta^p = 1 + \beta(p + \gamma) \) where \( \gamma \in N(RG) \). Hence \( \beta(p + \gamma) = 0 \). Since \( p \in \text{inv}(R) \), \( p + \gamma \) is a unit and so \( \beta = 0 \). Therefore \( \alpha = 1 \) proving the Lemma when \( |\text{supp}(G)| = 1 \).

If \( |\text{supp}(G)| > 1 \) write \( G = G_1 \oplus G_2 \) where \( \text{supp}(G_1) \cap \text{supp}(G_2) = \emptyset \) and \( |\text{supp}(G_2)| = 1 \). Set \( S = RG_1 \) and view \( RG \) as \( SG_2 \). Note that \( 1/p \in S \) and \( \text{char}(S) = 0 \). Moreover, \( \text{inv}(R) = \text{inv}(S) \) since \( S \) is integral over \( R \) and so \( S \) is indecomposable by Proposition 2.3.7.

Since \( |\text{supp}(G_2)| = 1 \), \( V(SG_2)_p \) is trivial by what was shown above. But \( U(SG_2) = V(SG_2) \) since \( G_2 \) is torsion. So, \( (SG_2)_p^* = S_p^* \oplus V(SG_2)_p \) by Lemma 2.3.1. Thus \( (SG_2)_p^* = S_p^* \oplus V(SG_2)_p \) and so \( \alpha \in (RG_1)_p^* \). Since \( \text{aug}_R(\alpha) = 1 \)
we have \( \alpha \in V(RG_1)_p \). But \( V(RG_1)_p \) is trivial by induction and so \( \alpha = 1 \).

Q.E.D.

As mentioned above, the following result was shown in May [3] without any restriction on \( \text{char}(R) \).

**Proposition 2.3.12.** Let \( R \) be an indecomposable ring with \( \text{char}(R) = 0 \) and suppose \( G \) is an abelian group...
with \( G_R \) trivial. If \( p \in \text{inv}(R) \) then \( V(RG)_p \) is a trivial group.

**Proof.** Suppose \( \alpha \in V(RG)_p \) with \( \alpha^p = 1 \). We need only show that \( \alpha = 1 \). Without loss of generality we may assume that \( G \) is finitely generated and write \( G = G_0 \oplus F \) where \( G_0 \) is finite and \( F \) is free. Set \( S = RG_0 \) and view \( RG \) as \( SF \).

By Proposition 2.3.7 \( S \) is indecomposable and so \( SF^* = S^* \oplus F \oplus (1 + N(SF)^0) \) by Lemmas 2.3.1 and 2.3.8. Thus \( (SF)_p^* = S^*_p \oplus (1 + N(SF)^0)_p \) and \( \alpha \) may be written uniquely as \( \alpha = \beta(1 + \gamma) \) with \( \beta \in S^*_p \) and \( \gamma \in N(SF)^0 \). Since \( \alpha^p = 1 \) we must have \( (1 + \gamma)^p = 1 \) also. Hence \( 1 = (1 + \gamma)^p = 1 + py + \ldots + \gamma^p = 1 + \gamma(p + \delta) \) where \( \delta \in N(SF) \). Therefore \( \gamma(p + \delta) = 0 \).

Since \( 1/p \in S \), \( p + \delta \) is a unit and so \( \gamma = 0 \). Thus \( \alpha = \beta \in S^*_p = (RG_0)^*_p \). In fact \( \alpha \in V(RG_0)_p \) since \( \text{aug}_R(\alpha) = 1 \). But \( V(RG_0)_p \) is trivial by Lemma 2.3.11 and so \( \alpha = 1 \) as required. \( \text{Q.E.D.} \)

We now give our characterization of the kernel of the natural map \( RG \rightarrow R(G/G_R) \).

**Corollary.** Let \( R \) be indecomposable with \( \text{char}(R) = 0 \) and suppose \( G \) is an abelian group. Let \( \phi:RG \rightarrow R(G/G_R) \) be the natural map and set \( V_R = \bigoplus (V(RG)_p | p \in \text{inv}(R)) \). Then
Proof. Note that $\phi(V_R) \subseteq \bigoplus V(R(G/G_R))_p | p \in \text{inv}(R)$. But whenever $p \in \text{inv}(R)$, $V(R(G/G_R))_p$ is trivial by Proposition 2.3.12. Thus $\phi(v) = 1$ for all $v \in V_R$ and so $\{v - 1 | v \in V_R \} \subseteq \ker(\phi)$. In view of Lemma 2.1.2 the proof is now complete if one observes that $\{g - 1 | g \in G_R \} \subseteq \{v - 1 | v \in V_R \}$. Q.E.D.
Let $R$ be a ring and suppose $G$ is an abelian group. Since $G$ is a subgroup of $U(RG)$ one might expect that at least some of the structure of $G$ is known if one has $U(RG)$ in hand. This seems even more reasonable in view of the pushout result of Theorem 2.3.10 for indecomposable rings $R$ with $G_R$ trivial. The point of view here is that if $G$ is a pushout factor of $U(RG)$ then $G$ is not "too far" from being a direct summand of $U(RG)$.

If $G$ happens to be a direct summand of $U(RG)$ the conclusion that $R$ satisfies the Isomorphism Theorem is close at hand, at least for some types of rings. In May [3] and May [4] it was shown that $G$ is a direct summand of $U(RG)$ provided that $G_R$ is trivial and $R$ is either an integral domain of characteristic 0 or a finitely generated indecomposable ring of characteristic 0. May was then able to exploit these direct sum decompositions to show that such $R$ satisfy the Isomorphism Theorem. By generalizing this idea, we will show in Theorem 3.6 that any indecomposable ring of characteristic 0 (no restrictions on generation) satisfies the Isomorphism Theorem.
Our first result is a generalization of Theorem 2.3.4.

**Proposition 3.1.** Let $R$ be an indecomposable ring with $\text{char}(R) = 0$ and suppose $G$ is an abelian group with $G_R$ trivial. Then, $G_0$ is a direct summand of $V(RG)$.

**Proof.** Set $V = V(RG)$ and let $p_1, p_2, \ldots$ be the collection of all prime numbers not in $\text{inv}(R)$. (If no such $p_i$'s exist then $G_0$ is trivial and there is nothing to prove.) Thus $p_i \not\in \text{inv}(R)$ for all $i \geq 1$ and $\text{supp}(G) \subseteq \{p_i | i \geq 1\}$. Use Lemma 2.1.3 to select for each $i$ a prime ideal $p_i$ of $R$ such that $\text{char}(R/p_i) = 0$ and $p_i \not\in \text{inv}(R/p_i)$. Set $H_i = \bigoplus_{p \neq p_i} G_p$, the sum being taken over all prime numbers $p$ different from $p_i$. Let $\phi_i : RG \rightarrow (R/p_i)(G/H_i)$ be the natural map and set $V_i = V((R/p_i)(G/H_i))$. Note that $\phi_i(V) \subseteq V_i$ for every $i$.

Since $R/p_i$ is an integral domain with $\text{char}(R/p_i) = 0$ and since $\text{supp}(G/H_i) \cap \text{inv}(R/p_i)$ is empty, it follows from Theorem 2.3.4 that $(G/H_i)_0$ is a direct summand of $V_i$ for each $i$. Since $(G/H_i)_0 = (G/H_i)_{p_i}$ we have for each $i$ a group-homomorphism $\eta_i : V_i \rightarrow (G/H_i)_{p_i}$ that is the identity map when restricted to $(G/H_i)_{p_i}$.

For each $i$ set $\phi_{i1} = \phi_i|V$ and $\phi_{i2} = \phi_i|G_{p_i}$. Note that $\phi_{i2} : G_{p_i} \rightarrow (G/H_i)_{p_i}$ is an isomorphism and set
\( \rho_i = \phi_i^{-1} \circ \pi_i \circ \phi_i : V \rightarrow G \). Let \( \rho = \prod_i \rho_i : V \rightarrow \prod_i G_{P_i} \) be the induced map into the product. Identifying \( G_0 = \bigoplus_i G_{P_i} \) inside of \( \prod_i G_{P_i} \) in the obvious fashion we see that \( \rho \) is the identity map when restricted to \( G_0 \). Thus to complete the proof it suffices to show that \( \rho(V) \subseteq G_0 \).

Suppose \( \alpha \in V \). To show that \( \rho(\alpha) \in G_0 \) it suffices to show that \( \phi_i(\alpha) = (1 + P_i)(eH_i) \), the multiplicative identity of \( (R/P_i)(G/H_i) \), for all but finitely many \( i \).

Since \( \alpha \) is integral over \( R \), \( \alpha = r_1 g_1 + \ldots + r_n g_n + \beta (r_1, \ldots, r_n \in R; g_1, \ldots, g_n \in G_0; \beta \in N(RG)) \) by Proposition 2.2.2. Since \( \phi_i(\beta) \) is nilpotent and since \( N((R/P_i)(G/H_i)) = 0 \) by the Corollary to Proposition 2.2.2, \( \phi_i(\beta) = 0 \) for all \( i \). Thus \( \phi_i(\alpha) = (r_1 + P_i)(g_1 H_i) + \ldots + (r_n + P_i)(g_n H_i) \) for every \( i \). But each \( g_j, 1 \leq j \leq n, \) is in all but finitely many \( H_i \). Hence \( \phi_i(\alpha) = ((r_1 + \ldots + r_n) + P_i)(eH_i) \) for all but finitely many \( i \). Since \( \text{aug}(\alpha) = 1 \) we conclude that \( \phi_i(\alpha) = (1 + P_i)(eH_i) \) for all but finitely many \( i \). Q.E.D.

Note that by combining Proposition 3.1 and the pushout result of Theorem 2.3.10 we may conclude that \( G \) is a direct summand of \( U(RG) \) provided that \( R \) is indecomposable with \( \text{char}(R) = 0 \) and \( G_R \) trivial. Thus we will be close to showing that such \( R \) satisfy the Isomorphism Theorem if we
can show that the complement of $G$ in $U(RG)$ can be chosen in a "nice" way.

Let us be more precise. Suppose $R$ is as above and let $G$ and $G'$ be abelian groups with $G_R$ and $G'_R$ trivial and with $RG \cong RG'$ as $R$-algebras. By Lemma 2.1.1 we can choose a normalized $R$-algebra isomorphism $f: RG \to RG'$. If we could choose subgroups $F$ and $F'$ of $V(RG)$ and $V(RG')$ respectively with $V(RG) = G_0 \oplus F$, $V(RG') = G'_0 \oplus F'$ and $f(F) = F'$ then $G \cong U(RG)/F \cong f(U(RG))/f(F) = U(RG')/F' \cong G'$. The next three Lemmas will be used to show that for any given $f$ such $F$ and $F'$ always exist.

**Lemma 3.2.** Suppose $I$ is an ideal of the ring $R$ and let $G$ and $G'$ be abelian groups. Let $\phi: RG \to (R/I)G$ and $\phi': RG' \to (R/I)G'$ be the natural maps. If $f: RG \to RG'$ is a normalized $R$-algebra isomorphism then there exists a normalized $(R/I)$-algebra isomorphism $f': (R/I)G \to (R/I)G'$ with $f' \circ \phi = \phi' \circ f$.

**Proof.** Let $IG$ (respectively, $IG'$) be the ideal of $RG$ (respectively, $RG'$) consisting of all elements with coefficients in $I$. Clearly $\ker(\phi) = IG$, $\ker(\phi') = IG'$ and $f(IG) = IG'$. Thus $f$, $\phi$ and $\phi'$ induce, in the usual ways, $(R/I)$-algebra isomorphisms $\bar{f}: (RG)/IG \to (RG')/(IG')$, $\bar{\phi}: (RG)/(IG) \to (R/I)G$ and $\bar{\phi'}: (RG')/(IG') \to (R/I)G'$.
respectively. Set $f' = \phi' \circ \overline{f} \circ \phi^{-1}$. It is now routine to check that $f'$ has the desired properties. Q.E.D.

Lemma 3.3. Suppose $R$ is an indecomposable ring with $\text{char}(R) = 0$ and let $G$ and $G'$ be abelian groups. Suppose $\phi : RG \to R(G/G_R)$ and $\phi' : RG' \to R(G'/G'_R)$ are the natural maps. If $f : RG \to RG'$ is a normalized $R$-algebra isomorphism then there exists a normalized $R$-algebra isomorphism $f^* : R(G/G_R) \rightarrow R(G'/G'_R)$ with $f^* \circ \phi = \phi' \circ f$.

Proof. Set $V_R = \bigoplus (V(RG)_p | p \in \text{inv}(R))$ and $V'_R = \bigoplus (V(RG'_p) | p \in \text{inv}(R))$. By the Corollary to Proposition 2.3.12, $\ker(\phi)$ (respectively, $\ker(\phi')$) is the ideal of $RG$ (respectively, $RG'$) generated by $\{v - 1 | v \in V_R \}$ (respectively, $\{v' - 1 | v' \in V'_R \}$). Since $f(V_R) = V'_R$, a set of generators for $\ker(\phi)$ is carried by $f$ onto a set of generators for $\ker(\phi')$. Consequently, $f(\ker(\phi)) = \ker(\phi')$.

Let $\overline{f} : (RG)/\ker(\phi) \rightarrow (RG'/\ker(\phi')$, $\overline{\phi} : (RG)/\ker(\phi) \rightarrow R(G/G_R)$ and $\overline{\phi'} : (RG'/\ker(\phi') \rightarrow R(G'/G'_R)$ be the $R$-algebra isomorphisms induced by $f$, $\phi$ and $\phi'$ respectively. Then $f^* = \overline{\phi'} \circ \overline{f} \circ \overline{\phi}^{-1}$ is the desired normalized $R$-algebra isomorphism. Q.E.D.

Lemma 3.4. Suppose $R$ is indecomposable with $\text{char}(R) = 0$. Let $p$ be a prime number with $p \notin \text{inv}(R)$ and suppose $P$ is a prime ideal of $R$ with $p \notin \text{inv}(R/P)$ and
char $\text{char}(R/P) = 0$. Let $G$ and $G'$ be abelian groups and set

$H = \bigoplus_{q \neq p} G_q$ and $H' = \bigoplus_{q \neq p} G'_q$, the sums being taken over all prime numbers $q$ different from $p$. Suppose $\phi : RG \rightarrow (R/P)(G/H)$ and $\phi' : RG' \rightarrow (R/P)(G'/H')$ are the natural maps. If $f : RG \rightarrow RG'$ is a normalized $R$-algebra isomorphism then there exists a normalized $(R/P)$-algebra isomorphism $\overline{f} : (R/P)(G/H) + (R/P)(G'/H')$ with $\overline{f} \circ \phi = \phi' \circ f$.

**Proof.** Set $K = \bigoplus \{ G_q \mid q \in \text{inv}(R/P) \}$. Using Lemmas 3.2 and 3.3 we obtain the commutative diagram

$$
\begin{array}{ccc}
RG & \longrightarrow & (R/P)(G/K) \\
\downarrow f & & \downarrow f^* \\
RG' & \longrightarrow & (R/P)(G'/K')
\end{array}
$$

where the horizontal maps are natural and $f^*$ is a normalized $(R/P)$-algebra isomorphism.

Set $V = V((R/P)(G/K))$ and $V' = V((R/P)(G'/K'))$. Since $R/P$ is an integral domain with $\text{char}(R/P) = 0$ and since both $\text{inv}(R/P) \cap \text{supp}(G/K)$ and $\text{inv}(R/P) \cap \text{supp}(G'/K')$ are empty we conclude from Theorem 2.3.3 that $V_0 = (G/K)_0$ and $V'_0 = (G'/K')_0$. In particular, $\bigoplus_{q \neq p} V_q = \bigoplus_{q \neq p} (G/K)_q = H/K$ and $\bigoplus_{q \neq p} V'_q = \bigoplus_{q \neq p} (G'/K')_q = H'/K'$. Consequently, $f^*(V) = V'$ implies that $f^*(H/K) = H'/K'$.

The group-homomorphisms $G/K \rightarrow (G/K)/(H/K) \cong G/H$ and $G'/K' \rightarrow (G'/K')/(H'/K') \cong G'/H'$ induce natural maps.
(R/P)(G/K) + (R/P)(G/H) and (R/P)(G'/K') + (R/P)(G'/H')

with kernels I and I' respectively. By Lemma 2.1.2 I
(respectively, I') is generated by \{x - 1| x \in H/K\} (re-
respectively \{x' - 1| x' \in H'/K'\}). Since \(f^*(H/K) = H'/K'\)
we conclude that \(f^*(I) = I'\)

As in the proofs of Lemmas 3.2 and 3.3 we now obtain
a normalized (R/P)-algebra isomorphism \(\tilde{f}:(R/P)(G/H)
+ (R/P)(G'/H')\) with the desired property. Q.E.D.

We can now show that if \(R\) is indecomposable with
\(\text{char}(R) = 0\) and if \(G_R\) is trivial then \(G_0\) has comple-
ments in \(V(RG)\) with sufficiently nice properties. This
generalizes Lemma 14 in May [3].

**Proposition 3.5.** Let \(R\) be an indecomposable ring
with \(\text{char}(R) = 0\) and suppose \(G\) and \(G'\) are abelian
groups with \(G_R\) and \(G'_R\) trivial. If \(f:RG \to RG'\) is a
normalized \(R\)-algebra isomorphism then there exist internal
direct sum decompositions \(V(RG) = G_0 \oplus F\) and \(V(RG') = G'_0 \oplus F'\) with \(f(F) = F'\).

**Proof.** Set \(V = V(RG), V' = V(RG')\) and let \(p_1, p_2, \ldots\)
be the prime numbers with \(p_i \notin \text{inv}(R); i = 1, 2, \ldots\). (If no such prime numbers exist set \(F = V, F' = V'\)
and the proof is complete.) For each \(i\) use Lemma 2.1.3 to
select a prime ideal \(p_i\) of \(R\) such that \(p_i \notin \text{inv}(R/P_i)\)
and \( \text{char}(R/P_i) = 0 \). Set \( H_i = \bigoplus_{p \neq p_i} G_p \) and \( H'_i = \bigoplus_{p \neq p_i} G'_p \),
the sums being taken over all primes \( p \neq p_i \). Let \( \phi_i : RG \rightarrow (R/P_i)(G/H_i) \) and \( \phi'_i : RG' \rightarrow (R/P_i)(G'/H'_i) \) be the natural maps. Thus, for each \( i \), Lemma 3.4 provides a normalized \((R/P_i)\)-algebra isomorphism \( f_i : (R/P_i)(G/H_i) \to (R/P_i)(G'/H'_i) \) such that the diagram

\[
\begin{array}{ccc}
RG & \longrightarrow & (R/P_i)(G/H_i) \\
f \downarrow & & \downarrow f_i \\
RG' & \longrightarrow & (R/P_i)(G'/H'_i)
\end{array}
\]

commutes for every \( i \).

For each \( i \) set \( V_i = V((R/P_i)(G/H_i)) \) and \( V'_i = V((R/P_i)(G'/H'_i)) \). Note that \( f_i(V_i) = V'_i \) for every \( i \).

By Proposition 3.1 we have group-homomorphisms \( \Pi_i : V_i \to (G/H_i)_0 = (G/H_i)_0 p_i \) and \( \Pi'_i : V'_i \to (G'/H'_i)_0 = (G'/H'_i)_0 p_i \) such that \( \Pi_i \) (respectively, \( \Pi'_i \)) is the identity when restricted to \((G/H_i)_0 p_i\) (respectively, \((G'/H'_i)_0 p_i\)). By Theorem 2.3.3 \( (V_i)_0 = (G/H_i)_0 p_i \) and \( (V'_i)_0 = (G'/H'_i)_0 p_i \) and since \( f_i((V_i)_0) = (V'_i)_0 \) we have \( f_i(ker(\Pi_i)) = ker(\Pi'_i) \) for every \( i \). Finally, for each \( i \), set \( \phi_{il} = \phi_i|V \) and \( \phi'_{il} = \phi'_i|V' \). Thus for each \( i \) we have a commutative diagram
of groups and group-homomorphisms where the vertical maps are obtained by restriction.

Set $F = \cap_i \ker(\Pi_i \circ \phi_{il})$ and $F' = \cap_i \ker(\Pi'_i \circ \phi'_{il})$. If one reviews the construction of the splitting $\rho:V \rightarrow G_0$ in the proof of Proposition 3.1 it is clear that $V = G_0 \oplus F$ and $V' = G'_0 \oplus F'$. To see that $f(F) = F'$ it suffices to show that $f(\ker(\Pi_i \circ \phi_{il})) \subseteq \ker(\Pi'_i \circ \phi'_{il})$ for every $i$.

Suppose $\alpha \in \ker(\Pi_i \circ \phi_{il})$ for some $i \geq 1$. Then $\phi_{il}(\alpha) \in \ker(\Pi_i)$ which implies that $(\bar{\Pi}_i \circ \phi_{il})(\alpha) \in \ker(\Pi'_i)$ since $\bar{\Pi}_i(\ker(\Pi_i)) = \ker(\Pi'_i)$. Thus $\alpha \in \ker(\Pi'_i \circ \bar{\Pi}_i \circ \phi_{il}) = \ker(\Pi'_i \circ \phi'_{il} \circ f)$ and so $f(\alpha) \in \ker(\Pi'_i \circ \phi'_{il})$ as required. Q.E.D.

**Theorem 3.6.** Let $R$ be an indecomposable ring with $\text{char}(R) = 0$. Then, $R$ satisfies the Isomorphism Theorem.

**Proof.** Let $G$ and $G'$ be abelian groups with $RG \cong RG'$ as $R$-algebras. Use Lemma 2.1.1 to select a normalized $R$-algebra isomorphism $f:RG \rightarrow RG'$. By Lemma 3.3
there is a normalized $R$-algebra isomorphism $\bar{f}: R(G/G_R) \to R(G'/G'_R)$ so we may assume that $G_R$ and $G'_R$ are trivial and show $G \cong G'$.

By Proposition 3.5 we may select $F$ and $F'$ such that $V(RG) = G_0 \oplus F$, $V(RG') = G'_0 \oplus F'$ and $f(F) = F'$.

By Theorem 2.3.10, $U(RG)$ (respectively, $U(RG')$) is the pushout of $G$ and $V(RG)$ (respectively, $G'$ and $V(RG')$) by $G_0$ (respectively, $G'_0$). Thus $U(RG) = G \oplus F$ and $U(RG') = G' \oplus F'$. Therefore, $G \cong U(RG)/F \cong f(U(RG))/f(F) = U(RG')/F' \cong G'$. Q.E.D.

In the next chapter we will show that there exist decomposable rings of characteristic 0 which do not satisfy the Isomorphism Theorem. In fact, we shall see that even finite products of characteristic 0 integral domains fail, in general, to satisfy the Isomorphism Theorem.

Suppose $R$ satisfies the Isomorphism Theorem and $RG \cong RG'$ as $R$-algebras for abelian groups $G$ and $G'$. Thus $G/G_R \cong G'/G'_R$. If $G_R$ is trivial we then conclude that $G \cong G'/G'_R$. If $G'_R$ is also trivial we have $G \cong G'$. This raises the following question: If $G_R$ is trivial is $G'_R$ necessarily trivial? We can answer this question in the affirmative if $R$ happens to be indecomposable of characteristic 0.
Corollary. Suppose \( R \) is an indecomposable ring of characteristic 0. Let \( G \) and \( G' \) be abelian groups with \( RG \cong RG' \) as \( R \)-algebras. If \( G_R \) is trivial then \( G'_R \) is also trivial. In particular, \( G \cong G' \).

Proof. Suppose \( p \in \text{inv}(R) \). In view of Theorem 3.6 it suffices to show that \( G'_p \) is trivial. By Proposition 2.3.12, \( V(RG)_p \) is trivial. Since \( G'_p \subseteq V(RG')_p \) we conclude that \( G'_p \) is trivial by Lemma 2.1.1. Q.E.D.

We remark that the Corollary remains true if the condition that \( \text{char}(R) = 0 \) is replaced by the condition that \( R \) satisfies the Isomorphism Theorem. This follows from a Corollary of Proposition 7 in May [3] which is our Proposition 2.3.12 with the restriction on \( \text{char}(R) \) removed.
CHAPTER 4

NOT ALL RINGS SATISFY THE ISOMORPHISM THEOREM

In May [4] it was conjectured that all rings satisfy the Isomorphism Theorem. In this chapter we show that there are rings of characteristic 0 containing nontrivial idempotents that do not satisfy the Isomorphism Theorem. In fact, we show that even finite products of indecomposable rings of characteristic 0 do not necessarily satisfy the Isomorphism Theorem. In Theorem 4.6 we give necessary and sufficient conditions for such a product to satisfy the Isomorphism Theorem. In particular, it will follow that the hypothesis of indecomposable cannot be omitted from Theorem 3.6.

The crucial result needed is our Proposition 4.5. However, we first need several preliminary results. All of our first four results are known in greater generality; however, it is easier to prove only the special cases which we will need.

**Lemma 4.1.** Let $\mathbb{Q}$ be the field of rational numbers and suppose $G$ is a finite abelian group of order $n$. If $\alpha = r_1g_1 + \ldots + r_kg_k \in \mathbb{Q}G$ is idempotent then $nr_i \in \mathbb{Z}$ for all $i$, $1 \leq i \leq k$. Thus $n\alpha \in \mathbb{Z}G$. 

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Proof. Regard $\mathbb{Q}G$ as a vector space over $\mathbb{Q}$ with basis consisting of the elements of $G$. For each $x \in \mathbb{Q}G$ define a linear transformation $T_x : \mathbb{Q}G \to \mathbb{Q}G$ by $T_x(y) = xy$ for all $y \in \mathbb{Q}G$. Let $tr$ denote the trace of linear transformations and observe that $tr(T_{g_i^{-1}}) = nr_i \in \mathbb{Q}$ for each $i$.

Suppose $j$ is an integer with $1 < j < k$ and select a positive integer $m > 1$ such that $(g_j^{-1})^m = g_j^{-1}$. Thus $(g_j^{-1})^m = g_j^{-1}$ and so the eigenvalues of $T_{g_j^{-1}}$ are roots of the polynomial $x^m - x$ and so all of these eigenvalues are algebraic integers. Thus $tr(T_{g_j^{-1}})$ is an algebraic integer in $\mathbb{Q}$. Therefore $nr_j = tr(T_{g_j^{-1}}) \in \mathbb{Z}$. Q.E.D.

Our next Lemma is a special case of Lemma 2 in May [4].

**Lemma 4.2.** (May [4]). Let $p$ be a positive integer and suppose $G$ is an abelian group of order $p^n$, $n \geq 0$. Set $R = \mathbb{Z}[1/p]$. Then:

1. $RG$ has only finitely many distinct minimal prime ideals, say $P_1', \ldots, P_r'$.
2. The map $\phi : RG \to (RG/P_1') \times \cdots \times (RG/P_r')$ induced by the quotient maps is an isomorphism.

Proof. Since $R$ is Noetherian and $G$ is finite $RG$ is also Noetherian. Thus $RG$ has only finitely many distinct minimal prime ideals and (1) is shown.
QG may be regarded as the localization of RG by the nonzero elements of R. Since $p_i \cap R = 0$ for all $i$, it follows from the correspondence between prime ideals in a ring and its localization that each $p_i$ extends to a minimal prime ideal $M_i$ of $QG$ with $M_i \cap RG = p_i$. Thus the map

$$\psi: QG \rightarrow (QG/M_1) \times \cdots \times (QG/M_r)$$

extends $\phi$.

Since $QG$ is semisimple, the minimal prime ideals are precisely the maximal ideals and so each $M_i$ is maximal and $M_1 \cap \cdots \cap M_r = 0$. Thus $\psi$ is an isomorphism by the Chinese Remainder Theorem. It follows from Lemma 4.1 that all idempotents of $QG$ lie in $RG$. Thus, since $\psi$ extends $\phi$, we conclude that $\phi$ is also an isomorphism. Q.E.D.

Suppose $G$ is an abelian group and let $p$ be a prime number. For a nonnegative integer $n$ let $G^p_n$ be the subgroup given by $G^p_n = \{g^p \mid g \in G\}$. Recall that the $p$-height of an element $x \in G$ is the smallest nonnegative integer $n$, if such exists, with $x \in G^p_n = G^p_{n+1}$. If no such $n$ exists then $x \in \cap \{G^p_n \mid n \geq 0\}$ and we say that $x$ has infinite $p$-height. (See Fuchs [1] for a reference.)

Our next result is a special case of Proposition 3 in May [4]. May proves this for the case when $R$ is any integrally closed domain with $p \in \text{inv}(R)$. The proof of the special case which follows is somewhat shorter because we are able to do without some of the preliminaries needed in handling the more general situation.
Proposition 4.3. (May [4]). Let \( p \) be a prime number and set \( R = \mathbb{Z}[1/p] \). Suppose \( G \) is an abelian group with \( G_0 = G_p \) and let \( V_{p,\infty} \) be the subgroup of \( V(RG)_p \) consisting of the elements with infinite \( p \)-height. Then, \( V_{p,\infty} \) is the maximal divisible subgroup of \( V(RG)_p \).

**Proof.** We note that \( V(RG) \subseteq RG_0 \) since \( A(RG) = RG_0 \) by the Corollary to Proposition 2.2.2.

Suppose \( x \in V_{p,\infty} \). It suffices to show that \( x \) is contained in a divisible subgroup of \( V(RG)_p \). Choose \( y_i \in V(RG)_p \) such that \( y_i^p = x \) \((i = 0, 1, 2, \ldots)\) and then select a sequence \( H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots \) of finite subgroups of \( G_0 \) such that \( y_i \in RH_i \) for every \( i \). Let \( P_{i1}, P_{i2}, \ldots \) be the minimal prime ideals of \( RH_i \). By Lemma 4.2 we have an isomorphism \( RH_i \to (RH_i/P_{i1}) \times (RH_i/P_{i2}) \times \ldots \) induced by the quotient maps for each \( i \). Consequently each element of \( RH_i \) has components relative to this isomorphism. If \( \alpha \in RH_i \) let \( (\alpha)_{ij} \) denote the component of \( \alpha \) in \( RH_i/P_{ij} \).

We now construct elements \( z_0, z_1, \ldots \) of \( RG^* \) such that the subgroup of \( RG^* \) generated by \( \{z_i \mid i \geq 0\} \) is a divisible subgroup of \( V(RG)_p \) containing \( x \).

Assume that \( z_0, \ldots, z_n \) have been constructed so that \( z_0 = x, z_i \in RH_i, z_i^p = z_{i-1} \) \((1 \leq i \leq n)\) and such that whenever \( P_{ij} \) \((1 \leq i \leq n)\) lies over \( P_{0k} \) and
(x)_{0k} = 1 then \( (z_i)_{ij} = 1 \). We now show how to construct \( z_{n+1} \) satisfying the appropriate conditions.

Suppose \( P_{n+1,i} \) lies over \( P_{nj} \). If \( (z_n)_{nj} = 1 \) choose \( (z_{n+1})_{n+1,i} = 1 \). If \( (z_n)_{nj} \neq 1 \) then

\[
z_n^{p^{n+1}} = x = y_n^{p^{n+1}} \quad \text{and so if } P_{nj} \text{ lies over } P_{0k} \text{ then}
\]

\[
(y_n^{p^{n+1}})^{n+1,i} = (z_n)^{p^n} = (x)_{0k} \neq 1. \quad \text{Since all of these components are } p^r \text{-th roots of unity in an integral domain (for some } r) \text{ we conclude that } (z_n)_{nj} \text{ has a } p \text{-th root in } RH_{n+1}/P_{n+1,i}. \text{ Choose } (z_{n+1})_{n+1,i} \text{ to be such a } p \text{-th root. Thus we have chosen all components of } z_{n+1} \text{ so that } z_{n+1}^p = z_n \text{ and the condition on certain components being 1 is satisfied. It is now clear that } \{z_i|i \geq 0\} \text{ generates a divisible subgroup of } (RG*)_p. \text{ Thus to show that this subgroup is actually in } V(RG)_p \text{ it suffices to show that } \text{aug}(z_i) = 1 \text{ for all } i.

The kernel of the augmentation mapping on } RH_0 \text{ is a minimal prime ideal of } RH_0, \text{ say } P_{0k}. \text{ Moreover, the elements of augmentation 0 in } RH_i \text{ form a minimal prime ideal } P_{ij} \text{ of } RH_i \text{ lying over } P_{0k}. \text{ Since } \text{aug}(x) = 1, \ (x)_{0k} = 1. \text{ Thus } (z_i)_{ij} = 1 \text{ and } \text{aug}(z_i) = 1 \text{ as required. Q.E.D.}

The next result is a special case of Proposition 4 in May [4].
Proposition 4.4. (May [4]). Let $R$ be an integral domain with $\text{char}(R) = 0$ and suppose $G$ is an abelian group. If $V(RG)$ is a direct summand of $G \cdot V(RG)$ then $RG \cong R(G_0 \times (G/G_0))$ as $R$-algebras.

Proof. Set $V = V(RG)$ and suppose $B$ is a subgroup of $GV$ with $GV = V \oplus B$. We claim that $B$ is a basis for $RG$ viewed as an $RG_0$-algebra. Since $V \subseteq RG_0$ by the Corollary to Proposition 2.2.2 and since $G \subseteq VB$ we see that $B$ spans $RG$ as an $RG_0$-algebra. Moreover, since $B \cap G_0 = 1$ and since $B \cong GV/V \cong G/(G \cap V) = G/G_0$ it is not too hard to see that $B$ is $RG_0$-independent. Therefore $B$ is a basis for $RG$ as an $RG_0$-algebra as claimed. Thus, $RG \cong (RG_0)B$ as $R$-algebras. Also, $(RG_0)B \cong (RG_0)(G/G_0) \cong R(G_0 \times (G/G_0))$ as $R$-algebras. Q.E.D.

In May [4] Propositions 4.3 and 4.4 were used to construct two examples. In the first example it was shown that if $R$ is any ring with $\text{inv}(R)$ nonempty then there exist nonisomorphic abelian groups $G$ and $H$ with $RG \cong RH$. In the second example nonisomorphic abelian groups $G$ and $H$ were constructed so that $FG \cong FH$ as $F$-algebras for all fields $F$. The basic idea in both of these examples was to construct a nonsplit abelian group $G$ such that $V(RG)$ is a direct summand of $G \cdot V(RG)$ for the appropriate rings.
R. Then Proposition 4.4 is applied to show that the group algebras of $G$ and the split group $G_0 \times (G/G_0)$ are isomorphic. We employ this idea in proving our next result.

**Proposition 4.5.** Let $R$ be a ring and suppose $p$ is a prime number with $p \in \text{inv}(R)$. Then, there exists a countable nonsplit reduced abelian group $G$ with

1. $G_0 = G_p$;
2. $G/G_0 \cong \mathbb{Q}$, the additive group of rationals;
3. $RG \cong R(G_0 \times (G/G_0))$ as $R$-algebras.

**Proof.** Let $G$ be the abelian group generated by elements $u_i, v_{ij}, g_k, g_{qj} (i \geq 1, j \geq 1, k > 0, q$ prime $\neq p)$ subject to the relations $u_i^{p^i} = e$, $v_{ij}^{p^j} = u_i$, $g_k^{p} = g_k u_{k+1}$, $g_{q1} = g_0$, $g_{q}^{q} = g_{q}, \ell - 1$ ($i \geq 1, j \geq 1, k > 0, \ell \geq 2, q$ prime $\neq p$). We claim that $G$ has all the desired properties. We first observe that $G$ is countable since it is the homomorphic image of a countably generated free abelian group.

Let $A$ be the subgroup of $G$ generated by $(u_i, v_{ij})$ ($i \geq 1, j \geq 1$). Note that $A$ is a $p$-group. We intend to show that $A = G_0$.

We first claim that $g_0$ has infinite order in $G$.

Let $F$ be the free abelian group generated by elements $u_i', v_{ij}', g_k', g_{qj}'$ ($i \geq 1, j \geq 1, k > 0, q$ prime $\neq p$). Then
G is a homomorphic image of F in the obvious fashion. If \( g_0 \) is a torsion element of G, say \( g_0^n = e \) \((n \geq 1)\), then \( (g_0')^n \) is in the kernel of the homomorphism \( F \to G \); that is, \( (g_0')^n \) is in the subgroup of F generated by various elements of the form \( (u_i')^{p_i^i}, (v_{ij}')^{p_j^j}(u_i')^{-1}, (g_{k+1}')^{p}(g_{k+1}u_{k+1}')^{-1}, (g_{q^l})'^{p}(g_{q^m})^{-1}, (g_{q^l}')^{p}(g_{q^m}^{-1})^{-1} \) \((i \geq 1, j \geq 1, k \geq 0, \ell \geq 2, q \text{ prime} \neq p)\). Writing out what it means for \( (g_0')^n \) to be in this subgroup and using the independence of the generators for F we see that \( n = 0 \), contradicting \( n \geq 1 \). Therefore \( g_0 \) has infinite order in G as claimed.

Next observe that \( G/A \) is generated by the cosets of \( g_k \) and \( g_{q^j} \) \((k \geq 0, j \geq 1, q \text{ prime} \neq p)\). Furthermore, \( g_0A \) has infinite order in \( G/A \) and \( g_{p^{k+1}}A = g_0A \), \( g_{q^j}A = g_0A \). Consequently, \( G/A \) is rank one torsion free and divisible. Thus \( G/A \not\subset \varnothing \). Since \( A \subsetneq G_0 \) and \( G/A \) is torsion free we conclude that \( A = G_0 \). Hence (1) and (2) are shown.

Next we will show that \( G \) does not split. This part of our proof is patterned after the argument in Example 1 of May [4].

Let \( G_\infty \) be the subgroup of \( G_0 \) consisting of all elements of infinite p-height. Using an independence argument in \( F \) (as above) one sees that \( G_\infty = \bigoplus \langle u_i \rangle \mid i \geq 1 \) where \( \langle u_i \rangle \) is the (cyclic) group generated by \( u_i \). From
Now, suppose to the contrary that $G$ splits. Then there is a group-homomorphism $f: G \to G_0$ which is the identity when restricted to $G_0$. Set $w_k = f(g_k)$, $k \geq 0$.

Thus $w_{k+t}^p = w_k u_{k+1}^p u_{k+2}^p \ldots u_{k+t}^p$ for all $t \geq 1$ and $k \geq 0$. Therefore $w_k \in G_\infty$ for all $k \geq 0$.

Select $m \geq 1$ such that $w_0$ is contained in the subgroup generated by $\{u_1, \ldots, u_m\}$. Then

$$w_m^p = w_0 u_1 u_2^p \ldots u_m^{p^{m-1}}.$$ Hence $w_0 u_1 u_2^p \ldots u_m^{p^{m-1}}$ is contained in both $G_\infty^p$ and $\langle u_1, \ldots, u_m \rangle$. Consequently

$$w_0 u_1 u_2^p \ldots u_m^{p^{m-1}} = e$$ and so $u_m^{p^m} = w_0 u_1 u_2^p \ldots u_m^{p^m}$

$$= w_{m+1}^p \in G_\infty^p,$$ a contradiction. Therefore $G$ is not a split group.

Suppose $G$ is not reduced. Then $G$ contains a divisible subgroup $D$ with either $D \cong \mathbb{Z}(p^\infty)$ or $D \cong \mathbb{Q}$.

If $D \cong \mathbb{Q}$ then $G = D \bigoplus T$ for some subgroup $T$. But $G$ has torsion free rank 1 and so $T = G_0$, contradicting the fact that $G$ does not split. Consequently $D \not\cong \mathbb{Q}$. If $D \cong \mathbb{Z}(p^\infty)$ then $D \subseteq G_\infty$, again a contradiction since $G_\infty$ is a direct sum of cyclic $p$-groups and hence reduced. Therefore $G$ is reduced.

It remains to show (3). Since $1/p \notin R$ there is a natural ring-homomorphism $\mathbb{Z}[1/p] \to R$ which endows $R$ with
the structure of a \( \mathbb{Z}[1/p] \)-algebra. If we can show that 
\( (\mathbb{Z}[1/p])G \cong (\mathbb{Z}[1/p])(G_0 \times (G/G_0)) \) as \( \mathbb{Z}[1/p] \)-algebras then
by applying \( R_{\mathbb{Z}[1/p]} \) to both sides we obtain \( RG \)
\( \cong R(G_0 \times (G/G_0)) \) as \( R \)-algebras. So, there is no loss in
assuming \( R = \mathbb{Z}[1/p] \).

Set \( V = V(RG) \) and let \( V_{p,\infty} \) be the subgroup of \( V_p \)
consisting of all elements of infinite \( p \)-height. By
Proposition 4.3, \( V_{p,\infty} \) is the maximal divisible subgroup
of \( V_p \). Let \( H \) be the subgroup of \( GV \) generated by
\( V_{p,\infty} \) and \( \{ g_k, g_{qj} | k \geq 0, j \geq 1, q \text{ prime } \neq p \} \). Then \( V_{p,\infty} \) is
a direct summand of \( H \), say \( H = B \oplus V_{p,\infty} \). Clearly \( BV = GV \)
since \( G_0 \subseteq V \) and \( H \subseteq BV \). Note that \( H \cap V = V_{p,\infty} \) since
the torsion subgroup of \( \langle g_k, g_{qj} \rangle \) is generated by
\( \{ u_i | i \geq 1 \} \). Thus \( B \cap V = B \cap H \cap V = B \cap V_{p,\infty} = \{1\} \). Hence
\( GV = B \oplus V \) and Proposition 4.4 yield \( RG \cong R(G_0 \times (G/G_0)) \)
as \( R \)-algebras. Q.E.D.

We now prove a Theorem which will enable us to con-
struct examples of rings which do not satisfy the Isomorphism
Theorem. If \( A \) is any abelian group we write \( \bigoplus_{\aleph_0} A \) to
denote the external direct sum of countably many copies of
\( A \).

Theorem 4.6. Let \( R = R_1 \times \ldots \times R_n \) where each \( R_i \)
\( (1 \leq i \leq n) \) is an indecomposable ring of characteristic 0.
Then, \( R \) satisfies the Isomorphism Theorem if and only if 
\[ \text{inv}(R_i) = \text{inv}(R) \] 
for some \( i \).

**Proof.** Suppose that 
\[ \text{inv}(R_i) = \text{inv}(R) \] 
and let \( G \) and \( H \) be abelian groups with \( RG \cong RH \) as \( R \)-algebras. Then \( R_iG \cong R_iH \) as \( R_i \)-algebras and so \( G/R_i \cong H/R_i \) by Theorem 3.6. But \( G_{R_i} = G_R \) and \( H_{R_i} = H_R \) and so \( R \) satisfies the Isomorphism Theorem.

Conversely, suppose \( \text{inv}(R_i) \neq \text{inv}(R) \) for all \( i \). Then we can choose prime numbers \( p_1, p_2, \ldots, p_n \) such that \( p_i \in \text{inv}(R_i) - \text{inv}(R), \ 1 \leq i \leq n \). For each \( i \) use Proposition 4.5 to obtain a (countable) reduced abelian group \( G_i \) such that \( \text{supp}(G_i) = \{p_i\}, \ G_i/(G_i)_0 \cong \emptyset \) and \( R_iG_i \cong R_i((G_i)_0 \times (G_i/(G_i)_0)) \) as \( R_i \)-algebras. Set 
\[ G = \prod_{i=1}^{n} (G_1 \times \cdots \times G_n \times (G_1)_0 \times (G_2)_0 \times \cdots \times (G_n)_0) \] 
and \( H = \emptyset \times G \). Note that \( \text{supp}(G) = \text{supp}(H) = \{p_1, p_2, \ldots, p_n\} \) so that \( G_R \) and \( H_R \) are trivial. Moreover, for each \( i \), 
\[ R_iG \cong R_i((G_i \times G) \cong (R_iG_i)G \cong (R_i((G_i)_0 \times (G_i/(G_i)_0))))G \] 
\[ \cong R_i((G_i)_0 \times (G_i/(G_i)_0) \times G) \cong (G_i/(G_i)_0) \times G) \cong R_i((G_i/(G_i)_0) \times G) \cong R_i(\emptyset \times G) = R_iH \] 
as \( R_i \)-algebras. Thus \( RG \cong RH \) as \( R \)-algebras. However, \( G \) and \( H \) are not isomorphic since \( G \) is reduced and \( H \) has a subgroup isomorphic to \( \emptyset \). Therefore \( R \) does not satisfy the Isomorphism Theorem.

Q.E.D.
It is now clear that even finite products of integral domains of characteristic 0 fail, in general, to satisfy the Isomorphism Theorem. For example, if \( p_1, p_2, \ldots, p_n \) are distinct prime numbers with \( n \geq 2 \) then \( R = \mathbb{Z}[1/p_1] \times \mathbb{Z}[1/p_2] \times \ldots \times \mathbb{Z}[1/p_n] \) does not satisfy the Isomorphism Theorem by Theorem 4.6.

Continuing with \( R \) as above, note that the constructions in the proofs of Proposition 4.5 and Theorem 4.6 can be used to explicitly produce nonisomorphic nonsplit abelian groups \( G \) and \( H \) with \( G_R \) and \( H_R \) trivial and \( RG \ncong RH \) as \( R \)-algebras. In fact, in the Appendix we will see that such \( G \) and \( H \) are necessarily not split (see Theorem A.3.7). This contrasts sharply with the examples in May [4] discussed in the paragraph preceding Proposition 4.5.
CHAPTER 5

ND-RINGS

In this chapter we continue our study of rings which satisfy the Isomorphism Theorem. We begin by finding a condition which holds for all such rings. This condition will then be used to define the notion of an ND-ring. Later in the chapter we will see that a great many ND-rings satisfy the Isomorphism Theorem leading us to conjecture that a ring satisfies the Isomorphism Theorem if and only if it is an ND-ring. First we need some notation.

Let $R$ be a ring with ideals $R_1, ..., R_n$. We write $R = R_1 + ... + R_n$ to mean that $R$ is the internal direct product of the ideals $R_1, ..., R_n$; that is, each $r \in R$ can be written uniquely as $r = r_1 + ... + r_n$ where $r_i \in R_i$ ($1 \leq i \leq n$). If each $R_i$ is nonzero note that $1 \in R$ can be written uniquely as $1 = e_1 + ... + e_n$ where each $e_i$ is a nonzero idempotent and $e_i e_j = 0$ if $n > 2$ and $i \neq j$. In this case $R_i = Re_i$, the principal ideal generated by $e_i$, and we say that $\{e_i | 1 \leq i \leq n\}$ is a complete set of orthogonal idempotents for $R$. Note that $R$ is indecomposable (i.e., has no idempotents different from 0 or 1) if and only if the singleton $\{1\}$ is the only complete set of orthogonal idempotents for $R$. 

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If \( \{e_i \mid 1 \leq i \leq n\} \) is a complete set of orthogonal idempotents for \( R \) then each ideal \( R e_i \) is a ring with identity \( e_i \). However, it should be noted that \( R e_i \) is not a subring of \( R \) unless \( n = 1 \).

We can now formulate a condition which necessarily holds for all rings satisfying the Isomorphism Theorem. The proof of this result is essentially copied from the second half of our proof of Theorem 4.6.

**Proposition 5.1.** Suppose \( R \) satisfies the Isomorphism Theorem. Then, whenever \( R = R_1 \oplus \ldots \oplus R_n \) for nonzero ideals \( R_1, \ldots, R_n \) there exists an \( R_i, 1 \leq i \leq n \), with \( \text{inv}(R_i) = \text{inv}(R) \).

**Proof.** Suppose there exist nonzero ideals \( R_1, R_2, \ldots, R_n \) of such that \( R = R_1 \oplus R_2 \oplus \ldots \oplus R_n \) with \( \text{inv}(R_i) \neq \text{inv}(R) \) for all \( i, 1 \leq i \leq n \). For each \( i \) select a prime number \( p_i \in \text{inv}(R_i) - \text{inv}(R) \). Then, by Proposition 4.5, we can select for each \( i \) a reduced abelian group \( G_i \) with \( \text{supp}(G_i) = \{p_i\} \), \( G_i/(G_i)_0 \cong \mathbb{Q} \) and \( R_i G_i \cong R_i((G_i)_0 \times (G_i/(G_i)_0)) \) as \( R_i \)-algebras.

Set \( G = \prod_{i=0}^n (G_1 \times G_2 \times \ldots \times G_n \times (G_1)_0 \times (G_2)_0 \times \ldots \times (G_n)_0) \) and \( H = \emptyset \times G \). Then \( \text{supp}(G) = \text{supp}(H) = \{p_1, p_2, \ldots, p_n\} \) and so \( G_R \) and \( H_R \) are trivial. As in the proof of Theorem 4.6, \( RG \cong RH \) as \( R \)-algebras. However, \( G \) and \( H \) are not isomorphic since \( G \) is reduced and \( H \)
has a nontrivial divisible subgroup. Therefore, \( R \) does not satisfy the Isomorphism Theorem. Q.E.D.

It will be convenient to have the following definition: Suppose \( R \) is a ring such that whenever \( R = R_1 + \ldots + R_n \) for nonzero ideals \( R_1, \ldots, R_n \) there exists an \( R_i, 1 \leq i \leq n \), with \( \text{inv}(R_i) = \text{inv}(R) \). If this is the case then we call \( R \) an ND-ring ("ND" standing for "nicely decomposing"). Thus, we may restate Proposition 5.1 as follows:

\textbf{Proposition 5.1.} Every ring satisfying the Isomorphism Theorem is an ND-ring.

It is perhaps natural to ask at this point if the converse of Proposition 5.1 is true; that is, does every ND-ring satisfy the Isomorphism Theorem? As a partial answer to this question we will see in Theorem 5.4(2) that if \( R \) is an ND-ring then \( R \) satisfies the Isomorphism Theorem if \( \text{inv}(R) \) is not the complement of a single prime. Before proving this result we need two Lemmas. Our first one is rather technical.

\textbf{Lemma 5.2.} Let \( R \) and \( S \) be rings with complete sets of orthogonal idempotents \( \{e_1, \ldots, e_m\} \subseteq R \) and \( F = \{f_1, \ldots, f_n\} \subseteq S \) respectively. Suppose \( Sf_j \) is indecomposable for \( 1 \leq j \leq n \) (i.e., suppose each \( f_j \) is a
primitive idempotent) and suppose \( \psi: R \to S \) is an injective ring-homomorphism. Then:

1. For each \( i, 1 \leq i \leq m \), the set \( E_i = \{ f \in F | \psi(e_i)f = f \} \) is nonempty and \( \psi(e_i) = \sum \{ f | f \in E_i \} \).

2. \( E_1, \ldots, E_m \) partition \( F \).

3. Each \( E_i \) is unique. That is, if \( F_i \subseteq F \) with \( \psi(e_i) = \sum \{ f | f \in F_i \} \) then \( F_i = E_i \).

**Proof.** To prove (1) fix \( i \) and note that \( \psi(e_i)f_j \) is an idempotent of \( Sf_j \) \( (1 \leq j \leq n) \). Moreover, \( \psi(e_i) = \psi(e_i)f_1 + \cdots + \psi(e_i)f_n \). Since each \( Sf_j \) is indecomposable, \( \psi(e_i)f_j = 0 \) or \( \psi(e_i)f_j = f_j \). However, since \( \psi \) is injective, \( \psi(e_i) \neq 0 \). Thus \( \psi(e_i)f_j = f_j \) for at least some \( j \). Therefore \( E_i = \{ f \in F | \psi(e_i)f = f \} \) is nonempty and \( \psi(e_i) = \sum \{ f | f \in E_i \} \).

If \( m \geq 2 \) we claim that \( E_i \cap E_j = \emptyset \) if \( i \neq j \). Suppose to the contrary that \( f \in E_i \cap E_j \). Then \( \psi(e_i) = f + e \) and \( \psi(e_j) = f + e' \) where \( e = \sum \{ s | s \in E_i \setminus \{ f \} \} \) and \( e' = \sum \{ s | s \in E_j \setminus \{ f \} \} \). Consequently \( 0 = \psi(e_ie_j) = \psi(e_i)\psi(e_j) = f^2 + fe' + ef + ee' = f + ee' \). Thus \( 0 = f^2(f + ee') = f + (ef)(e'f) = f \), contradicting \( f \neq 0 \) and establishing our claim.

Now, \( 1 = \psi(1) = \psi(e_1 + \cdots + e_m) = \sum \{ \psi(e_i) | 1 \leq i \leq m \} = \sum \{ f | f \in E_i \} = \sum \{ f | f \in F \} \). Since it is also true that \( 1 = f_1 + \cdots + f_n \) we must have \( F = U_i E_i \) by the
orthogonality of the $f_j$'s $(1 \leq j \leq n)$. Therefore the $E_i$'s partition $F$ and (2) is shown.

Finally, if $F_i \subseteq F$ is such that $\psi(e_i) = \bigcup \{f | f \in F_i \}$ then $\bigcup \{f | f \in E_i \} = \bigcup \{f | f \in F_i \}$. Since $F$ is a set of non-zero orthogonal idempotents it follows easily that $E_i = F_i$ and (3) is established. Q.E.D.

We assume that the reader is somewhat familiar with the notions of directed sets, direct and inverse systems and the elementary properties of direct and inverse limits. (See Jacobson [1] for a reference.)

Let $I$ be a directed set. A direct system over $I$ will be written as $\{S_\alpha (\alpha \in I); \psi_{\alpha \beta} \}$ where for each pair $\alpha, \beta \in I$ with $\alpha \leq \beta$, $\psi_{\alpha \beta} : S_\alpha \to S_\beta$ is a morphism in the appropriate category (for us either the category of sets or the category of rings) such that if $\alpha, \beta, \gamma \in I$ with $\alpha \leq \beta \leq \gamma$ then $\psi_{\alpha \alpha} = \text{id}_\alpha$, the identity morphism on $S_\alpha$, and $\psi_{\beta \gamma} \circ \psi_{\alpha \beta} = \psi_{\alpha \gamma}$. On the other hand $\{S_\alpha (\alpha \in I); \psi_{\alpha \beta} \}$ is an inverse system over $I$ if for all $\alpha, \beta, \gamma \in I$ with $\alpha \leq \beta \leq \gamma$, $\psi_{\beta \alpha} : S_\beta \to S_\alpha$ is a morphism with $\psi_{\alpha \alpha} = \text{id}_\alpha$ and $\psi_{\beta \alpha} \circ \psi_{\gamma \beta} = \psi_{\gamma \alpha}$. We use the notation $\lim_{\alpha} S_\alpha$ for the direct limit of the direct system $\{S_\alpha (\alpha \in I); \psi_{\alpha \beta} \}$ and $\lim_{\alpha} S_\alpha$ for the inverse limit of the inverse system $\{S_\alpha (\alpha \in I); \psi_{\beta \alpha} \}$.
Lemma 5.3. Let \( \{ S_\alpha (\alpha \in I); \psi_{\alpha \beta} \} \) be a direct system of rings and ring-homomorphisms over a directed set \( I \). Then:

1. If each \( S_\alpha \) is indecomposable then \( \lim_I S_\alpha \) is indecomposable.

2. If \( \text{char}(S_\alpha) = 0 \) for all \( \alpha \) then \( \text{char}(\lim_I S_\alpha) = 0 \).

Proof. Recall that \( S = \lim_I S_\alpha \) may be defined as follows: Define an equivalence relation \( \sim \) on the disjoint union \( \hat{S} = \bigcup \{ S_\alpha | \alpha \in I \} \) by \( a \sim b \) \((a \in S_\alpha, b \in S_\beta)\) if and only if there exists \( \gamma \geq \alpha, \beta \) such that \( \psi_{\alpha \gamma}(a) = \psi_{\beta \gamma}(b) \). Then, as a set, \( S = \hat{S}/\sim \). If \( a \in \hat{S} \) let \( \bar{a} \) denote the equivalence class of \( a \). \( S \) becomes a ring with zero element \( 0 \) and identity element \( 1 \) by setting \( 0 = \bar{0}_\alpha \), \( 1 = \bar{1}_\alpha \) for any \( \alpha \in I \) (where for each \( \alpha \), \( 0_\alpha \) and \( 1_\alpha \) are respectively the zero and identity elements of \( S_\alpha \)).

If \( a \in S_\alpha \) and \( b \in S_\beta \) the ring operations are defined by \( \bar{a} + \bar{b} = \psi_{\alpha \gamma}(a) + \psi_{\beta \gamma}(b) \) and \( \bar{a} \bar{b} = \psi_{\alpha \gamma}(a) \psi_{\beta \gamma}(b) \) where \( \gamma \in I \) satisfies \( \gamma \geq \alpha, \beta \). We now proceed with the proof:

First suppose that \( e \in S \) is idempotent. Choose \( \alpha \in I \) and \( a \in S_\alpha \) with \( e = \bar{a} \). Then \( a^2 - a = \bar{0}_\alpha \). Thus there exists \( \beta \geq \alpha \) with \( \psi_{\alpha \beta}(a^2 - a) = 0_\beta \) and so \( \psi_{\alpha \beta}(a) \) is an idempotent in \( S_\beta \). Hence \( \psi_{\alpha \beta}(a) = 0_\beta \) or \( 1_\beta \) and so \( e = \bar{a} = \psi_{\alpha \beta}(a) = 0 \) or \( 1 \). Therefore \( S \) is indecomposable.
Suppose \( \text{char}(S) = m \). Then \( ml = 0 \) and so \( ml_\alpha = 0 \) for some (and, in fact, all) \( \alpha \in \mathcal{I} \). If \( \beta \geq \alpha \) is such that \( \psi_\alpha\beta(ml_\alpha) = \psi_\alpha\beta(0_\alpha) \) then \( ml_\beta = \psi_\alpha\beta(ml_\alpha) = \psi_\alpha\beta(0_\alpha) = 0_\beta \).

Therefore \( m = 0 \) since \( \text{char}(S_\beta) = 0 \). Q.E.D.

We now state the main result of this chapter:

**Theorem 5.4.** Let \( R \) be a ring.

1. If \( R \) is an ND-ring then \( R \) has an indecomposable homomorphic image \( S \) with \( \text{inv}(S) = \text{inv}(R) \).

2. Suppose \( R \) is an ND-ring such that \( \text{inv}(R) \) is not the complement of a single prime. Then \( R \) satisfies the Isomorphism Theorem.

3. Suppose \( \text{char}(R) = 0 \) and whenever \( R = R_1 + \ldots + R_n \) for nonzero ideals \( R_1, \ldots, R_n \) then there exists an \( i, 1 \leq i \leq n \), such that both \( \text{inv}(R_i) = \text{inv}(R) \) and \( \text{char}(R_i) = 0 \). Then \( R \) satisfies the Isomorphism Theorem.

A few comments are in order. Note that if \( R \) satisfies the hypothesis of (2) then \( \text{inv}(R) \) contains all prime numbers or excludes at least two prime numbers. Note further that if \( R \) satisfies the hypotheses of (3) then \( R \) is an ND-ring but not conversely. For example, consider \( R = F \times \emptyset \) where \( F \) is a field of prime characteristic. Then \( \text{char}(R) = 0 \) but \( R \) does not satisfy the additional hypotheses of
(3). However, \( R \) is an ND-ring since \( \text{inv}(R) = \text{inv}(F) \).

Before giving the proof of Theorem 5.4 we will show that (2) follows from (1) and so it will suffice to prove (1) and (3). This follows from the Corollary of our next result.

**Lemma 5.5.** Suppose \( \psi: R \rightarrow S \) is a ring-homomorphism with \( \text{inv}(R) = \text{inv}(S) \). If \( S \) satisfies the Isomorphism Theorem then so does \( R \).

**Proof.** Suppose \( G \) and \( H \) are abelian groups with \( RG \cong RH \) as \( R \)-algebras. The homomorphism \( \psi: R \rightarrow S \) endows \( S \) with the structure of an \( R \)-algebra and so \( SG \cong S \psi\cdot RG \cong S\psi\cdot RH \cong SH \) as \( S \)-algebras. Thus \( G/G_S \cong H/H_S \). Since \( G_S = G_R \) and \( H_S = H_R \) the result follows. Q.E.D.

**Corollary.** Suppose \( \text{inv}(R) \) is not the complement of a single prime number. If \( \psi: R \rightarrow S \) is a ring-homomorphism with \( S \) indecomposable and \( \text{inv}(R) = \text{inv}(S) \) then \( R \) satisfies the Isomorphism Theorem.

**Proof.** Since \( S \) is indecomposable, \( \text{char}(S) = 0 \) or a prime power \( p^k \) with \( k \geq 1 \). If \( \text{char}(S) = p^k \) then all primes except \( p \) invert in \( S \) and hence in \( R \). Thus \( \text{char}(S) = 0 \). By Theorem 3.6 \( S \) satisfies the Isomorphism Theorem and the result now follows from Lemma 5.5. Q.E.D.
Hence we see that Theorem 5.4(2) follows from Theorem 5.4(1). We now give the

**Proof of Theorem 5.4.** Let \( \{R_\alpha | \alpha \in I\} \) be the set of all finitely generated subrings of \( R \) indexed by a set \( I \). Partially order \( I \) by defining \( \alpha \leq \beta \) (\( \alpha, \beta \in I \)) if and only if \( R_\alpha \subseteq R_\beta \). \( I \) is then a directed set and for all pairs \( \alpha, \beta \in I \) with \( \alpha \leq \beta \) let \( \psi_{\alpha \beta} : R_\alpha \rightarrow R_\beta \) be the inclusion map. Then, \( \{R_\alpha (\alpha \in I); \psi_{\alpha \beta}\} \) is a direct system and we may view \( R \) as the direct limit \( \lim_{\longrightarrow} R_\alpha \).

Since each \( R_\alpha \) is Noetherian we can write \( R_\alpha = R_\alpha e_{\alpha 1} + \ldots + R_\alpha e_{\alpha n(\alpha)} \) where \( n(\alpha) \) is a positive integer and \( E_\alpha = \{e_{\alpha 1}, \ldots, e_{\alpha n(\alpha)}\} \) is a complete set of orthogonal idempotents for \( R_\alpha \) (and hence for \( R \)) with \( R_\alpha e_{\alpha i} \) indecomposable for all \( e \in E_\alpha \).

For each pair \( \alpha, \beta \in I \) with \( \alpha \leq \beta \) use Lemma 5.2 to obtain a partition \( E_{\alpha 1}^\alpha, \ldots, E_{n(\alpha)}^\alpha \) of \( E_\beta \) such that \( \psi_{\alpha \beta}(e_{\alpha i}) = \sum \{f | f \in E_{\beta i}^\alpha \} \) for all \( i, 1 \leq i \leq n(\alpha) \). Define a function \( \lambda_{\beta \alpha} : E_\beta \rightarrow E_\alpha \) as follows: Given \( e \in E_\beta \) select the unique \( E_{\beta j}^\alpha, 1 \leq j \leq n(\alpha) \), with \( e \in E_{\beta j}^\alpha \) and define \( \lambda_{\beta \alpha}(e) = e_{\alpha j} \).

Note that \( \lambda_{\beta \alpha} \) is defined for all \( \alpha, \beta \in I \) with \( \alpha \leq \beta \) and that \( \lambda_{\alpha \alpha} \) is the identity map on \( E_\alpha \). Now suppose that \( \alpha, \beta, \gamma \in I \) with \( \alpha \leq \beta \leq \gamma \). We claim that \( \lambda_{\beta \alpha} \circ \lambda_{\gamma \beta} = \lambda_{\gamma \alpha} \). Indeed, given \( e \in E_\gamma \) select integers \( i \)
and \( j \) such that \( e \in E_{\gamma_1}^\alpha \) and \( e \in E_{\gamma_j}^\beta \). Then \( \lambda_{\gamma_\alpha}(e) = e_{\alpha_i} \) and \( \lambda_{\gamma_\beta}(e) = e_{\beta_j} \). Choose \( k \) such that \( e_{\beta_j} \in E_{\lambda_k}^\alpha \). Then \((\lambda_{\beta_\alpha} \circ \lambda_{\gamma_\beta})(e) = e_{\alpha_k}\). So, to show that \( \lambda_{\beta_\alpha} \circ \lambda_{\gamma_\beta} = \lambda_{\gamma_\alpha} \), it suffices to show that \( i = k \). But it is not too hard to see that \( e \in E_{\gamma_i}^\alpha \cap E_{\gamma_k}^\alpha \) and so \( i = k \) by Lemma 5.2. Consequently we see that \( \{E_{\alpha}(\alpha \in I) : \lambda_{\beta_\alpha}\} \) is an inverse system of nonempty finite sets.

We now must consider two cases. We say that we are in case (1) if \( R \) satisfies the hypothesis of Theorem 5.4(1) (i.e., if \( R \) is an ND-ring). We say that we are in case (3) if \( R \) satisfies the hypotheses of Theorem 5.4(3) (i.e., \( \text{char}(R) = 0 \) and whenever \( R = R_1 + \ldots + R_n \) for nonzero ideals \( R_1, \ldots, R_n \), there exists \( i, 1 < i < n \), such that both \( \text{inv}(R_i) = \text{inv}(R) \) and \( \text{char}(R_i) = 0 \).

For each \( \alpha \in I \) set \( F_{\alpha} = \{e \in E_{\lambda_\alpha} \mid \text{inv}(R_\alpha e) \subseteq \text{inv}(R)\} \) and \( F'_{\alpha} = \{e \in E_{\lambda_\alpha} \mid \text{inv}(R_\alpha e) \subseteq \text{inv}(R) \) and \( \text{char}(R_\alpha e) = 0 \} \). We claim that \( F_{\alpha} \) is nonempty in case (1) and that \( F'_{\alpha} \) is nonempty in case (3). (Note that in any case \( F'_{\alpha} \subseteq F_{\alpha} \).) Observe that \( e_{\alpha_1} + \ldots + e_{\alpha_n}(a) = 1 \) implies that \( R = R_1 + \ldots + R_{\alpha_n}(a) \). Thus in case (1) \( \text{inv}(R_{\alpha_1} e_1) \subseteq \text{inv}(R_{\alpha_1}) = \text{inv}(R) \) for some \( i, 1 < i < n \). Therefore \( F_{\alpha} \neq \emptyset \) in case (1). In case (3), \( \text{inv}(R_{\alpha_1} e_j) \subseteq \text{inv}(R_{\alpha_1}) = \text{inv}(R) \) and \( \text{char}(R_{\alpha_j}) = 0 \) for some \( j, 1 < j < n(a) \). But \( \text{char}(R_{\alpha_j}) = 0 \) implies that \( \text{char}(R_{\alpha_j} e_{\alpha_j}) = 0 \). Therefore \( F'_{\alpha} \neq \emptyset \) in case (3) and our claim is established.
We next claim that \( \lambda_{\beta \alpha}(F_\beta) \subseteq F_\alpha \) in case (1) and 
\( \lambda_{\beta \alpha}(F'_\beta) \subseteq F'_\alpha \) in case (3) for all \( \alpha, \beta \in I \) with \( \alpha \leq \beta \).

To see this suppose that \( e \in F_\beta \); that is, \( e \in F_\beta \) and 
\( \text{inv}(R_\beta e) \subseteq \text{inv}(R) \). Let \( j \) be the unique integer with 
\( 1 \leq j \leq n(\alpha) \) and \( e \in E^\alpha_{\beta j} \). Then, 
\( \lambda_{\beta \alpha}(e) = e_{\alpha j} \) and 
\( \psi_{\alpha \beta}(e_{\alpha j}) = e + \sum\{f | f \in E^\alpha_{\beta j} - \{e\} \} \). Now, if \( p \in \text{inv}(R_{\alpha e_{\alpha j}}) \) 
then there exists \( r \in R_{\alpha e_{\alpha j}} \) with \( pr = e_{\alpha j} \). Hence, 
\( \psi_{\alpha \beta}(r) = \psi_{\alpha \beta}(pr) = \psi_{\alpha \beta}(e_{\alpha j}) = e + \sum\{f | f \in E^\alpha_{\beta j} - \{e\} \} \). Thus, 
\( p \psi_{\alpha \beta}(r) = e \) and so \( p \in \text{inv}(R_{\alpha e_{\alpha j}}) \subseteq \text{inv}(R) \). Therefore 
\( \text{inv}(R_{\alpha e_{\alpha j}}) \subseteq \text{inv}(R) \) and \( e_{\alpha j} \in F_\alpha \). Consequently, 
\( \lambda_{\beta \alpha}(F_\beta) \subseteq F_\alpha \). In case (3) suppose further that \( \text{char}(R_{\alpha e_{\alpha j}}) = 0 \) 
(i.e., suppose \( e \in F'_\beta \)). Then \( \psi_{\alpha \beta} \) maps the ring \( R_{\alpha e_{\alpha j}} \) 
injectively into the characteristic 0 ring \( R_{\alpha e_{\alpha j}} \) 
\( (\{R_{\alpha e_{\alpha j}} f | f \in E^\alpha_{\beta j} - \{e\}\}) \). Thus \( \text{char}(R_{\alpha e_{\alpha j}}) = 0 \). Therefore 
e_{\alpha j} \in F'_\alpha \) in case (3) and so \( \lambda_{\beta \alpha}(F'_\beta) \subseteq F'_\alpha \).

For all \( \alpha, \beta \in I \) with \( \alpha \leq \beta \) set 
\( \mu_{\beta \alpha} = \lambda_{\beta \alpha}|F'_\beta \) 
and \( \mu'_{\beta \alpha} = \lambda_{\beta \alpha}|F'_\beta \). Thus in case (1) (respectively, case (3)) 
\( \{F_\alpha(\alpha \in I); \mu_{\beta \alpha} \} \) (respectively, \( \{F'_\alpha(\alpha \in I); \mu'_{\beta \alpha} \} \)) is an 
inverse system of nonempty finite sets. Thus in cases (1) 
and (3) respectively the inverse limits \( \varprojlim F_\alpha \) and 
\( \varprojlim F'_\alpha \) are nonempty (for a reference see Eilenberg and 
Steenrod [1]). Consequently in case (1) (respectively, case 
(3)) we can select for each \( \alpha \in I \) an idempotent \( e_\alpha \in F_\alpha \) 
(respectively, \( e'_\alpha \in F'_\alpha \)) such that 
\( \mu_{\beta \alpha}(e_\beta) = e_\alpha \) (respectively, 
\( \mu'_{\beta \alpha}(e'_\beta) = e'_\alpha \)) for all \( \beta \geq \alpha \).
For each \( \alpha \in I \) let \( \Pi_\alpha : R_\alpha + R_\alpha e_\alpha \) and \( \Pi'_\alpha : R_\alpha + R_\alpha e'_\alpha \) be the projection maps in cases (1) and (3) respectively. Note that \( \Pi_\alpha (r) = re_\alpha \) and \( \Pi'_\alpha (r) = re'_\alpha \) for all \( r \in R_\alpha \) and \( \alpha \in I \). For each pair \( \alpha, \beta \in I \) with \( \alpha \leq \beta \) define \( \eta_{\alpha \beta} : R_\alpha e_\alpha + R_\beta e_\beta \) in case (1) and \( \eta'_{\alpha \beta} : R_\alpha e'_\alpha + R_\beta e'_\beta \) in case (3) by \( \eta_{\alpha \beta} = \Pi_\beta \circ (\psi_{\alpha \beta} | R_\alpha e_\alpha) \) and \( \eta'_{\alpha \beta} = \Pi'_\beta \circ (\psi_{\alpha \beta} | R_\alpha e'_\alpha) \).

In case (1) we claim that \( \{ R_\alpha e_\alpha (\alpha \in I) ; \eta_{\alpha \beta} \} \) is a direct system of rings and ring-homomorphisms. Indeed, since \( \mu_{\beta \alpha} (e_\beta) = e_\alpha \) when \( \alpha \leq \beta \) we conclude that \( \eta_{\alpha \beta} (e_\alpha) = e_\beta \). Thus each \( \eta_{\alpha \beta} \) is a ring-homomorphism and \( \eta_{\alpha \alpha} \) is the identity on \( R_\alpha e_\alpha \). Finally, to show that \( \eta_{\beta \gamma} \circ \eta_{\alpha \beta} = \eta_{\alpha \gamma} \) whenever \( \alpha \leq \beta \leq \gamma \) observe that \( (\psi_{\alpha \beta} (e_\alpha)) e_\beta = e_\alpha e_\beta = e_\beta \). Thus, for all \( r e_\alpha \in R_\alpha e_\alpha \) \( (r \in R_\alpha) \), \( (\eta_{\beta \gamma} \circ \eta_{\alpha \beta}) (r e_\alpha) = (\eta_{\beta \gamma} \circ \Pi_\beta) (r e_\alpha) = \eta_{\beta \gamma} (r e_\beta) = \eta_{\beta \gamma} (r e_\alpha) = r e_\alpha e_\gamma = r e_\gamma = \eta_{\alpha \gamma} (r e_\alpha) \). Hence \( \eta_{\beta \gamma} \circ \eta_{\alpha \beta} = \eta_{\alpha \gamma} \) whenever \( \alpha \leq \beta \leq \gamma \). Therefore, in case (1), \( \{ R_\alpha e_\alpha (\alpha \in I) ; \eta_{\alpha \beta} \} \) is a direct system as claimed. Likewise, \( \{ R_\alpha e'_\alpha (\alpha \in I) ; \eta'_{\alpha \beta} \} \) is a direct system of rings and ring-homomorphisms.

In case (1) set \( S = \varprojlim I R_\alpha e_\alpha \) and in case (3) set \( S' = \varprojlim I R_\alpha e'_\alpha \). Since each \( R_\alpha e_\alpha \) and \( R_\alpha e'_\alpha \) is indecomposable and since \( \text{char}(R_\alpha e'_\alpha) = 0 \) for all \( \alpha \in I \), we conclude that \( S \) is indecomposable and that \( S' \) is indecomposable of characteristic 0 by Lemma 5.3.
Note that for each pair $\alpha, \beta \in I$ with $\alpha \leq \beta$ the diagrams

\[
\begin{array}{ccc}
R & \xrightarrow{\psi_{\alpha\beta}} & R' \\
\downarrow{\eta_{\alpha\beta}} & & \downarrow{\eta'_{\alpha\beta}} \\
R e_\alpha & \xrightarrow{\eta_{\alpha\beta}} & R' e_{\alpha'}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
R & \xrightarrow{\psi_{\alpha\beta}} & R' \\
\downarrow{\eta_{\alpha\beta}} & & \downarrow{\eta'_{\alpha\beta}} \\
R e_\alpha & \xrightarrow{\eta_{\alpha\beta}} & R' e_{\alpha'}
\end{array}
\]

commute in cases (1) and (3) respectively. Since $R = \lim_{\alpha} R_\alpha$ we have induced ring-homomorphisms $\Pi : R \to S$ in case (1) and $\Pi' : R \to S'$ in case (3). Moreover, $\Pi$ is surjective since each $\Pi_\alpha$ is surjective.

We claim that $\text{inv}(R) = \text{inv}(S)$ and $\text{inv}(R) = \text{inv}(S')$. Since $\Pi : R \to S$ and $\Pi' : R \to S'$ are ring-homomorphisms, $\text{inv}(R) \subseteq \text{inv}(S)$ and $\text{inv}(R) \subseteq \text{inv}(S')$. We now show that $\text{inv}(S) \subseteq \text{inv}(R)$. Suppose $p \in \text{inv}(S)$ and view $S$ as $\bigcup \{R_\alpha e_\alpha | \alpha \in I\}/\sim$ as in the proof of Lemma 5.3. Thus $p a = 1$ for some $a \in S$ and $p r_\alpha = e_\alpha$ where $\alpha \in I$ and $r_\alpha \in R e_\alpha$ are such that $\overline{r_\alpha} = a$. So, there exists $\beta \geq \alpha$ such that $p \eta_{\alpha\beta}(r_\alpha) = \eta_{\alpha\beta}(e_\alpha) = e_{\beta}$. Thus $p \in \text{inv}(R_\beta e_\beta) \subseteq \text{inv}(R)$ showing $\text{inv}(S) \subseteq \text{inv}(R)$. Therefore $\text{inv}(S) = \text{inv}(R)$ in case (1).
In a similar fashion we see that \( \text{inv}(S') = \text{inv}(R) \) in case (3).

Now the proof of case (1) is complete and so (2) follows from the Corollary to Lemma 5.5. Since \( S' \) satisfies the Isomorphism Theorem by Theorem 3.6, Lemma 5.5 completes the proof of (3). Q.E.D.

We now draw several corollaries from Theorem 5.4. Our first one gives us a characterization of ND-rings.

**Corollary 1.** \( R \) is an ND-ring if and only if \( R \) has an indecomposable homomorphic image \( S \) with \( \text{inv}(R) = \text{inv}(S) \).

**Proof.** The "only if" part is Theorem 5.4(1). Now suppose \( \phi: R \to S \) is a surjective ring-homomorphism with \( \text{inv}(R) = \text{inv}(S) \) and \( S \) is indecomposable. Suppose

\[
R = R_1 \oplus \ldots \oplus R_n
\]

for nonzero ideals \( R_1, \ldots, R_n \) of \( R \). Since each ideal of \( R \) is of the form \( I_1 \oplus \ldots \oplus I_n \) where \( I_i \) is an ideal of \( R_i \) \( (1 \leq i \leq n) \) and since \( R/\ker(\phi) \ncong S \) is indecomposable we conclude that \( \ker(\phi) = R_1 \oplus \ldots \oplus R_{i-1} \oplus I_i \oplus R_{i+1} \oplus \ldots \oplus R_n \) for some \( i \) where \( I_i \) is an ideal of \( R_i \). Thus \( R_i/I_i \cong S \) and so \( \text{inv}(R_i) \subseteq \text{inv}(S) = \text{inv}(R) \). On the other hand it is clear that \( \text{inv}(R) \subseteq \text{inv}(R_i) \). Therefore \( \text{inv}(R_i) = \text{inv}(R) \) and \( R \) is an ND-ring. Q.E.D.
If we use Proposition 2.3.12 with the restriction on char(R) removed (see the third Corollary to Proposition 7 in May [3] for a reference) we obtain the following generalization of the Corollary to Theorem 3.6.

**Corollary 2.** Let R be a ring and suppose G and H are abelian groups with RG ≅ RH as R-algebras. Suppose further that \( G_R \) is trivial. Then:

1. If R is an ND-ring then \( H_R \) is also trivial.
2. If R satisfies the Isomorphism Theorem then \( G \cong H \).

**Proof.** In view of Proposition 5.1 it suffices to prove (1). By Theorem 5.4(1) there exists an indecomposable ring S with SG ≅ SH as S-algebras and with inv(S) = inv(R). If \( p \) inverts in S then \( V(SG)_p \) is trivial by the third Corollary to Proposition 7 in May [3]. Therefore \( H_p \subseteq V(SH)_p \) is trivial by Lemma 2.1.1. Since inv(R) = inv(S) it follows that \( H_R \) is also trivial. Q.E.D.

**Corollary 3.** Suppose R is a ring such that \( \text{inv}(R) \) contains all primes or excludes at least two primes. Then R satisfies the Isomorphism Theorem if and only if R is an ND-ring.

**Proof.** Simply apply Proposition 5.1 and Theorem 5.4(2). Q.E.D.
Note that Theorem 4.6 says that if $R$ is a finite product of indecomposable rings of characteristic 0 then $R$ satisfies the Isomorphism Theorem if and only if $R$ is an ND-ring. This result together with Corollary 3 above suggests that there is a strong connection between rings satisfying the Isomorphism Theorem and ND-rings. In fact, there is now reason to conjecture that rings satisfying the Isomorphism Theorem and ND-rings are one and the same. This conjecture will be investigated further in Chapter 6.

To demonstrate the usefulness of Corollary 3 we close this chapter by giving an example: Let $p_1', p_2', \ldots$ be an infinite sequence of distinct prime numbers. If $n > 2$ then $\mathbb{Z}[1/p_1'] \times \ldots \times \mathbb{Z}[1/p_n']$ does not satisfy the Isomorphism Theorem since it is not an ND-ring. However, the infinite direct product $\prod\{\mathbb{Z}[1/p_i']|i \geq 1\}$ satisfies the Isomorphism Theorem since it is easy to verify that it is an ND-ring satisfying the hypothesis of Corollary 3.
CHAPTER 6

EQUIVALENT FORMULATIONS OF THE CONJECTURE

In Chapter 5 we saw that every ring $R$ that satisfies the Isomorphism Theorem is necessarily an NO-ring. That is, whenever $R = R_1 + \ldots + R_n$ for nonzero ideals $R_1, \ldots, R_n$ there exists an $R_i$, $1 \leq i \leq n$, such that $\text{inv}(R_i) = \text{inv}(R)$. In Corollary 3 of Theorem 5.4 we saw that if $\text{inv}(R)$ contains all prime numbers or excludes at least two prime numbers then $R$ satisfies the Isomorphism Theorem if and only if $R$ is an NO-ring. This result together with some other fragmentary evidence led us to conjecture that the NO-rings are precisely the rings which satisfy the Isomorphism Theorem.

In the present chapter we will find several equivalent formulations of this conjecture. Our final goal will be to prove the following result:

**Theorem 6.3.3.** The following statements are equivalent.

1. Every NO-ring satisfies the Isomorphism Theorem.
2. Every NO-ring of characteristic 0 satisfies the Isomorphism Theorem.
3. Every field of prime characteristic satisfies the Isomorphism Theorem.
(4) Every algebraically closed field of prime characteristic satisfies the Isomorphism Theorem.

Of course that (1) implies (2), (3) and (4) is immediate. It is also clear that (3) implies (4). The difficult parts of the proof will be to show that (2) implies (3) and that (4) implies (1).

In the first two sections of this chapter we will show that (2) implies (3). In Section 3 we will complete the proof of Theorem 6.3.3 by showing that (4) implies (1) and then discuss the significance of the Theorem in view of known isomorphism results.

§1. Isomorphism of Group Algebras over Fields of Prime Characteristic

Let $F$ be a field with $\text{char}(F) = p > 0$ and suppose $G$ and $G'$ are abelian groups. Our goal in this section will be to show that if $FG \cong FG'$ as $F$-algebras then the cardinal numbers $|G_p|$ and $|G'_p|$ are the same provided that $G_p$ is infinite.

Let $A$ be a ring and suppose $I$ is an $A$-module. If $S \subseteq I$ we write $\langle S \rangle_A$ or simply $\langle S \rangle$ for the $A$-submodule of $I$ generated by $S$. $S \subseteq I$ is called a minimal generating set for $I$ if $I = \langle S \rangle$ and whenever $T$ is a proper subset of $S$ then $\langle T \rangle \neq I$. By a minimal generating set for an abelian group $G$ we mean a minimal
generating set for $G$ viewed as a multiplicative $\mathbb{Z}$-module (with the module action written exponentially). We note that if $G$ is a direct sum of cyclic groups then $G$ has a minimal generating set formed by selecting one generator from each of the cyclic summands.

The proof of our first result mimics the well known proof for the corresponding result for vector spaces.

**Lemma 6.1.1.** Let $A$ be a ring and suppose $I$ is an $A$-module. If $S_1$ and $S_2$ are minimal generating sets for $I$ then $S_1$ is finite if and only if $S_2$ is finite. If $S_1$ is infinite then $|S_1| = |S_2|$.

**Proof.** For every $x \in S_1$ there is a finite subset $S_2(x)$ of $S_2$ such that $x$ can be written as an $A$-linear combination of the elements of $S_2(x)$. By the minimality of $S_2$, $S_2 = \bigcup \{S_2(x) | x \in S_1\}$. Thus $S_2$ is finite if $S_1$ is finite. By symmetry we conclude that $S_1$ is finite if and only if $S_2$ is finite.

If $S_1$ is infinite then $|S_2| = \left| \bigcup \{S_2(x) | x \in S_1\} \right| \leq \aleph_0 |S_1| = |S_1|$. Using a similar argument we also see that $|S_1| \leq |S_2|$. Therefore $|S_1| = |S_2|$ if $S_1$ is infinite.

Q.E.D.

Let $F$ be a field with $\text{char}(F) = p > 0$ and let $G$ be an abelian group. The map $\varrho : FG \to FG$ given by $\varrho(\alpha) = \alpha^p$
for all \( a \in FG \) is a ring-homomorphism called the Frobenius endomorphism of \( FG \). Note that if \( a = r_1g_1 + \ldots + r_kg_k \) \((r_i \in F, \ g_i \in G, \ 1 \leq i \leq k)\) then \( \theta(a) = r_1^{p}g_1^{p} + \ldots + r_k^{p}g_k^{p} \). If \( n \) is a positive integer we write \( \theta^n \) for the \( n \)-fold iteration of \( \theta \). That is, \( \theta^1 = \theta \) and \( \theta^n = \theta \circ \theta^{n-1} \) for \( n \geq 2 \).

If \( G \) is an abelian group and if \( p^n \) is a prime power we write \( G[p^n] \) for the subgroup \( \{g \in G \mid g^{p^n} = e\} \). Note that \( G[p] \subseteq G[p^2] \subseteq \ldots \) is an ascending sequence of subgroups of \( G_p \) with \( G_p = \bigcup \{G[p^n] \mid n \geq 1\} \).

Our next result would follow directly from Proposition 2 in May [1] which is a more general form of our Lemma 2.1.2.

**Lemma 6.1.2.** Let \( F \) be a field with \( \text{char}(F) = p > 0 \) and suppose \( G \) is an abelian group. Let \( \theta \) be the Frobenius endomorphism of \( FG \). For every integer \( n \geq 1 \) set \( I_n = \ker(\theta^n) \) and let \( \phi_n:FG \to F(G/G[p^n]) \) be the natural map. Then, as ideals of \( FG \), both \( I_n \) and \( \ker(\phi_n) \) are generated by \( S_n = \{h-1 \mid h \in G[p^n]\} \). Thus \( I_n = \ker(\phi_n) \) for all \( n \geq 1 \).

**Proof.** By Lemma 2.1.2 we know that \( \ker(\phi_n) = \langle S_n \rangle \) so it remains to show that \( I_n = \langle S_n \rangle \).

Clearly \( \langle S_n \rangle \subseteq I_n \) so suppose \( a \in I_n \). Let \( \{g_i \mid i \in \Lambda \} \) be a complete set of coset representatives for
for \( G[p^n] \) in \( G \) indexed by a set \( \Lambda \) and write

\[
\alpha = \sum_{i \in \Lambda} (\sum_{h \in G[p^n]} r_i, h) g_i
\]

where \( r_i, h \in F \) for all \( i \in \Lambda \) and \( h \in G[p^n] \). Then

\[
0 = \theta^n(\alpha) = \alpha p^n = \sum_i (\sum_h r_i, h) p^n g_i.
\]

Note that if \( i, j \in \Lambda \) with \( i \neq j \) then \( g_i^p \neq g_j^p \).

Thus \( \sum_h r_i, h)^p = 0 \) and so \( \sum_h r_i, h = 0 \) for all \( i \in \Lambda \).

Hence for each \( i \) we may write \( r_{i,e} = \sum_{h \neq e} (-r_i, h) \) and so

\[
\sum_{h \neq e} r_i, h(h - 1) \in \langle S_n \rangle.
\]

Thus \( \alpha = \sum_i (\sum_h r_i, h) g_i \in \langle S_n \rangle \). Consequently, \( I_n \subseteq \langle S_n \rangle \) and the proof is complete. Q.E.D.

We now establish a connection between minimal generating sets for \( I_n = \ker(\theta^n) \) and \( G[p^n] \). As a corollary we will see that minimal generating sets for \( I_n \) exist.

**Lemma 6.1.3.** Let \( F \) be a field with \( \text{char}(F) = p > 0 \) and let \( G \) be an abelian group. Let \( \theta \) be the Frobenius endomorphism of \( FG \) and for \( n \geq 1 \) set \( I_n = \ker(\theta^n) \) and let \( S_n \) be a subset of \( G[p^n] \). Then, \( S_n \) generates \( G[p^n] \) if and only if \( \{g - 1 | g \in S_n\} \) generates \( I_n \). Consequently \( S_n \) is a minimal generating set for \( G[p^n] \) if and only if \( \{g - 1 | g \in S_n\} \) is a minimal generating set for \( I_n \).
Proof. Suppose $S_n$ generates $G[p^n]$. By Lemma 2.1.2, \( \{g - 1 | g \in S_n \} \) generates the kernel of the natural map \( FG \to F(G/G[p^n]) \) which is \( I_n \) by Lemma 6.1.2. Therefore, if $S_n$ generates $G[p^n]$ then \( \{g - 1 | g \in S_n \} \) generates \( I_n \).

Now suppose \( \{g - 1 | g \in S_n \} \) generates \( I_n \) and let $H$ be the subgroup of $G[p^n]$ generated by $S_n$. By Lemma 2.1.2 \( I_n = \ker(\phi) \) where \( \phi: FG \to F(G/H) \) is the natural map. Suppose $x \in G[p^n]$. Then \( (x - 1)^{p^n} = 0 \) and so \( xH = eH \). Thus $x \in H$ and $H = G[p^n]$. Therefore if \( \{g - 1 | g \in S_n \} \) generates \( I_n \) then $S_n$ generates $G[p^n]$. Q.E.D.

Corollary. Let $F$, $G$, $\theta$ and $I_n$ be as in Lemma 6.1.3. Then, there exist minimal generating sets for $I_n$.

Proof. Since $G[p^n]$ is a bounded group it is a direct sum of cyclic groups (see Kaplansky [1] for a reference). Thus $G[p^n]$ has minimal generating sets and the result follows from the Lemma. Q.E.D.

We are now ready to prove the main result of this section.

Proposition 6.1.4. Let $F$ be a field with \( \text{char}(F) = p > 0 \) and suppose $G$ and $G'$ are abelian groups with
$\mathbf{F}_G \cong \mathbf{F}_G'$ as $\mathbf{F}$-algebras. If $G_p$ is infinite then $|G_p| = |G_p'|$.

Proof. Let $f : \mathbf{F}_G \to \mathbf{F}_G'$ be an $\mathbf{F}$-algebra isomorphism and let $\theta$ (respectively, $\theta'$) be the Frobenius endomorphism of $\mathbf{F}_G$ (respectively, $\mathbf{F}_G'$). For each $n \geq 1$ let $I_n = \ker(\theta^n)$ and $I'_n = \ker((\theta')^n)$ with minimal generating sets $T_n$ and $T'_n$ respectively and let $S_n$ (respectively, $S'_n$) be a minimal generating set for $G[p^n]$ (respectively, $G'[p^n]$).

First suppose $S_k$ is infinite for some $k \geq 1$. Then, for every $n \geq k$, $S_n$ is infinite and every minimal generating set for $G[p^n]$ has cardinality $|S_n|$ by Lemma 6.1.1. Thus Lemmas 6.1.1 and 6.1.3 imply that $T_n$ is infinite and $|T_n| = |S_n|$ for each $n \geq k$. Since $f(I_n) = I'_n$ for each $n$, $f$ carries $T_n$ onto a minimal generating set for $I'_n$. Hence each minimal generating set for $I'_n$ is infinite of cardinality $|T_n|$ by Lemma 6.1.1. Thus $|T_n| = |T'_n|$ for all $n \geq k$. Moreover, $|T'_n| = |S'_n|$ by Lemmas 6.1.1 and 6.1.3. Thus $|S_n| = |T_n| = |T'_n| = |S'_n|$ for each $n \geq k$. Therefore, for all $n \geq k$, $|G[p^n]| = |S_n| = |S'_n| = |G'[p^n]|$ and so $|G_p| = \sum_{n \geq k} |G[p^n]| = \sum_{n \geq k} |G'[p^n]| = |G'|$.

We may now assume that $S_n$ is finite for all $n \geq 1$. Thus, each $G[p^n]$ is finite. We consider two cases.
Case 1. Suppose that for each \( n > 1 \) there exists \( k > n \) such that \( I_n \) is properly contained in \( I_k \). Then Lemma 6.1.3 implies that for each \( n > 1 \) there exists \( k > n \) such that \( G[p^n] \) is properly contained in \( G[p^k] \). Thus \( |G_p| = |U_{n\geq 1} G[p^n]| = \aleph_0 \). Since \( f(I_n) = I'_n \) for each \( n > 1 \) it follows that \( |G'_p| = \aleph_0 \) also.

Case 2. Suppose that there exist \( k > 1 \) such that \( I_n = I_k \) for all \( n \geq k \). Then it follows from Lemma 6.1.3 that \( G[p^n] = G[p^k] \) for all \( n \geq k \). Thus \( G_p = G[p^k] \), a finite group, contradicting the hypothesis that \( G_p \) is infinite. This completes the proof. Q.E.D.

§2. ND-Rings of Characteristic Zero and Fields of Prime Characteristic

In this section we will show that if every ND-ring of characteristic 0 satisfies the Isomorphism Theorem then every field of prime characteristic must also satisfy the Isomorphism Theorem. We first need several preliminary results.

Lemma 6.2.1. (May [1]). Let \( F \) be a field with \( \text{char}(F) = p > 0 \) and suppose \( G \) and \( G' \) are abelian groups with \( FG \cong FG' \) as \( F \)-algebras. Then, \( F(G/G_p) \cong F(G'/G'_p) \) as \( F \)-algebras.

Proof. Let \( \theta \) (respectively, \( \theta' \)) be the Frobenius endomorphism of \( FG \) (respectively, \( FG' \)). For each integer
n \geq 1 \text{ set } I_n = \ker(\theta^n) \text{ and } I'_n = \ker((\theta')^n). \text{ Then } \\
I = \bigcup_{n \geq 1} I_n \text{ and } I' = \bigcup_{n \geq 1} I'_n \text{ are ideals of } FG \text{ and } FG' \text{ respectively.} \\

By Lemma 6.1.2 \ I_n \text{ and } I'_n \text{ are generated by} \\
\{g - 1|g \in G[p^n]\} \text{ and } \{g' - 1|g' \in G'[p^n]\} \text{ respectively.} \text{ Thus } I \text{ and } I' \text{ are generated by} \\
\{g - 1|g \in G_p\} \text{ and } \{g' - 1|g' \in G'_p\} \text{ respectively. By Lemma 2.1.2, } I \text{ (respectively, } I') \text{ is the kernel of the natural map } FG \\
\rightarrow F(G/G_p) \text{ (respectively, } FG' \rightarrow F(G'/G'_p)). \text{ Since } f(I) = I' \text{ for any } F\text{-algebra isomorphism } f:FG \rightarrow FG', \text{ the result follows.} \text{ Q.E.D.} \\

\text{Lemma 6.2.2. Let } F \text{ be a field with } \text{char}(F) = p > 0 \text{ and let } G \text{ be an abelian group. Set } V_F = \bigoplus (V(FG)_q | q \in \text{inv}(F)). \text{ Then, } V_F \subseteq F(G_F). \\

\text{Proof. Suppose } q \text{ is a prime different from } p. \text{ It suffices to show that } V(FG)_q \subseteq F(G_F). \\

\text{We first show that } V(FG)_q \subseteq FG_0. \text{ Suppose } \\
\alpha \in V(FG)_q \text{, say } \alpha^q = 1. \text{ Note that } \alpha, \alpha^{-1} \in FG_0 + N(FG) \text{ by Proposition 2.2.2. Select } \beta, \beta' \in FG_0 \text{ and } \gamma, \gamma' \in N(FG) \text{ such that } \alpha = \beta + \gamma \text{ and } \alpha^{-1} = \beta' + \gamma'. \text{ Choose an integer } s \geq 1 \text{ sufficiently large so that both } \gamma^p = 0 \text{ and } (\gamma')^p = 0. \text{ Select integers } a \text{ and } b \text{ such that } \\
a\alpha^q + bp^s = 1.
If \( b > 0 \) then \( \alpha = \alpha^{bp^s} = (\beta + \gamma)^{bp^s} = \beta^{bp^s} \in FG_0 \). If \( b < 0 \) then \( \alpha = \alpha^{bp^s} = (\alpha^{-1})^{bp^s} = (\beta' + \gamma')^{-bp^s} = (\beta')^{-bp^s} \in FG_0 \). Thus, in either case \( \alpha \in FG_0 \) and \( V(FG)_q \subseteq FG_0 \). Hence we may select an integer \( t \) so that \( \alpha^t \in F(G_F) \) and \( (\alpha^{-1})^t \in F(G_F) \).

Select integers \( c \) and \( d \) so that \( cp^t + dq^r = 1 \).

Then, if \( c > 0 \) then \( \alpha = (\alpha^p)^c \in F(G_F) \) and if \( c < 0 \) then \( \alpha = (\alpha^{-1})^{cp^t} \in F(G_F) \). Thus, in either case \( \alpha \in F(G_F) \). Therefore \( V(FG)_q \subseteq F(G_F) \) for all primes \( q \in \text{inv}(F) \).

Q.E.D.

Lemma 6.2.3. Let \( F \) be a field with \( \text{char}(F) = p > 0 \) and suppose that \( G \) and \( G' \) are abelian groups with \( FG \cong FG' \) as \( F \)-algebras. Then \( F(G/G_F) \cong F(G'/G_F') \) as \( F \)-algebras.

Proof. By Lemma 2.1.1 we may select a normalized \( F \)-algebra isomorphism \( f:FG \to FG' \). Let \( I \) and \( I' \) be the kernels of the natural maps \( FG \to F(G/G_F) \) and \( FG' \to F(G'/G_F') \) respectively. We will be done if we can show that \( f(I) = I' \).

Set \( V_F = \bigoplus \{ V(FG)_q | q \in \text{inv}(F) \} \) and \( V'_F = \bigoplus \{ V(FG')_q | q \in \text{inv}(F) \} \). From Lemma 6.2.2 we have \( G_F \subseteq V_F \subseteq F(G_F) \) and \( G_F' \subseteq V'_F \subseteq F(G_F') \). Thus \( f(F(G_F)) = F(G_F') \).

From Lemma 2.1.2 and its Corollary we know that \( I \) (respectively, \( I' \)) is generated by the elements of \( F(G_F) \)
(respectively, $F(G_1')$) of zero augmentation. Thus, since
$f$ is normalized, a set of generators for $I$ is carried by
$f$ onto a set of generators for $I'$. Therefore $f(I) = I'$
and the proof is complete. Q.E.D.

Our next result is a slight generalization of Lemma
2 in May [1].

Lemma 6.2.4. (May [1]). Let $R$ be an integral
domain and suppose $G$ and $G'$ are torsion free abelian
groups with $RG \cong RG'$ as $R$-algebras. Then, $G \cong G'$.

Proof. By Lemma 2.1.1 we may select a normalized $R$-
algebra isomorphism $f:RG \to RG'$. By Lemma 2.2.1(3) every
unit of $RG$ and $RG'$ is trivial. Thus $U(RG) = G$ and
$U(RG') = G'$. Since $f(U(RG)) = U(RG')$ we have $G \cong G'$.
Q.E.D.

In proving the final result of this section we will
use the following Theorem of May which we state without
proof.

Theorem. (May [2]). Let $K$ be an algebraically
closed field and let $G$ be an abelian group such that if
$\text{char}(K) \neq 0$ then $\text{char}(K) \not\subseteq \text{supp}(G)$. Then a complete set
of invariants for $KG$ as a $K$-algebra is the cardinal
number $|G_0|$ and the isomorphism class of $G/G_0$. 
We are now ready to prove the main result of this section.

**Proposition 6.2.5.** Suppose every ND-ring of characteristic 0 satisfies the Isomorphism Theorem. Then every field of prime characteristic satisfies the Isomorphism Theorem.

**Proof.** Let \( F \) be a field with \( \text{char}(F) = p > 0 \) and suppose \( G \) and \( G' \) are abelian groups with \( FG \cong FG' \) as \( F \)-algebras. We need to show that \( G/G_F = G'/G'_F \). Note that \( F(G/G_F) \cong F(G'/G'_F) \) by Lemma 6.2.3. So, we may assume that \( G_0 = \mathfrak{c}_p \) and \( G'_0 = \mathfrak{c}'_p \) and show that \( G \cong G' \).

Set \( H = G \times \mathbb{Z}[p^\infty] \) and \( H' = G' \times \mathbb{Z}[p^\infty] \). Thus \( FH \cong (FG)(\mathbb{Z}[p^\infty]) \cong (FG')(\mathbb{Z}[p^\infty]) \cong FH' \). By Proposition 6.1.4, \( |H_0| = |H_p| = |H'_p| = |H'_0| \). Moreover, \( F(H/H_0) \cong F(H'/H'_0) \) by Lemma 6.2.1 and so \( H/H_0 \cong H'/H'_0 \) by Lemma 6.2.4. Therefore by May's Theorem in [2] we have \( \mathfrak{c}H \cong \mathfrak{c}H' \) as \( \mathfrak{c} \)-algebras and so \( (\mathfrak{c} \times F)H \cong (\mathfrak{c} \times F)H' \) as \( (\mathfrak{c} \times F) \)-algebras.

Note that \( \mathfrak{c} \times F \) is an ND-ring of characteristic 0 and so \( \mathfrak{c} \times F \) satisfies the Isomorphism Theorem by hypothesis. Since \( \text{inv}(\mathfrak{c} \times F) \cap \text{supp}(H) \) and \( \text{inv}(\mathfrak{c} \times F) \cap \text{supp}(H') \) are empty we have \( H \cong H' \). Thus \( G \times \mathbb{Z}[p^\infty] \cong G' \times \mathbb{Z}[p^\infty] \) and so \( G \cong G' \) as required. Q.E.D.
§3. Final Theorem

In this section we provide the last step needed to complete the proof of Theorem 6.3.3. In particular we will show in Proposition 6.3.2 that if every algebraically closed field of prime characteristic satisfies the Isomorphism Theorem then every ND-ring satisfies the Isomorphism Theorem. We separate off part of the proof of Proposition 6.3.2 in the following Lemma.

**Lemma 6.3.1.** Let $R$ be an ND-ring with $\text{char}(R) = 0$ such that there is a unique prime number $p$ with $p \not\in \text{inv}(R)$. Suppose further that every algebraically closed field of prime characteristic satisfies the Isomorphism Theorem. Then $R$ satisfies the Isomorphism Theorem.

**Proof.** We consider two cases.

**Case 1.** Suppose that whenever $R = R_1 \oplus \ldots \oplus R_n$ for nonzero ideals $R_1, \ldots, R_n$ there exists an $i$ ($1 \leq i \leq n$) such that both $\text{inv}(R_i) = \text{inv}(R)$ and $\text{char}(R_i) = 0$. Then, $R$ satisfies the Isomorphism Theorem by Theorem 5.4(3).

**Case 2.** Suppose there exist nonzero ideals $R_1, \ldots, R_n$ such that $R = R_1 \oplus \ldots \oplus R_n$ and if $\text{inv}(R_1) = \text{inv}(R)$ then $\text{char}(R_1) \neq 0$ ($1 \leq i \leq n$). Thus, since $\text{char}(R) = 0$ and since $R$ is an ND-ring we can write $R = S_1 \oplus S_2$ for nonzero ideals $S_1$ and $S_2$ of $R$ such
that char(S₁) = 0, char(S₂) ≠ 0, inv(S₁) contains all prime numbers and inv(S₂) = inv(R). Because we have the projection map R → S₂ with inv(S₂) = inv(R) it suffices by Lemma 5.5 to show that S₂ satisfies the Isomorphism Theorem. Since p does not invert in S₂ we may select a maximal ideal M of S₂ with p ∈ M. Thus S₂/M is a field of characteristic p. Let K be an algebraic closure of S₂/M. Thus we have the composition S₂ → S₂/M → K with the first map being the quotient map and the second map being inclusion. Moreover, inv(S₂) = inv(K) and K is an algebraically closed field of prime characteristic. From our hypothesis and Lemma 5.5 we conclude that S₂ satisfies the Isomorphism Theorem.

We note for future reference that the hypothesis that every algebraically closed field of prime characteristic satisfies the Isomorphism Theorem is not used in proving case 1 above.

Proposition 6.3.2. Let R be an ND-ring and suppose every algebraically closed field of prime characteristic satisfies the Isomorphism Theorem. Then R satisfies the Isomorphism Theorem.

Proof. If inv(R) is not the complement of a single prime then R satisfies the Isomorphism Theorem by Theorem
5.4(2). Thus we may assume that there is a unique prime number $p$ with $p \notin \text{inv}(R)$. Hence either $\text{char}(R) = 0$ or $\text{char}(R) = p^k$ ($k \geq 1$).

If $\text{char}(R) = 0$ then $R$ satisfies the Isomorphism Theorem by Lemma 6.3.1. If $\text{char}(R) = p^k$ select a maximal ideal $M$ of $R$. Then $R/M$ is a field of characteristic $p$. Let $K$ be an algebraic closure of $R/M$. Then we have the composition $R \rightarrow R/M \rightarrow K$ with $\text{inv}(R) = \text{inv}(K)$. Therefore, since $K$ satisfies the Isomorphism Theorem by hypothesis, $R$ must satisfy the Isomorphism Theorem by Lemma 5.5. Q.E.D.

We are now ready to prove the long awaited Theorem 6.3.3.

Theorem 6.3.3. The following statements are equivalent.

1. Every ND-ring satisfies the Isomorphism Theorem.
2. Every ND-ring of characteristic 0 satisfies the Isomorphism Theorem.
3. Every field of prime characteristic satisfies the Isomorphism Theorem.
4. Every algebraically closed field of prime characteristic satisfies the Isomorphism Theorem.

Proof. Trivially (1) implies (2) and (3) implies (4). Now, (2) implies (3) by Proposition 6.2.5 and (4) implies (1) by Proposition 6.3.2. Q.E.D.
Reviewing the proof of Lemma 6.3.1 and taking into account Theorem 5.4(3) we note that we have proven that ND-rings of characteristic 0 satisfy the Isomorphism Theorem in all cases except for a particularly troublesome one. The only ND-rings \( R \) with \( \text{char}(R) = 0 \) for which we have no definitive Isomorphism Theorem information are the rings that can be written in the form \( R = S_1 \oplus S_2 \) with \( \text{char}(S_1) = 0 \), \( \text{char}(S_2) = p^k \) for some prime \( p \) and integer \( k \geq 1 \), and \( \text{inv}(S_1) \) contains all prime numbers. An example of a ring of this type is \( \mathbb{Q} \times F \) where \( F \) is a field of characteristic \( p \).

The question of whether fields of prime characteristic satisfy the Isomorphism Theorem seems difficult and very little is known in this direction. Suppose \( F \) is a field with \( \text{char}(F) = p > 0 \) and let \( G \) and \( G' \) be abelian groups with \( FG \cong FG' \) as \( F \)-algebras. We would like to conclude that \( G/G_p \cong G'/G'_p \). In view of Lemma 6.2.3 it suffices to consider the case where \( G \) and \( G' \) have only \( p \)-torsion and show that \( G \cong G' \). In this situation it is known that \( G/G_p \cong G'/G'_p \), the divisible parts of \( G_p \) and \( G'_p \) are isomorphic, and the reduced parts of \( G_p \) and \( G'_p \) have the same Ulm invariants (see May [1]). Thus if the reduced parts of \( G_p \) and \( G'_p \) are totally projective then it follows from Hill's version of Ulm's Theorem (see Griffith [1] for a
reference) that $G_p \supsetneq G'_p$. In the case when the reduced parts of $G_p$ and $G'_p$ are not totally projective we can at least deduce that $|G_p| = |G'_p|$ from Proposition 6.1.4. However, in general it is not known whether $G_p \supsetneq G'_p$. Moreover, if $G$ is a split group it is not known whether $G'_p$ must also split.

In conclusion we reiterate our conjecture: The rings which satisfy the Isomorphism Theorem are precisely the ND-rings. It is now apparent in view of Theorem 6.3.3 that considerably more information must be obtained concerning isomorphism questions over fields of prime characteristic before our conjecture can be verified or rejected.
Let $R = R_1 \times \ldots \times R_n$ where each $R_j$, $1 \leq j \leq n$, is an indecomposable ring with $\text{char}(R_j) = 0$. In Theorem 4.6 we gave necessary and sufficient conditions for such a ring to satisfy the Isomorphism Theorem. In particular, we saw that $R$ does not necessarily satisfy the Isomorphism Theorem. Thus if $G$ and $H$ are abelian groups with $G_R$ and $H_R$ trivial and $RG \cong RH$ as $R$-algebras we cannot conclude in general that $G \cong H$. However we shall soon see that some relationships do exist between $G$ and $H$ in certain circumstances.

Let $R$ be as above and once again suppose that $G$ and $H$ are abelian groups with $G_R$ and $H_R$ trivial and $RG \cong RH$ as $R$-algebras. In Section 1 we will see that the torsion subgroups $G_0$ and $H_0$ are necessarily isomorphic. In Section 2 we continue our study of units of group algebras over indecomposable rings begun in the third section of Chapter 2. We will then be able to show that if each $R_j$ ($1 \leq j \leq n$) is an integral domain with $\text{char}(R_j) = 0$ and if $G$ is a split group then $G \cong H$. In particular, the
§1. $G_0$ is Determined up to Isomorphism

In this section it will be shown that if $R = R_1 \times R_2 \times \cdots \times R_n$ where each $R_j$ ($1 \leq j \leq n$) is an indecomposable ring with $\text{char}(R_j) = 0$ then the group algebra $RG$ determines $G_0$ up to isomorphism provided that $G_R$ is trivial.

Suppose $R = R_1 \times R_2 \times \cdots \times R_n$ where each $R_j$ is a ring and let $G$ be an abelian group. There is an explicit isomorphism of $RG$ with $R_1G \times R_2G \times \cdots \times R_nG$ given as follows: $\sum_{g \in G}(r_{1g}, r_{2g}, \ldots, r_{ng})g \in RG$ corresponds to $(\sum_{g \in G}r_{1g}g, \sum_{g \in G}r_{2g}g, \ldots, \sum_{g \in G}r_{ng}g) \in R_1G \times R_2G \times \cdots \times R_nG$ where $r_{jg} \in R_j$ for all $g \in G$ and $1 \leq j \leq n$. When convenient we shall identify $RG$ with $R_1G \times R_2G \times \cdots \times R_nG$ via this isomorphism. It should be noted that if $R$ happens to be an infinite product and if $G$ is infinite then the above isomorphism, extended in the obvious fashion to infinite products, identifies $RG$ with a proper subalgebra of the corresponding infinite product of group algebras.

Once again let $R = R_1 \times R_2 \times \cdots \times R_n$ be a finite product of rings and let $G$ be any abelian group. Under the isomorphism $RG \rightarrow R_1G \times R_2G \times \cdots \times R_nG$ described above observe that $G \subseteq RG$ is identified with $\{(g, g, \ldots, g) | g \in G\}$. 

groups $G$ and $H$ in the example at the end of Chapter 4 cannot be split groups.
Under these circumstances we write \( \hat{G} \) for this image of \( G \) in \( R_1G \times R_2G \times \ldots \times R_nG \). Note that \( \hat{G} \subseteq G \times G \times \ldots \times G \subseteq R_1G \times R_2G \times \ldots \times R_nG \) with \( \hat{G} = G \times G \times \ldots \times G \) if and only if \( G \) is trivial or \( n = 1 \).

Our first two Lemmas show that the identification of \( RG \) with \( R_1G \times \ldots \times R_nG \) is, in some sense, natural.

**Lemma A.1.1.** Let \( R = R_1 \times R_2 \times \ldots \times R_n \) where each \( R_j \) is a ring \((1 \leq j \leq n)\) and suppose \( G \) is an abelian group. If \( RG \) is identified with \( R_1G \times R_2G \times \ldots \times R_nG \) as above then:

1. \( U(RG) = U(R_1G) \times U(R_2G) \times \ldots \times U(R_nG) \);
2. \( V(RG) = V(R_1G) \times V(R_2G) \times \ldots \times V(R_nG) \).

**Proof.** (1) is clear since \( 1_R = (1_{R_1}, 1_{R_2}, \ldots, 1_{R_n}) \) and so \( \alpha \in RG \) is a unit of augmentation 1 (= \( 1_R \)) if and only if each component of \( \alpha \) is a unit of augmentation 1. A moment of reflection reveals that if \( \alpha \) is integral over \( R \) then its \( j \)-th component is integral over \( R_j \). Thus \( V \subseteq V_1 \times V_2 \times \ldots \times V_n \) where \( V = V(RG) \) and \( V_j = V(R_jG) \). To complete the proof we need only show that if \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) has \( j \)-th component integral over \( R_j \) for every \( j \) then \( \alpha \) is integral over \( R = R_1 \times R_2 \times \ldots \times R_n \).
For each \( j \) choose a monic polynomial \( f_j \in R_j[X] \) of degree \( d(j) \geq 1 \) such that \( f_j(a_j) = 0 \). Say \( f_j(X) = \sum_{i=0}^{d(j)-1} r_{ij} X^i \) where \( r_{ij} \in R_j \) \((0 \leq i \leq d(j) - 1, 1 \leq j \leq n)\). For each \( j \) define \( \hat{f}_j \in R[X] \) by \( \hat{f}_j(X) = S_{0j} \sum_{i=1}^{d(j)} X^{d(j)-1} + \sum_{i=1}^{d(j)} X^{d(j)-1} + \sum_{i=1}^{d(j)} X^{d(j)-1} \) where \( S_{ij} \in R \) has \( k \)-th component 0 if \( j \neq k \) and \( j \)-th component \( r_{ij} \). Then \( f(X) = \prod_{j=1}^{n} \hat{f}_j(X) \) is monic of degree \( \sum_{j=1}^{n} d(j) \). Moreover, \( f(\alpha) = 0 \) since the \( j \)-th component of \( f_j(\alpha) \) is 0 for all \( j \).

Thus \( \alpha \) is integral over \( R \).

Q.E.D.

Lemma A.1.2. Let \( R = R_1 \times \ldots \times R_n \) be a finite product of rings and suppose \( G \) and \( H \) are abelian groups. Identify \( RG \) (respectively, \( RH \)) with \( R_1G \times \ldots \times R_nG \) (respectively, \( R_1H \times \ldots \times R_nH \)) as above and let \( \Pi_j:RG \rightarrow R_jG \) and \( \Pi_j':RH \rightarrow R_jH \) be the natural projections. If \( f:RG \rightarrow RH \) is an \( R \)-algebra homomorphism then for each \( j \) there exists an \( R_j \)-algebra homomorphism \( f_j:R_jG \rightarrow R_jH \) such that \( f_j \circ \Pi_j = \Pi_j' \circ f \). Moreover, if \( f \) is a normalized \( R \)-algebra isomorphism then each \( f_j \) can be chosen to be a normalized \( R_j \)-algebra isomorphism.

Proof. Define \( f_j:R_jG \rightarrow R_jH \) as follows: Given \( r \in R_j \) let \( r_j \in R \) have \( k \)-th component 0 if \( j \neq k \) and \( j \)-th component \( r \). Now if \( r \in R_j \) and \( g \in G \) define \( f_j(rg) = (\Pi_j' \circ f)(r_jg) \) and then extend linearly to
all of \( R_j G \). A routine check reveals that these \( f_j \)'s work. Q.E.D.

We now proceed with our proof of the desired isomorphism result. Our program will be similar to the procedure used in Chapter 3 to prove that indecomposable rings of characteristic 0 satisfy the Isomorphism Theorem (see Theorem 3.6). More precisely, we shall show that if \( R = R_1 \times \ldots \times R_n \) where each \( R_j \) is indecomposable with \( \text{char}(R_j) = 0 \) then \( \hat{\mathcal{C}}_0 \) is a direct summand of \( V(RG) \) in a particularly nice way provided that \( G_R \) is trivial. However, in this more general setting it is not necessarily true that \( U(RG) \) is the pushout of \( \hat{\mathcal{C}} \) and \( V(RG) \) by \( \hat{\mathcal{C}}_0 \). Thus \( \hat{\mathcal{C}} \) will not necessarily be a direct summand of \( U(RG) \) as before and we must content ourselves with retrieving \( \hat{\mathcal{C}}_0 \) from \( RG \). Since \( \hat{\mathcal{C}}_0 \cong G_0 \) we will then obtain the desired result.

**Proposition A.1.3.** Suppose \( R = R_1 \times \ldots \times R_n \) where each \( R_j \), \( 1 \leq j \leq n \), is an indecomposable ring with \( \text{char}(R_j) = 0 \). Let \( G \) be an abelian group with \( G_R \) trivial. Then, \( \hat{\mathcal{C}}_0 \) is a direct summand of \( V(RG) \).

**Proof.** Set \( V = V(RG) \) and let \( p_1, p_2, \ldots \) be the prime numbers such that \( p_i \not\in \text{inv}(R) \), \( i \geq 1 \). (If no such \( p_i \)'s exist, \( \hat{\mathcal{C}}_0 \) is trivial and the proof is complete.)
For each $i$ use the Corollary to Lemma 2.1.3 to select a prime ideal $P_i$ of $R$ such that $P_i \notin \operatorname{inv}(R/P_i)$ and $\operatorname{char}(R/P_i) = 0$. Set $H_i = \bigoplus_{p \neq P_i} G_p$, the sum being taken over all prime numbers $p$ different from $P_i$. Let $\phi_i: RG \to (R/P_i)(G/H_i)$ be the natural maps and set $V_i = V((R/P_i)(G/H_i))$. Note that $\phi_i(V) \subseteq V_i$. By Theorem 2.3.4 there is a group-homomorphism $\Pi_i: V_i \to (G/H_i)p_i$ for each $i$ such that $\Pi_i$ restricts to the identity map on $(G/H_i)p_i$.

Next set $\phi_{i1} = \phi_i|V$ and $\phi_{i2} = \phi_i|G_{P_i}$ and note that $\phi_{i2}: G_{P_i} \to (G/H_i)p_i$ is an isomorphism for all $i$. Now set $\rho_i = \phi_{i2}^{-1} \circ \Pi_i \circ \phi_{i1}: V \to G_{P_i}$ and let $\rho = \prod_i \rho_i: V \to \prod_i G_{P_i}$ be the induced map into the (external) direct product.

We claim that $\rho(V) \subseteq \bigsqcup_i G_{P_i}$, the external direct sum, where $\bigsqcup_i G_{P_i}$ is identified with a subgroup of $\prod_i G_{P_i}$ in the obvious fashion. To show this note that it suffices to show that if $a \in V$ then $\phi_i(a)$ is the multiplicative identity of $(R/P_i)(G/H_i)$ for all but finitely many $i$.

Since $a \in V$, $a \in RG_0 + N(RG)$ by Proposition 2.2.2. Thus we may write $a = r_1g_1 + \ldots + r_kg_k + \beta$ where $r_1, \ldots, r_k \in R; g_1, \ldots, g_k \in G_0; \beta \in N(RG)$. Since $\beta$ is
nilpotent so is $\phi_i(\beta)$ for each $i$. Thus $\phi_i(\beta)$
$\in N((R/P_i)(G/H_i)) = 0$ for each $i$ by the Corollary to
Proposition 2.2.2. Consequently $\phi_i(\alpha) = (r_1 + P_i)(g_1H_i)$
+ $\ldots + (r_k + P_i)(g_kH_i)$ for every $i$. But each $g_\ell$,
$1 \leq \ell \leq k$, is in all but finitely many $H_i$ and so $\phi_i(\alpha)$
$= [(r_1 + \ldots + r_k) + P_i](eH_i)$ for all but finitely many $i$.
Since $\text{aug}_R(\alpha) = 1$ we conclude that $\phi_i(\alpha) = (1 + P_i)(eH_i)$
for all but finitely many $i$. Thus $\rho(V) \subseteq \prod G_{P_i}$ as
claimed.

Let $\mu: \prod G_{P_i} \to G_0$ be given by $\mu(g_1, g_2, \ldots)$
$= g_1g_2 \ldots$ where $g_i \in G_{P_i}$. Note that $\mu$ makes sense since
all but finitely many $g_i$ are equal to $e$. Thus $\mu$ is
clearly a group-isomorphism. Finally, let $\lambda: G_0 \to \tilde{G}_0$ be
such that every component of $\lambda(g)$ is $g$ ($g \in G_0$). Thus
$\lambda \circ \mu \circ \rho: V \to \tilde{G}_0$ is a group-homomorphism which splits the
inclusion map $\tilde{G}_0 \to V$. Therefore $\tilde{G}_0$ is a direct summand
of $V$. Q.E.D.

We note that Proposition A.1.3 is also true if $R$
is an infinite product of indecomposable rings of characteristic $0$ provided that the definition of $\tilde{G}$ is extended
(in the obvious fashion) to the case of infinite products.
The proof is virtually identical to the above. It should
also be pointed out that Proposition A.1.3 generalizes
Proposition 3.1.
We will find the following technicality useful in proving Proposition A.1.5.

**Lemma A.1.4.** Suppose \( R = R_1 \times \ldots \times R_n \) where each \( R_j \) \((1 \leq j \leq n)\) is an indecomposable ring with \( \text{char}(R_j) = 0 \). Let \( p \) be a prime number with \( p \not\in \text{inv}(R) \). Suppose \( G \) and \( G' \) are abelian groups and set \( H = \bigoplus_{q \neq p} G_q \) and \( H' = \bigoplus_{q \neq p} G'_q \), the sums being taken over all prime numbers \( q \) different from \( p \). If \( f: RG \rightarrow RG' \) is a normalized \( R \)-algebra isomorphism then there exist a prime ideal \( P \) of \( R \) and a normalized \( (R/P) \)-algebra isomorphism \( \tilde{f}: (R/P)(G/H) \rightarrow (R/P)(G'/H') \) satisfying:

1. \( p \not\in \text{inv}(R/P) \) and \( \text{char}(R/P) = 0 \);
2. If \( \phi: RG \rightarrow (R/P)(G/H) \) and \( \phi': RG' \rightarrow (R/P)(G'/H') \) are the natural maps then \( \tilde{f} \circ \phi = \phi' \circ f \).

**Proof.** The existence of \( P \) satisfying (1) follows from the Corollary to Lemma 2.1.3. The existence of \( \tilde{f} \) satisfying (2) is guaranteed by Lemmas 3.2 and 3.4. Q.E.D.

We can now show that \( G_0 \) has complements in \( V(RG) \) with particularly nice properties.

**Proposition A.1.5.** Suppose \( R = R_1 \times \ldots \times R_n \) where each \( R_j \), \( 1 \leq j \leq n \), is an indecomposable ring with \( \text{char}(R_j) = 0 \). Let \( G \) and \( G' \) be abelian groups with \( G_R \)
and \( G'_R \) trivial. Suppose \( f: RG \to RG' \) is a normalized \( R \)-algebra isomorphism. Then, there exist direct sum decompositions \( V(RG) = G_0 \oplus F \) and \( V(RG') = G'_0 \oplus F' \) with \( f(F) = F' \).

**Proof.** Set \( V = V(RG) \) and \( V' = V(RG') \) and let \( p_1, p_2, \ldots \) be the prime numbers such that \( p_i \not\in \text{inv}(R) \), \( i \geq 1 \). (If no such \( p_i \)'s exist set \( F = V \) and \( F' = V' \) and the proof is complete.) For each \( i \) set \( H_i = \bigoplus_{p \neq p_i} G_p \) and \( H'_i = \bigoplus_{p \neq p_i} G'_p \), the sums being taken over all prime numbers \( p \neq p_i \). For each \( i \) use Lemma A.1.4 to select a prime ideal \( p_i \) of \( R \) and a normalized \((R/P_i)\)-algebra isomorphism \( \bar{\phi}_i : (R/P_i)(G/H_i) \to (R/P_i)(G'/H'_i) \) such that \( \bar{\phi}_i = \phi_i \circ f \) where \( \phi_i : RG \to (R/P_i)(G/H_i) \) and \( \phi'_i : RG' \to (R/P_i)(G'/H'_i) \) are the natural maps.

Set \( V_i = V((R/P_i)(G/H_i)) \) and \( V'_i = V((R/P_i)(G'/H'_i)) \). By Theorems 2.3.3 and 2.3.4 we may select group-homomorphisms \( \Pi_i : V_i \to (G/H_i)_{P_i} \) and \( \Pi'_i : V'_i \to (G'/H'_i)_{P_i} \) for each \( i \) such that \( \Pi_i \) (respectively, \( \Pi'_i \)) is the identity when restricted to \((G/H_i)_{P_i}\) (respectively, \((G'/H'_i)_{P_i}\)) and such that \( \bar{\phi}_i (\ker(\Pi_i)) = \ker(\Pi'_i) \) for every \( i \).

Set \( \phi_{i1} = \phi_i|_V \), \( \phi_{i1} = \phi_i|_{V'} \), \( \phi_{i2} = \phi_i|_{G_{P_i}} \) and \( \phi'_{i2} = \phi'_i|_{G'_P} \). Note that \( \phi_{i2} : (G/H_i)_{P_i} \to (G'/H'_i)_{P_i} \) and \( \phi'_{i2} : G'_P \to (G'/H'_i)_{P_i} \) are isomorphisms. Set \( \rho_i = \phi_{i2}^{-1} \circ \Pi_i \circ \phi_{i1} \).
\[ \rho_i = (\phi_{i2})^{-1} \circ \Pi_i \circ \phi_{i1} \] and let \[ \rho = \prod_i \rho_i : V \rightarrow \prod_i G_{p_i} \] and \[ \rho' = \prod_i \rho'_i : V_i \rightarrow \prod G'_{p_i} \] be the induced maps into the products.

In view of the proof of Proposition A.1.3 we may set \[ F = \ker(\rho) \] and \[ F' = \ker(\rho') \] and conclude that \[ V = G_0 \oplus F \] and \[ V' = G'_0 \oplus F'. \] Thus, to complete the proof it suffices to show that \[ f(\ker(\rho_i)) \subseteq \ker(\rho'_i) \] for every \( i \).

From the above discussion we have for each \( i \) the following commutative diagram of groups and group-homomorphisms where \[ \psi_i = (\phi_{i2})^{-1} \circ \overline{f}_i \circ \phi_{i2}. \]

Thus if \( \alpha \in \ker(\rho_i) \) then \[ \rho'_i(f(\alpha)) = (\psi_i \circ \rho_i)(\alpha) = \psi_i(e) = e. \] Therefore \( f(\alpha) \in \ker(\rho'_i) \) and \( f(\ker(\rho_i)) \subseteq \ker(\rho'_i) \) as required. Q.E.D.
We are now ready to prove the main result of this section:

**Theorem A.1.6.** Suppose $R = R_1 \times \cdots \times R_n$ where each $R_j$, $1 \leq j \leq n$, is an indecomposable ring with $\text{char}(R_j) = 0$. Let $G$ and $G'$ be abelian groups such that $G_R$ and $G'_R$ are trivial. If $RG \cong RG'$ as $R$-algebras then $G_0 \cong G'_0$.

**Proof.** By Lemma 2.1.1 we can choose a normalized $R$-algebra isomorphism $f:RG \rightarrow RG'$. Set $V = V(RG)$ and $V' = V(RG')$ and note that $f(V) = V'$. By Proposition A.1.5 we have direct sum decompositions $V = \hat{\bigoplus} F$ and $V' = \hat{\bigoplus} F'$ with $f(F) = f'$. Therefore, 

$$G_0 \cong \hat{\bigoplus} F \cong f(V)/f(F) = V'/F' \cong \hat{\bigoplus} F' \cong G'_0.$$  

Q.E.D.

We note that Theorem A.1.6 has added significance in view of a result in May [1]: Suppose $R$ is a ring and $G$ and $G'$ are abelian groups with $G_R$ and $G'_R$ trivial and $RG \cong RG'$ as $R$-algebras. Then the maximal divisible subgroups of $G_p$ and $G'_p$ are isomorphic and the Ulm $p$-invariants of the reduced parts of $G_p$ and $G'_p$ are the same for all primes of $p$.

In May [1] it was then asked whether in fact $G_p \cong G'_p$ for all primes $p$. Theorem 3.6 answers this question in the affirmative for indecomposable rings of
characteristic 0. Theorem A.1.6 gives an affirmative answer for an additional class of rings; namely, finite products of indecomposable rings of characteristic 0.

§2. Units of Group Algebras over Indecomposable Rings

Let \( R \) be a ring and suppose \( G \) is an abelian group. If \( H \) is a subgroup of \( G \) let \( \phi_H \) denote the natural map \( R G \rightarrow R(G/H) \). Define \( V_H(RG) \), a subgroup of \( U(RG) \), by \( V_H(RG) = U(RG) \cap \phi_H^{-1}(V(R(G/H))) \). Note that \( V(RG) \subseteq V_H(RG) \subseteq U(RG) \). We will see that \( V_H(RG) \) has several nice properties if \( R \) is indecomposable and if \( H \) is selected judiciously.

Our first result can be viewed as a generalization of Theorem 2.3.10.

**Proposition A.2.1.** Let \( R \) be an indecomposable ring and suppose \( G \) is an abelian group. If \( H \) is a subgroup of \( G \) with \( G_R \subseteq H \subseteq G_0 \) then \( U(RG) \) is the pushout of \( G \) and \( V_H(RG) \) by \( G_0 \).

**Proof.** Set \( \phi = \phi_H \) and \( V = V_H(RG) \). Clearly \( G_0 \subseteq V \cap G \) and if \( g \in V \cap G \) then \( \phi(g) = gH \in V(R(G/H)) \cap (G/H) = (G/H)_0 \). Thus \( g \in G_0 \) and so \( V \cap G = G_0 \).

Clearly \( GV \subseteq U(RG) \) and if \( \alpha \in U(RG) \) then \( \phi(\alpha) \in U(R(G/H)) \). By Theorem 2.3.10 \( U(R(G/H)) = (G/H) \cdot V(R(G/H)) \).
So, \( \phi(\alpha) = (gH) \bar{v} \) for some \( g \in G \) and \( \bar{v} \in V(R(G/H)) \).

Thus \( \phi(g^{-1}) \in V(R(G/H)) \) and so \( g^{-1} \in \mathcal{V} \). Therefore \( \alpha \in \mathcal{Gv} \) and \( U(RG) = \mathcal{Gv} \) as required. Q.E.D.

**Corollary 1.** Let \( R \) be an indecomposable ring and suppose \( G \) is an abelian group. Let \( H_1 \) and \( H_2 \) be subgroups of \( G \) with \( G_R \subseteq H_i \subseteq G_0, \ i = 1, 2 \). Then, \( \mathcal{V}_{H_1}(RG) = \mathcal{V}_{H_2}(RG) \).

**Proof.** Set \( \mathcal{V}_i = \mathcal{V}_{H_i}(RG), \ i = 1, 2 \), and set \( \mathcal{V}_H = \mathcal{V}_{H_i}(RG) \) where \( H = H_1 \cap H_2 \). By Proposition A.2.1 \( \mathcal{Gv}_H = \mathcal{Gv}_i \) and \( G \cap \mathcal{V}_H = G \cap \mathcal{V}_i \) for each \( i \). Since \( H \subseteq H_i, \ i = 1, 2 \), it is clear that \( \mathcal{V}_H \subseteq \mathcal{V}_i \). Therefore \( \mathcal{V}_1 = \mathcal{V}_H = \mathcal{V}_2 \). Q.E.D.

Suppose \( R \) is an indecomposable ring and let \( G \) be an abelian group. If \( H \) is any subgroup of \( G \) satisfying \( G_R \subseteq H \subseteq G_0 \) we define \( \mathcal{V}(RG) = \mathcal{V}_H(RG) \). Corollary 1 above shows that \( \mathcal{V}(RG) \) is independent of the choice of \( H \). We can now restate Proposition A.2.1 as follows.

**Corollary 2.** Let \( R \) be an indecomposable ring and let \( G \) be an abelian group. Then \( U(RG) \) is the pushout of \( G \) and \( \mathcal{V}(RG) \) by \( G_0 \).

Our next result shows that \( \mathcal{V}(RG) \) is an algebraic invariant of \( RG \) if \( \text{char}(R) = 0 \).
Proposition A.2.2. Let $R$ be an indecomposable ring with $\text{char}(R) = 0$ and suppose $G$ and $G'$ are abelian groups. If $f: RG \rightarrow RG'$ is a normalized $R$-algebra isomorphism then $f(V(RG)) = V(RG')$.

Proof. Set $\phi = \phi_{G_R}$, $\phi' = \phi_{G'_R}$, $\tilde{V} = V(R(G/G_R))$ and $\tilde{V}' = V(R(G'/G_R))$. Since $f(U(RG)) = U(RG')$ we need only show that $f(\phi^{-1}(\tilde{V})) \subseteq (\phi')^{-1}(\tilde{V}')$.

By Lemma 3.3 there exists a normalized $R$-algebra isomorphism $f^* : R(G/G_R) \rightarrow R(G'/G_R)$ with $f^* \circ \phi = \phi' \circ f$. Now, suppose $a \in \phi^{-1}(\tilde{V})$. Then $\phi(a) \in \tilde{V}$ and so $(f^* \circ \phi)(a) \in f^*(\tilde{V}) = \tilde{V}'$. Thus $(\phi' \circ f)(a) \in \tilde{V}'$ and $f(a) \in (\phi')^{-1}(\tilde{V}')$. Therefore $f(\phi^{-1}(\tilde{V})) \subseteq (\phi')^{-1}(\tilde{V}')$ as required. Q.E.D.

Our next result generalizes Proposition 3.1

Proposition A.2.3. Suppose $R$ is indecomposable with $\text{char}(R) = 0$ and let $G$ be an abelian group. Then $K = \bigoplus_{p \in \text{inv}(R)} G_p$ is a direct summand of $V(RG)$.

Proof. Set $\tilde{V} = V(R(G/G_R))$. By Proposition 3.1 there is a group-homomorphism $\Pi: \tilde{V} \rightarrow (G/G_R)_0$ which restricts to the identity on $(G/G_R)_0$. Set $\phi = \phi_{G_R}$ and note that $(\phi|K): K \rightarrow (G/G_R)_0$ is an isomorphism since $G_0 = K \bigoplus G_R$. Therefore the composition $(\phi|K)^{-1} \circ \Pi \circ (\phi|_V(RG)): V(RG) \rightarrow K$
splits the inclusion map \( K \rightarrow \mathbb{V}(RG) \). 

Q.E.D.

Suppose \( R \) is an indecomposable ring of characteristic 0 and let \( G \) and \( K \) be as above. We now show that \( K \) has a complement in \( \mathbb{V}(RG) \) with a useful mapping property. For the convenience of the reader we note that it is important to distinguish between \( K \) and \( G_R \). In fact, \( G_0 = K \oplus G_R \).

**Proposition A.2.4.** Suppose \( R \) is indecomposable with \( \text{char}(R) = 0 \) and let \( G \) and \( G' \) be abelian groups. Set \( K = \bigoplus \{ G_p | p \notin \text{inv}(R) \} \) and \( K' = \bigoplus \{ G'_p | p \notin \text{inv}(R) \} \). If \( f: RG \rightarrow RG' \) is a normalized \( R \)-algebra isomorphism then there exist direct sum decompositions \( \mathbb{V}(RG) = K \oplus F \) and \( \mathbb{V}(RG') = K' \oplus F' \) with \( f(F) = F' \).

**Proof.** Set \( \phi = \phi_{G_R} \) \( \phi' = \phi_{G'_R} \) \( \tilde{V} = V(R(G/G_R)) \), \( \tilde{V}' = V(R(G'/G'_R)) \), \( V = \mathbb{V}(RG) \) and \( V' = \mathbb{V}(RG') \). By Lemma 3.3 there is a normalized \( R \)-algebra isomorphism \( f^*: R(G/G_R) \rightarrow R(G'/G'_R) \) such that \( f^* \circ \phi = \phi' \circ f \). By Proposition 3.5 there exist direct sum decompositions \( \tilde{V} = (G/G_R)_0 \oplus \bar{F} \) and \( \tilde{V}' = (G'/G'_R)_0 \oplus \bar{F}' \) with \( f^*(\bar{F}) = \bar{F}' \). Thus we can choose group-homomorphisms \( \Pi \) and \( \Pi' \) such that the diagram
commutes and \( f^*(\ker(\Pi)) = \ker(\Pi') \).

Set \( F = \ker(\Pi \circ (\phi|_V)) \) and \( F' = \ker(\Pi' \circ (\phi'|_V')) \).

In view of the proof of Proposition A.2.3 it is clear that \( V = K \oplus F \) and \( V' = K' \oplus F' \). To complete the proof we need only show that \( f(F) \subseteq F' \).

Suppose \( \alpha \in F \). Then \( \Pi(\phi(\alpha)) = eG_R \) and so \( \phi(\alpha) \in \ker(\Pi) \). Thus \( \phi'(f(\alpha)) = f^*(\phi(\alpha)) \in f^*(\ker(\Pi)) = \ker(\Pi') \). Therefore \( f(\alpha) \in \ker(\Pi' \circ (\phi'|_V')) = F' \) and \( f(F) \subseteq F' \).

Q.E.D.

We include one further result in this section which will be of some use in Section 3.

**Proposition A.2.5.** Suppose \( R \) is an integral domain with \( \text{char}(R) = 0 \) and let \( S \) be a set of prime numbers with \( \text{inv}(R) \subseteq S \). Let \( G \) and \( G' \) be abelian groups and suppose \( f:RG \to RG' \) is a normalized \( R \)-algebra isomorphism. Set \( K = \bigoplus \{ G_p | p \not\in S \} \) and \( K' = \bigoplus \{ G'_p | p \not\in S \} \). Then, there exist direct sum decompositions \( V(R(G)) = K \oplus F \) and \( V(R(G')) = K' \oplus F' \) with \( f(F) = F' \).
Proof. Set $\phi = \phi_{G_R}$, $\phi' = \phi_{G'_R}$, $V = V(RG)$, $V' = V(RG')$, $L = \bigoplus \{G_p | p \in S \}$ and $L' = \bigoplus \{G'_p | p \in S \}$. Note that $G_R \subseteq L$ and $G'_R \subseteq L'$.

By Lemma 3.3 there exists a normalized R-algebra isomorphism $f^* : R(G/G_R) \rightarrow R(G'/G'_R)$ with $f^* \circ \phi = \phi' \circ f$.

Set $V = V(R(G/G_R))$ and $V' = V(R(G'/G'_R))$. By Theorems 2.3.3 and 2.3.4 there exist group-homomorphisms $\Pi : V \rightarrow (G/G_R)_0$ and $\Pi' : V' \rightarrow (G'/G'_R)_0$ such that $\Pi$ (respectively, $\Pi'$) is the identity when restricted to $(G/G_R)_0$ (respectively, $(G'/G'_R)_0$) and $f^*(\ker(\Pi)) = \ker(\Pi')$.

Let $\psi : G/G_R \rightarrow G/L$ and $\psi' : G'/G'_R \rightarrow G'/L'$ be the natural quotient maps. Note that $\phi_L|K$ and $\phi_L'|K'$ are isomorphisms and set $\eta = (\phi_L|K)^{-1} \circ \psi \circ \Pi \circ (\phi|V)$ and $\eta' = (\phi_L'|K')^{-1} \circ \psi' \circ \Pi' \circ (\phi'|V')$. Then $\eta$ (respectively, $\eta'$) restricts to the identity on $K$ (respectively, $K'$).

Set $F = \ker(\eta)$ and $F' = \ker(\eta')$. Then $V = K \bigoplus F$ and $V' = K' \bigoplus F'$. Finally, a routine computation reveals that $f(F) = F'$.

Q.E.D.

§3. Group Algebras of Split Groups Over Finite Products of Integral Domains

In this section our main result is Theorem A.3.7:

Suppose $R = R_1 \times \ldots \times R_n$ where each $R_j$, $1 \leq j \leq n$, is an integral domain with $\text{char}(R_j) = 0$. Let $G$ and $G'$ be abelian groups with $G_R$ and $G'_R$ trivial and suppose
RG \cong RG' as R-algebras. If G is a split group then G \cong G'. In particular, G' must also split.

Our approach in proving Theorem A.3.7 will be somewhat similar to that used in proving the Isomorphism Theorem for indecomposable rings of characteristic 0 (Theorem 3.6). If R is indecomposable with \text{char}(R) = 0 and if G is an abelian group with \( C_R \) trivial we saw in Chapter 3 that \( U(RG) = G \bigoplus F \) where the complement F can be chosen to have a nice mapping property. This then was enough to determine G when RG was known up to isomorphism. Similarly, if \( R = R_1 \times \ldots \times R_n \) as above and if G is split we will see that \( \overset{\sim}{G} \) is a direct summand of a subgroup of \( U(RG) \) that can be algebraically determined from RG. Moreover, we will see that \( \overset{\sim}{G} \) has a complement in this subgroup with a nice enough mapping property so that \( \overset{\sim}{G} \) (and hence G) is determined if RG is known up to isomorphism.

Lemma A.3.1. Let R be an integral domain with \text{char}(R) = 0 and let G be an abelian group. Then the ideal I of RG generated by \( \{g - e | g \in G_0\} \) is the ideal \( \langle S \rangle \) of RG generated by \( S = \{a \in RG | \text{aug}(a) = 0 \text{ and } a \in A(RG)\} \).

Proof. Clearly \( I \subseteq \langle S \rangle \). Suppose \( a \in S \). To complete the proof it suffices to show that \( a \in I \). Now, \( a \in A(RG) \) and so \( a \in RG_0 \) by the Corollary to Proposition 2.2.2. The Corollary to Lemma 2.1.2 says that the ideal of
RG₀ generated by \( \{g - e|g \in G₀\} \) is the kernel of the augmentation mapping on \( RG₀ \). Since \( \text{aug}(a) = 0 \) it follows that \( a \in I \). Q.E.D.

Our next result was essentially shown in May [1] for the case when \( R \) is an algebraically closed field (with no restriction on \( \text{char}(R) \)).

**Lemma A.3.2.** Let \( R \) be an integral domain with \( \text{char}(R) = 0 \) and suppose that \( G \) and \( G' \) are abelian groups. Let \( \phi:RG \rightarrow R(G/G₀) \) and \( \phi':RG' \rightarrow R(G'/G'₀) \) be the natural maps and suppose \( f:RG \rightarrow RG' \) is a normalized \( R \)-algebra isomorphism. Then, there exists a normalized \( R \)-algebra isomorphism \( \overline{f}:R(G/G₀) \rightarrow R(G'/G'₀) \) with \( \overline{f} \circ \phi = \phi' \circ f \).

**Proof.** Set \( I = \ker(\phi) \) and \( I' = \ker(\phi') \). By Lemma 2.1.2 \( I \) (respectively, \( I' \)) is the ideal of \( RG \) (respectively, \( RG' \)) generated by \( \{g - e|g \in G₀\} \) (respectively, \( \{g' - e|g' \in G'₀\} \)). Thus \( f(I) = I' \) by Lemma A.3.1.

Let \( f*:RG/I \rightarrow RG'/I' \), \( \phi*:RG/I \rightarrow R(G/G₀) \) and \( \phi':RG'/I' \rightarrow R(G'/G'₀) \) be the \( R \)-algebra isomorphisms induced by \( f \), \( \phi \) and \( \phi' \) respectively. Then \( \overline{f} = \phi' \circ f* \circ \phi⁻¹ \) is the desired isomorphism. Q.E.D.

If \( X \) is any group we let \( \text{Aut}(X) \) denote the group of automorphisms of \( X \).
We will find it convenient to have the following group-theoretic result which deals with the lifting of automorphisms. Its proof is a triviality and will be omitted.

**Lemma A.3.3.** Let $G$ be a split abelian group and let $\phi : G \to G/G_0$ be the quotient map. If $\psi \in \text{Aut}(G/G_0)$ then there exists $\eta \in \text{Aut}(G)$ such that $\psi \circ \phi = \phi \circ \eta$.

Until further notice let $R = R_1 \times \ldots \times R_n$ where each $R_i$ $(1 \leq i \leq n)$ is an integral domain with $\text{char}(R_i) = 0$; let $G$ and $G'$ be abelian groups with $G_R$ and $G'_R$ trivial; and, suppose $f : RG \to RG'$ is a normalized $R$-algebra isomorphism. Also, we identify $RG$ (respectively, $RG'$) with $R_1G \times \ldots \times R_nG$ (respectively, $R_1G' \times \ldots \times R_nG'$) as in Section A.1 above.

For each $i$ let $\Pi_i : RG \to R_iG$ and $\Pi'_i : RG' \to R_iG'$ be the projection maps. By Lemma A.1.2 we may choose for each $i$ a normalized $R_i$-algebra isomorphism $f_i : R_iG \to R_iG'$ such that $f_i \circ \Pi_i = \Pi'_i \circ f$.

If $\phi_i : R_iG \to R_i(G/G_0)$ and $\phi'_i : R_iG' \to R_i(G'/G'_0)$ are the natural maps then Lemma A.3.2 gives, for each $i$, a normalized $R_i$-algebra isomorphism $f_i : R_i(G/G_0) \to R_i(G'/G'_0)$ such that $f_i \circ \phi_i = \phi'_i \circ f_i$ for every $i$. Moreover, since each $R_i$ is an integral domain and since $G/G_0$ and $G'/G'_0$
are torsion free, \( U(R_i(G/G_0)) = G/G_0 \) and \( U(R_i(G'/G'_0)) = G'/G'_0 \) by Lemma 2.2.1(3).

In summary, for each \( i \), we have the following commutative diagram of abelian groups with all homomorphisms obtained by restriction.

\[
\begin{array}{c}
\text{U}(R G) \xrightarrow{\pi_i} \text{U}(R_i G) \xrightarrow{\phi_i} \text{U}(R_i (G/G_0)) = G/G_0 \\
\downarrow \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \downarrow \\
\text{U}(RG') \xrightarrow{-\pi_i} \text{U}(R_i G') \xrightarrow{\phi_i} \text{U}(R_i (G'/G'_0)) = G'/G'_0
\end{array}
\]

Proposition A.3.4. With notation as above set \( V_i = V(R_i G) \) and \( V_i' = V(R_i G') \). Then, if \( G' \) is a split group there exists a normalized \( R \)-algebra isomorphism \( \tilde{\psi}: R G \rightarrow R G' \) such that \( \tilde{\psi}((V_1 \times \ldots \times V_n)_{G'}) = (V_1' \times \ldots \times V_n')_{G'} \).

**Proof.** Assume \( n \geq 2 \). As the proof progresses the case \( n = 1 \) will become clear.

For \( 2 \leq i \leq n \) define \( \psi_i \in \text{Aut}(G'/G'_0) \) by \( \psi_i = (\tilde{\psi} \circ -\tilde{\psi}_i^{-1})|_{(G'/G'_0)} \). From Lemma A.3.3 we obtain \( \eta_2, \ldots, \eta_n \in \text{Aut}(G') \) such that \( \psi_i \circ \phi = \phi \circ \eta_i \) (\( 2 \leq i \leq n \)) where \( \phi: G' \rightarrow G'/G'_0 \) is the quotient map.

Each \( \eta_i \) induces a normalized \( R_i \)-algebra automorphism \( \gamma_i \) of \( R_i G' \) which restricts to an element (which we again
call $\gamma_i$ of $\text{Aut}(U(R_i G'))$ such that the diagram

$$\begin{array}{ccc}
U(R_i G') & \xrightarrow{\phi_i} & U(R_i (G'/G'_0)) = G'/G'_0 \\
\downarrow{\gamma_i} & & \downarrow{\psi_i} \\
U(R_i G') & \xrightarrow{\phi_i} & U(R_i (G'/G'_0)) = G'/G'_0
\end{array}$$

commutes for $2 \leq i \leq n$.

Define $\hat{\mathfrak{f}}: RG \to RG'$ by $\hat{\mathfrak{f}}((\alpha_1, \alpha_2, \ldots, \alpha_n)) = (f_1(\alpha_1), (\gamma_2 \circ f_2)(\alpha_2), \ldots, (\gamma_n \circ f_n)(\alpha_n))$ where $\alpha_i \in R_i G$, $1 \leq i \leq n$. Clearly $\hat{\mathfrak{f}}$ is a normalized $R$-algebra isomorphism. To show that $\hat{\mathfrak{f}}((V_1 \times \ldots \times V_n) G') = (V_1' \times \ldots \times V_n') G'$ it suffices, by Proposition A.2.2, to show that $\hat{\mathfrak{f}}(\overline{G}) \subseteq (V_1' \times \ldots \times V_n') \overline{G'}$.

So, suppose $(g, \ldots, g) \in \overline{G}$. Then $\hat{\mathfrak{f}}(g, \ldots, g) = (v_1 g_1, v_2 g_2, \ldots, v_n g_n) \in U(RG')$ where $v_i \in V_i'$, $g_i \in G'$ by Corollary 2 of Proposition A.2.1 and Lemma A.1.1(1).

Note that by using the definitions of $\hat{\mathfrak{f}}$ and $\psi_i$ together with the diagrams (*) and (***) we have $$(\phi_i' \circ \Pi_i' \circ \hat{\mathfrak{f}})(g, \ldots, g) = (\phi_i' \circ \Pi_i' \circ \hat{\mathfrak{f}})(g, \ldots, g)$$ for all $i$, $1 \leq i \leq n$. On the other hand $$(\phi_i' \circ \Pi_i' \circ \hat{\mathfrak{f}})(g, \ldots, g) = \phi_i'(v_i g_i) = [\phi_i'(V_i')](g_i G'_0)$. But $\phi_i'(v_i) \in \phi_i'(V_i') \subseteq V(R_i (G'/G'_0))$ which is trivial for all $i$ by Lemma
2.2.1(2). Hence \((\phi_1^i \circ \Pi_i^i \circ \psi)(g, \ldots, g) = g_i G_0^i\) for all \(i\). We conclude then that \(g_i G_0^i = g_i G_0^i\) for all \(i \geq 2\).

For each \(i \geq 2\) choose \(g_i^i \in G_0^i\) such that
\[g_i = g_1 g_i^i.\]
So finally, \(\psi(g, \ldots, g) = (v_1 g_1, v_2 g_2, \ldots, v_n g_n)\)
\[= (v_1 g_1, v_2 g_1 g_2, \ldots, v_n g_1 g_2 \ldots g_n) = (v_1, v_2 g_2', \ldots, v_n g_n')(g_1, g_1', \ldots, g_1) \in (V_1' \times \ldots \times V_n')G'\] as required. Q.E.D.

We remark that the group \((V_1' \times \ldots \times V_n')G'\) of Proposition A.3.4 is the subgroup of \(U(RG)\) referred to at the beginning of this section. Thus, the remainder of our work in this section will be directed towards showing that \(\hat{G}'\) is a direct summand of this subgroup and that whenever \(f:RG \to RG'\) is a normalized \(R\)-algebra isomorphism with \(G'\) split then a complement for \(\hat{G}'\) can be chosen which has a certain nice mapping property with respect to the isomorphism \(\hat{f}\) constructed in Proposition A.3.4.

**Proposition A.3.5.** Suppose \(R = R_1 \times \ldots \times R_n\) where each \(R_i\) \((1 \leq i \leq n)\) is an integral domain with \(\text{char}(R_i) = 0\). Let \(G\) and \(G'\) be abelian groups with \(G_R\) and \(G_R'\) trivial and suppose \(f:RG \to RG'\) is a normalized \(R\)-algebra isomorphism. Then, if \(G'\) splits there exist direct sum decompositions
\[V(R_1 G) \times \ldots \times V(R_n G) = G_0^i \oplus F\] and
\[V(R_1 G') \times \ldots \times V(R_n G') = G_0^i \oplus F'\] with \(f(F) = F'\).
Proof. Set $V_i = V(R_i G)$, $V'_i = V(R_i G')$, $V = V_1 \times \ldots \times V_n$ and $V' = V'_1 \times \ldots \times V'_n$.

By Theorem A.1.6, $G_0 \cong G'_0$. In particular, $\text{supp}(G) = \text{supp}(G')$. Partition $\text{supp}(G)$ into $n$ disjoint sets $S_1, \ldots, S_n$ such that $S_i \cap \text{inv}(R_i)$ is empty, $1 \leq i \leq n$. (It should be noted that we do not require that each $S_i$ is nonempty.) Set $H_i = \bigoplus \{G_p | p \in S_i\}$ and $H'_i = \bigoplus \{G'_p | p \in S_i\}$ and observe that $G_0 = \bigoplus_i H_i$ and $G'_0 = \bigoplus_i H'_i$.

For each $i$ use Lemma A.1.2 to select a normalized $R_i$-algebra isomorphism $f_i : R_i G \cong R_i G'$ such that if $\Pi_i : R G \cong R_i G$ and $\Pi'_i : R G \cong R_i G'$ are the projection maps then $f_i \circ \Pi_i = \Pi'_i \circ f_i$.

Since $\text{inv}(R_i)$ is in the complement of $S_i$ in the set of all prime numbers we have from Propositions A.2.3 and A.2.5 direct sum decompositions $V_i = H_i \bigoplus F_i$ and $V'_i = H'_i \bigoplus F'_i$ such that $f_i(F_i) = F'_i$ for each $i$. Set $F = F_1 \times \ldots \times F_n$ and $F' = F'_1 \times \ldots \times F'_n$.

Clearly $V = (H_1 \times \ldots \times H_n) \bigoplus F$ and $V' = (H'_1 \times \ldots \times H'_n) \bigoplus F'$. Moreover, $f(F) = f(F_1 \times \ldots \times F_n) = f_1(F_1) \times \ldots \times f_n(F_n) = F'_1 \times \ldots \times F'_n = F'$. So, all that remains is to show that $V = G_0 \bigoplus F$ and $V' = G'_0 \bigoplus F'$.

Let $\Pi : V \rightarrow H_1 \times \ldots \times H_n$ and $\Pi' : V' \rightarrow H'_1 \times \ldots \times H'_n$ be the projections along $F$ and $F'$ respectively. Let $\mu : H_1 \times \ldots \times H_n \rightarrow G_0$ and $\mu' : H'_1 \times \ldots \times H'_n \rightarrow G'_0$ be the multiplication maps (e.g., $\mu(h_1, h_2, \ldots, h_n) = h_1 h_2 \ldots h_n)$. 
Let \( \psi: G_0 \to \hat{G}_0 \) and \( \psi': G_0 \to \hat{G}'_0 \) be the natural injections (e.g., \( \psi(g) = (g, g, \ldots, g) \)). Clearly \( \mu, \mu', \psi \) and \( \psi' \) are all group-isomorphisms.

Now \( \psi \circ \mu \circ \Pi: \hat{V} \to \hat{G}_0 \) and \( \psi' \circ \mu' \circ \Pi': \hat{V}' \to \hat{G}'_0 \) are group-homomorphisms which restrict to the identity on \( \hat{G}_0 \) and \( \hat{G}'_0 \) respectively. Moreover, \( \ker(\psi \circ \mu \circ \Pi) = \ker(\Pi) = F \) and \( \ker(\psi' \circ \mu' \circ \Pi') = \ker(\Pi') = F' \). Thus \( \hat{V} = \hat{G}_0 \oplus F \) and \( \hat{V}' = \hat{G}'_0 \oplus F' \). Q.E.D.

Lemma A.3.6. Suppose \( R = R_1 \times \cdots \times R_n \) where each \( R_i \) (\( 1 \leq i \leq n \)) is indecomposable and let \( G \) be an abelian group. Then \( (\hat{V}(R_1G) \times \cdots \times \hat{V}(R_nG)) \cap \hat{G}_0 = \hat{G}_0 \).

Proof. Clearly \( \hat{G}_0 \subseteq (\hat{V}(R_1G) \times \cdots \times \hat{V}(R_nG)) \cap \hat{G} \) and the other inclusion follows from Corollary 2 of Proposition A.2.1. Q.E.D.

We are now ready to prove the main result of this section.

Theorem A.3.7. Let \( R = R_1 \times \cdots \times R_n \) where each \( R_i \) (\( 1 \leq i \leq n \)) is an integral domain with \( \text{char}(R_i) = 0 \). Suppose \( G \) and \( G' \) are abelian groups with \( G_R \) and \( G'_R \) trivial. If \( RG \cong RG' \) as \( R \)-algebras and if \( G \) splits then \( G \cong G' \). In particular \( G' \) must also split.
Proof. Set \( V = V(R_1 G) \times \ldots \times V(R_n G) \) and 
\( V' = V(R_1 G') \times \ldots \times V(R_n G') \). By Lemma 2.1.1 and by
Proposition A.3.4 we can choose a normalized \( R \)-algebra isomorphism 
\( f: RG' \rightarrow RG \) such that \( f(V'G') = \overline{V'} \). By Proposition
A.3.5 we can write \( V = G_0 \oplus F \) and \( V' = G'_0 \oplus F' \) with 
\( f(F') = F \).

Note that \( \overline{V'} = G_0 F' G = F G' \) and \( F \cap G \subseteq F \cap G_0 = G_0 \)
by Lemma A.3.6. Thus \( F \cap G = F \cap G_0 = F \cap G_0 \) which is trivial.
Hence \( \overline{V'} = G_0 F \). Likewise \( \overline{V'} = G'_0 \oplus F' \).
Since \( f(V'G') = \overline{V'} \) and \( f(F') = F \) it follows that 
\( G' \subseteq G \). Therefore \( G' \cong G \). Q.E.D.

We should point out that Theorem A.3.7 remains true
if the hypothesis that \( G \) splits is replaced by the require-
ment that any automorphism of \( G/G_0 \) can be lifted to \( G \)
(in the sense of Lemma A.3.3). This is clear after inspecting
the proof of Proposition A.3.4.

We conclude by giving an example. Let \( p_1, p_2, \ldots, p_n \) (\( n \geq 2 \)) be prime numbers such that at least
two of the \( p_i \)'s are different. Set \( R = \mathbb{Z}[1/p_1] \times \ldots \times \mathbb{Z}[1/p_n] \).
Note that \( \text{inv}(R) \) is empty so that \( G_R \) is
trivial for all abelian groups \( G \).

Since \( R \) is not an \( ND \)-ring, \( R \) does not satisfy the
Isomorphism Theorem by Proposition 5.1. However Theorems
A.1.6 and A.3.7 imply that the isomorphism class of the
commutative group algebra $RG$ determines the isomorphism class of $G$ if $G$ is torsion, torsion free or split.
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