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APPROXIMATING REACHABLE SETS FOR A CLASS OF LINEAR SYSTEMS SUBJECT TO BOUNDED CONTROL

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APPROXIMATING REACHABLE SETS FOR A CLASS OF LINEAR SYSTEMS SUBJECT TO BOUNDED CONTROL

by

Jonathan Edward Gayek

A Dissertation Submitted to the Faculty of the

PROGRAM IN APPLIED MATHEMATICS

In Partial Fulfillment of the Requirements
For the Degree of

DOCTOR OF PHILOSOPHY

In the Graduate College

THE UNIVERSITY OF ARIZONA

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THE UNIVERSITY OF ARIZONA
GRADUATE COLLEGE

As members of the Final Examination Committee, we certify that we have read

the dissertation prepared by Jonathan Edward Gayek

entitled Approximating Reachable Sets for a Class of Linear Systems Subject to Bounded Control

and recommend that it be accepted as fulfilling the dissertation requirement

for the Degree of Doctor of Philosophy.

Thomas L. Vincent	April 6, 1984 Date
Wilfred M. Greenlee	6 April 1984 Date
David S. Schultz	6 April 1984 Date
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Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

Thomas L. Vincent Dissertation Director	April 6, 1984 Date

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SIGNED: Jonathan E. Gayek

Kookaburra sits in the old gum tree;
Merry, merry King of the Bush is he.
Laugh, Kookaburra, laugh;
Kookaburra, gay your life must be.

Australian Childrens Song

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To each and every one of you,

May you have the times,
May you have the controls,
To reach, whatever,
Your life's goals.

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ABSTRACT

A method is proposed for approximating the reachable set from the origin for a class of n first order linear ordinary differential equations subject to bounded control. The technique involves decoupling the system equations into 1- and 2-dimensional linear subsystems, and then finding the reachable set of each of the subsystems. Having obtained bounds on each of the decoupled state variables, a n-dimensional parallelpiped is constructed which contains the reachable set from the origin for the original system. Several illustrative examples are presented for the case where the control is a scalar. The technique is also compared to a Lyapunov approach of approximating the reachable set in a simple 2-dimensional example.

CHAPTER 1

INTRODUCTION

Within the past decade reachable set theory has been used in several fields. In ecology, Goh (1976), Vincent and Anderson (1979), Fisher and Goh (1980), Grantham (1980a), and Vincent (1980b) have found or approximated reachable sets for ecosystems under bounded disturbances (harvesting, for example) in order to examine the vulnerability (Goh 1975, 1976) of the system. In the area of engineering design, Ritter and Vincent (1981) used reachable set analysis in their solution to the inverted beam problem and Vincent (1980a) utilized reachable sets to determine regions of motion for a magnetically suspended rotor under the influence of a bounded external force. Reachable sets are applicable to the field of optimal control theory (Pontryagin et al., 1962) where solutions to many problems are made under the proviso that the final state is attainable in finite time. In a different context these sets are used to characterize regions of controllability with capture to a target (Vincent and Skowronski, 1979).

A number of techniques have been proposed either to find or approximate the reachable set from a nonempty, connected initial set where the system dynamics are governed by n first order ordinary differential equations subject to bounded control. The first method to appear in the literature is based upon a reachability maximum principle (Grantham, 1973; Grantham and Vincent , 1975). Under the hypothesis that a trajectory on the boundary of the reachable set exists, the Reachability Maximum Principle (RMP) provides a necessary condition which must be satisfied by the boundary trajectory. The method has been applied to 2-dimensional systems where it is assumed that the boundary of the reachable set intersects the boundary of a smooth initial set having nonzero normal vectors. In this instance, the boundary trajectory, if it exists, must satisfy a transversality condition (Grantham and Vincent, 1975). Examples of this procedure may be found in the works of Grantham and Vincent (1975) and Vincent and Skowronski (1979). If the initial set is a single point in the interior of its reachable set, then Vincent (1980b) conjectured that the boundary of the reachable set could be found by generating a forward solution from the point using the abnormal control law as provided by the RMP. The unproven claim is that the resulting trajectory will approach the boundary of the reachable set asymptotically. This method of finding candidate reachable sets has been used for 2-dimensional systems (Vincent, 1980a, 1980b; Ritter and Vincent, 1981) where the resulting sets can be described graphically.

A second method of constructing the reachable set is to employ time optimal control theory. Vincent and Anderson (1979) suggest generating a minimum time trajectory because the trajectory will asymptotically approach the boundary of the reachable set as the minimum time approaches positive infinity. This procedure is used in

Vincent and Anderson (1979), Vincent and Skowronski (1979), and Fisher and Goh (1980). Unlike the RMP method with its unproven assumptions (the existence and asymptotic stability of boundary trajectories), the minimal time procedure will work provided a minimum time trajectory and the boundary of the reachable set both exist. However, just as the RMP method works best in 2-dimensions (so that the boundary of the reachable set can be defined graphically), the minimal time procedure is also best suited for 2-dimensional systems.

Both the RMP technique and the minimal time procedure are attempts to determine the boundary of the reachable set. Due to their characterization of the boundary of the reachable set by graphical means, they are restricted to, at most, 3-dimensional systems. It is for this reason that researchers must consider alternate methods, methods which trade-off accuracy with the ability to deal with problems in higher dimensions.

The first attempts to overestimate the reachable set are based on the "second" or "direct" method of Lyapunov (LaSalle and Lefschetz, 1961). In the papers of Goh (1976) and Grantham (1980a, 1980b, 1981) continuously differentiable "Lyapunov-like" functionals are used to define regions which are guaranteed to contain the reachable set. This technique is attractive because we can find upper bounds to reachable sets resulting from n-dimensional nonlinear systems. The serious drawback, however, is that an appropriate

Lyapunov function must be found, a task which can be quite difficult, even for 2-dimensional problems. Furthermore, a nonlinear optimization problem must be solved if we are to find the best approximation from a given class of Lyapunov-like functionals. However, these problems are reduced if the system is linear since Lyapunov theory is well established for such systems, and the resulting optimization problem is one involving quadratic forms.

An alternative way of approximating the reachable set for an n-dimensional linear dynamical system subject to bounded control is to use the "box" procedure introduced in this dissertation. The method begins with decoupling the system into 1- and 2-dimensional modal systems through a similarity transformation. Next, the reachable set of each of the modal systems is approximated so that bounds on each of the transformed variables can be established. With these bounds we can enclose the reachable set of the decoupled system in a box (hence, the name of the method). Transforming the faces of the box into the original space results in a parallelpiped which encloses the reachable set of the original system. Through this process, not only is the reachable set approximated, but upper and lower bounds on each of the original state space variables are found as well. This procedure relies on the use of linear algebra (eigenvalues, eigenvectors, and matrix inversion) as well as applications of the Reachability Maximum Principle to 2-dimensional linear systems. (As will be shown, many of the pitfalls of the RMP techniques can be overcome in this instance.)

The purpose of this dissertation is to present the box method of approximating the reachable set from the origin for a class of n-dimensional linear systems. Material relevant to reachable sets and the box procedure is also presented and is grouped in the following manner:

Chapter 2: Preliminary Definitions. For a system of n nonlinear first order ordinary differential equations under bounded control, we define the reachable set, the controllable set, and the retrosystem. Other key definitions include positive invariant sets, manoeuvrable sets, and the concepts of stability which are used throughout this dissertation.

Chapter 3: Properties of Controllable and Reachable Sets. A mixture of new and old results concerning these sets are presented. The key results, which are found in the literature and are presented here, deal with transitive properties (Section 3.1), the relationship between the reachable and controllable set (Section 3.2), the positive invariance of the reachable set and its interior (Section 3.4), and the Reachability Maximum Principle (Section 3.5). The new material deals with points sharing the same reachable and controllable sets (Section 3.3), the existence and asymptotic stability of boundary trajectories for some 2-dimensional problems

(Sections 3.6 and 3.7), and upper bounds on the reachable set from the origin (Section 3.8).

Chapter 4: Critical Points and the Boundary of the Reachable Set for Linear Systems. Throughout this chapter we investigate the location of critical points of linear systems subject to bounded control in relation to the boundary of the reachable set from the origin. Defining the abnormal control law and the switching function for linear systems, together with what is meant by a bang-bang control law (Section 4.1), we go on to prove a number of results concerning the abnormal bang-bang control law for linear systems (Section 4.2). In Section 4.3 it is proven that if a boundary trajectory exists and if the eigenvalues of the linear system matrix are real and negative, then at least one controlled equilibrium point exists on the boundary of the reachable set. For certain 2-dimensional linear systems it is shown that either two critical points can be found on the boundary of the reachable set or all the critical points are on the interior of the reachable set (Section 4.4).

Chapter 5: Reachable Sets from the Origin for 1- and 2-Dimensional Linear Systems with a Scalar Control. The constrained control set for these examples is taken to be a compact interval with the zero control in the interior. For the 1-dimensional linear system the reachable set from the origin is simply described (Section 5.1). For a class of 2-dimensional linear systems the reachable set from the origin is found using the theory of Chapters 3 and 4 (Section 5.2).

Chapter 6: Approximating the Reachable Set from the Origin for Linear Systems. In this chapter we discuss the box procedure. After outling the method (Section 6.1), we show how to decouple the system into 1- and 2-dimensional modal systems (Section 6.2). In Section 6.3 we prove that under an invertible linear transformation the points of the reachable set in the original space map onto the points of the reachable set in the transformed space. The final part of the chapter (Section 6.4) is devoted to applying the box procedure to several examples.

CHAPTER 2

PRELIMINARY DEFINITIONS

We begin by considering dynamical systems of the form

$$\frac{dx}{dt} = f(x,u) \tag{2.1}$$

where $x \varepsilon E^n$ represents the state of the system, $u \varepsilon \Omega \subset E^m$ (Ω compact) is the constrained control vector, $f:E^n \times E^m \to E^n$ is continuously differentiable with respect to x and continuous in u, and t represents the monotone increasing independent variable which, for convenience, we will call time. The sets E^n and E^m are respectively the sets R^n and R^m with the Euclidean norm, i.e., if $x \varepsilon E^n$, then $\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}$.

2.1. Admissable Control Laws

A control law $u(t)$ is said to be admissible if and only if it is a piecewise continuous function of time and $u:R^+ \to \Omega$. With this definition we can speak of the existence and uniqueness of the forward solutions to (2.1) in a neighborhood of the initial point $x_0 \varepsilon E^n$ on some positive time interval $[0,t_1)$ (Coddington and Levinson, 1955).

2.2. Reachable and Controllable Sets

Let $\theta \subset E^n$ be a nonempty, connected, and compact set. We say that a point $x_1 \varepsilon E^n$ is reachable from θ if and only if there exists an admissable control law $u(t)$, a time $T > 0$, and a point $x_0 \varepsilon \theta$ such that if $\xi(t,x_0)$ is the forward solution to (2.1) with $\xi(0,x_0) = x_0$, then $\xi(T,x_1) = x_1$. We call the set of all points reachable from θ for some fixed time $T > 0$ the reachable set from θ in time T and designate it by $\mathbf{R}_T(\theta)$. The set of all points reachable from θ for any time $T > 0$ is said to be the reachable set from θ, $\mathbf{R}(\theta)$. Observe that $\mathbf{R}(\theta) = \bigcup_{T>0} \mathbf{R}_T(\theta)$.

Closely related to the idea of a reachable set (where the system can go) is the concept of a controllable set (where the system could have come from). We say that a point $x_1 \varepsilon E^n$ is controllable to θ if and only if there exists an admissable control law $u(t)$, a time $T > 0$, and a point $x_0 \varepsilon \theta$ such that if $\xi(t,x_1)$ is the forward solution to (2.1) with $\xi(0,x_1) = x_1$, then $\xi(T,x_1) = x_0$. We call the set of all points controllable to θ for a fixed $T > 0$ the controllable set to θ in time T, $\mathbf{C}_T(\theta)$. Lastly, we define the set of points controllable to θ for any time $T > 0$ to be the controllable set to θ, $\mathbf{C}(\theta)$. As with reachable sets, note that we can write $\mathbf{C}(\theta) = \bigcup_{T>0} \mathbf{C}_T(\theta)$.

Remarks: 1) By setting $T = 0$ in the definitions of $\mathbf{R}(\theta)$ and $\mathbf{C}(\theta)$, it is a simple matter to see that $\theta \subseteq \mathbf{R}(\theta)$ and $\theta \subseteq \mathbf{C}(\theta)$. Consequently, the set $\mathbf{C}(\theta) \cap \mathbf{R}(\theta)$ is nonempty.

2) Comparing the definitions of reachable and controllable sets we suspect there is some relationship between these two sets. As proven in Section 3.2 this is indeed the case; in fact, $\mathbf{C}(\theta)$ can be thought of as the reachable set from θ for the retrosystem to (2.1)

$$\frac{dx}{dt} = -f(x,u). \tag{2.2}$$

A similar relationship exists between $C_T(\theta)$ and the corresponding reachable set of the retrosystem (2.2). We will represent controllable and reachable sets of the retrosystem by "^", i.e., $\hat{\mathbf{R}}(\theta)$ is the reachable set from θ for the system (2.2).

Notation: We will designate the reachable set from the origin by $\mathbf{R}$ and the controllable set to the origin by $\mathbf{C}$.

2.3. Boundary Controls and Boundary Trajectories

Much of our attention in Chapters 3-6 will focus upon the boundary of the reachable set and forward solutions to (2.1) from points on $\partial\mathbf{R}(\theta)$. In particular we will be interested in boundary trajectories. Given an admissable control law $u(t)$ and a point $x_1 \in \partial\mathbf{R}(\theta)$, a boundary trajectory is a forward solution $\xi(t,x_1)$ to (2.1) with $\xi(0,x_1) = x_1$ which satisfies $\xi(t,x_1) \in \partial\mathbf{R}(\theta)$ for all $t > 0$. An admissable control which generates a boundary trajectory is said to be a boundary control.

2.4. Positive Invariant Sets

Let M be an open, nonempty subset of E^n. We say that M is positive invariant if and only if for every $x_0 \in M$ there exists an admissable control law $u(t)$ such that if $\xi(t,x_0)$ is the forward solution to (2.1) with $\xi(0,x_0) = x_0$, then $\xi(t,x_0) \in M$ for all $t \geq 0$. Two very special positive invariant sets are now examined.

Let u_c be a constant admissable control. The point $x_{u_c} \in E^n$ is said to be a controlled equilibrium point (critical point) with respect to u_c if and only if $f(x_{u_c}, u_c) = 0$. The set $M = \{x_{u_c}\}$ is positive invariant.

Let $u(t)$ be an admissable contol law and let $x_0 \in E^n$. A forward solution $\xi(t,x_0)$ to (2.1) is said to be periodic if and only if there exists a time $p > 0$ such that $\xi(t + p,x_0) = \xi(t,x_0)$ for all $t \geq 0$. We call the least p for which this is true the period. Defining $M \subset E^n$ to be the set of points of the periodic trajectory $\xi(t,x_0)$ generated by $u(t)$, we see that M is a positive invariant set.

2.5. Manoeuvrable Sets

A nonempty, connected set $\theta \subset E^n$ is said to be manoeuvrable if and only if for every pair of points x, $y \in \theta$ there exists an admissable control law $u(t)$ and a time $T \geq 0$ such that if $\xi(t,x)$ is the forward solution to (2.1) with $\xi(0,x) = x$ and $\xi(T,x) = y$, then $\xi(t,x) \in \theta$ for all $t \in (0,T)$. An immediate consequence of this definition is that if θ is manoeuvrable, then every $x \in \theta$ is controllable to and reachable from every point $y \in \theta$; furthermore, there exists forward solutions to (2.1) which transfer the state from x to y and back again while remaining in θ. From this it follows that if θ is manoeuvrable, then θ is positive invariant and $y \in \mathbf{C}(x) \cap \mathbf{R}(x)$ for all $x, y \in \theta$.

Remark. A controlled equilibrium point and a periodic trajectory are two examples of a manoeuvrable set.

2.6. Stability

Given an admissable control law $u(t)$ we will examine the stability of forward solutions to (2.1). In the definitions which follow, all of which have their origin in the stability of ordinary differential equations (Hahn, 1967), we let $\xi_1(t,x_0)$ and $\xi_2(t,y_0)$ to be the forward solutions to (2.1) from x_0 and y_0, respectively.

We say that $\xi_1(t,x_0)$ is stable if and only if given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|\xi_1(t,x_0) - \xi_2(t,y_0)\| < \varepsilon$$

for all $t \geq 0$ whenever

$$\|x_0 - y_0\| < \delta.$$

The solution $\xi_1(t,x_0)$ is asymptotically stable if and only if it is stable and $\|\xi_1(t,x_0) - \xi_2(t,y_0)\| \to 0$ as $t \to +\infty$. If $\xi_1(t,x_0)$ is asymptotically stable for any choice $\delta > 0$, then we say that it is globally asymptotically stable.

Let $u(t) = u_c$ be a constant admissable control and let x_0 be a controlled equilibrium point with respect to u_c. We say that x_0 is a stable equilibrium point if and only if given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|x_0 - \xi_2(t,y_0)\| < \varepsilon$$

for all $t \geq 0$ whenever

$$\|x_0 - y_0\| < \delta.$$

We say that x_0 is asymptotically stable if and only if x_0 is stable and $\|x_0 - \xi_2(t,y_0)\| \to 0$ as $t \to +\infty$. Furthermore, the point x_0 is globally asymptotically stable if and only if x_0 is asymptotically stable for all choices $\delta > 0$.

2.7. The Velocity Vectogram and the Convex Cone of Velocity Vectors

Let $x \in E^n$ and let u_c be a constant admissable control law. We define the n-dimensional velocity vector associated with u_c at x of (2.1) to be the directed line segment from x to the point $x + f(x,u_c)$ and represent it by $\overrightarrow{f(x,u_c)}$. Physically, $\overrightarrow{f(x,u_c)}$ represents the direction of motion that the system (2.1) would travel from x given $u = u_c$. The collection of all velocity vectors at x is said to be the <u>velocity vectogram at</u> x $F(x,\Omega) = \{\overrightarrow{f(x,u_c)} | u_c \in \Omega\}$ (Isaacs, 1965). We will assume $F(x,\Omega)$ is closed. Note that since $f(x,u)$ is continuously differentiable in x, it follows that $F(x,\Omega)$ varies continuously in x. The physical interpretation of $F(x,\Omega)$ is that it represents all achievable directions of motion from x for the system (2.1).

Let q be a positive integer and let u_{c_j}, $j = 1, \ldots, q$, be constant admissable control laws. The <u>convex cone of velocity vectors at</u> x <u>generated by</u> $\{f(x,u_{c_1}), \ldots, f(x,u_{c_q})\}$ is given by

$$\{z \in E^n | z = \sum_{j=1}^{q} a_j f(x,u_{c_j}), \quad a_j > 0\}$$

(Vincent and Grantham, 1980). Note that the convex cone generated is dependent upon the choice of q and $\{f(x,u_{c_1}), \ldots, f(x,u_{c_q})\}$. For this reason we will assume the existence of a positive integer q^* and constant admissable control laws $u^*_{c_j}$, $j = 1, \ldots, q^*$, such that for any positive integer q and constant admissable control laws u_{c_j}, $j = 1, \ldots, q$,

$$C(x,\Omega) = \{z \in E^n | \sum_{j=1}^{q^*} a_j f(x,u^*_{c_j}), a_j > 0\}$$

$$\supseteq \{z \in E^n | \sum_{j=1}^{q} b_j f(x,u_{c_j}), b_j > 0\}.$$

We call $C(x,\Omega)$ the convex cone of velocity vectors at x. An example of systems where $C(x,\Omega)$ can be defined is the class of 2-dimensional systems under scalar control, $f(x,u) = g(x) + Bu$, where $-\infty < u_{min} \leq u_{max} < +\infty$. Since the control entries $f(x,u)$ linearly, we can choose $q^* = 2$, $u^*_{c_1} = u_{min}$, and $u^*_{c_2} = u_{max}$.

From the definition of $C(x,\Omega)$ it follows that for every $u \in \Omega$, $f(x,u) \in C(x,\Omega)$. Consequently, $F(x,\Omega) \subseteq C(x,\Omega)$. Furthermore, since $F(x,\Omega)$ is closed, $C(x,\Omega)$ is closed. Figure 2.1 illustrates the relationship between $F(x,\Omega)$ and $C(x,\Omega)$ for some 2-dimensional situations. Physically, $C(x,\Omega)$ indicates regions where the system (2.1) could travel in the vicinity of x. For example, suppose the velocity vectogram at x is given by Figure 2.2a. From this diagram we see that the system could not immediately move into the region A. However, as $F(x,\Omega)$ varies continuously in x, we see that if we move from x to x_1, then we are in a position to move into A (Figure 2.2b). Alternatively, we can obtain the same information by noting that the vectors of $C(x,\Omega)$ point in every direction about x (Figure 2.2c).

For 2-dimensional systems of the form (2.1) we can speak of the angle of $C(x,\Omega)$. The angle of the convex cone of vectors at x $\angle C(x,\Omega)$ is defined to be the maximum angle between vectors of $C(x,\Omega)$ measured so as to include the interior of $C(x,\Omega)$. To help

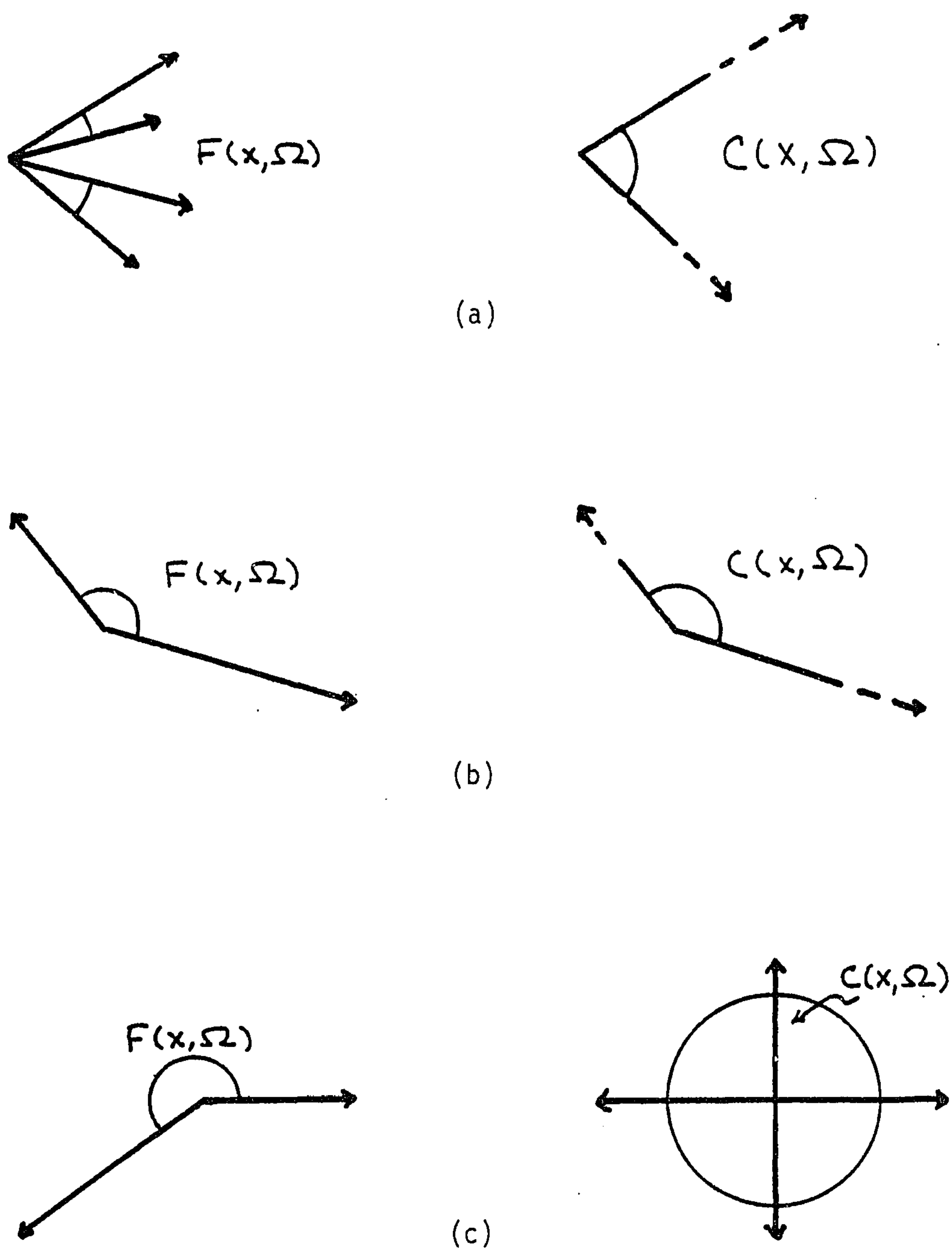


Figure 2.1. The geometric relationship between $F(x,\Omega)$ and $C(x,\Omega)$.

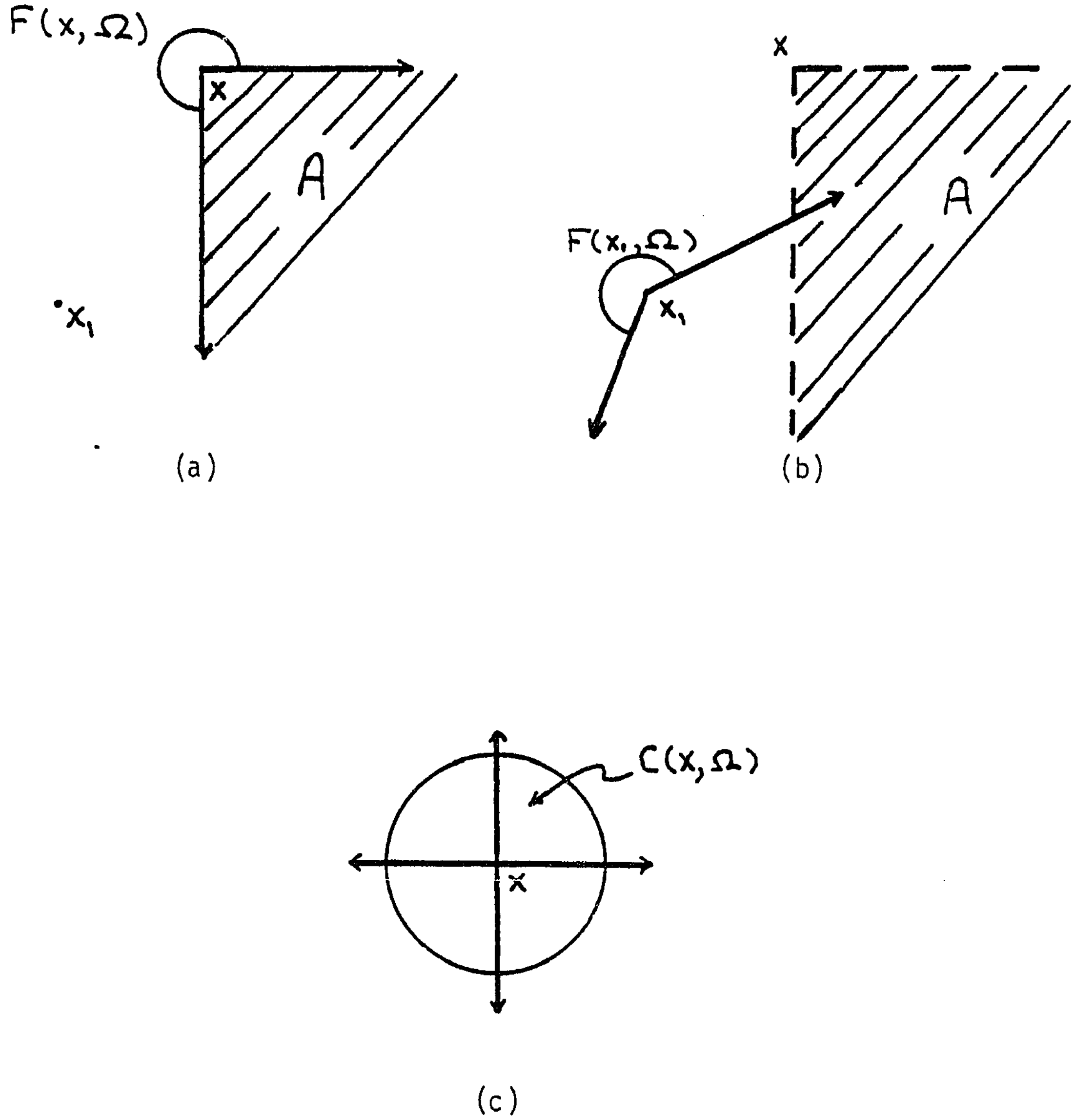


Figure 2.2. A graphical interpretation of $C(x,\Omega)$.

(a) Region A cannot be immediately entered into from x.

(b) Region A can be entered into from a point x_1 sufficiently close to and attainable from x.

(c) The convex cone of velocity vectors at x for the velocity vectogram in Figure 2.2a.

illustrate this concept, consider the 2-dimensional velocity vectograms and their respective convex cone of velocity vectors in Figure 2.3. Note that while the directions of possible motion (as indicated by $F(x,\Omega)$) are not directed into more than half of the plane (Figures 2.3a, b, and c), $C(x,\Omega)$ is a convex subset of E^2. Consequently, $C(x,\Omega)$ and $\angle C(x,\Omega)$ vary continuously in x since $F(x,\Omega)$ varies continuously in x. Furthermore, vectors on $\partial C(x,\Omega)$ are on $\partial F(x,\Omega)$. However, after $F(x,\Omega)$ is directed into more than half of the plane (Figure 2.3d), then because $C(x,\Omega)$ is convex, it follows that $C(x,\Omega)$ has vectors pointing in every direction and $\angle C(x,\Omega) = 2\pi$.

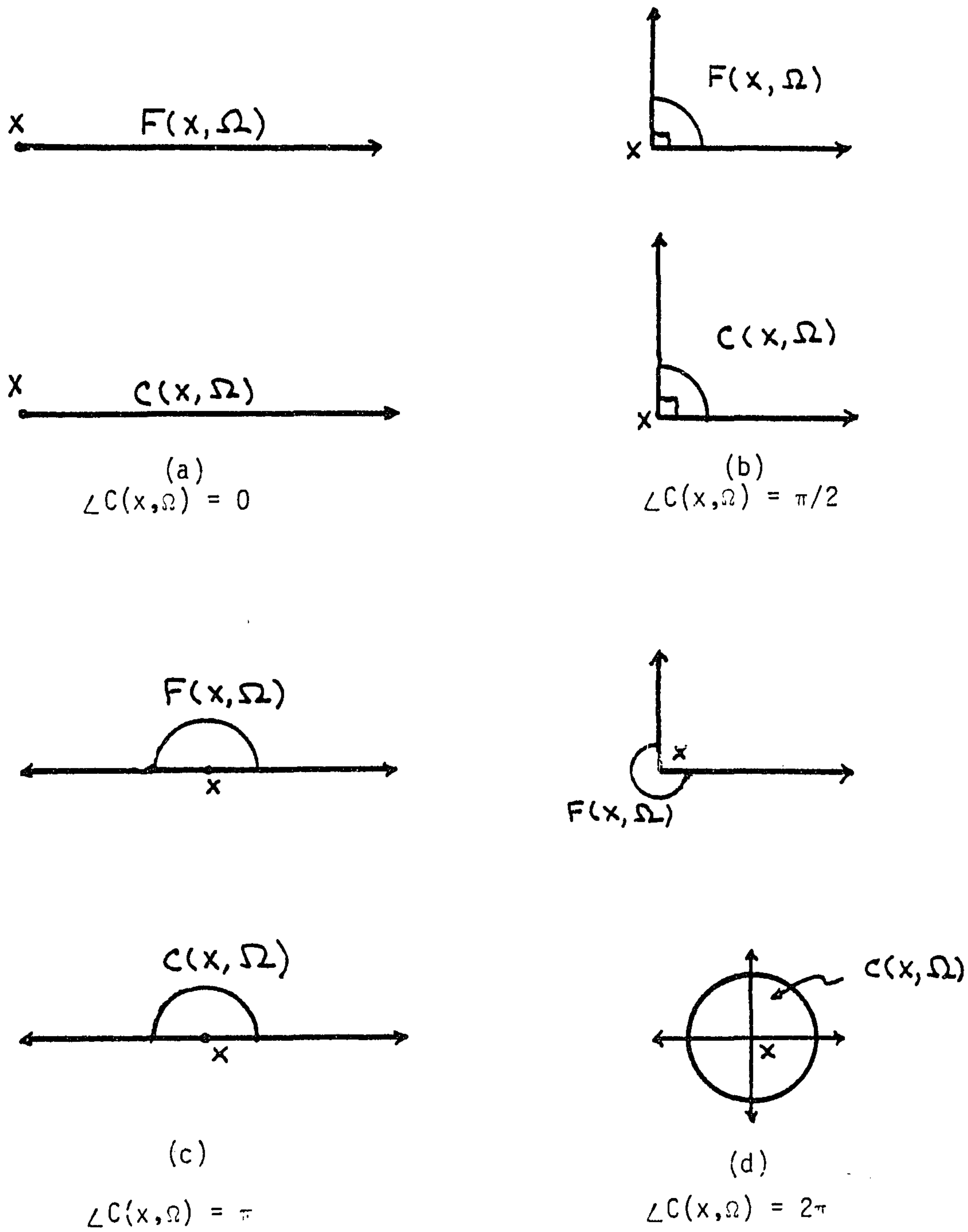


Figure 2.3. Some velocity vectograms, their corresponding convex cone of velocity vectors, and the resulting angle of the convex cone of velocity vectors.

CHAPTER 3

PROPERTIES OF CONTROLLABLE AND REACHABLE SETS

In this chapter we discuss a number of properties of reachable and controllable sets. In particular, we review the transitive properties of reachable and controllable sets, the close relationship between $R(x_0)$ and $\hat{C}(x_0)$ (see Section 2.2 for definitions), the nonpenetration of $R(x_0)$, the reachability maximum principle, and a number of results concerning the reachable set of linear systems. In addition, a number of new results are presented. These concern the boundedness of $R(x_0)$ for a class of nonlinear control systems. An examination of the properties of $C(x_0) \cap R(x_0)$ is also included.

3.1. Transitive Properties

Lemma 3.1. (Roxin and Spinadel, 1963). Let $R(x_0)$ and $R(x_1)$ denote the reachable sets from x_0 and x_1, respectively. If $x_1 \varepsilon R(x_0)$ and $x_2 \varepsilon R(x_1)$ then $x_2 \varepsilon R(x_0)$.

Proof. We will follow the proof of Roxin and Spinadel. Since $x_1 \varepsilon R(x_0)$ there is an admissable control $u_1(t)$ and a time $T_1 > 0$ such that if $\xi_1(t,x_0)$ is the forward solution to (2.1) with $\xi_1(0,x_0) = x_0$, then $\xi_1(T_1,x_0) = x_1$. Similarly, since $x_2 \varepsilon R(x_1)$ there is an admissable control $u_2(t)$ and a time $T_2 > 0$ such that if $\xi_2(t,x_1)$ is the forward solution to (2.1) with $\xi_2(0,x_1) = x_1$, then $\xi_2(T_2,x_1) = x_2$.

Consider the admissable control

$$u_3(t) = \begin{cases} u_1(t), & 0 \leq t \leq T_1 \\ u_2(t - T_1), & T_1 < t \leq T_1 + T_2 \end{cases} \tag{3.1}$$

With this admissable control we can reach z from x in time $T_1 + T_2$. Let $\xi_3(t,x_0)$ be the forward solution to (2.1) under (3.1). Then

$$\xi_3(t,x_0) = \begin{cases} \xi_1(t,x_0), & 0 \leq t \leq T_1 \\ \xi_2(t - T_1,x_1), & T_1 < t \leq T_1 + T_2 \end{cases} \tag{3.2}$$

so that $\xi_3(T_1 + T_2,x_0) = \xi_2(T_2,x_1) = x_2$. Hence, $x_2 \in \mathbf{R}(x_0)$. Q.E.D.

Lemma 3.2. Let $C(x_0)$ and $\mathbf{C}(x_1)$ denote the controllable sets to x_0 and x_1, respectively. If $x_2 \in \mathbf{C}(x_1)$ and $x_1 \in \mathbf{C}(x_0)$ then $x_2 \in \mathbf{C}(x_0)$.

Proof. As $x_2 \in \mathbf{C}(x_1)$ there exists an admissable control $u_1(t)$ and a time $T_1 \geq 0$ such that if $\xi_1(t,x_2)$ is the forward solution to (2.1) under $u_1(t)$ with $\xi_1(0,x_2) = x_2$, then $\xi_1(T_1,x_2) = x_1$. In like manner, as $x_1 \in \mathbf{C}(x_0)$ there exists an admissable control $u_2(t)$ and a time $T_2 \geq 0$ such that if

$\xi_2(t,x_1)$ is the forward solution to (2.1) with $\xi_2(0,x_1) = x_1$, then $\xi_2(T_2,x_1) = x_0$.

Consider the admissable control law

$$u_3(t) = \begin{cases} u_1(t), & 0 \leq t \leq T_1 \\ u_2(t - T_1), & T_1 < t \leq T_1 + T_2. \end{cases} \tag{3.3}$$

Under this control law x_2 can be controlled to x_0 in time $T_1 + T_2$. To see this, let $\xi_3(t,x_2)$ be the forward solution to (2.1) under (3.3). Then

$$\xi_3(t,x_2) = \begin{cases} \xi_1(t,x_2), & 0 \leq t \leq T_1 \\ \xi_2(t - T_1,x_1), & T_1 < t \leq T_1 + T_2 \end{cases} \tag{3.4}$$

so that $\xi_3(T_1 + T_2,x_2) = \xi_2(T_2,x_1) = x_0$. Consequently, x_2 is controllable to x_0. Q.E.D.

Simply put, Lemma 3.1 states that if x_1 is reachable from x_0 and x_2 is reachable from x_1, then x_2 is reachable from x_0. Likewise, Lemma 3.2 states that if x_2 is controllable to x_1 and x_1 is controllable to x_0, then x_2 is controllable to x_0.

Remark. In using the transitivity properties (Lemmas 3.1 and 3.2) it is important to note that $x_0 \varepsilon \mathbf{C}(x_1)$ is the the same as saying $x_1 \varepsilon \mathbf{R}(x_0)$. This is based upon the fact that the definition of $x_0 \varepsilon \mathbf{C}(x_1)$ can also be read as the definition of $x_1 \varepsilon \mathbf{R}(x_0)$. To see this, consider a point $x_0 \varepsilon \mathbf{C}(x_1)$. Then by definition there exists an admissable control $u(t)$ and a time $T > 0$ such that if $\xi(t,x_0)$ is the forward solution to (2.1) with $\xi(0,x_0) = x_0$, then $\xi(T,x_0) = x_1$. Close examination of this statement will reveal that it is the same as the definition for $x_1 \varepsilon \mathbf{R}(x_0)$; hence, $x_0 \varepsilon \mathbf{C}(x_1)$ also means $x_1 \varepsilon \mathbf{R}(x_0)$.

3.2. The Relationship Between $\mathbf{R}(x_0)$ and $\hat{\mathbf{C}}(x_0)$

Theorem 3.1. (Snow, 1967). Let $x_0 \varepsilon E^n$ and $T > 0$. Let $\mathbf{R}_T(x_0)$ be the reachable set from x_0 in time T for the system (2.1) and let $\hat{\mathbf{C}}_T(x_0)$ be the controllable set to x_0 in time T for the retrosystem (2.2). Then $\mathbf{R}_T(x_0) = \hat{\mathbf{C}}_T(x_0)$.

Proof. We follow a proof similar to Snow's. Let $x_1 \varepsilon \mathbf{R}_T(x_0)$. Then there exists an admissable control $u(t)$ such that if $\xi_1(t,x_0)$ is the forward solution to (2.1) with $\xi_1(0,x_0) = x_0$, then $\xi_1(T,x_0) = x_1$. Define $\tau = T - t$ and let $\xi_2(\tau,x_1) = \xi_1(T - t,x_0)$. Then

$$\frac{d\xi_2(\tau,x_1)}{d\tau} = \frac{d\xi_1(T-t,x_0)}{dt} \cdot \frac{dt}{d\tau}$$

$$= -f(\xi_1(T - t, x_0), u)$$

$$= -f(\xi_2(\tau, x_1), u)$$

and $\xi_2(0,x_1) = x_1$. Noting that $\xi_2(T,x_1) = x_0$ we conclude that $x_1 \in \hat{C}_T(x_0)$. Therefore, $R_T(x_0) \subseteq \hat{C}_T(x_0)$.

Conversely, let $x_1 \in \hat{C}_T(x_0)$ and let $\xi_2(t,x_1)$ be the forward solution to (2.2) under $u(t)$ where $\xi_2(0,x_1) = x_1$ and $\xi_2(T,x_1) = x_0$. Defining $t = T - \tau$ and $\xi_1(t,x_0) = \xi_2(T - \tau, x_1)$ we have that

$$\frac{d\xi_1(t,x_0)}{dt} = \frac{d\xi_2(T-t,x_1)}{dt} \quad \frac{dt}{d\tau}$$

$$= f(\xi_2(T - t, x_1), u)$$

$$= f(\xi_1(\tau, x_1), u),$$

where $\xi_1(0,x_0) = \xi_2(T,x_1) = x_0$ and $\xi_1(T,x_0) = \xi_2(0,x_1) = x_1$. Thus, $x_1 \in R_T(x_0)$ so that $\hat{C}_T(x_0) \subseteq R_T(x_0)$ and the proof is complete. Q.E.D.

Corollary 3.1. Let $x_0 \in E^n$ and $T > 0$. Let $C_T(x_0)$ be the controllable set from x_0 for the system (2.1) and let $\hat{R}_T(x_0)$ be the reachable set from x_0 in time T for the retrosystem (2.2). Then $C_T(x_0) = \hat{R}_T(x_0)$.

Proof. Follows immediately from the theorem with the obvious modifications. Q.E.D.

Corollary 3.2. Let $x_0 \varepsilon E^n$. Let $\mathbf{R}(x_0)$ be the reachable set from x_0 for the system (2.1) and let $\hat{\mathbf{C}}(x_0)$ be the controllable set to x_0 for the system (2.2). Then $\mathbf{R}(x_0) = \hat{\mathbf{C}}(x_0)$.

Proof. Let $x_1 \varepsilon \mathbf{R}(x_0)$. Then there exists an admissable control $u_1(t)$ and a time $T_1 > 0$ such that if $\xi_1(t,x_0)$ is the forward solution to (2.1) with $\xi_1(0,x_0) = x_0$, then $\xi_1(T_1,x_0) = x_1$. Consequently, $x_1 \varepsilon \mathbf{R}_{T_1}(x_0) = \hat{\mathbf{C}}_{T_1}(x_0) \subseteq \bigcup_{T>0} \hat{\mathbf{C}}_T(x_0) = \hat{\mathbf{C}}(x_0)$.

Therefore, $\mathbf{R}(x_0) \subseteq \hat{\mathbf{C}}(x_0)$.

Conversely, let $x_1 \varepsilon \mathbf{C}(x_0)$. Then there exists an admissable control $u_2(t)$ and a time $T_2 > 0$ such that if $\xi_2(0,x_1) = x_1$, then $\xi_2(T_2,x_1) = x_0$. Hence, $x_1 \varepsilon \hat{\mathbf{C}}_{T_2}(x_0) = \mathbf{R}_{T_2}(x_0) \subseteq \bigcup_{T>0} \mathbf{R}_T(x_0)$ $= \mathbf{R}(x_0)$ so that $\hat{\mathbf{C}}(x_0) \subseteq \mathbf{R}(x_0)$. Combining the set inclusions establishes the corollary. Q.E.D.

Corollary 3.3. Let $x_0 \varepsilon E^n$. Let $\mathbf{C}(x_0)$ be the controllable set to x_0 for the system (2.1) and let $\hat{\mathbf{R}}(x_0)$ be the reachable set from x_0 for the retrosystem (2.2). Then $\mathbf{C}(x_0) = \hat{\mathbf{R}}(x_0)$.

Proof. Same as that for Corollary 3.2 with the obvious modifications. Q.E.D.

The usefulness of Theorem 3.1 and its corollaries is that if we prove a result for $R_T(x_0)$ or $R(x_0)$ an analogous statement can be made for $C_T(x_0)$ or $C(x_0)$, respectively, by making the obvious modifications.

3.3. The Set $C(x_0) \cap R(x_0)$

In this section we establish the fact that $C(x_0) \cap R(x_0)$ is connected and that points of this set have the same controllable and reachable set.

Lemma 3.3. Let $R(x_0)$ and $C(x_0)$ be the reachable set from x_0 and the controllable set to x_0, respectively. Then $C(x_0) \cap R(x_0)$ is connected.

Proof. Let us assume that $C(x_0) \cap R(x_0)$ is not connected. Then there exists nonempty sets $A_0, \ldots, A_r$ such that $\bigcup_{j=0}^{r} A_j = C(x_0) \cap R(x_0)$ and $\overline{A}_j \cap A_k = \phi$, for $j, k = 0, \ldots, r$ and $j \neq k$. (Goldberg, 1976). Without loss of generality let us suppose $x_0 \in A_0$ and let $x_1 \in A_1$. As $x_1 \in A_1 \subset C(x_0) \cap R(x_0)$ there exists an admissable control law which generates a forward solution from x_1 to x_0 in finite time. The resulting trajectory must remain in $C(x_0)$; however, in order to do so and reach x_0 it must leave A_1 (Figure 3.1). In fact, the trajectory must spend time outside of $C(x_0) \cap R(x_0)$ due to the supposition that $C(x_0) \cap R(x_0)$ is not connected. Thus, there exists a point x_2 on the trajectory joining x_1 to x_0 such that $x_2 \notin C(x_0) \cap R(x_0)$. By virtue of

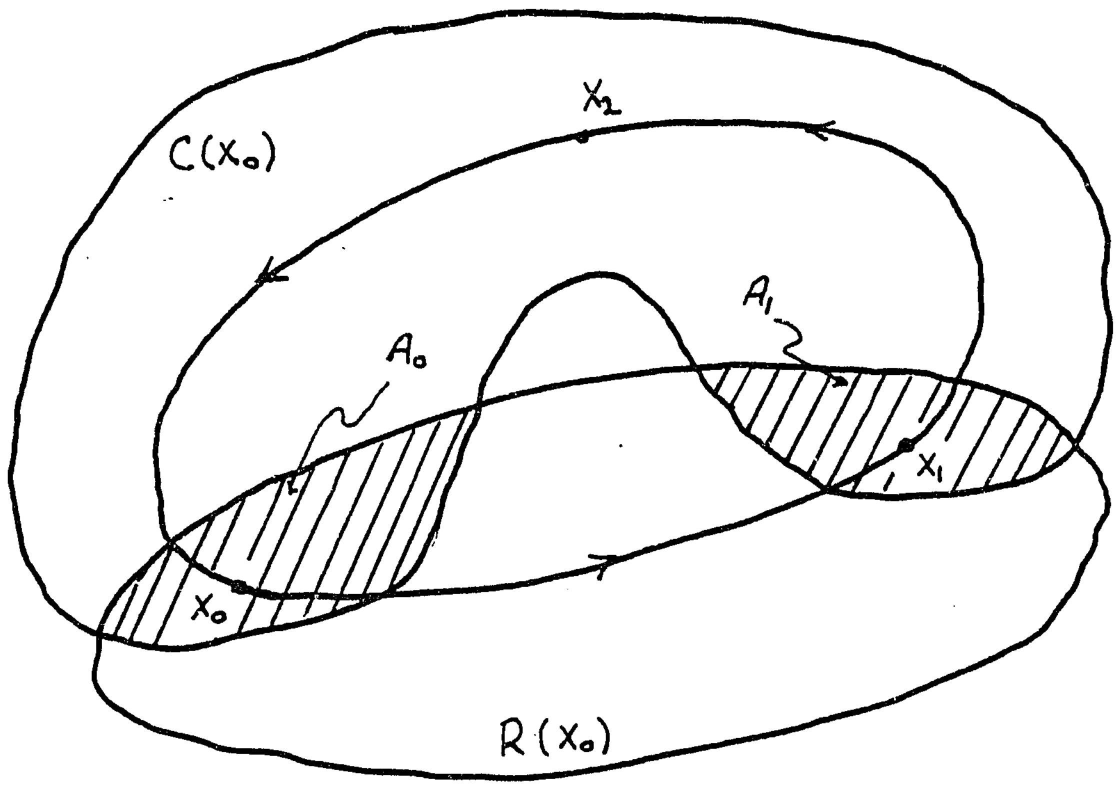


Figure 3.1. A trajectory linking the assumed disconnected components of $C(x_0) \cap R(x_0)$ contains a point x_2 exterior to $R(x_0)$.

definition $x_2 \in C(x_0)$. Observe that as $x_1 \in R(x_0)$ and $x_2 \in R(x_1)$ we have, by the transitive property of reachable sets, $x_2 \in R(x_0)$. Thus, $x_2 \in C(x_0) \cap R(x_0)$ which is a contradiction. Therefore our assumption is false and we must conclude that $C(x_0) \cap R(x_0)$ is connected. Q.E.D.

<u>Theorem 3.2</u>. Let $C(x_0)$ and $R(x_0)$ denote the controllable set to x_0 and the reachable set from x_0, respectively. If $x_1 \in C(x_0) \cap R(x_0)$, then $C(x_1) = C(x_0)$.

<u>Proof</u>. Let $x_2 \in C(x_0)$. As $x_1 \in R(x_0)$ is the same as saying $x_0 \in C(x_1)$, the transitive property of controllable sets leads to $x_2 \in C(x_1)$; hence, $C(x_0) \subseteq C(x_1)$.

Conversely, let $x_2 \in C(x_1)$. Since $x_1 \in C(x_0)$ we have that $x_2 \in C(x_0)$. Consequently $C(x_1) \subseteq C(x_0)$. Combining our set inclusions establishes the theorem. Q.E.D.

<u>Theorem 3.3</u>. Let $C(x_0)$ and $R(x_0)$ denote the controllable set to x_0 and the reachable set from x_0, respectively. If $x_1 \in C(x_0) \cap R(x_0)$, then $R(x_1) = R(x_0)$.

<u>Proof</u>. Let $x_2 \in R(x_1)$. By hypothesis $x_1 \in R(x_0)$; hence, an application of Lemma 3.1 leads us to conclude that $x_2 \in R(x_0)$. Therefore, $R(x_1) \subseteq R(x_0)$.

Conversely, let $x_2 \in R(x_0)$. Since $x_1 \in C(x_0)$ is the same as saying $x_0 \in R(x_1)$, we have $x_2 \in R(x_1)$ by Lemma 3.1.

Consequently, $\mathbf{R}(x_0) \subseteq \mathbf{R}(x_1)$. Combining our results establishes the theorem. Q.E.D.

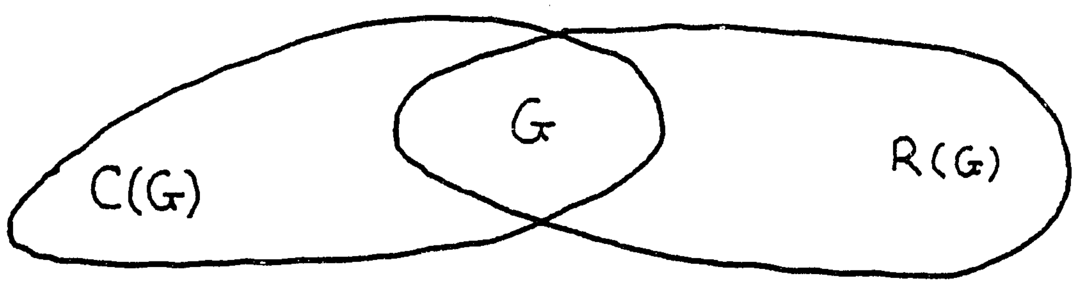
Observe that by combining Theorems 3.2 and 3.3 we have that $x_1 \in \mathbf{C}(x_0) \cap \mathbf{R}(x_0)$ implies $\mathbf{C}(x_0) \cap \mathbf{R}(x_0) = \mathbf{C}(x_1) \cap \mathbf{R}(x_1)$. That is to say the set $G = \mathbf{C}(x_1) \cap \mathbf{R}(x_1)$ is independent of its defining point provided $x_1 \in \mathbf{C}(x_0) \cap \mathbf{R}(x_0)$. It follows from these same theorems that every point in G has the same controllable and reachable sets. Consequently, we can say $\mathbf{C}(G) = \mathbf{C}(x_1)$ and $\mathbf{R}(G) = \mathbf{R}(x_1)$ for all $x_1 \in G$, where $\mathbf{C}(G)$ is the controllable set to the set G and $\mathbf{R}(G)$ is the reachable set from G. Using this and the definition of G, it follows that $G = \mathbf{C}(G) \cap \mathbf{R}(G)$. Since $\mathbf{C}(G) \cap \mathbf{R}(G)$ is connected (as $\mathbf{C}(x_0) \cap \mathbf{R}(x_0)$ is connected), there are only three possible configurations for the sets G, $\mathbf{C}(G)$, and $\mathbf{R}(G)$ (Figure 3.2). These configurations represent the cases

1) $G \subset \mathbf{C}(G)$ and $G \subset \mathbf{R}(G)$,
2) $G = \mathbf{R}(G) \subseteq \mathbf{C}(G)$, and
3) $G = \mathbf{C}(G) \subseteq \mathbf{R}(G)$.

Several one dimensional examples are now given to demonstrate the existence of each of these possibilities.

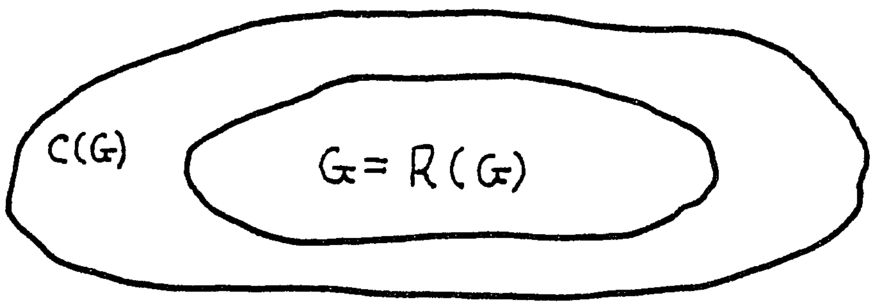
<u>Example 3.1</u>. ($G \subset \mathbf{C}(G)$ and $G \subset \mathbf{R}(G)$). Consider the dynamical system

$$\dot{x} = x^2 + u, \qquad |u| \leq 1. \tag{3.5}$$



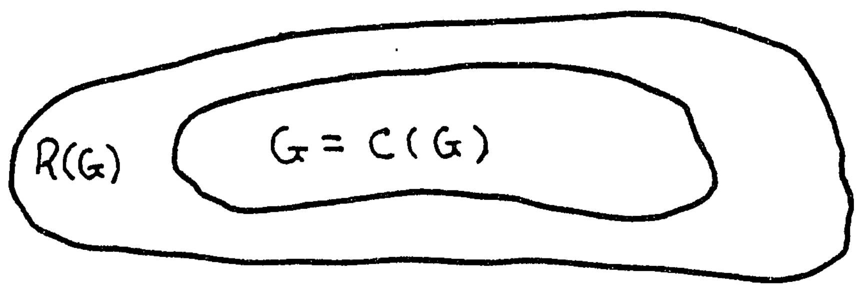
(a)

$G \subset C(G)$ and $G \subset R(G)$



(b)

$G = R(G) \subseteq C(G)$



(c)

$G = C(G) \subseteq R(G)$

Figure 3.2. The three possible relationships between the set G and its controllable and reachable sets.

Examination of the system reveals that the controllable set to the origin $\mathbf{C} = \{x \in E^1 | -\infty < x < 1\}$ and that the reachable set from the origin $\mathbf{R} = \{x \in E^1 | -1 < x < \infty\}$. Hence, $G = \mathbf{C} \cap \mathbf{R}$ $= \{x \in E^1 | -1 < x < 1\}$ so that $G \subset \mathbf{C}$ and $G \subset \mathbf{R}$.

Example 3.2. $(G = \mathbf{R}(G) \subseteq \mathbf{C}(G))$. Consider the dynamical system

$$x = -x + u, \qquad |u| \leqslant 1. \tag{3.6}$$

It is a simple matter to see that the controllable set to the origin $\mathbf{C} = E^1$ and that the reachable set from the origin $\mathbf{R} = \{x \in E^1 | -1 < x < 1\}$. Hence, $G = \mathbf{R} \subseteq \mathbf{C}$.

Example 3.3. $(G = \mathbf{C}(G) \subseteq \mathbf{R}(G))$. Consider the retrosystem to (3.6), i.e.,

$$x = x - u, \qquad |u| \leqslant 1. \tag{3.7}$$

We have that the controllable set to the origin $\mathbf{C} = \{x \in E^1 | -1 < x < 1\}$ and that the reachable set from the origin $\mathbf{R} = E^1$; hence, $G = \mathbf{C} \subseteq \mathbf{R}$.

Note that in each of the preceding examples G is manoeuvrable. This will always be the case provided G is not a set which contains only x_0.

Theorem 3.4. Let $x_0 \in E^n$ and consider the system (2.1). If the set $G = \mathbf{C}(x_0) \cap \mathbf{R}(x_0) \neq \{x_0\}$, then G is manoeuvrable.

Proof. Let G contain points other than x_0 and suppose that G is not manoeuvrable. Then either there is a point in G which is not reachable from, or controllable to, all the other points of G, or there exists points x_1, $x_2 \in G$ for which no admissable control exists which generates a trajectory from x_1 to x_2 while remaining in G during the transition. From Theorems 3.2 and 3.3 we can eliminate the possibility that there is a point in G which is not reachable from, or controllable to, the remaining points of G. Hence, if G is not manoeuvrable, then there exists distinct points x_1 and $x_2 \in G$ such that if $u(t)$ is an admissable control, $T > 0$, and $\xi(t,x_1)$ is the forward solution to (2.1) with $\xi(0,x_1) = x_1$ and $\xi(T,x_1) = x_2$, then $\xi(\tau,x_1) = x_3 \notin G$ for some $\tau \in (0,T)$. By construction $x_3 \in \mathbf{R}(x_1)$ and $x_3 \in C(x_2)$. However, from Theorems 3.2 and 3.3 it follows that $x_3 \in \mathbf{R}(x_0)$ and $x_3 \in \mathbf{C}(x_0)$, i.e., $x_3 \in \mathbf{C}(x_0) \cap \mathbf{R}(x_0) = G$, which is a contradiction. Therefore, G is manoeuvrable. Q.E.D.

Given a set $\theta \subset E^n$ which is nonempty, connected, and compact, it is reasonable to question whether or not $\mathbf{C}(\theta) \cap \mathbf{R}(\theta)$ is manoeuvrable. The answer is no it is not, unless the points of θ are controllable to each other. Thus, we will prove that if θ is manueoeuvrable, then $\mathbf{C}(\theta) \cap \mathbf{R}(\theta)$ is manoeuvrable. To do so, we must first consider the following lemmas.

Lemma 3.4. If $\theta \subset E^n$ is manoeuvrable, then for all $x_0 \in \theta$, $C(x_0) = C(\theta)$.

Proof. Let θ be a manoeuvrable set, $x_0 \in \theta$, and $x_1 \in C(\theta)$. Then there exists a $x_2 \in \theta$ such that $x_1 \in C(x_2)$. Since θ is manoeuvrable, $x_2 \in C(x_0)$. Hence, from the transitivity property of controllable sets (Lemma 3.2), it follows that $x_1 \in C(x_0)$. Therefore, $C(\theta) \subseteq C(x_0)$. Sinc $C(x_0) \subseteq C(\theta)$ from the definition of controllability to a set, we conclude that $C(x_0) = C(\theta)$. As $x_0 \in \theta$ is arbitrary, we have that $C(x_0) = C(\theta)$ for all $x_0 \in \theta$. Q.E.D.

Lemma 3.5. If $\theta \subset E^n$ is manoeuvrable, then for all $x_0 \in \theta$, $R(x_0) = R(\theta)$.

Proof. Let θ be a manoeuvrable set, $x_0 \in \theta$, and $x_1 \in R(\theta)$. Then there exists a point $x_2 \in \theta$ such that $x_1 \in R(x_2)$. Since θ is a manoeuvrable set, $x_2 \in R(x_0)$. From the transitivity property of reachable sets (Lemma 3.1), it follows that $x_1 \in R(x_0)$. Therefore, $R(\theta) \subseteq R(x_0)$. Since $R(x_0) \subseteq R(\theta)$ from the definition of reachability from a set, it follows that $R(x_0) = R(\theta)$ for all $x_0 \in \theta$. Q.E.D.

Theorem 3.5. If $\theta \subset E^n$ is manoeuvrable, then $C(\theta) \cap R(\theta)$ is a manoeuvrable set.

Proof. It follows from Lemma 3.4 that $C(x_0) = C(\theta)$ for all $x_0 \in \theta$. Similarly, from Lemma 3.5 we have that $R(x_0) = R(\theta)$ for all $x_0 \in \theta$. Consequently, $C(x_0) \cap R(x_0) = C(\theta) \cap R(\theta)$ for all $x_0 \in \theta$. If $C(x_0) \cap R(x_0) \neq \{x_0\}$, then Theorem 3.4 results in $C(x_0) \cap R(x_0) = C(\theta) \cap R(\theta)$ being a manoeuvrable set. However, if $C(x_0) \cap R(x_0) = \{x_0\}$, then it follows that $\theta = \{x_0\}$ (otherwise points of θ do not have the same reachable and controllable sets), and $C(\theta) \cap R(\theta) = \theta$ is manoeuvrable since θ is manoeuvrable. Q.E.D.

One might be led to believe that we could keep applying Theorem 3.5 over and over again and each time obtain a new manoeuvrable set. The following corollary establishes that this is not the case.

Corollary 3.4. If θ is a manoeuvrable set, then $C(\theta) \cap R(\theta)$ is the largest manoeuvrable set containing θ in the sense that there is no set S containing $C(\theta) \cap R(\theta)$ which is manoeuvrable.

Proof. Let θ be a manoeuvrable set. Suppose $S \subset E^n$ is such that S is manoeuvrable and $C(\theta) \cap R(\theta) \subset S$. Let $x_0 \in C(\theta) \cap R(\theta)$ and let $x_1 \in S \cap \text{ext}(C(\theta) \cap R(\theta))$. Since S is manoeuvrable $x_0 \in C(x_1)$. Furthermore, from Lemma 3.5, we have $x_0 \in R(x_2)$ for all $x_2 \in \theta$. Since $x_0 \in C(x_1)$ is the same as saying $x_1 \in R(x_0)$ (see Remark, Section 3.1), we can use Lemma 3.1 to establish that $x_1 \in R(x_2) = R(\theta)$.

Similarly, as S is manoeuvrable $x_0 \in R(x_1)$. Since $x_0 \in \mathbb{C}(\theta) \cap R(\theta)$, it follows that for every $x_2 \in \theta$, $x_0 \in C(x_2)$. From the Remark of Section 3.1, $x_0 \in R(x_1)$ is the same as saying $x_1 \in \mathbb{C}(x_0)$. Consequently, $x_1 \in C(x_0)$ and $x_0 \in C(x_2)$; hence, $x_1 \in \mathbf{C}(x_2)$ (Lemma 3.2). Since $C(x_2) = \mathbf{C}(\theta)$ it follows that $x_1 \in \mathbf{C}(\theta) \cap \mathbf{R}(\theta)$. Combining our results leads to the statement $x_1 \in \mathbf{C}(\theta) \cap \mathbf{R}(\theta)$ which is a contradiction. Q.E.D.

3.4. The Positive Invariance of $R(x_0)$ and int $R(x_0)$

The two results of this section establish the positive invariance of the sets $\mathbf{R}(x_0)$ and int $\mathbf{R}(x_0)$.

Lemma 3.6. (Grantham, 1973). Let $\mathbf{R}(x_0)$ denote the reachable set from x_0 for the system (2.1). Let $u_1(t)$ be an admissable control law which generates the forward solution $\xi_1(t,x_1)$ to (2.1) where $\xi_1(0,x_1) = x_1$. If $x_1 \in \mathbf{R}(x_0)$, then $\xi_1(t,x_1) \in R(x_0)$ for all $t > 0$.

Proof. Suppose that this were not the case. Then there exists a point $x_2 \in R(x_1)$ for which $x_2 \notin R(x_0)$. But from Lemma 3.1, $x_1 \in R(x_0)$ and $x_2 \in R(x_1)$ implies that $x_2 \in R(x_0)$, which is a contradiction. Q.E.D.

Theorem 3.6. (Grantham, 1973). Let $\mathbf{R}(x_0)$ denote the reachable set from x_0 for the system (2.1) and let $u_1(t)$ be an admissable control law which generates the forward solution $\xi(t,x_1)$

to (2.1) where $\xi_1(0,x_1) = x_1$. If $x_1 \in \text{int } R(x_0)$, then $\xi_1(t,x_1) \in \text{int } R(x_0)$ for all $t > 0$.

Proof. We will follow the proof due to Grantham. Suppose this were not the case. Then there exists a minimum time $T_1 > 0$ such that $\xi_1(T_1,x_1) \in \partial R(x_0)$. Using the continuity of initial conditions for ordinary differential equations (i.e., open balls about x_1 are sent into open balls about $\xi_1(T_1,x_1)$), there exists a forward solution $\xi_2(t,x_2)$ to (2.1) satisfying $\xi_2(0,x_2) = x_2 \in B_\delta(x_1) \cap \text{int } R(x_0)$ and $\xi_2(T_1,x_2) \in B_\varepsilon(\xi_1(T_1,x_2)) \cap \text{ext } R(x_0)$. where $B_\eta(y)$ represents an open ball of radius η about the point y (Figure 3.3). But the trajectory $\xi_2(t,x_2)$, $t \in [0,T_1]$ violates Lemma 3.6, thus yielding a contradiction. Q.E.D.

3.5. The Reachability Maximum Principle and the Abnormal Control Law

Theorem 3.7. (Grantham, 1973; Grantham and Vincent, 1975). Let $u^*(t)$ be an admissable control which generates a boundary trajectory $x^*(t)$ to the system (2.1). It is necessary that there exists a nonzero continuous vector $\lambda(t) \in E^n$ such that for almost all $t > 0$

$$H(x^*,\lambda,u^*) = \max_{u \in \Omega} H(x^*,\lambda,u) = 0 \tag{3.8}$$

and

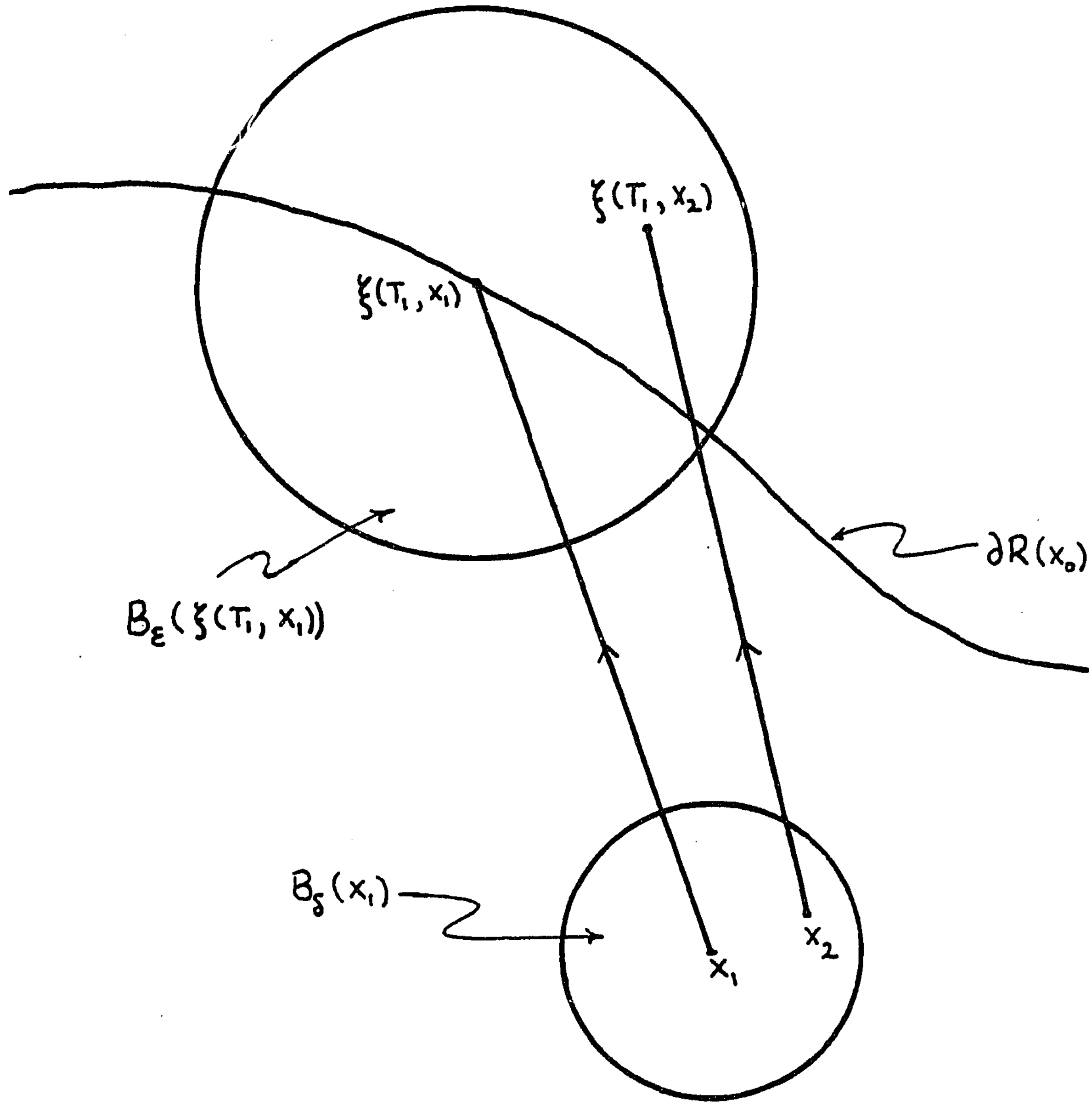


Figure 3.3. If $x_1 \varepsilon$ int $R(x_0)$ and $\xi_1(T_1,x_1) \varepsilon \partial R(x_0)$, then the continuity of initial conditions requires $\xi_2(T_1,x_2)$ $\varepsilon B_{\varepsilon}(\xi_1(T_1,x_1))$ for some $x_2 \varepsilon B_{\delta}(x_1)$.

$$\lambda^T = -\frac{\partial H(x^*,\lambda,u^*)}{\partial x} \tag{3.9}$$

where

$$H(x,\lambda,u) = \lambda^T f(x,u). \tag{3.10}$$

Remarks. The Reachability Maximum Principle is the basis behind approximating the reachable set for several 2-dimensional problems (Vincent, 1980a, 1980b; Ritter and Vincent, 1981). A brief explanation of the technique is as follows: Let $x_0 \in E^n$ be the point whose reachable set is desired. Use (3.8)-(3.10) to generate a forward solution $\xi_1(t,x_0)$ to (2.1) from x_0. The conjecture of Vincent (1980b) is that $\xi_1(t,x_0) \to x^*(t)$ as $t \to +\infty$, where $x^*(t)$ is a boundary trajectory also satisfying (3.8)-(3.10). This method of finding $\partial\mathbf{R}(x_0)$ is best suited for 2-dimensional problems as $\partial\mathbf{R}(x_0)$ is described graphically and not analytically. Furthermore this technique is applicable to systems whose boundary trajectories exist and are asymptotically stable from within $\mathbf{R}(x_0)$. A discussion on the existence and asymptotic stability of boundary trajectories for 2-dimensional systems is presented in Section 3.6 and 3.7, respectively.

Equations (3.8)-(3.10) represent the abnormal case (Blaquière and Leitmann, 1967) of the Pontryagin maximum principle of optimal control theory (Pontryagin et al., 1962). An admissable control which is a solution to the abnormal case is said to be an abnormal control. Thus boundary trajectories, if they exist, are generated by abnormal control laws; however, an abnormal control law does not necessarily generate a boundary trajectory.

In obtaining an abnormal control law (3.8) plays an important role. Observe that substituting (3.10) into (3.8) results in the expression

$$\max_{u \in \Omega} \lambda^T f(x,u) = 0. \tag{3.11}$$

The maximization of (3.11) with respect to the kth component of u leads us to define the switching function for u_k

$$\begin{aligned} \sigma_k(t) &= \frac{\partial H(x,\lambda,u)}{\partial u_k} \\ &= \lambda^T \frac{\partial f(x,u)}{\partial u_k}. \end{aligned} \tag{3.12}$$

An alternative representation of the switching function can be made for the 2-dimensional systems (2.1) where $f(x,u) = g(x) + Bu$, with $g:E^n \to E^n$ being continuously differentiable and B being a 2×1 real constant matrix, and $\Omega = [u_{min}, u_{max}]$, where $-\infty < u_{min} \leq u \leq u_{max} < +\infty$. If $u(t)$ is an abnormal control law and $\sigma(t) \not\equiv 0$ for all intervals of positive time, then (3.8)-(3.10) yield the conditions under which the control is to switch, i.e.,

$$H(x,\lambda,u) = \lambda_1(t)[g_1(x) + b_1u] + \lambda_2(t)[g_2(x) + b_2u] = 0 \quad (3.13)$$

and

$$\frac{\partial H(x,\lambda,u)}{\partial u} = \lambda_1(t)b_1 + \lambda_2(t)b_2 = 0. \quad (3.14)$$

where $g_1(x)$ and $g_2(x)$ are the first and second components of $g(x)$, respectively. As (3.13) and (3.14) must be satisfied simultaneously with $\lambda(t) \not\equiv 0$, we have the requirement that

$$\det\begin{vmatrix} g_1(x) + b_1u & g_2(x) + b_2u \\ b_1 & b_2 \end{vmatrix} = 0. \quad (3.15)$$

Expansion of (3.15) results in $s(x) = 0$ where

$$s(x) = b_2g_1(x) - b_1g_2(x) \quad (3.16)$$

is the abnormal bang-bang state space switching function. Consequently, for this special system the scalar abnormal control $u(t)$ switches from one extreme to the other whenever $s(x)$ changes sign.

For these 2-dimensional systems it is computationally convenient, and greater insight is obtained, if we use the abnormal bang-bang state space switching function rather than switching functions of the form (3.12). The reason for this is that we do not have to solve the adjoint equation (3.9) and we can actually plot the switching arcs in the state space, thus obtaining valuable information. One such piece of information is given by

Lemma 3.7. Consider the 2-dimensional dynamical system

$$\dot{x} = g(x) + Bu$$

where $g:E^n \to E^n$ is continuously differentiable, B is a 2×1 real constant matrix, and $-\infty < u_{min} \leq u \leq u_{max} < +\infty$. Let u_c be a constant admissable control law and let x_{u_c} be a controlled equilibrium point associated with u_c. Then $s(x_{u_c}) = 0$, i.e., x_{u_c} lies on the abnormal bang-bang state space switching arc.

Proof. Under the stated hypothesis

$$g_1(x_{u_c}) + b_1 u_c = 0$$

and

$$g_2(x_{u_c}) + b_2 u_c = 0.$$

If $b_1 \neq 0$ and $b_2 \neq 0$, we can solve for u_c and find that

$$u_c = -\frac{g_1(x_{u_c})}{b_1} = -\frac{g_2(x_{u_c})}{b_2}.$$

From this it follows that $s(x_{u_c}) = 0$.

Next, suppose that $b_1 = 0$ and $b_2 \neq 0$ (a similar argument exists for the case $b_1 \neq 0$ and $b_2 = 0$). For this situation we have that $g_1(x_{u_c}) = 0$. Substituting this into (3.16) we find that $s(x_{u_c}) = 0$. Q.E.D.

3.6. An Existence Theorem for Boundary Trajectories

In later sections of this work we will use the Reachability Maximum Principle to find the boundary of 2-dimensional reachable sets. In order to do this we must first eliminate the need to assume the existence of boundary trajectories. For a class of systems this is accomplished with

Theorem 3.8. Consider the 2-dimensional version of (2.1). Let $\angle x_0 \in E^2$ and let $\mathbf{R}(x_0)$ represent the reachable set from x_0. Let $\angle C(x,\Omega)$ be the angle of the convex cone of velocity vectors at x. Suppose $\partial\mathbf{R}(x_0)$ exists and that the boundary points for which $\angle C(x,\Omega) = 0$ are isolated. Let $x_1 \in \partial\mathbf{R}(x_0)$. If there exists constant admissable controls v_c and w_c such that

i) the angle between $f(x,v_c)$ and $f(x,w_c)$ defines $\angle C(x,\Omega)$ in some ball about x_1,

ii) no zeros of $f(x,v_c)$ and $f(x,w_c)$ exist in a ball about x_1, with the possible exception at $x = x_1$, and

iii) there exists a ball about x_1 such that the forward solutions to (2.1) with $u = v_c$ and $u = w_c$ each cross the boundary of the ball at a single point (perhaps different points),

then a boundary trajectory from x_1 exists with boundary control $u = v_c$ or $u = w_c$.

Proof. We will break the proof into two parts, each part based upon whether or not $\angle C(x_1,\Omega) = 0$.

Part i). We begin with the case $\angle C(x_1,\Omega) \neq 0$ and suppose that there is no boundary trajectory from x_1 resulting from $u = v_c$ and $u = w_c$.

Let $\xi_1(t,x_1)$ be the forward solution to (2.1) with $u = v_c$ and let $\xi_2(t,x_1)$ be the forward solution to (2.1) with $u = w_c$. By supposition $\xi_1(t,x_1)$ and $\xi_2(t,x_1)$ are not boundary trajectories so they must travel into the interior of $R(x_0)$.

By hypothesis there exists a $\delta_1 > 0$ such that $\xi_1(t,x_1)$ and $\xi_2(t,x_1)$ each intersect the ball about x_1 of radius δ_1 $B_{\delta_1}(x_1)$ at a single point (possibly different). Call these points P_1 and P_2, respectively. Also by hypothesis, there exists a $\delta_2 > 0$ such that $f(x,v_c)$ and $f(x,w_c)$ define $\angle C(x,\Omega)$ when $x \in B_{\delta_2}(x_1)$.

By hypothesis the zeros of $f(x,v_c)$ and $f(x,w_c)$ are isolated; hence, there exists a $\delta_3 > 0$ such that $f(x,v_c) \neq 0$ and $f(x,w_c) \neq 0$ for all $x \in B_{\delta_3}(x_1)$.

Since $f(x_1,v_c)$ and $f(x_1,w_c)$ define $\angle C(x,\Omega)$, and the trajectories $\xi_1(t,x_1)$ and $\xi_2(t,x_1)$ are not a boundary trajectories, it follows that $\angle C(x_1,\Omega) < \pi$. Hence, by the continuity of $\angle C(x,\Omega)$ (Section 2.8), there exists a $\delta_4 > 0$ such that $0 < \angle C(x,\Omega) < \pi$ whenever $x \in B_{\delta_4}(x_1)$.

Let $\delta = \min(\delta_1,\delta_2,\delta_3, \delta_4,\|x_1 - x_0\|/2)$. Let $t_j > 0$ be the minimum time for which $\xi_i(t_i,x_1) = P_i$, $i = 1$ and 2. Let

$$\textstyle\sum_i = \{x \in B_\delta(x_1) | x = \xi_1(t,x_1), 0 < t < t_1\}, \quad \text{for } i = 1 \text{ and } 2.$$

Observe that $\sum_1$ is a semipermeable surface (Isaacs, 1965) in that all forward trajectories to (2.1) which cross $\sum_1$ do so from the same side. Similarly $\sum_2$ is a semipermeable surface. Let Γ be the open region of space bounded by $\sum_1$, $\sum_2$, and $\partial B_\delta(x_1)$ which includes all possible forward trajectories from x_1. We will show that there exists a point $x_2 \in \partial\mathbf{R}(x_0) \cap B_\delta(x_1)$ which is not found in $\overline{\Gamma}$. First of all, note that for every ball about x_1 there exists a point of $\partial\mathbf{R}(x_0) \cap B_\delta(x_1)$ which is not on $\partial\Gamma$. If this were not the case then $\xi_1(t,x_1)$ or $\xi_2(t,x_1)$ would travel along $\partial\mathbf{R}(x_0)$ for some time $\tau > 0$ thus violating the supposition that no boundary trajectory from x_1 exists with $u = v_c$ and $u = w_c$.

Consequently, let us choose a point $x_2 \in \partial\mathbf{R}(x_0) \cap B_\delta(x_1)$ not on $\partial\Gamma$. Observe that $x_2 \notin \Gamma$, for if it were, then there must be some point $Q \in \partial\mathbf{R}(x_0)$ from which we can penetrate the boundary of the reachable set and violate the positive invariance of $\partial\mathbf{R}(x_0)$ (Theorem 3.6). In Figure 3.4 we demonstrate this impossibility. If $x_2 \in \Gamma$, then at $Q \in \partial\mathbf{R}(x_0) \cap \sum_2$ the boundary of the reachable set can be penetrated using the control $u = w_c$, i.e., the control which generated $\sum_2$. Consequently, we conclude that we can choose $x_2 \in \partial\mathbf{R}(x_0) \cap B_\delta(x_1)$ such that $x_2 \notin \overline{\Gamma}$.

We are now prepared to complete part (i) of the proof. Choose a time $\tau \in (0,t_1)$ so that $x_3 = \xi_1(\tau,x_1) \in \sum_1$. Choose $\varepsilon > 0$ such that $B_\varepsilon(x_3) \cap \sum_2 = \emptyset$ and $B_\varepsilon(x_3) \subset B_\delta(x_1)$. From the continuity of initial conditions for ordinary differential equations, there exists $\delta_5 > 0$ such that whenever $x_2 \in B_{\delta_5}(x_1)$, then $\xi_3(\tau,x_2) \in B_\varepsilon(x_1)$, where $\xi_3(t,x_2)$ is the forward solution to (2.1) from x_2. Let us choose $x_2 \in B_{\delta_5}(x_1) \cap \partial\mathbf{R}(x_0) \cap \text{ext}\ \overline{\Gamma}$ such that $\xi_3(t,x_2)$ must cross $\sum_2$ (Figure 3.5). Let us define $\sum_3 = \{x \in B_\delta(x_1) | x = \xi_3(t,x_2), 0 < t < t_1\}$. Observe that $\sum_3$ is a semipermeable surface with the directon of permeability indicated at the point Q. Consider the region Ξ bounded by $\sum_2$, $\sum_3$, and $\partial\mathbf{R}(x_0)$. By noting the directions of permeability through $\sum_2$ and $\sum_3$ (Figure 3.6) we see that points of Ξ cannot be reached by any trajectory from x_0. Hence $\Xi \not\subset \mathbf{R}(x_0)$, which is a contradiction. Consequently, for the case of $\angle C(x_1,\Omega) \neq 0$ a

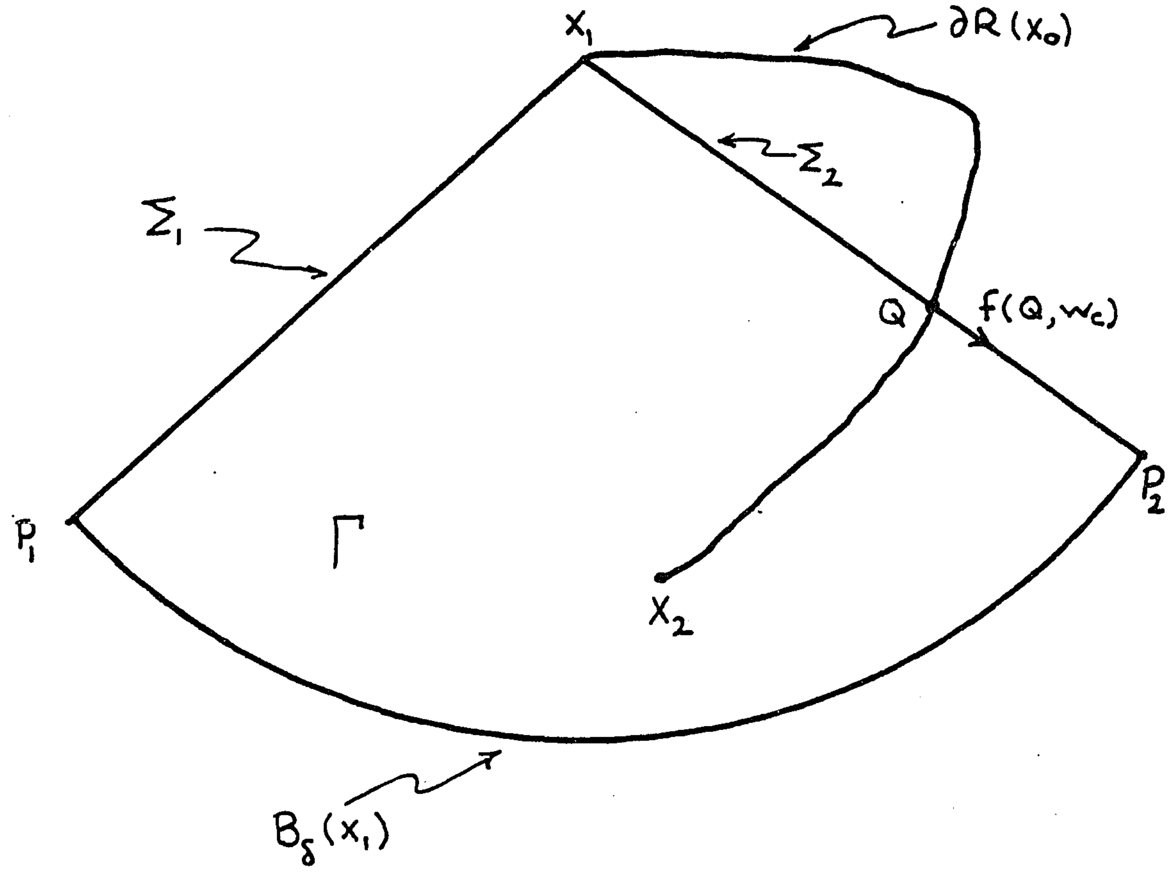


Figure 3.4. If there is a point $x_2 \in \partial R(x_0) \cap \Gamma$, then it is possible to penetrate $R(x_0)$ at some point Q.

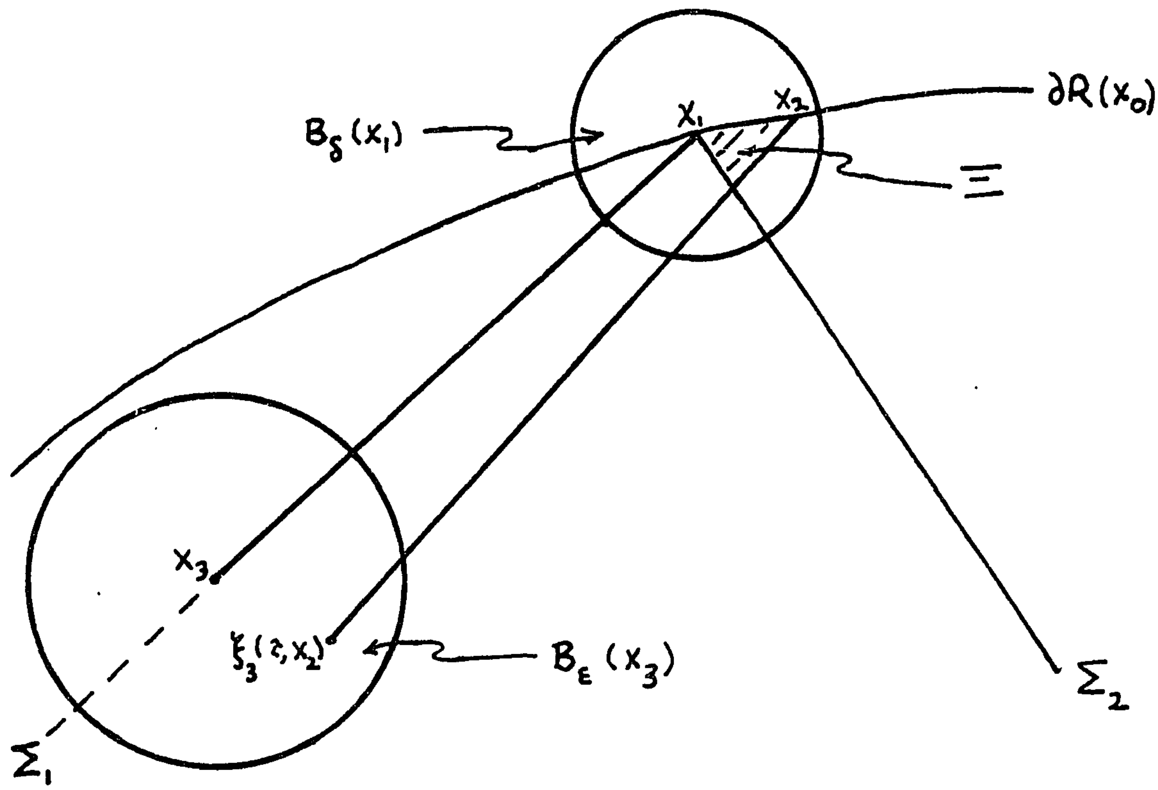


Figure 3.5. From the continuity of initial conditions we can bound a region Ξ by semipermeable surfaces and $\partial R(x_0)$.

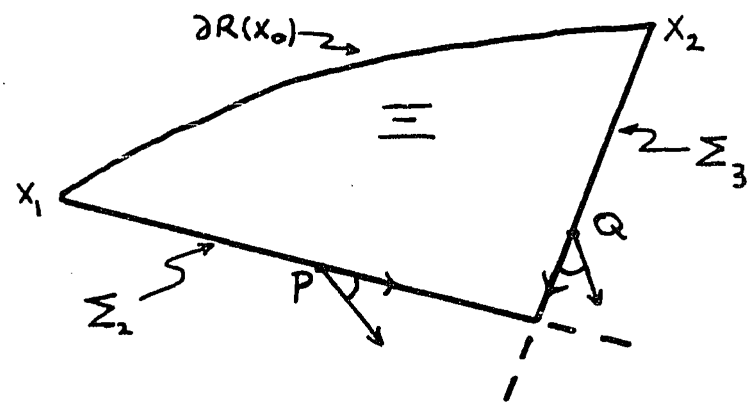


Figure 3.6. The semipermeability of Σ_2 and Σ_3 (as indicated at points P and Q, respectively) result in points of Ξ not being reachable from x_0.

boundary trajectory from x_1, resulting from $u = v_c$ or $u = w_c$, exists.

Part ii). Let us consider the case $\angle C(x_1,\Omega) = 0$. Suppose there is no boundary trajectory from x_1 resulting from $u = v_c$ and $u = w_c$. By hypothesis the boundary points for which $\angle C(x,\Omega) = 0$ are isolated. Hence, there exists $\delta_6 > 0$ such that $\angle C(x,\Omega) \neq 0$ whenever $x \in \partial \mathbf{R}(x_0) \cap B_{\delta_6}(x_1)$ and $x \neq x_1$. From above, we know that we can find $B_\delta(x_1)$ such that the hypotheses (i)-(iii) are met, $\angle C(x,\Omega)$ is continuous, and $B_\delta(x_1) \subset B_{\delta_6}(x_1)$. Let $\{y_n\}$ be a sequence of points taken from $B_\delta(x_1) \cap \partial \mathbf{R}(x_0)$ such that $y_n \to x_1$. As $\angle C(x,\Omega)$ is continuous, it follows that $\lim \angle C(y_n,\Omega) = 0$ as $n \to +\infty$. That is to say, the angle between $f(y_n,v_c)$ and $f(y_n,w_c)$ is decreasing to zero as $n \to +\infty$. This can occur either by having $f(x_1,v_c) = 0$, $f(x_1,w_c) = 0$, or by having $f(y_n,v_c)$ and $f(y_n,w_c)$ directed into $\mathbf{R}(x_0)$ for n sufficiently large and becoming parallel in the limit. Since all three possibilities lead to contradictions, it follows that the supposition is false and the proof is complete. Q.E.D.

<u>Corollary 3.5.</u> Consider the linear dynamical system

$$\dot{x} = f(x,u) = Ax + Bu \tag{3.17}$$

where $x \in E^2$, $u \in \Omega = [u_{min},u_{max}]$ with $-\infty < u_{min} < 0 < u_{max} < +\infty$, A is a 2×2 real constant matrix whose eigenvalues have negative real part, and B is a 2×1 real constant matrix.

Let **R** represent the reachable set from the origin for (3.17). If rank $[B \quad AB] = 2$, then there exists a boundary trajectory through every point of $\partial\mathbf{R}$.

Proof. We begin by showing that **R** is bounded as this leads to the existence of $\partial\mathbf{R}$. Let $u(t)$ be an admissable control law. Then the forward solution to (3.17) from the origin is given by

$$x(t) = \int_0^t e^{A(t-s)} Bu(s)ds. \tag{3.18}$$

Taking the Euclidean norm of both sides of (3.18) results in

$$\|x(t)\| \le \int_0^t \|e^{A(t-s)}\| \, \|Bu(s)\| ds. \tag{3.19}$$

Since the eigenvalues of A are negative, there exists positive constants K and μ such that $\|e^{At}\| \le Ke^{-\mu t}$ (Brauer and Nohel, 1969). Since Ω is compact, there exists a positive constant L such that $\|Bu\| \le L$. Using this information in (3.19) results in

$$\|x(t)\| \le KL\int_0^t e^{-\mu(t-s)} ds \le \frac{KL}{\mu}, \tag{3.20}$$

for all $t \ge 0$. Hence, **R** is bounded and $\partial\mathbf{R}$ exists.

Let $\angle C(x,\Omega)$ be the angle of the convex cone of velocity vectors at x to (3.17). Note that the boundary points for which $\angle C(x,\Omega) = 0$ are isolated. To see this, let us first find all points such that $\angle C(x,\Omega) = 0$. Since u enters (3.17) linearly, $\angle C(x,\Omega) = 0$ whenever the vector Ax is a parallel to the vector Bu, i.e., when $x = cA^{-1}B$, where $c \in R$. These points define the line $\ell = \{x \in E^2 | x = cA^{-1}B, c \in R\}$. If we are to show that the boundary points for which $\angle C(x,\Omega) = 0$ are isolated, then we must show that the system (3.17) does not travel along ℓ. Thus, let us consider $x_1 \in \partial \mathbf{R} \cap \ell$. Then at $x = x_1$ we have

$$\dot{x} = Ax_1 + Bu$$

$$= A(cA^{-1}B) + Bu$$

$$= \lambda B,$$

where $\lambda = c + u$. Since rank $[B \quad AB] = 2$ by hypothesis, it follows that the vector λB is not parallel to the vector $A^{-1}B$; hence, the system does not move along ℓ. Consequently, the boundary points for which $\angle C(x,\Omega) = 0$ must be isolated.

Let x_{min} and x_{max} be the unique controlled equilibrium points associated with the constant admissable controls u_{min} and u_{max}, respectively. Since x_{min} and x_{max} are unique they must be isolated. Consequently, if $x_1 \varepsilon \mathbf{R}$, then no zeros of $f(x,u_{min})$ and $f(x,u_{max})$ exist in a deleted ball about x_1. Thus we have fulfilled condition (ii) Theorem 3.8.

Next, observe that since u enters (3.17) linearly, the angle between $f(x,u_{min})$ and $f(x,u_{max})$ defines $\angle C(x,\Omega)$ for all $x \varepsilon E^2$. Hence, condition (i) of Theorem 3.8 is satisfied.

Finally, since the system (3.17) is linear and the eigenvalues of A have negative real part (thus eliminating the chance that u_{min} or u_{max} leads to a periodic forward solution of (3.17)), it follows that condition (iii) of Theorem 3.8 is satisfied. Having established that (3.17) satisfies all the conditions of Theorem 3.8, it follows that there exists a boundary trajectory through every point of $\partial\mathbf{R}$. Q.E.D.

3.7. On the Asymptotic Stability of Boundary Trajectories

The use of the Reachability Maximum Principle is predicated upon knowing a boundary trajectory exists and a point on the boundary of the reachable set from which the trajectory can begin. Clearly, if a boundary trajectory exists but a suitable initial condition cannot be found, then another approach must be taken. One method is to make use of the asymptotic stability of the boundary trajectory. This technique can be used for a class of dynamical systems and is a special case of the following theorem.

Theorem 3.9. Consider the dynamical system

$$\dot{x} = Ax + g(x,u) \tag{3.21}$$

where $x \in E^n$, $u \in \Omega \subset E^n$ (Ω compact), A is a $n \times n$ real constant matrix with all of its eigenvalues having negative real part, and $g: E^n \times E^m \to E^n$ is continuously differentiable in x and continuous in u.

Let $R(x_0)$ be the reachable set from x_0 for the system (3.21) and let θ be a nonempty, connected set such that $\theta \subseteq R(x_0)$. Let X be an open convex subset of E^n such that $R(x_0) \subseteq X$. If

i) $$\|\partial g(x,u)/\partial x\| < L$$

for all $x \in X$ and $u \in \Omega$,

ii) there exists a trajectory $x^*(t)$ generated by an admissable control law $u^*(t)$ such that $x^*(t) \in \partial\theta$ for all $t > 0$ and

iii) there exists positive constants K and μ such that $-\mu + KL < 0$ and $\|e^{At}\| < Ke^{-\mu t}$,

then all solutions to (3.21), with $u = u^*(t)$, which begin in $R(x_0)$ approach $x^*(t)$ asymptotically as $t \to +\infty$.

Proof. Let $x_0^* \in \partial\theta$ and let $x^*(t)$ be the trajectory generated by $u^*(t)$ such that $x^*(0) = x_0^*$ and $x^*(t) \in \partial\theta$ for all

$t \geq 0$. That is to say, $x^*(t)$ satisfies

$$\dot{x}^* = Ax^* + g(x^*,u^*) \quad \text{and} \quad x^*(0) = x_0^*. \tag{3.22}$$

Let $y_0 \in \mathbf{R}(x_0)$ but such that $y_0 \neq x_0^*$. Let $y(t)$ be the forward solution to (3.21) from y_0 generated by $u^*(t)$. Then

$$\dot{y} = Ay + g(y,u^*) \quad \text{and} \quad y(0) = y_0. \tag{3.23}$$

Subtracting (3.22) from (3.23) and letting $\phi = y - x^*$ we have

$$\phi = A\phi + [g(y,u^*) - g(x^*,u^*)] \tag{3.24}$$

where

$$\phi(0) = y_0 - x_0^*.$$

In integral form (3.24) becomes

$$\phi(t) = e^{At}\phi(0) + \int_0^t e^{A(t-s)}[g(y,u^*) - g(x^*,u^*)]ds. \tag{3.25}$$

Taking the Euclidean norm of both sides of (3.25), and using the fact that there exists positive constants K and μ such that $\|e^{At}\| \leq Ke^{-\mu t}$, we find that

$$\|\phi(t)\| \leq Ke^{-\mu t}\|\phi(0)\| + \int_0^t Ke^{-\mu(t-s)}\|g(y,u^*) - g(x^*,u^*)\|ds. \tag{3.26}$$

Since $y(t)$ and $x^*(t)$ remain in $R(x_0) \subseteq X$, it follows that

$$\begin{aligned}\|g(y,u^*) - g(x^*,u^*)\| &\leq L\|y - x^*\| \\ &= L\|\phi\|\end{aligned} \tag{3.27}$$

(Cartan, 1967; Edwards, 1973). Substituting (3.27) into (3.26) yields

$$\|\phi(t)\| \leq Ke^{-\mu t}\|\phi(0)\| + KL\int_0^t e^{-\mu(t-s)}\|\phi(s)\|ds. \tag{3.28}$$

Multiplying (3.28) by $e^{\mu t}$ and letting $\psi(t) = e^{\mu t}\|\phi(t)\|$ we have

$$\psi(t) \leq K\|\phi(0)\| + KL\int_0^t \psi(s)ds. \tag{3.29}$$

A direct application of Gronwall's inequality leads to

$$\psi(t) \leq K\|\phi(0)\|e^{KLt}. \tag{3.30}$$

In terms of $\|\phi(t)\|$, (3.30) becomes

$$\|\phi(t)\| \leq K\|\phi(0)\|e^{(-\mu+KL)t}. \tag{3.31}$$

By hypothesis $-\mu + KL < 0$, hence, $\|\phi(t)\| \to 0$ exponentially as $t \to +\infty$. Therefore, $x^*(t)$ is asymptotically stable. Since $y_0 \in R(x_0)$ is arbitrary, the proof is complete. Q.E.D.

Corollary 3.6. Let $R(x_0)$ be the reachable set from x_0 for the system (3.21) with $g(x,u) = Bu$, where B is a $n \times m$ real constant matrix. Let θ be a nonempty connected subset of $R(x_0)$. If there exists a trajectory $x^*(t)$ generated by an admissable control law $u^*(t)$ such that $x^*(t) \in \partial\theta$ for all $t \geq 0$, then $x^*(t)$ is globally asymptotically stable under $u^*(t)$.

Proof. Let $x_0^* \in \partial\theta$ and let $x^*(t)$ be the trajectory generated by $u^*(t)$ with $x^*(0) = x_0^*$ and $x^*(t) \in \partial\theta$ for all $t \geq 0$. Then $x^*(t)$ satisfies

$$\dot{x}^* = Ax^* + Bu^* \quad \text{and} \quad x^*(0) = x^*_0. \tag{3.32}$$

Let $y_0 \in E^n$, but such that $y_0 \neq x_0^*$. Let $y(t)$ be the forward solution to

$$\dot{y} = Ay + Bu^* \quad \text{and} \quad y(0) = y_0. \tag{3.33}$$

Subtracting (3.33) from (3.32) and letting $\phi = x^* - y$ results in

$$\dot{\phi} = A\phi \quad \text{and} \quad \phi(0) = x_0^* - y_0. \tag{3.34}$$

Since all of the eigenvalues of A have negative real part, $\|\phi\| \to 0$ as $t \to +\infty$ exponentially. Since $y_0 \varepsilon E^n$ is arbitrary, it follows that $x^*(t)$ is globally asymptotically stable under $u^*(t)$. Q.E.D.

3.8. Upper Bounds on the Reachable Set from the Origin

In this section we present several results dealing with the overestimation of the reachable set from the origin R. We begin by proving two theorems, each of which guarantee that R is bounded. The first of these theorems is based on the "second" or "direct" method of Lyapunov (LaSalle and Lefschetz, 1961). The second theorem relies on the definition of the reachable set and the system dynamics. Following these results we state conditions under which the reachable set from the origin for a linear system will be the entire state space.

Theorem 3.10. (Grantham, 1980a). Let R be a reachable set from the origin for the system (2.1). If there exists a continuously differentiable function $V(x)$ with $V:E^n \to E^1$ and a real number V^* such that

$$V(0) < V^*, \tag{3.35}$$

$$\{x \varepsilon E^n | V(x) < V^*\} \subset \{x \varepsilon E^n | V(x) < k\} \quad \text{for all} \quad k > V^*, \tag{3.36}$$

and

$$\{x \in E^n | \dot{V}(x,u) \geq 0, u \in \Omega\} \subset \{x \in E^n | V(x) \leq V^*\}, \tag{3.37}$$

where

$$\dot{V}(x,u) = \frac{\partial V(x)}{\partial x} f(x,u), \tag{3.38}$$

then $R \subseteq \{x \in E^n | V(x) \leq V^*\}$.

Proof. We will follow Grantham's proof. From (3.35) and (3.36) the origin is in the region where $V(x) \leq V^*$ and this region is inside the set for which $V(x) < k$ for $k > V^*$. In order for the system to leave $V(x) \leq V^*$ it must penetrate $V(x) = V^*$ in the direction of increasing $V(x)$. This cannot happen, however, as (3.37) indicates that all regions for which $\dot{V}(x,u) > 0$ are contained in the set for which $V(x) \leq V^*$. This completes the theorem. Q.E.D.

Example 3.4. Use Theorem 3.10 to find an upper bound on the reachable set from the origin R for the system

$$\dot{x} = \begin{vmatrix} 0 & 1 \\ -2 & -3 \end{vmatrix} x + \begin{vmatrix} 2 \\ -3 \end{vmatrix} u$$

where $u \in [-1,1]$.

We proceed as in Summers (1984). Let

$$V(x) = p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2$$

where p_{11}, p_{12}, and p_{22} are positive constants. The time derivatives of $V(x)$ along forward solutions is given by

$$\dot{V}(x,u) = (\nabla V \dot{x}) = -[4p_{12}x_1^2 + 4(p_{22} + 6p_{12} - 2p_{12})x_1x_2$$

$$+ (6p_{22} - 2p_{12})x_2^2 + (4p_{11} - 6p_{12})x_1u + (4p_{12} - 6p_{22})x_2u].$$

A natural choice for p_{11}, p_{12}, and p_{22} comes from letting

$$4p_{12} = 1,$$

$$4p_{22} + 6p_{12} - 2p_{11} = 0,$$

and

$$6p_{22} - 2p_{12} = 1.$$

Solving these equations results in $p_{11} = 5/4$, $p_{12} = 1/4$, and $p_{22} = 1/4$. Hence,

$$\{x \in E^2 | \dot{V}(x,u) > 0\} = \{x \in E^2 | (x_1 - \frac{7u}{4})^2 + (x_2 + \frac{u}{4})^2 < (\frac{5}{4}\sqrt{2})^2u^2\}$$

$$= \{x \in E^2 | (x_1 - \tfrac{7}{4})^2 + (x_2 + \tfrac{1}{4})^2 \leq (\tfrac{5}{4}\sqrt{2})^2\}$$

$$\cup \{x \in E^2 | (x_1 + \tfrac{7}{4})^2 + (x_2 - \tfrac{1}{4})^2 < (\tfrac{5}{4}\sqrt{2})^2\}.$$

Maximizing

$$V(x) = \tfrac{5}{4}x_1^2 + \tfrac{1}{2}x_1x_2 + \tfrac{1}{4}x_2^2$$

subject to the constraint

$$(x_1 - \tfrac{7}{4})^2 + (x_2 + \tfrac{1}{4}) = (\tfrac{5}{4}\sqrt{2})^2$$

results in the solution $x_1 \doteq 3.482$ and $x_2 \doteq 0.1047$. Hence, $V^* \doteq 15.34$. From Theorem 3.10 we conclude that

$$R \subseteq \{x \in E^2 | 5x_1^2 + 2x_1x_2 + x_2^2 \leq 61.36\}.$$

As no procedure exists for constructing $V(x)$, Theorem 3.10 can be extremely difficult to use, even for 2-dimensional problems. It is for this reason that alternative ways of bounding **R** are desired, ways which rely more upon the system dynamics and not on auxilary functions. One such method of doing this is found in

Theorem 3.11. Consider the dynamical system

$$\dot{x} = Ax + g(x,u) \qquad (3.39)$$

where $x \in E^n$, $u \in \Omega \subset E^m$ (Ω compact), $g: E^n \times E^m \to E^n$ is continuously differentiable in x and continuous in u, and A is a $n \times n$ real constant matrix, the eigenvalues of which have negative real part.

Let R denote the reachable set from the origin for the system (3.39). Let K and μ be positive constants such that $\|e^{At}\| \leq Ke^{-\mu t}$. If

i) there exists a compact set $X \subset E^n$ such that $0 \in \text{int } X$ and $\|g(x,u)\| \leq L$ for all $x \in X$ and $u \in \Omega$, and

ii) $d(0,\partial X) \geq KL/\mu$, where

$$d(0,\partial X) = \inf_{x \in \partial X} \|x\|,$$

then $R \subseteq X$.

Proof. Let $u(t)$ be an admissable control law. Then any forward solution to (3.39) from the origin can be represented by the integral equation

$$x(t) = \int_0^t e^{A(t-s)} g(x(s),u(s))ds. \qquad (3.40)$$

Taking the Euclidean norm of both sides of (3.40) and using the fact that $\|e^{At}\| \leq Ke^{-\mu t}$ results in

$$\|x(t)\| \leq Ke^{-\mu t}\int_0^t e^{\mu s}\|g(x(s),u(s))\|ds. \qquad (3.41)$$

Requiring $x(s) \in X$ for all $s \in [0,t]$ allows us to write

$$\|x(t)\| \leq Ke^{-\mu t}\int_0^t e^{\mu s}L\,ds$$

$$= \frac{KLe^{-\mu t}}{\mu}(e^{\mu t} - 1) \qquad (3.42)$$

$$< \frac{KL}{\mu}.$$

Thus we conclude that $\|x(t)\| < KL/\mu$ for all $t \geq 0$ provided $x(t) \in X$ for all $t \geq 0$. Observe that we can guarantee $x(t) \in X$ for all $t \geq 0$ if $\mathbf{R} \subseteq X$. Since $x(t) \in \mathbf{R}$, for all $t \geq 0$, and $d(0,\partial X) \geq KL/\mu$ by hypothesis, it follows from (3.42) that $\mathbf{R} \subseteq X$. Q.E.D.

Example 3.5. Consider the dynamical system

$$\dot{x} = \begin{vmatrix} -6 & 0 \\ 0 & -8 \end{vmatrix} x + \begin{vmatrix} u \\ x_1^2 \end{vmatrix} \qquad (3.43)$$

where $u \in [-1,1]$. We can find an upper bound to the reachable set from the origin for (3.43) using Theorem 3.11. Note that $\|e^{At}\| \leq \sqrt{2}e^{-6t}$ so that we can choose $K = \sqrt{2}$ and $\mu = 6$. Since $\|g(x,u)\| = \sqrt{u^2 + x_1^2} \leq \sqrt{1 + x_1^2}$, we can let $L = \max_{x \in \partial X} \sqrt{1 + x_1^2}$. Let

X be the circle centered at the origin having radius 1. Then $KL/\mu = 1/3$. Since $d(0,\partial X) = 1$, it follows from Theorem 3.11 that $\mathbf{R} \subseteq X$.

Tighter bounds on the reachable set from the origin for (3.43) can be obtained. Restricting X to be the family of circles centered at the origin having radius r, we see that $L = \sqrt{1 + r^4}$. Hence, if we use the values of K and μ listed above, then the requirement that $KL/\mu < d(0,\partial X)$ reduces to

$$\frac{\sqrt{2}}{6}\sqrt{1 + r^4} < r^2. \tag{3.44}$$

Since we are looking for the smallest value of r which satisfies (3.44), let us consider the equation

$$\frac{\sqrt{2}}{6}\sqrt{1 + r^4} = r^2 \tag{3.45}$$

Solving (3.45) results in $r \doteq 0.492$. Therefore, the smallest circle centered at the origin which contains $\mathbf{R}$ has radius 0.492 (approximately).

<u>Corollary 3.7</u>. Let $\mathbf{R}$ be the reachable set from the origin for the system (3.39) with $g(x,u) = Bu$, where B is a $n \times m$ real constant matrix. Then $\mathbf{R}$ is bounded.

<u>Proof</u>. Since $g(x,u) = Bu$ it follows that there exists a positive constant L such that $\|g(x,u)\| < L$ for all $u \in \Omega$

independent of x. Since the eigenvalues of A have negative real part it follows that there exists constants $K > 0$ and $\mu > 0$ such that $\|e^{At}\| \leq Ke^{-\mu t}$. Let X be the ball centered at the origin having radius KL/μ. From Theorem 3.11 it follows that $R \subseteq X$, thus establishing the corollary. Q.E.D.

Example 3.6. We will use Theorem 3.11 to approximate the reachable set from the origin for the system in Example 3.4. Note that for this sytem $\|Bu\| = \sqrt{9u^2 + 4u^2} \leq \sqrt{13}$; hence, set $L = \sqrt{13}$. Since

$$e^{At} = \begin{vmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{vmatrix},$$

it follows that $\|e^{At}\| = \sqrt{10e^{-2t} - 14e^{-3t} + 10e^{-4t}}$. We wish to find positive constants K and μ such that $\|e^{At}\| \leq Ke^{-\mu t}$ for all $t \geq 0$. Alternatively, we examine the expression $\|e^{At}\|^2 \leq K^2e^{-2\mu t}$ for all $t \geq 0$. To find satisfactory values of K and μ we require

$$h(t) = 10e^{-2t} - 14e^{-3t} + 10e^{-4t} - K^2e^{-2\mu t} \leq 0$$

for all $t \geq 0$. Letting $v = e^{-t}$ we see that $h(t) \leq 0$ for all $t \geq 0$ is the same as requiring

$$w = 10v^2[1 - \frac{7v}{5} + v^2 - \frac{K^2}{10}v^{2(\mu-1)}] \leq 0$$

for all $v \in (0,1]$. Consequently, $w \leq 0$ for $v \in (0,1]$ provided

$$1 - \frac{7v}{5} + v^2 < \frac{K^2}{10}v^{2(\mu-1)}$$

for $v \in (0,1]$. If we choose $\mu = 1$, then the inequality is satisfied provided $k > \sqrt{10}$. Since we seek to minimize KL/μ, we let $K = \sqrt{10}$. Therefore, $KL/\mu = \sqrt{130} \doteq 11.4$. Using Theorem 3.11, we conclude that the reachable set is contained in the circle centered at the origin with radius 11.4.

Remark. In Figure 3.7 we illustrate the regions bounding the reachable set from the origin as found in Examples 3.4 and 3.6. It is not surprising to see that the Lyapunov approach gives a better approximation than the technique given in Theorem 3.11. The reason for this is simple: the Lyapunov method uses information about the direction of $F(x,\Omega)$ along level curves, whereas the bounds in Theorem 3.11 are obtained using the magnitude of $f(x,u)$ and the definition of a reachable set. However, along with the better approximation comes the work in finding a Lyapunov function and the optimization of the Lyapunov function subject to constraint.

The following theorem, which is stated without proof, presents conditions for the controllable set of linear system with bounded control to be the entire state space. We will use this

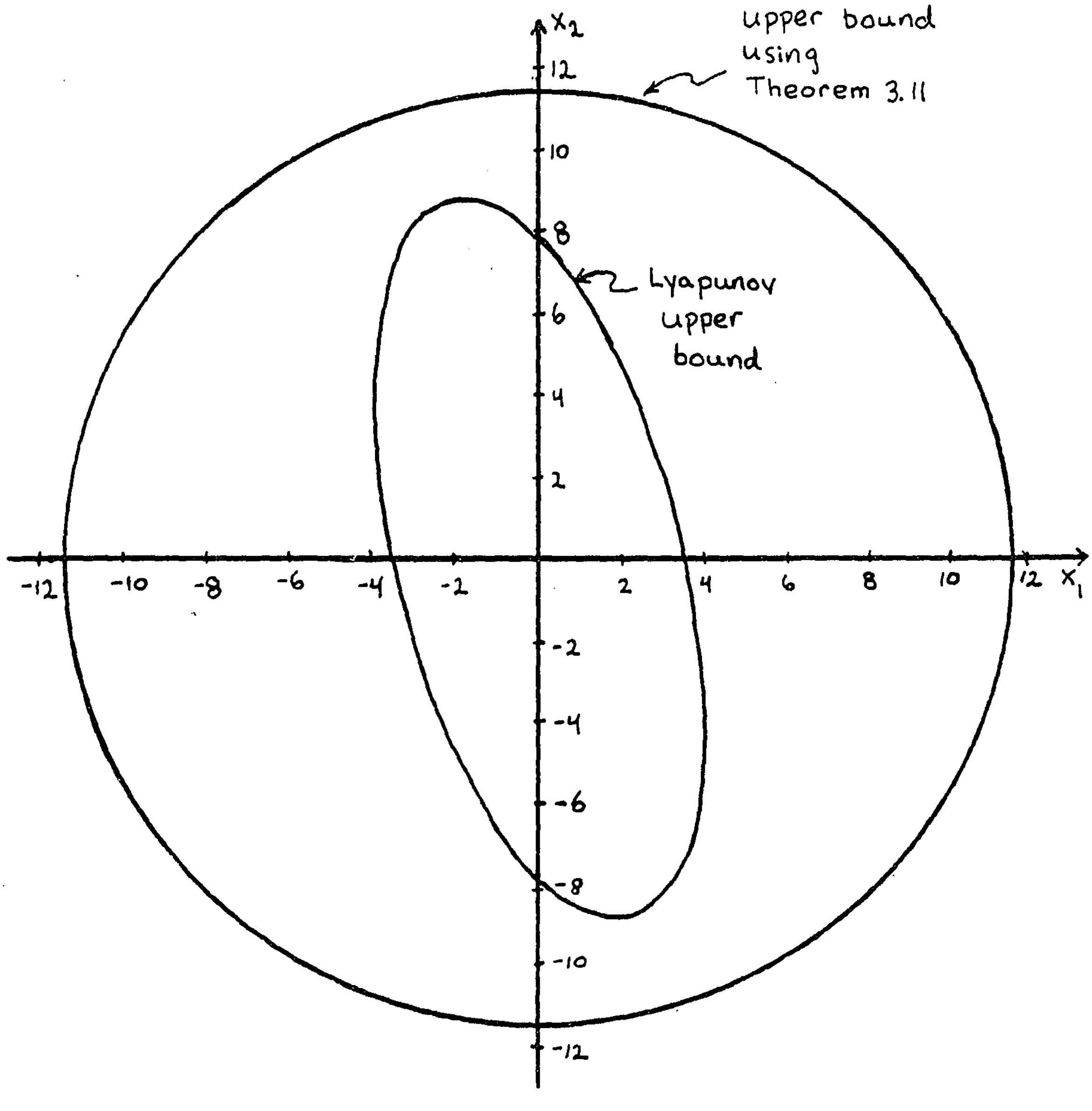


Figure 3.7. A comparison of the reachable set approximations for the system discussed in Examples 3.4 and 3.6.

result to formulate conditions under which the reachable set from the origin for a linear system under bounded control is the entire state space.

Theorem 3.12. (Lee and Markus, 1967). Consider the linear system

$$\dot{x} = Ax + Bu \qquad (3.46)$$

where $x \in E^n$, $u \in \Omega \subset E^m$ and A and B are $n \times n$ and $n \times m$ real constant matrices, respectively. If

i) $u = 0$ lies in the interior of Ω,

ii) rank $[B\ AB\ \ldots\ A^{n-1}B] = n$, and

iii) every eigenvalue of A has a negative real part, then the controllable set to the origin $\mathbf{C} = E^n$.

Corollary 3.8. Consider the system (3.46). If

i) $u = 0$ lies in the interior of Ω,

ii) rank $[B\ AB\ \ldots\ A^{n-1}B] = n$, and

iii) every eigenvalue of A has a positive real part, then the reachable set from the origin $\mathbf{R} = E^n$.

Proof. Consider the retrosystem to (3.46)

$$\dot{x} = -Ax - Bu. \qquad (3.47)$$

of $-A$ has a negative real part. Thus under the stated hypotheses, and using Theorem 3.11, we have that the controllable set to the origin of the retrosystem (3.47) $\hat{\mathbf{C}} = E^n$. From Corollary 3.2 we have that $\hat{\mathbf{C}} = \mathbf{R}$ so that $\mathbf{R} = E^n$, thus concluding the proof. Q.E.D.

Remarks. 1) Theorem 3.12 and its corollary requires the Kalman controllability criterion. In a paper by Kalman, Ho, and Narendra (1962) it is established that for the system (3.46) with $u \in \Omega = E^m$, if rank $[B \; AB \; \ldots \; A^{n-1}B] = n$, then $\mathbf{C} = E^n$. Under these same hypotheses the matrices of the retrosystem (3.44) satisfy rank $[-B \; (-A)(-B) \; \ldots \; (-A)^{n-1}(-B)] = \text{rank } [B \; AB \; \ldots \; A^{n-1}B] = n$. Hence, if we use the result of Kalman et al., it follows that for (3.47) $\hat{\mathbf{C}} = \mathbf{R}$. But as Corollary 3.2 states that $\hat{\mathbf{C}} = \mathbf{R}$, we conclude that $\mathbf{R} = E^n$. That is to say, under the hypotheses of Kalman et al. the reachable set from the origin for (3.46) $\mathbf{R} = E^n$. Although this conclusion is the same as that for Corollary 3.8 it is important to note the differences: Corollary 3.8 requires a bounded control set Ω and that the eigenvalues of A have a positive real part, whereas Kalman et al. make no such restriction on A and assumes the control can take on any value in E^m.

2) If a reachable set is a proper subset of E^n, then we can speak of a boundary to the reachable set. Thus the hypotheses of Theorems 3.10 and 3.11 not only result in bounded reachable sets, they also guarantee the existence of the boundary to the reachable set.

3.9. Two Results on the Reachable Set for Linear Systems

In this section we present two results which will be of use to us later.

Theorem 3.13. (Ryan, 1982). Consider the linear dynamical system

$$\dot{x} = Ax + Bu \tag{3.48}$$

where $x \in E^n$, $u \in \Omega \subset E^m$, and A and B are $n \times n$ and $n \times m$ real constant matrices, respectively.

Let $x_0 \in E^n$. Let $\mathbf{R}_T(x_0)$ and $\mathbf{R}(x_0)$ denote the reachable set from x_0 in time T and the reachable set from x_0 for (3.48), respectively. If Ω is convex, then $\mathbf{R}_T(x_0)$ and $\mathbf{R}(x_0)$ are convex.

Proof. We will follow Ryan's proof. Let $u_1(t)$ and $u_2(t)$ be two admissable control laws which generate the forward solutions to (3.48) from x_0

$$x_1(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu_1s(s)ds \tag{3.49}$$

and

$$x_2(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu_2(s)ds, \tag{3.50}$$

respectively. Hence, for $t = T > 0$ we have $x_1(T)$ and $x_2(T) \in \mathbf{R}_T(x_0)$ where

$$x_1(T) = e^{AT}x_0 + \int_0^T e^{A(T-s)}Bu_1(s)ds \tag{3.51}$$

and

$$x_2(T) = e^{AT}x_0 + \int_0^T e^{A(T-s)}Bu_2(s)ds. \tag{3.52}$$

Let $0 < \mu < 1$ and consider the point $\mu x_1(T) + (1 - \mu)x_2(T)$. From (3.51) and (3.52) we have that

$$\begin{aligned} \mu x_1(T) + (1 - \mu)x_2(T) &= e^{AT}x_0 \\ &+ \int_0^T e^{A(T-s)}B[\mu u_1(s) + (1 - \mu)u_2(s)]ds \end{aligned} \tag{3.53}$$

As Ω is convex, the control law $u_3(t) = \mu u_1(t) + (1 - \mu)u_2(t)$ is an admissable control, which at time $t = T$, generates the forward solution to (3.48) from x_0, $x_3(T) = \mu x_1(T) + (1 - \mu)x_2(T)$. Consequently, $x_3(T) \in \mathbf{R}_T(x_0)$, and $R_T(x_0)$ is convex.

Recalling that $\mathbf{R}(x_0) = \bigcup_{t>0} \mathbf{R}(x_0)$ we conclude that $\mathbf{R}(x_0)$ is convex as $\mathbf{R}_T(x_0)$, $T > 0$, is convex. Q.E.D.

<u>Theorem 3.14.</u> (Hermes and LaSalle, 1969). Consider the linear system

$$\dot{x} = Ax + Bu \qquad (3.54)$$

where $x \in E^n$, $u \in \Omega = \{u \in E^m \mid |u_j| < u_j^{max}, j = 1, \ldots, m < n\}$, and A and B are $n \times n$ and $n \times m$ real constant matrices, respectively.

Let $\mathbf{R}_T$ and $\mathbf{R}$ denote the reachable set form the origin in time T and the reachable set from the origin for the system (3.54), respectively. Then $\mathbf{R}_T$ and $\mathbf{R}$ are symmetric about the origin.

Proof. We will follow the proof of Hermes and LaSalle. Let $x_1 \in \mathbf{R}_T$. Then there exists an admissable control law $u(t)$ such that the forward solution to (3.54) from the origin at $t = T$ can be expressed as

$$x(T) = \int_0^T e^{A(T-s)} Bu(s)ds = x_1. \qquad (3.55)$$

By hypothesis $|u_j| < u_j^{max}$, $j = 1, \ldots, m$, so that if $u(t)$ is an admissable control so is $-u(t)$. Consequently, at $t = T$ the forward solution generated from the origin to (3.54) by $-u(t)$ is given by

$$\int_0^T e^{A(T-s)} B\text{-}u(s))ds = -\int_0^T e^{A(T-s)} Bu(s)ds = -x_1. \qquad (3.56)$$

Therefore, $-x_1 \in \mathbf{R}_T$, thus proving that $\mathbf{R}_T$ is symmetric about the origin.

To prove that **R** is symmetric about the origin note that for any $x_1 \varepsilon \mathbf{R}$ there exists a time $T > 0$ such that $x_1 \varepsilon \mathbf{R}_T$. From the above we have that $-x_1 \varepsilon \mathbf{R}_T \subseteq \mathbf{R}$; hence, **R** is symmetric about the origin. Q.E.D.

CHAPTER 4

CRITICAL POINTS AND THE BOUNDARY OF REACHABLE SETS FOR LINEAR SYSTEMS

Throughout this chapter we will consider the reachable set from the origin **R** for the linear dynamical system

$$\dot{x} = Ax + Bu \qquad (4.1)$$

where $x \in E^n$, $u \in \Omega = \{u \in E^m | -\infty < u_j^{min} \leq u_j \leq u_j^{max} < +\infty,$ $j = 1, \ldots, m \leq n\}$, A and B are $n \times n$ and $n \times m$ real constant matrices, respectively. As we shall be speaking at great length of the boundary of the reachable set for (4.1), let us begin by restating the reachability maximum principle (Theorem 3.7) in terms of linear systems.

4.1. The Reachability Maximum Principle for Linear Systems

Theorem 4.1. (Grantham 1973; Grantham and Vincent, 1975). Let $u^*(t)$ be an admissable control law which generates a boundary trajectory $x^*(t)$ to the system (4.1). It is necessary that there exist a nonzero continuous vector $\lambda(t) \in E^n$ such that for almost all $t > 0$

$$H(x^*,\lambda,u^*) = \max_{u \in \Omega} H(x^*,\lambda,u) = 0 \qquad (4.2)$$

and

$$\dot{\lambda}^T = -\lambda^T A \tag{4.3}$$

where

$$H(x,\lambda,u) = \lambda^T(Ax + Bu). \tag{4.4}$$

Equations (4.2)-(4.4) represent the abnormal case of the Pontryagin maximum principle for linear optimal control problems. In accordance with the terminology of Section 3.5 we say that an admissable control law which is a solution to the abnormal case is an abnormal control law. With this in mind, we see that boundary trajectories, if they exist, are generated by abnormal control laws; however, an abnormal control law does not necessarily generate a boundary to the reachable set. Using (3.12), the abnormal switching function associated with u_k for the linear system (4.1) is given by

$$\sigma_k(t) = \sum_{j=1}^{n} \lambda_j(t) b_{jk}. \tag{4.5}$$

If $\sigma_k(t) \equiv 0$ for some interval of time I, then (4.4) is independent of u_k on I; hence u_k can take on _any_ value in the interval $[u_k^{min}, u_k^{max}]$ whenever $t \in I$. Over such intervals of time we say that u_k is _singular_. However, if $\sigma_k(t) = 0$ only at isolated points of time, then u_k is said to be a bang-bang control and only takes on the values u_k^{min} or u_k^{max} depending upon whether $\sigma_k(t) < 0$ or $\sigma_k(t) > 0$, respectively. If $u_k(t)$ is a bang-bang control for every $k = 1, \ldots, m$, then we say that the vector $u(t)$ is a bang-bang control law.

Since a great deal of future attention will focus upon the system (4.1) where the control is both abnormal and bang-bang, we will now present a number of useful results for this situation.

4.2. Abnormal Bang-Bang Control Laws

We will now prove a number of results about abnormal bang-bang control laws. These theorems are similar to theorems in the theory of time optimal control; in fact, the statements and proofs follow directly from the works referenced by using the abnormality assumption.

Theorem 4.2. (Lee and Markus, 1967; Leitmann, 1981). If $u(t)$ is an abnormal control for the system (4.1) and if rank $[b_k \; Ab_k \; \ldots \; A^{n-1}b_k] = n$, where b_k is the kth column vector of B, then $u_k(t)$ is a bang-bang control.

Proof. We will follow Leitman's proof. Suppose that $u_k(t)$ is not a bang-bang control. Then there exists an interval of time $t_1 < t < t_2$ for which $\sigma_k(t) \equiv 0$. Consequently we have that

$$\sigma_k(t) = \frac{d\sigma_k(t)}{dt} = \ldots = \frac{d^{n-1}\sigma_k(t)}{dt} \equiv 0 \qquad (4.6)$$

whenever $t \in (t_1, t_2)$.

Reexpressing (4.5) in vector form results in

$$\sigma_k(t) = \lambda^T(t) b_k . \qquad (4.7)$$

Taking the first $n - 1$ time derivatives of (4.7) by using (4.3), and then appealing to (4.6), leads to the system of equations

$$\begin{aligned} \lambda^T(t) b_k &= 0 \\ \lambda^T(t) A b_k &= 0 \\ &\vdots \\ \lambda^T(t) A^{n-1} b_k &= 0 \end{aligned} \qquad (4.8)$$

whenever $t \in (t_1, t_2)$. In matrix form (4.8) can be written as

$$\lambda^T(t)[b_k \;\; Ab_k \;\; \ldots \;\; A^{n-1}b_k] = \underline{0} \qquad (4.9)$$

whenever $t \in (t_1, t_2)$, where $\underline{0}$ is the n-dimensional zero vector. As the coefficient matrix is nonsingular by hypothesis, the solution to (4.9) is $\lambda^T(t) \equiv \underline{0}$ for $t \in (t_1, t_2)$. However, by hypothesis

$u(t)$ is an abnormal control law; hence, we must have $\lambda^T(t) \not\equiv \underline{0}$ for almost all $t > 0$. Thus we have a contradiction and we conclude that $u_k(t)$ must be a bang-bang control. Q.E.D.

Theorem 4.3. (Lee and Markus, 1967; Leitmann, 1981; Ryan, 1982). Let $u(t)$ be an abnormal control law for the system (4.1). If rank $[b_k \ Ab_k \ \ldots \ A^{n-1}b_k] = n$, where b_k is the kth column vector of B, and if all the eigenvalues of A are real, then $u_k(t)$ switches at most $n - 1$ times on $0 < t < +\infty$.

Proof. We will follow Leitmann's proof. Let $\mu_1, \ldots, \mu_r$ be the distinct eigenvalues of $-A$ and let $m_1, \ldots, m_r$ be their respective multiplicities. From the theory of ordinary differential equations, if $\lambda^T(t)$ is a solution to (4.2), then its components are of the form

$$p_1(t)e^{\mu_1 t} + p_2(t)e^{\mu_2 t} + \ldots + p_r(t)e^{\mu_r t} \tag{4.10}$$

where $p_j(t)$ is a polynomial with degree not exceeding $m_j - 1$, for all $j = 1, \ldots, r$. Hence, the switching function (4.5) is also of the form (4.10). By hypothesis rank $[b_k \ Ab_k \ \ldots \ A^{n-1}b_k] = n$; hence, $u_k(t)$ is a bang-bang control by Theorem 4.2. Thus, in order to establish the theorem we must show that the continuous switching function $\sigma_k(t)$ has at most $n - 1$ zeros. We do this by first observing that multiplication of (4.10) by $e^{-\mu_r t}$ does not change the number of roots of (4.10). Thus, let us consider the expression

$$p_1(t)e^{(\mu_1-\mu_r)t} + \dots + p_{r-1}(t)e^{(\mu_{r-1}-\mu_r)t} + p_r(t). \qquad (4.11)$$

As (4.11) represents a smooth function we know that between any two roots there is a root of the derivative. Hence, every time we differentiate (4.11) the resulting expression has one less zero than the expression preceeding it. Consequently, if we differentiate (4.11) m_r times the resulting expression

$$\tilde{p}_1(t)e^{(\mu_1-\mu_r)t} + \dots + \tilde{p}_{r-1}(t)e^{(\mu_{r-1}-\mu_r)t}, \qquad (4.12)$$

where $\tilde{p}_j(t)$ is a polynomial of degree less than or equal to $m_j - 1$, $j = 1, \dots, r - 1$, has m_r fewer roots than (4.11). Note that (4.12) is the same type of expression as (4.11) so that we can repeat the argument. Hence, multiplying (4.12) by $e^{-(\mu_{r-1}-\mu_r)t}$ results in a function which has the same zero structure as (4.12):

$$\tilde{p}_1(t)e^{(\mu_1-\mu_{r-1})t} + \dots + \tilde{p}_{r-2}(t)e^{(\mu_{r-2}-\mu_{r-1})t} + \tilde{p}_{r-1}(t). \qquad (4.13)$$

Differentiating (4.13) m_{r-1} times leaves us with a function with m_{r-1} fewer roots than (4.12) and thus $m_r + m_{r-1}$ fewer roots than (4.10). This process is repeated until we are left with a function of the form

$$\overline{p}_1(t)e^{(\mu_1-\mu_2)t} \tag{4.14}$$

where $\overline{p}_1(t)$ is a polynomial whose degree does not exceed $m_1 - 1$. Note that (4.14) will have $m_2 + m_3 + \ldots + m_r$ fewer zeros than (4.10). Consequently, we conclude that the maximum number of roots to (4.10) is $m_1 + m_2 + \ldots + m_r - 1 = n - 1$. Q.E.D.

Corollary 4.1. Let $u(t)$ be an abnormal control law for the system (4.1). If all the eigenvalues of A are real and rank $[b_k \; Ab_k \; \ldots \; A^{n-1}b_k] = n$ for all $k = 1, \ldots, m$, then there exists a time $T > 0$ such that $u(t) \equiv u_c$ for all $t > T$, where u_c is a constant admissable control.

Proof. Under the stated hypotheses, Theorem 4.2 guarantees that the control component $u_k(t)$ is bang-bang. Hence, over any positive time interval $u_k(t) = u_k^{min}$ or $u_k(t) = u_k^{max}$ for all $k = 1, \ldots, m$. Furthermore, Theorem 4.3 states that $u_k(t)$ will switch at most $n - 1$ times on $0 \leq t < \infty$; consequently, for every $k = 1, \ldots, m$, there exists a time $T_k > 0$ such that $u_k(t) \equiv u_k^*$ for all $t > T_k$, where u_k^* is either u_k^{min} or u_k^{max}. If we define $T = \max\{T_1, T_2, \ldots, T_m\}$ and $u_c = [u_1^* \; u_2^* \ldots u_m^*]^T$, then $u(t) \equiv u_c$ for all $t > T$. Q.E.D.

4.3. Controlled Equilibrium Points of n-Dimensional Linear Systems on $\partial\mathbb{R}$

In this section we consider the linear system (4.1) where the eigenvalues of A are real and negative. For such a system we know that the reachable set from the origin is bounded (Corollary 3.4); consequently, we can speak of the existence of $\partial \mathbf{R}$. With this in mind we will prove that if the abnormal bang-bang control law generates a boundary trajectory, then there is at least one controlled equilibrium point (Section 2.4) on $\partial \mathbf{R}$.

Theorem 4.4. Let $\mathbf{R}$ denote the reachable set from the origin for the system (4.1) where the elements of A are real and negative. If there exists a boundary trajectory $x^*(t)$ generated by an abnormal control law $u^*(t)$ and if rank $[b_k \; Ab_k \; \ldots \; A^{n-1}b_k] = n$, $k = 1, \ldots, m$, where b_k is the kth column of B, then there exists at least one controlled equilibrium point on $\partial \mathbf{R}$.

Proof. As the eigenvalues of A are real and rank $[b_k \; Ab_k \; \ldots \; A^{n-1}b_k] = n$, Corollary 4.1 guarantees the existence of a constant admissable control law u_c and a time $t \geqslant T$ such that $u^*(t) \equiv u_c$ for all $t \geqslant T$. Consequently, the boundary trajectory obeys the differential equation

$$\dot{x} = Ax + Bu_c \tag{4.15}$$

for all $t \geqslant T$. Since all of the eigenvalues of A are negative $\det A = \prod_{j=1}^{n} \mu_j \nsupseteq 0$, where μ_j, $j = 1, \ldots, n$, is an eigenvalue of A. Hence, A^{-1} exists and the controlled equilibrium point

$$x_{u_c} = - A^{-1}Bu_c \tag{4.16}$$

is globally asymptotically stable. Consequently, $x^*(t) \to x_{u_c}$ as $t \to +\infty$ and we conclude that $x_{u_c} \in \overline{\partial \mathbf{R}}$. Since $\partial \mathbf{R}$ is the intersection of the closure of $\mathbf{R}$ with the closure of the compliment of $\mathbf{R}$ (Hoffman, 1975), and the intersection of two closed sets is closed, it follows that $\partial \mathbf{R} = \overline{\partial \mathbf{R}}$. Therefore, $x_{u_c} \in \partial \mathbf{R}$ and the proof is complete. Q.E.D.

Corollary 4.2. Let R denote the reachable set from the origin for the system (4.1) where the eigenvalues of A are real and negative. If

i) there exists a boundary trajectory $x^*(t)$ generated by an abnormal control law $u^*(t)$,

ii) rank $[b_k \quad Ab_k \quad \dots \quad A^{n-1}b_k] = n$ for all $k = 1, \dots, m$, where b_k is the kth column vector of B, and

iii) $|u_k| \leq u_k^{max}$ for all $k = 1, \dots, m$,

then there are at least two controlled equilibrium points on $\partial \mathbf{R}$.

Proof. Follows directly from Theorem 3.13 and Theorem 4.4. Q.E.D.

The usefulness of Theorem 4.4 and its corollary is that if we wish to generate portions of $\partial \mathbf{R}$ we can first find all the solutions to

$$x_{u_c} = -A^{-1}Bu_c, \qquad (4.17)$$

where the kth component of u_c is either u_k^{min} or u_k^{max}, and then use Theorem 4.1 to generate candidate boundary trajectories from x_{u_c}. (We emphasise the fact that these are candidate boundary trajectories as Theorem 4.4 guarantees at least one solution to (4.17) is on $\partial\mathbf{R}$ provided a boundary trajectory exists.) In a state space with dimension greater than or equal to three it is extremely difficult to visualize the candidate boundary trajectories and their relationship to one another (which ones are remaining on the boundary, which ones are moving in the interior of $\mathbf{R}$, etc.). The difficulty is compounded further for state spaces of dimension greater than two since through every point on the boundary, many boundary trajectories can pass through it. However, if we restrict attention to 2-dimensional state spaces these problems do not arise as motion is restricted to the plane.

4.4. Controlled Equilibrium Points of 2-Dimensional Linear Systems on $\partial\mathbf{R}$

In the previous section we presented a number of results concerning the location of controlled equilibria in relation to the boundary of the reachable set for n-dimensional linear systems. In each of these results it is assumed that a boundary trajectory exists. Present theory, however, only guarantees the existence of boundary trajectories for a class of 2-dimensional systems (Section

3.6). It is for this reason that we now restrict attention to linear systems in the plane.

Theorem 4.5. Let **R** designate the reachable set from origin for the system

$$\dot{x} = Ax + Bu \tag{4.18}$$

where $x \in E^2$, $u \in [u_{min}, u_{max}]$ with $-\infty < u_{min} < 0 < u_{max} < +\infty$, and A and B are 2×2 and 2×1 real constant matrices, respectively. Let x_{min} and x_{max} be the controlled equilibrium points associated with u_{min} and u_{max}, respectively. If all the eigenvalues of A are negative real numbers and rank $[B \;\; AB] = 2$, then x_{min} and x_{max} both lie on $\partial\mathbf{R}$.

Proof. From Corollary 3.7 we know that **R** is bounded as the eigenvalues of A have negative real part; hence, $\partial\mathbf{R}$ exists. Furthermore, as rank $[B \;\; AB] = 2$, it follows from Corollary 3.5 that a boundary trajectory passes through every point of $\partial\mathbf{R}$. Consequently, from Theorem 4.4 we know that at least one of the points x_{min} or x_{max} lies on $\partial\mathbf{R}$. Without loss of generality, let us suppose that $x_{min} \in \partial\mathbf{R}$ but $x_{max} \notin \partial\mathbf{R}$.

Observe that under the hypothesis rank $[B \;\; AB] = 2$, it follows that the control law governing the boundary trajectory is bang-bang; therefore, we can use Lemma 3.8 and (3.16) to draw the straight line state space switching arc through x_{min} and x_{max}. As we will be discussing points on the state space switching arc in relation to x_{min} and x_{max}, it will be convenient to parameterize this line in the following manner:

$$\ell = \{p(\mu) \in E^2 | p(\mu) = x_{min} + \mu(x_{max} - x_{min}), \mu \in R\}. \quad (4.19)$$

Since $x_{min} \in \partial\mathbf{R}$ and the control law used on $\partial\mathbf{R}$ is bang-bang it follows that to generate a boundary trajectory from x_{min} we must set $u(t) = u_{max}$. (Setting $u = u_{min}$ results in $\dot{x} \equiv 0$ and a trivial boundary trajectory is generated.) The resulting boundary trajectory must either

1) intersect ℓ with $\mu < 0$,
2) intersect ℓ with $0 < \mu < 1$,
3) intersect ℓ with $\mu > 1$, or
4) approach x_{max} asymptotically without first crossing ℓ.

We can immediately eliminate the possibility that the boundary trajectory intersects ℓ for values $\mu < 0$ based upon the fact that x_{max} is a globally asymptotically stable controlled equilibrium point for the linear system (4.18).

Let us now examine the possibility of the boundary trajectory intersecting ℓ with $0 < \mu < 1$ (Figure 4.1). For this to be the case we must have $x_{max} \notin \bar{\mathbf{R}}$. To see this, note that once the boundary trajectory intersects ℓ with $0 < \mu < 1$ the control is switched to $u(t) = u_{min}$ and the boundary trajectory must return to

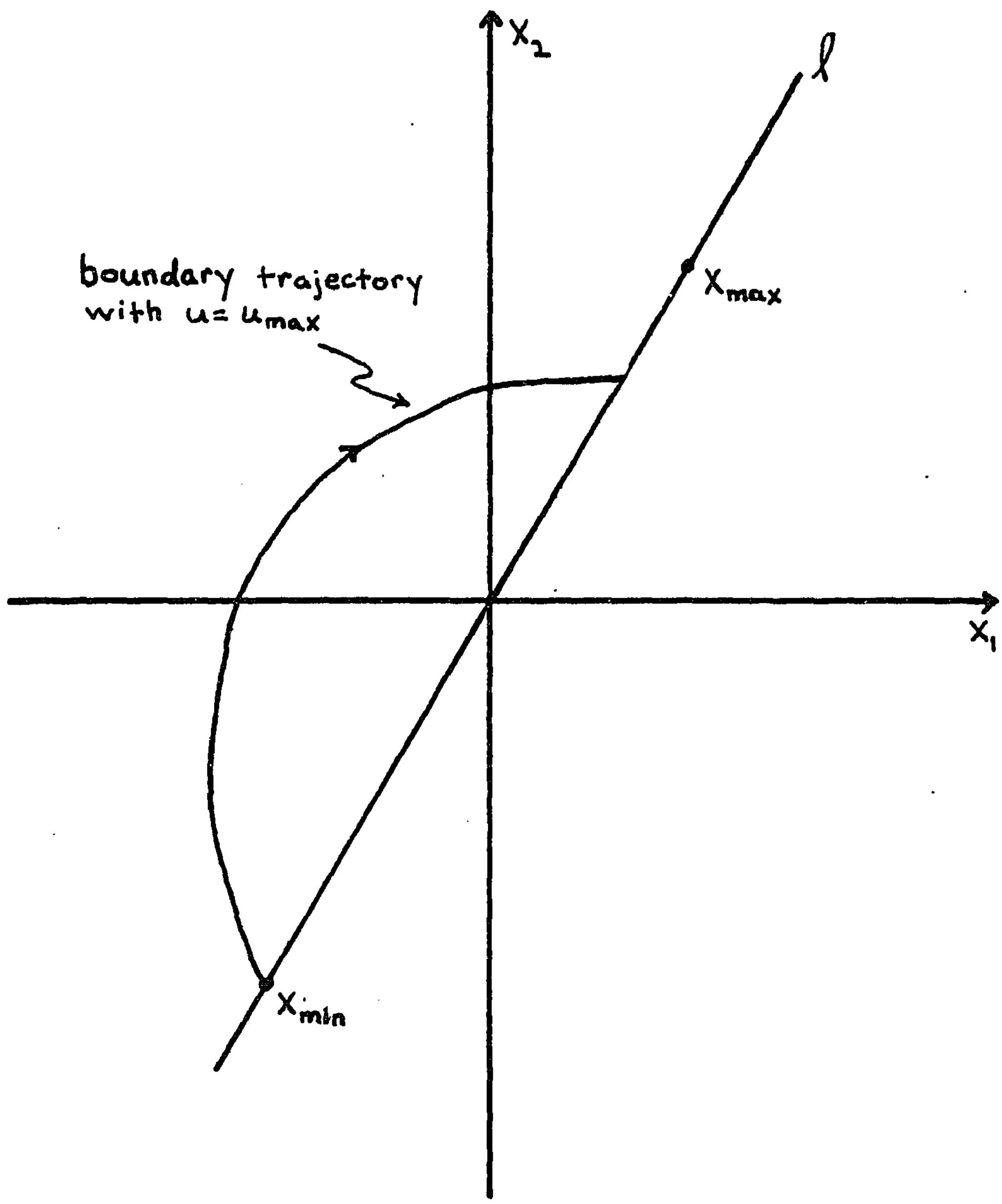


Figure 4.1. If the boundary trajectory from x_{min} intersects ℓ along the segment where $0 < \mu < 1$, then $x_{max} \notin \partial R$.

$x_{min} \in \partial\mathbf{R}$. Consequently, $x_{max} \notin \mathbf{R}$; however, we know that under the control program $u(t) \equiv u_{max}$ the forward solution $x(t)$ of (4.18) from the origin satisfies $x(t) \to x_{max}$ as $t \to +\infty$. Thus, we have a contradiction and we conclude that $\partial\mathbf{R}$ does not intersect ℓ with $0 < \mu < 1$.

Continuing, we investigate the possibility of the boundary trajectory intersecting ℓ with $\mu > 1$. To help visualize what is happening, let us make the change of variable $z = x - x_{max}$. Then

$$\dot{z} = Az$$
$$z(0) = x_{min} - x_{max} \tag{4.20}$$

where use was made of the fact that $Ax_{max} + bu_{max} = 0$ (i.e., x_{max} is the controlled equilibrium point to (4.18) for $u = u_{max}$). As the matrix A has negative real eigenvalues, the origin in z-space is globally asymptotically stable. Figure 4.2 illustrates the eigenvector solutions of (4.20) about the origin and the boundary trajectory in z-space. (If there is a single eigenvector solution to (4.20) the argument follows in a similar manner.) From Figure 4.2 it is apparent that the only way the boundary trajcectory can intersect the line ℓ with $\mu > 1$ is for the trajectory to first intersect an eigenvector solution. This, however, cannot occur or else the

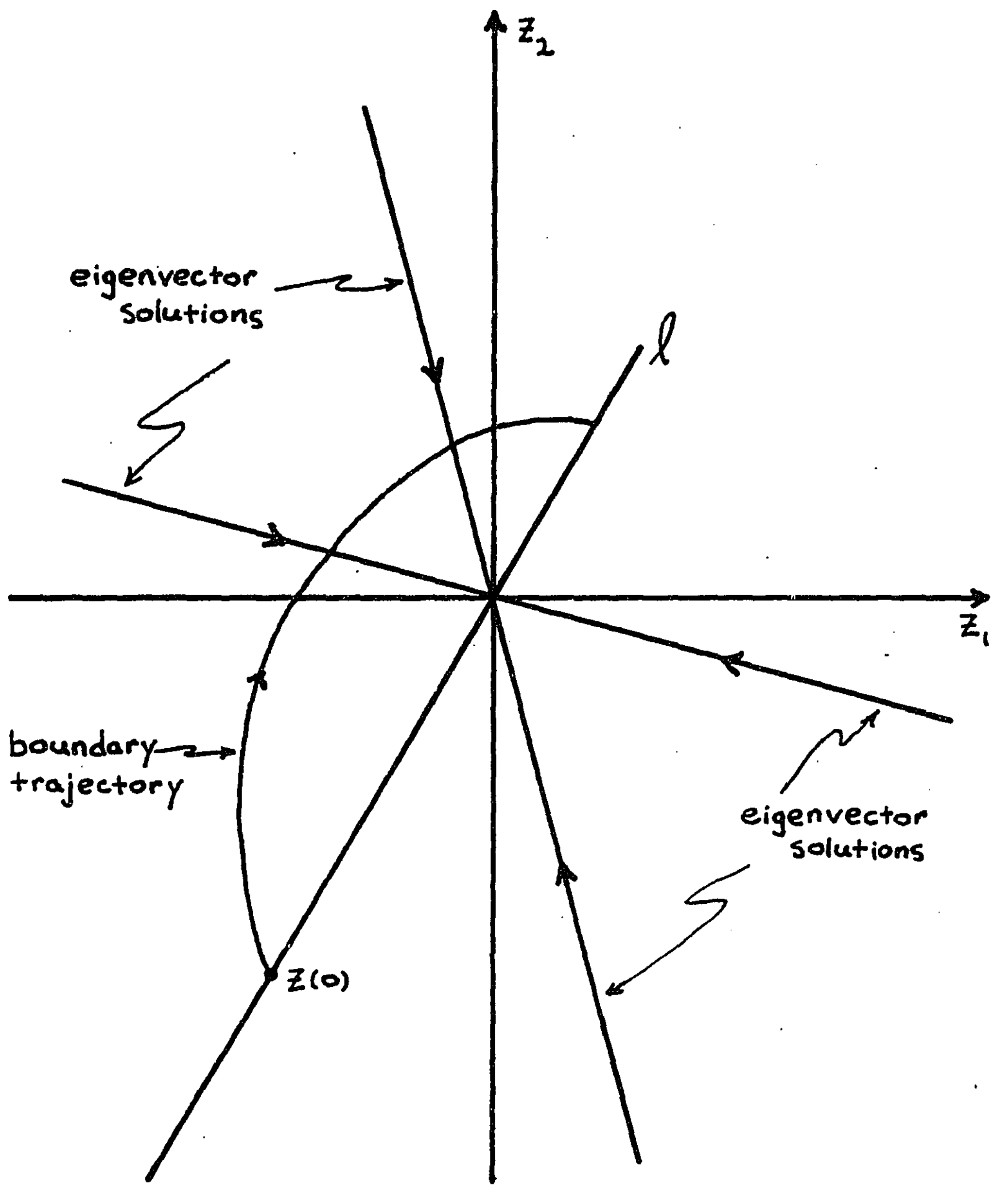


Figure 4.2. If the boundary trajectory from $z(0)$ intersects ℓ along the segment where $\mu > 1$, then the trajectory crosses eigenvector solutions.

uniqueness of solutions to (4.20) is violated. Hence, the boundary trajectory cannot intersect ℓ with $\mu > 1$.

In summary, we have shown that the boundary trajectory to the system (4.18) cannot intersect ℓ for $\mu \neq 1$. The only possible solution is that the boundary trajectory approach x_{max} asymptotically without first crossing ℓ. Hence, $x_{max} \in \partial\mathbf{R}$. Q.E.D.

Up to this point we have focused our attention on the location of controlled equilibrium points of linear systems where A has all of its eigenvalues real and negative. If A has complex eigenvalues with negative real part, then we ask "Where are the controlled equilibrium points of such a linear system to be found in relation to $\partial\mathbf{R}$?" The answer to this question appears to be difficult for state spaces with dimension greater than two. However, for a 2-dimensional state space the question is simply answered.

Lemma 4.1. Consider the 2-dimensional system (4.1). Let $\mathbf{R}$ denote the reachable set from the origin. If the eigenvalues of A have negative real part and nonzero imaginary part, then the controlled equilibria of (4.1) are in the interior of $\mathbf{R}$.

Proof. Under the stated hypothesis the controlled equilibria are asymptotically stable spiral points; hence, every controlled equilibrium point can be surrounded by points belonging to $\mathbf{R}$ (if x_{u_c} is the controlled equilibrium point associated with the constant admissable control u_c, then consider the points of the forward solution to (4.1) from the origin using $u(t) \equiv u_c$). Since Ω is convex, it follows that **R** is convex (Theorem 3.12); hence we must have that these equilibria are in the interior of **R**. Q.E.D.

For 2-dimensional systems (4.1) where the eigenvalues of A have a negative real part and nonzero imaginary part, and $0 \varepsilon \text{ int } \Omega$, we can not directly determine any point on ∂**R** from which to begin a boundary trajectory. Consequently, we cannot find ∂**R** by a direct application of the Reachability Maximum Principle. This does not stop us from finding ∂**R**, however, as we can employ the asymptotic stability of the boundary trajectory (Corollary 3.6). To do this we must know a priori an admissable control law which will generate a boundary trajectory. This information is provided in

Theorem 4.6. Consider the 2-dimensional system (4.1) where the eigenvalues of A are complex and have negative real part and suppose $0 \varepsilon \text{ int } \Omega$. If

i) there exists a boundary trajectory $x^*(t)$ and

ii) rank $[b_j \; Ab_j] = 2$, where b_j is the jth column of B, for $j = 1$ and 2,

then $u_j^*(t)$, $j = 1$ and 2, switches with frequencey $\omega = \text{Im}(\mu)$, where μ is an eigenvalue of A, and $u^*(t) = [u_1^*(t) \; u_2^*(t)]^T$ is the boundary control law. Furthermore the phase difference in switching is given by $\theta_1 - \theta_1$, where

$$\theta_1 = \tan^{-1}\{\frac{(b_{21}r-b_{11})a_{12}\omega}{[b_{12}(\omega^2+a_{22}^2)-b_{21}a_{12}a_{22}]r+(b_{11}a_{11}a_{22}-b_{21}a_{12}^2)}\}$$

$$\theta_2 = \tan^{-1}\left\{\frac{(b_{22}r-b_{12})a_{12}\omega}{[b_{12}(\omega^2+a_{22}^2)-b_{22}a_{12}a_{22}]r+(b_{12}a_{11}a_{22}-b_{22}a_{12}^2)}\right\}$$

and

$$r = -\frac{[a_{11}x_1^*(0)+a_{12}x_2^*(0)+b_{11}u_1^*(0)+b_{12}u_2^*(0)]}{a_{21}x_1^*(0)+a_{22}x_2^*(0)+b_{21}u_1^*(0)+b_{22}u^*{}_2(0)}.$$

Proof. Note that we cannot have $a_{12} = 0$ and $a_{21} = 0$ as A has complex eigenvalues. Without loss of generality let us assume $a_{12} \neq 0$.

By Theorem 4.1, if $u^*(t)$ is a boundary control law which generates the boundary trajectory $x^*(t)$, then there exists a nonzero continuous vector $\lambda(t) \in E^2$ such that for almost all $t > 0$

$$H(x^*,\lambda,u^*) = \max_{u\in\Omega} H(x^*,\lambda,u) = 0 \tag{4.21}$$

and

$$\dot{\lambda}^T = -\lambda^T A \tag{4.22}$$

where

$$H(x,\lambda,u) = \lambda^T(Ax + Bu). \tag{4.23}$$

Expanding (4.22) results in the expressions

$$\dot{\lambda}_1 = -a_{11}\lambda_1 - a_{21}\lambda_2 \tag{4.24}$$

$$\dot{\lambda}_2 = -a_{12}\lambda_1 - a_{22}\lambda_2. \tag{4.25}$$

From (4.24) and (4.25) we can see that neither $\lambda_1(t) \equiv 0$ nor $\lambda_2(t) \equiv 0$ as this would lead to $\lambda(t) \equiv 0$ for all $t > 0$. Differentiating (4.25) with respect to time and using (4.24) we have

$$\ddot{\lambda}_2 = a_{11}a_{12}\lambda_1 + a_{12}a_{21}\lambda_2 - a_{22}\dot{\lambda}_2. \tag{4.26}$$

Solving for $a_{12}\lambda_1$ in (4.25) and substituting this into (4.26) results in the expression

$$\ddot{\lambda}_2 + (a_{11} + a_{22})\dot{\lambda}_2 + (a_{11}a_{22} - a_{12}a_{21})\lambda_2 = 0. \tag{4.27}$$

The solution of (4.27) is found to be

$$\lambda_2(t) = C_2\sin \omega t + D_2\cos \omega t \tag{4.28}$$

where $\omega = \mathrm{Im}(\mu)$ and μ is an eigenvlaue of A. Differentiating (4.28) with respect to time we have

$$\dot{\lambda}_2(t) = \omega(C_2\cos \omega t - D_2\sin \omega t). \tag{4.29}$$

Substituting (4.28) and (4.29) into (4.25) and solving for λ_1 we have

$$\lambda_1(t) = \frac{1}{a_{12}}(\omega D_2 - a_{22}C_2)\sin \omega t - (\omega C_2 + a_{22}D_2)\cos \omega t \tag{4.30}$$

or, more simply,

$$\lambda_1(t) = C_1\sin \omega t + D_1\cos \omega t. \tag{4.31}$$

Recall that for linear systems (4.23) is equivalent to (4.5). Consequently, we can write the switcing function for $u_1^*(t)$ and $u_2^*(t)$ as

$$\sigma_1(t) = b_{11}\lambda_1(t) + b_{21}\lambda_2(t) \tag{4.32}$$

and

$$\sigma_2(t) = b_{12}\lambda_1(t) + b_{22}\lambda_2(t), \tag{4.33}$$

respectively. Substituting (4.28) and (4.31) into (4.32) and (4.33) we have equations of the form

$$\sigma_j(t) = c_j \sin \omega t + d_j \cos \omega t$$
$$= K_j \sin(\omega t + \theta_j) \tag{4.34}$$

for $j = 1$ and 2. As rank $[b_j \;\; Ab_j] = 2$, for $j = 1$ and 2, each component of $u^*(t)$ is bang-bang; hence, $\sigma_j(t)$ has isolated zeros, meaning $K_j \neq 0$, $j = 1$ and 2. Therefore, $u_1^*(t)$ and $u_2^*(t)$ switch with frequency ω.

To determine the phase difference of switchings, let us first make use of (4.28) and (4.31) to note that

$$D_2 = \lambda_2(0) \tag{4.35}$$

$$-\frac{(\omega C_2 + a_{22} D_2)}{a_{12}} = \lambda_1(0). \tag{4.36}$$

Using (4.35) and (4.36) we solve for C_2 to find

$$C_2 = -\frac{(a_{12}\lambda_1(0) + a_{22}\lambda_2(0))}{\omega}. \tag{4.37}$$

Consequently, we can write

$$\lambda_1(t) = \frac{[(\omega^2 + a_{22}^2)\lambda_2(0) + a_{11}a_{22}\lambda_1(0)]}{a_{12}\omega} \sin \omega t - \lambda_1(0) \cos \omega t \tag{4.38}$$

$$\lambda_2(t) = -\frac{[a_{12}\lambda_1(0) + a_{22}\lambda_2(0)]}{\omega} \sin \omega t + \lambda_2(0) \cos \omega t. \tag{4.39}$$

Substituting (4.38) and (4.39) into (4.32) results in

$$\sigma_1(t) = \{\frac{b_{11}[(\omega^2+a_{22}^2)\lambda_2(0)+a_{11}a_{22}\lambda_1(0)]-b_{21}a_{12}[a_{12}\lambda_1(0)+a_{22}\lambda_2(0)]}{a_{12}\omega} \tag{4.40}$$

$$\times \sin \omega t + [b_{21}\lambda_2(0) - b_{11}\lambda_1(0)]\cos \omega t.$$

We can write (4.40) in the form (4.34) provided

$$\theta_1 = \tan^{-1} \cdot \{\frac{[b_{21}\lambda_2(0)-b_{11}\lambda_1(0)]a_{12}\omega}{[b_{11}(\omega^2+a_{22}^2)-b_{21}a_{12}a_{22}]\lambda_2(0)+(b_{11}a_{11}a_{22}-b_{21}a_{12}^2)\lambda_1(0)}\}. \tag{4.41}$$

Following a similar procedure we get

$$\theta_2 = \tan^{-1} \cdot \{\frac{[b_{22}\lambda_2(0)-b_{12}\lambda_1(0)]a_{12}\omega}{[b_{12}(\omega^2+a_{22}^2)-b_{22}a_{12}a_{22}]\lambda_2(0)+(b_{12}a_{11}a_{22}-b_{22}a_{12}^2)\lambda_1(0)}\}. \tag{4.42}$$

Equation (4.21) at $t = 0$ gives conditions for $\lambda_2(0)$ in terms of $\lambda_1(0)$, $u^*(0)$, and $x^*(0)$. That is to say, $\lambda_2(0) = r\lambda_1(0)$ where

$$r = -\frac{a_{11}x_1^*(0)+a_{12}x_2^*(0)+b_{11}u_1^*(0)+b_{12}u_2^*(0)}{a_{21}x_1^*(0)+a_{22}x_2^*(0)+b_{21}u_1^*(0)+b_{22}u_2^*(0)}.$$

Hence, (4.41) and (4.42) become

$$\theta_1 = \tan^{-1}\{\frac{(b_{21}r-b_{11})a_{12}\omega}{[b_{11}(\omega^2+a_{22}^2)-b_{21}a_{12}a_{22}]r+(b_{11}a_{11}a_{22}-b_{21}a_{12}^2)}\} \quad (4.43)$$

$$\theta_2 = \tan^{-1}\{\frac{(b_2r-b_{12})a_{12}\omega}{[b_{12}(\omega^2+a_{22}^2)-b_{22}a_{12}a_{22}]r+(b_{12}a_{11}a_{22}-b_{22}a_{12}^2)}\}, \quad (4.44)$$

respectively. Therefore, the phase difference in switching is $\theta_1 - \theta_2$. Q.E.D.

Corollary 4.3. Consider the 2-dimensional system (4.1) where the eigenvalues of A have negative real part and nonzero imaginary part, and $\Omega = [u_{min}, u_{max}]$, where $-\infty < u_{min} < 0 < u_{max} < +\infty$. If rank [B AB] = 2, then $u^*(t)$ is a bang-bang boundary control law which switches with frequency $\omega = Im(\mu)$, where μ is an eigenvalue of A.

Proof. Under the stated hypothesis, a boundary trajectory passes through every point on the boundary of the reachable set (Corollary 3.5). Since the control is scalar there is no phase differences in switching to worry about. Hence, a direct application of Theorem 4.6 establishes the corollary. Q.E.D.

Remark. The abnormal control law for the 2-dimensinal system (4.1) with scalar control is independent of the initial state and $\lambda(t)$. Consequently, the abnormal control law is the boundary control law. Since the boundary trajectories for such systems are asymptotically stable (Corollary 3.6), it follows that the conjecture of Vincent (1980b) (using abnormal control for points interior to the reachable set will transfer the system to $\partial\mathbf{R}$ asymptotically) is correct in this case.

CHAPTER 5

REACHABLE SETS FROM THE ORIGIN FOR 1- AND 2-DIMENSIONAL LINEAR SYSTEM WITH A SCALAR CONTROL

In this chapter we demonstrate techniques for constructing the reachable set from the origin for 1-dimensional linear systems and for a class of 2-dimensional linear systems. In both instances we assume that control is scalar and that the zero control $(u = 0)$ is in the interior of the constraint set Ω; consequently, if **R** represents the reachable set from the origin, then $0 \varepsilon \text{ int } \mathbf{R}$.

5.1. The One Dimensional Case

Consider the system

$$\dot{x} = ax + bu \tag{5.1}$$

where $x \varepsilon E^1$, $u \varepsilon [u_{min}, u_{max}]$ with $-\infty < u_{min} < 0 < u_{max} < +\infty$, and the coefficients a and b are real numbers with $b > 0$. For such a system the reachable set from the origin is dependent on the constant a:

Case 1. $a \geqslant 0$. For this situation $\mathbf{R} = E^1$. To see this, note that choosing $u(t) \equiv u_{max}$ results in our driving the system to the right without approaching a stable equilibrium point; hence, all points to the right of the origin are in **R**. Similarly, if we choose

$u(t) \equiv u_{min}$, then we are continually forcing the system to the left without approaching a stable equilibrium point; consequently, all points to the left of the origin are in **R**. Since $0 \in \mathbf{R}$ by definition, it follows that $\mathbf{R} = E^1$.

Case 2. $a < 0$. In this instance all the controlled equilibria are stable. By setting $u(t) \equiv u_{max}$ we continually force the system to the right, thus causing it to asymptotically approach $x = bu_{max}/|a|$. Similarly, if we set $u(t) \equiv u_{min}$, then the system asymptotically approaches $x = bu_{min}/|a|$. Hence, $R = \{x \in E^1 | bu_{min}/|a| < x < bu_{max}/|a|\}$.

Example 5.1. The reachable set from the origin for the system

$$\dot{x} = -2x + u$$

where $u \in [-\frac{1}{2}, 2]$ is found to be $\mathbf{R} = \{x \in E^1 | -\frac{1}{4} < x < 1\}$.

5.2. Two Dimensional Systems

Consider the dynamical system

$$\dot{x} = Ax + Bu \tag{5.2}$$

where $x \in E^2$, $u \in \Omega = [u_{min}, u_{max}]$ with $-\infty < u_{min} < 0 < u_{max} < +\infty$, and A and B are 2×2 and 2×1 real constant matrices, respectively. We assume that rank $[B \;\; AB] = 2$. We will restrict our attention to the case where the eigenvalues of A have negative real part as Corollary 3.7 guarantees that such systems have bounded reachable sets. This leads us to examine the cases (i) the eigenvalues of A are real and negative and (ii) the eigenvalues have negative real part and nonzero imaginary part.

Case 1. Eigenvalues real and negative. The technique used to construct $\mathbf{R}$ for this case is based upon Theorems 4.4 and 4.5 and is best illustrated by means of the following example.

Example 5.2. Consider (5.2) with

$$A = \begin{vmatrix} 0 & 1 \\ -2 & -3 \end{vmatrix}, \qquad B = \begin{vmatrix} 2 \\ -3 \end{vmatrix}, \quad \text{and} \quad u \in [-1,1].$$

A quick check reveals that the eigenvalues of A are -2 and -3. Therefore, Corollary 3.7 insures us that $\mathbf{R}$ is bounded; hence, $\partial\mathbf{R}$ exists. Furthermore, as rank $[B \;\; AB] = 2$ a boundary trajectory exist (Corollary 3.5) and the boundary control is bang-bang (Theorem 4.2). Consequently, Theorem 4.5 tells us that the controlled equilibrium points $(-3/2,2)^T$ and $(3/2,-2)^T$, associated with $u \equiv -1$ and $u \equiv +1$, respectively, lie on $\partial\mathbf{R}$. With this information we construct boundaries to the reachable set in segments. The first segment is found by integrating the system equations from $(-3/2,2)^T$ with $u = +1$ (recall that we cannot use

$u = -1$ as $(-3/2,2)^T$ is a controlled equilibrium point for $u = -1$). The second segment is found in a similar manner except we begin at $(3/2,-2)^T$ and use $u = -1$. Figure 5.1 illustrates the joining of these segments to form **R**.

Case 2. Eigenvalues having negative real part and nonzero imaginary part. The method of finding **R** for (5.2) under these conditions is based upon Corollaries 3.6 and 4.3.

Example 5.3. Consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ &= \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} x + \begin{vmatrix} 0 \\ 1 \end{vmatrix} u \end{aligned} \tag{5.3}$$

where $u \varepsilon [-1,1]$. Find **R**.

A quick calculation shows that the eigenvalues of A are $(-1 \pm i\sqrt{3})/2$, where $i = \sqrt{-1}$. Since the eigenvalues have a negative real part, it follows from Corollary 3.7 that **R** is bounded; hence, ∂**R** exists. As rank $[B \ \ AB] = 2$ and the eigenvalues of A have a negative real part, it follows that a boundary trajectory passes through every point of ∂**R** (Corollary 3.5). Furthermore, this trajectory is asymptotically stable (Corollary 3.6). Using Corollary 4.3, the boundary control law is given by $u^*(t) = \text{sgn}(\sin(\sqrt{3}t/2))$, where

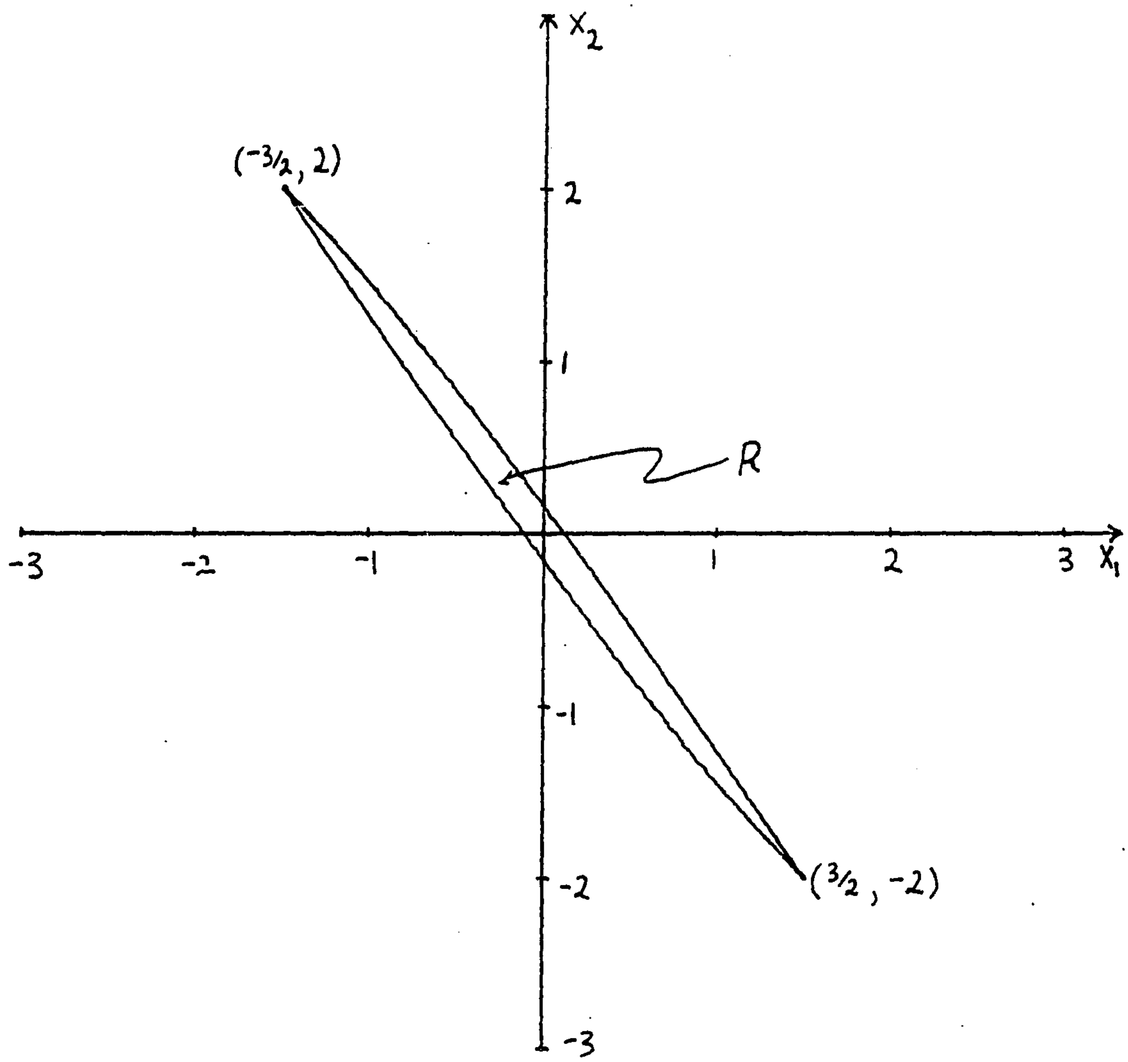


Figure 5.1. The reachable set from the origin for the system in Example 5.2.

$$\operatorname{sgn}(z) = \begin{cases} 1, & \text{if } z > 0 \\ y \in [-1,1], & \text{if } z = 0. \\ -1, & \text{if } z < 0 \end{cases}$$

Thus if we use $u^*(t)$ we can transfer the state from the origin to $\partial \mathbf{R}$ asymptotically. In Figure 5.2 we illustrate the forward solution from the origin to (5.3) where $u = u^*(t)$. From this diagram and the theory stated above we see that **R** is enclosed in the region ABCD.

It is interesting to note that the forward solution from the origin corresponding to (5.3) with $u = \sin(\sqrt{2}t/2)$ (Figure 5.3) is found entirely in **R**. This is important because $\omega = \sqrt{2}/2$ represents the <u>peak frequency</u> (Raven, 1978). (In classical control theory, if we convert (5.3) to a single second order differential equation in x_1 and set $u = \sin \omega t$, then the peak frequency ω_p maximizes the amplitude of x_1 resulting from sinusoidal control.) Note that if we set $u = \operatorname{sgn}(\sin(\sqrt{2}t/2))$, i.e., switch at the peak frequency while using extemal control, the resulting forward solution (Figure 5.4) yields a larger x_1 amplitude than that for $u = \sin(\sqrt{2}t/2)$; however, the solution remains in **R**. Consequently, for the linear system (5.3) $u^*(t)$ is the periodic control law which maximizes the amplitude of x_1.

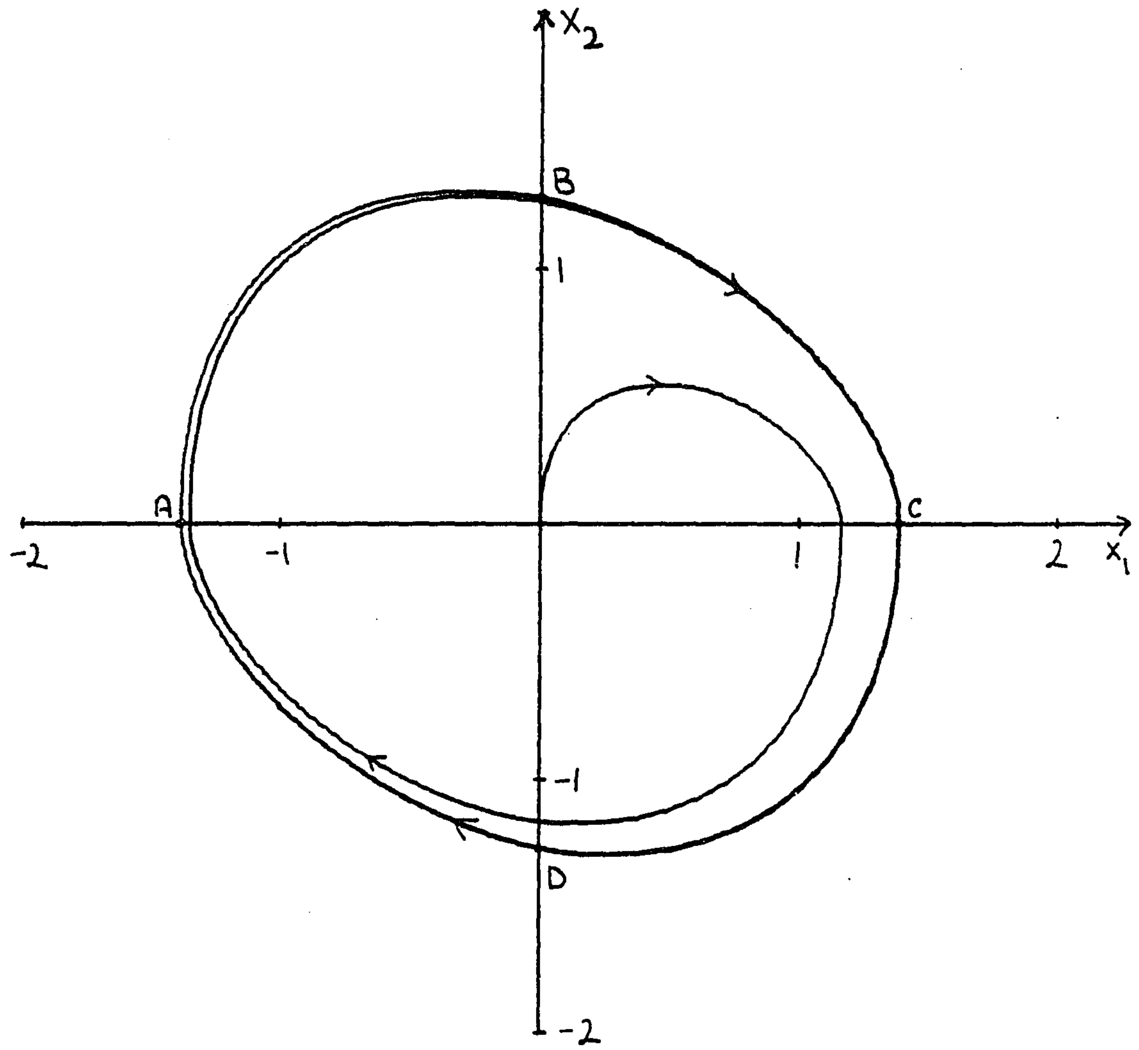


Figure 5.2. The forward trajectory to (5.3) with $u = \text{sgn}(\sin(\sqrt{3}t/2))$ approaches ∂R asymptotically. Hence, the reachable set is enclosed in the region ABCD.

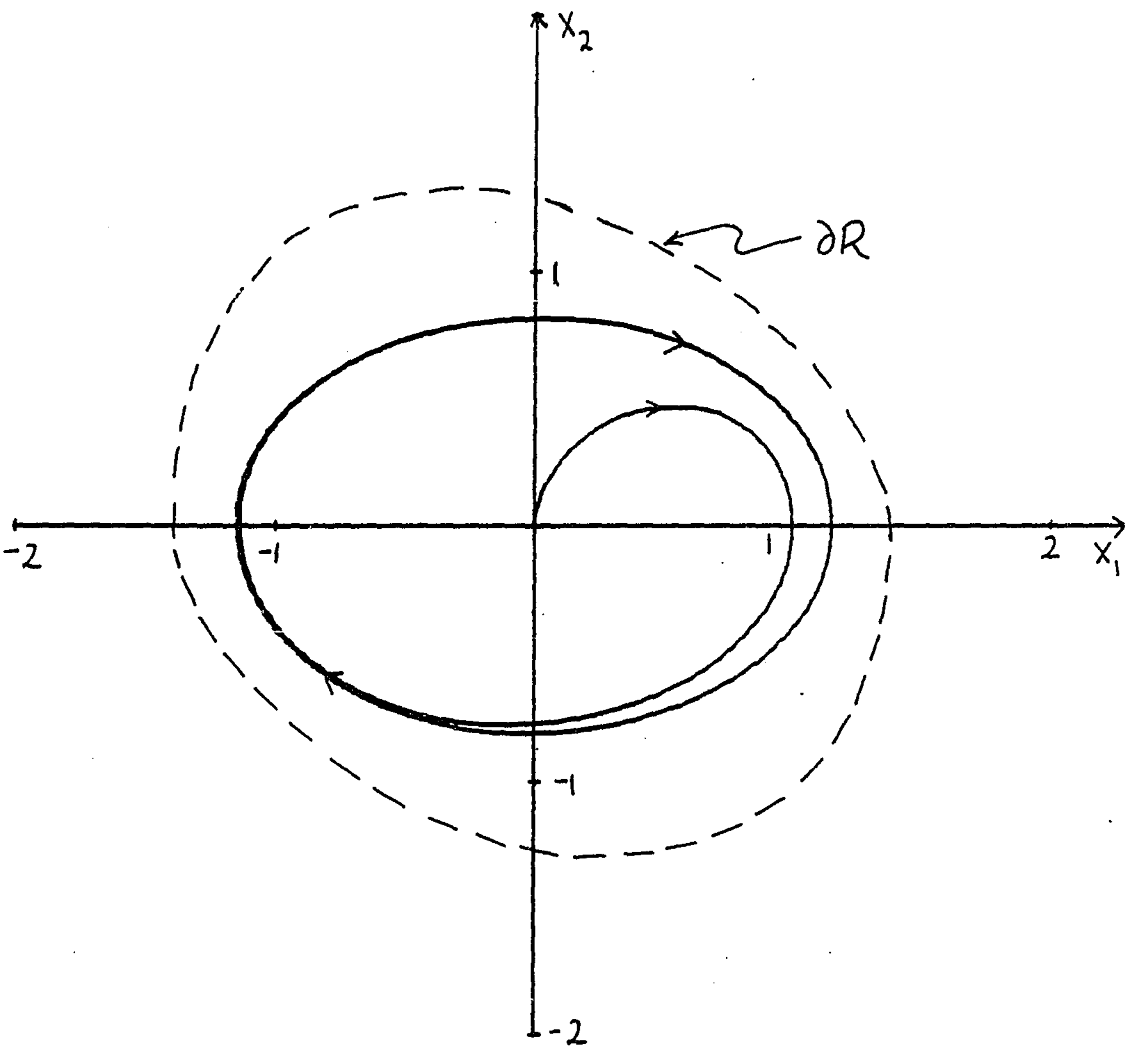


Figure 5.3. The forward trajectory to (5.3) with $u = \sin(\sqrt{2}t/2)$ remains well within reachable set.

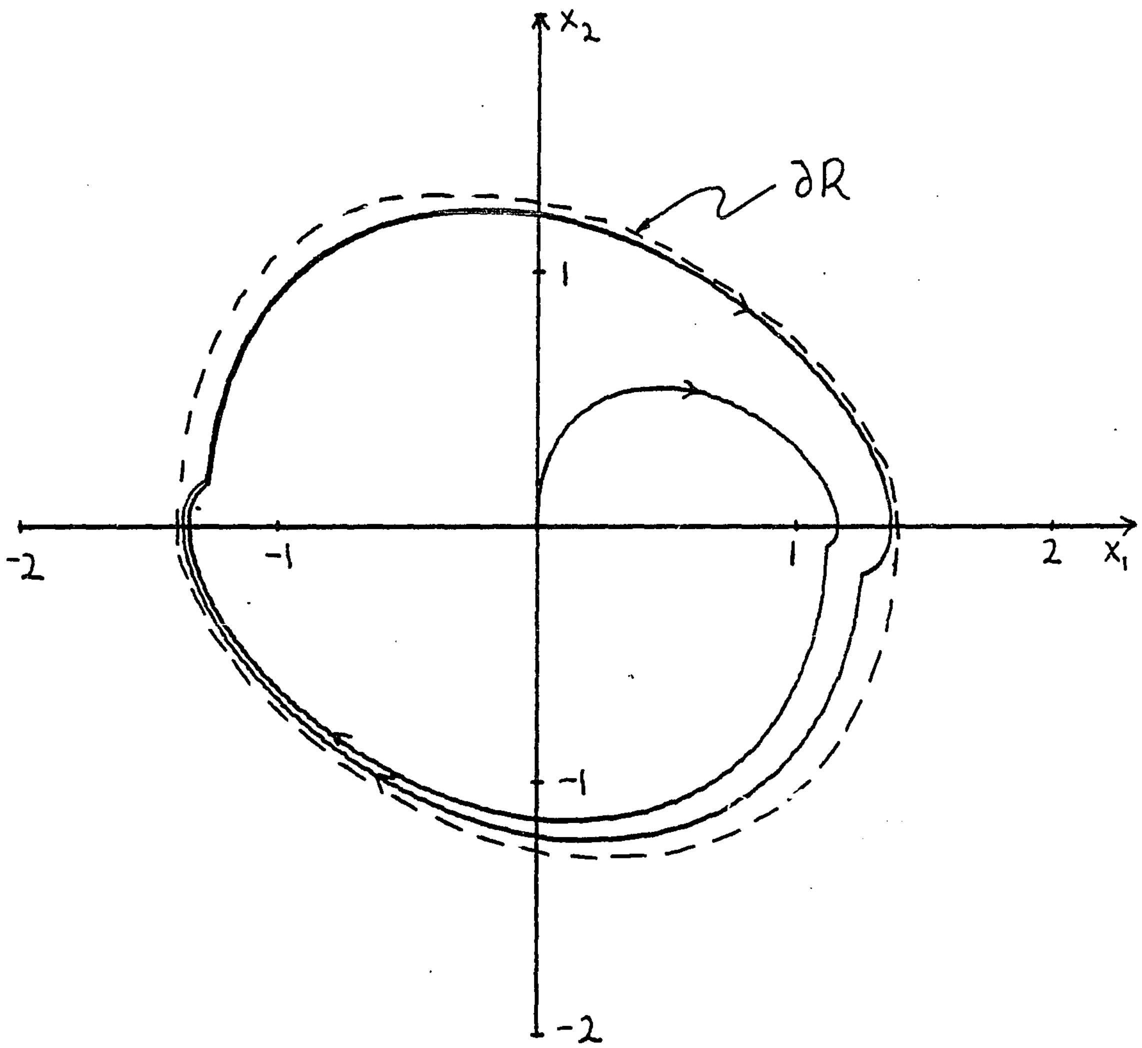


Figure 5.4. The forward trajectory to (5.3) with $u = \text{sgn}(\sin(\sqrt{2}t/2))$ does not approach ∂R.

CHAPTER 6

APPROXIMATING THE REACHABLE SET FROM THE ORIGIN FOR LINEAR SYSTEMS

In this chapter we present a method of approximating the reachable set from the origin **R** for the system

$$\dot{x} = Ax + Bu \qquad (6.1)$$

where $x \in E^n$, $u \in \Omega = \{u \in E^m | -\infty < u_j^{min} \leq u_j \leq u_j^{max} < +\infty$, with $u_j^{min} < 0 < u_j^{max}$, for $j = 1, \ldots, m \leq n\}$, A is a $n \times n$ real constant matrix with distinct eigenvalues, each of which have a negative real part, and B is a $n \times m$ real constant matrix. We assume that A and B are such that rank $[B \;\; AB \;\; \ldots \;\; A^{n-1}B] = n$ thereby insuring all the modes of the system are influenced by the control.

The essence of the technique is to reduce the original n-dimensional problem into a series of 1- and 2-dimensional problems. Not only will this allow us to use the theory developed in Chapters 3-5 to their full extent, it also enables us to visualize to some degree what is happening in the original setting. The manner in which we carry out this procedure is described in the following section.

6.1. Outline of the Procedure

Let R_x denote the reachable set from the origin for the system (6.1). The algorithm for approximating R_x proceeds as follows:

1. Define $z = Lx$, where L is a $n \times n$ real invertible constant matrix such that LAL^{-1} is a block diagonal matrix with each block being a 1×1 (scalar) or a 2×2 real constant matrix. Furthermore, L is such that each 1×1 block of LAL^{-1} corresponds to a real eigenvalue of A and each 2×2 block of LAL^{-1} has one of the complex conjugate pairs of eigenvalues of A as its own eigenvalues. This enables us to write

$$\dot{z} = LAL^{-1}z + LBu$$

$$= \begin{vmatrix} A_1 & & & & & \\ & \ddots & & & & \\ & & A_r & & & \\ & & & a_1 & & \\ & & & & \ddots & \\ & & & & & a_s \end{vmatrix} z + LBu \qquad (6.2)$$

where $A_1, \ldots, A_2$ are 2×2 real constant matrices and $a_1, \ldots, a_s$ are negative real constants. Note that $2r + s = n$. In the discussion which follows, let R_z denote the reachable set from the origin for the system (6.2).

2. Use the methods of Chapter 5 to find the reachable sets of each of the decoupled systems

$$\frac{d}{dt}\begin{vmatrix} z_{2j-1} \\ z_{2j} \end{vmatrix} = A_j \begin{vmatrix} z_{2j-1} \\ z_{2j} \end{vmatrix} + \begin{vmatrix} v_{2j-1} \\ v_{2j} \end{vmatrix}, \tag{6.3}$$

for $j = 1, \ldots, r$, and

$$\frac{dz_{2r+k}}{dt} = a_k z_{2r+k} + v_{2r+k}, \tag{6.4}$$

$k = 1, \ldots, s$, where the v_j, $j = 1, \ldots, n$, are treated as independent control variables with bounds

$$\min_{u\varepsilon\Omega} \sum_{k=1}^{n} \sum_{i=1}^{m} \ell_{jk} b_{ki} u_i \leq v_j \leq \max_{u\varepsilon\Omega} \sum_{k=1}^{n} \sum_{i=1}^{m} \ell_{jk} b_{ki} u_i. \tag{6.5}$$

Denoting the closure of the reachable set for each individual components of z by $\overline{\mathbf{B}}_{z_j} = [z_j^{min}, z_j^{max}]$, $j = 1, \ldots, n$, we can enclose R_z in the n-dimensional box $\overline{\mathbf{B}}_z = \overline{\mathbf{B}}_{z_1} \times \overline{\mathbf{B}}_{z_2} \times \ldots \times \overline{\mathbf{B}}_{z_n}$, i.e., $\mathbf{R}_z \subseteq \overline{\mathbf{B}}_z$.

3. Using the fact that under the linear transformation $x = L^{-1}z$ parallel hyperplanes are mapped to parallel hyperplanes, we form a parallelpiped which encloses $\mathbf{R}_x$ by using $x = L^{-1}z$ on $\partial\overline{\mathbf{B}}_z$.

In some instances further refinement to the approximating set can be made in each coordinate direction. Denoting $L^{-1} = [\ell^*_{jk}]$ we have that

$$x_j = \sum_{k=1}^{n} \ell^*_{jk} z_k \tag{6.6}$$

for $j = 1, \ldots, n$. Upper and lower bounds to the x_j are made by summing the respective upper and lower bounds of the coupled terms of the right hand side of (6.6) over the boundaries of the reachable sets to (6.3) and (6.4), i.e.,

$$\begin{aligned}\sup_{R_x} x_j \leq b_j = &\sum_{k=1}^{r} \sup_{\partial R_{z(k)}} \left(\ell^*_{j,2k-1} z_{2k-1} + \ell^*_{j,2k} z_{2k}\right) \\ &+ \sum_{k=1}^{s} \sup_{B_{z_{2k+s}}} \left(\ell^*_{j,2r+k} z_{2r+k}\right)\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}\inf_{R_x} x_j \geq a_j = &\sum_{k=1}^{r} \inf_{\partial R_{z(k)}} \left(\ell^*_{j,2k-1} z_{2k-1} + \ell^*_{j,2k} z_{2k}\right) \\ &+ \sum_{k=1}^{s} \sup_{B_{z_{2k+s}}} \left(\ell^*_{j,2r+k} z_{2r+k}\right)\end{aligned} \tag{6.8}$$

where $R_{z(j)}$ is the reachable set to (6.3), $j = 1, \ldots, r$.

<u>Remark</u>. Observe that the reachable set of (6.3) will contain the reachable set for the system

$$\frac{d}{dt}\begin{vmatrix} z_{2j-1} \\ \\ z_{2j} \end{vmatrix} = A_j \begin{vmatrix} z_{2k-1} \\ \\ z_{2j} \end{vmatrix} + \begin{vmatrix} \sum_{k=1}^{m} \sum_{i=1}^{m} \ell_{2j-1,k} b_{ki} u_i \\ \\ \sum_{k=1}^{m} \sum_{i=1}^{m} \ell_{2j,k} b_{ki} u_i \end{vmatrix} \tag{6.9}$$

as we are allowing more control to be utilized at any given instant in time (note that the controls in (6.3) are independent, whereas the controls in (6.9) are coupled). However, if u is a scalar or the components of u found in the expression

$$\sum_{k=1}^{n} \sum_{i=1}^{m} \ell_{2j-1,k} b_{ki} u_i$$

are not to be found in the sum

$$\sum_{k=1}^{n} \sum_{i=1}^{m} \ell_{2j,k} b_{ki} u_i ,$$

and vice-versa, then no coupling of controls in (6.9) exists; consequently, the reachable set of (6.3) will be the same as that for (6.9) and will constitute the acutal reachable set for the $z_{2j-1}z_{2j}$ system. If this is true for all $j = 1, \ldots, r$, then $\mathbf{B}_z$ not only encloses $\mathbf{R}_z$, it circumscribes $\mathbf{R}_z$. Consequently, there are points on each face of $\mathbf{B}_z$ which belong to $\partial\mathbf{R}_z$.

6.2. A Similarity Transformation

In this section we prove the existence of a $n \times n$ invertible real constant matrix L with the property that if A is a specified $n \times n$ real constant matrix with distinct eigenvalues, then LAL^{-1} is block diagonal with each block being either a 1×1 (scalar) or 2×2 real constant matrix. Furthermore, the 1×1 blocks correspond to the real eigenvalues of A, whereas each 2×2 block corresponds to a distinct pair of complex conjugate eigenvalues of A.

Theorem 6.1. If A is a $n \times n$ real constant matrix with distinct eigenvalues, then there exists a $n \times n$ invertible real constant matrix L such that $A = LAL^{-1}$ is block diagonal and each block is either a 1×1 (scalar) or 2×2 real constant matrix with eigenvalues being real or complex conjugate pairs respectively.

Proof. We will prove this theorem by actually constructing the matrix L and then show it is real and invertible. In the discussion which follows, let α_j and $\overline{\alpha}_j$, $j = 1, \ldots, r$, be the r complex pairs of eigenvalues of A and let a_j, $j = 1, \ldots, s$, be the s real eigenvalues of A. Note that $2r + s = n$. Associated with each of these distinct eigenvalues, we have the independent eigenvecors of A, e_j and $\overline{e}_j$, $j = 1, \ldots, r$, and e_{r+k}, $k = 1, \ldots, s$, where

$$Ae_j = \alpha_j e_j, \qquad j = 1, \ldots, r,$$

$$A\bar{e}_j = \bar{\alpha}_j\bar{e}_j, \qquad j = 1, \ldots, r,$$

and

$$Ae_{r+k} = a_k e_{r+k}, \qquad k = 1,\ldots, s.$$

The construction of L proceeds in the following manner:

1. Since A has distinct eigenvalues we can perform the similarity transformation $D = M^{-1}AM$, where

$$D = \begin{bmatrix} \alpha_1 & & & & & & & & & \\ & \bar{\alpha}_1 & & & & & & & & \\ & & \alpha_2 & & & & & & & \\ & & & \bar{\alpha}_2 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & \alpha_r & & & & \\ & & & & & & \bar{\alpha}_r & & & \\ & & & & & & & a_1 & & \\ & & & & & & & & a_2 & \\ & & & & & & & & & \ddots \; a_s \end{bmatrix} \tag{6.10}$$

and M is the $n \times n$ complex matrix whose columns are the eigenvectors of A, i.e.,

$$M = [e_1\ \overline{e}_1\ e_2\ \overline{e}_2\ \dots\ e_r\ e_{r+1}\ e_{r+2}\ \dots\ e_{r+s}]. \qquad (6.11)$$

2. For each complex conjugate pair of eigenvalues consider the 2×2 block

$$D_j = \begin{vmatrix} \alpha_j & 0 \\ 0 & \overline{\alpha}_j \end{vmatrix}, \qquad j = 1, \dots, r. \qquad (6.12)$$

We can transform D_j to a matrix of the form

$$\tilde{A}_j = \begin{vmatrix} 0 & 1 \\ -|\alpha_j|^2 & 2\mathrm{Re}(\alpha_j) \end{vmatrix}, \qquad j = 1, \dots, r, \qquad (6.13)$$

by setting $\tilde{A}_j = \tilde{M}_j D_j \tilde{M}_j^{-1}$, where $\tilde{M}_j$ has the eigenvecotrs of $\tilde{A}_j$ as columns (the first column is the eigenvector associated with α_j and the second column is the eigenvector for $\overline{\alpha}_j$). Explicitly, we can write

$$\tilde{M}_j = \begin{vmatrix} 1 & 1 \\ \alpha_j & \overline{\alpha}_j \end{vmatrix}, \qquad j = 1, \dots, r, \qquad (6.14)$$

and

$$\tilde{M}_j^{-1} = \frac{-1}{2i\ \mathrm{Im}(\alpha_j)} \begin{vmatrix} \bar{\alpha}_j & -1 \\ -\alpha_j & 1 \end{vmatrix}, \qquad j = 1, \ldots, r, \tag{6.15}$$

where $i = \sqrt{-1}$.

3. Form the block diagonal matrices

$$\tilde{M} = \begin{vmatrix} \tilde{M}_1 & & & & 0 \\ & \tilde{M}_2 & & & \\ & & \ddots & & \\ & 0 & & \tilde{M}_r & \\ & & & & I_s \end{vmatrix} \tag{6.16}$$

and

$$\tilde{M}^{-1} = \begin{vmatrix} \tilde{M}_1^{-1} & & & & 0 \\ & \tilde{M}_2^{-1} & & & \\ & & \ddots & & \\ & 0 & & \tilde{M}_r^{-1} & \\ & & & & I_s \end{vmatrix}, \tag{6.17}$$

where I_s is the $s \times s$ identity matrix. Then we have that $\tilde{A} = \tilde{M}D\tilde{M}^{-1}$, where

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 & & & & \\ & \ddots & & & \\ & & \tilde{A}_r & & \\ & & & a_1 & \\ & & & & \ddots \\ & & & & & a_s \end{bmatrix} \tag{6.18}$$

Thus $\tilde{A}$ is the block diagonal form that we desire. Combining our steps we see that $\tilde{A} = \tilde{M}D\tilde{M}^{-1} = \tilde{M}(M^{-1}AM)\tilde{M}^{-1} = (\tilde{M}M^{-1})A(M\tilde{M}^{-1}) = LAL^{-1}$, where $L = \tilde{M}M^{-1}$. This completes the construction of L.

We complete the proof by showing that L is a real matrix. To do this we will show that L^{-1} is real. By definition

$$L^{-1} = M\tilde{M}^{-1} = \begin{vmatrix} e_1^1 & \bar{e}_1^1 & \cdots & e_r^1 & \bar{e}_r^2 & e_{r+1}^1 & \cdots & e_{r+s}^1 \\ \vdots & & & & & & & \\ e_n^n & \bar{e}_1^n & \cdots & e_r^n & \bar{e}_r^n & e_{r+1}^n & \cdots & e_{r+s}^n \end{vmatrix}$$

$$\cdot \begin{vmatrix} \tilde{M}_1^{-1} & & & & \\ & \tilde{M}_2^{-1} & & & 0 \\ & & \ddots & & \\ & 0 & & \tilde{M}_r^{-1} & \\ & & & & I_s \end{vmatrix}, \tag{6.19}$$

where we represent the jth component of the eigenvectors of A by the superscript j, $j = 1, \ldots, n$. Careful expansion of (6.19) using (6.15), results in

$$L^{-1} = [p_1 \; p_2 \; \cdots \; p_{2r} \; e_{r+1} \; e_{r+2} \; \cdots \; e_{r+s}] \tag{6.20}$$

where

$$p_j = \frac{-1}{\mathrm{Im}(\alpha_j)} \begin{vmatrix} \mathrm{Im}(e_j^1\bar{\alpha}_j) \\ \mathrm{Im}(e_j^2\bar{\alpha}_j) \\ \vdots \\ \mathrm{Im}(e_j^n\bar{\alpha}_j) \end{vmatrix}, \quad \text{for} \quad j = 1, 3, \ldots, 2r - 1, \tag{6.21}$$

and

$$p_j = \frac{-1}{\mathrm{Im}(\alpha_j)} \begin{vmatrix} \mathrm{Im}(\bar{e}_j^1) \\ \mathrm{Im}(\bar{e}_j^2) \\ \vdots \\ \mathrm{Im}(\bar{e}_j^n) \end{vmatrix}, \quad \text{for} \quad j = 2, 4, \ldots, 2r. \tag{6.22}$$

Consequently, we see that L^{-1} is real; hence, L is real.

Q.E.D.

Remarks. Equations (6.20)-(6.22) give a simple algorithm for constructing L^{-1} once the eigenvalues and eigenvectors of A are known. It is this task, together with inverting L^{-1}, that consumes the greatest amount of time and effort in applying the box procedure of Section 6.1. In some instances this time factor can be reduced as it will not be necessary to find L in its entirety. To see this, note that LAL^{-1} can be constructed from (6.13) and (6.18) once the eigenvalues of A are known. Consequently, the only reason we need L is to form LB. However, if $B = [0 \; 0 \ldots 0 \; 1]^T$, then $LB = [\ell_{1n} \; \ell_{2n} \cdots \ell_{nn}]^T$ so that we need only find the last column of L and not the entire matrix. This can be done by using the fact that the nth column of $L^{-1}L$ is $L^{-1}L_n = [0 \ 0 \ldots 0 \ 1]^T$, where L_n is the nth column of L. Hence, in this case, we need only solve a linear system of equations to find L_n instead of finding L in its completeness.

Example 6.1. Let

$$A = \begin{vmatrix} 2 & 5 & 5 \\ -2 & -4 & -3 \\ 0 & 0 & -1 \end{vmatrix}.$$

Find a nonsingular matrix L such that LAL^{-1} is in block diagonal form and the blocks are either 1×1 (scalar) or 2×2 real constant matrices.

Solution. One posible represeation of LAL^{-1} is determined by (6.13) and (6.18), i.e.,

$$\begin{vmatrix} 0 & 1 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & -1 \end{vmatrix}.$$

To find the matrix L which leads to this representation we use (6.20)-(6.22) to find L^{-1}. Denoting the eigenvalues of A as $\alpha_1 = -1 + i$, $\overline{\alpha}_1 = -1 - i$, and $a_1 = -1$, and their respective eigenvectors by $e_1 = [5 \quad -3 + i \quad 0]^T$, $\overline{e}_1 = [5 \quad -3 - i \quad 0]^T$, and $e_2 = [0 \quad -1 \quad 1]^T$, equations (6.20)-(6.22) lead to

$$L^{-1} = \begin{vmatrix} 5 & 0 & 0 \\ -2 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix}.$$

Inverting L^{-1} yields

$$L = \begin{vmatrix} 1/5 & 0 & 0 \\ 2/5 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}.$$

An alternate way of finding L, without first finding L^{-1} is to use the procedure outlined in equations (6.10)-(6.17).

6.3. Reachable Sets Under a Similarity Transformation

An essential part of the procedure outlined in Section 6.1 is to relate the reachable set of (6.1) to the reachable set to (6.2). Using the terminology of Section 6.1, we will establish that R_Z is obtainable from R_X under the transformation L, i.e., $L(R_Z) = R_X$. These results follow from

Theorem 6.2. Consider the linear system

$$\dot{x} = Ax + Bu \tag{6.23}$$

where $x \in E^n$, $u \in \Omega \subset E^m$, and A and B are $n \times n$ and $n \times m$ real constant matrices, respectively. Let $z = Lx$, where L is an invertible $n \times n$ real constant matrix. Let R_x be the reachable set from the origin for (6.23), and let R_z be the reachable set from the origin for the system

$$\dot{z} = LAL^{-1}z + LBu. \tag{6.24}$$

Then $L(R_x) = R_z$, $L(\partial R_x) = \partial R_z$, and $L(\text{ext } \overline{R_x}) = \text{ext } \overline{R_z}$.

Proof. Let $x_1 \in R_x$. Then there exists an admissable control law $u(t)$ and time $T \geq 0$ such that if $\xi(t,0)$ is the forward solution to (6.23) with $\xi(0,0) = 0$, then $\xi(T,0) = x_1$. Explicitly, we have that

$$\xi(t,0) = \int_0^t e^{A(t-s)}Bu(s)ds, \tag{6.25}$$

for $0 \leq t \leq T$, and

$$x_1 = \xi(T,0) = \int_0^T e^{A(T-s)}Bu(s)ds. \tag{6.26}$$

We will show that $z_1 = Lx_1 \varepsilon R_z$. To see this, first observe that an equivalent way of writing (6.24) is

$$\dot{z} = L\dot{x} = L(Ax + Bu). \tag{6.27}$$

As $\xi(t,0)$ is a solution to (6.23) under the admissable control law $u(t)$ for $0 \leq t \leq T$, we can substitute $\xi(t,0)$ for x in (6.27) to get

$$\dot{z} = L[A\xi(t,0) + Bu]. \tag{6.28}$$

Integrating both sides of (6.28) with respect to t between $t = 0$ and $t = T$ leads to the expression

$$z(T) = L\int_0^T [A\xi(t,0) + Bu(t)]dt. \tag{6.29}$$

Substituting (6.25) into (6.29) results in

$$z(T) = L\int_0^T \int_0^t Ae^{A(t-s)}Bu(s)dsdt + L\int_0^T Bu(t)dt. \tag{6.30}$$

Switching order of integration of the first term on the right hand side of (6.30) yields the expression

$$z(T) = L\int_0^T \int_s^T Ae^{A(t-s)}Bu(s)dtds + L\int_0^T Bu(t)dt. \tag{6.31}$$

Integrating (6.31) leads us to conclude that

$$z(T) = L[\int_0^T e^{A(T-s)}Bu(s)ds]. \tag{6.32}$$

Using (6.25) results in

$$z(T) = L\xi(T,0) = Lx_1. \tag{6.33}$$

Consequently, we have demonstrated that for the system (6.28), if we use the admissable control law $u(t)$ and integrate the system equations forward to $t = T$, then the point $z_1 = Lx_1 \in \mathbf{R}_z$. Since $x_1 \in \mathbf{R}_x$ is arbitrary, it follows that $L(\mathbf{R}_x) \subseteq \mathbf{R}_z$.

Using a similar argument we can show that if $z_2 \in \mathbf{R}_z$, then $x_2 = L^{-1}z_2 \in \mathbf{R}_x$. Combining this with the above enables us to write $Lx_2 = L(L^{-1}z_2) = z_2 \in L(\mathbf{R}_x)$ so that $\mathbf{R}_z \subseteq L(\mathbf{R}_x)$. Thus we conclude that $L(\mathbf{R}_x) = \mathbf{R}_z$.

Next we show that, under the transformation L, points on the boundary of $\mathbf{R}_x$ are mapped onto the boundary of $\mathbf{R}_z$. To do this, suppose that points of $\partial\mathbf{R}_x$ are not mapped to points of $\partial\mathbf{R}_z$. Then there exists some point $x_1 \in \partial\mathbf{R}_x$ such that $Lx_1 \notin \partial\mathbf{R}_z$. Thus we have $Lx_1 \in \text{int } \mathbf{R}_z$ or $Lx_1 \in \text{ext } \overline{\mathbf{R}_z}$. Let us first examine the case of $Lx_1 \in \text{ext } \overline{\mathbf{R}_z}$. Consider the open ball $B_\varepsilon(Lx_1)$ where $\varepsilon > 0$ is chosen such that $B_\varepsilon(Lx_1) \cap \overline{\mathbf{R}}_z = \emptyset$. Since Lx is continuous, there exists a $\delta > 0$ such that $Lx_2 \in B_\varepsilon(Lx_1)$ whenever

$x_2 \in B_\delta(x_1)$. In particular, if we choose $x_2 \in B_\delta(x_1) \cap \mathbf{R}_x$, we have $Lx_2 \in B_\varepsilon(Lx_1) \subset \text{ext } \overline{R_z}$ which contradicts the fact that $L(\mathbf{R}_x) = \mathbf{R}_z$. Thus we conclude that if $x_1 \in \partial R_x$, then $Lx_1 \notin \text{ext } R_z$.

Now let us investigate the possibility $Lx_1 \in \text{int } R_z$ when $x_1 \in \partial R_x$. Consider the ball $B_\varepsilon(x_1)$. Since $L^{-1}z$ is a continuous function, there exists a $\delta > 0$ such that $L^{-1}z_2 \in B_\varepsilon(x_1)$ whenever $z_2 \in B_\delta(Lx_1)$. Consequently, we can find $z_2 \in B_\delta(Lx_1) \cap \text{int } R_z$ such that $L^{-1}z_2 \in \text{ ext } \overline{R_z}$ which contradicts the fact that $L^{-1}(\mathbf{R}_z) = R_x$. Hence, we cannot have $x_1 \in \partial R_x$ lead to $Lx_1 \in \text{int } R_z$. Combining this with the result of the preceeding paragraph, we conclude that if $x_1 \in \partial R_x$, then $Lx_1 \in \partial \mathbf{R}_z$, i.e., $L(\partial R_x) \subseteq \partial R_z$. Repeating the argument from z-space to x-space we have that $L^{-1}(\partial R_z) \subseteq \partial R_x$. Hence $L(\partial R_x) = \partial R_z$.

Having shown that $L(\mathbf{R}_z) = R_z$ and $L(\partial \mathbf{R}_x) = \partial R_z$, it follows from the continuity of Lx and the invertibility of L, together with arguments similar to the above, that $L(\text{ext } \overline{R_x}) = \text{ext } \overline{R_z}$.

Q.E.D.

<u>Remark.</u> Coupling the Remark of Section 6.1 with Theorem 6.2 we have an interesting result. In referemce to the box procedure, if $\overline{\mathbf{B}}_z$ circumscribes R_z, then there are points on each face of $\overline{\mathbf{B}}_z$ which belong to ∂R_z. Hence, by Theorem 6.2 these points map to ∂R_x under the transformation $x = L^{-1}z$. Thus, we conclude that the hyperplanes enclosing R_x also circumscribe $\mathbf{R}_x$.

6.4. Illustrative Examples

Example 6.2. Approximate R_X for the system

$$\dot{x} = \begin{vmatrix} 0 & 1 \\ -2 & -3 \end{vmatrix} x + \begin{vmatrix} 2 \\ -3 \end{vmatrix} u \tag{6.34}$$

where $u \in [-1,1]$.

Although we can find R_X immediately from the results in Chapter 5, we will approximate R_X using the technique proposed in this chapter.

First let us observe that rank $[B \quad AB] = 2$ and that the eigenvalues of A are -1 and -2; hence, the conditions underlying the "box procedure" are satisfied. Thus, let us diagonalize the system by performing the transformation $z = Lx$, where

$$L = \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix}.$$

This results in

$$\dot{z} = \begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} z + \begin{vmatrix} 1 \\ 1 \end{vmatrix} u. \qquad (6.35)$$

Using the result of Section 5.1, the closure of the reachable set for z_1 and z_2, respectively, are

$\overline{R}_{z_1} = \{z_1 \varepsilon E^1 | -1 \leq z_1 \leq 1\}$ and $\overline{R}_{z_2} = \{z_2 \varepsilon E^1 | -\frac{1}{2} \leq z_2 \leq \frac{1}{2}\}$.

Consequently, the reachable set of (6.35) form the origin $R_z \subseteq \overline{R}_{z_1} \times \overline{R}_{z_2}$ (Figure 6.1). To find a 2-dimensional parallelpiped containing R_x we transform the points of $\partial(\overline{R}_{z_1} \times \overline{R}_{z_2})$ into x-space using the transformation $x = L^{-1}z$. Specifically, we have

$$x = z_1 + z_2$$

$$x_2 = -z_1 - 2z_2.$$

Applying this transformation to each of the sides of B_z we find that R_x is bounded by the lines

$$2x_1 + x_2 = 1$$

$$2x_1 + x_2 = -1$$

$$x_1 + x_2 = \frac{1}{2}$$

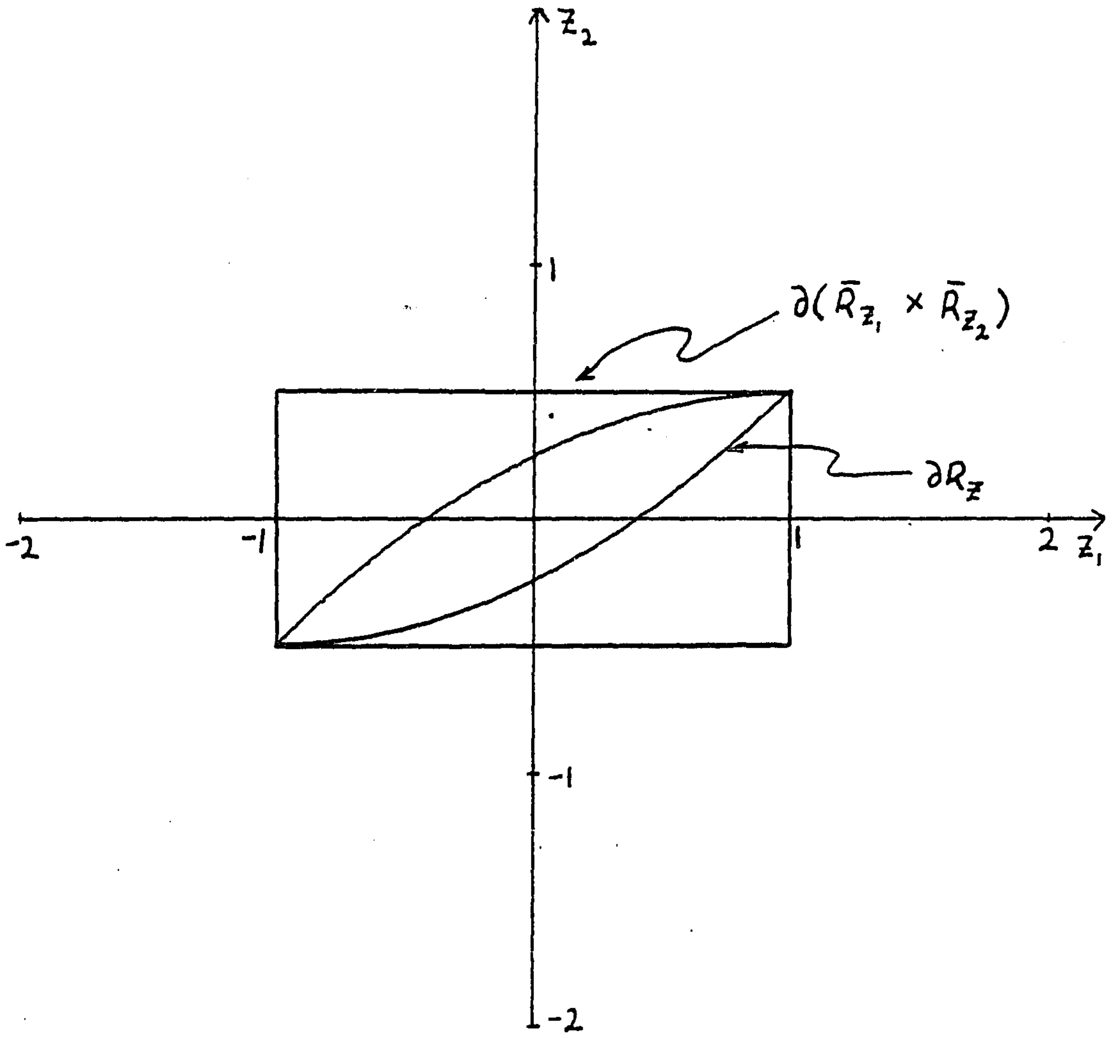


Figure 6.1. Relationship between ∂R_Z and $\partial(\bar{R}_{Z_1} \times \bar{R}_{Z_2})$ for system (6.35).

$$x_1 + x_2 = -\frac{1}{2},$$

as seen in Figure 6.2. As we are also interested in the supremum and infimum values of both x_1 and x_2 in R_x, we find that

$$\sup_{R_x} x_1 = \sup_{R_z} (z_1 + z_2) \leq \sup z_1 + \sup z_2$$

$$\inf_{R_x} x_1 = \inf_{R_z} (z_1 + z_2) \geq \inf z_1 + \inf z_2$$

$$\sup_{R_x} x_2 = \sup_{R_z} (-z_1 - 2z_2) \leq \sup (-z_1) + \sup (-2z_2)$$

$$\inf_{R_x} x_2 = \inf_{R_z} (-z_1 - 2z_2) \geq \inf (-z_1) + \inf (-2z_2).$$

Thus we have that $-3/2 \leq x_1 \leq 3/2$ and $-2 \leq x_2 \leq 2$.

Remark. It is interesting to compare the approximation to **R** for the system (6.35) as given by the box method and the Lyapunov approach due to Grantham (1980a). (Refer to Section 3.8, Example 3.4) In Figure 6.3 we illustrate the relationship of the two approximations with each other and to the actual reachable set. Clearly, for the Lyapunov function used, the box method offers a better approximation to **R**. This would appear to be the case, in general, since there is such a variety of Lyapunov functions which could be used, with no guide as to which is "best". Consequently, when dealing with linear systems, the box method should be employed.

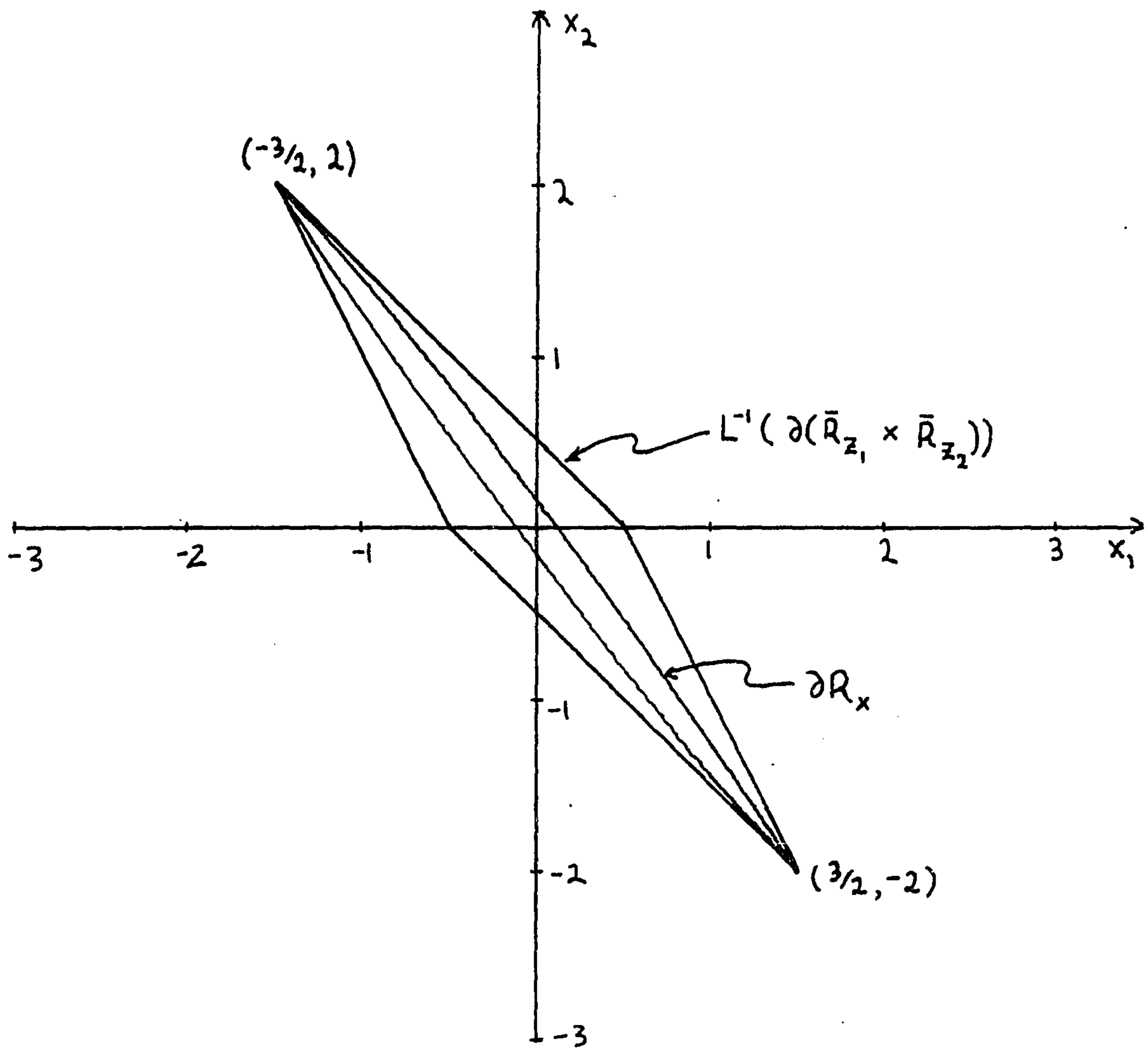


Figure 6.2. Relationship between R_X and the approximating parallelpiped $L^{-1}(\partial(\bar{R}_{Z_1} \times \bar{R}_{Z_2}))$.

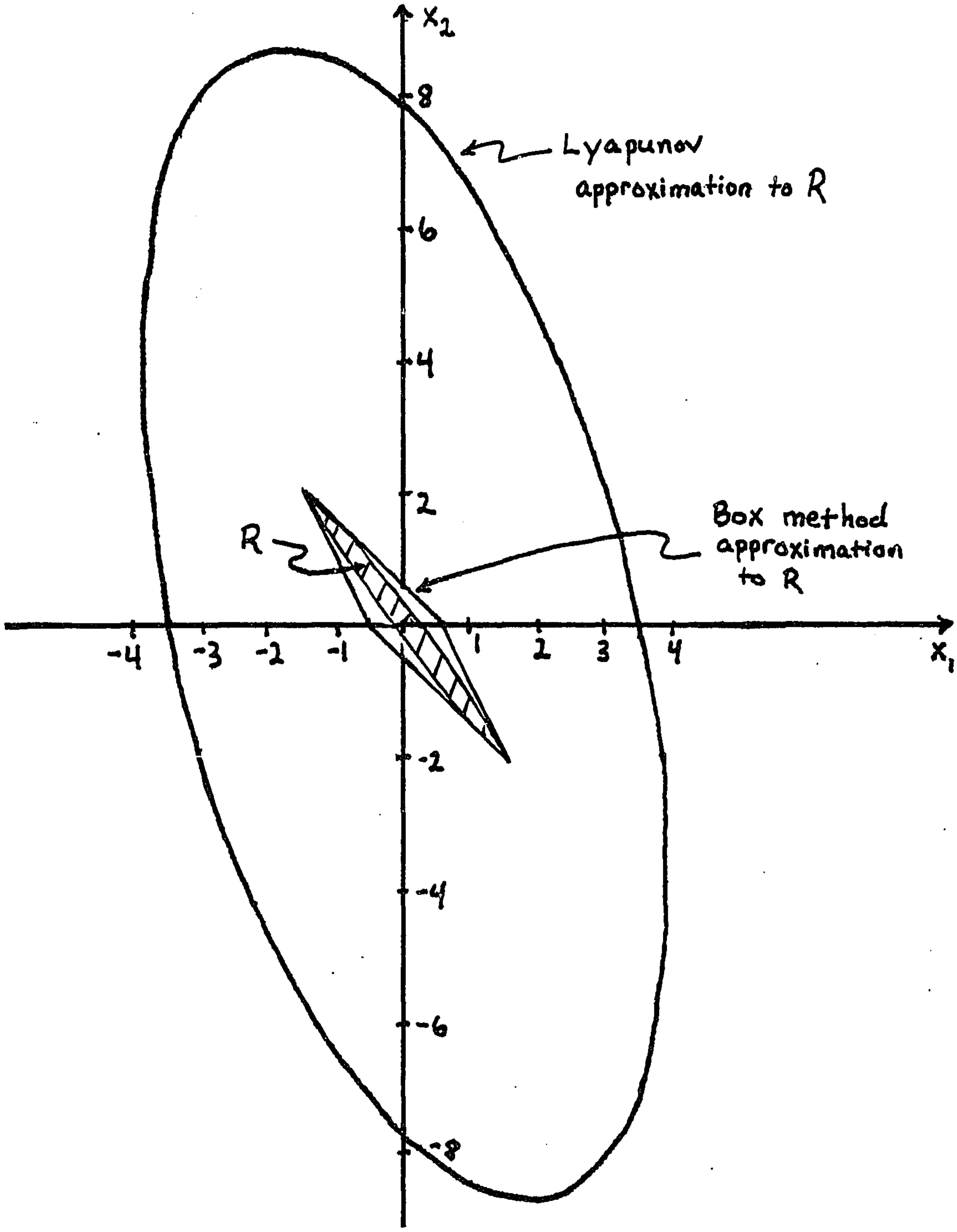


Figure 6.3. A comparison of a Lyapunov estimate and the box method approximation for the system in Example 6.2.

Example 6.3. Find a parallelpiped which contains R_x for the system described by

$$\dot{x} = \begin{vmatrix} 2 & 5 & 5 \\ -2 & -4 & -3 \\ 0 & 0 & 1 \end{vmatrix} x + \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} u \tag{6.36}$$

with $|u| \leq 1$.

Observing that the A matrix in (6.36) is that of Example 6.1 we, can use the result of that example to write

$$\dot{z} = \begin{vmatrix} 0 & 1 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 1 \end{vmatrix} z + \begin{vmatrix} 0 \\ 1 \\ 1 \end{vmatrix} u. \tag{6.37}$$

Hence, we may consider the decoupled systems

$$\frac{d}{dt}\begin{vmatrix} z_1 \\ z_2 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -2 & -2 \end{vmatrix} \begin{vmatrix} z_1 \\ z_2 \end{vmatrix} + \begin{vmatrix} 0 \\ 1 \end{vmatrix} u \tag{6.38}$$

and

$$\dot{z}_3 = -z_3 + u. \tag{6.39}$$

Using the 1- and 2-dimensional theory on (6.38) and (6.39) (Section 5.2 and 5.2, respectively) we find that $|z_1| \leq 0.5451$, $|z_2| \leq 0.6737$, and $|z_3| \leq 1$. Consequently, $R_z \subseteq \{z \in E^3 \,|\, |z_1| \leq 0.5451, \ |z_2| \leq 0.6737, \text{ and } |z_3| \leq 1\}$. From the comments at the end of Section 6.1 we see that $\bar{B}_z$ circumscribes $\bar{R}_z$.

Using the transformation $x = L^{-1}z$ we can write

$$x_1 = 5z_1$$

$$x_2 = -2z_1 + z_2 - z_3$$

$$x_3 = z_3.$$

Under this transformation the faces of $\bar{B}_z$ are mapped into the faces of the parallelpiped circumscribing R_x:

$$x_1 = 2.73$$

$$x_1 = -2.73$$

$$2x_1 + 5x_2 - 5x_3 = 3.37 \qquad (6.40)$$

$$2x_1 + 5x_2 - 5x_3 = -3.37$$

$$x_3 = 1$$

$$x_3 = -1$$

We can achieve more information by realizing that in the x_2 direction we have

$$\sup_{\mathbf{R}_x} x_2 \leq \sup_{\partial \mathbf{R}_{z(1)}} (-2z_1 + z_2) + \sup_{\mathbf{B}_{z_3}} (-z_3)$$

and

$$\inf_{\mathbf{R}_x} x_2 \geq \inf_{\partial \mathbf{R}_{z(1)}} (-2z_1 + 2z_2) + \sup_{\mathbf{B}_{z_3}} (-z_3)$$

where $\partial \mathbf{R}_{z(1)}$ is the boundary of the reachable set for (6.38) and $\mathbf{B}_{z_3}$ is the reachable set for (6.39). This results in $|x_2| \leq 2.40$. Hence, $\mathbf{R}_x$ is enclosed in the region bounded by the planes $x_2 = 2.40$, $x_2 = -2.40$, and (6.40).

Example 6.4. Approximate the reachable set from the origin for the system

$$\dot{x} = \begin{vmatrix} 759 & -442 & 828 & -2442 & 9246 \\ 790 & 358 & 1750 & 1840 & 670 \\ -330 & 193 & -360 & 1066 & -4020 \\ -159 & -72 & -352 & -370 & -137 \\ -33 & 19 & -36 & 105 & -402 \end{vmatrix} x + \begin{vmatrix} -207 \\ 895 \\ 90 \\ -175 \\ 9 \end{vmatrix} u \qquad (6.41)$$

where $u \in [-3,6]$.

Let $z = Lx$, where

$$L = \begin{vmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 5 & 0 \\ 6 & 0 & 13 & 0 & 8 \\ 0 & 9 & 0 & 46 & 0 \\ 11 & 0 & 12 & 0 & 134 \end{vmatrix}.$$

Then in terms of the variable z (6.41) becomes

$$\dot{z} = \begin{vmatrix} 0 & 1 & 0 & 0 & 0 \\ -5 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -34 & -10 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{vmatrix} z + \begin{vmatrix} 0 \\ 20 \\ 0 \\ 50 \\ 9 \end{vmatrix} u. \qquad (6.42)$$

Thus we have the subsystems

$$\frac{d}{dt}\begin{vmatrix} z_1 \\ z_2 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -5 & -2 \end{vmatrix}\begin{vmatrix} z_1 \\ z_2 \end{vmatrix} + \begin{vmatrix} 0 \\ 20 \end{vmatrix} u, \qquad (6.43)$$

$$\frac{d}{dt}\begin{vmatrix} z_3 \\ z_4 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -34 & -10 \end{vmatrix}\begin{vmatrix} z_3 \\ z_4 \end{vmatrix} + \begin{vmatrix} 0 \\ 50 \end{vmatrix} u, \qquad (6.44)$$

and

$$\dot{z}_5 = -3z_5 + 9u, \qquad (6.45)$$

with the respective system eigenvalues $-1 \pm 2i$, $-5 \pm 3i$, and -3. Applying the methods of Chapter 5, we find that the bounds on each of the compontnts of z are

$$-21.45 < z_1 < 33.44$$

$$-58.41 < z_2 < 58.36$$

$$-4.48 < z_3 < 8.89$$

$$-31.21 < z_4 < 31.21$$

$$-3 < z_5 < 3.$$

Hence, R_z is enclosed in the 5-dimensional box

$$\overline{B}_z = [-21.45,33.44] \times [-58.41,58.36] \times [-4.48,8.89]$$

$$\times [-31.21,31.21] \times [-3,3];$$

in fact, as the control is a scalar, $\overline{B}_z$ circumscribes $\overline{R}_z$ (see Remark at end of Section 6.1).

Using the transformation

$$x_1 = 1646z_1 - 232z_3 - 23z_5$$

$$x_2 = 46z_2 - 5z_4$$

$$x_3 = -716z_1 + 101z_3 + 10z_5$$

$$x_4 = -9z_2 + z_4$$

$$x_5 = -71z_1 + 10z_3 + z_5,$$

we transform the faces of $\mathbb{B}_z$ into the faces of a parallelpiped which circumscribes $\mathbb{R}_x$. These faces are described by the equations

$$x_1 + 2x_3 + 3x_5 = 236$$

$$x_1 + 2x_3 + 3x_5 = -368$$

$$x_2 + 5x_4 = 58.4$$

$$x_2 + 5x_4 = -58.4$$

$$9x_2 + 46x_4 = 31.2$$

$$9x_2 + 46x_4 = -31.2$$

$$36x_1 - 6454x_3 + 65{,}920x_5 = 29{,}500$$

$$36x_1 - 6454x_3 + 65{,}920x_5 = -58{,}400$$

$$55x_1 + 60x_3 + 670x_5 = 5610$$

$$55x_1 + 60x_3 + 670x_5 = -5610.$$

Furthermore, we have the bounds on the individual components of x:

$$-37,400 < x_1 < 56,200$$

$$-2,840 < x_2 < 2,840$$

$$-24,400 < x_3 < 16,300$$

$$-560 < x_4 < 560$$

$$-2,420 < x_5 < 1,620.$$

CHAPTER 7

CONCLUSIONS AND FUTURE WORK

A method of approximating the reachable set from the origin for the linear system (6.1) subject to bounded control has been proposed. The "box" method is based upon decoupling the system into 1- and 2-dimensional subsystems, and then finding the reachable set of each of these subsystems. For the resulting 1-dimensional subsystems this is accomplished in a straightforward manner as described in Section 5.1. For 2-dimensional subsystems with two control variables no technique is proposed to find the reachable set; however, if there is a single control variable, then we can use the Reachability Maximum Principle (Grantham, 1973; Grantham and Vincent, 1975) to find the boundary of the reachable set since a boundary trajectory exists (Corollary 3.5) and is asymptotically stable (Corollary 3.6). Having approximated the reachable sets of each of the decoupled systems, a n-dimensional parallelpiped is constructed which contains the reachable set from the origin of the original system. No measure of the "goodness-of-fit" is given to this approximation; however, based upon a 2-dimensional example (Examples 3.4 and 6.1), the box method appears to offer a better and more straightforward approximation to the reachable set than does Grantham's (1980a) Lyapunov approach.

Extension of this work would include a way of finding the reachable set from the origin of 2-dimensional linear systems with two control variables. (If it can be shown that the a boundary trajectory for such a system exists and is nontrivial, then Theorem 4.6 can be used to transfer the state asymptotically from the origin to the boundary of the reachable set.) Also, if a boundary control law for nonlinear systems could be found, then asymptotic stability results could be used to transfer the state from the origin to the boundary of the reachable set.

SYMBOLS AND NOTATION

$=$	Equals
$\neq$	Does not equal
$\doteq$	Approximately equal to
$\equiv$	Equals identically
$\geq$ ($\leq$)	Greater (less) than or equal to
$>$ ($<$)	Greater (less) than
ε	Is an element of
$\not\varepsilon$	Is not an element of
$\cup$	Union
$\cap$	Intersection
$\subseteq$	Is contained in or all of
$\subset$	Is contained in
$\supseteq$	Contains or all of
$A\backslash B$	Set of elements in A but not in B
$\emptyset$	Empty set
$[a,b]$	Closed interval $a \leq x \leq b$
(a,b)	Open interval $a < x < b$
$[a,b)$	Half closed interval $a \leq x < b$
E^m	Euclidean space of dimension m
$\Vert\cdot\Vert$	Euclidean norm. If $x \in E^n$, then $\Vert x\Vert = \sqrt{x_1^2 + \ldots + x_n^2}$. If $A = [a_{ij}]$ is a $n \times n$ real constant matrix, then

$$\|A\| = [\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2]^{1/2}.$$

$\lvert\cdot\rvert$	Absolute value
$\{A \mid B\}$	The set of all A such that B holds
$G: A \to B$	G maps A into B
int A	Interior of set A
ext A	Exterior of set A
$\bar{A}$	Closure of set A
∂A	Boundary of set A
$B_\delta(x)$	Ball about x of radius δ

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