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EFFICIENT METHODS FOR MECHANICAL AND STRUCTURAL RELIABILITY ANALYSIS AND DESIGN

by

Yih-Tsuen Wu

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As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Yih-Tsuen Wu

entitled EFFICIENT METHODS FOR MECHANICAL AND STRUCTURAL RELIABILITY ANALYSIS AND DESIGN

and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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	Date
	Date

Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

Paul H. Wirsching
Dissertation Director

April 16, 1984
Date

STATEMENT BY AUTHOR

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NOMENCLATURE

A	Scale factor of a three-parameter equivalent normal (subscript indicate variables); constant (area, cyclic strength coefficient, scale factor)
$\underset{\sim}{a}$	Expanding point of the Taylor's series; coefficients of the variable in a linear limit state
B	Scale factor
b	Fatigue strength exponent
C	Damping coefficient
C_X	Coefficient of variation of the X variable
c	Fatigue ductility exponent
cdf	Cumulative distribution function
COV	Coefficient of variation
cr	Subscript, critical value (slope = 0)
D	A random variable (dead load)
E	Modulus of elasticity, error sum
F	Cumulative distribution function (cdf); subscript indicates variable
f	Probability distribution function (pdf); subscript indicates variable
$f(\cdot)$	Function of
$g(\underset{\sim}{X})$	Failure or limit state function of $\underset{\sim}{X}$
$g_1(\underset{\sim}{u})$	Transformed or reduced limit state
$h(\cdot)$	Function of
I	A random variable with unit nominal value
i	ith variable (i = 1,n)

j	jth approximation point ($j = 1,N$)
K	Sample size of Monte Carlo simulation; from S-N curve $NS^m=K$
k_t	Stress concentration factor
k	Number of normal equations; integer
L	Initial approximation range on a linear limit state; random variable (live load)
ℓ	Integer
M	Mass
m	Number of observations or data sets; integer; from S-N curve $NS^m=K$
N	Cycles to failure; number of initial approximation points
N_A, N_B	End points of the reduced approximation points
N_o	Service life in cycles
n	Number of design random variables; subscript, nominal value
n_o	Number of observations of $g<0$ in a Monte Carlo simulation
$P(\cdot)$	Probability of
P	Pressure; design point
pdf	Probability density function
$\hat{p}$	Probability estimate using Monte Carlo method
p_L	Lower confidence limit of $\hat{p}$
p_o	Target probability of failure
p_U	Upper confidence limit of $\hat{p}$
Q	Loading
R	Strength random variable
$\underset{\sim}{R}$	A vector of the design factors governing design strength

R_n	Nominal value of R
r_j	Distance measured from design point, on L, of jth approximation point
S	Stress random variable
$\underset{\sim}{S}$	A vector of the design factors governing applied stress
S_n	Nominal stress
s	Cyclic strain hardening exponent
T	Torque
t	Time
U	A random variable
u	Reduced variable; standard normal variate
V	A stress variable
$\underset{\sim}{X}$	A vector of the design variables
x	Real number associates with random variable X
Y	A function of $\underset{\sim}{X}$; a random variable
y	Real number associates with Y; displacement function (Chapter 5)
Z	Defined as $Z = g(\underset{\sim}{X})$
z	Reduced variable when using Rackwitz-Fiesslwr algorithm (Chapter 4)
α	Coefficient in polynomial equation
$\hat{\alpha}$	Least square estimator of α
β	Safety index, generalized safety index, equivalent safety index
β_o	Target safety index
γ	Partial safety factor; load resistance factor
Δ	Distance measured from the origin on the u space

δ	Minimum distance using the three-parameter equivalent normal concept
ε	Strain range
ε_a	Cyclic strain amplitude
ε_f'	Fatigue ductility coefficient
θ	Directional angle
κ	Main curvature at the design point of $g_1(\underset{\sim}{u}) = 0$
λ	Direction cosine
μ	Mean value; subscript indicates variables
μ_N	Equivalent normal mean
ν	Safety factor
Π	Product
Σ	Summation
σ	Standard deviation (subscript indicates variable); stress range
σ_f'	Fatigue strength coefficient
σ_N	Equivalent normal standard deviation
σ_o	Mean Stress
$\Phi(x)$	Normal cdf with mean μ_N and standard deviation σ_N
$\Phi'(x)$	Normal cdf using Rackwitz-Fiessler algorithm
$\Phi(u)$	Standard normal cdf with zero mean and unity standard deviation
$\phi(x)$	Normal pdf with mean μ_N and standard deviation σ_N
$\phi'(x)$	Normal pdf using Rackwitz-Fiessler algorithm
$\phi(u)$	Standard normal pdf with zero mean and unity standard deviation

χ_n^2 Chi-square distribution with n degrees of freedom

Ω Failure domain

* Supercript denotes design point value

~ Supercript denotes median value; subscript denotes a vector

ABSTRACT

Three fundamental problems of mechanical reliability are addressed. (1) computing the probability of failure, p_f, of a component having design factors with known statistical distributions and a limit state with a closed form algebraic expression (2) computing the probability of failure of a component having design factors with known distributions and a limit state which can only be expressed by a computer algorithm, and (3) deriving safety check expressions in a "design by reliability" approach.

An algorithm for generating estimates of p_f is presented. The method is an extension of, and demonstrated to be a significant improvement to, the widely used Rackwitz-Fiessler (R-F) method -- a fast and efficient numerical method for performing reliability analysis. Comparisons were made for numerous examples, it was found that the error in p_f, using the proposed method, is typically about half of the error in R-F estimates.

A method was proposed for computing p_f when the relationship between design factors can be defined only using a computer algorithm, e.g., finite element analysis. A second order polynomial is constructed, using a simple curve fitting routine, to approximate the limit state in the neighborhood of the design point (i.e., a point

close to the most likely value of the design variables at failure). Then the R-F method can be applied easily. It is demonstrated that this scheme is much faster than the Monte Carlo method in producing reasonable estimates of p_f.

Methods of deriving safety check expressions for design codes and design criteria documents are studied. A Level I format employing partial safety factors derived from Level II methods is used to construct the safety check expressions which are suitable for code development. The procedures are demonstrated using numerous examples which include the problems where the limit states are complicated, i.e., the limit states are not explicitly defined.

CHAPTER 1

INTRODUCTION

1.1 Evolution of the Probabilistic Design Approach

It has been traditional in design practice, particularly in published codes and requirements, to use a safety check expression for a structural or mechanical component of the form

$$S_n < \frac{R_n}{\nu} \tag{1.1}$$

in which S_n and R_n are nominal stress and strength, respectively. ν is a factor of safety ($\nu > 1$). It is a single factor which accounts for all sources of uncertainty. Some examples are given in Table 1.1. The ratio R_n/ν is sometimes called working stress, or allowable stress and a code which employs such forms is called working stress design (WSD) or allowable stress design (ASD). In fact, this term is really the design strength. In general, safety factors can be applied either to strength, as in Eq. 1.1, or to the load (or stress). In the case of fatigue, factors of safety have been introduced by increasing the design life.

Factors of safety are used simply to insure safety in light of the uncertainties which are known to exist in a mechanical or structural design problem. Uncertainty can be roughly divided into statistical uncertainty (e.g., where data is available on material

TABLE 1.1

TYPICAL VALUES OF SAFETY FACTORS

	Safety Factor, ν (applied to yield strength)
Machinery	Between 1.5 and 2.0
Pressure Vessels*	1.6
Civil Structures	1.7 steel 6.0 wood

* ν = 4.0 relative to ultimate strength.

behavior, or loading environment) and nonstatistical uncertainty, i.e. that associated with assumptions made in the mathematical model. Typically, values of safety factors are based on engineering experience and intuition. WSD is generally effective and has served the engineering profession well, but it is suspected that it often results in over-designed components, an indication of the caution exercised by designers and the responsible authorities. Moreover, WSD produces systems in which components may have significantly different levels of risk. As a result, an optimum balance between reliability and economy is not always achieved.

In order that design criteria may be formulated more rationally to produce "optimum designs", a probabilistic and statistical approach has been suggested more than three decades ago by Freudenthal (1947). Serious interest in the systematic application of probabilistic and statistical methods to design developed after the mid-1950's. For example, the ASCE Task Committee on Factors of Safety were formed in 1955. This committee attempted to incorporate statistics into considerations for deriving safety factors. A summary of the final report of the Committee was published in 1966. Summaries which trace the evolution of thought in probabilistic design include the papers by Freudenthal (1956), Kececioglu and Cormier (1964), Freudenthal, Garrelt and Shinozuka (1966), Cornell (1969), Ang and Cornell (1974), Haugen and Wirsching (1975), Ravindra and Galambos (1978), Shinozuka (1983), and the books by Thoft-Christensen and Baker (1983) and Ang and Tang (1984).

Some of the advantages of adopting the probabilistic design approach are:

(1) Reliability is a more reasonable index of structural performance than the conventional safety factor.

(2) The effect on risk of making a design modification can be quantified.

(3) Factors which have nonstatistical uncertainty can be treated as random variables. Their effect on design can be quantified and their relative importance assessed. A rational basis for decisions regarding research program is available.

(4) All components can be designed to a balanced level of risk, thereby producing a more efficient system.

(5) Probability based information on mechanical performance can be used to develop rational policies towards pricing, warranties, spare parts requirements, etc.

In conclusion, probabilistic design approach has the promise of producing better engineered design, i.e., a design that is more safe, reliable and cost effective.

1.2 Basic Definitions

A limit state is a state beyond which a structure, or component, can no longer fulfill function for which it was designed. The limit states may be divided into tw(main groups, i.e., (1) ultimate limit states corresponding to the maximum load-carrying capacity, and (2) service limit states related to the normal use and durability.

Ultimate limit states can be reached as a result of, e.g., buckling due to instability, fatigue, etc. Service limit states can be reached as a result of excessive deflections, vibrations, etc. For a detailed description of structural behavior all possible limit states should be taken into consideration (Leporati, 1979).

Let $\underset{\sim}{X} = (X_1, X_2, \ldots X_n)$ denote the vector of design factors which are, in general, random variables. The limit state function or failure function, $g(\underset{\sim}{X})$, of the basic design variables can be defined such that the event of failure is,

$$g(\underset{\sim}{X}) \leq 0 \tag{1.2}$$

The limit state or failure surface defines the boundary between the "safe" and "failed" region in the design parameter space; $g(\underset{\sim}{X}) = 0$. The probability of failure may be defined as

$$p_f = P\{g(\underset{\sim}{X}) \leq 0\} \tag{1.3}$$

In general, the probability of failure can be written as

$$p_f = \int_\Omega f_{\underset{\sim}{X}}(\underset{\sim}{x})d\underset{\sim}{x} \tag{1.4}$$

where

$f_{\underset{\sim}{X}}(\underset{\sim}{x}) = f_{\underset{\sim}{X}}(x_1, x_2, \ldots x_n)$

= the joint probability density function

Ω = the failed region

Computation of p_f by Eq. 1.4 is called the "full distributional approach". The complexity of this multiple integral depends upon the number of the design variables, the form of the joint probability density function (pdf) and the form of the limit state which defines the integration domain. In a general design problem, however, solution to Eq. 1.4 is a practical impossibility. The theme of this dissertation is to explore efficient schemes for evaluating p_f.

Consider an alternative formulation of this problem. The condition of failure can be written as

$$\text{Strength } (\underset{\sim}{R}) \leq \text{Stress } (\underset{\sim}{S}) \tag{1.5}$$

where $\underset{\sim}{R}$ is a vector of the design parameters governing design strength and $\underset{\sim}{S}$ is a vector of the design parameters governing applied stress. Note that it is often an arbitrary decision which side of the expression a single variable should be placed. The vector of all the design factors is

$$\underset{\sim}{X} = (\underset{\sim}{R}, \underset{\sim}{S}) \tag{1.6}$$

and the limit state function is

$$g(\underset{\sim}{X}) = \text{Strength } (\underset{\sim}{R}) - \text{Stress } (\underset{\sim}{S}) \tag{1.7}$$

A fundamental limit state in probabilistic design involves a single strength and a single stress variable. The limit state function is

$$g(\underset{\sim}{X}) = R - S \tag{1.8}$$

and the limit state is

$$R = S \tag{1.9}$$

The probability of failure can be computed as (Eq. 1.4)

$$p_f = \int_{R\leq S} f_{RS}(r,s)drds \tag{1.10}$$

where f_{RS} is the joint pdf of R and S. Assume that R and S are statistically independent, Eq. 1.10 can be simplified as

$$p_f = \int_{-\infty}^{\infty} f_S(s)\{\int_{-\infty}^{s} f_R(r)dr\}ds = \int_{-\infty}^{\infty} F_R(s)f_S(s)ds \tag{1.11}$$

where

$$F_R(s) = \int_{-\infty}^{s} f_R(r)dr \tag{1.12}$$

is the cumulative distribution function (cdf) of the random variable R. Similarly, p_f can also be formulated as

$$p_f = \int_{-\infty}^{\infty} (1-F_S(r))\ f_R(r)dr \tag{1.13}$$

The dimension of the integral reduces to one. Some closed form solutions exist, and simple integration is always possible. In the special case where R and S are both normally distributed, a closed form solution is

$$p_f = \Phi\ (-\beta) \tag{1.14}$$

in which

$$\beta = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \tag{1.15}$$

where Φ is the standard normal cdf, μ and σ denote the mean and standard deviation, respectively; the subscripts identify the variable. β is defined as the safety index which is inversely related to p_f, i.e., larger β implies smaller p_f or higher reliability. A closed form solution is also available when R and S are both lognormally distributed, and is widely applied.

But as indicated above, both the strength and the stress may be a complicated function of several design parameters. Moreover, the limit state may not have an explicit form. Therefore, approximation techniques are usually required to estimate p_f.

1.3 Classification of Reliability Methods

Reliability methods are used for two basic problems, (1) to assess the safety of an existing design, and (2) to develop safety checking algorithms for design codes. The two are intimately related. Methods of reliability and safety checking have been divided into three broad cases depending upon the degree of sophistication applied to the treatment of the problems (Rackwitz, 1976), Level 3: The full distributional approach. The statistical distributions of all design factors are given. The exact probability of failure is computed as Eq. 1.4. If $p_f < p_o$, the maximum allowable p_f, the design is considered safe.

Level 2: Approximations are made to either p_f or an index of reliability. Distributions of the design factors may be known; perhaps only the mean and standard deviation are given. Generally, an idealization of the failure domain is required.

Level 1: An appropriate degree of reliability is provided for an element by applying "partial safety factors" to nominal values of the major variables. In the simplest form, a single factor of safety is used, e.g., Eq. 1.1. A simple example illustrating the partial safety factors format is $\gamma S_n \leq \phi R_n$ where γ is a load factor and ϕ is a resistance factor (Ravindra and Galambos, 1978).

Typically, a full distributional approach is impractical for all but the simplest cases. A Monte Carlo simulation is possible, but the simulation can become very costly particularly for the low values of p_f typical of structural problems. For this reason, research efforts have been emphasized in recent years to develop approximate methods, i.e., Level 2 and Level 1 methods.

1.4 The Goal and the Scope of the Present Study

The general goal of this study is to develop improved Level 2 and Level 1 methods for design of a single element structure. In general, it will be assumed that all design factors have known distribution functions. In Chapter 2 which follows, the reliability approximation methods are reviewed and their performance (in particular, the Rackwitz-Fiessler and Chen-Lind algorithms) are examined. In order to provide "exact" solutions for comparison purposes, a computer program was also developed to compute p_f by numerical integration for two variable ($n = 2$) cases. When $n \geq 3$, a Monte Carlo program (written by Colleen F. Kelly) was used to estimate p_f.

In practical design problems, it is not uncommon that the limit state function cannot be expressed by a closed form algebraic expression. The design parameters may be related through a differential equation, or a system of equations, etc. Very often, the limit state can only be defined using a numerical algorithm, e.g., Runge-Kutta method, local strain analysis, finite element analysis, etc. Reliability analysis using existing Level 2 methods would be extremely difficult or costly. Presented in Chapter 3 is a method to solve this class of problem, i.e., the cases where $g(\underset{\sim}{X})$ is not explicit. An example involving fatigue reliability analysis is solved using the proposed method.

In Chapter 4, a new algorithm for Level 2 reliability analysis is developed. The method attempts to produce improved p_f estimates relative to the Rackwitz-Fiessler and Chen-Lind methods based on the assumption of a linear limit state at the design point. A computer code was developed using this proposed numerical algorithm.

Methods of constructing partial safety factors (the Level 1 reliability method) are investigated in Chapter 5. This chapter describes procedures of establishing design requirements which employ advanced Level 2 methods. A code was written to compute partial safety factors based on the "design point" (at or near a point where the failure is most likely to occur) concept. Numerous examples are studied which include the cases where the limit states are not explicitly defined.

It is noted that statistical independence between each design random variable is assumed in this study. This, however, does not limit the methods to be applied to dependent variable cases because an orthogonal transformation can be made to replace the correlated variables by uncorrelated variables (Hasofer and Lind, 1974).

Recognizing the problems associated with the uncertainties, more experimental data are now available for statistical analysis. In analyzing the cycles to failure fatigue test data for many materials, it has been observed that enormous scatter of the data exists with coefficient of variations (standard deviation/mean) ranging typically from 20 to 70% (Wirsching and Wu, 1983). This observation strongly supports the relevance of reliability methods by suggesting that the approximate mathematical model to describe fatigue design factor is a probabilistic one rather than a deterministic one. This is a major reason that fatigue reliability problem is emphasized in this study. However, the methods and procedures developed are general, and can be equally applied to any mode of mechanical or structural failure.

CHAPTER 2

REVIEW OF THE RELIABILITY ANALYSIS METHODS

2.1 Introduction

Numerous methods have been suggested to describe structural reliability using a "reliability index" or "safety index" analogous to factor of safety. For example, a "safety index", β, based on the Mean Value First Order Second Moment (MVFOSM) theory was suggested (Cornell, 1969) and adopted in Mexican (Rosenblueth and Esteva, 1971) and Canadian (Parimi and Lind, 1976) code proposals. Subsequently, a principal criticism of this method was that the safety index is not invariant with respect to the mechanical formulation of the limit state. The invariance problem was later solved by Hasofer and Lind (1974) who introduced the concept of generalized safety index, as a "better" measure of the reliability. However, because the method requires only the mean and standard deviation of each design variable as input, no direct relationship between reliability and the safety index can be described. An exception is when the limit state is linear and the random variables are all normally distributed.

To extend the concept of the safety index, Rackwitz and Fiessler (1978) suggest that non-normally distributed variables can be transformed to "equivalent" normal variables in the Hasofer-Lind analysis. The refined safety index β is obtained through a convergent numerical process. An approximation to the actual probability of failure is provided by letting $p_f = \Phi(-\beta)$, where Φ is the standard normal distribution. The theory behind this well-known R-F algorithm was later formally investigated by Ditlevsen (1981) who called it the "principle of normal tail approximation". Chen and Lind (1982) refined the Rackwitz-Fiessler method by employing an additional scale parameter for the equivalent normal distribution. A more accurate approximation of the original distribution is promised.

The performance of the R-F and the C-L algorithm, both methods of "fast probability integration", are considered to be effective and accurate for general reliability purposes in design. The methods are fast in that numerical integration of p_f is avoided and require much less computation time compared with a Monte Carlo solution. Both algorithm may be categorized as "first order" reliability analysis method because the estimation of p_f has assumed that the limit state is linear at the design point.

"Second order" reliability analysis methods which estimate p_f by assuming a quadratic (second order) limit state on the standardized normal coordinates have also been suggested (Fiessler, Neumann and Rackwitz, 1979; Breitung, 1983; Tredt, 1983). Because a more accurate description of the limit state is used, the improvement of p_f estimation over first order method is promised. However, the techniques are in general very complicated and are relatively inefficient compared with first order methods.

In the remaining of this chapter, all the methods mentioned above will be reviewed. In particular, the performance of the R-F and C-L methods will be carefully examined.

2.2 Monte Carlo Method

In this work, Monte Carlo simulation will be used primarily as a research tool to provide a comparison basis, the procedures used in this study will be described herein.

Define

$$Z = g(\underset{\sim}{X}) \tag{2.1}$$

The event of failure is $g \leq 0$, where g is the failure function. The vector of n design factors is $\underset{\sim}{X}$. The procedure of obtaining an estimate of p_f is as follows: (1) generate a random sample of size K for each of the n design factors; (2) obtain a random sample of size K of g_i from the sampled $\underset{\sim}{X}$'s using the functional relationship, $g(\underset{\sim}{X})$; (3) sort the g's and construct the empirical distribution function of g; (4) then $p_f = P(g \leq 0)$. The estimate of p_f can be established by using a probability plot. The probability plot allows extrapolations to be made into low probability regions. But the cost of Monte Carlo can be substantially reduced by using as an estimate of p_f

$$p_f = n_o/K \tag{2.2}$$

where n_o is number of observations of g less than zero.

In theory, very accurate p_f estimates may be obtained by generating large samples (K). However, for structural and mechanical problems, it is usually not economical to do so because the exact p_f is typically small (e.g., 10^{-3} to 10^{-6}). A very large sample size would be required to yield sufficiently small confidence limits on p_f. The cost also increase depending upon (1) the number of design factors, and (2) the complexity of $g(\underset{\sim}{X})$. If $g(\underset{\sim}{X})$ is not explicitly defined, the cost becomes especially excessive. Nevertheless, the method is easy to apply and is able to provide p_f estimates for very complicated problems. Therefore, Monte Carlo is a valuable tool for research and for design.

2.3 Mean Value First Order Second Moment Method

Evaluation of p_f using the multiple integral (Eq. 1.4) requires that all the distributions of X are known. In many cases, only the mean and standard deviation are known to a reasonable degree of accuracy. Often, there is not enough evidence to define the distribution type. To provide a "measure" of reliability, "mean value first order second moment reliability method" (MVFOSM) may be used (Cornell, 1969). The method utilizes the mean and standard deviation (relates to second moment) of each design variable and a Taylor series expansion to linearize the failure function at the "mean" values. The safety index is defined as

$$\beta = \frac{\mu_Z}{\sigma_Z} \tag{2.3}$$

where

$$\mu_Z = g(\underset{\sim}{\mu}) \tag{2.4}$$

$$\sigma_Z = \sqrt{\sum_{i=1}^{n} \left(\frac{\partial g}{\partial X_i}\right)^2_{\underset{\sim}{\mu}} \sigma_i^2} \tag{2.5}$$

and where $\underset{\sim}{\mu}$ is a vector of mean values, σ_i is the standard deviation of the X_i variable.

Although only mean μ_Z and standard deviation σ_Z of Z are considered, the safety index defined in Eq. 2.3 provides a reasonable reliability measure. In general,a large value of β implies a larger reliability or smaller risk, p_f, as illustrated in Fig. 2.1. Therefore, the safety index plays the same role in probabilistic design as does the safety factor in conventional design. However, the relationship between β and p_f is not known exactly except for some special cases.

For example, assume that (1) all X_i are normally distributed, and (2) the limit state function has the linear form. Let

$$Z = g(\underset{\sim}{X}) = a_o + \sum_{i=1}^{n} a_i X_i \tag{2.6}$$

where a_o and a_i are constants. From probability theory, Z is also a normal variable with

$$\mu_Z = a_o + \sum_{i=1}^{n} a_i \mu_i \tag{2.7}$$

$$\sigma_Z = \sqrt{\sum_{i=1}^{n} a_i^2 \sigma_i^2} \tag{2.8}$$

which are the same as Eq. 2.4 and Eq. 2.5. The failure condition is $Z < 0$ and the probability of failure is

$$p_f = P(Z \leq 0) = P\left(\frac{Z - \mu_Z}{\sigma_z} \leq \frac{\mu_Z}{\sigma_Z}\right) \tag{2.9}$$

and

$$p_f = \Phi(-\beta) = P\left(-\frac{\mu_Z}{\sigma_Z}\right) \tag{2.10}$$

which provides a convenient "normal format". Thus β is inversely related to p_f given by Eq. 2.10. A plot using Eq. 2.10 is provided in Fig. 2.2 to show the relationship between β and p_f.

An exact functional relationship between β and p_f can also be obtained when (1) all X_i have lognormal distributions, and (2) $g(\underset{\sim}{X})$ is a multiplicative function of $\underset{\sim}{X}$,

$$g(\underset{\sim}{X}) = \prod_{i=1}^{n} a_o X_i^{a_i} \tag{2.11}$$

where a_o and all a_i are constants. Note that $g(\underset{\sim}{X})$ is defined such that the failure condition is $g \leq 1$. For example, if a limit state is R(strength) = S(stress), two equally valid formulations of $g(\underset{\sim}{X})$ are

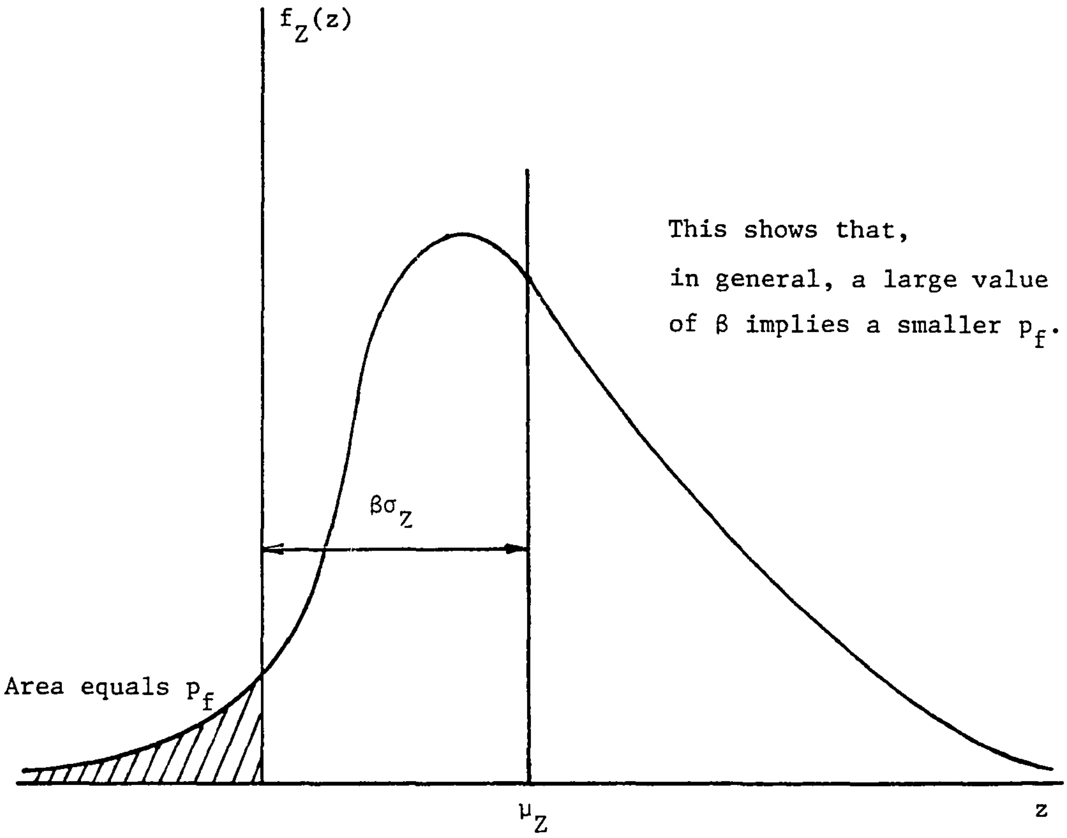


Figure 2.1 Density Function for Z.

$p_f = \Phi(-\beta)$

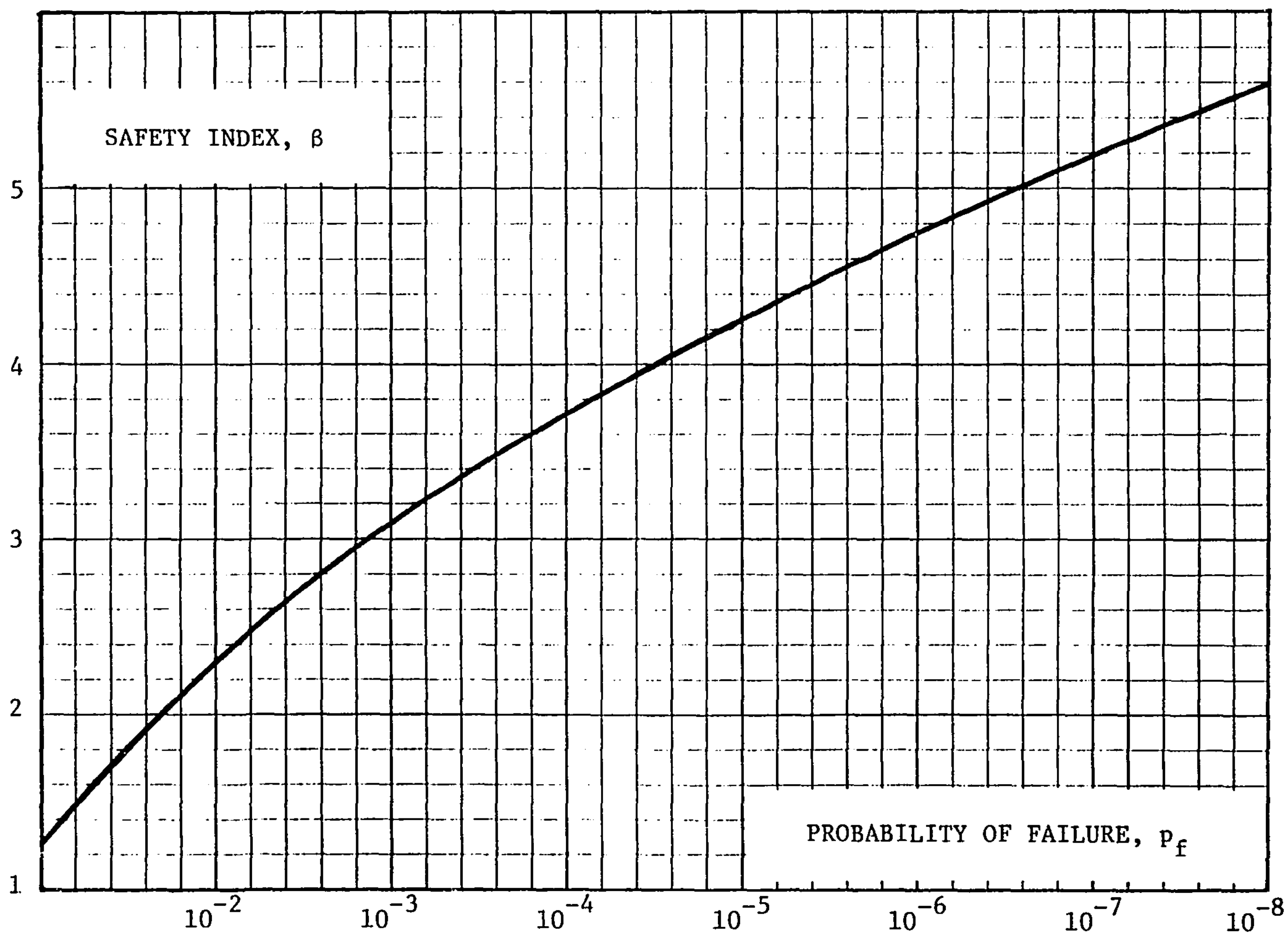


Figure 2.2 Failure Probability as a Function of Safety Index, β.

$$g(\underset{\sim}{X}) = R - S = 0 \tag{2.12}$$

and

$$g(\underset{\sim}{X}) = \frac{R}{S} = 1 \tag{2.13}$$

where Eq. 2.13 is suitable in the present derivation. To find the safety index, let $Z = \ln g$,

$$Z = \ln a_o + \sum_{i=1}^{n} a_i \ln X_i \tag{2.14}$$

The failure condition is $Z \leq 0$. Because X_i is lognormal, it follows that $\ln X_i$ is normal. Therefore, this "lognormal format" becomes identical to the normal format. Let $Y_i = \ln X_i$, the mean and standard deviation of Y_i are

$$\mu_{y_i} = \ln \tilde{X}_i \tag{2.15}$$

$$\sigma_{y_i} = \sqrt{\ln(1 + C_i^2)} \tag{2.16}$$

where $\tilde{X}_i$ and C_i are the median and coefficient of variation (COV) of X_i. Using Eq. 2.15 and Eq. 2.16,

$$\mu_Z = \ln \left(\prod_{i=1}^{n} a_o \tilde{X}_i^{a_i} \right) \tag{2.17}$$

$$\sigma_Z = \sqrt{\ln[\prod_{i=1}^{n} (1 + C_i^2)^{a_i^2}]} \tag{2.18}$$

and p_f is given by Eq. 2.10 with $\beta = \mu_Z/\sigma_Z$. Because the lognormal distribution plays an important role in probabilistic design, the lognormal format is particularly useful. For example, in fatigue design, some of the important equations tend to be multiplicative (e.g., Wirsching, 1981).

Owing to its operational simplicity, the MVFOSM method has advantages in practice. However, it has been criticized that the distributional information (i.e., the type) of the design variable can not be used even if available. Consequently, the relation between β and p_f is not known. The other criticism, and the major one, was that β fails to be invariant to equally valid formulations of the limit state.

To illustrate the invariance problem, consider that a simple tension member with cross-sectional area A has to withstand an applied loading Q. The strength is R and the stress is S = Q/A. The limit state can be formulated in two ways.

$$Z = R - Q/A = 0 \tag{2.19}$$

or

$$Z = RA - Q = 0 \tag{2.20}$$

Assume that Q is deterministic, R and A are random variables, it can be shown (Fig. 2.3) that evaluation of β using MVFOSM method may produce a different value in each case. Therefore, β is dependent upon how the designer formulates the limit state. The lack of invariance problem was overcome by the following advanced method.

2.4 The Hasofer-Lind "Minimum Distance" Method

Hasofer and Lind (1974) introduced the "generalized safety index" as a solution to the lack of invariance problem. The method is described as follows:

1. Define reduced variables

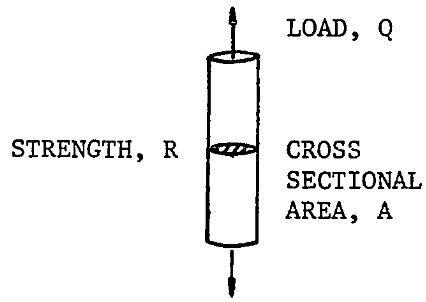
$$u_i = \frac{X_i - \mu_i}{\sigma_i} \quad , \quad i = 1, n \tag{2.21}$$

   in which μ_i and σ_i are the mean values and standard deviations. u_i is therefore a random variable with zero mean and unity standard deviation.

2. Using Eq. 2.21, transform the original limit state, $g(\underset{\sim}{X}) = 0$, to the reduced limit state

$$g_1(\underset{\sim}{u}) = 0 \tag{2.22}$$

3. Measure the "minimum distance" (defined as the generalized safety index, β) from the origin of the axes, on the reduced coordinate system, to the failure surface, i.e.,



STATISTICS OF DESIGN VARIABLES

$R \sim (62,\ 6.2)$

$A \sim (2.80,\ .14)$

$Q = 100$

NOTE: No distribution is assumed for the design variables.

<table>
<tr><th colspan="2">FAILURE FUNCTION</th></tr>
<tr><td>FORMULATION 1
$Z = R - Q/A$</td><td>FORMULATION 2
$Z = RA - Q$</td></tr>
<tr><th colspan="2">MEAN AND STANDARD DEVIATION OF THE FAILURE SURFACE</th></tr>
<tr><td>$\mu_Z = \mu_R - Q/\mu_A$
$= 26.286$
$\sigma_Z = (\sigma_R^2(1) + (\frac{Q}{\mu_A^2})^2\sigma_A^2)^{\frac{1}{2}}$
$= 6.452$</td><td>$\mu_Z = \mu_R\mu_A - Q$
$= 73.60$
$\sigma_Z = (\sigma_R^2(\mu_A)^2 + \sigma_A^2(\mu_R)^2)^{\frac{1}{2}}$
$= 19.41$</td></tr>
<tr><th colspan="2">PROBABILITY OF FAILURE</th></tr>
<tr><td>$\beta = \mu_Z/\sigma_Z = \frac{26.286}{6.452}$
$= 4.07$</td><td>$\beta = \mu_Z/\sigma_Z = \frac{73.60}{19.41}$
$= 3.79$</td></tr>
<tr><td colspan="2">Note: Both expressions of Z are equally valid descriptions of the failure surface (or limit state). But each gives a different reliability.</td></tr>
</table>

Figure 2.3 Example of Lack of Invariance in MVFOSM Format.

$$\beta = \min \sqrt{\sum_{i=1}^{n} u_i^2} \tag{2.23}$$

The point on the failure surface which defines β is called the "design point" (or safety-checking point).

Consider a special case in which all X_i are normally distributed and the limit state is linear as defined in Eq. 2.6, the reduced limit state applying Eq. 2.21 becomes

$$g_1(\underset{\sim}{u}) = a_o + \sum_{i=1}^{n} a_i(u_i\sigma_i + \mu_i)$$

$$= (a_o + \sum_{i=1}^{n} a_i\mu_i) + \sum_{i=1}^{n} a_i\sigma_i u_i \tag{2.24}$$

From theory of calculus, the minimum distance, β,from the origin to the "hyperplane" is

$$\beta = \frac{a_o + \sum_{i=1}^{n} a_i\mu_i}{\sqrt{\sum_{i=1}^{n} a_i^2 \sigma_i^2}} \tag{2.25}$$

where the numerator and the denominator are precisely μ_Z and σ_Z, respectively, as defined in Eq. 2.7 and Eq. 2.8. Therefore, β is the same as derived from MVFOSM method and an exact relationship between p_f and β is given by Eq. 2.10. Under other conditions, however, p_f computed using Eq. 2.10 can only be interpreted as a "notional" probability of failure.

The Hasofer-Lind safety index is invariant because the exact failure surface are considered, while linearization of the failure surface at mean values is required using MVFOSM method. To find the minimum distance for a nonlinear limit state, an optimization routine may be used to solve Eq. 2.22 and Eq. 2.23. For a simple two variable case, β may be determined graphically. An example is provided in Fig. 2.4, where R and T are the random variables having mean values and standard deviations of (45, 3) and (20, 2), respectively. β may also be found by iteratively solving the following equations (Leporati, 1979, p. 46).

$$\lambda_i = \frac{\left(\frac{\partial g}{\partial u_i}\right)_{\underset{\sim}{u}^*}}{\sqrt{\sum_{j=1}^{n} \left(\frac{\partial g}{\partial u_j}\right)^2_{\underset{\sim}{u}^*}}} \tag{2.26}$$

$$u_i^* = \beta\lambda_i \tag{2.27}$$

$$g(\lambda_1\beta, \lambda_2\beta, \ldots \lambda_n\beta) = 0 \tag{2.28}$$

where the partial derivatives are calculated at the design point, $\underset{\sim}{u}^*$, λ_i are the direction cosines of the vector that defines the design point. Applying the method, a design would be considered as safe, if $\beta > \beta_o$, the target safety index. The target safety index is the minimum allowable, specified for a safe design. The method is justified as follows: it is assumed that most of the joint probability

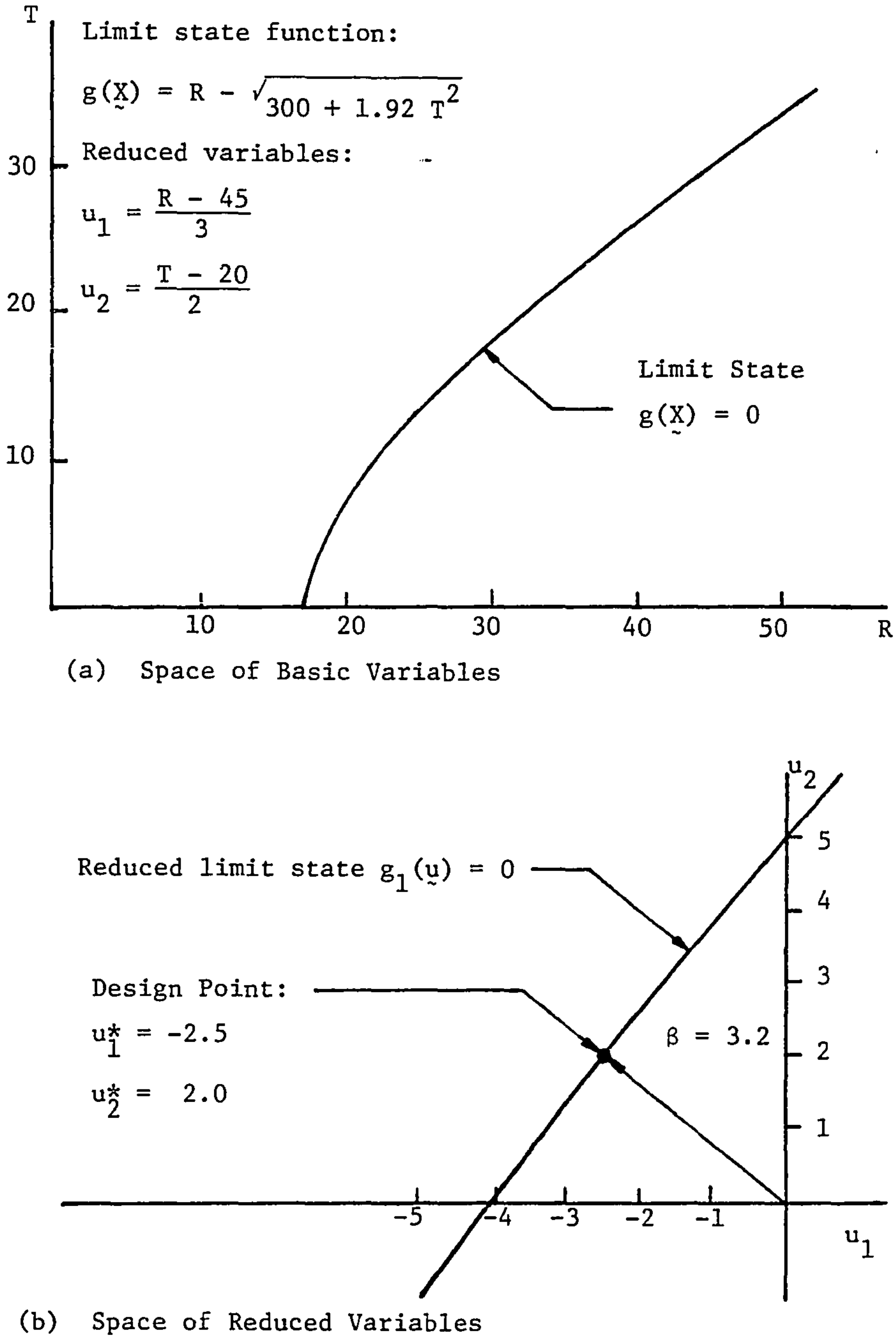


Figure 2.4 An Example of Computing the Generalized Safety Index.

density will be concentrated in the "hypersphere", in the space of reduced coordinates, having radius β (within the sphere is the safety domain). Therefore, β relates closely to the reliability. Moreover, the use of the reduced variables ensures that the measurements along the different directions in the reduced coordinates will be nondimensional and comparable.

In conclusions, the Hasofer-Lind method provides a reasonable index of reliability which is invariant to the formulation of the limit state. However, this method uses only the mean and standard deviation. Even if distributional information is available, it must be discarded. It is precisely this criticism of the Hasofer-Lind method which motivated the development by Rackwitz and Fiessler described in the next section.

2.5 Rackwitz-Fiessler (R-F) and Chen-Lind (C-L) Algorithms

Rackwitz and Fiessler (1978) suggest a method which extends the Hasofer-Lind safety index concept to accommodate distributional information of the design factors. This method transforms a non-normal distribution into "equivalent" normal distribution by adjusting the mean and standard deviation so that the cdf and pdf of the original and the equivalent distributions are equal at the design point. The detailed scheme as implemented in this study is summarized in Appendix A. The original University of Arizona computer program for the Hasofer-Lind safety index was written by Vannoni (1979), and extended by Kelly (1981) to include the R-F equivalent distribution concept. Limit states with design factors having normal, lognormal, Weibull and extreme value distributions can be analyzed using this program. The program produces a safety index β which in turn produces reasonably accurate estimates of p_f using Eq. 2.10 (Ellingwood et al., 1980).

An extension of the R-F scheme was proposed by Chen and Lind (1981). This method uses a three-parameter (scaled) equivalent normal distribution. The scale factor was used to match the slope of the pdf, in addition to cdf and pdf, at the design point. It was anticipated that this C-L method can produce more accurate estimates of p_f than does R-F. The detailed scheme as implemented in this study is summarized in Appendix B. The University of Arizona program mentioned above can also accommodate the C-L method.

The performance of the R-F and the C-L algorithms for estimating probability of failure, was carefully examined (Wu et al., 1983). Examples which include failure functions having linear or nonlinear limit states with design factors having various combinations of distributions are presented in Appendix C. Following are the conclusions drawing from the observations.

In general, the Rackwitz-Fiessler (R-F) and Chen-Lind (C-L) estimates of the safety index β and probability of failure p_f are reasonably close to the exact values. Results for linear and nonlinear limit states suggest that the R-F and C-L methods are practical alternatives for computing probabilities of failure.

Because the C-L algorithm uses an additional parameter, the scheme has the promise of improving R-F estimates. In some cases, C-L performed better than R-F. But, in general, one cannot conclude from the evidence presented here that one method consistently produces better p_f estimates than the other.

Perhaps the most important attribute of both R-F and C-L is its efficiency in terms of computer time as compared to the numerical integration or Monte Carlo technique. In making such comparisons, it is valid only to compare central processing unit (CPU) time as the measure of efficiency because the compilation and input-output time is a fixed overhead in the computational process. Running on the CYBER 175, R-F is approximately seven times as efficient as the equivalent numerical integration method for problems involving two variables. For complex non-linear limit states, examples have shown that R-F is approximately 50 times as efficient as Monte Carlo. In practice, values of p_f lower than those used in some examples would be more typical, and Monte Carlo may even be substantially slower.

2.6 First Order Versus Second Order Reliability Methods

In general, reliability analysis can be performed by transforming the basic variables, X_i, to the standard normal variables, u_i. The transformation is (Rosenblatt, 1952)

$$u = \Phi^{-1}[F_X(X)] \tag{2.29}$$

and the inverse transformation is

$$X = F_X^{-1}[\Phi(u)] \tag{2.30}$$

For example, if X has a type I extreme value distribution with mean μ and standard deviation σ, Eq. 2.29 is

$$u = \Phi^{-1}\left[e^{-e^{-\frac{1.283(X-\mu)}{\sigma} - 0.5772}}\right] \tag{2.31}$$

The inverse tranformation is

$$X = \left[\frac{-\ln[-\ln\Phi(u)] - 0.5772}{1.283}\right]\sigma + \mu \tag{2.32}$$

By substituting X_i into the original limit state function, $g(\underset{\sim}{X})$, the transformed limit state function, $g_1(\underset{\sim}{u})$, can be formulated. However, the inverse transformation is generally nonlinear; its form can be extremely complicated. The mathematical and numerical problem may be extremely complicated and for non-continuous distribution functions, it is even impossible.

By a Taylor series expansion of Eq. 2.29 at the design point, (considering linear terms only), it can be shown (e.g., Leporati, 1977, p. 66) that the transformation is approximated by equating the cdf's and the pdf's at the design point, precisely the R-F scheme. To obtain the minimum distance, β, the R-F algorithm is recognized as efficient because it does not require the knowledge of the exact limit state in the u space. More importantly, the equivalent normal concept avoids the difficulties associated with the inverse transformation.

Note that the R-F and C-L methods are called "first order" methods because they assume that the limit state is linear at the design point in computing p_f. The shortcoming of the R-F method is that $p_f = \Phi(-\beta)$ is exact only if the "transformed" limit state is linear. The error increases if, (1) the original distribution function deviates significantly from the normal, and (2) the original limit state is highly nonlinear. As suggested by Fiessler et al., (1979), the curvature caused by the distribution transformation may overrule those given by the mechanical problem. Fig. 2.5 shows how a linear limit state, on the x space, may become very nonlinear, on the u space, due to a non-normal variable. The problem with the R-F method motivated studies of the quadratic limit state formulations.

The linear limit state, $g_1 = 0$, and the "hypersphere", with radius, β, as illustrated in Fig. 2.6 for a two dimensional case, construct the bounds for the exact p_f for convex failure surface,

$$\Phi(-\beta) < p_f < 1 - \chi_n^2\ (\beta^2) \tag{2.33}$$

where χ_n^2 is the chi-squared distribution with n degrees of freedom. Because the upper bound is usually too conservative, improvements to the first order method have been suggested by various authors to use the higher order limit state approximation. For example, Ditlevsen

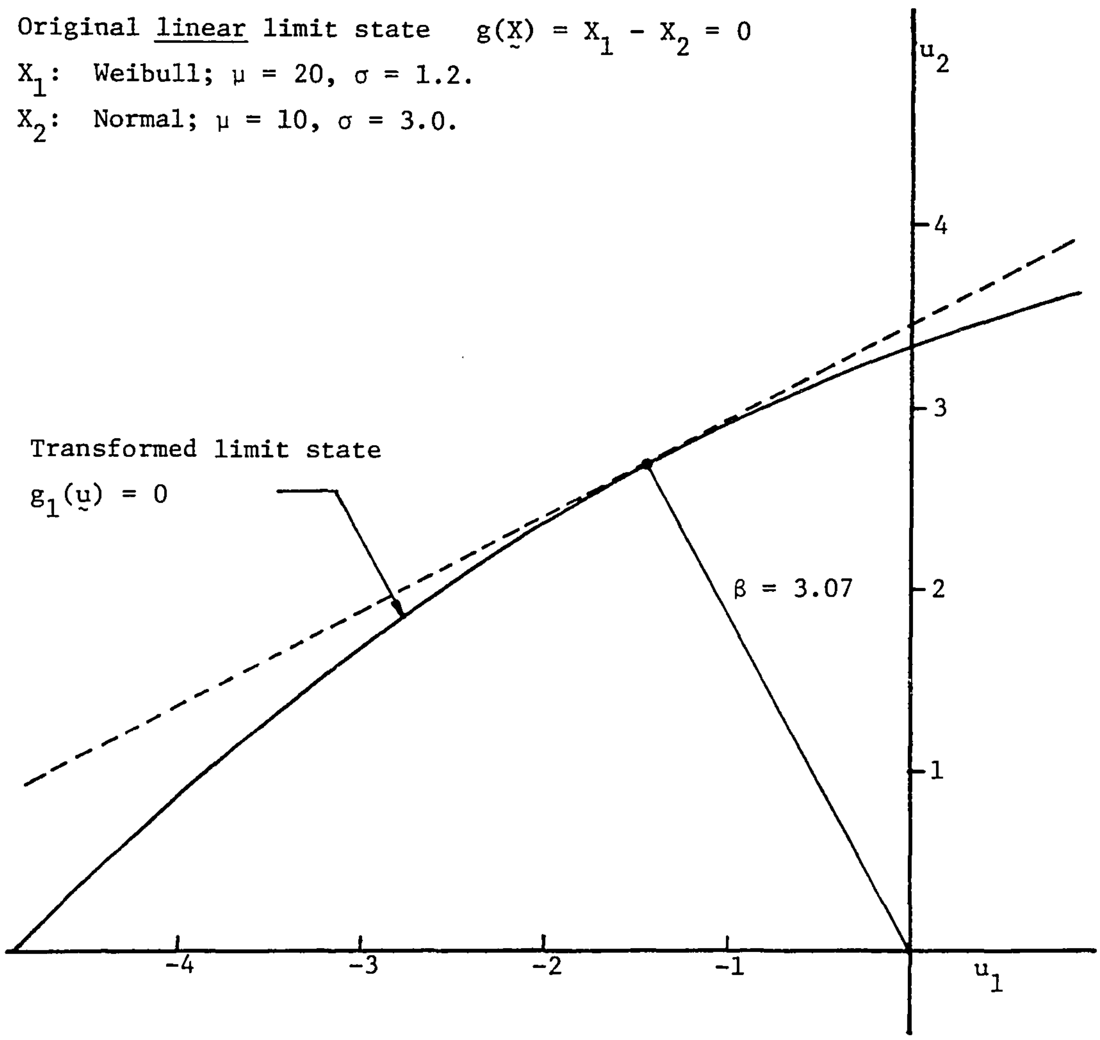


Figure 2.5 Exact Transformation of a Linear Limit State.

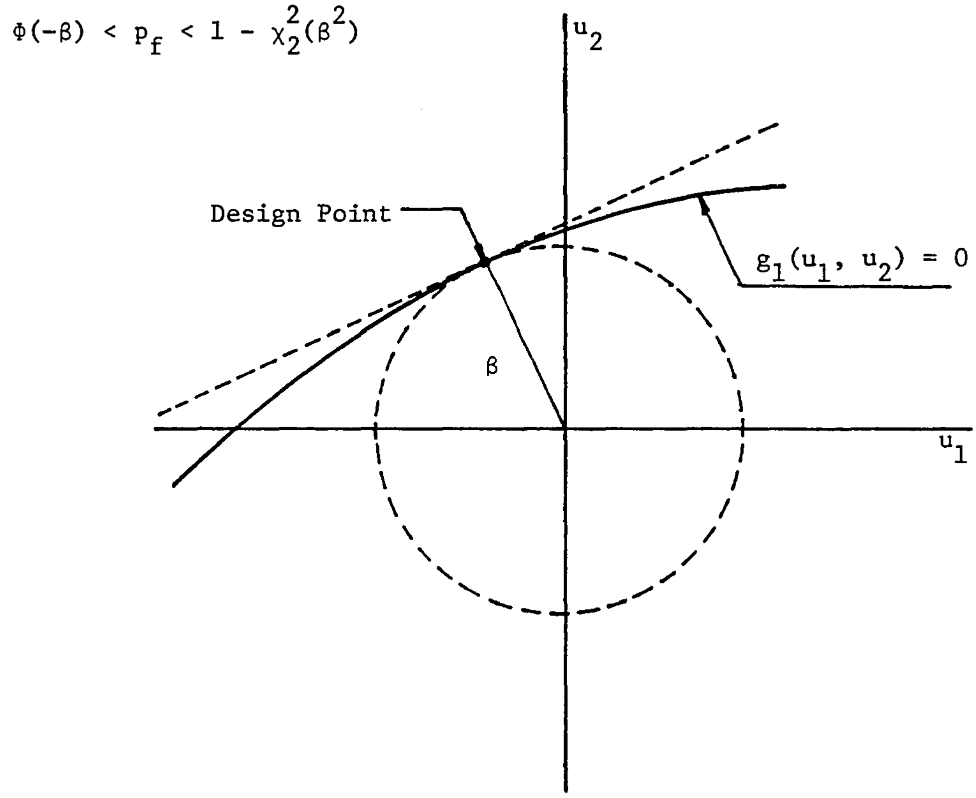


Figure 2.6 The Bounds of the Probability of Failure for Convex Failure Surface.

(1976) developed sharper bounds by inscribing and circumscribing the limit state with rotational paraboloids. Horn and Price (1977) investigated the error of the linear approximation by studying an approximating hypersphere with radius corresponding to the mean curvature in the design point. In general, if the shape of g_1 is completely known, it is possible to choose a most suitable approximating limit state (e.g., a parabolic, elliptical or hyperbolic) to generate sharper bounds.

To avoid the arbitrariness of the choice of a suitable approximating limit state, Fiessler et al., (1979) investigated several possible forms of the quadratic limit state,

$$g_1(\underset{\sim}{u}) = \sum_{i=1}^{n} \left.\frac{\partial g(\underset{\sim}{u})}{\partial u_i}\right|_{\underset{\sim}{u}^*} (u_i - u_i^*)$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \left.\frac{\partial^2 g(\underset{\sim}{u})}{\partial u_i^2}\right|_{\underset{\sim}{u}^*} (u_i - u_i^*)^2$$

$$+ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left.\frac{\partial^2 g(\underset{\sim}{u})}{\partial u_i \partial u_j}\right|_{\underset{\sim}{u}^*} (u_i - u_i^*)(u_j - u_j^*) = 0 \qquad (2.34)$$

where the partials are evaluated at the design point, $\underset{\sim}{u}^*$. The general formulations of computing p_f are provided on the basis of the noncentral chi-square distribution and other distributions of quadratic forms (Imhof, 1961; Johnson and Kotz, 1970; Ruben, 1962 and 1963; Press, 1966). Approximation solutions for rotational paraboloids and hyperspheres have been provided which require the curvature information of the limit state in addition to the minimum distance.

Breitung (1983) derived an asymptotic formula

$$\lim_{\beta\to\infty} p_f = \Phi(-\beta) \prod_{i=1}^{n-1} (1-\beta\kappa_i)^{-\frac{1}{2}} \tag{2.35}$$

where κ_i are the main curvatures at the design point. Tvedt (1983) derived two approximation formulas by approximating the failure surface with a parabolic surface at the design point which give more closer bound than Eq. 2.35.

Because a more accurate information of the limit state is used, these second order methods have the promises of consistently producing better estimates of p_f relative to the first order methods. However, these schemes are more complicated because the formulations are made on the transformed space and require second partial derivatives of g_1. Therefore, the methods are expected to be more time-consuming than the first order methods, particularly when the limit state is complicated and the number of the design variables is high. Furthermore, the inverse cdf transformation may be computationally difficult for some statistical distribution functions.

In the literature, only several simple examples have been presented (Fiessler et al.,1979; Breitung, 1983; Tvedt, 1983). The evidence is not entirely convincing that quadratic methods could produce consistently accurate results relative to other methods. As an example, it has been shown that due to the "non-normality" of the basic variables, an originally linear limit state can become drastically distorted on the u space such that more than one "local" minimum distance exist which then requires the method of "multiple point checking" (Hohenbichler and Rackwitz, 1981). This suggests that,sometimes,a higher than second order approximation may be required to accurately describe the limit state even if there is only one local minimum distance.

On the other hand, the first order methods of R-F and C-L seem to perform well in many engineering applications in spite of the linear approximation. First order methods are easy to program and computer costs are negligible. The more complicated inverse problem of computing partial safety factors (discussed in Chapter 5) for different design is easily programmed. It appears that it would be a formidable task to produce a general purpose code which would be easy for designer to use. Moreover, a partial safety factor program to be used for code development may be even an order of magnitude more complicated when using the second order methods. Therefore, first order reliability methods, featured by conceptual simplicity and efficiency, may be more practical than the second order reliability methods for general engineering design purpose.

In Chapter 4,an algorithm will be proposed which provides improvement to the R-F procedure for estimating p_f so long as the "original" limit state is approximately linear in the neighborhood of the design point. Nonlinearity problems resulting from non-normal distributions are reduced relative to R-F. The proposed method of the design factors retains the important feature of the first order methods, i.e., no inverse transformation is involved.

CHAPTER 3

A METHOD OF RELIABILITY ANALYSIS WHEN THE LIMIT STATE HAS NO EXPLICIT FORM

3.1 Introduction

Addressed in Chapter 2 was the problem of estimating the probability of failure (p_f) for a structural or mechanical component whose design factors are modelled as random variables, and where the failure function had an explicit algebraic form. The goal of this chapter is to perform reliability analysis for the case where relationship between the design factors for a particular mode of failure is so complicated that it can be defined only using a computer algorithm, e.g., finite element analysis, local strain analysis. Methods described in Chapter 2 i.e., the Rackwitz-Fiessler and Chen-Lind cannot be applied unless the failure function is explicit. Monte Carlo simulation is able to provide approximate solutions to p_f for this difficult problem. Unfortunately, Monte Carlo is costly, particularly when additional computation must be made to define the limit state. To compound the problem, very large sample sizes are required in order to get sufficient resolution in the tail areas to approximate the low probabilities typical of structural application.

To overcome the practical limitations of Monte Carlo, a simple, and easily constructed, second degree polynomial which approximates the complicated limit state in the neighborhood of the design point is suggested. A simple curve fitting routine provides an analytical form which relates the design variables at selected points. Then the R - F scheme can be used to estimate p_f.

An example presented herein is a computation of reliability for a fatigue problem for which the local strain approach is used to determine cycles to failure in a notched member to which a constant amplitude oscillatory load is applied. A closed form expression of cycles to failure N as a function of the remaining design factors does not exist, and a computer program is the only practical way of computing N. The details of computing fatigue initiation life is described by Collins (1981) and by Fuchs and Stephens (1980).

The failure function of the local strain analysis could be expressed implicitly as

$$S = h(\sigma_f', \varepsilon_f', N, ..) \tag{3.1}$$

in which S is the applied stress, σ_f' and ε_f' are fatigue strength and fatigue ductility coefficient respectively. Given that the design life is N_o, the event of failure is $(N \leq N_o)$ and the limit state is $(N=N_o)$. If the closed form expression of h were given the R-F algorithm could be used effectively to estimate p_f given $(N=N_o)$. The scheme presented here is to construct an approximation of h in the neighborhood of the design point and then use the R-F method. But the procedure is general and can be applied to any mode of structural or mechanical failure.

Computer costs for a Monte Carlo solution would be excessive because the local strain program would have to be run for each random sample point (maybe 10,000 times). A limited Monte Carlo analysis is employed later to check the results of the proposed method.

3.2 Polynomial Approximation to Failure Function

If the implicitly defined failure function can be approximated by a simple (e.g., polynomial) equation, β can be determined effectively using R-F alogrithm. The entire process of computing β and then p_f is summarized by the block diagram of Fig. 3.1. The method will be developed and demonstrated by solving a sample problem.

Assume, in general, that there are n + 1 variables involved in a failure function expression

$$g(X_o, X_1, \ldots X_n) = 0 \tag{3.2}$$

Assume also that the failure function is differentiable with continuous partial derivatives in the whole domain. By expanding it at a point $\underset{\sim}{a} = (a_o, a_1, \ldots a_n)$ by Taylor series, the following polynomial results:

$$g(\underset{\sim}{X}) = g(\underset{\sim}{a}) + \sum_{i=0}^{n} \left.\frac{\partial g}{\partial X_i}\right|_{\underset{\sim}{a}} (X_i - a_i) + \frac{1}{2} \sum_{i=0}^{n} \left.\frac{\partial^2 g}{\partial X_i^2}\right|_{\underset{\sim}{a}} \cdot$$

$$(X_i - a_i)^2 + \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} \left.\frac{\partial^2 g}{\partial X_i \partial X_j}\right|_{\underset{\sim}{a}} (X_i - a_i)(X_j - a_j)$$

$$+ \text{H.O.T. (higher order terms).} \tag{3.3}$$

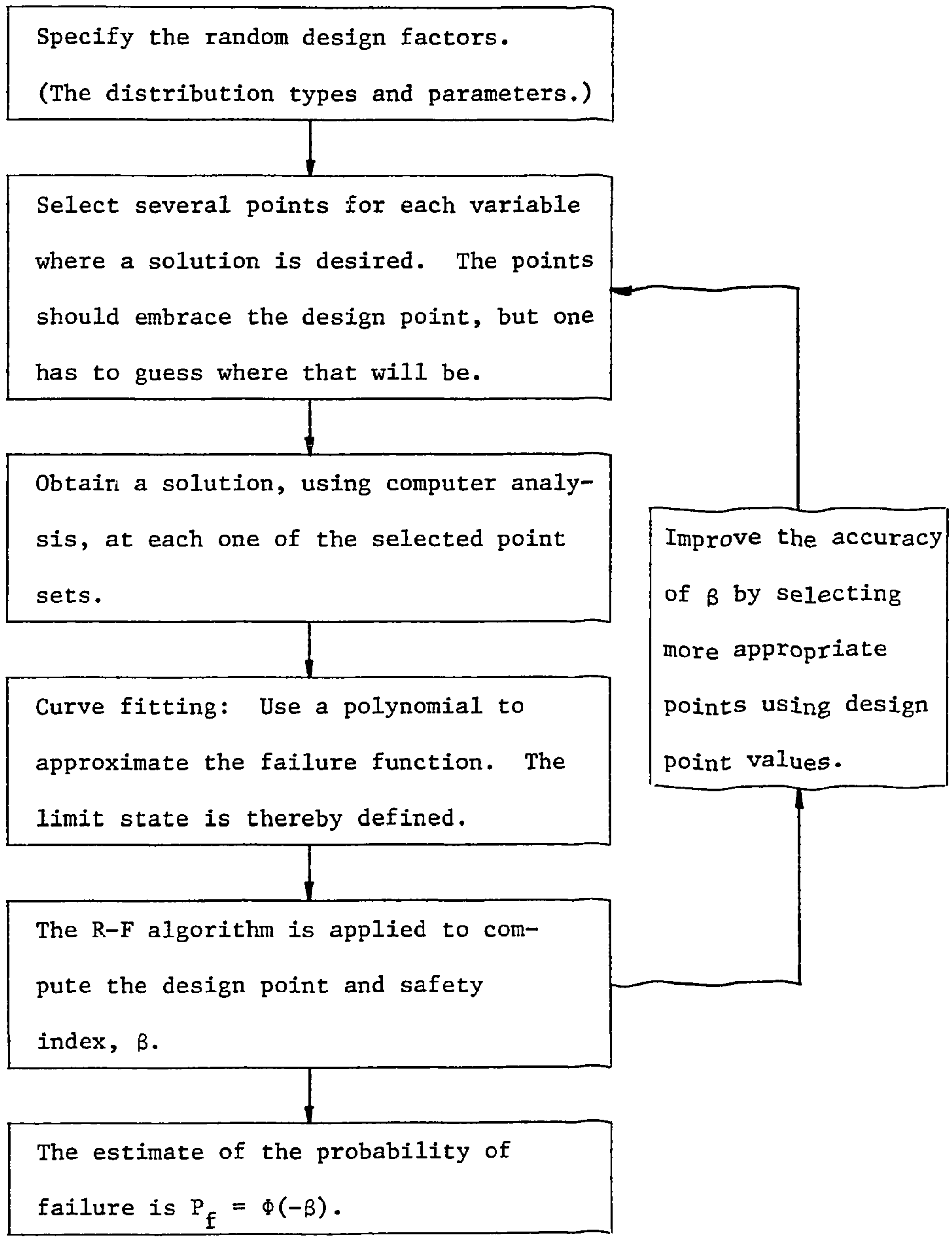


Figure 3.1 Block Diagram for Computation of the Safety Index, β, when the Failure Function is not Available as a Closed Algebraic Form.

In order to compute β using an optimization scheme of the R-F algorithm (Appendix A), it is necessary to rewrite the equation such that any one of the variables is a function of the remaining variables. Thus Eq. 3.2 becomes, when one chooses X_o,

$$X_o = h\,(X_1, X_2, \ldots X_n) \tag{3.4}$$

such that the failure function is

$$g(\underset{\sim}{X}) = X_o - h\,(X_1, X_2, \ldots X_n) = 0 \tag{3.5}$$

By substituting Eq. 3.5 into Eq. 3.3, it follows that

$$X_o = h(\underset{\sim}{a}) + \sum_{i=1}^{n} \left.\frac{\partial h}{\partial X_i}\right|_{\underset{\sim}{a}} (X_i - a_i) + \frac{1}{2} \sum_{i=1}^{n} \left.\frac{\partial^2 h}{\partial X_i^2}\right|_{\underset{\sim}{a}} \cdot$$

$$(X_i - a_i)^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left.\frac{\partial^2 h}{\partial X_i \partial X_j}\right|_{\underset{\sim}{a}} (X_i - a_i)(X_j - a_j)$$

$$+ \text{H.O.T.} \tag{3.6}$$

Deleting the higher order terms and simplifying,

$$X_o = \alpha_o + \sum_{i=1}^{n} \alpha_i\,(X_i - a_i) + \sum_{i=1}^{n} \alpha_{n+i}(X_i - a_i)^2$$

$$+ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \alpha_{(n-\frac{i}{2})(i+1)+j}(X_i - a_i)(X_j - a_j) \tag{3.7}$$

in which the α's are defined by comparing Eq. 3.6 and Eq. 3.7, each α (except α_o) relates to each partial-derivative term in Eq. 3.6. In the last term of Eq. 3.7, the subscript of α, denoted by k, is $k = n(n + 3)/2$. Therefore, the total number of terms, $k + 1$, equals $(n + 2)(n + 1)/2$. Eq. 3.7 can be further simplified as

$$X_o = \alpha_o + \sum_{i=1}^{k} \alpha_i Y_i \tag{3.8}$$

in which
$$\begin{aligned} Y_i &= X_i - a_i \quad (\text{for } i = 1, n) \\ &= (X_i - a_i)^2 \quad (\text{for } i = n + 1, 2n) \\ &= (X_i - a_i)(X_j - a_j) \quad (\text{for } i = 2n + 1, k) \end{aligned}$$

This is the familiar multiple linear regression model in which Y_i are independent random variables. [The model is linear in the parameters (the α's) regardless of the shape of the surface it generates.] The least square normal equation may be written in matrix notation as (see e.g., Montgomery, 1982)

$$[Y]^T[Y]\,[\alpha] = [Y]^T[X_o] \tag{3.9}$$

in which

$$[Y] = \begin{bmatrix} 1 & Y_{11} & Y_{12} & \cdots & Y_{1k} \\ \cdot & Y_{21} & {}_{22} & \cdots & Y_{2k} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & Y_{m1} & {}_{m2} & \cdots & Y_{mk} \end{bmatrix}, \qquad (3.10)$$

$$[\alpha] = \begin{bmatrix} \alpha_o \\ \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_k \end{bmatrix}, [X_o] = \begin{bmatrix} (X_o)_1 \\ (X_o)_2 \\ \cdot \\ \cdot \\ \cdot \\ (X_o)_m \end{bmatrix}$$

Note that there are k + 1 normal equations, one for each of the unknown α_i. m is the number of observations or data sets. Given the data sets Y_{ij} and $(X_o)_i$, the solution will be the least square estimators, α_i.

3.3 Selection of Data Points

Assume that the failure function is to be approximated by a truncated polynomial equation. In a general data analysis problem, data obtained from experiments are subject to random errors.

A statistical analysis of estimators α_i are required to validate the model. In such cases, accuracy of the model will be heavily dependent upon the sample size, m. In this study, however, the data sets $Y_{ij}, (X_o)_i$ are those values which satisfy the exact failure function. Therefore, smooth functions are anticipated and statistical analysis is not necessary. The number of data sets for curve fitting can be chosen relatively small. More precisely, the minimum number of data sets (m) is equal to the number of α_i.

The key to constructing an effective approximating model is to choose, for curve fitting, data points (centered around the expanding point $\underset{\sim}{a}$) from a domain that is relatively small to give satisfactory agreement between the exact and the approximating limit state, but wide enough to cover the unknown design point. When R-F algorithm is used, the model need be accurate only in the region of the design point rather than the whole domain. If the expanding point, $\underset{\sim}{a}$, which must be chosen at the outset, is relatively close to the design point, "accurate" results can be expected.

In the general case, the design point may be difficult to predict in advance. However, it is reasonable to assume that design point values of stress variables would be greater (may be much greater) than its mean or median. Similarly, design point values of strength variables would be less than its mean. Hence, one can at least make a reasonable first guess of the neighborhood of the design point. Another observation which is useful is that the failure functions are expected to be relatively smooth and unlikely to have discontinuities or other significant nonlinearities in the neighborhood of the design point. Therefore, it seems reasonable to use a quadratic polynomial model to approximate the failure function. On the other hand, it is worth mentioning that $p_f = \Phi(-\beta)$ is a good approximation when the limit state surface can be approximated by a tangent hyperplane at the design point. In such cases it is evident that a quadratic model will be sufficient to approximate the limit state in the region close to the design point.

As an example of solving α_i using a mimimum of data points, consider the case where there are four design variables involved. If the quadratic polynomial without interaction terms (or mixed terms, e.g., $(X_1 - a_1) \cdot (X_2 - a_2)$, . . . etc.) is chosen, the model is

$$X_o = \alpha_o + \alpha_1(X_1 - a_1) + \alpha_2(X_2 - a_2) + \alpha_3(X_3 - a_3) + \alpha_4(X_1 - a_1)^2 + \alpha_5(X_2 - a_2)^2 + \alpha_6(X_3 - a_3)^2 \quad (3.11)$$

The expanding point $\underset{\sim}{a}$ has been chosen and it is required to compute the estimates, $\hat{\alpha}_o$ . . . $\hat{\alpha}_6$. These estimates can be computed by calculations relating the design parameters at the following specific points.

X_1	X_2	X_3	X_o
a_1	a_2	a_3	$(X_o)_1$
$a_1 + \Delta X_1$	a_2	a_3	$(X_o)_2$
$a_1 - \Delta X_1$	a_2	a_3	$(X_o)_3$
a_1	$a_2 + \Delta X_2$	a_3	$(X_o)_4$
a_1	$a_2 - \Delta X_2$	a_3	$(X_o)_5$
a_1	a_2	$a_3 + \Delta X_3$	$(X_o)_6$
a_1	a_2	$a_3 - \Delta X_3$	$(X_o)_7$

The ΔX_i's must be chosen. They must be large enough to embrace the design point, but not too large so as to introduce significant errors in the polynomial approximation. Here the analyst must again use his intuition.

Finally, the estimates of α's are computed by noting the following: $\hat{\alpha}_o$ is the value of X_o at the expanding point; $\hat{\alpha}_1$ and $\hat{\alpha}_4$ can be determined when $X_2 = a_2$ and $X_3 = a_3$, . . . etc. Thus, $\hat{\alpha}_i$ can be determined from the above seven data sets by choosing appropriate values of $\pm\ \Delta X_i$ which define the range of data for each variable.

In general, one can solve $\hat{\alpha}_i$ using matrix operations (Eq. 3.9), where Y matrix for the example is

$$[\,Y\,] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \Delta X_1 & 0 & 0 & \Delta X_1^2 & 0 & 0 \\ 1 & -\Delta X_1 & 0 & 0 & \Delta X_1^2 & 0 & 0 \\ 1 & 0 & \Delta X_2 & 0 & 0 & \Delta X_2^2 & 0 \\ 1 & 0 & -\Delta X_2 & 0 & 0 & \Delta X_2^2 & 0 \\ 1 & 0 & 0 & \Delta X_3 & 0 & 0 & \Delta X_3^2 \\ 1 & 0 & 0 & -\Delta X_3 & 0 & 0 & \Delta X_3^2 \end{bmatrix} \tag{3.12}$$

If the interaction terms are included, a minimum of three more data sets are required to determine $\hat{\alpha}_7$ to $\hat{\alpha}_9$. As an example,

X_1	X_2	X_3	X_o
$a_1 + \Delta X_1$	$a_2 - \Delta X_2$	0	$(X_o)_8$
$a_1 - \Delta X_1$	0	$a_3 + \Delta X_3$	$(X_o)_9$
0	$a_2 + \Delta X_2$	$a_3 - \Delta X_3$	$(X_o)_{10}$

Similar tables can be easily constructed when the number of design variables $(n + 1)$ is greater than four. In general, the minimum number of data sets required is $2n + 1$ if the interaction are excluded, and is $(n + 2)(n + 1)/2$ for a complete quadratic equation.

When using the minimum number of data points to generate the model, the values at the data points of the model are exact. Then the curve fitting procedure described above is actually an effort of trying to approximate the exact surface by interpolation. To determine suitable values of the three data points for each variable, namely, the expanding point and the other two points which define the "interpolation range," consider the following example. Assume that a stress random variable has a known distribution, but no information is available about the value of the design point. As shown in Fig. 3.2, a decision is made to choose the interpolation range from the median of X (data point 1) to the value of X where its cdf is 0.9999 (data point 3), and let the expanding point to be in the middle (data point 2). For a strength variable, points 2 and 3 should be chosen at the left tail region. In doing this, it is anticipated (as a first approximation) that the design point will lie within the interpolation range. To improve the approximation, a new range may be chosen if the design point lies outside this range in the first reliability computation by the fast probability integration technique. As demonstrated later, it is thought that one or two tries will be sufficient for generating satisfactory results.

In the first attempt, in general, one would like to choose a wide interpolation range to ensure that the range will cover the design point. However, it is obvious that a wider interpolation range may result in poorer approximation. In addition, if significant nonlinearities exist in the range, the quadratic curve or surface may turn into an inappropriate direction very fast which may result in a problem of obtaining unreasonable design point. It is helpful to have a good physical feel of the problem. Consider the example of fatigue reliability analysis mentioned before. For a given life N, fatigue strength (defined by ε) should increase if σ_f' being increased. In other words, $\partial\varepsilon/\partial\sigma_f'$ should always be positive. Suppose a linear

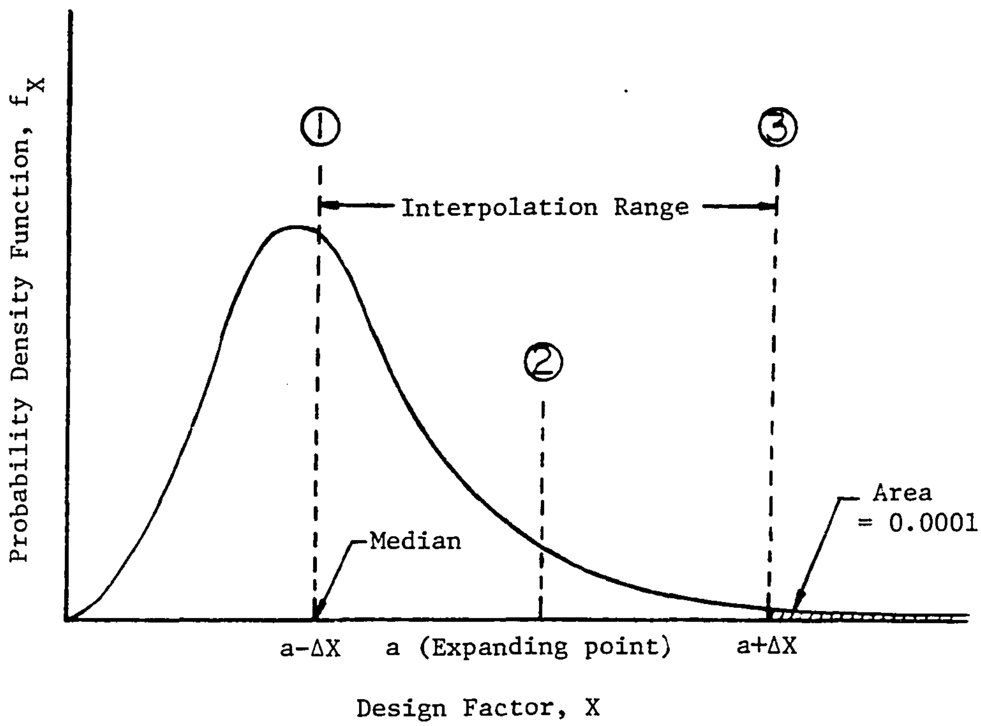


Figure 3.2 An Example of Choosing Data Points for Generating Curve Fitting Model (Stress Variable Case).

polynomial model is chosen. As a first approximation, the data points (ε_i, σ'_{fi}, . . .) will be fitted by a straight line on the two dimensional coordinates plane (e.g., ε - σ'_f plane). But the data are exact, and it follows that the sign of $\partial\varepsilon/\partial\sigma'_f$ will be preserved. Hence, the linear model is excellent in preserving the basic functional relationships between strength and stress variables.

In general, however, the quadratic model is more appropriate because of the wide range of good fit of the failure surface. But it is also clear that the slopes or (first) partial derivatives of a second degree equation can be positive, negative, or zero. Therefore, when a "strictly monotone" nonlinear function is approximated by a quadratic equation, it is no longer a strictly monotone function. As a result, there exist certain domains in which a strength variable will behave like a stress variable, and vice versa. Consequently, there is a possibility that the design point will converge to the undesirable domain where very poor approximation of the limit state is unavoidable, resulting in unreasonable design point. Therefore, it is important to examine the model before calculating β.

3.4 Validating and Improving Polynomial Approximating Model

Instead of checking the result of the curve fitting by plotting, it is more efficient to calculate the "effective domain" of the model where the basic functional relationships mentioned above are preserved. As illustrated in Fig. 3.3 the effective range can be determined for each variable by calculating the value at which the partial derivative is zero. For instance, recall Eq. 3.11 for a

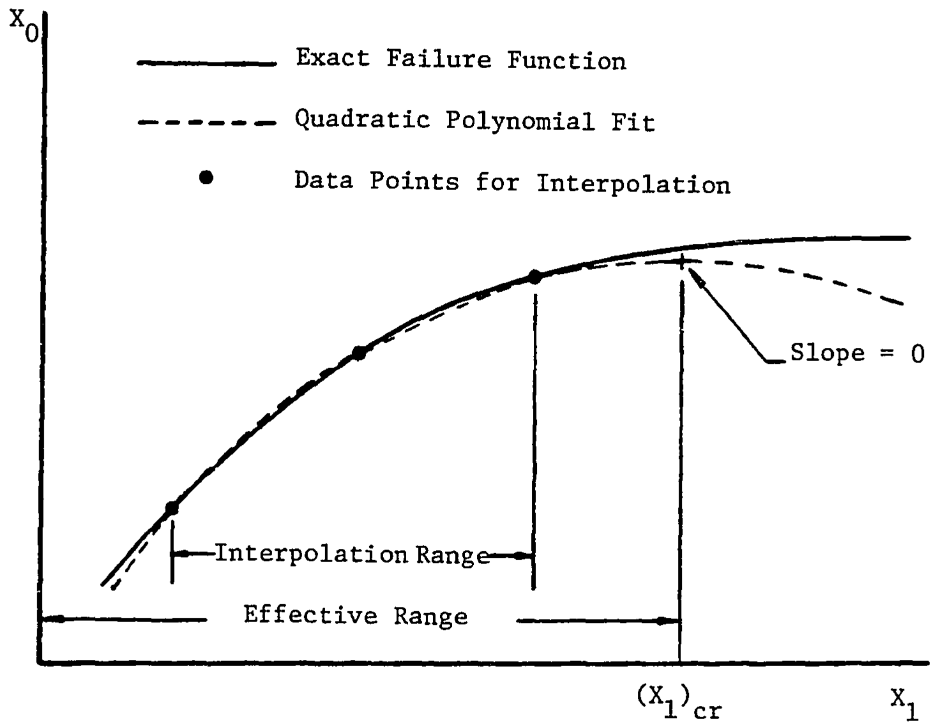


Figure 3.3 Definition of the Effective Range for the Quadratic Polynomial Curve Fitting Model.

quadratic polynomial with no interaction terms. Consider X_o as the dependent variable, and X_1, X_2, X_3 as the independent variables. From the conditions of $\partial X_o/\partial X_i = 0$, the "critical points" of X_i denoted by $(X_i)_{cr}$ are

$$(X_1)_{cr} = a_1 - \frac{\alpha_1}{2\alpha_4} \tag{3.13}$$

$$(X_2)cr = a_2 - \frac{\alpha_2}{2\alpha_5} \tag{3.14}$$

$$(X_3)_{cr} = a_3 - \frac{\alpha_3}{2\alpha_6} \tag{3.15}$$

Thus, the effective range can be easily determined. Similar results can be immediately inferred when more design variables are involved.

By introducing interaction terms to form a complete quadratic equation, Eq. 3.13 is replaced by

$$(X_1)_{cr} = (a_1 - \frac{\alpha_1}{2\alpha_4}) - \frac{\alpha_7}{2\alpha_4}(X_2 - a_2)$$

$$- \frac{\alpha_8}{2\alpha_4}(X_3 - a_3) \tag{3.16}$$

Considering the extremes of the interpolation range where $X_i = a_i \pm \Delta X_i$ for X_2 and X_3, Eq. 3.16 becomes

$$(X_1)cr = (a_1 - \frac{\alpha_1}{2\alpha_4}) - \frac{\alpha_7}{2\alpha_4}(\pm \Delta X_2) - \frac{\alpha_8}{2\alpha_4}(\pm \Delta X_3) \tag{3.17}$$

therefore, at least at one of the extremes, $(X_1)_{cr}$ will be

$$(X_1)_{cr} = (a_1 - \frac{\alpha_1}{2\alpha_4}) - \left| \frac{\alpha_7}{2\alpha_4} \Delta X_2 \right| - \left| \frac{\alpha_8}{2\alpha_4} \Delta X_3 \right| \qquad (3.18)$$

which is at best, equal to the value obtained from Eq. 3.13. Similar arguments can be used for the other X_i, and for the case of more variables. Thus, while the introduced interaction terms have the promises of improving the accuracy of the model, the effective range becomes smaller.

A general rule regarding the quality of the polynomial expression is that the model is good if all $(X_i)_{cr}$ are located well outside the interpolation range (See Fig. 3.3). It is clear that the effective range will be much wider than the interpolation range if the failure function is approximately linear in the interpolation range. On the other hand, when $(X_i)_{cr}$ lies inside the data range, it is implied that significant nonlinearity exists in the interpolation range. A "bad fit" results in some region of the interpolation range. Likely the fit is worse in the "extrapolation range".

Using the information of effective range, there are several ways of overcoming this "nonlinearity problem": (a) Restrict the value of the variable within its effective range. This will be demonstrated in the following example of the fatigue life reliability analysis; (b) Use linear model as a first approximation followed by a quadratic model using more appropriate interpolation range; (c) Use a piecewise (splines) model by splitting the interpolation range.

In the limiting case, any curve can be considered as connected by a series of straight lines.

These methods are useful when a range of p_f need be estimated, and a single quadratic curve is unable to provide a good fit in the relatively wide domain because of the nonlinearity of the failure function. For the case of single design-point checking problem, "range reduction" can be very effective simply because a narrower interpolation range always provides a better effective range.

In conclusion, it is thought that generally the quadratic polynomial curve fitting is an appropriate method to approximate the failure function. Furthermore, becuase the interaction terms tend to reduce the effective range and more data point sets are required for modelling, it is recommended that a quadratic polynomial model without the interaction terms may be adopted for an efficient estimation of p_f and the design point values. Using a minimum amount of data points, this model will be applied in the following example to demonstrate its effectiveness. It will be shown that critical values $(X_i)_{cr}$, are useful for validating and improving the model. Finally, it is believed that accurate β can be effectively obtained through the "design-point searching" and the "interpolation-range reduction" processes regardless of the complexity and nonlinearity of the failure function. As mentioned previously, $p_f = \Phi\ (-\beta)$ provides a reasonable estimate of p_f. For the case of single design-point checking problem, it is possible, after several iterations, to obtain better approximation of p_f by calculating the probability contents bounded by quadratic surface.

3.5 Example - Fatigue Life Reliability Analysis

Demonstrated here is the general method of producing a quadratic polynomial model where interaction terms are neglected. The sample problem involves fatigue reliability using local strain analysis to predict life. A zero-to-peak constant amplitude load producing the nominal stress history as shown in Fig. 3.4 is applied to the notched member. The peak stress S is not known exactly, and is considered to be a random variable.

To simplify the reliability problem, assume that only σ_f', ε_f' and S (applied nominal stress range of a constant amplitude process) are random variables. The remaining parameters are constant. It will be assumed also that the mean stress effect is negligible due to stress relaxation; $\sigma_o = 0$. It is important to note that in the example which follows, it is possible, with no extention of the theory, to include, for example, k_t, A, s, and even σ_o as random variables. A depiction of the basic equations of local strain analysis is provided in Figs. 3.5 and 3.6. Fig. 3.5 shows that σ and ε are determined from Neuber's rule and the cyclic stress-strain curve. Fig. 3.6 illustrates the relationship between strain amplitude at the notch (ε_a) and cycles to failure (N) for a given σ_f' and ε_f'. Both σ_f' and ε_f' are considered to be random variables, describing the scatter in fatigue data.

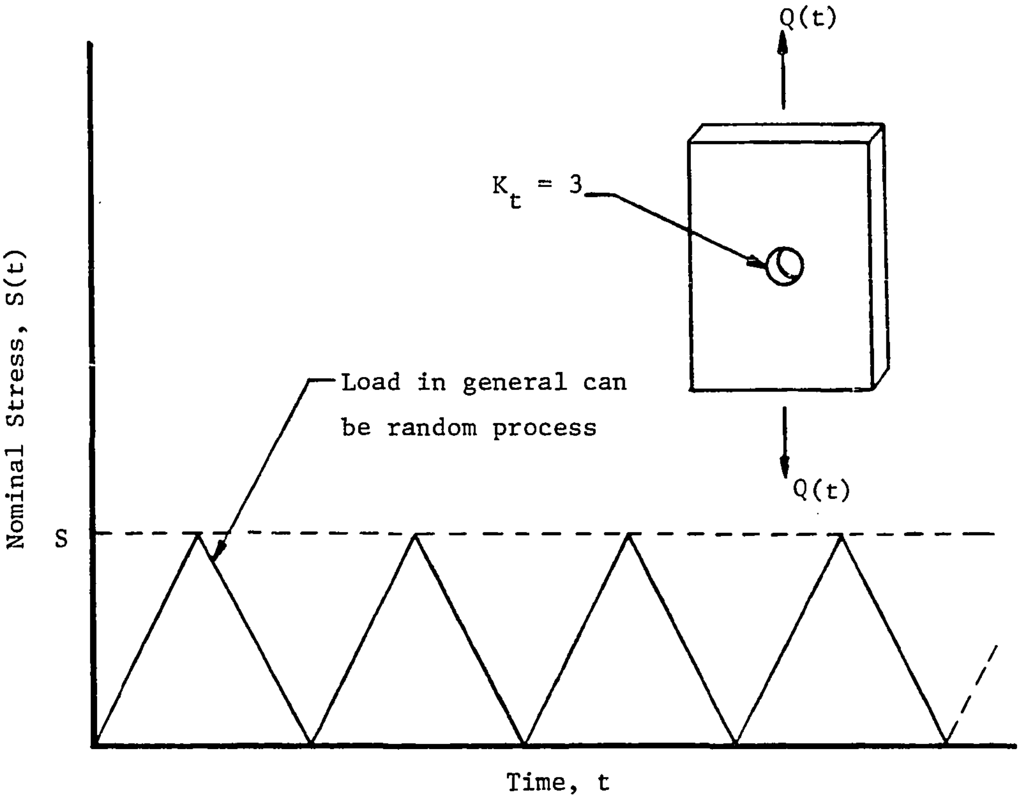


Figure 3.4 Nominal Stress Applied to Notched Member.

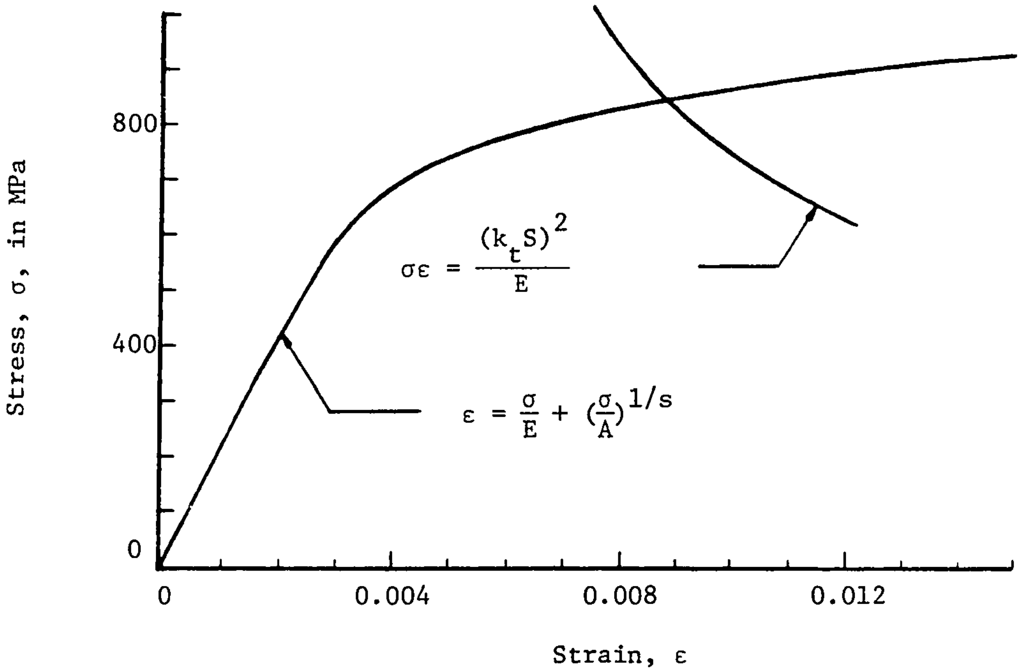


Figure 3.5 Cyclic Stress-Strain Curve and Neuber's Rule (S = 414 MPa).

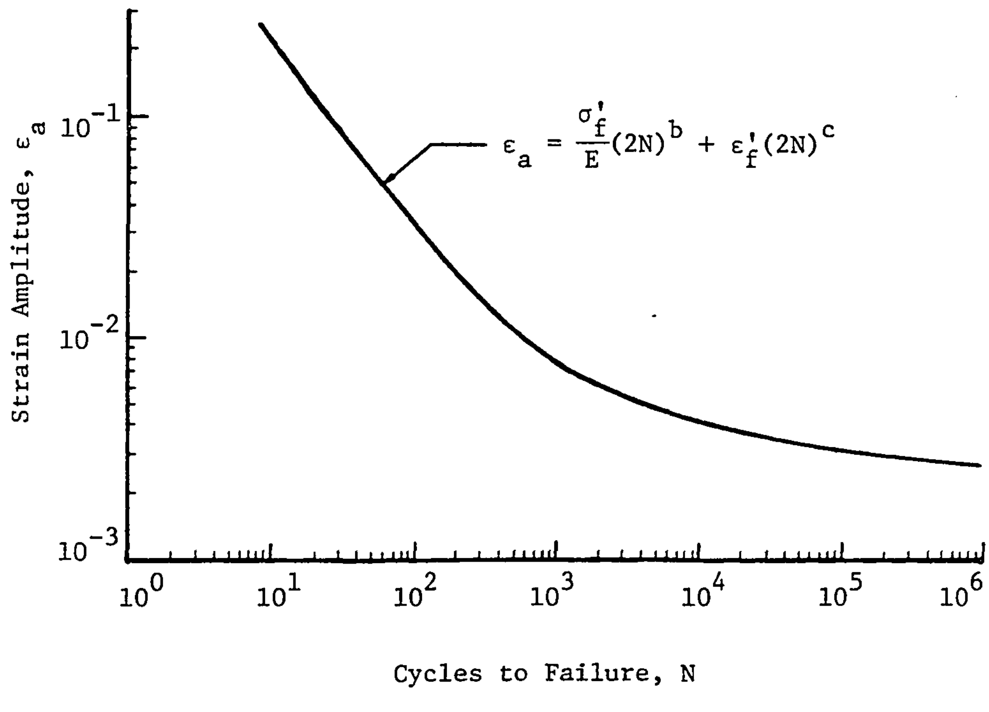


Figure 3.6 Strain Life Curve for Case A (Using Median Value of σ_f' and ε_f').

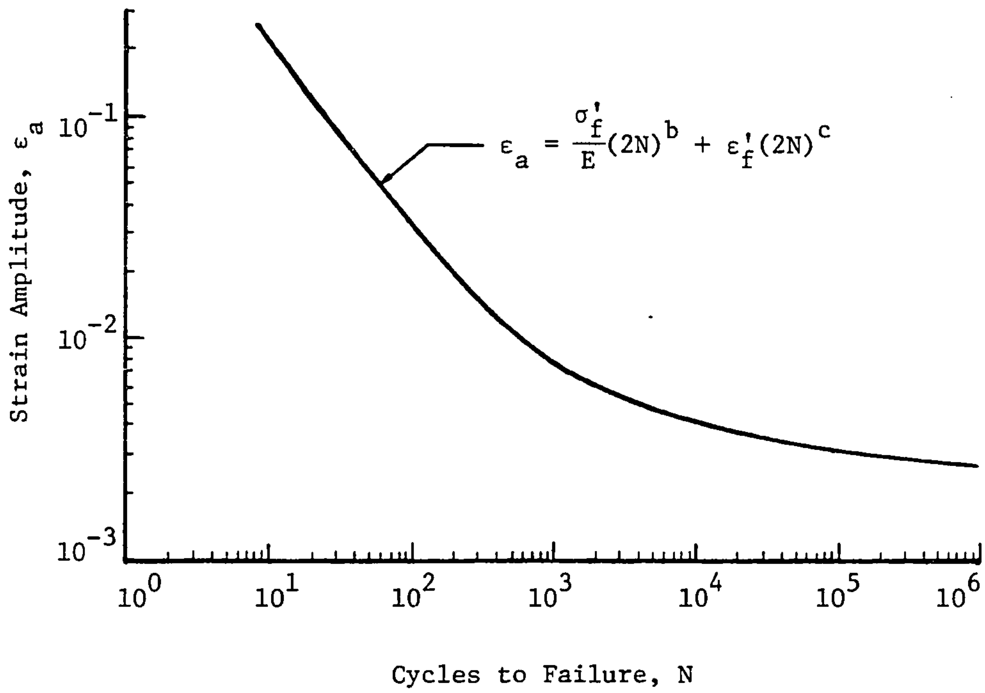


Figure 3.6 Strain Life Curve for Case A (Using Median Value of σ_f' and ε_f').

Three different applied stress conditions are examined using the proposed method described herein. The results are compared with those using Monte Carlo method where 100 samples are generated in each case to plot the empirical cdf.

Case A:

The material analyzed is Waspaloy B at 1000 degrees F. The median nominal stress is $\tilde{S}$ = 310 MPa (45 ksi). The data is tabulated in Table 3.1. To obtain wide range of effectiveness of the polynomial fit, data points are chosen as described in Fig. 3.7.

Cycles to failure N is computed using a local strain analysis program. Seven data sets for curve fitting (analysis is repeated seven times) are given in Table 3.2. Because the relation between S and N is extremely nonlinear, log N, instead of N, will be defined as a basic variable to "linearize" the function. Once the data sets are generated, any variable can be treated as the dependent variable. In this example, S was chosen

$$S = \alpha_o + \alpha_1 (\sigma_f' - a_1) + \alpha_2 (\varepsilon_f' - a_2) + \alpha_3 (\log N - a_3) + \alpha_4 (\sigma_f' - a_1)^2 + \alpha_5 (\varepsilon_f' - a_2)^2 + \alpha_6 (\log N - a_3)^2 \tag{3.19}$$

where the expanding point values, a_i, can be found in Table 3.2. By the transformation of $Y_1 = \sigma_f' - a_1$, etc., as defined in Eq. 3.8, S can also be written as

Table 3.1 Data Information for Example

(a) Random Variables

Parameter	Distribution Type	Median	Coefficient of Variation
S	lognormal	310.2 Mpa[a] (45 ksi)	0.20
σ_f'	lognormal	1841 MPa (267 ksi)	0.05
ε_f'	lognormal	3.47	0.43

[a] $\tilde{S}$ = 414 MPa (60 ksi) for Case B, $\tilde{S}$ = 551 MPa (80 ksi) for Case C

(b) Constants

Parameter	Value	Parameter	Value
K_t	3.0	s	0.108
b	-0.0843	c	-0.9126
E	206820 MPa (30,000 ksi)	A	1518 MPa (220.4 ksi)

Note: $\sigma_o = 0$

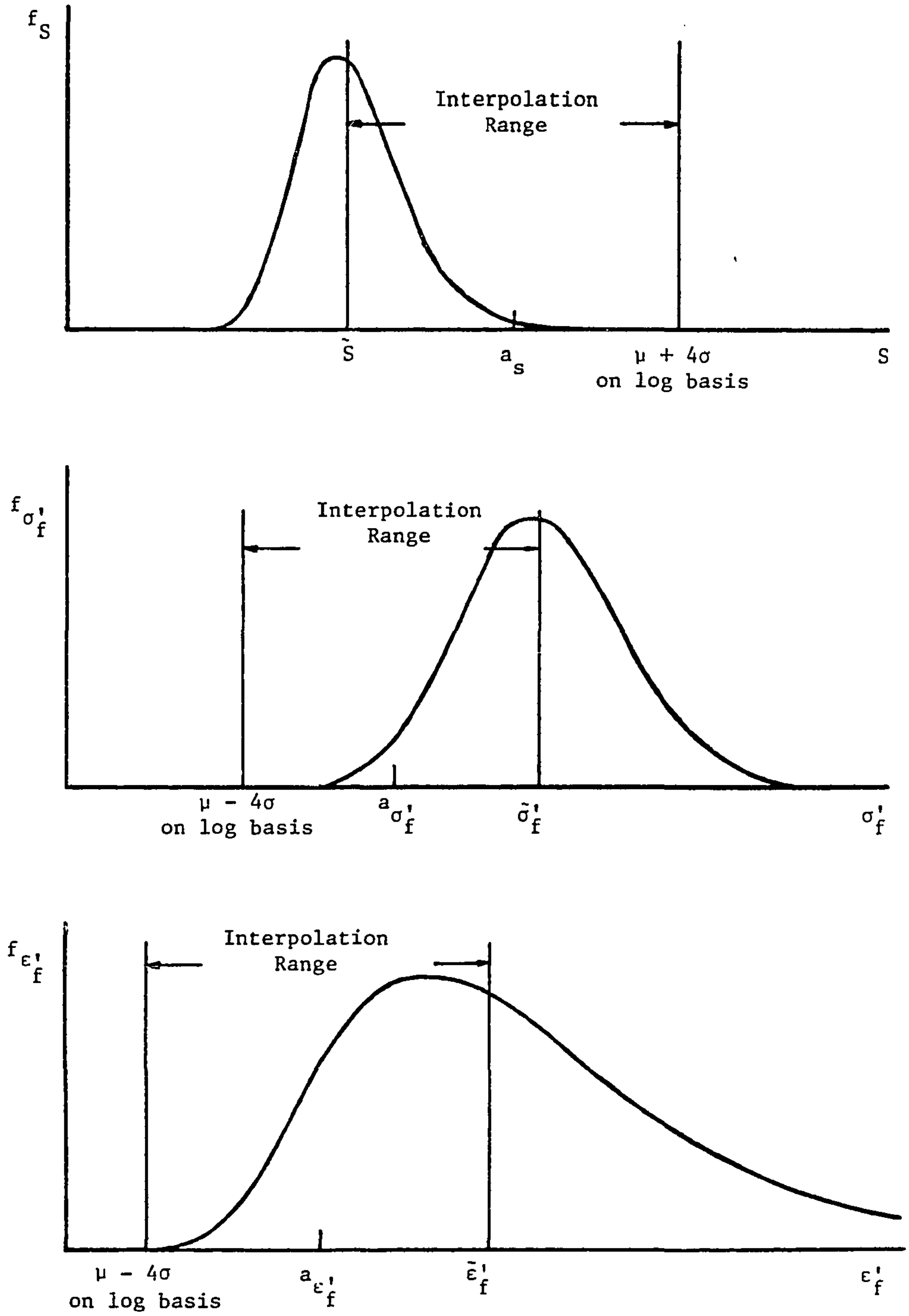


Figure 3.7 How the Expanding Point $\underset{\sim}{a}$ and the Data Points were Chosen for Generating the Quadratic Polynomial Model.

Table 3.2 Data Point Sets for Curve Fitting

Set Number	S, in MPa	σ_f', in MPa	ε_f'	$\log_{10}N$, in cycles [(b)]
1[a]	497.7	1674	2.069	3.859
2	310.2	1674	2.069	6.191
3	685.1	1674	2.069	2.935
4	497.7	1507	2.069	3.611
5	497.7	1841	2.069	4.149
6	497.7	1674	0.668	3.644
7	497.7	1674	3.470	3.984
Interpolation Range	310.2-685.1	1507-1841	0.668-3.47	2.935-6.191

[a]Expanding point

[b]Computed using a local strain analysis program originally written by W. R. Brose.

$$S = \alpha_o + \alpha_1 Y_1 + \dots \alpha_6 Y_6 \tag{3.20}$$

Therefore, α_i can be solved simultaneously from Eq. 3.9 where the Y matrix can be constructed using data in Table 3.2. The result is:

$$\begin{aligned} S &= 497.7 + 0.268\ (\sigma_f' - a_1) + 20.8\ (\varepsilon_f' - a_2) \\ &- 168\ (\log N - a_3) + 0.000027\ (\sigma_f' - a_1)^2 \\ &- 4.52\ (\varepsilon_f' - a_2)^2 + 37.5\ (\log N - a_3)^2 \end{aligned} \tag{3.21}$$

Now examine the quality of the polynomial expression. Consider the effective range for each independent variable. Using Eqs. 3.13 to 3.15, the values at $\partial S/\partial\sigma_f' = 0$, etc., are

$$(\sigma_f', \varepsilon_f', \log N)_{cr} = (-3290, 4.4, 6.1)$$

The value of $(\sigma_f')_{cr}$ is negative implying the nonlinear effect of σ_f' is very small. The value of $(\varepsilon_f')_{cr}$ when compared with the interpolation range of ε_f' shown in Table 3.2 also suggests that the nonlinear effect of ε_f' is not significant. Thus, the quadratic model seems to be satisfactory for σ_f' and ε_f' in the chosen ranges of interpolation. The value of $(\log N)_{cr}$ is close to the interpolation range implying a moderate nonlinear effect of log N in its chosen range.

The R-F algorithm was used to compute p_f for several values of N_o. The results are shown in Table 3.3. It is noted that the highest value of log N_o is chosen to be 5.5, a values less than $(\log N)_{cr}$, 6.1. This point is inside the interpolation range and

Table 3.3 Results of the Reliability Analysis for Case A Using the Proposed Method

		Design Point			Probability of failure, %
N_o (cycles)	Log N_o	S, in MPa	σ'_f, in MPa	ε'_f	
3,160	3.5	605	1779	2.88	0.026
10,000	4.0	520	1785	2.95	0.34
31,600	4.5	454	1793	3.04	2.2
100,000	5.0	407	1802	3.13	7.3
316,000	5.5	378	1810	3.21	14.2

reasonable p_f estimates are expected. There is no limitation for choosing lower log N_o. The values listed in Table 3.3 are in the range where comparisons can be made with Monte Carlo results, i.e., where the lowest possible estimate of p_f is about 1%. Observing that all the estimated design point values in Table 3.3 are well within the interpolation range shown in Table 3.2 one notes that perhaps accuracy could be further improved by reducing the interpolation ranges.

Monte Carlo was used as an independent check on the quality of results of the proposed method. Unfortunately, each trial requires running the local strain analysis program with a random sample of S, σ_f', and ε_f'. This process is extremely costly and inefficient. Therefore, only a sample of size 100 was obtained.

Computed values of p_f from the R-F algorithm are plotted in Fig. 3.8. The empirical cdf generated from Monte Carlo method is also plotted on the same lognormal probability paper. Assuming that the p_f of the present method are accurate, the upper and lower 1% points for the maximum likelihood estimator (MLE), $\hat{p}$, of p are given. The calculation is similar to that of a confidence interval for the binomial. If the circled points in Fig. 3.8 are accurate, then there is a 98% chance that $\hat{p}$ obtained from an experiment would lie in the interval shown. Even though the algorithm used to estimate p_f was really $(i - \frac{1}{2})/n$, not exactly the MLE, the evidence of Fig. 3.8 suggests good agreement between the present method and Monte Carlo.

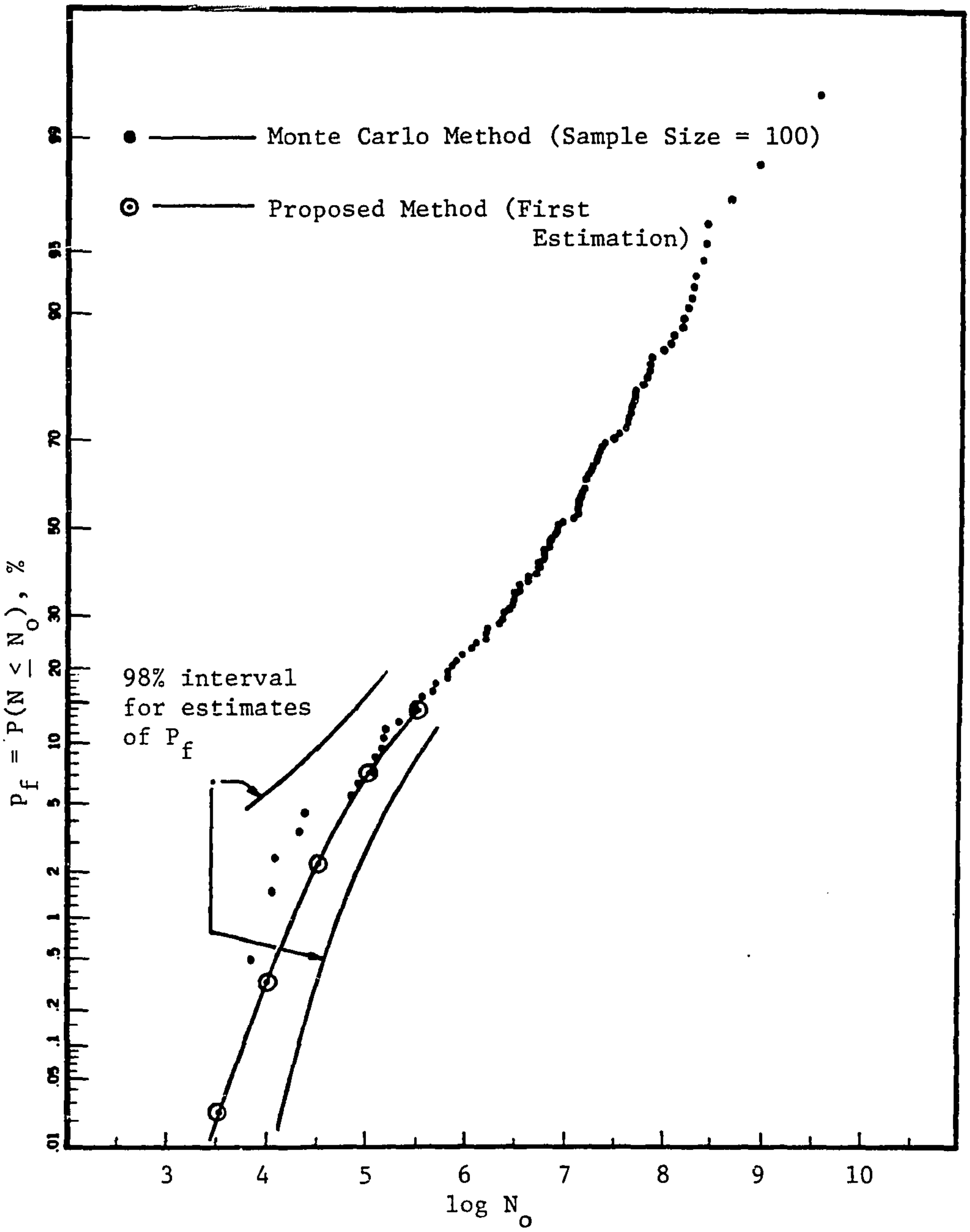


Figure 3.8 Reliability Analysis of Fatigue Life (Case A); A Comparison of the Present Method with Monte Carlo.

The median of S in this case was sufficiently small so that generally the deformation of the material is in the elastic range. However, because S has a large coefficient of variation, some plasticity is expected for samples having a higher than average stress. The plasticity effect (ε_f' term) could be important in highly stress specimens. Hence, the lower tail of the cdf of N is not a straight line as it would be on lognormal paper if S were deterministic. This exercise demonstrates that extrapolating p_f using Monte Carlo results will likely produce inaccurate estimates of p_f. This is a principal reason why the number of simulations should be larger when the p_f to be estimated is smaller and why Monte Carlo can be so expensive.

Case B:

The data is the same as in Case A (See Table 3.1) but the median of nominal stress is increased to 414 MPa (60 ksi) so that more cyclic plasticity is expected at the notch.

The data points for S are (414, 663, 913) MPa. The method for computing these was described in Fig. 3.7. Following the same procedures as in Case A, seven data points were obtained. The expanding point of log N is 3.0, and the data of log N ranges from 2.5 to 4.7.

The result of the curve fit, choosing S as the dependent variable, is

$$S = 663 + 0.217\,(\sigma_f' - a_1) + 106\,(\varepsilon_f' - a_2)$$
$$- 425\,(\log N - a_3) + 0.000074\,(\sigma_f' - a_1)^2$$
$$- 34.1\,(\varepsilon_f' - a_2)^2 + 164\,(\log N - a_3)^2 \qquad (3.22)$$

The values at $\partial S/\partial\sigma_f' = 0$, etc., are

$$(\sigma_f', \varepsilon_f', \log N)_{cr} = (207, 3.6, 4.3)$$

Again, both $(\sigma_f')_{cr}$ and $(\varepsilon_f')_{cr}$ values are well outside the interpolation range shown in Table 3.2 implying satisfactory effective ranges for σ_f' and ε_f'. However, $(\log N)_{cr}$ is inside the interpolation range (2.5, 4.7), suggesting the possibility of a poor fit in some regions because of significant nonlinearity between S and log N in the interpolation region. Therefore, in forming the limit states of Eq. 3.22, values of $\log N_o$ should be limited to, say, 4.0 as higher errors may result for larger $\log N_o$.

The design points and p_f for several values of $\log N_o$ were computed using the same method as described for Case A. The results of this first approximation is shown in Table 3.4(a). Values of the p_f are plotted on Fig. 3.10.

The fact that the $(\log N)_{cr}$ value lies inside the interpolation range implies that the degree of nonlinearity between S and log N is significant in this range. A higher degree polynomial equation is required to improve the fit unless the range can be reduced. Suppose, for instance, the goal of the reliability analysis of $\log N_o$ ranges from 3.0 to 4.5. The design point information

Table 3.4 Results of the Reliability Analysis Using the First Approximating Model, and the Data Range for Generating Second Approximating Model (Case B).

(a) Results Using First Approximating Model

Design Point					Probability of failure, %
N_o (cycles)	$\log N_o$	S, in MPa	σ_f', in MPa	ε_f'	
1,000	3.0	658	1806	1.75	0.19
1,778	3.25	587	1811	1.95	1.14
3,162	3.5	539	1816	2.20	3.97
5,623	3.75	508	1820	2.47	8.93
10,000	4.0	490	1823	2.69	14.2

(b) Data Range for Generating Second Approximating Model

Variables	Data Range	Expanding Point
S, in MPa	655 - 448	551
σ_f', in MPa	1806 - 1834	1820
ε_f'	1.75 - 2.80	2.275
[a]log N	3.15 - 4.58	3.686
[a]Corresponding Values		

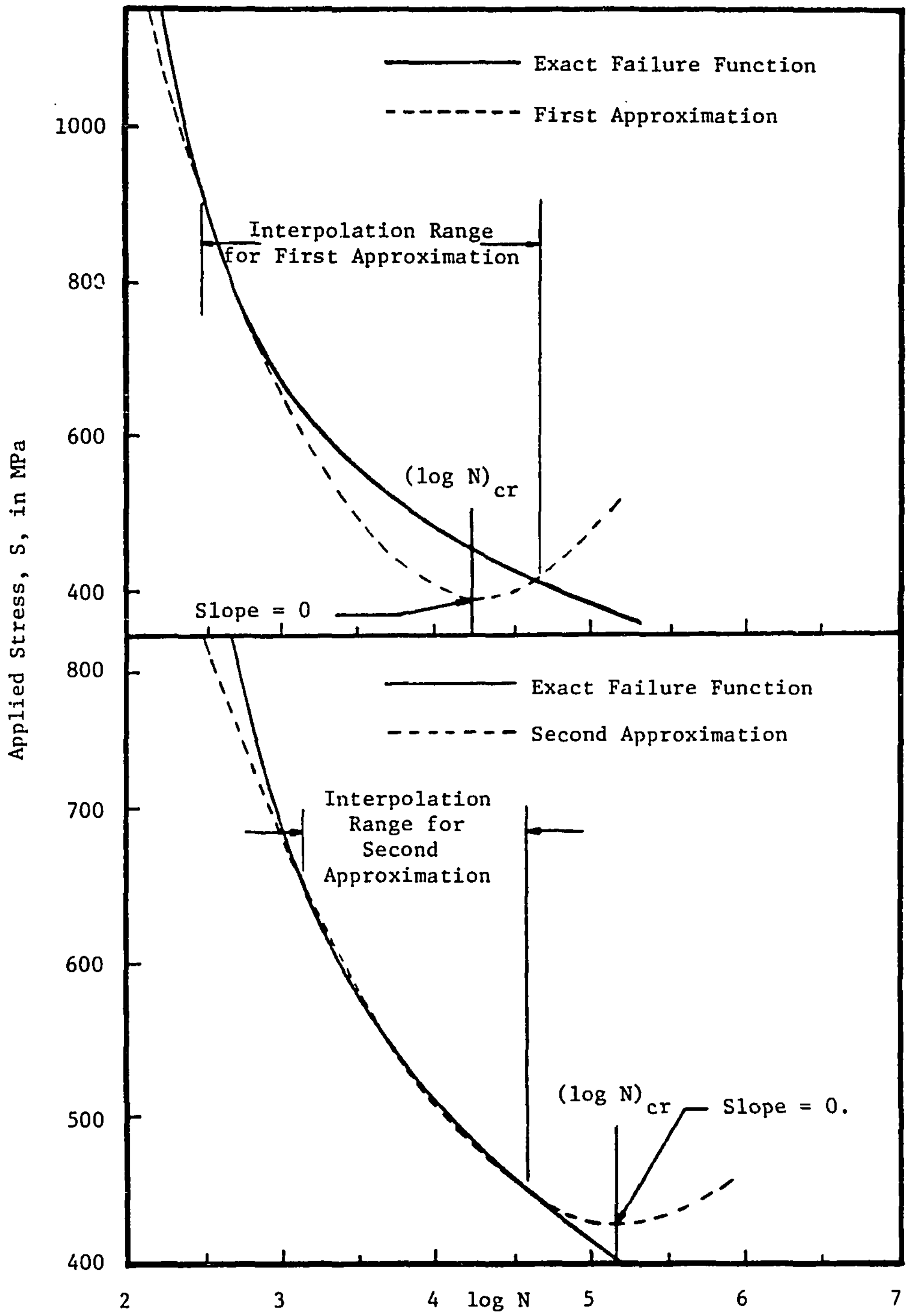


Figure 3.9 Comparison of the Exact and the Approximating Failure Function.

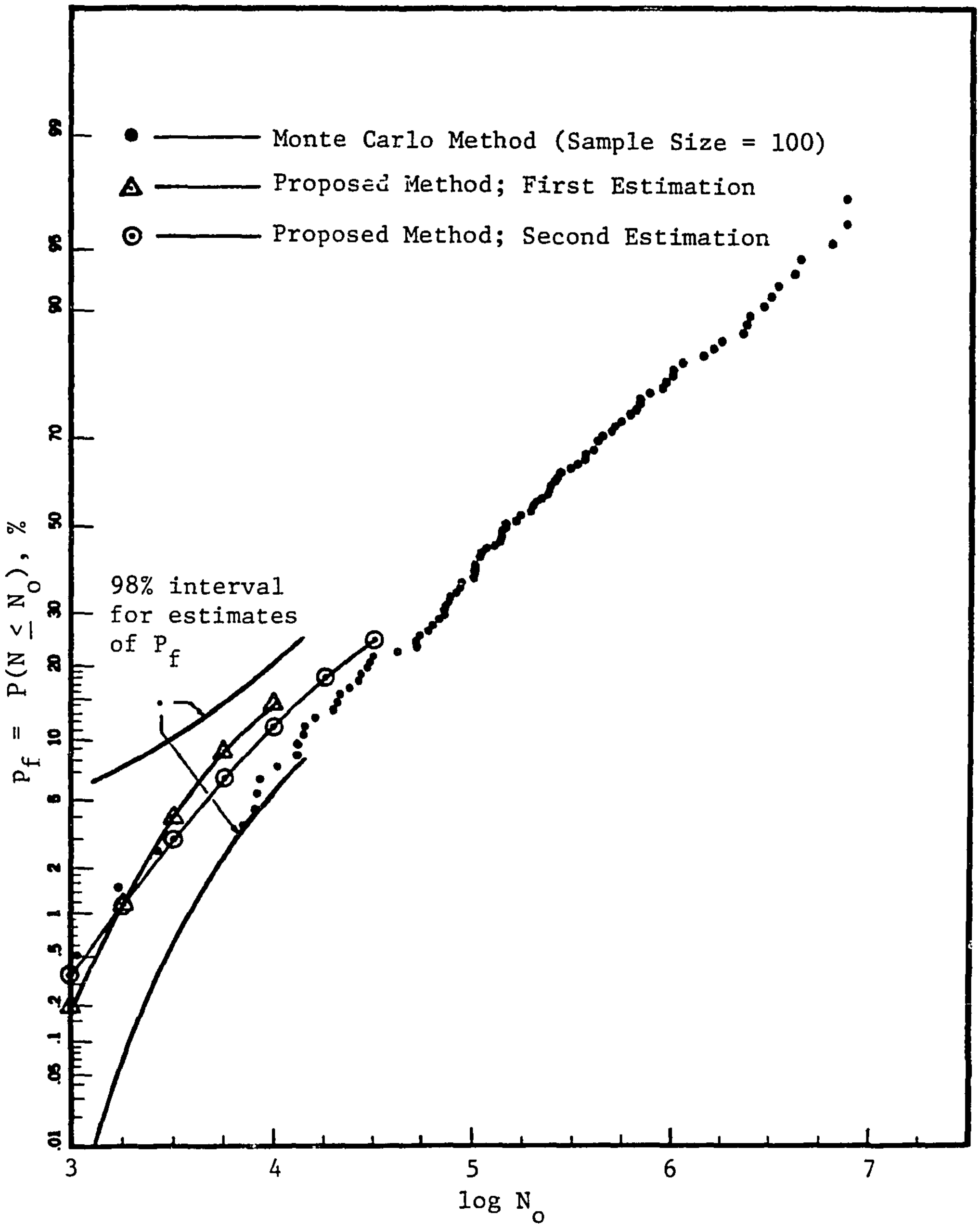


Figure 3.10 Reliability Analysis of Fatigue Life (Case B); A Comparison of the Present Method with Monte Carlo.

generated from the first approximation (Table 3.4(a)) reveals that the interpolation ranges may be reduced to generate improved, second approximation. It should be emphasized that when only point estimate is required (e.g., $\log N_o = 3.0$), the interpolation range for the second approximation can be made very small so that the quadratic model becomes a good approximation. In this study, however, the narrower interpolation range is chosen to be wide enough to cover the region of $\log N_o$ of interest in this study.

Using the design point information in Table 3.4(a), the new and reduced interpolation ranges are determined as shown in Table 3.4(b). The improved quadratic model becomes:

$$\begin{aligned} S &= 551 + 0.212\,(\sigma_f' - a_1) + 19.3\,(\varepsilon_f' - a_2) \\ &- 165\,(\log N - a_3) + 0.00074\,(\sigma_f' - a_1)^2 \\ &- 4.08\,(\varepsilon_f' - a_2)^2 + 55.1\,(\log N - a_3)^2 \end{aligned} \tag{3.23}$$

where the expanding point values can be found in Table 3.4(b). The values at $\partial S/\partial \sigma_f' = 0$, etc., are

$$(\sigma_f',\ \varepsilon_f',\ \log N)_{cr} = (390,\ 4.6,\ 5.2)$$

By comparing these values with the interpolation ranges shown in Table 3.4(b) it is found that the effective ranges for all three variables are satisfactory, suggesting that the model has been improved. To demonstrate the improvement made in the model, the exact and the approximate S - N curves are drawn in Fig. 3.9 for both the first and the second approximation (Note: Values of σ_f' and ε_f' are at the expanding point). The figure clearly shows that the model is much better when $(\log N)_{cr}$ lies outside the interpolation range suggesting that $(\log N)_{cr}$, etc., which defines the effective range, is a convenient index of "measuring" the quality of the model, and the degree of nonlinearity in the failure function.

The new design points and the associated p_f using Eq. 3.23 are shown in Table 3.5. The design point values of ε_f' as well as the value of S when $\log N_o = 3.0$ are outside the data ranges indicating the extrapolating values of Eq. 3.23 are being used. However, these values are well within the effective ranges and closer to the interpolation range. Therefore, it is not unreasonable to assume that good approximation can still be obtained.

Values of p_f of the improved model are also plotted on Fig. 3.10. Following the same procedure as in Case A, a 98% interval on the estimates of p_f is constructed which covers the Monte Carlo results. The improvement of the model seems justified. By using more appropriate data ranges or splitting the data range into two parts, accurate Rackwitz-Fiessler values of β can be obtained for a wide range of fatigue life. For a specified single value of design life, it is believed that no more than two tries are necessary to obtain satisfactory results.

Table 3.5 Improved Results of the Reliability Analysis for Case B Using the Proposed Method.

		Design Point			Improved	First Estimates
N_o (cycles)	log N_o	S, in MPa	σ'_f, in MPa	ε'_f	p_f, %	of p_f, %
1,000	3.0	700	1808	3.05	0.34	0.19
1,778	3.25	645	1810	3.08	1.12	1.14
3,162	3.5	596	1814	3.13	2.96	3.97
5,623	3.75	555	1817	3.18	6.43	8.93
10,000	4.0	520	1821	3.23	11.7	14.2
17,782	4.25	493	1825	3.28	18.2	-----
31,622	4.5	472	1828	3.32	24.7	-----

Case C:

For the previous examples, the median nominal stress is further increased to 551 MPa (80 ksi) so that even more cyclic plasticity is expected. Otherwise, the parameters are the same as in Cases A and B.

Applying the same methods as described in previous cases, the data points for S are (551, 885, 1218) MPa. The corresponding expanding point value of log N is 2.55, and the interpolation range of log N is from 2.05 to 3.48. The result of the curve fit is:

$$\begin{aligned} S &= 885 + 0.133\,(\sigma_f' - a_1) + 210\,(\varepsilon_f' - a_2) \\ &\quad - 702\,(\log N - a_3) + 0.000012\,(\sigma_f' - a_1)^2 \\ &\quad - 77.1\,(\varepsilon_f' - a_2)^2 + 370\,(\log N - a_3)^2 \end{aligned} \tag{3.24}$$

The values at $\partial S/\partial \sigma_f' = 0$, etc., are

$$(\sigma_f', \varepsilon_f', \log N)_{cr} = (-3870, 3.43, 3.5)$$

Thus, satisfactory effective ranges of σ_f' and ε_f' result. The value of $(\log N)_{cr}$ is approximately equal to the value at right end of the interpolation range suggesting that the range of interpolation of S is too wide. To improve the model, a decision can be made to reduce the interpolation range either before or after the design points are estimated using Eq. 3.24. The method of improving the accuracy through range reduction process has been demonstrated in Case B, therefore it is not repeated here.

To obtain the first estimation of reliability analysis of N, several values of log N_o are chosen to form the limit states. The highest value is restricted to 3.25 which is somewhat less than the value of $(\log N)_{cr}$. The design points and p_f using R-F algorithm are shown in Table 3.6. The values of ε_f' and σ_f' indicate that cyclic plasticity indeed plays the dominant role in the region of high p_f. p_f values are plotted in Fig. 3.11. A 98% interval on the estimates of p_f, and the Monte Carlo results are also plotted. It is noted that for this high stress case, lower tail cdf of N_o should approach a straight line on the lognormal paper if the elasticity effect is being neglected. Thus, the result of the first estimation is reasonable.

3.6 Summary and Conclusion

A numerical procedure is proposed for estimating risk for the case when the failure function is not available as a closed algebraic form. The safety index β is computed by the R-F algorithm after the failure function is approximated as a simple polynomial equation. The proposed method is described by the following step-by-step procedures (refer to Fig. 3.1).

1. Specify the random design factors.
2. Make a somewhat conservative but reasonable first guess of the design point to define the range of interpolation. Select several points from the range for each variable where a solution is required.

Table 3.6 Results of the Reliability Analysis for Case C Using the Proposed Method.

		Design Point			Probability of failure, %
N_o (cycles)	log N_o	S, in MPa	σ'_f, in MPa	ε'_f	
178	2.25	927	1825	1.25	0.015
316	2.5	786	1828	1.46	0.28
562	2.75	706	1831	1.75	1.8
1000	3.0	663	1833	2.06	5.7
1778	3.25	641	1834	2.30	10.5

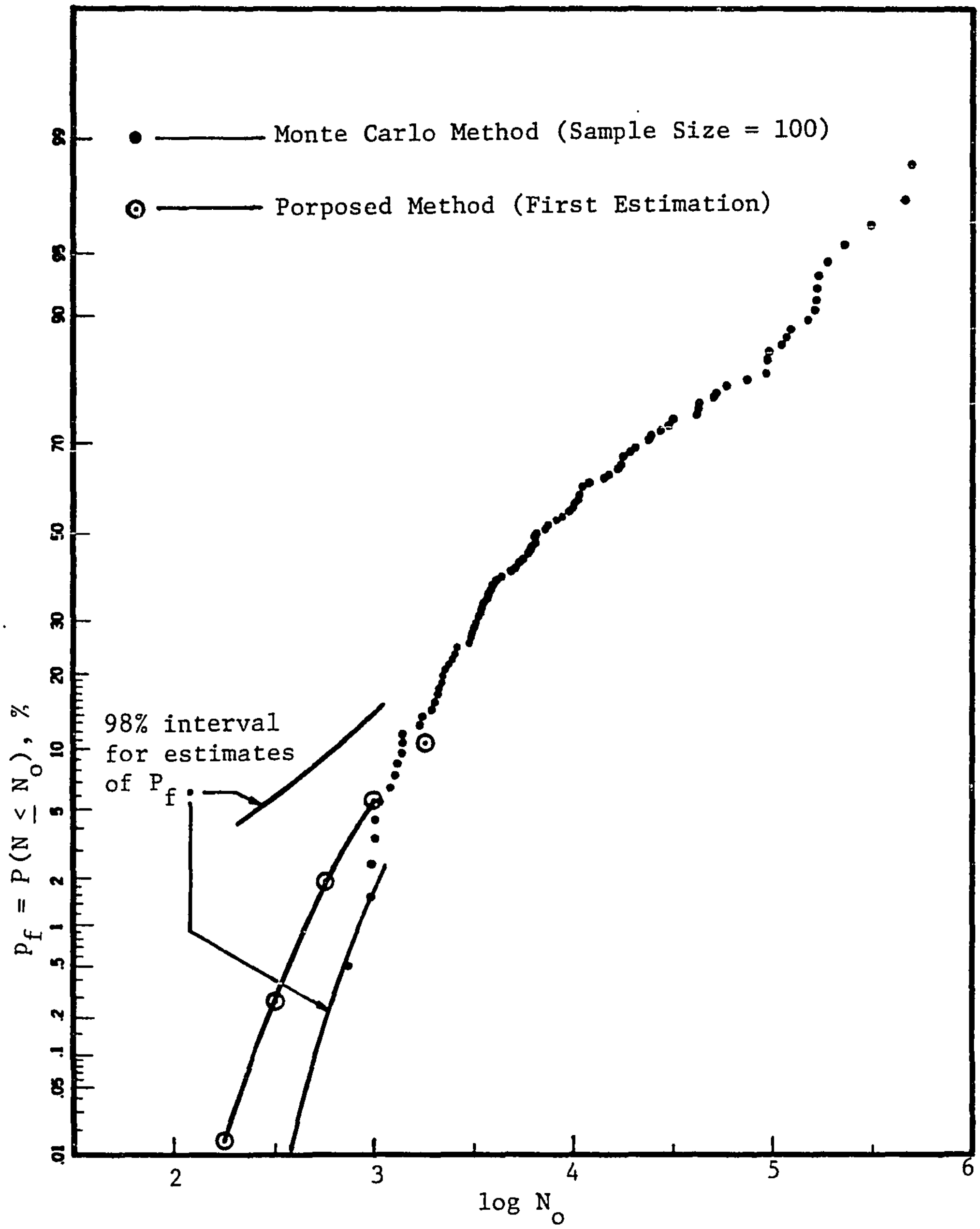


Figure 3.11 Reliability Analysis of Fatigue Life (Case C); A Comparison of the Present Method with Monte Carlo.

3. Solve the failure function, using computer analysis, to obtain a solution at each one of the point sets selected. The data sets for curve fitting are thereby provided.
4. Obtain a polynomial equation using a curve fitting technique. It is recommended that a quadratic equation without the interaction terms may be adopted for an efficient estimation.
5. Check the quality of the polynomial (e.g., define effective ranges) as an aid in forming the limit state where a reasonable agreement betweeen the exact and the approximating equation can be anticipated. A reduction of the ranges of the selected points may be required if the failure function is highly nonlinear in the selected interpolation range.
6. Apply R-F algorithm to the approximating limit state to compute the design point and β.
7. Compare the design point values with the ranges of the selected points. To improve the approximation in the region of design point, go to step 2, but adjust the values and ranges of the points for each variable. In general, accuracy can be improved by reducing the ranges which shouid still embrace the design point.
8. Estimate the probability of failure using $p_f = \Phi(-\beta)$.

As demonstrated by the example, agreement of the proposed method with Monte Carlo is considered to be good. Because the procedure is much faster than Monte Carlo, the method is thought to be particularly useful in any mechanical or structural problem where computer analysis is required to relate the random design variables.

CHAPTER 4

A NEW ALGORITHM OF RELIABILITY ANALYSIS

4.1 Introduction

Assume that the failure function, $g(\underset{\sim}{X})$, which describes the event of failure for a given mode (e.g., buckling, fatigue) has an explicit form of the design variables. It is assumed that each design variable X has a known distribution function (cdf), F(x) and density function (pdf), f(x). The variables are independent. It is required to estimate the probability of failure, p_f, (or reliability). Methods of estimating p_f were summarized in Chapter 2. A new method of computing p_f is proposed here. It is demonstrated that this scheme will consistently produce p_f estimates as good as Rackwitz-Fiessler (R-F). In many cases, the estimates are significantly better than R-F.

As described in Chapter 2, and in Appendix A, the R-F method, an extension of the Hasofer-Lind technique, transforms the non-normal design variables into equivalent standard uncorrelated normal variables at the design point. Assuming a linear limit state, at the design point, the probability of failure can be estimated using $p_f = \Phi(-\beta)$. An extension of the R-F method proposed by Chen and Lind (C-L) (1982), employs three, instead of two, parameters to construct the equivalent normal distribution. In a study of the performance of the approximation algorithms (Appendix C), it was observed that C-L does not seem to consistently produce better estimates of p_f than R-F, despite the fact that the equivalent normal, based on three parameters, is able to approximate the original distribution more closely in the neighborhood of the design point. A possible reason for the apparent inconsistency is that when the original distribution cannot be fitted well over a wide region, the forcing of the more exact fitting at the design point, in some cases, may cause larger errors in the cdf in the region not too far away from the design point; the error introduced in those regions may outweigh the accuracy gained in the vicinity of the design point.

A different approach is proposed herein. An approximation method which fits the functions over a sufficiently wide region using a least square technique is studied. The function involved in the method includes not only the cdf of the variable which is to be transformed, but also the joint pdf of the rest of the variables. Such a function is more closely related to the probability of failure. Therefore, an "optimum" estimate may be obtained.

The proposed scheme was applied to numerous examples, including linear, as well as non-linear limit state functions with design factors having various combinations of the distributions (normal, lognormal, Weibull and Type I extreme value). It was demonstrated that the method produces an equivalent normal such that the error of p_f estimate due to the "normalization" approximation may be significantly reduced. The additional cost (over R-F) for applying this "new" algorithm, in terms of the computer time, is estimated to be less than 0.1 second CPU time for each non-normal variable when running on the CYBER 175 system.

4.2 Approximating Known Distribution Functions

Consider a non-normal random variable with cdf of F(x) and pdf of f(x). The R-F method constructs equivalent normals by equating the distribution and density function at the design point, x^*,

$$\Phi(x^*) = F(x^*) \tag{4.1}$$

$$\phi(x^*) = f(x^*) \tag{4.2}$$

where Φ and ϕ are the equivalent normal cdf and pdf.

Chen and Lind introduced a three parameter normal distribution function $A\Phi(x)$, where A is a constant, to fit F(x), f(x) and $\frac{df(x)}{dx}$, at x^* if x^* is at the lower tail of the pdf. When the design point is at the upper tail, $A\Phi(x)$ and F(x) are replaced by $A(1-\Phi(x))$ and 1-F(x), respectively. The equivalent normal function should fit the original function better in the region near the design point. Because the design point is near the maximum joint pdf of the design factors, a better fit near the design point, in some cases, is sufficient to produce a better estimate of p_f. On the other hand, it is not uncommon that the equivalent normal distribution must be accurate over a relatively wide region around the design point to ensure a good estimate of p_f. In such cases,a better fit using C-L algorithm cannot guarantee an improvement of p_f estimate over R-F.

To illustrate the problem, consider a simple linear limit state function

$$g(\underset{\sim}{x}) = R - S \tag{4.3}$$

Where R and S are strength and stress variables, respectively. Assuming R is Weibull with $\mu_R = 20$, $\sigma_R = 1.2$, and S is normal with $\mu_S = 10$, $\sigma_S = 3.0$. The probability of failure, assuming R and S are mutually independent, is

$$p_f = \int_{-\infty}^{\infty} F_R(x)\phi_S(x)dx \tag{4.4}$$

The integrand $F_R\phi_S$ is plotted in Fig. 4.1 where the area under the curve is p_f. It is shown that R-F transformation of F_R to Φ_R produces an exact fit of $F_R\phi_S$ and its slope at the design point. The fit, however, is effective only over a relatively small range of x. But the fit can be considered only fair over the whole range of $F_R\phi_S$ values. As a result, there is 17% error in estimating p_f using R-F. It is interesting to note that in order to have an accurate p_f estimate, the equivalent normal Φ_R should fit the original distribution F_R equally well even when x is greater that μ_R, thus demonstrating the importance of a good fit over a "wide" range.

By fitting also $f'(x^*)$, C-L algorithm produces a better fit at the design point, consequently the error have been cut by

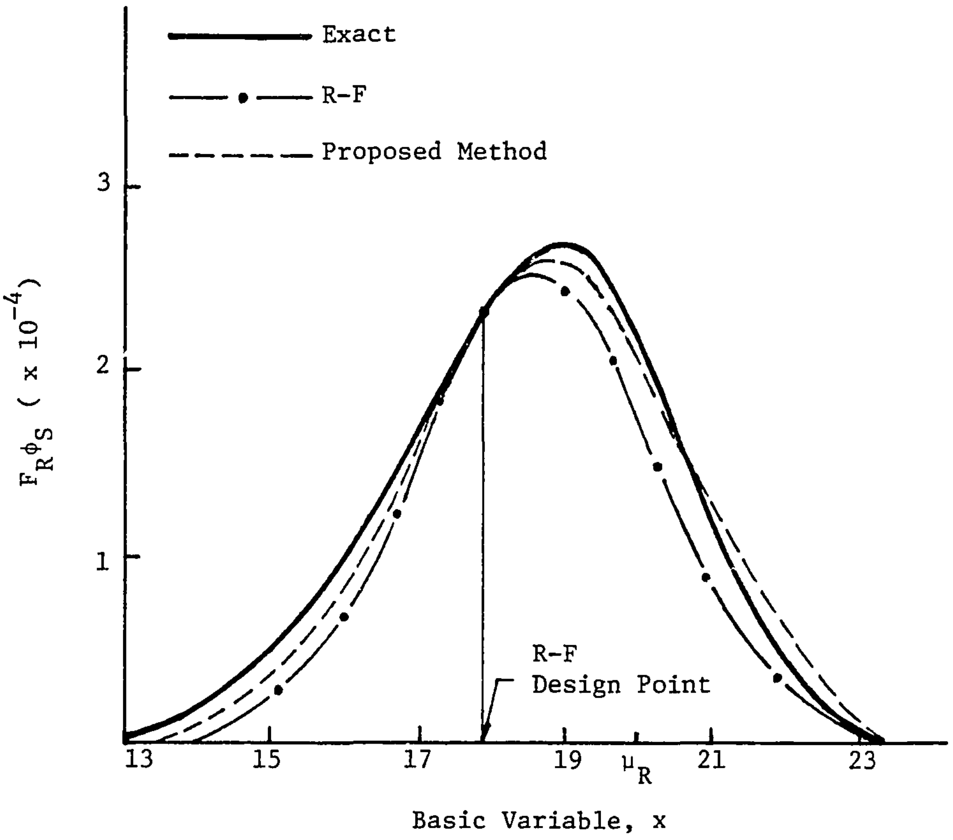


Figure 4.1 Illustration of the R-F Method and the Proposed Method in Estimating Probability of Failure.

approximately half of the R-F estimates (Appendix C). It is noted that in this example R-F result under-estimates the exact p_f while C-L over-estimates the p_f. In some other cases, e.g., when the stress, S, has an extreme value distribution with small coefficient of variation, the region that contributes to p_f (Eq. 4.4) becomes even wider and the performance of both R-F and C-L becomes worse. It can be concluded that it is, in general, impossible to find an equivalent normal distribution which fits the original distribution closely over a sufficiently wide region. It is possible, however, to reduce the accuracy of the fit near the design point in exchange for a generally better fitting over the whole region, and therefore, a better p_f approximation. It is this policy which is pursued in the present method.

A better fit of F_R by Φ_R should consider the whole $F_R\phi_S$ function which is more closely related to p_f. This can be done by fitting a curve, $A\Phi_R\phi_S$, using the least squares criterion, which minimizes the total deviations from the $F_R\phi_S$ function. The goal is not to directly minimize the error under the $F_R\phi_S$ curve, but to eliminate the possibilities of the high deviations in the region other than that near the design point. Using the least square scheme described below, an improved approximation to the integrand is established. This curve is also drawn on Fig. 4.1 Due to the cancellation effect of the positive and negative deviations in the fitting, the error of p_f has been reduced to 2.5%.

In the following, the method proposed is basically the first order method, i.e., a linear limit state $g(\underset{\sim}{X}) = 0$ is assumed and no inverse cdf formulations, as described in Chapter 2, are required. Similar to the C-L algorithm, the method attempts to find better equivalent normal distributions. The method is, however, different from the C-L method in that a "weighting function" is introduced, and a least-squares scheme is used to fit the original cdf in a relatively wide region.

4.3 A New Method of Determining Equivalent Normal Parameters

The method suggested here for constructing the equivalent normals utilizes the R-F results. Consider a limit state function $g(\underset{\sim}{X})$ involving n independent random variables which is linearized at the design point,

$$g(\underset{\sim}{X}) = a_o + \sum_{i=1}^{n} a_i X_i \tag{4.5}$$

Three equivalent normal parameters (A, μ_N, σ_N) for each X_i are found, one by one, by applying a numerical scheme such that

$$p_f = \int_{\Omega_1} f_{\underset{\sim}{X}}(\underset{\sim}{x}) \;\tilde{=}\; \left(\prod_{i=1}^{n} A_i\right) \int_{\Omega_2} \phi_{\underset{\sim}{u}}(\underset{\sim}{u}) d\underset{\sim}{u} \tag{4.6}$$

where Ω_1 is the failure region on the original coordinates, $\underset{\sim}{x}$, and Ω_2 is the corresponding failure region on the reduced coordinates, $\underset{\sim}{u}$, in which

$$u_i = \frac{X_i - \mu_{Ni}}{\sigma_{Ni}} \tag{4.7}$$

Using Eq. 4.7, the transformed limit state function $g_1(\underset{\sim}{u})$ is also linear. A minimum distance, denoted as δ, on the u space can be found, and p_f is estimated by

$$p_f = (\prod_{i=1}^{n} A_i)\Phi(-\delta) \tag{4.8}$$

A generalized safety index, β, can be defined as

$$\beta = -\Phi^{-1}(p_f) \tag{4.9}$$

In applying the least squares scheme, the original function is the product of the original cdf and a "weighting function" that is constructed from the initial R-F results. The approximating function is the product of the new equivalent normal cdf and the same weighting function.

In the following, the method will be developed starting from a two-variable case and then extends to the high dimensional cases. Each variable, X_i, will be identified either as a "strength" variable or a "stress" variable using R-F results. When the reduced design point value u_i^* is negative (i.e., $x_i^* < \mu_{Ni}$), X_i is defined as a strength variable, otherwise X_i is a stress variable.

Consider a limit state function involving a strength variable X_1 and a stress variable X_2, the failure probability is

$$p_f = \int_{-\infty}^{\infty} F_1(x_1) f_2(x_2) dx_1 \tag{4.10}$$

in which the integration is to be performed on x_1 axis, and x_2 is evaluated according to the limit state. In the first step, assume that the R-F results are obtained, and Eq. 4.10 is approximated by

$$p_f' = \int_{-\infty}^{\infty} \Phi_1'(x_1)\phi_2'(x_2)dx_1 \tag{4.11}$$

in which the "prime" denotes that the normal functions are evaluated using R-F equivalent normal parameters. Another approximation of p_f, denoted as p_f'', uses F_1 instead of Φ_1' in Eq. 4.11,

$$p_f'' = \int_{-\infty}^{\infty} F_1(x_1)\phi_2'(x_2)dx_1 \tag{4.12}$$

Thus only one approximation, ϕ_2', is involved in Eq. 4.12. Now consider reconstructing Φ_1' by a three-parameter normal cdf, $A_1\Phi_1$. The integrand $F_1\phi_2'$ in Eq. 4.12, as illustrated in Fig. 4.1, will be curve-fitted by $A_1\Phi_1\phi_2'$ such that Eq. 4.12 becomes

$$p_f'' = \int_{-\infty}^{\infty} A_1\Phi_1(x_1)\phi_2'(x_2)dx_1 \tag{4.13}$$

where A_1 may be considered as a "scale factor" of Φ_1. It seems obvious that $A_1\Phi_1$ has more flexibility than Φ_1 to fit F_1 more closely in a region that is critical for p_f calculation.

The error sum, E, of two curves, $F_1\phi_2'$ and $A\Phi_1\phi_2'$, is

$$E = \int_{-\infty}^{\infty} [(A_1\Phi_1(x) - F_1(x))\phi_2'(x_2)]^2 dx_1 \tag{4.14}$$

which is a function of A_1, μ_{N1}, σ_{N1}. These three parameters need be found to minimize the error sum,

$$\text{Min: } E$$

$$\text{Subject to } g(\underset{\sim}{X}) = 0 \qquad (4.15)$$

and may be solved using suitable minimization routine. However, the process may be somewhat time consuming. In the following, the problem is simplified by imposing two constraints to Eq. 4.14.

A reasonable choice of putting constraints to E is to fit cdf and pdf at the design point, i.e.,

$$A\Phi(u^*) = F(x^*) \qquad (4.16)$$

$$A\,\frac{\Phi(u^*)}{\sigma_N} = f(x^*) \qquad (4.17)$$

where $\Phi(u^*)$ and $\phi(u^*)$ are the "standardized" normal cdf and pdf, respectively, evaluated at $u^* = (x^* - \mu_N)/\sigma_N$. From Eq. 4.16, the standardized normal variate, u^*, is computed first for any given A,

$$u^* = \Phi^{-1}\left(\frac{F(x^*)}{A}\right) \qquad (4.18)$$

σ_{N1} can be found from Eq. 4.17,

$$\sigma_N = A\,\frac{\phi(u^*)}{f(x^*)} \qquad (4.19)$$

μ_{N1} can be calculated using Eq. 4.7 after u^* and σ_N are determined,

$$\mu_N = x^* - u^*\sigma \qquad (4.20)$$

μ_N and σ_N are used to evaluate $\Phi(x)$. Therefore the error sum (Eq. 4.14) reduces to a function of one variable only. A value of A_1 with the corresponding μ_{N1} and σ_{N1} which minimize E are the resulting three "optimum" parameters.

The above formulation has assumed that X_1 is a strength variable. If X_1 is instead, a stress variable, Eq. 4.12 should be written as

$$p_f'' = \int_{-\infty}^{\infty} (1 - F_1(x_1))\phi_2'(x_2)dx_1 \tag{4.21}$$

which will be approximated by

$$p_f'' = \int_{-\infty}^{\infty} A(1 - \Phi_1(x_1))\phi_2'(x_2)dx_1 \tag{4.22}$$

Therefore, the only changes that have to be made are to replace F_1 by $(1 - F_1)$ and replace Φ_1 by $(1 - \Phi_1)$ in Eq. 4.14 and Eq. 4.16. Also Eq. 4.18 becomes

$$u^* = \Phi^{-1} \left[\frac{A - (1 - F(x^*))}{A} \right] \tag{4.23}$$

After $A_1\Phi_1$ is generated, the same procedure is applied to the X_2 variable. Afterwards, the minimum distance, δ can be generated using the derived equivalent normal mean values and standard deviations. The probability of failure and the generalized safety index are computed using Eq. 4.8 and Eq. 4.9.

The procedures suggested above can be easily extended to the cases involving more variables. Assume that the initial R-F program has been run and the R-F equivalent normal parameters are computed. To find three parameters for X_1, X_2 through X_n terms in Eq. 4.5 may be added together to form a new variable, defined as Y. Let

$$Y = a_2X_2 + \ldots + a_nX_n \tag{4.24}$$

Because X_2 to X_n are "normal", Y is normal. Eq. 4.5 becomes

$$g(\underset{\sim}{X}) = a_o + a_1X_1 + Y \tag{4.25}$$

and the probability of failure, assuming X_1 is a strength variable, is approximated by

$$p_f'' = \int_{-\infty}^{\infty} F_1(x_1)\phi_2'(y)dx_1 \tag{4.26}$$

Therefore the problem can still be solved on a two dimensional coordinates. Furthermore, it will be shown later that ϕ_2' can be easily established using design point information.

Before a numerical scheme is devised to evaluate the error sum conviently, it must be pointed out that the error of $\phi_2'(y)$ in Eq. 4.26 does not necessarily affect, proportionally, the p_f estimate. The reason is that Eq. 4.26 (or a modified form for stress variables) is applied to all the n variables, and p_f is not computed until all the equivalent normal parameters are obtained.

In R-F method, only the value of f_2 at the design point plays a role in the analysis. In the present method, it is suggested that the entire f_2 function approximated by an equivalent normal pdf ϕ_2' obtained from R-F analysis be considered. The reason for including ϕ_2' is that $F_1\phi_2'$ over a wide region of x_1 is closely related to p_f and is able to provide more information, compared with a single design point, for a better fitting of F_1.

The R-F result provides a good fit in the neighborhood of the design point. Therefore, it can be expected that ϕ_2' would be close to f_2 in the region near the design point. ϕ_2' may be less accurate outside the region, but F_2f_2 is also getting smaller and less important. In general, it is expected that $F_1\phi_2'$ is able to describe F_1f_2 closely in a relatively wide region, for example, a region which contains 80% of the total p_f. In applying the least squares scheme the fitting range should then cover a sufficiently wide region of x_1.

Although the goal is to fit F_1, ϕ_2' is included for the curve fitting. Therefore, ϕ_2' is the "weighting function" mentioned above. It is a function which provides the information on the relative importance of the F_1 function; e.g., a high value of F_1 is not necessarily considered important if ϕ_2' is very small, etc.

4.4 A Numerical Scheme of Finding Equivalent Normal Parameters

The error sum for a strength variable X is

$$E = \int_{-\infty}^{\infty} [\ (A\Phi(x) - F(x))\ \phi'(y)]^2 dx \qquad (4.27)$$

in which F and $A\Phi$ denote the "original" and the "transformed" cdf, respectively, and ϕ' is the R-F estimate of the joint pdf of the other variables.

The problem is solved numerically by choosing N "equally spaced" x values to approximate E by a summation,

$$E = \sum_{j=1}^{N} [(A\Phi(x_j) - F(x_j))\phi'(y_j)]^2 \, \Delta x \tag{4.28}$$

where Δx may be deleted in the minimization procedure because it is a constant. Instead of choosing x_i directly, one can pick equally spaced points from the R-F limit state on the reduced coordinates as shown in Fig. 4.2 in which the reduced variable of X is defined as

$$z = \frac{x - \mu}{\sigma} \tag{4.29}$$

where μ and σ are the R-F equivalent normal mean and standard deviation of X. Note that if μ_N and σ_N are used in Eq. 4.29 then the reduced variable is defined as u (Eq. 4.7). In Fig. 4.2, P is the design point, OP defines the R-F safety index β. The limit state in the figure can be constructed using only β and the directional angle, θ, which can be computed as

$$\theta = \cos^{-1}\left(\frac{|z^*|}{\beta}\right) \tag{4.30}$$

in which z^* is the design point value. For any point j on the limit state, the corresponding z_j can be computed as

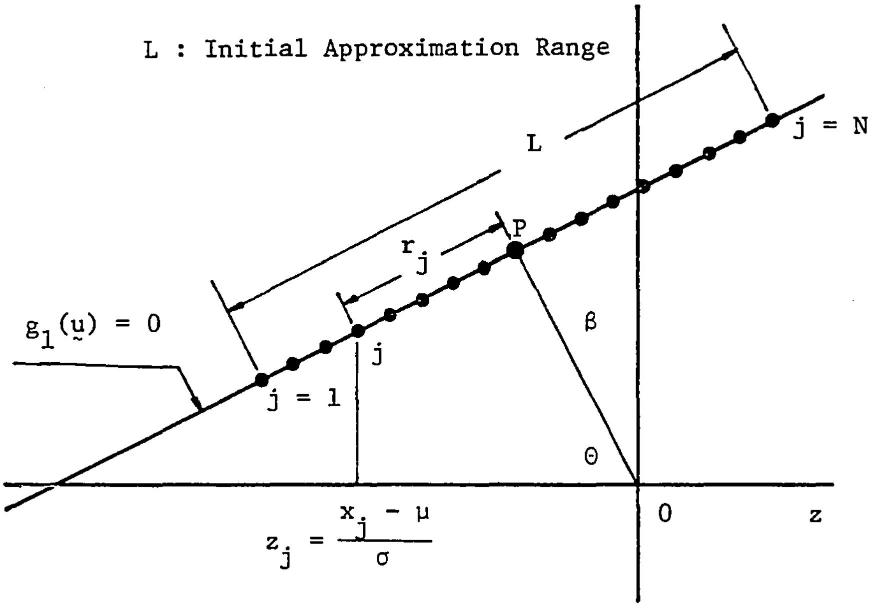


Figure 4.2 Determining the Initial Approximation Range and Points on the Limit State using R-F Results.

$$z_j = z^* \pm |r_j| \sin\theta \tag{4.31}$$

in which r_j is the distance measured from the design point P to the jth point on the limit state as shown in Fig. 4.2. The corresponding $\phi'(y_j)$ is

$$\phi'(y_j) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\beta^2 + r_j^2 - z_j^2}{2}} \tag{4.32}$$

and the corresponding x_j, using Eq. 4.29, is

$$x_j = \mu + z_j\sigma \tag{4.33}$$

Because the limit state is linear, it is clear that x_j will be equally spaced.

The "initial" approximating range on the limit state, denoted as L, and the number of points, N, as shown in Fig. 4.2 should be chosen such that the "discrete" $F_j\phi_j'$ function can fully describe the "continuous" $F\phi'$ function. The approximating range is determined as follows. The joint pdf at the design point, which is assumed to be near maximum on the limit state, is

$$(\underset{\sim}{\phi}')^* = \frac{1}{2\pi} e^{-\frac{\beta^2}{2}} \tag{4.34}$$

and the joint pdf at a point j is

$$(\underset{\sim}{\phi}')_j = \frac{1}{2\pi} e^{-\frac{\beta^2 + r_j^2}{2}} \tag{4.35}$$

Dividing Eq. 4.35 by Eq. 4.34,

$$\frac{(\underset{\sim}{\phi}')_j}{(\underset{\sim}{\phi}')^*} = e^{-\frac{r_j^2}{2}} \tag{4.36}$$

For j = 1 or j = N (the two end points), the above ratio should be sufficiently small to ensure a sufficiently wide region that contains essentially the whole p_f contents. For example, if L = 8, at the ends of the approximating range $r_j = \pm 4$, the ratio is 0.003, which is considered small enough. For L = 16, the ratio is 1.2×10^{-14}. A more accurate requirement for having a sufficiently wide L is that the ratio of $F_j\phi_j'$ to $(F\phi')_{max}$ for j = 1 and j = N should be small, say less than 20%. In this study, the "reduced" approximation range and points are defined by the following criteria

$$\frac{F_j\phi_j'}{(F\phi')^*} \begin{cases} \geq 0.2 \text{ for } j = N_A, N_B \\ < 0.2 \text{ for } j = 1 \text{ or } N \end{cases} \tag{4.37}$$

which is illustrated in Fig. 4.3, where $(F\phi')^*$ is evaluated at j = (N + 1)/2; N_A and N_B are the end points of the reduced approximation points. N is chosen as N - 1 = 5L. For example, if L = 8, then N = 41. When the directional angle θ approaches $\pi/2$, the approximation range of z approaches L, and Δx is the widest for a fixed N points, therefore maximum Δz (i.e., L/(N - 1)) is 0.2 and maximum Δx is 0.2σ (0.2σ per point).

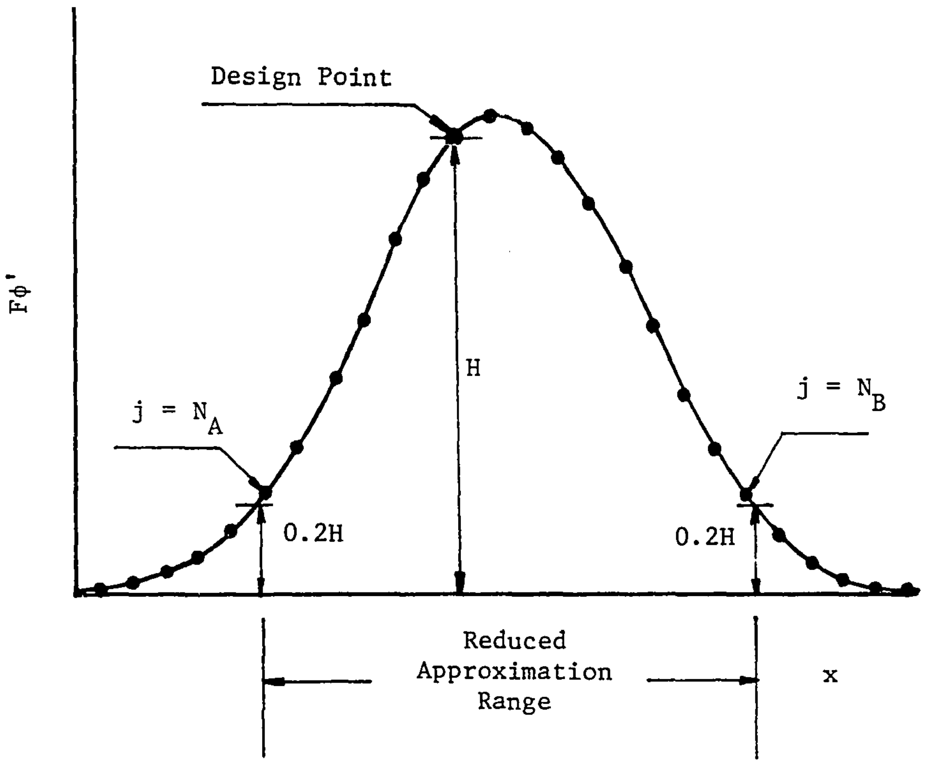


Figure 4.3 Define Reduced Approximating Range and Points.

It is noted that in Eq. 4.37, $F\phi'$ evaluated at the design point is in general not the peak value. The value, 0.2, in Eq. 4.37 is considered as a "reasonable" choice. It should be small enough to cover most of the area under the $F\phi'$ curve. On the other hand, less points are needed to be included in minimizing E if the value is chosen large. It has been observed that the resulting p_f estimates are nearly the same when the value changes to 0.1.

It is important to note that in the present method, either Eq. 4.37 must be satisfied or R-F equivalent normal parameters be used. When z^* is very small such that θ near $\pi/2$, it may require a very large L (e.g. L > 16) to satisfy Eq. 4.37, however, this implies that the variable has very low sensitivity; it does not contribute to β. In this study L is first chosen as 8 for each variable. If $N_A = 1$ or $N_B = N$ (Eq. 4.37 is not satisfied), the next step is to choose L = 16 and N = 81. If the criteria is still violated, it will be assumed that R-F result for the variable is already reasonable; no further improvement is necessary. It was found that L = 8 is usually an effective choice and L = 16 is wide enough for almost every example examined.

In general not all of the N points have to be included in minimizing E. For example, if N = 41, the reduced approximation points may be determined as, e.g., from $N_A = 10$ to $N_B = 30$. Only these points need to be included in Eq. 4.28, i.e., Eq. 4.28 reduces to

$$E = \sum_{j=N_A}^{N_B} [(A\Phi(x_j) - F(x_j))\phi_j']^2 \tag{4.38}$$

In the final step, the values of E are computed for different values of A. Because $F(x_j)$ and ϕ_j' are computed immediately after R-F results are obtained, only $\Phi(x_j)$ have to be recomputed using Eq. 4.18 to Eq. 4.20 in the minimization process. The value of A which minimizes E can be found using a suitable optimization routine or other efficient search procedures. As described before, minor modifications are required if X is a stress variable.

It should be noted that there is a lower limit for A. For a strength variable, because

$$\Phi(u^*) = \frac{F(x^*)}{A} < 1 \tag{4.39}$$

therefore

$$A > F(x^*) \tag{4.40}$$

Similarly, for a stress variable,

$$A > 1 - F(x^*) \tag{4.41}$$

When a basic variable is not significantly non-normal, the scale A may be close to unity. In application, A's as small as 10^{-2} have been observed for strength variables, large A's such as 10^8 have been generated for a stress (EVD) variable. In both instances, the variables are significantly non-normal. However, in such extreme cases, it was observed that the accuracy of A is not very critical and it is more efficient to stop the minimization process as soon as E stabilized.

4.5 Summary Procedures

The above numerical scheme was employed to develop a user-oriented reliability program. The input of the new program (Appendix D) are the distributional information (type code and the distribution parameters) of the n design variables and the limit state. The automated numerical procedures for computing the generalized safety index and p_f estimate may be summarized as follows:

1. Generate R-F β and z_i^*; $z_i^*(i = 1,n)$ are used to identify strength ($z_i^* < 0$) and stress ($z_i^* > 0$) variables.
2. Calculate r_j (Fig. 4.2) for L = 8 and N = 41; j = 1,N.
3. Calculate the directional angle, θ (Eq. 4.30).
4. Compute z_j and ϕ_j' (Eq. 4.31 and Eq. 4.32).
5. Compute x_j (Eq. 4.33) and $F(x_j)$; replace F by 1 - F for a stress variable defined in Step 1.
6. Calculate $F_j\phi_j'$ and $f(x^*)$, (j = (N + 1)/2 is the design point); compare each $F_j\phi_j'$ with $(F\phi')^*$ to determine the end points, N_A and N_B, of the reduced approximation points using Eq. 4.37. If $N_A = 1$ or $N_B = 41$, L and N increase to L = 16 and N = 81, then go to step 2. In the second iteration, if $N_A = 1$ or $N_B = 81$, skip step 7, and use R-F equivalent normal parameters in step 9.

7. Use an optimization routine to minimize E (Eq. 4.38) where x_j are defined in step 5, and $\Phi(x_j)$ are evaluated using μ_{Ni} and σ_{Ni} determined from Eq. 4.18 to Eq. 4.20 for any assigned A (replace Φ by $1 - \Phi$ in Eq. 4.38, and replace Eq. 4.18 by Eq. 4.23 for a stress variable defined in step 1). Three parameters A_i, μ_{Ni} and σ_{Ni} are found after E stabilized.
8. Repeat step 2 through 7 for each non-normal variable.
9. Find the minimum distance, δ, using μ_{Ni} and σ_{Ni} $(i = 1, n)$.
10. $p_f = \prod_{i=1}^{n} A_i \Phi(-\delta)$ where $A_i = 1$ for normal variables.
11. The generalized safety index is computed as $\beta = -\Phi^{-1}(p_f)$.

4.6 Examples

A computer program was developed (Appendix D) to examine the performance of the proposed numerical scheme. Three non-normal statistical distributions can be transformed to equivalent normal distributions in the current version of the program. For the purpose of brevity, the following notation is used to represent the distributions throughout; Normal (N), Lognormal (LN), Weibull (WEI), Type I Extreme Value distribution of maximum (EVD).

The following examples compare p_f estimates as computed by R-F and the present method with the values obtained by numerical integration and Monte Carlo. Numerical integration is used for two variables; Monte Carlo is used for three or more variables.

The estimate of p_f using Monte Carlo is

$$\hat{p} = \frac{n_o}{K} \tag{4.43}$$

where K is the sample size and n_o is the number of samples within the failure domain. Noting that $\hat{p}$ is the maximum likelihood estimator (MLE) for the binomial parameter, the upper and lower 1% confidence limits of $\hat{p}$ denoted as p_U and p_L respectively, are calculated. Using $\hat{p}$, an "equivalent β" is evaluated as $\beta = -\Phi^{-1}(\hat{p})$.

The examples include linear as well as nonlinear limit states with design variables having various combinations of the four distributions.

Example 1. The limit state function is

$$g(\underset{\sim}{X}) = R - (L + D) \tag{4.43}$$

where R is a strength variable, L and D are the stress variables. The distributions of R, L and D are shown in Table 4.1.

For a Monte Carlo solution, K = 100,000 samples are generated. The number of g values less than zero is 350. According to Eq. 4.42, the estimate of p_f is $\hat{p}$ = .0035. The 98% confidence limits are calculated as p_L = .0031 and p_U = .0040. The R-F estimate is .0028, and the present method generates an estimate of .00335. Comparisons of the above three methods are summarized in Table 4.1.

Example 2. This example is a fatigue problem involving the general strain - life relationship to define fatigue strength

$$\varepsilon_a = \frac{\sigma_f'}{E} (2N)^b + \varepsilon_f'(2N)^c \tag{4.44}$$

Table 4.1 A Summary of Example 1

-DATA

Failure Function $g = R - (L + D)$

Failure Event ($g \leq 0$)

Variable	Distribution	Mean	Standard Deviation
R	WEI	50.	5.0
L	EVD	10.	2.0
D	LN	20.[a]	0.15[b]

[a] Median

[b] COV

-MONTE CARLO RESULTS

Sample Size, K	100,000
Number of values of $g \leq 0$, n_o	350
Estimate of p_f, $f_o = (n_o/K)$	0.0035

-DESIGN POINT (point on failure surface which is approximately the "most probable" value)

Variable	R-F	Proposed
R	34.60	34.39
L	11.13	10.98
D	23.46	23.41

-COMPARISONS OF SAFETY INDEX AND PROBABILITY OF FAILURE

	Safety Index, β	Prob. of Failure, p_f
R-F	2.77	2.82 E-3
Proposed	2.71	3.35 E-3
Monte Carlo	2.70	3.50 E-3[a]

a. 98% confidence limits: p_L = 3.1 E-3, p_U = 4.0 E-3

The failure function may be written as

$$g(\underset{\sim}{X}) = \varepsilon_a - \varepsilon_S \tag{4.45}$$

where the strain amplitude ε_a is identified as a strength variable, and the service strain amplitude ε_S is identified as a stress variable. Substituting Eq. 4.44 into Eq. 4.45, the failure function is of the form

$$g(\underset{\sim}{X}) = \frac{\sigma_f'}{E} (2N)^b + \varepsilon_f'(2N)^c - \varepsilon_S \tag{4.46}$$

For this case, three design parameters ε_S, σ_f' and ε_f' are considered as random variables with the distributional information given in Table 4.2. The constants are b = -0.1007, c = -1.017 and E = 30 ksi. Also the fatigue life is N = 10,000.

For a Monte Carlo solution, K = 500,000 samples are generated. Comparisons of R-F, present method and Monte Carlo are summarized in Table 4.2.

Example 3. This example involves a cylindrical pressure vessel having an external torque, T, and internal pressure, P. Failure occurs when Von Mises stress exceeds the yield strength, R. When numerical values of diameter, thickness are introduced into the stress, the limit state function is of the form (McLain, Kelly and Wirsching, 1982)

$$g(\underset{\sim}{X}) = R - \sqrt{300P^2 + 1.92T^2} \tag{4.47}$$

TABLE 4.2

A Summary of Example 2

-FAILURE FUNCTION

$$g = \frac{\sigma_f'}{E} (2N)^b + \varepsilon_f' (2N)^c - \varepsilon_S$$

-DESIGN VARIABLES

Variable	Distribution	Mean	Std. Dev.
ε_S	EVD	.0015	.00015
σ_f'	LN	310.*	.43*
ε_f'	LN	9.14*	.05*

*Median and COV

-DESIGN POINT

Variable	R-F	Proposed
ε_S	1.641 E-3	1.591 E-3
σ_f'	1.022 E+2	9.818 E+1
ε_f'	9.094 E+0	9.097 E+0

-COMPARISONS OF SAFETY INDEX AND PROBABILITY OF FAILURE

	Safety Index, β	Prob. of Failure, p_f
R-F	2.88	1.98 E-3
Proposed	2.87	2.07 E-3
Monte Carlo[a]	2.86	2.13 E-3[b]

a. Sample Size = 500,000
b. 98% Confidence limits, p_L = 1.98 E-3, p_U = 2.29 E-3

where the design parameters R, P and T have the distributional information given in Table 4.3.

K = 100,000 samples are generated for a Monte Carlo solution. Comparisons of R-F, present method and Monte Carlo are summarized in Table 4.3.

Example 4. This is an example used to demonstrate a second order reliability method developed by Tvedt (1983). The failure function which is associated with a simple structure can be written as

$$g(\underset{\sim}{X}) = X_1 + 2X_2 + 2X_3 + X_4 - 5(X_5 + X_6) \tag{4.48}$$

All six variables are lognormally distributed with medians and coefficient of variations given in Table 4.4.

The result of a Monte Carlo simulation with K = 200,000 yields a p_f estimate of $\hat{p}$ = .0121. The 98% confidence limits are calculated as p_L = .00115 and p_U = .00127. R-F estimate is p_f = .0094, and the present method has an estimate of p_f = .0121. Summary results are given in Table 4.4 (the results of the second order methods are p_f = .01233 using a "three term formula", and p_f = .01215 using a quadrature approximation).

Example 5. Following is an analysis of the basic case where the limit state is linear.

$$g(\underset{\sim}{X}) = R - S \tag{4.49}$$

Comparison of R-F, present method and the numerical integration computations is made for two different cases. The results are presented

TABLE 4.3

A Summary of Example 3

-FAILURE FUNCTION

$$g = R - \sqrt{300\,P^2 + 1.92T^2}$$

-DESIGN VARIABLES

Variable	Distribution	Mean	Std. Dev.
R	WEI	48.0	3.0
P	LN	.987*	.16*
T	EVD	20.0	2.0

*Median and COV

-DESIGN POINT

Variable	R-F	Proposed
R	39.31	38.60
P	1.092	1.104
T	24.87	24.20

-COMPARISONS OF SAFETY INDEX AND PROBABILITY OF FAILURE

	Safety Index, β	Prob. of Failure, p_f
R-F	3.09	.988 E-3
Proposed	2.95	1.60 E-3
Monte Carlo[a]	2.96	1.54 E-3[b]

a. Sample Size = 100,000
b. 98% Confidence limits: p_L = 1.28 E-3, p_U = 1.86 E-3

Table 4.4 A Summary of Example 4

-DATA

Failure Function; $g = X_1 + 2X_2 + 2X_3 + X_4 - 5(X_5 + X_6)$

Variables	Distributions	Median	C.O.V.
X_1	LN	119.4	.1
X_2	LN	119.4	.1
X_3	LN	119.4	.1
X_4	LN	119.4	.1
X_5	LN	383.1	.3
X_6	LN	478.9	.3

-DESIGN POINT

Variable	R-F	Proposed
X_1	117.3	117.0
X_2	115.2	114.5
X_3	115.2	114.5
X_4	117.3	117.0
X_5	55.53	55.53
X_6	83.57	82.89

-COMPARISONS OF SAFETY INDEX AND PROBABILITY OF FAILURE

	Safety Index, β	Prob. of Failure, p_f
R-F	2.35	.941 E-2
Proposed	2.25	1.21 E-2
Monte Carlo [a]	2.25	1.21 E-2 [b]

a. Sample Size = 200,000
b. 98% Confidence Limits, p_L = 1.15 E-2, p_U = 1.29 E-2

in Table 4.5. Each case represents a different combination of strength and stress distributions. The numerical integration results were computed using Simpsons's rule over a sufficiently wide region, and are considered as "exact" solution.

Example 6. The simple strength-stress relationship is extended to the cases where the stress is a non-linear function.

$$g(\underset{\sim}{X}) = R - V^k \quad , \quad k = 2, 3, 4 \tag{4.50}$$

Where V^k is identified as the "stress" function, S. The relationship between μ's and σ's for S and V is constructed using the truncated Taylor series expansion,

$$\mu_S = (\mu_V)^k + \frac{1}{2}\frac{\partial^2(V^k)}{\partial V^2} \cdot \sigma_V{}^2 \tag{4.51}$$

$$\sigma_S = \frac{\partial(V^k)}{\partial V} \cdot \sigma_V \tag{4.52}$$

where the partials are evaluated at μ_V.

The following cases are the same as in the previous study (Appendix C), but only part of the cases where R-F have larger errors are examined here for comparison purpose. Values of β and p_f for various combinations of distributions and different k are summarized in Table 4.6.

4.7 Conclusions

The Rackwitz-Fiessler algorithm is efficient in estimating probability of failure because the exact transformation, $\Phi(u) = F(x)$,

TABLE 4.5

Example 5; Comparisons of Reliability Calculations

Limit State Function $g = R - S$; $\mu_R = 20$ and $\mu_S = 10$ for both cases

Case A; R = Weibull, S = EVD

σ_R	σ_S	Safety Index, β: Numerical Integ.	R-F	Proposed	Probability of Failure, p_f: Numerical Integ.	R-F	% Error	Proposed	% Error
3.0	3.0	2.048	2.136	2.041	2.031 E-02	1.633 E-02	19.6	2.064 E-02	1.7
2.0	2.0	2.757	2.855	2.754	2.920 E-03	2.152 E-03	26.3	2.946 E-03	0.9
1.2	1.2	3.897	3.992	3.914	4.873 E-05	3.269 E-05	32.9	4.546 E-05	6.7
3.0	1.2	2.614	2.681	2.644	4.479 E-03	3.667 E-03	18.1	4.095 E-03	8.5
1.2	3.0	2.365	2.404	2.381	9.027 E-03	8.102 E-03	10.2	8.637 E-03	4.3

Case B; R = Weibull, S = Normal

σ_R	σ_S	Safety Index, β: Numerical Integ.	R-F	Proposed	Probability of Failure, p_f: Numerical Integ.	R-F	% Error	Proposed	% Error
3.0	3.0	2.207	2.217	2.206	1.366 E-02	1.329 E-02	2.7	1.369 E-02	0.3
2.0	2.0	3.086	3.093	3.086	10.14 E-04	9.878 E-04	2.6	10.13 E-04	0.0
1.2	1.2	4.502	4.504	4.503	3.353 E-06	3.332 E-06	0.6	3.346 E-05	0.2
3.0	1.2	2.649	2.647	2.649	4.038 E-03	4.061 E-03	0.6	4.037 E-03	0.0
1.2	3.0	3.020	3.075	3.028	1.263 E-03	1.050 E-03	16.9	1.232 E-03	2.5

Table 4.6 Example 6; Comparisons of Reliability Calculations

Failure Function, $g = R - S = R - V^k$; $\mu_R = 20$ and $\mu_S = 10$ in all cases

(a) Distributions

Case	R dist.	V dist.	k	σ_R	σ_S
1	LN	EVD	2	3.0	1.2
2	WEI	EVD	2	3.0	3.0
3	WEI	EVD	2	1.2	1.2
4	WEI	EVD	3	3.0	3.0
5	WEI	EVD	3	3.0	1.2
6	WEI	EVD	4	3.0	3.0
7	WEI	EVD	4	3.0	1.2

(b) Comparisons of Safety Index, β, and Probability of Failure, p_f

	Safety Index, β			Prob. of Failure, p_f				
Case	Numerical Integ.	R-F	Proposed	Numerical Integ.	R-F	% Error	Proposed	% Error
1	3.178	3.205	3.192	7.432 E-4	6.755 E-4	9.1	7.080 E-4	4.7
2	1.931	2.029	1.959	2.676 E-2	2.126 E-2	20.5	2.506 E-2	6.4
3	3.619	3.695	3.649	1.478 E-4	1.100 E-4	25.6	1.317 E-4	10.9
4	1.187	1.232	1.210	1.175 E-1	1.090 E-1	7.2	1.131 E-1	3.7
5	2.005	2.120	2.044	2.248 E-2	1.700 E-2	24.3	2.047 E-2	8.9
6	0.903	0.933	0.921	1.831 E-1	1.753 E-1	4.3	1.785 E-1	2.5
7	1.603	1.681	1.641	5.441 E-2	4.643 E-2	14.7	5.051 E-2	7.3

is approximated by finding the equivalent normal parameters consistent with the Hasofer-Lind safety index concept. As described in Chapter 2, the estimate of p_f is sufficiently accurate for practical engineering problems as long as the design variables are not significantly deviate from the normal and the limit state is not extremely nonlinear.

The proposed method is conceptually as simple as the first order reliability method, however, more reasonable equivalent normals may be constructed by solving an optimization problem. As demonstrated by the examples, much better p_f estimates than the R-F method may be produced for the originally linear limit states even when the design variables are significantly non-normal. For the more complicated nonlinear limit states, better p_f estimates are also observed.

Because the present method has assumed that the original limit state is linear, the estimate of p_f may be less accurate for highly nonlinear limit states. However, highly nonlinear limit states seems to be rare in many engineering applications (as suggested by Fiessler et al., 1979, the curvature at the design point seldom exceeds 0.05). Furthermore, modelling errors (non-statistical uncertainty) are difficult to assess; therefore, extreme accuracy does not seem to be necessary. In general, the error of p_f estimate due to the "normalization" approximation may be significantly reduced applying the present method. For highly nonlinear limit state, the differences in p_f estimates between the R-F and the present method may be considered roughly as a "measure" of the possible error introduced by the R-F equivalent normals.

The proposed algorithm is simple; minor programming work was required to modify a R-F program. The additional cost (over R-F), in terms of CPU time, is estimated to be less that 0.1 second for each non-normal variable when running on the CYBER 175 system.

CHAPTER 5

ESTABLISHING THE DESIGN REQUIREMENT

5.1 Introduction

Methods described in the earlier chapters provide estimates of the probability of failure p_f of an existing design. The limit state function and statistics of the design variables must be known. The reliability analysis produces the safety index β, an estimate of p_f, and the design point which is at or near the point where the failure is most likely to occur.

The inverse problem is to construct a design rule which provides a specified reliability or target safety index. Such rules or statements typically reside in design codes and design criteria documents. As described earlier, methods of safety checking have been divided into three broad classes, namely Level 3, Level 2 and Level 1. This chapter addresses the problem of developing Level 1 safety check expressions using Level 2 methods to derive partial safety factors. In general, the goal of this chapter is to provide a framework for constructing prescriptions for safe design which are in a familiar form for designers. As a ground rule, it is established that designers are not going to be required to apply probability theory.

The conventional working stress design (WSD), uses a single factor of safety, ν, to establish a design rule or safety check expression,

$$S_n < \frac{R_n}{\nu} \tag{5.1}$$

in which S_n and R_n are nominal values, and $\nu > 1$. A reliability based safety check expression which employs partial safety factors (PSF's) applied to each variable has been developed by Ravindra and Galambos (1978). Load and resistance factor design (LRFD) has the basic form

$$\gamma S_n \leq \phi R_n \tag{5.2}$$

where γ and ϕ are the load and resistance factors respectively. In theory, a format in which each design factor has its own PSF has the promise of providing more efficient designs relative to WSD. Methods of probabilistic design are employed to derive the PSF's, γ and ϕ. More sophisticated safety check expressions using PSF's have been discussed in CIRIA report (1977), and by Thoft-Christensen and Baker (1983).

Consider the general "design by reliability" problem in which the maximum probability of failure is specified. In such cases, it is usually necessary to iterate (performing reliability computations) with respect to one of the design parameter, e.g., beam width. But in general, there may be several major design parameters in the limit state function. As a result, there are infinite possible combinations of the design values which have the same target reliability;

the design problem becomes intractable. It is, therefore, necessary to establish a systematic and simplified procedure for safety checking. It would be desirable to be able to derive a safety check expression of the form

$$g(\underset{\sim}{\gamma}\underset{\sim}{X}_n) \geq 0 \tag{5.3}$$

where $\underset{\sim}{\gamma}$ is a vector of partial safety factors, and $\underset{\sim}{X}_n$ is a vector of nominal values of the design factors. The PSF's should be derived from reliability considerations.

Using design point values resulting from reliability analysis, the partial safety factors can be derived such that there is a PSF associated with each basic variable, e.g., $\underset{\sim}{\gamma}$ as in Eq. 5.3. PSF's so derived should be constants. In general, however, each PSF depends on the values of the parameters of all the design variables, and on the form of the limit state function. Therefore, if the design requirements were to be expressed in terms of PSF's, it would be necessary to express the PSF's in terms of parameters. While the PSF approach enjoys current popularity, other possible forms of the safety check expression are also investigated in this study. Because PSF's are not all independent, it was found that a safety check expression relating only the independent design parameters seems more appropriate than the one using PSF's. In an example involving limit state function R = L + D, where R, L and D are design random variables, it was demonstrated that only one design curve is required using the suggested procedure; three PSF curves are required to define the safety check expressions. A generalized procedure is suggested to define the independent design parameters. The resulting safety check expression may be constructed using the "design surface" which separates the design parameter space into safe and failed region.

The study was extended to the design problems where the limit state function cannot be expressed in closed form. Applying the method suggested in Chapter 3, the limit state function is approximated by one or more second order polynomials, and then used to derive the safety check expressions. The suggested procedures are demonstrated using examples including dynamics problems and fatigue reliability problem. But the procedures are general, and may be applied to other design problems which require a complicated computer program to relate the design parameters.

5.2 The Design Point

Define the limit state as $g(\underset{\sim}{X}) = 0$, where $\underset{\sim}{X} = (X_1, X_2, .., X_n)$ is a vector of basic design variables with mean μ_i and standard deviation σ_i, $g(\underset{\sim}{X})$ is defined so that

$$g(\underset{\sim}{X}) < 0 \quad \text{defines failure region}$$

$$g(\underset{\sim}{X}) \geq 0 \quad \text{defines safe region}$$

A goal of design is to select $\underset{\sim}{X}$ so that $g \geq 0$.

Define the reduced coordinates as

$$u_i = \frac{X_i - \mu_i}{\sigma_i} \tag{5.4}$$

The u_i all have zero mean and a unit standard deviation. The corresponding limit state on the reduced coordinates system is

$$g_1(\underset{\sim}{u}) = 0 \tag{5.5}$$

which divides the safe and failed region in the u space. The design point is defined as the point on the reduced limit state which is closest to the origin, i.e., the design point is associated with the minimum distance, β, which is defined as the generalized safety index.

Assume that all X_i are normally distributed, the joint probability density can be expressed as, using Eq. 5.4,

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}) = \left(\frac{1}{2\pi}\right)^{n/2} \cdot \frac{1}{\sigma_1 \cdot \sigma_2 \cdots \sigma_n} \cdot e^{-\frac{\Delta^2}{2}} \tag{5.6}$$

where

$$\Delta^2 = u_1^2 + u_2^2 + \ldots u_n^2 \tag{5.7}$$

If $(u_1, u_2, \ldots u_n)$ is a point on the reduced limit state, Δ becomes the geometric distance measured from the origin to $g_1=0$. From Eq. 5.6, maximum $f_{\underset{\sim}{X}}(\underset{\sim}{x})$ occurs when Δ is a minimum, i.e., when $\Delta = \beta$. Thus, it can be concluded that the design point, assuming all X_i are normal variables, is a point of <u>maximum probability of failure density</u>.

(or maximum likelihood point). Because the design point defines the set of variables at which the failure is most likely to occur, it provides a rational means of choosing partial safety factors.

If one or more design variables are non-normal, the design point is usually defined after the variables are "normalized" using e.g., Rackwitz-Fiessler (R-F) algorithm. The design point found using R-F scheme is in general not the maximum likelihood point. In fact, Horne and Price (1976) have shown that for determining of a true maximum probability density point for the non-normal cases, the original distribution should be approximated by a normal distribution such that (1) the pdf, and (2) the derivative of pdf, are equal. It has been suggested (Shinozuka, 1983) that the point of maximum liklihood be used as the design point for non-normal cases; a procedure was suggested to find the safety index using maximum likelihood point.

In this study, procedures of deriving the partial safety factors and the safety check expression will be addressed. The R-F design point, which is an approximating maximum likelihood point, will be used to define partial safety factors as described in the following section.

5.3 Definition of Partial Safety Factors (PSF's)

In general all design parameters involved in a limit state function can be considered as random variables. In practice, some parameters such as geometric variables are usually treated as deterministic because they have small variances. The random variables may be separated into either the strength variables (R_i, failure when too small) or the stress variables (S_i, failure when too large), and the limit state function can be expressed as

$$g(\underset{\sim}{X}) = g(\underset{\sim}{R}, \underset{\sim}{S}, \underset{\sim}{A}) \tag{5.8}$$

where

$$\left.\begin{aligned} \underset{\sim}{R} &= (R_1, R_2, .., R_\ell) \\ \underset{\sim}{S} &= (S_1, S_2, .., S_m) \end{aligned}\right\} \text{random}$$

$$\underset{\sim}{A} = (A_1, A_2, .., A_k) \text{ ; deterministic}$$

The limit state is $g(\underset{\sim}{X}) = 0$, and a safe design results when $g(\underset{\sim}{X}) \geq 0$.

Assume that nominal values (R_{ni} and S_{ni}) are given for all design variables. Nominal values can be defined in several ways (Wirsching, 1981).

1. As a central tendency; e.g., mean, median, most likely value, or "best estimate".
2. As the extreme value in a sample, e.g., for strength, the smallest value observed.
3. As the extreme value in a sample plus a little "breathing room".
4. By a lower (or upper) tolerance limit or prediction interval. A simplified version of this form for two variables R and S is

$$R_n = \mu_R (1 - K_R C_R) \tag{5.9}$$

$$S_n = \mu_S (1 + K_S C_S) \tag{5.10}$$

where μ_R, μ_S are the means, K_R and K_S are the specified constants (in design practice, often values of 2 or 3 are chosen). C_R, C_S are the coefficient of variations of R and S, respectively. Thus in general R_n (strength) has a value less than μ_R and S_n (stress) is greater than μ_S because nominal values are always chosen on the safe side of a distribution.

In general, the safety check expression can have the form

$$g\left(\frac{R_{n1}}{\gamma_{R1}}, \ldots \frac{R_{n\ell}}{\gamma_{R\ell}}, \gamma_{S1} S_{n1}, \ldots \gamma_{Sm} S_{nm}, A_1, \ldots A_k\right) \geq 0 \tag{5.11}$$

for a safe design where

R_{ni} = nominal value of R_i

S_{ni} = nominal value of S_i

The PSF's (γ's) are defined so that they will tend to be greater than unity. Note that a PSF is identified with each variable.

If all variables in Eq. 5.8 have known distributions, a safety index β as well as the design point can be determined by the reliability analysis methods described before. Conversely, by adjusting the variables such as A_i, it is possible to obtain a design that satisfies a given design goal (i.e., given a β or p_f). The design point is on the failure surface and therefore, satisfies,

$$g(\underset{\sim}{R}^*, \underset{\sim}{S}^*, \underset{\sim}{A}^*) = 0 \tag{5.12}$$

Comparing Eq. 5.11 with Eq. 5.12, the PSF's are defined as

$$\gamma_{Ri} = \frac{R_{ni}}{R_i^*} \tag{5.13}$$

$$\gamma_{Si} = \frac{S_i^*}{S_{ni}} \tag{5.14}$$

A computer program has been developed to determined the PSF's (Appendix E). An undetermined design parameter can be automatically found to achieve a given design goal. PSF's are determined using design point information.

To illustrate how the PSF's can be derived as used in the safety check expression consider the following simple linear limit state cases.

Example 1. A bar having a cross sectional area A is subjected to a harmonic loading Q(t) as shown in Fig. 5.1. Assume that Q(t) causes only elastic deformation and that no endurance limit exists, fatigue strength can be defined as

$$NS^m = K \tag{5.15}$$

The material is Waspoloy B with m = 11.86; K is a random variable having lognormal distribution with $K = 3.04 \times 10^{28}$ ksi and $C_K = 0.65$. The fatigue stress amplitude is given as

$$S = \frac{Q}{A} \tag{5.16}$$

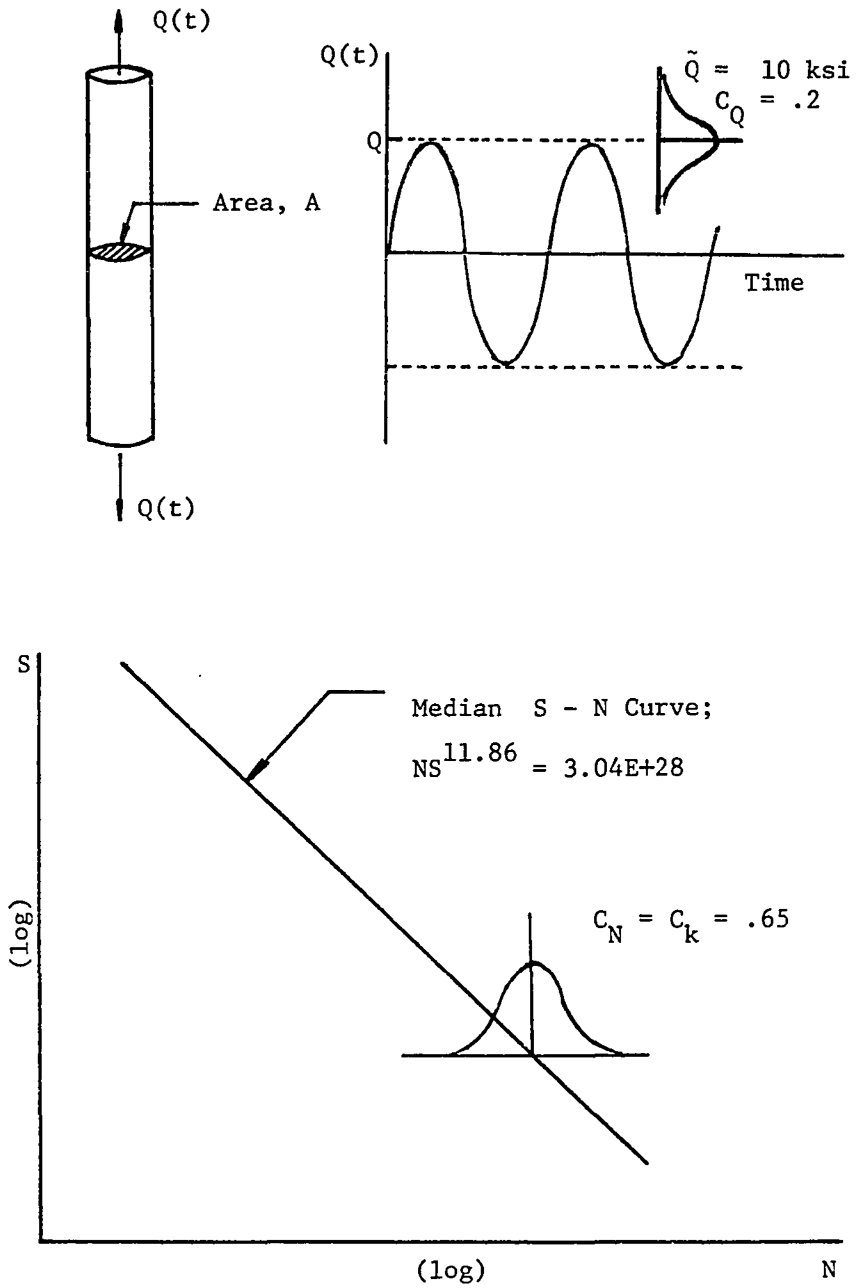


Figure 5.1 Median S - N Curve for Example 1.

where Q is the loading amplitude. Q is assumed to be lognormally distributed with $\bar{Q}$ = 10 ksi and $C_Q = 0.2$. Substituting Eq. 5.16 into Eq. 5.15 provides an expression for cycles to failure, N, which is a random variable because K and Q are random variables.

The design life is given as N_o, and the probability of failure is

$$p_f = P(N \leq N_o) \tag{5.17}$$

The target safety index is given as $\beta_o = 3$. The goal is to find the PSF's and derive the safety check expression.

To solve the problem, assume, for the moment, that the design life equals 2×10^6 cycles. The limit state becomes

$$2 \times 10^6 (\frac{Q}{A})^{11.86} = K \tag{5.18}$$

By adjusting A (a deterministic parameter), $\beta_o = 3$ can be achieved. The result of the reliability analysis is shown in Table 5.1 where A = 0.249.

Substituting the design point values of Table 5.1 into Eq. 5.18,

$$2 \times 10^6 \left(\frac{1.78\ \tilde{Q}}{0.249}\right)^{11.86} = \frac{\tilde{K}}{1.55} \tag{5.19}$$

where it has been assumed that $\tilde{Q}$ (stress) and $\tilde{K}$ (strength) are defined as nominal values Q_n and K_n.

TABLE 5.1

RESULTS OF EXAMPLE 1

Failure Function: (2.E+06)*(Q/A)**11.86 = K

Design Variables

Variable	Distribution	Nominal	Mean/Median	COV
K	log	.30400E+29	.30400E+29	.65000
Q	log	.10000E+02	.10000E+02	.20000

Note: The median is specified for a lognormal variable only.

Design Point

Variable	Reduced Value	Basic Value	Partial Safety Factor
K	XR(1) = -.73579	X(1) = .19642E+29	1.5477
Q	XR(2) = 2.90845	X(2) = .17788E+02	1.7788

Safety Index, Beta = 3.00

Scale Factor, A = .24878E+00

The design is considered safe for any choice of design parameters which satisfy

$$N_o\left(\frac{\gamma_Q \tilde{Q}}{A}\right)^{11.86} \leq \frac{\tilde{K}}{\gamma_K} \tag{5.20}$$

where

$$\gamma_Q = 1.78$$

$$\gamma_K = 1.55$$

Note that K is a strength variable. γ_K is defined as a factor which divides this variable so that it will tend to be greater than one. If the nominal values are defined differently, then the γ's should be adjusted accordingly.

For a chosen material, $\tilde{K}$ is determined. If it is assumed that C_Q and C_K are fixed, then for other values of $\tilde{Q}$ or N_o, the minimum required area A can be calculated directly from Eq. 5.19 without having to perform additional reliability analysis. In other words, the γ's would remain the same for other values of $\tilde{Q}$ and N_o. This conclusion is obvious if N_o, $\tilde{Q}$ and A in Eq. 5.20 are grouped together and treated as a single variable,

$$\left[N_o\left(\frac{\tilde{Q}}{A}\right)^{11.86}\right] \cdot (\gamma_Q)^{11.86} \leq \frac{\tilde{K}}{\gamma_K} \tag{5.21}$$

It is known that when $N_o = 2 \times 10^6$ and $\tilde{Q} = 10$ ksi, then A = 0.249, therefore

$$N_o(\frac{\tilde{Q}}{A})^{11.86} = 2.12 \times 10^{25} \qquad (5.22)$$

The safety index is 3 so long as Eq. 5.22 is satisfied, and the PSF's will be the same. It is noted that Eq. 5.22 actually provides an alternative design equation which does not involve PSF's.

A design S - N curve may be constructed using Eq. 5.20 which can be written as

$$N_o = (\frac{1}{\gamma_Q \tilde{S}})^{11.86}(\frac{\tilde{K}}{\gamma_K}) \qquad (5.23)$$

where $\tilde{S} = \tilde{Q}/A$. By substituting values of γ_Q, γ_K and $\tilde{K}$, and taking logarithms on both sides of Eq. 5.23,

$$\log N_o = 25.29 - 11.86 \log \tilde{S} \qquad (5.24)$$

Thus the S - N design curve based on PSF's is a straight line on the log - log coordinates as shown in Fig. 5.2. By substituting $\tilde{K}$ and $\tilde{S}$ into Eq. 5.15 (or by letting the γ's = 1 in Eq. 5.23), the "median line" is derived as

$$\log N_o = 28.5 - 11.86 \log \tilde{S} \qquad (5.25)$$

Given a S, N is lognormally distributed with $C_N = C_K = 0.65$. Define

$$Y = \log N \qquad (5.26)$$

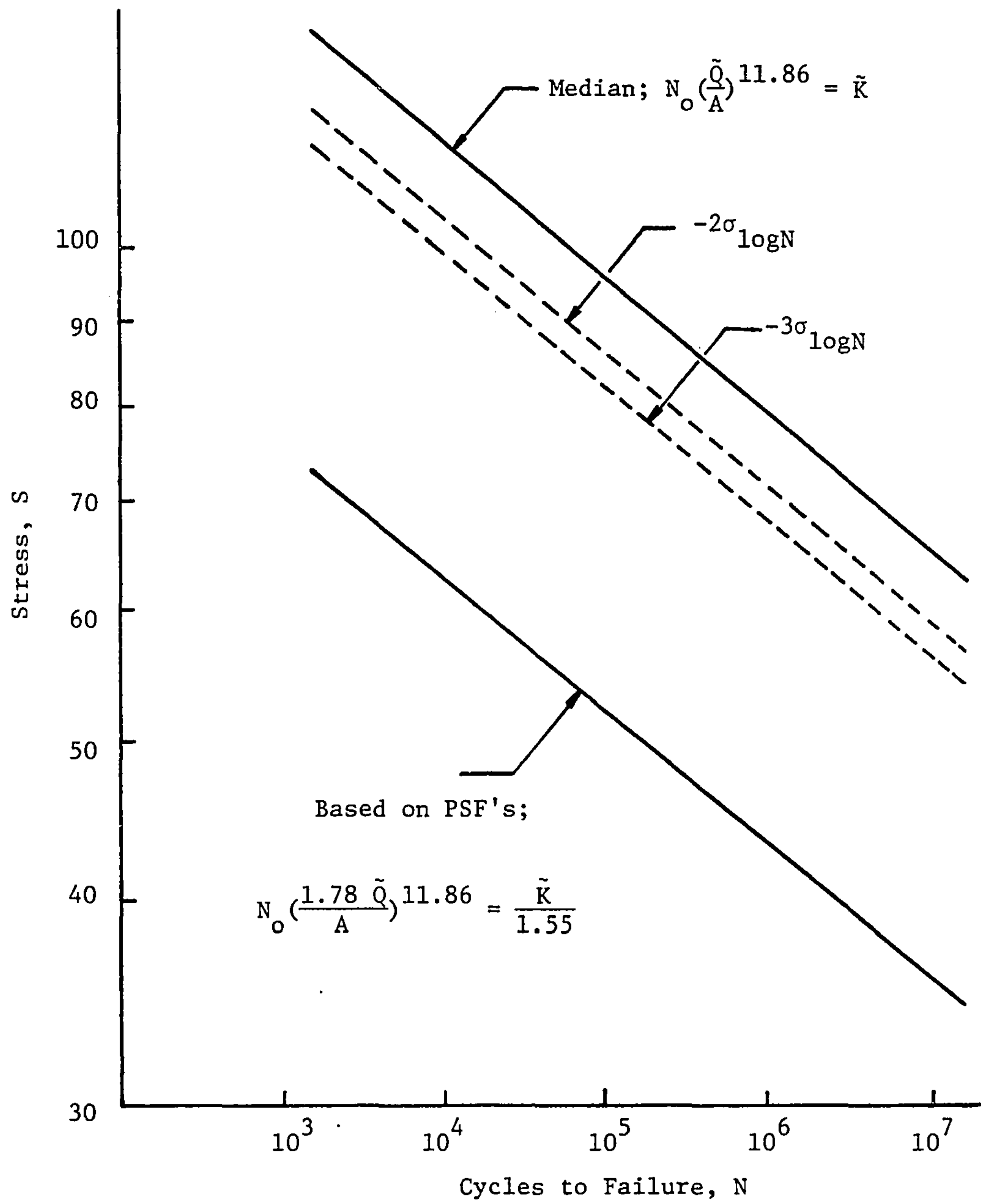


Figure 5.2 Design (S - N) Curve of Example 1.

the standard deviation of Y can be calculated as (Wirsching, 1981)

$$\sigma_Y = \sqrt{.434 \log(1 + C_N^2)} = 0.2577 \tag{5.27}$$

With the median curve as a reference, the lower 2σ design curve, using Eq. 5.27, is determined as

$$Y = \log N - 0.5154 \tag{5.28}$$

Similarly, a 3σ design curve may be found. The median, 2σ and 3σ design curves are also drawn in Fig. 5.2 for comparisons.

Example 2. This example involves three design random variables. The limit state is

$$AR = BD + L \tag{5.29}$$

where R is a strength variable, D and L are stress variables representing "dead" load and "live" load, respectively. A and B are undetermined deterministic parameters. A can be thought of as a member size where R is unit strength. B can be thought of as a factor defines the ratio of D to L. But in general, A and B can be considered as design parameters (or scale factors) to be determined in a design process.

Assume that R, D and L are specified, then A can be adjusted to achieve the reliability target if B (a constant) is also specified. Using the procedures as illustrated in Example 1, PSF's can be found for each variable and a safety check expression can be found as

$$A\left(\frac{R_n}{\gamma_R}\right) \geq B(\gamma_D D_n) + \gamma_L L_n \tag{5.30}$$

But the PSF's will be functions of B. This problem will be examined further in the next section.

In general, PSF's may be determined using reliability analysis by varying one design parameter. If there are several design parameters, it is necessary to fix (assume) all except one of the parameters. A procedure must be devised to systematically change the parameters to establish design requirement.

5.4 Other Forms of Safety Check Expression

5.4.1 Formulation of Safety Check Expression when COV's are Known

In previous section, it has been assumed that the nominal values of the design random variables (R_{ni} and S_{ni}) are given and the design target is achieved by adjusting one of the deterministic parameters (A_i). In many design problems, the coefficient of variations are known. Then it is also possible to adjust the nominal values to achieve the reliability target.

When the COV of a design variable X is specified, X can be defined as

$$X = \mu I \tag{5.31}$$

where μ is the mean of X, I is a random variable with unit mean and the same COV as X. Consider a limit state which involves n random variables and k design constants,

$$g(X_1, \ldots X_n, A_1, \ldots A_k) = 0 \tag{5.32}$$

Using the definition of Eq. 5.31, it can be expressed as

$$g(I_1, \ldots I_n, A_1, \ldots A_k, \mu_1, \ldots \mu_n) = 0 \tag{5.33}$$

Note that all μ_i are deterministic values and may be adjusted to achieve the reliability target. Therefore, μ's have similar meaning as A's and it is convenient to express Eq. 5.33 as

$$g(I_1, \ldots I_n, A_1, \ldots A_k, A_{k+1}, \ldots A_{k+n}) = 0 \tag{5.34}$$

where $(\mu_1, \ldots \mu_n)$ are replaced by $(A_{k+1}, \ldots A_{k+n})$.

Using Eq. 5.34, the safety check expression may be derived in the same way as described in the previous section, but more scale factors, A_i, are involved. An example is provided in the following.

Example. The limit state is

$$R = L + D \tag{5.35}$$

which can be written as, using Eq. 5.31,

$$\mu_R R_1 = \mu_D D_1 + \mu_L L_1 \tag{5.36}$$

where R_1, D_1 and L_1 are random variables having unit mean values. μ_R, μ_D and μ_L are mean values, but can be considered as scale factors.

By adjusting μ's, the design goal may be reached. Considering the mean values (all unity) as the nominal values, the safety check expression is

$$\frac{\mu_R}{\gamma_R} \geq \gamma_D \mu_D + \gamma_L \mu_L \tag{5.37}$$

the γ's are found by performing reliability analysis. If nominal values are other than mean values, only γ's will be different.

5.4.2 Safety Check Expressions Without Using PSF's

In general, a PSF, γ_i, associated with a basic variable X_i depends not only on the statistical parameters of X_i, but also on the parameters of the other design variables. γ_i also depends on the form of the limit state function; moreover, there is a PSF on each basic variable. Therefore, the safety check expression may not always be convenient for practical use in design. It may be more suitable to use the safety check expressions that involve only the design parameters.

To demonstrate the problem, consider the example of Eq. 5.35, dividing by μ_L, Eq. 5.36 becomes

$$\frac{\mu_R}{\mu_L} R_1 = \frac{\mu_D}{\mu_L} D_1 + L_1 \tag{5.38}$$

Define

$$A = \frac{\mu_R}{\mu_L} \tag{5.39}$$

and

$$B = \frac{\mu_D}{\mu_L} \tag{5.40}$$

Then

$$AR_1 = BD_1 + L_1 \tag{5.41}$$

Assume that R, D and L are normally distributed with $C_R = 0.1$, $C_D = 0.2$ and $C_L = 0.3$. To obtain the PSF's, specify either A or B first and adjust the other one to achieve the reliability target $\beta_0 = 3$. The results of γ_i as a function of B are shown in Fig. 5.3 where the relationship between A and B is also drawn. It can be concluded from the figure that, in general, γ_i may be complicated continuous function of B. Therefore, it may be necessary to discretize the three PSF functions in a published formal version of the design rule. In the following, an alternative approach for determining safety check expression will be discussed.

Consider Eq. 5.3, the safety check expression is

$$\frac{A}{\gamma_R} \geq B\gamma_D + \gamma_L \tag{5.42}$$

Multiply the equation by γ_R, it is noted that the equivalent of two PSF's are required,

$$A \geq (\gamma_R\gamma_D)B + (\gamma_R\gamma_L) \tag{5.43}$$

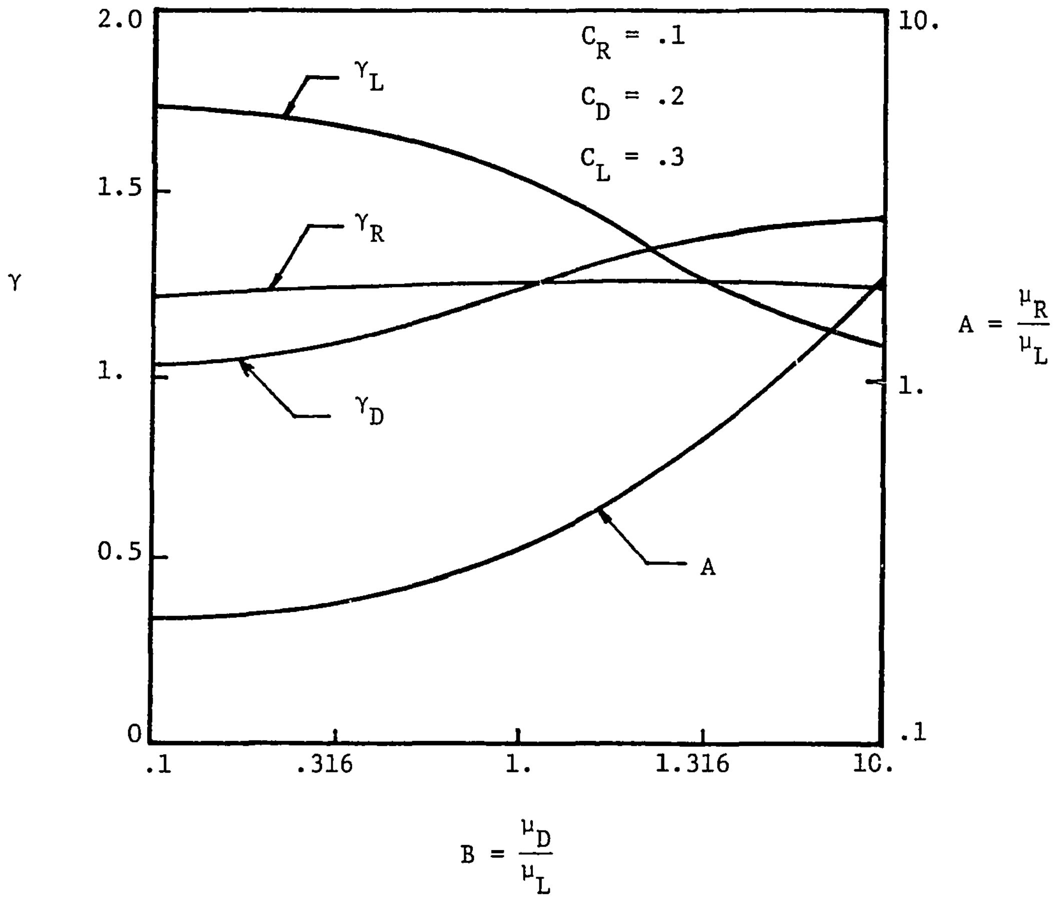


Figure 5.3 The Relationship Between the Partial Safety Factors and the Design Parameters.

γ_R, γ_D and γ_L can be found from A and B. It is equivalent to say that the safety index can be determined using only A and B. This suggests a possibility of expressing the design requirement in terms of the design parameters without using PSF's. Also the expression is expected to be simpler because it involves only independent design parameters (in this example there are two).

Thus to determine the design requirement it is only necessary to relate A and B. A curve of A vs. B as shown in Fig. 5.4 can be considered as a design curve where each point on the curve represents a design of $\beta = 3$. The design curve separates the A, B space into safety and failed regions. A safety check expression may be formulated as

$$A \geq f(B) \tag{5.44}$$

or

$$\mu_R \geq f(\mu_D, \mu_L) \tag{5.45}$$

The above procedures can be applied to other cases. For example, consider the limit state

$$\frac{R}{U} = D + L \tag{5.46}$$

where there are four random design factors, R, U, L and D. The safety check expression can be formulated as

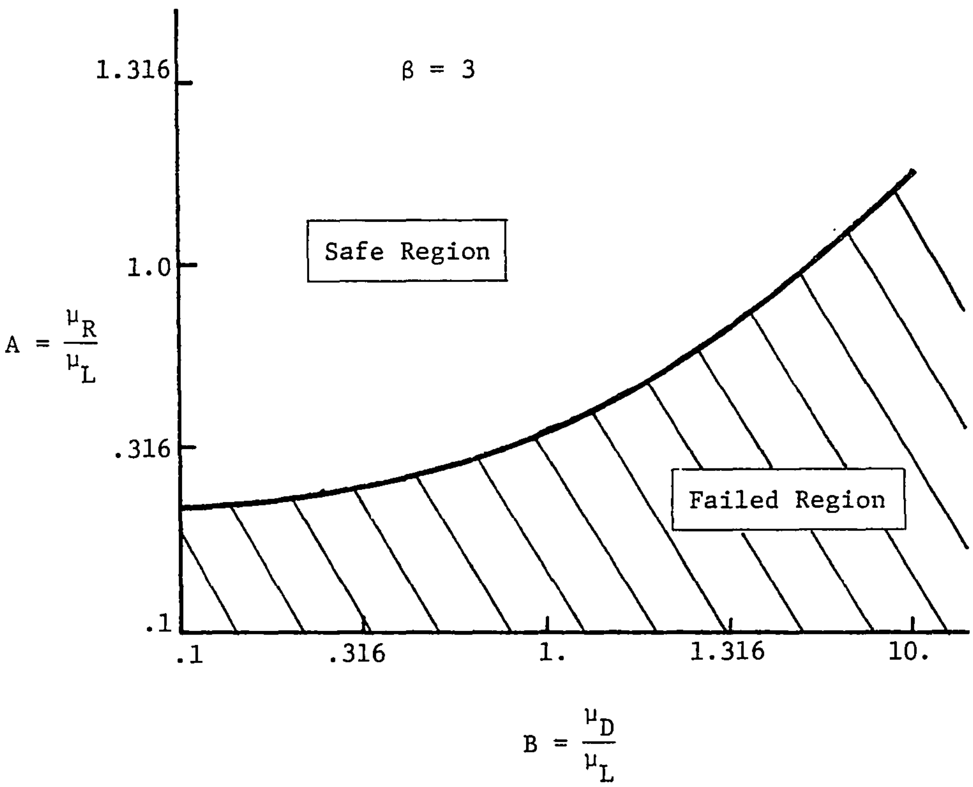


Figure 5.4 A Design Curve.

$$\frac{R_n}{\gamma_R \gamma_U U_n} \geq \gamma_D D_n + \gamma_L L_n \tag{5.47}$$

Multiplying by γ_R/L_n,

$$\frac{R_n}{U_n L_n} \geq \gamma_R \gamma_U \gamma_D \left(\frac{R_n}{L_n}\right) + \gamma_R \gamma_U \gamma_L \tag{5.48}$$

or

$$A \geq (\gamma_R \gamma_U \gamma_D) B + \gamma_R \gamma_U \gamma_L \tag{5.49}$$

where A and B are the scale factors. Note that there are four PSF's, but Eq. 5.49 requires only the relationship between two parameters, A and B. Therefore, only one design curve is required.

The above exercises provide a procedure by which the number of independent scale factors can be defined. And, in general, the safety check expression can be formulated as

$$f(\underset{\sim}{A}) \geq 0 \tag{5.50}$$

where

$$\underset{\sim}{A} = (A_1, A_2, \ldots A_n)$$

m = number of scale factors needed

$f(\underset{\sim}{A}) = 0$ may be considered as a "design surface" which separates the design space into safe and failed region. The number of scale factors m is established on an individual case basis.

The safety check expression needs to be written as illustrated by the above examples. If m is large, $f(\underset{\sim}{A})$ will become complicated. In practice, it may be necessary to simplify the process by identifying several "major" design parameters (in the limit, only one design parameter need to be determined) and specify the values for the other minor parameters, thereby reducing the number of m.

5.5 Design Requirement for More Complicated Limit States

In practical design, it is not uncommon that the limit state function cannot be expressed by a closed form expression. The design parameters may be related through a differential equation which has no closed form solution, or related by a system of linear or non-linear equations; the limit state function can be defined only using a numerical algorithm, e.g., Runge-Kutta method, local strain analysis, etc. In such cases, reliability analysis might become extremely difficult or inefficient.

Applying the method developed in Chapter 3, a procedure is suggested and illustrated by the following examples to establish design requirements for the complicated limit state cases.

Example 1. Consider a mass, M, subjected to a suddenly applied force, $Qt^2e^{-t^2}$, where Q is the amplitude and t denotes time. The equation of motion is

$$M\ddot{y} = Qt^2e^{-t^2} \tag{5.51}$$

with initial conditions $y(0) = \dot{y}(0) = 0$, where y is the displacement.

Assume that Q and M are random variables and define the design target as

$$P(y \geq y_o \mid t = t_o) < p_o \tag{5.52}$$

That is, the probability that the displacement of the mass greater than a specified value, y_o, at a specified time, t_o, should be less than p_o.

The displacement function can be expressed implicitly as

$$y = f(M, Q, t) \tag{5.53}$$

But it can be related only using a numerical algorithm. To find the design requirement, a limit state relating the design variables can be formulated as

$$\frac{Q}{M} = f(y_o, t_o) \tag{5.54}$$

Assume that for $y_o = 10$ and $t_o = 5$, there is a corresponding value of Q/M. Applying the Runge-Kutta method to integrate the equation of motion and a trial-and-error procedure to force the solution through $(y_o,\ t_o)$, it can be found that the solution is Q/M = 5.835. This is illustrated in Fig. 5.5. Thus, the limit state which originally is in the form of a differential equation reduces to a simple linear limit state,

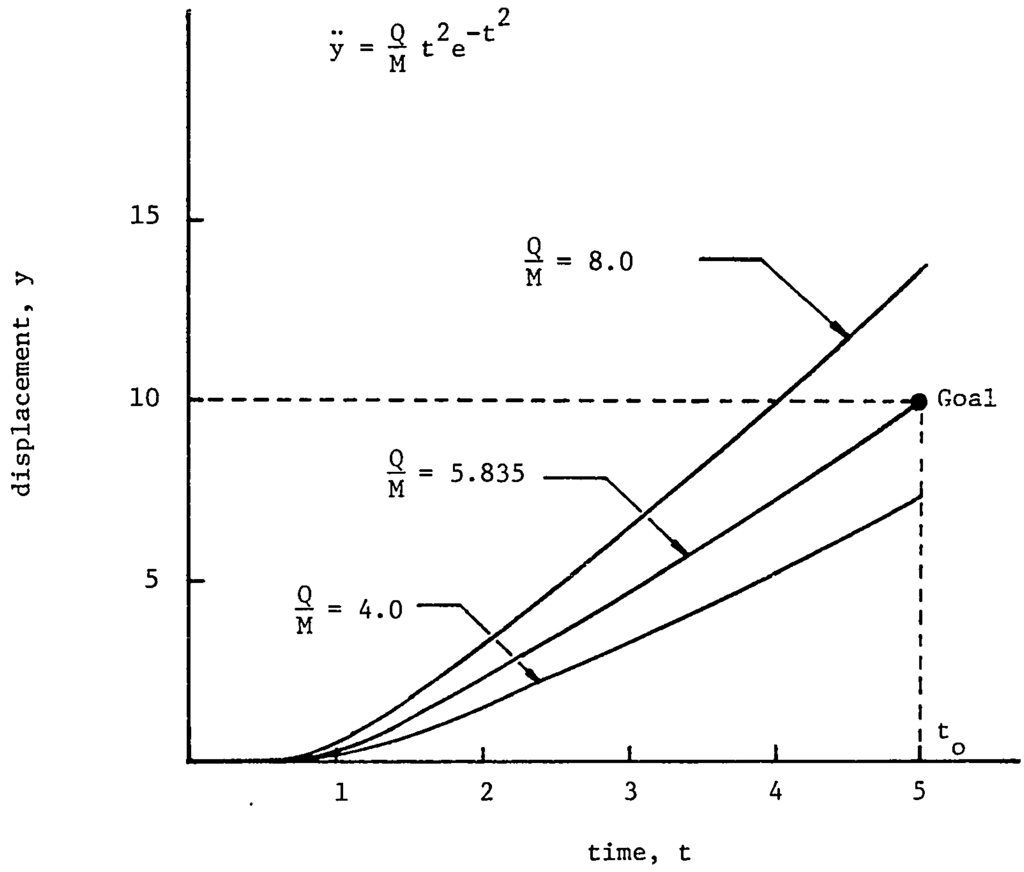


Figure 5.5 Displacement as a Function of Time Using the Runge-Kutta Method.

$$5.835\ M = Q \tag{5.55}$$

which relates the random design variables Q and M.

Assume that the COV's of Q and M, denoted as C_Q and C_M, are known. The mean (or nominal) values of both variables may be adjusted to achieve the design goal. Applying the procedure described before, Eq. 5.55 is written as

$$5.835\ A_1M_1 = A_2Q_1 \tag{5.56}$$

where M_1 and Q_1 have unit mean values, A_1 and A_2 are scale factors representing the mean values of M and Q. Eq. 5.56 may be further simplified as

$$AM_1 = Q_1 \tag{5.57}$$

where

$$A = \frac{5.835\ A_1}{A_2} \tag{5.58}$$

thus only one scale factor involved in Eq. 5.57. The PSF's are determined by adjusting A. For example, assume that Q, M are normally distributed with $C_M = 0.05$, $C_Q = 0.2$ and $\beta_o = 3$, it can be found, by performing reliability analysis, that $\gamma_M = 1.061$, $\gamma_Q = 1.554$ and $A = 1.649$. The safety check expression considering nominal values as the mean values, is

$$\frac{5.835\ \mu_M}{\mu_Q}\ \gamma_M \geq \gamma_Q \tag{5.59}$$

or, in the form similar to the limit state,

$$5.835\ (\gamma_M \mu_M) \geq (\gamma_Q \mu_Q) \tag{5.60}$$

An alternative safety check expression without using PSF's, can be written by requiring,

$$A \geq 1.649 \tag{5.61}$$

Or, in terms of A_1 and A_2 (Eq. 5.58),

$$\frac{5.835\ A_1}{A_2} \geq 1.649 \tag{5.62}$$

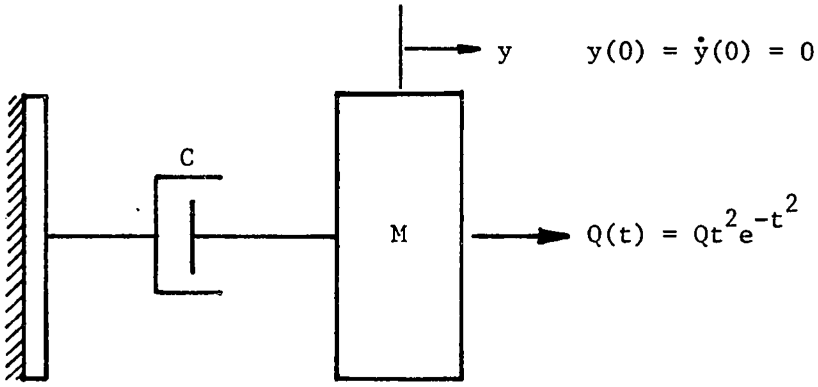
or, after replacing the mean values for A_1 and A_2

$$\frac{\mu_M}{\mu_Q} \geq 0.282 \tag{5.63}$$

This is the simplest form of $f(\underset{\sim}{A}) \geq 0$ as defined in Eq. 5.50. The number of scale factors, m, is one.

<u>Example 2.</u> Assume that a damper with damping coefficient C is attached to the simple mass system described in Example 1. The equation of motion of the system (Fig. 5.6) is

$$M\ddot{y} + C\dot{y} = Qt^2e^{-t^2} \tag{5.64}$$



Design Goal: $p_f = P(y \geq 10 \mid t = 5) \leq 0.0014$

Figure 5.6 Illustration of Example 2.

The initial conditions and the reliability goal are the same as in Example 1.

The purpose of the analysis is to find the design requirement of C, given M and Q. Assume all three variables are normally distributed with $\mu_M = 2$, $C_M = 0.05$ and $\mu_Q = 10$, $C_Q = 0.2$; also $C_C = 0.1$. The basic design requirement is that the target safety index $\beta_o = 3$. This translates to a maximum allowable probability of failure, $p_o = \Phi(-\beta_o) = 0.0014$. In this example, the requirement is,

$$p_f = P(y \geq 10 \mid t = 5) \leq 0.0014 \tag{5.65}$$

It is noted that the assumption of normal distributions are made for easier illustration. The procedures of the analyses are similar if non-normal distributions are involved.

M, Q and C are related by the differential equation (Eq. 5.64). The limit state is non-linear and cannot be expressed in a closed form, i.e.,

$$g(M, Q, C) = 0 \tag{5.66}$$

is not available explicitly.

As suggested in Chapter 3, an approximating second order polynomial limit state can be constructed easily which relates M, Q and C in the region near the design point. Although the design point is difficult to predict accurately, a reasonable narrow region may be estimated from the following analysis.

Transform the basic variables to the reduced variables,

$$u_i = \frac{X_i - \mu_i}{\sigma_i} \qquad i = 1, 3 \tag{5.67}$$

where i = 1, 2 and 3 denote M, Q and C respectively. At the design point,

$$(u_1^*)^2 + (u_2^*)^2 + (u_3^*)^2 = \beta_o^2 \tag{5.68}$$

therefore

$$|u_i^*| < \beta_o \qquad i = 1, 3 \tag{5.69}$$

i.e., the reduced design point values, u_i^*, for each variable cannot be greater than β_o (usually much smaller than β_o). Another condition which is in general true for design point values of the reduced coordinates is

$$u^* < 0 \qquad \text{for a strength variable} \tag{5.70}$$

and

$$u_i^* > 0 \qquad \text{for a stress variable} \tag{5.71}$$

Using Eq. 5.69 to Eq. 5.71 and Eq. 5.67, the possible design point (basic) values are limited as

$$\mu - \beta_o\sigma < x^* < \mu \qquad \text{for a strength variable} \tag{5.72}$$

and

$$\mu < x^* < \mu + \beta_o \sigma \quad \text{for a stress variable} \tag{5.73}$$

Eq. 5.72 and Eq. 5.73 define the bounds that bracket the design point. On the basis of engineering judgement and experience, further reasonable reduction of the range over which the approximating polynomial should function is possible. As an example, there is only one stress variable Q, but two strength variables, M and C. Also Q has the largest COV. Therefore, it is expected that u_Q^* is likely to be greater than, say 1. Then the range of x_Q^* becomes

$$\mu_Q + \sigma_Q < x_Q^* < \mu_Q + \beta_o \sigma \tag{5.74}$$

Thus, the "guessing" range of x_Q^* reduced by one standard deviation. It is suggested that the initial guessing of the ranges should be chosen carefully. If the limit state is highly nonlinear, the range should be chosen relatively narrow and it should still cover, approximately, the design point, so that the approximating limit state will be reasonably close to the exact limit state in the chosen range.

Consider Q and M as the independent parameters and C as the dependent parameter, the limit state (Eq. 5.66) can be expressed as

$$C = f(Q, M) \tag{5.75}$$

To construct a second order polynomial equation to approximate

Eq. 5.75, three points for both M and Q are chosen which are shown in Fig. 5.7. Point 1 and point 3 are the bounds of Eq. 5.72 (for M) or Eq. 5.74 (for Q); point 2 is in the middle for both variables. Using three data points of both Q and M, five sets of Q and M are determined as shown in Table 5.2. For each selected set, a corresponding value of C is solved numerically from the equation of motion (Eq. 5.64). A trial-and-error procedure is required as described in Example 1. The results of five solutions of C are shown in Table 5.2.

Using the data sets of Table 5.2, a second order polynomial equation (without the mixed terms) can be constructed using curve fitting method,

$$C = 0.259 - 0.43\,(M - 1.85) - 0.0222(M - 1.85)^2 + 0.074(Q - 14) - 0.01(Q - 14)^2 \tag{5.76}$$

where the values "1.85" and "14" are the Taylor's "expanding points", and are chosen as the midpoints defined in Fig. 5.7.

Define

$$C = AC_1 \tag{5.77}$$

where C_1 has unit mean value, and A is the mean value of C. By performing the reliability analysis, it is found that A = 0.371 is the solution for $\beta_o = 3.0$. The design point and the PSF's when nominal values are the mean values are listed in Table 5.3.

TABLE 5.2

DATA POINT SET FOR GENERATING FIRST APPROXIMATING MODEL

Set Number	Q	M	C[b]
1[a]	$\mu + 2\sigma = 14$	$\mu - 1.5\sigma = 1.85$	0.259
2	$\mu + 2\sigma = 14$	$\mu - 3\sigma = 1.70$	0.323
3	$\mu + 2\sigma = 14$	$\mu = 2.00$	0.194
4	$\mu + 3\sigma = 16$	$\mu - 1.5\sigma = 1.85$	0.403
5	$\mu + \sigma = 12$	$\mu - 1.5\sigma = 1.85$	0.103

[a] Expanding Points

[b] Computed Using Runge-Kutta Method

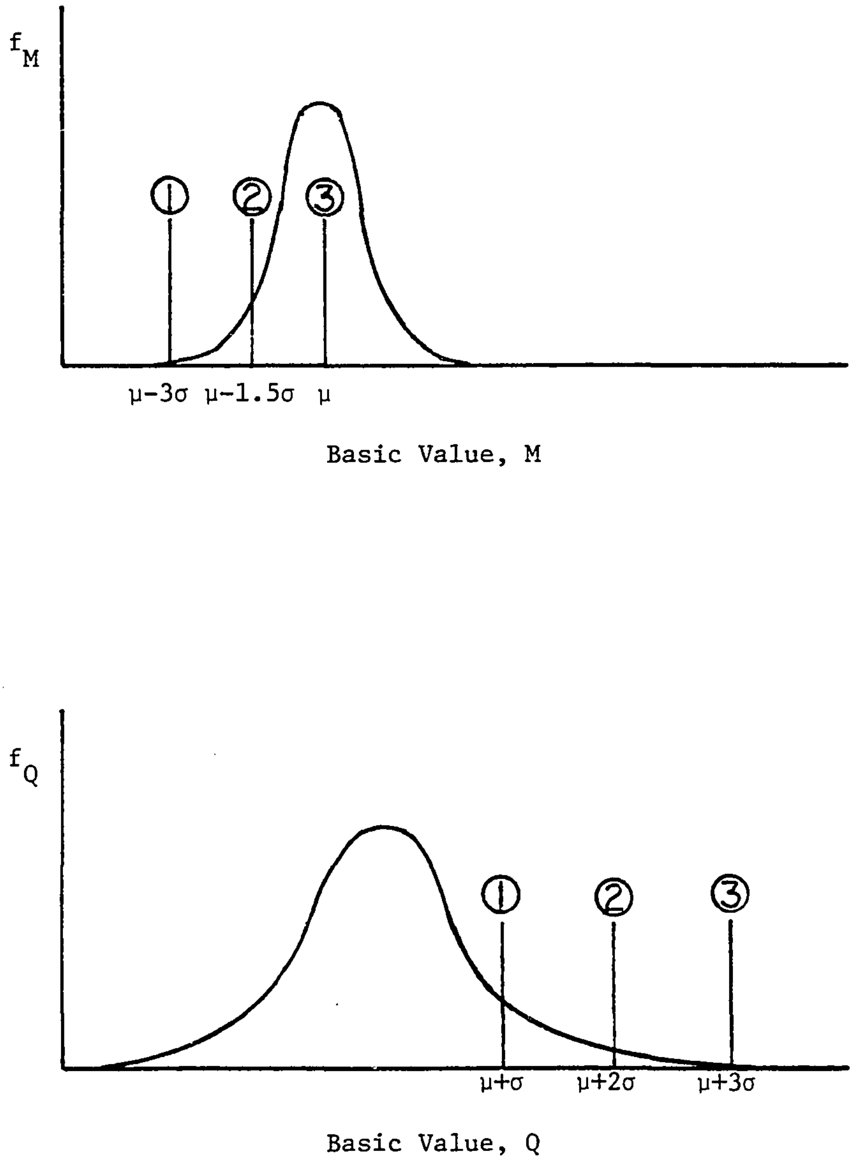


Figure 5.7 Define Three Points for Each Design Parameter to Construct Approximating Limit State.

TABLE 5.3

RESULTS OF THE RELIABILITY ANALYSIS USING FIRST APPROXIMATING MODEL

	Q	M	C
Design Point	15.57	1.915	0.3444
Partial Safety[(a)] Factor	1.557	1.044	1.078
Mean Value	10.0	2.0	0.371

[a]Nominal Value = Mean Value

From the results presented in Table 5.3, it is seen that the chosen interpolation range (the range covered by points 1 and 3 in Fig. 5.7) indeed cover the design point for all three design variables. However, the expanding points do not correspond to the design point values very well. Therefore, there should be some question regarding the accuracy of the reliability analyses based on Eq. 5.76.

In general, the approximating model can be improved by reducing the interpolation range using the "old" design point as the midpoint. To demonstrate this, new data sets are established as shown in Table 5.4 in which the interpolation range of Q and M are chosen 0.44σ and 1.0σ, respectively. Applying the curve fitting scheme again, the "new" model for C is

$$C = 0.35 - 0.373(M - 1.915) - 0.66(M - 1.915)^2$$
$$+ 0.076(Q - 15.57) - 0.01(Q - 15.57)^2 \qquad (5.78)$$

It is noted that the expanding points are the design point values of the old model (Table 5.2).

Using the new model, the results of the reliability analysis are shown in Table 5.5. The mean value of C is A = 0.378, which is very close to the previous value (0.371). The PSF's are also approximately the same as of the first model. Moreover, the design point values are near the new expanding points. Thus, the present model seems to be accurate at least in the region near the design point.

TABLE 5.4

DATA POINT SET FOR GENERATING SECOND APPROXIMATING MODEL

Set Number	Q	M	C[b]
1[a]	15.57	1.915	0.3504
2	15.57	1.865	0.3674
3	15.57	1.965	0.3301
4	15.13	1.915	0.3179
5	16.00	1.915	0.3792

[a]Expanding Points.

[b]Computed Using Runge-Kutta Method.

TABLE 5.5

RESULTS OF THE RELIABILITY ANALYSIS USING SECOND APPROXIMATING MODEL

	Q	M	C
Design Point	15.60	1.923	0.3496
Partial Safety[(a)] Factor	1.560	1.040	1.08
Mean Value	10.0	2.0	0.378

[a]Nominal Value = Mean Value

The results suggest that even the first approximating model is effective. A check was made to confirm that the "effective ranges", defined in Chapter 3 covers the chosen interpolation ranges for the first approximating model. Therefore, the construction of the second model here was unnecessary. It was used only to demonstrate the effectiveness of the second order polynomial model and the scheme of reducing the interpolation ranges.

A safety check expression, using Eq. 5.78 and Table 5.5, can be written as

$$\frac{\mu_C}{1.08} \geq 0.35 - 1.373\left(\frac{\mu_M}{1.04} - 1.915\right) - 0.66\left(\frac{\mu_M}{1.04} - 1.915\right)^2 + 0.076(1.56\mu_Q - 15.57) - 0.01(1.56\mu_Q - 15.57)^2 \qquad (5.79)$$

where it is noted that the expression is derived based on $\mu_M = 2.0$ and $\mu_Q = 10$. For additional sets of μ_M and μ_Q, different PSF's are expected. Therefore, depending on the different design situations, the possible range of μ_M and μ_Q should be defined, and then discretized. Thus, more than one safety check expression, in the forms similar to Eq. 5.79, are required. Each corresponds to certain applicable ranges of μ_M and μ_Q.

Instead of using multiple checking expressions, it is possible to use a single design curve in the form of

$$\mu_C \geq f(\frac{\mu_M}{\mu_Q}) \tag{5.80}$$

As just found, $\mu_C = 0.378$ if $\mu_M/\mu_Q = 0.2$. For other possible values of μ_M/μ_Q corresponding values of μ_C must be found, and a design curve of μ_C vs. μ_M/μ_Q may be constructed. Also it is possible to approximate Eq. 5.80 by, e.g., a polynomial equation, which may be more convenient for design checking.

Example 3. The procedure described in Example 2 can be extended to the general one degree of freedom system with the equation of motion

$$M\ddot{y} + C\dot{y} + Ky = Q(t) \tag{5.81}$$

where K is the spring constant (or stiffness). The design requirement is related to the displacement, y, or the stress generated in the spring. If Eq. 5.85 is nonlinear, a step-by-step integration scheme may be required to evaluate y. The limit state function, given $y = y_o$ at $t = t_o$, can be written as

$$K = f(M, C, Q) \tag{5.82}$$

which is only implicitly defined. Applying the same procedures as in Example 2, a second order equation can be generated using seven points. Therefore, Eq. 5.81 needs to be solved seven times. The safety check expression can be generated using the derived PSF's. An alternative safety check expression in the form of (or other suitable forms),

$$\mu_K \geq f(\frac{\mu_M}{\mu_C}, \frac{\mu_M}{\mu_Q}) \qquad (5.83)$$

may be found similarly as described in the previous example. In general, several approximating second order limit states may be required to cover the design parameter space effectively.

It is natural to extend the problem to a multiple degree of freedom system where the equations of motion, in matrix form, are

$$[M]\{\ddot{y}\} + [C]\{\dot{y}\} + [K]\{y\} = \{Q(t)\} \qquad (5.84)$$

In general, a finite element program has to be used to relate the design parameters. The entire process of generating the design requirement may become very complicated because too many design parameters involved. However, if a relatively small number of design parameters can be identified as the major design factors, and specify other design parameters, then the procedures described may be applied to relate those chosen variables and to establish design rules.

Example 4. Consider the fatigue reliability problem described in Chapter 3 where a harmonic load (zero-to-peak) is applied to a component. The stress in the bulk of the material remains elastic, but there is cyclic plasticity at the notch (Fig. 3.1). Random design factors include (a) the stress amplitude S reflecting statistical scatter in load data and modelling errors, and (b) the fatigue strength coefficient σ_f' and fatigue ductility coefficient ϵ_f' representing scatter in material fatigue behavior.

The limit state can be implicitly expressed as

$$S = f(\sigma_f', \varepsilon_f', N) \tag{5.85}$$

but the design parameters can only be related using a complicated local strain analysis program. Note that cycles to crack initiation, N, is uncertain because it depends on other factors which are uncertain. Given the service life, N_o, the probability of failure is

$$p_f = P[N \leq N_o] \tag{5.86}$$

The data of the design parameter is assumed to be the same as in Chapter 3. (Table 3.1). However, in the present example, the median of S, $\tilde{S}$, is not specified and may be adjusted to achieve a reliability goal.

Assume that it is required to establish design requirements for $\beta_o = 4$. The service life N_o specified can range from 10^2 to 10^6 cycles. Several approximating limit states may be needed because when the range of N_o is wide, the exact limit state, which is highly nonlinear in the present example, cannot be approximated closely using a single quadratic equation.

The procedure of establishing the design requirements may be devised as follows:

(1) Choose several median values of S, denoted as $\tilde{S}_j$.

(2) For each $\tilde{S}_j$, select a data set of S, σ_f' and ε_f'.

(3) Solve N using a local strain program for each set in step 2.

(4) Construct a second order polynomial equation using data in step 2 and step 3.

(5) Repeat step 2 to step 4 to construct j equations.

(6) For each equation, substitute appropriate N_o values and perform reliability analysis to find $\tilde{S}$ which satisfies the reliability goal.

To illustrate the above procedure, the analysis performed in Chapter 3 may be used. Three values of $\tilde{S}$, (310, 414, 551) MPa, are chosen. For each $\tilde{S}$, three points were selected for each of $\dot{S}$, σ_f' and ε_f' where the chosen interpolation ranges are $\mu \pm 4\sigma$ (on log basis) for all three variables. These ranges are the bounds of the design point values for $\beta_o = 4$. These may be too conservative and may be made smaller as described in Example 2.

Seven points of the limit state are needed to construct a second order equation. Using a local strain program, seven corresponding N values are found. The smallest and the largest N values define the interpolation range of N. The results of the curve fitting are summarized in Table 5.6 where for each $\tilde{S}_j$ (j = 1, 3) the interpolation ranges of N are also shown.

Because each of the constructed quadratic limit state is responsible for a certain range of the design parameters, the ranges of N in Table 5.6 provide the bounds within which the corresponding quadratic limit state is a better approximation.

TABLE 5.6

INTERPOLATION RANGE OF N AND APPROXIMATING LIMIT STATE FOR THREE STRESS LEVELS

(a)

S	Interpolation Range of N
310 MPa (45 Ksi)	$8.7 \times 10^2 - 1.5 \times 10^6$
414 MPa (60 Ksi)	$1.4 \times 10^3 - 3.8 \times 10^4$
551 MPa (80 Ksi)	$1.1 \times 10^2 - 3.0 \times 10^3$

(b)

$\tilde{S}$	Approximating Limit State (First Approximation)
310 Mpa	$S = 498 + 0.268(\sigma_f' - a_1) + 20.8(\varepsilon_f' - a_2) - 168(\log N - 3.86) + 0.000027(\sigma_f' - a_1)^2 - 4.52(\varepsilon_f' - a_2)^2 + 37.5(\log N - 3.86)^2$
414 MPa	$S = 663 + 0.217(\sigma_f' - a_1) + 106(\varepsilon_f' - a_2) - 425(\log N - 3.0) + 0.000074(\sigma_f' - a_1)^2 - 34.1(\varepsilon_f' - a_2)^2 + 164(\log N - 3.0)^2$
551 MPa	$S = 885 + 0.133(\sigma_f' - a_1) + 210(\varepsilon_f' - a_2) - 702(\log N - 2.55) + 0.000012(\sigma_f' - a_1)^2 - 77.1(\varepsilon_f' - a_2)^2 + 370(\log N - 2.55)^2$
Note: $a_1 = 1674$; $a_2 = 2.069$	

For example, let $N_o = 200$ cycles. A better quadratic limit state appears to be the case of $\tilde{S} = 551$ MPa because the range of N for this case well covers N_o. Therefore, the third equation in Table 5.6 should be used for $N_0 = 200$. Similarly, the second equation is a better choice when $N_o = 10^4$ cycles.

To derive the design requirement, let

$$S = AS_1 \tag{5.87}$$

where A is the median of S and S_1 has unit median and the same coefficient of variation as $S(C_S = 0.2)$. Thus, A is the scale factor to be adjusted, $\tilde{S} = A$. As an example, assume $\beta_o = 3.0$, a reliability analysis using R-F algorithm may be used iteratively for finding A. It was found that A = 621.5 MPa for $N_o = 200$ and $\beta_o = 3.0$. The design point values and the PSF's, based on median values, are listed in Table 5.7. The safety check expression is

$$\begin{aligned} \gamma_S\tilde{S} \le\ & 885 + 0.133\left(\frac{\tilde{\sigma}_f'}{1.007} - 1674\right) + 210\left(\frac{\tilde{\varepsilon}_f'}{2.364} - 2.07\right) \\ & - 702(\log N_o - 2.55) + 0.000012\left(\frac{\tilde{\sigma}_f'}{1.007} - 1674\right)^2 \\ & - 77.1\left(\frac{\tilde{\varepsilon}_f'}{2.364} - 2.07\right)^2 + 370(\log N_o - 2.55)^2 \end{aligned} \tag{5.88}$$

By substituting $\tilde{\sigma}_f'$ and $\tilde{\varepsilon}_f'$ into Eq. 5.88.

$$\tilde{S} \le 621.5 \tag{5.89}$$

TABLE 5.7

RESULTS OF THE RELIABILITY ANALYSIS FOR
$N_o = 200$ AND $\beta_o = 3.0$

	S	σ_f'	ε_f'
Design Point	951 MPa	1828 MPa	1.467
Partial Safety[(a)] Factor	1.53	1.007	2.364
Median Value	621.5 MPa	1841 MPa	3.47

[a]Nominal Value = Mean Value

which is an alternative design requirement of $\tilde{S}$ for $N_o = 200$ and $\beta_o = 3$.

By substituting different values of N_o to the appropriate limit states, using Table 5.6, PSF's as a function of N can be determined. Alternatively, the S - N design curves for different β_o (e.g., $\beta_o = 3$, $\beta_o = 4$) may be constructed using results similar to Eq. 5.89. An improvement of the results may be necessary, the procedures are the same as described in Example 2 where the design point information is used to determine the "new" interpolation ranges and the approximating limit states.

5.6 Summary

The design point derived using Level 2 reliability analyses methods, e.g., Rackwitz-Fiessler, provides a rational basis of defining partial safety factors and constructing the corresponding safety check expression. In this chapter, the R-F method was used to demonstrate procedures for establishing the safety check expression. A computer program was developed (Appendix E) which automates the process. Other Level 2 reliability methods (e.g., the numerical algorithm proposed in Chapter 4) may also be used.

It has been demonstrated that each PSF is ,in general , dependent on all the design variables, and it may be necessary to discretize the PSF's in a published formal version of the design rule.

In general, safety check expressions using PSF's are desirable because consistent standards of reliability can be reasonably achieved with statements in a safety factor format familiar to designers.

Because PSF's are in general related, it is possible to simplify the design rule by using a safety check expression which involves a smaller number (compared with the number of PSF's) of independent parameters or scale factors defined earlier. Given a target safety index, a design surface may be constructed which separates the design parameter space into safe or failed design regions. Several examples have been presented which showed that a design rule using a design surface may be more convenient than a safety check expressions using PSF's.

When COV's are known, it has been shown in the examples that several design factors may be grouped as a single scale factor. This is especially convenient when a limit state is in a multiplicative form,

$$g(\underset{\sim}{X}) = a_o \prod_{i=1}^{n} X_i^{a_i} = 1 \qquad (5.90)$$

where a's are constants and COV's of X_i are given. By letting $X_i = \mu_i I_i$, where each I_i has unit mean value, Eq. 5.90 becomes

$$a_o \prod_{i=1}^{n} \mu_i^{a_i} I_i^{a_i} = 1 \qquad (5.91)$$

or

$$A \prod_{i=1}^{n} I_i^{a_i} = 1 \tag{5.92}$$

where A is a scale factor which is a function of all the μ's,

$$A = a_o \prod_{i=1}^{n} \mu_i^{a_i} \tag{5.93}$$

Because all I_i are completely defined, there is a value of A, denoted as A_o, such that $\beta = \beta_o$. Therefore, the design surface, using Eq. 5.93, is

$$a_o \prod_{i=1}^{n} \mu_i^{a_i} = A_o \tag{5.94}$$

which is an equation similar to the form of the limit state, and may be called a "deterministic design equation" for $\beta = \beta_o$. For different β_o there is a different A_o. Note that a safety checking equation, using Eq. 5.94 can be defined by identifying whether A is a "strength" or a "stress" variable. A corresponding equation using PSF's is

$$a_o \prod_{i=1}^{n} (\gamma_i \mu_i)^{a_i} = 1 \tag{5.95}$$

By comparing Eq. 5.93 and Eq. 5.95,

$$A \prod_{i=1}^{n} \gamma_i^{a_i} = 1 \tag{5.96}$$

Therefore, an equivalent of only one PSF is required. It may be more convenient to use Eq. 5.94 for reliability design without using all the n PSF's. For general design problems, however, the method of approach for establishing the design requirement has to be tailored to the problem.

Finally, for the complicated limit states which have no closed form expressions, the method proposed in Chapter 3 has been applied to approximate the limit state; then PSF's and the design requirement are derived using Level 2 reliability method. Because a Monte Carlo solution would be extremely costly in searching for a design, the procedure suggested is thought to be particularly useful for the problem where a costly computer analysis is required to relate the design variables. For example, in the fatigue reliability design problem, if the specimen differs from cases for which Neuber's rule has been verified, elastic-plastic finite element analysis may be required to relate the stress and strain in a local strain analysis (Dowling, 1983).

CHAPTER 6

SUMMARY AND CONCLUSIONS

The theme of this study was to develop and apply the efficient reliability algorithms for component design problems, in particular, the fatigue reliability analysis and design problem. The study has been focused on the first order reliability methods because of their efficiency.

A major problem addressed in this study is that of computing the probability of failure p_f for a structural component whose design factors are random variables. The relationship between the design factors is not explicitly defined. Level 2 reliability methods cannot be applied directly. A method is proposed to approximate the limit state by a second order polynomial constructed using a simple curve fitting routine. An effective procedure is suggested to select the points relating the design variables. A fatigue reliability problem (Chapter 3), which requires a complicated local strain analysis program to relate the design variables, was presented. Also investigated were the dynamics problems (Chapter 5) in which the design variables must be related using a numerical algorithm for solving a differential equation. Because the method suggested can be applied to any mode of structural or mechanical failure, and is much faster than the Monte Carlo method in producing good estimates of p_f, it is particularly useful for the problem where a costly computer analysis is required to relate the design variables. As an example, non-linear, finite element structural analysis may have to be used for computing the cyclic stress-strain response of hot-section components such as turbine blades and vanes (Kaufman and Hunt, 1982).

There appears to be no substantial operational difficulties in applying the above method to the "design by reliability" problem. In fact, if the target safety index is not too high, the bounds of the "guessing" design point can be chosen relatively narrow, and the approximating quadratic limit state would be effective for generating accurate results. Note that in the reliability design problem, the method is even more efficient than the Monte Carlo because in general the reliability analysis must be performed many times to reach the design target. Furthermore, it would be very difficult to find the design point using Monte Carlo method.

Assume that the limit state has a closed form algebraic form and that all the design variables have known statistical distributions, the performances of the first order reliability methods (i.e., R-F and C-L methods) for estimating the generalized safety index and p_f have been examined. Upon investigating a large number of cases, it was found that these methods do well for realistic problems resulting in usually less than 20% error in p_f and much less in β.

Another first order method is proposed in Chapter 4. The method is developed as an improvement to the R-F and C-L methods. It was thought that when a variable is non-normal, a more reasonable way of fitting the original cdf by an equivalent normal cdf would be to use a least square technique. In addition, the estimate of the joint pdf of the other variables is included in the scheme to provide the range (which contains most of the p_f contents) for an effective curve fitting. As demonstrated by the examples, the error in p_f is usually less than 10% (typically about half of the error in R-F estimates).

It seems that use of quadratic limit states have the promise of consistently producing better estimates of p_f relative to first order methods. However, there is no strong motivation for extending the above proposed method to a second order code for the following practical reasons:

1. First order methods seems to perform well. They are easy to program and computer costs are negligible. The more complicated inverse problem of computing partial safety factors (PSF's) is easily programmed.
2. By contrast, the second order computational algorithms seems to be extremely complicated and relatively inefficient. It seems that it would be a formidable task to produce a general purpose code which would be easy for designers to use. Then a partial safety factor program to be used for code development may be even an order of magnitude more complicated.

3. As described in Chapter 2, the evidence on behalf of the second order method is not entirely convincing. Monte Carlo simulation must be used for "exact" values of p_f for complicated limit states and for three or more variables. But Monte Carlo tends to be inaccurate and checking results of second order methods is risky.
4. Modelling error (non-statistical uncertainty) plays an important role in the design problem which is certainly true of fatigue. Statistical models for modelling error are always arbitrarily chosen. Also, even when data is available, there is seldom a sufficiently large sample to point to a particular statistical model. Therefore, efforts spent for obtaining extreme accuracy may not be justified.

The R-F algorithm has been applied to demonstrate the procedure of establishing PSF's and the safety check expression using the design point information. Because PSF's are functions of all the design variables, it is suggested that sets of PSF's should be provided to establish a design code. Each set of PSF's represents an equal safety design. Other forms of the safety check expression are investigated. When COV's are known, it has been shown that it is possible to construct a simplified design surface by grouping several design factors into one design parameter or scale factor, thereby reducing the number of the independent design parameters. However, reducing the scale factors must be established in an individual case basis.

In conclusion, it is demonstrated that the reliability analysis algorithms are well developed to provide good p_f estimate efficiently for complicated limit states. The reliability by design problem using design point concept is also well defined. However, it is thought that a design code using PSF's may not be convenient when there are too many PSF's involved. It is,therefore, suggested that further investigation may be necessary to provide a general, reasonable and efficient method of generating design codes for the components.

APPENDIX A

THE RACKWITZ-FIESSLER (R-F) ALGORITHM

APPENDIX A

THE RACKWITZ-FIESSLER (R-F) ALGORITHM

The algorithm proposed by Rackwitz and Fiessler (1978) has been extensively described in recent literature (Ellingwood et al., 1980; Thoft-Christensen and Baker, 1982). The procedure for calculating the R-F algorithm safety index can be summarized as follows:

1. Define each design factor, X_i (i = 1, n) and its corresponding probability distribution F_i and f_i denotes to cdf and pdf of X_i respectively.
2. Define reduced variables

$$u_i = \frac{X_i - \mu_i}{\sigma_i} \qquad i = 1, n \tag{A.1}$$

where (μ_i, σ_i) = mean and standard deviation of X_i respectively.

3. Define the limit state in reduced variables

$$g_1(\underset{\sim}{u}) = 0 \tag{A.2}$$

where $\underset{\sim}{u} = (u_1, u_2, \ldots u_n)$.

4. Make an initial estimate of the safety index

$$\beta = \min\sqrt{u_1^2 + u_2^2 + \ldots + u_n^2} \tag{A.3}$$

subject to $g_1(\underset{\sim}{u}) = 0$.

5. Calculate the corresponding design point, x*,

$$x_i^* = u_i^*\sigma_i + \mu_i \qquad i = 1, n \tag{A.4}$$

6. Calculate the mean and standard deviation of the equivalent normal distribution for each non-normal variable

$$\sigma_{Ni} = \frac{\phi[\Phi^{-1}(F_i(x_i^*))]}{f_i(x_i^*)} \tag{A.5}$$

$$\mu_{Ni} = x_i^* - \Phi^{-1}[F_i(x_i^*)]\sigma_{Ni}$$

where ϕ = standard normal pdf and Φ = standard normal cdf.

7. Define the new reduced variables

$$u_i' = \frac{X_i - \mu_{Ni}}{\sigma_{Ni}} \tag{A.6}$$

8. Calculate a new estimate of the safety index

$$\beta_1 = \min \sqrt{(u_1')^2 + \ldots + (u_N')^2} \qquad (A.7)$$

subject to $g_1(\underset{\sim}{u}') = 0$.

9. Repeat steps 6 through 10 until the difference

$$|\beta_N - \beta_{N-1}| \leq t \qquad (A.8)$$

where t is the "error". In this study, the value $t = 0.001$ was used.

10. The probability of failure is calculated using $\hat{\beta} = \beta_N$

$$p_f = \Phi(-\hat{\beta}) \qquad (A.9)$$

APPENDIX B

THE CHEN-LIND (C-L) ALGORITHM

APPENDIX B

THE CHEN-LIND (C-L) ALGORITHM

Chen and Lind (1982) proposed an extension of the R-F algorithm by using a three parameter approximation. The third parameter is established by imposing the condition that at the design point the slopes of the pdf are equal (in addition to cdf and pdf) for both the original and transformed distribution.

As in the case of the R-F method, the C-L scheme requires an equivalent normal distribution to be fitted to the non-normal variate by an appropriate choice of parameters. In addition to μ and σ, a third parameter A is introduced by the C-L method.

The three conditions used to compute μ, σ, and A are:

1. cdf's are equal

$$F(x^*) = A\Phi(u) \quad \text{, For Lower Tail Approx.} \tag{B.1}$$

$$1 - F(x^*) = A[1 - \Phi(u)], \text{ For Upper Tail Approx.}$$

2. pdf's are equal

$$f(x^*) = \frac{A}{\sigma} \phi(u) \tag{B.2}$$

3. Slopes of the pdf's are equal

$$f'(x^*) = -\frac{Au}{\sigma^2} \phi(u) \tag{B.3}$$

where x^* is the design point, u is the reduced variable, and "A" is the third parameter that introduced to "scale" the normal distribution.

A relationship can be found between reduced value u and design point x^* as:

$$\frac{u\Phi(u)}{\phi(u)} = -\frac{F(x^*)f'(x^*)}{[f(x^*)]^2} = q \text{ , For Lower Tail Approx.}$$

$$\frac{u\Phi(-u)}{\phi(u)} = -\frac{[1 - F(x^*)]\ f'(x^*)}{[f(x^*)]^2} = q\text{, For Upper Tail Approx.} \tag{B.4}$$

Given a design point x^*, a corresponding u can be found from Eq. B.4. Three parameters are determined as:

$$\sigma = -\frac{uf(x^*)}{f'(x^*)} \tag{B.5}$$

$$\mu = x^* - u\sigma \tag{B.6}$$

$$A = \frac{F(x^*)}{\Phi(u)} \text{ , For Lower Tail Approx.}$$

$$= \frac{1-F(x^*)}{\Phi(-u)} \text{ , For Upper Tail Approx.} \tag{B.7}$$

The iteration process is the same as described in R-F algorithm. The probability of failure is approximated by

$$p_f = \Phi(-\beta) \prod_{i=1}^{n} A_i \qquad i = 1, 2, \ldots n \tag{B.8}$$

where $A_i = 1$ for the design variables that have normal distributions.

Note that here β is not the safety index, but the minimum distance measured from the origin to the failure surface on the "scaled" reduced coordinates. However, an "equivalent safety index" can be found from p_f, i.e.,

$$\beta_{eq} = -\Phi^{-1}(p_f) \tag{B.9}$$

Because in some cases, α_i could be a very large number, β will also become large (e.g. $\beta \simeq 10^2$, or even higher). To avoid numerical problems the stop criteria of iteration process is defined as

$$|\beta_N - \beta_{N-1}| \leq 0.0001 \qquad \text{when} \qquad \beta < 4$$

and

$$\frac{|\beta_N - \beta_{N-1}|}{\beta_{N-1}} \leq 0.001 \qquad \text{when} \qquad \beta \geq 4$$

An option can be added to the R-F program for problems having variables with Weibull and EVD distributions. In order to compute reduced value u from the design point x^* more efficiently, Chen and Lind introduced a polynomial approximation formula shown below.

1. For $|q| \leq 0.656$

$$u = \text{sgn}\,(q) \sum_{i=1}^{7} a_i |q|^i \tag{B.10}$$

where

$a_1 = 0.8009,\quad a_2 = 0.4338,\quad a_3 = 1.14,\quad a_4 = -3.46$

$a_5 = 10.186,\quad a_6 = -13.241,\quad a_7 = 7.9528$

2. For $|q| > 0.656$

$$u = \text{sgn}(q)\ \left(t - \frac{b_o + b_1 t}{1 + b_2 t + b_3 t^2} + b_4\right) \tag{B.11}$$

where

$t = \sqrt{-2\ln(1 - |q|)}$

and

$b_o = 5.343,\quad b_1 = -1.8521,\quad b_2 = -0.414$

$b_3 = 0.0442,\quad b_4 = 4.925$

B.10 and B.11 were used in developing the program, but later it was found that B.11 is not appropriate for the case when $|q|$ approaches 1 which could happen in some cases (e.g. when an EVD variate has a small COV such that, at the approximation point, the slope of the pdf is very small). Because in these cases the corresponding $|u|$ is very large ($|u| > 10$), the following formula was proved to be very useful in avoiding numerical difficulty,

$$\Phi(-|u|) = \frac{\phi(u)}{u} \left(1 - \frac{1}{u^2} + \frac{3}{u^4} - \frac{15}{u^6} + \frac{105}{u^8} - + \ldots\right) \quad \text{(B.12)}$$

By using Eq. B.12,

$$|q| = \frac{u\Phi(-|u|)}{\phi(u)} = 1 - \frac{1}{u^2} + \frac{3}{u^4} - \frac{15}{u^6} + - \ldots \quad \text{(B.13)}$$

When Eq. B.13 is truncated at u^4, the following approximation can be obtained,

$$u = \text{sgn}(q) \sqrt{\frac{1 + \sqrt{1 - 12(1 - |q|)}}{2(1 - |q|)}} \quad \text{For} \quad |u| >> 1 \quad \text{(B.14)}$$

Eq. B.14 was employed in the program when $|q| \geq 0.985$ where the corresponding $|u| \approx 8$. As a comparison, let q = .999998. The value of u using Eq. B.14 is u = 707 while u = 106 if Eq. B.11 is used.

Another numerical problem was faced when values of u (and therefore β) became large. When computing Φ(u) or φ(u), one needs to calculate $e^{-u_i^2/2}$, which the computer set to zero when u > 21.5. Again, this problem was solved by using Eq. B.12, expressed as

$$\Phi(-u) = \frac{e^{-u^2/2}}{\sqrt{2\pi}} \left(\frac{1}{u} - \frac{1}{u^3} + \frac{3}{u^5} - \frac{15}{u^7} + - \ldots\right) \quad \text{(B.15)}$$

By holding all the $e^{-u_i^2/2}$ and $e^{-\beta^2/2}$ computations, the probability of failure is modified to

$$p_f = \Phi(-\beta) \prod_{i=1}^{n} A_i = \lambda e^{-(\beta^2 - u_1^2 - u_2^2 - \ldots)/2} \quad (B.16)$$

and can be easily calculated because $(\beta^2 - u_1^2 - u_2^2 - \ldots)$ is approximately zero.

The C-L algorithm cannot be applied to the uniform distribution because the slope of the pdf is zero. Thus it is not surprising that in some cases where the slope $f'(x^*) \simeq 0$, numerical difficulties occur. This situation can only be improved by using double precision computation.

APPENDIX C

THE PERFORMANCE OF RACKWITZ-FIESSLER (R-F) AND CHEN-LIND (C-L) ALGORITHMS FOR COMPUTING PROBABILITY OF FAILURE

APPENDIX C

THE PERFORMANCE OF RACKWITZ-FIESSLER (R-F) AND CHEN-LIND (C-L) ALGORITHMS FOR COMPUTING PROBABILITY OF FAILURE

Numerous examples have been examined to investigate the performance of R-F and C-F algorithms for estimating the safety index β and probability of failure p_f (Wu et al, 1983). Examples included the linear as well as nonlinear limit states with design factors having various combinations of distributions. It was concluded that the algorithms are efficient compared with numerical integration or Monte Carlo simulation. The accuracy is considered excellent in many cases. However, in some cases the agreement can only be considered as fair. It was also found that C-L algorithm, which uses an additional parameter to construct the equivalent normal, did not show improvement over R-F in many cases. The problems of inconsistency motivated the development of an algorithm described in Chapter 4. Included herein are those examples examined which are related to the examples used in Chapter 4.

In the following, the examples compare β and p_f as computed by the R-F and C-L methods with the values obtained by numerical integration and Monte Carlo. Monte Carlo was described in Chapter 2. Numerical integration is used for cases which involve two (stress and strength) variables. A computer program using the Romberg integration method (e.g., Hornbeck, 1975) was used to evaluate

$$p_f = \int_o^\infty f_S(x)\ F_R(x)\ dx \tag{C.1}$$

The lower limit value of x = 0.1 was used to avoid singularities in the function to be integrated. The upper limit value was arbitrarily chosen as ten times the standard deviation of the "strength" term plus the mean strength, or ten times the standard deviation of the "stress" term plus the mean stress, whichever is the greater. Upon obtaining p_f, the "equivalent" β is computed as the inverse of the standard normal cdf. This will be compared to the generalized β as computed by R-F or C-L.

A demonstration of the quality of the numerical integration program is provided in Table C.1. For the case where the failure function is g = R - S, and R and S are normal,

$$\beta = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \tag{C.2}$$

This is the exact solution of Table C.1.

The examples involve four distributions of the design variables; Normal, Lognormal, Weibull and Type I Extreme Value Distribution of maxima (EVD).

Example 1. Consider a failure function of the form

$$g(\underset{\sim}{X}) = R - (L + D) \tag{C.3}$$

TABLE C.1

DEMONSTRATION OF THE QUALITY OF THE NUMERICAL INTEGRATION ROUTINE

Failure Function; $g(R, S) = R - S$

$R \sim$ Normal; $S \sim$ Normal

$\mu_R = 20$; $\mu_S = 10$

		Safety Index, β		Probability of Failure		
σ_S	σ_S	Numerical Integ.	Exact	Numerical Integ.	Exact $\Phi(-\beta)$	% Error
3.0	3.0	2.3575	2.3570	9.211 E-03	9.222 E-03	0.1
2.0	2.0	3.5357	3.5355	2.035 E-04	2.035 E-04	0.0
1.2	1.2	5.8921	5.8926	1.903 E-09	1.908 E-09	0.3
3.0	1.2	3.0952	3.0949	9.844 E-04	9.844 E-04	0.0
1.2	3.0	3.0953	3.0949	9.839 E-04	9.844 E-04	0.1

The event of failure is $g \leq 0$. Table C.2 defines the distribution of each of the random variables R, L and D.

For a Monte Carlo solution, K = 10,000 random values of g were generated. The number of g values less than zero is denoted as n_o. The fraction of the number of values less than zero,

$$f_o = n_o/K \tag{C.4}$$

is an estimate of the probability of failure. The results are given in Table C.2.

The empirical distribution function of g is plotted in Fig. C.1; this process is automated. The scales are those of normal probability paper, with the ordinate being values of the standard normal variate z. Thus, the cdf of g is $F_g = \Phi(z)$. The value of z at g = 0 can be interpreted as the negative value of the safety index, $-\beta$. Then $p_f = \Phi(-\beta)$. Results of this example are presented in Table C.2 and Fig. C.1.

Example 2. Following is an analysis of the basic case where the limit state is linear.

$$g(\underset{\sim}{X}) = R - S = 0 \tag{C.5}$$

Comparison of R-F, C-L, and the numerical integration computation is made for two cases. The results are presented in Table C.3. Each case represents a different combination of strength and stress distributions.

Table C.2 A Summary of Example 1

DATA-

Failure Function: $g = R - (L + D)$; Failure Event ($g \leq 0$)

Variable	Distribution	Mean	Standard Deviation
R	Weibull	50.	5.0
L	EVD	10.	2.0
D	Lognormal	20.[a]	0.15[b]

[a] Median [b] COV

MONTE CARLO RESULTS-

Sample Size, K	10,000
Number of values of $g \leq 0$, n_o	35
Estimate of p_f, $f_o = (n_o/K)$	0.0035
Safety Index β from Probability Plot (Figure C.1)	2.70
Estimate of p_f from chart $p_f = \Phi(-\beta)$	0.0035

DESIGN POINT-

Variable	R-F	C-L
R	34.60	33.91
L	11.13	10.20
D	23.46	23.72

COMPARISONS OF SAFETY INDEX AND PROBABILITY OF FAILURE-

	Safety Index, β	Prob. of Failure, p_f
R-F	2.77	.0028
C-L	2.66	.0039
Monte Carlo	2.70	.0035

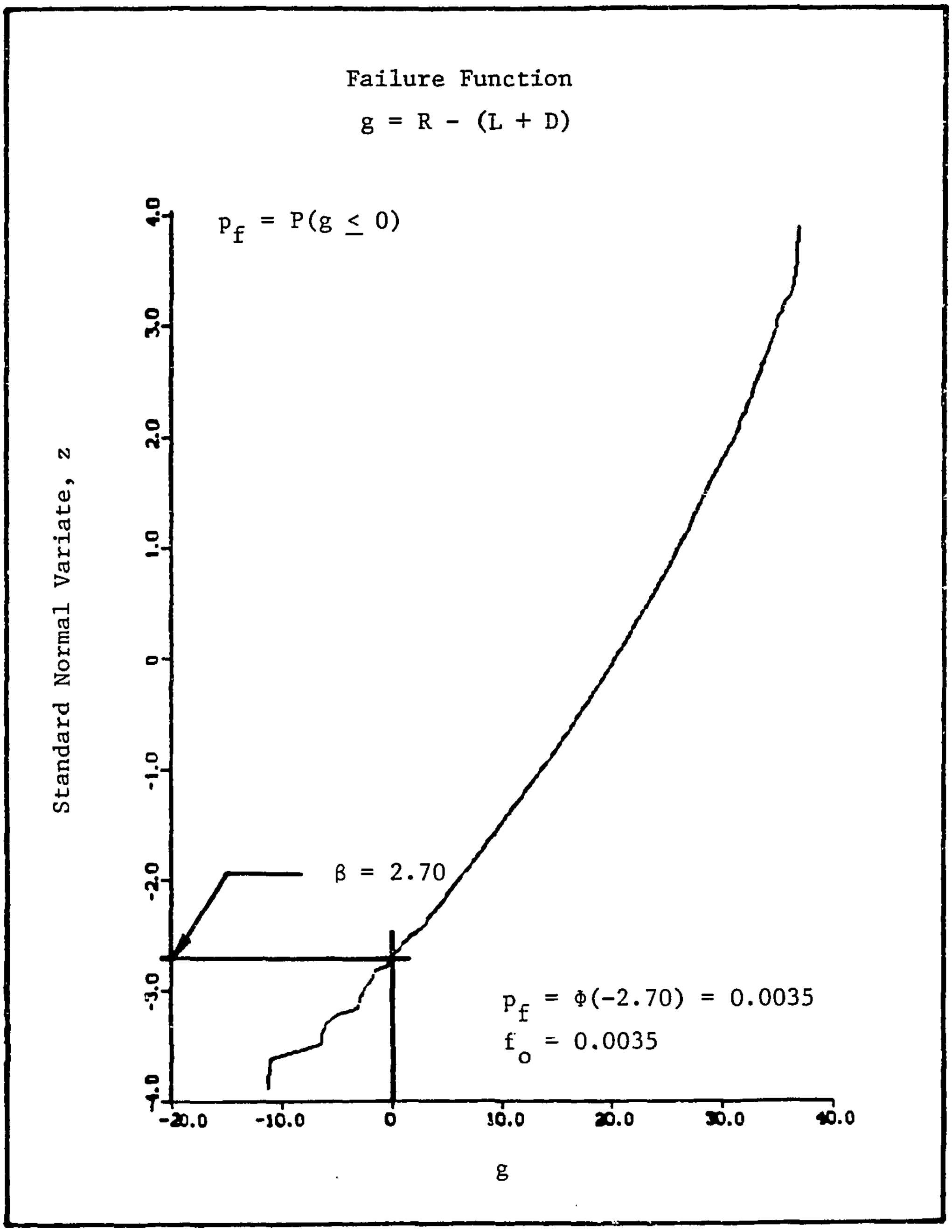


Figure C.1 Example of Monte Carlo Simulation; Empirical cdf of g.

TABLE C.3

EXAMPLE 2; COMPARISONS OF RELIABILITY CALCULATIONS

Limit State Functions $g = R - S$; $\mu_R = 20$ and $\mu_S = 10$ for all cases

σ_R	σ_S	SAFETY INDEX, β			PROBABILITY OF FAILURE, p_f				
		Case A; R = Weibull, S = EVD							
		Numerical Integ.	R-F	C-L	Numerical Integ.	R-F	% Error	C-L	% Error
3.0	3.0	2.048	2.136	2.004	2.026 E-02	1.633 E-02	19.4	2.252 E-02	11.1
2.0	2.0	2.757	2.855	2.701	2.915 E-03	2.152 E-03	26.2	3.455 E-03	18.5
1.2	1.2	3.897	3.992	3.845	4.866 E-05	3.269 E-05	32.8	6.007 E-05	23.4
3.0	1.2	2.616	2.681	2.588	4.447 E-03	3.667 E-03	17.5	4.823 E-03	8.5
1.2	3.0	2.365	2.404	2.298	9.026 E-03	8.102 E-03	10.2	10.79 E-03	19.5
		Case B; R = Weibull, S = Normal							
		Numerical Integ.	R-F	C-L	Numerical Integ.	R-F	% Error	C-L	% Error
3.0	3.0	2.207	2.217	2.205	1.366 E-02	1.329 E-02	2.7	1.373 E-02	0.5
2.0	2.0	3.086	3.093	3.080	10.14 E-04	9.878 E-04	2.6	10.15 E-04	0.07
1.2	1.2	4.502	4.504	4.502	1.353 E-04	3.332 E-06	0.6	3.351 E-05	0.06
3.0	1.2	2.649	2.647	2.648	4.038 E-03	4.061 E-03	0.6	4.042 E-03	0.1
1.2	3.0	3.020	3.075	2.993	1.263 E-03	1.050 E-03	16.9	1.383 E-03	9.5

Example 3. The simple strength-stress relationship is extended to the case where the stress is a non-linear function.

$$g(\underset{\sim}{X}) = R - V^2 \tag{C.6}$$

where V^2 is identified as the "stress" function, S. The relationship between μ's and σ's for S and V for this example and the ones to follow is constructed using the truncated Taylor series expansion. Considering $S = f(V)$, then

$$\mu_S = f(\mu_V) + \frac{1}{2}\frac{\partial^2 f}{\partial V^2} \cdot \sigma_V^2 \tag{C.7}$$

$$\sigma_S = \frac{\partial f}{\partial V} \cdot \sigma_V \tag{C.8}$$

where the partials are evaluated at μ_V. In this example, $S = V^2$ and

$$\mu_S = \mu_V^2 + \sigma_V^2 \qquad \sigma_S = 2\mu_V\sigma_V \tag{C.9}$$

Comparisons of the β and p_f computations are summarized in Table C.4.

Example 4. Consider the non-linear failure function

$$g(\underset{\sim}{X}) = R - V^3 \tag{C.10}$$

Table C.4 Example 3; Comparisons of Reliability Calculations Limit State Function $g = R - V^2$, $\mu_R = 20$ and $\mu_S = 10$ in all cases ($S = V^2$)

Case A; R = Normal, V = Normal

σ_R	σ_S	Safety Index, β: Numerical Integ.	Safety Index, β: R - F	Probability of Failure, p_f: Numerical Integ.	Probability of Failure, p_f: R - F	% Error
3.0	3.0	2.237	2.256	1.263 E-02	1.203 E-02	4.8
2.0	2.0	3.335	3.347	4.263 E-04	4.096 E-04	3.9
1.2	1.2	5.538	5.547	1.520 E-08	1.483 E-08	2.4
3.0	1.2	3.081	3.090	1.033 E-03	1.004 E-03	2.8
1.2	3.0	2.696	2.699	3.511 E-03	3.477 E-03	1.0

Case B; R = Lognormal, V = EVD ($S = V^2$)

σ_R	σ_S	Safety Index, β: Numerical Integ.	Safety Index, β: R - F	Safety Index, β: C - L	Probability of Failure, p_f: Numerical Integ.	Probability of Failure, p_f: R - F	% Error	Probability of Failure, p_f: C - L	% Error
3.0	3.0	2.004	2.013	1.991	2.252 E-02	2.206 E-02	2.0	2.323 E-02	3.1
2.0	2.0	2.653	2.659	2.644	3.991 E-03	3.925 E-03	1.7	4.090 E-03	2.5
1.2	1.2	3.676	3.679	3.668	1.181 E-04	1.176 E-04	0.4	1.221 E-04	3.4
3.0	1.2	3.177	3.204	3.156	7.432 E-04	6.792 E-04	8.6	8.003 E-04	7.7
1.2	3.0	2.137	2.139	2.134	1.629 E-02	1.623 E-02	0.4	1.640 E-02	0.7

Case C; R = Weibull, V = EVD ($S = V^2$)

σ_R	σ_S	Safety Index, β: Numerical Integ.	Safety Index, β: R - F	Safety Index, β: C - L	Probability of Failure, p_f: Numerical Integ.	Probability of Failure, p_f: R - F	% Error	Probability of Failure, p_f: C - L	% Error
3.0	3.0	1.931	2.028	1.905	2.676 E-02	2.126 E-02	20.6	2.837 E-02	6.0
2.0	2.0	2.578	2.672	2.546	4.967 E-03	3.769 E-03	24.1	5.467 E-03	10.1
1.2	1.2	3.617	3.694	3.591	1.478 E-04	1.108 E-04	25.6	1.646 E-04	10.5
3.0	1.2	2.593	2.682	2.593	4.750 E-03	3.658 E-03	23.0	4.754 E-03	0.1
1.2	3.0	2.133	2.163	2.060	1.646 E-02	1.526 E-02	7.3	1.969 E-02	19.6

where V^3 is identified as the "stress" function, S. Relationships between the means and standard deviations of S and V are established by Eqs. C.7 and C.8. Comparisons of β and p_f computations are provided in Table C.5.

Example 5. Consider the non-linear failure function,

$$g(\underset{\sim}{X}) = R - V^4 \tag{C.11}$$

where V^4 is identified as the stress function S. Relationships between the means and standard deviations of S and V are established by Eqs. C.7 and C.8. Comparisons of β and p_f computations are provided in Table C.6.

Table C.5 Example 4; Comparisons of Reliability Calculations Limit State Function $g = R - V^3$, $\mu_R = 20$ and $\mu_S = 10$ in all cases; $(S = V^3)$

Case A; R = Normal, V = Normal

σ_R	σ_S	SAFETY INDEX, β: Numerical Integ.	SAFETY INDEX, β: R - F	PROBABILITY OF FAILURE, p_f: Numerical Integ.	PROBABILITY OF FAILURE, p_f: R - F	% Error
3.0	3.0	1.179	1.193	1.192 E-01	1.163 E-01	2.4
2.0	2.0	1.750	1.759	4.008 E-02	3.927 E-02	2.0
1.2	1.2	2.900	2.905	1.868 E-03	1.840 E-03	1.5
3.0	1.2	2.348	2.374	9.430 E-03	8.801 E-03	6.7
1.2	3.0	1.233	1.235	1.087 E-01	1.082 E-01	0.5

Case B; R = Weibull, V = EVD

σ_R	σ_S	SAFETY INDEX, β: Numerical Integ.	SAFETY INDEX, β: R - F	SAFETY INDEX, β: C - L	PROBABILITY OF FAILURE, p_f: Numerical Integ.	PROBABILITY OF FAILURE, p_f: R - F	% Error	PROBABILITY OF FAILURE, p_f: C - L	% Error
3.0	3.0	1.187	1.231	1.123	1.175 E-01	1.090 E-01	7.2	1.307 E-01	11.2
2.0	2.0	1.581	1.622	1.508	5.688 E-02	5.240 E-02	7.9	6.571 E-02	15.5
1.2	1.2	2.242	2.273	2.173	1.250 E-02	1.150 E-02	8.0	1.488 E-02	19.0
3.0	1.2	2.005	2.120	1.996	2.248 E-02	1.701 E-02	24.3	2.298 E-02	2.2
1.2	3.0	1.232	1.247	1.115	1.089 E-01	1.061 E-01	2.6	1.324 E-01	21.6

Table C.6 Example 5; Comparisons of Reliability Calculations Limit State Function $g = R - V^4$, $\mu_R = 20$ and $\mu_S = 10$ in all cases; $(S = V^4)$

Case A; R = Normal, V = Normal

σ_R	σ_S	SAFETY INDEX, β: Numerical Integ.	SAFETY INDEX, β: R - F	PROBABILITY OF FAILURE, p_f: Numerical Integ.	PROBABILITY OF FAILURE, p_f: R - F	% Error
3.0	3.0	0.803	0.814	2.107 E-01	2.077 E-01	1.4
2.0	2.0	1.188	1.195	1.173 E-01	1.159 E-01	1.2
1.2	1.2	1.968	1.972	2.453 E-02	2.430 E-02	0.9
3.0	1.2	1.776	1.800	3.787 E-02	3.588 E-02	5.3
1.2	3.0	0.824	0.826	2.047 E-01	2.043 E-01	0.2

Case B; R = Weibull, V = EVD

σ_R	σ_S	SAFETY INDEX, β: Numerical Integ.	SAFETY INDEX, β: R - F	SAFETY INDEX, β: C - L	PROBABILITY OF FAILURE, p_f: Numerical Integ.	PROBABILITY OF FAILURE, p_f: R - F	% Error	PROBABILITY OF FAILURE, p_f: C - L	% Error
3.0	3.0	0.903	0.933	0.817	1.831 E-01	1.753 E-01	4.3	2.068 E-01	12.9
2.0	2.0	1.198	1.226	1.103	1.154 E-01	1.101 E-01	4.6	1.350 E-01	17.0
1.2	1.2	1.713	1.735	1.623	4.337 E-02	4.131 E-02	4.7	5.228 E-02	20.5
3.0	1.2	1.603	1.680	1.571	5.441 E-02	4.643 E-02	14.7	5.808 E-02	6.7
1.2	3.0	0.926	0.936	0.783	1.772 E-01	1.745 E-01	1.5	2.165 E-01	22.2

APPENDIX D

A COMPUTER PROGRAM FOR COMPUTING PROBABILITY OF FAILURE AND GENERALIZED SAFETY INDEX USING A CURVE FITTING ALGORITHM

APPENDIX D

A COMPUTER PROGRAM FOR COMPUTING PROBABILITY AND FAILURE OF GENERALIZED SAFETY INDEX USING A CURVE FITTING ALGORITHM

A proposed numerical algorithm presented in Chapter 4 has been used to estimate the probability of failure p_f and generalized safety index β. A block diagram of a computer program which automates the calculations is provided in Fig. D.1.

The input of the program includes, (1) the distribution information, i.e., the type code, mean (or median) and coefficient of variation (COV) for each of the design factors, and (2) the limit state $g(\underset{\sim}{X}) = 0$ which need be written as

$$X_i = f(X_1,..X_{i-1}, X_{i+1},..X_n) \tag{D.1}$$

where X_i can be any of the variables involved in the limit state. Four distributions are included in the current version of the program, namely, normal, lognormal, Weibull and Type I extreme value distribution of maxima (EVD). It is noted that the program has been designed in a format such that the inclusions of other distributions is easy.

The principal output of the program includes p_f, β and the design point information as shown in Table D.1 for an example with limit state

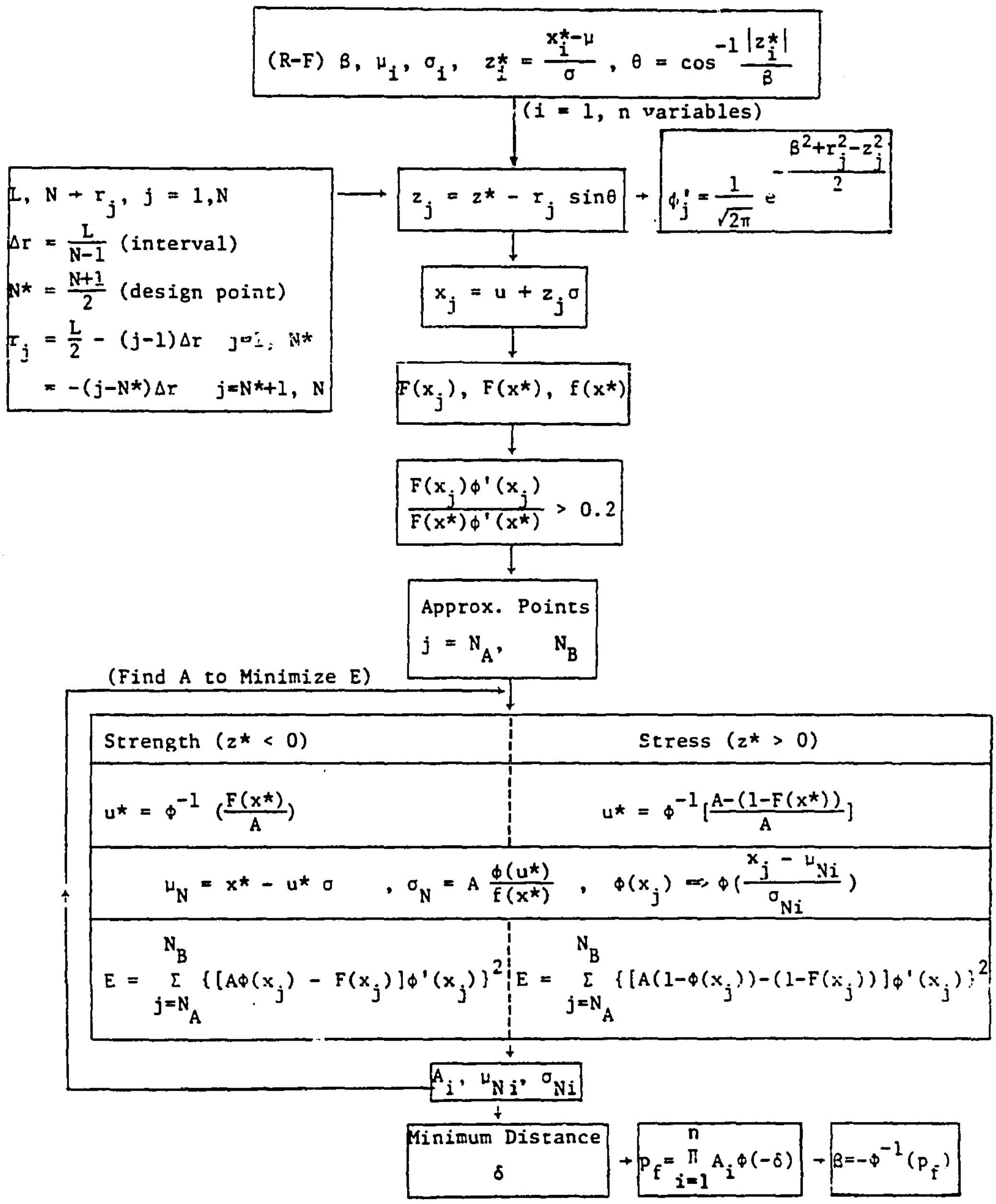


Figure D.1 Summary Numerical Procedure for the New Algorithm.

TABLE D.1

Example of Program Output

Limit State G(R,S) = 0. : R=L+D

Design Variables

Variable	Transformation	Mean/Median	COV
R	WEIBULL	.50000 E+02	.10000 E+00
L	EVD	.10000 E+02	.20000 E+00
D	LOG	.20000 E+02	.15000 E+00

Note: The median is specified for a lognormal variable only.

Design Point

Variable	Reduced Value	Basic Value
R	XR(1) = -2.99143	X(1) = .34388 E+02
L	XR(2) = .82044	X(2) = .10979 E+02
D	XR(3) = 1.20737	X(3) = .23409 E+02

Safety Index, Beta = 2.7112

Probability of Failure = .33556 E-02

$$R = L + D \tag{D.2}$$

This example has been presented in Chapter 4.

Another print-out is the information regarding the original, R-F equivalent normal and the three-parameter equivalent normal cdf (cumulative distribution function). The information can be used to observe the significance of the "non-normality" of the variable and to compare the fitting of cdf using R-F and the new algorithm. For example, Table D.2 is the cdf information of variable R in Eq. D.2. The REL.ERR. (relative error) is defined as

$$\text{REL.ERR.} = \frac{\Phi'(x) - F(x)}{F(x)} \times 100\% \text{ , R-F algorithm} \tag{D.3}$$

$$\text{REL.ERR.} = \frac{A\Phi(x) - F(x)}{F(x)} \times 100\% \text{ , New algorithm} \tag{D. 4}$$

where F(x) is the original cdf, $\Phi'(x)$ and $\Phi(x)$ are the normal cdf using the equivalent normal mean and standard deviation of R-F and present method. It can be seen that in the reduced approximation range (from $N_A = 13$ to $N_B = 31$), REL.ERR. using the R-F method are all negative. Therefore, the R-F estimate of p_f will tend to be less than the exact value (assuming L and D are already normalized) due to the non-normal distribution of R. On the other hand, REL.ERR. using the proposed method have both negative and positive signs which tend to cancel the error of the p_f estimate. Furthermore, it is obvious that the absolute errors of REL.ERR. are less than R-F

Table D.2 Comparisons of the Cumulative Distribution Function

R-F Method				
NX.[a]	X[b]	CDF(X)[c]	CDFNOR(X)[d]	REL.ERR.[e]
13	.28635 E+02	.73715 E-03	.69979 E-03	-.50685 E+01
14	.29380 E+02	.10039 E-02	.96205 E-03	-.41641 E+01
15	.30126 E+02	.13565 E-02	.13120 E-02	-.32787 E+01
16	.30871 E+02	.18194 E-02	.17749 E-02	-.24464 E+01
17	.31617 E+02	.24230 E-02	.23819 E-02	-.16992 E+01
18	.32362 E+02	.32052 E-02	.31710 E-02	-.10669 E+01
19	.33107 E+02	.42125 E-02	.41882 E-02	-.57686 E+00
20	.33853 E+02	.55018 E-02	.54879 E-02	-.25365 E+00
21	.34598 E+02	.71428 E-02	.71343 E-02	-.11865 E+00
22	.35344 E+02	.92195 E-02	.92020 E-02	-.18972 E+00
23	.36089 E+02	.11833 E-01	.11776 E-01	-.48076 E+00
24	.36834 E+02	.15105 E-01	.14954 E-01	-.10014 E+01
25	.37580 E+02	.19180 E-01	.18343 E-01	-.17568 E+01
26	.38325 E+02	.24225 E-01	.23560 E-01	-.27471 E+01
27	.39070 E+02	.30441 E-01	.29234 E-01	-.39678 E+01
28	.39816 E+02	.38057 E-01	.35999 E-01	-.54094 E+01
29	.40561 E+02	.47336 E-01	.43995 E-01	-.70574 E+01
30	.41307 E+02	.58576 E-01	.53367 E-01	-.88930 E+01
31	.42052 E+02	.72109 E-01	.64254 E-01	-.10893 E+02
Scale = .10000 E+01	Mean = .54229 E+02	Std. = .80112 E+01		

Proposed Method				
NX.	X	CDF(X)	CDFNOR(X)	REL.ERR.
13	.28635 E+02	.73715 E-03	.73309 E-03	-.55074 E+00
14	.29380 E+02	.10039 E-02	.99504 E-03	-.87837 E+00
15	.30126 E+02	.13565 E-02	.13425 E-02	-.10329 E+01
16	.30871 E+02	.18194 E-02	.18003 E-02	-.10503 E+01
17	.31617 E+02	.24230 E-02	.23997 E-02	-.96264 E+00
18	.32362 E+02	.32052 E-02	.31796 E-02	-.79883 E+00
19	.33107 E+02	.42125 E-02	.41878 E-02	-.58509 E+00
20	.33853 E+02	.55018 E-02	.54828 E-02	-.34508 E+00
21	.34598 E+02	.71428 E-02	.71356 E-02	-.10011 E+00
22	.35344 E+02	.92195 E-02	.92315 E-02	.13088 E+00
23	.36089 E+02	.11833 E-01	.11872 E-01	.33133 E+00
24	.36834 E+02	.15105 E-01	.15179 E-01	.48719 E+00
25	.37580 E+02	.19180 E-01	.19292 E-01	.58705 E+00
26	.38325 E+02	.24225 E-01	.24376 E-01	.62236 E+00
27	.39070 E+02	.30441 E-01	.30620 E-01	.58766 E+00
28	.39816 E+02	.38057 E-01	.38240 E-01	.48080 E+00
29	.40561 E+02	.47336 E-01	.47480 E-01	.30319 E+00
30	.41307 E+02	.58576 E-01	.58611 E-01	.60057 E-01
31	.42052 E+02	.72109 E-01	.71936 E-01	-.23930 E+00
Initial Approx. Range = 8.0	Initial Points = 41			
Design Point NX = 21				
Scale = .47750 E+01	Mean = .62242 E+02	Std. = .93110 E+01		

Notes: (a) Numbers of the approximation points.
(b) Basic values of the design variable.
(c) F(x), cdf of X.
(d) $\Phi'(x)$ for R-F method and $A\Phi(x)$ for the proposed method.
(e) Percent error of the equivalent normal cdf relative to F(x).

results in the approximation range, suggesting that a better fit of $A\Phi(x)$ than $\Phi'(x)$ (of R-F) has been achieved. The results of L and D variables, not included herein, showed the same tendency. Consequently, the R-F estimate of p_f (0.0028) is less than the present method (0.0033). A Monte Carlo solution (100,000 samples) produces an estimate of 0.0035, the 98% confidence limits are (0.0031, 0.0040).

The relatively time-consuming part of the computation in applying the new algorithm is the minimization routine which determines the three equivalent normal parameters, A, μ_N, and σ_N. An optimization routine (Converse, 1970) was modified and used in the program which starts with A = 1, and automatically goes back and forth to search for an optimum A which minimizes the error sum E (Fig. D.1). The scheme is effective but may not be the most efficient one, particularly when A is very different from unity. It is possible to cut down the computation time, if necessary, by using a quadratic estimation procedure (e.g., Powell, 1964) which successively predicts the solution of A. However, unless a variable is significantly non-normal, the number of iterations for finding A is usually reasonably small in the present program. For example, the values of the scale factors, A_i, for R, L and D of Table D.1 are 4.77, 1.25 and 1.28 respectively. The corresponding numbers of iteration are 31, 16 and 19, respectively. The CPU time including

R-F computation is 0.3 second. After observing numerous examples, it was estimated that the additional cost (over R-F) for applying the present algorithm is less than 0.1 second for each non-normal variable.

APPENDIX E

A COMPUTER PROGRAM FOR COMPUTING PARTIAL SAFETY FACTORS

APPENDIX E

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Consider a limit state function

$$g(X_1, X_2, \ldots, AX_i, \ldots X_n) = 0 \tag{E.1}$$

A is a scale factor and is deterministic, e.g., a geometric parameter. X_i are the design random variables having known distributional information. A computer program was developed which uses a bracketing procedure followed by a quadratic curve fitting scheme to find the value of A such that the safety index, β, equals the desired target safety index, denoted as β_o. Design variables can have any one of four distributions; normal, lognormal, Weibull, or Type I extreme value (EVD).

The program computes β and the associated design point using the R-F (Rackwitz-Fiessler) algorithm which employs an optimization routine (Converse, 1970).

Partial safety factors (PSF's) defined as

$$\gamma_i = \frac{\text{Design point value of } X_i}{\text{Nominal value of } X_i} \tag{E.2}$$

are obtained after <u>design point</u> values are found.

By assigning a value to the scale factor, A, the limit state is completely defined, and the safety index can be obtained by solving the following constrained optimization problem:

$$\beta = \text{min.} \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2} \tag{E.3}$$

$$\text{subject to} \quad g(X_1, \ldots, AX_i, \ldots X_n) = 0$$

in which

$$u_i = \frac{X_i - \mu_i}{\sigma_i} \qquad i = 1, n \tag{E.4}$$

μ_i and σ_i are the equivalent normal means and standard deviations, calculated using R-F algorithm. Note that Eq. E.3 can be reduced to an unconstrained optimization problem if the limit state were written as

$$X_1 = f(X_2, \ldots, AX_i, \ldots X_n) \tag{E.5}$$

The set of values of X_i which minimize the safety index are the design point values, and are used to compute the PSF's.

Procedures of Calculating Scale Factor, A

According to the definition (Eq. E.3), β can only be positive or zero. The relationship between β and A can in general be described by Fig. E.1, in which it is shown that given a target safety index,

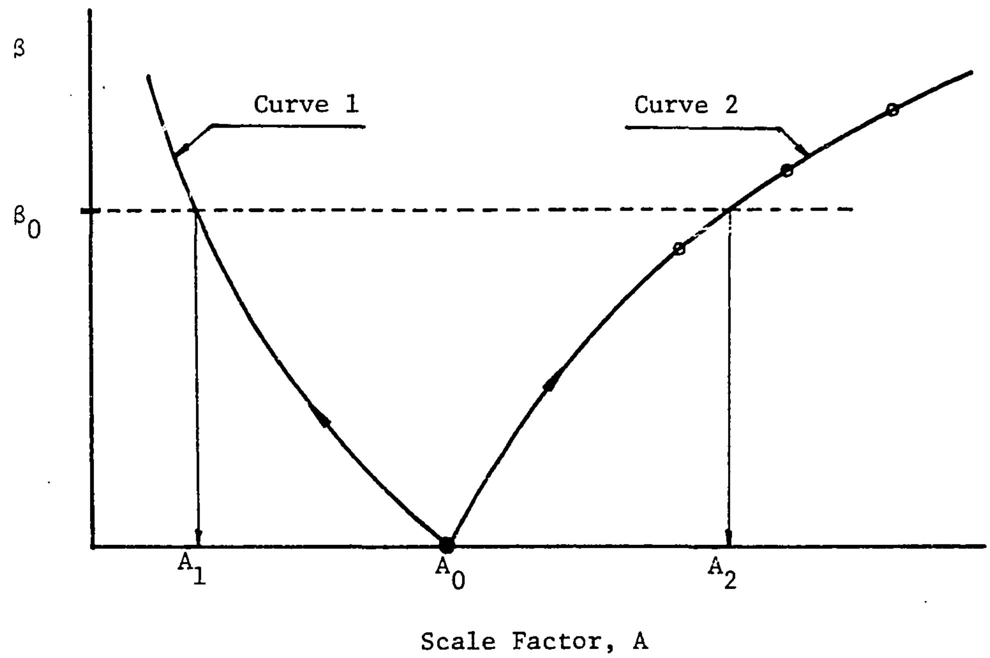

Figure E.1 The Relationship between Safety Index and Scale Factor.

β_o, there are two solutions, A_1 and A_2. However, only one of A_i is correct, depending upon whether A is identified with a "strength" variable or a "stress" variable. For example, if A multiplies a strength variable, β will be an increasing function of A (curve 2), then A_2 is the solution.

In order to adopt a curve fitting scheme to find the scale factor more efficiently, it is necessary that the points sampled are all from the correct curve, i.e., from curve 1 or curve 2, but not both. From Fig. E.1 it seems reasonable to start from A_o, a value of A associates with zero safety index β.

The procedures of calculating the scale factor using a quadratic curve fitting model are as follows:

1. Calculate starting scale factor A_o.

   Assume that all X_i are normally distributed. If the design point values, X_i^*, are the mean values, i.e.,

$$x_i^* = \mu_i \tag{E.6}$$

   then $u^* = 0$ for all i (Eq. E.4), and $\beta = 0$ (Eq. E.3). Substituting μ_i into Eq. E.5

$$\mu_1 = f(\mu_2, \mu_3, \ldots A_o\mu_i, \ldots\mu_n) \tag{E.7}$$

   therefore, A_o is the value which satisfies Eq. E.7. A_o may be solved using any suitable trial-and-error method. An efficient solution is to use the same optimization routine adopted for finding the safety index β. The only difference is that in the present problem there is only one unknown variable, A_o. By varying A, there is a value, A_o, associates with "minimum distance", β, in which $\beta = 0$. Finally, if X_i is non-normal, it is assumed that Eq. E.6 is a good approximation and may be used to generate equivalent normal mean which replaces μ_i in Eq. E.7.

2. Bracketing the target safety index. After A_o is found, A is continuously doubled (or halved) to increase β until $\beta \geq \beta_o$. The last two values of A which bracket β_o define the initial interpolation range for the quadratic curve fitting.

3. Quadratic curve fitting. A third value of A which is inside the interpolation range must be chosen. Three values of A and the corresponding β's are used to construct a second order equation,

$$\beta = \alpha_o + \alpha_1 A + \alpha_2 A^2 \qquad \text{(E.8)}$$

After α_i are found, the interpolated (or guessed) value of A_o, for $\beta = \beta_o$ are

$$A_o = \frac{-\alpha_1 \pm \sqrt{\alpha_1^2 - 4\alpha_2(\alpha_o - \beta_o)}}{2\alpha_2} \qquad \text{(E.9)}$$

in which one of the two roots, inside the interpolation range, is the desired answer. A safety index β using the guessed A_o value is then calculated and compared with β_o. If two values of β are approximately equal (defined by a stop criteria), A_o is the solution. If not, the interpolation range of A is reduced, and an improved guessing value of A_o is found using Eq. E.8 and Eq. E.9. The process is repeated until the stop criteria is satisfied. In general, only several interations are required because the interpolation range is continuously reduced (which still brackets β_o), and therefore, the interpolated A_o converges to the exact A_o very fast. Examples of this program and the subsequent construction of a safety check expression are provided in Chapter 5.

REFERENCES

Ang, A. H.-S., and Cornell, C. A., 1974, "Reliability Bases of Structural Safety and Design," Journal of the Structural Div. ASCE, Vol. 100, Sept., pp. 1755-1769.

Ang, A. H. S., and Tang, W. H., 1984, Probability Concepts in Engineering Planning and Design, Vol. II, Wiley.

Breitung, K., 1983, "Asymptotic Approximations for Multi-Normal Domain and Surface Integrals," Fourth International Conference on Applications of Statistics and Probability in Soil and Structural Engineering. Università di Firenze, Pitagora Editrice.

Chen, X., and Lind, N. C., 1982, "A New Method of Fast Probability Integration," Solid Mechanics Division, University of Waterloo, Canada. Paper No. 171, June.

Collins, J. A., 1981, Failure of Materials in Mechanical Design, Wiley.

Converse, A. O., 1970, Optimization, Holt, Rinehart and Winston, Inc.

Cornell, C. A., 1969, "A Probability-Based Structural Code," Journal of the American Concrete Institute, Vol. 66, No. 12, Dec.

Ditlevsen, O., 1976, "Evaluation of the Effect on Structural Reliability of Slight Deviations from Hyperplane Limit State Surfaces," Proc., 2nd Int'l. Wrkshp. on Code Formats, Mexico City.

Ditlevsen, O., 1981, "Principle of Normal Tail Approximation," Journal of Engineering Mechanics Division, ASCE, Vol. 107, Dec., pp. 1191-1208.

Dowling, N. E., 1982, "A Discussion of Methods for Estimating Fatigue Life," SAE Technical Paper Series 820691, pp. 161-173.

Ellingwood, B., Galambos, T. V., MacGregor, J. G., and Cornell, C.A., 1980, "Development of a Probability Based Load Criterion for American National Standard A58." NBS Special Publication 577, June.

Fiessler, B., Neumann H. J., and Rackwitz, R., 1979, "Quadratic Limit States in Structural Reliability," Journal of the Engineering Mechanics Division, ASCE, Vol. 105, Aug., pp. 661-676.

Freudenthal, A. M., 1947, "The Safety of Structures," Transactions, ASCE, Vol. 112, pp. 125-180.

Freudenthal, A. M., 1956, "Safety and the Probability of Structural Failure," Transactions, ASCE, Vol. 121, pp. 1337-1397.

Freudenthal, A. M., Garrelts, J. M. and Shinozuka, M., 1966, "The Analysis of Structural Safety," Journal of the Structural Division, ASCE, Vol. 92, No. ST1. Feb., pp. 267-325.

Fuchs, H. O., and Stephens, R. I., 1980, Metal Fatigue in Engineering, Wiley.

Hasofer, A. M., and Lind, N. C., 1974, "Exact and Invariant Second-Moment Code Format," Journal of the Engineering Mechanics Division, ASCE, Vol. 100, No. EM1, Feb., pp. 111-121.

Haugen, E. G., and Wirsching, P. H., 1975, "Probabilistic Design," Machine Design, Apr. 17 - June 12.

Hohenbichler, M., Rackwitz, R., 1981, "Non-normal Dependent Vectors in Structural Safety," Journal of the Engineering Mechanics Division, ASCE, Vol. 100, No. EM6, Dec., pp. 1227-1238.

Hornbeck, R. W., 1975, Numerical Methods, Quantum Publishers, Inc.

Horne, R., and Price, P. H., 1977, "Commentary on the Level-II-Procedure," Rationalization of Safety and Serviceability Factors in Structural Codes, CIRIA Report 63, Construction Industry Research and Information Association, London.

Imhof, J. P., 1961, "Computing the Distribution of Quadratic Forms in Normal Variables," Biometrika, Vol. 48, Nos. 3 and 4, pp. 419-426.

Johnson, N. I., and Kotz, S., 1970, Distributions in Statistics--Continuous Univariate Distributions--2, Houghton Mifflin Co., Boston, Mass.

Kaufman, A. and Hunt, L.E., 1982, "Material Constitutive Models for Non-linear Analysis of Thermally Cycled Structures," NASA Technical Paper 2055, Oct.

Kececioglu, D.B. and Cormier, D., 1964, "Designing a Specified Reliability into a Component," Proceedings of the Third Reliability and Maintainability Conference, Washington, D.C.

Kelly, C.F., 1981, "A New Computer Algorithm for Predicting Structural Reliability," Master's Report, Aerospace and Mechanical Engineering, The University of Arizona.

Leporati, E., 1979, "The Assessment of Structural Safety," Research Studies Press.

McLain, S.D., Kelly, C.F. and Wirsching, P.H., 1982, "On the Performance of a New Algorithm for Computing Structural Reliability," Aerospace and Mechanical Engineering, The University of Arizona.

Montgomery, D. C., 1982, Introduction to Linear Regression Analysis, Wiley.

Parimi, S. R., Lind, N. C., 1976, "Limit States Basis for Cold-formed Steel Design," Journal of the Structural Division, ASCE, March.

Powell, M. J. D., 1964, "An Efficient Method for Finding the Minimum of a Function of Several Variables Without Calculating Derivatives," Computer Journal, Vol. 7, pp. 155-162.

Press, S. J., 1966, "Linear Combination of Non-Central χ^2 -Variates," Annals of Mathematical Statistics, Vol. 37, pp. 480-487.

Rackwitz, R., and Fiessler, B., 1978, "Structural Reliability Under Combined Random Load Sequences," Journal of Computers and Structures, Vol. 9, pp. 489-494.

Rackwitz, R., 1976, "Practical Probabilistic Approach to Design," First Order Reliability Concepts for Design Codes, Comite European du Beton, Bulletin D'Information, No. 112, July.

Rationalisation of Safety and Serviceability Factors in Structural Codes, 1977, Report 63, CIRIA, Construction Industry Research and Information Association, 6 Storey's Gate, London SWIP 3AU, July.

Ravindra, M. K., and Galambos, T. V., 1978, "Load and Resistance Factor Design for Steel," Journal of the Structural Division, ASCE, Vol. 104, No. ST9, Sept., pp. 1337-1353.

Rosenblatt, M., 1952, "Remarks on a Multivariate Transformation," Annals of Mathematical Statistics, Vol. 23, No. 3, Sept., pp. 470-472.

Rosenblueth, E., and Esteva, L., 1972, "Reliability Bases for Some Mexican Codes," Probabilistic Design of Reinforced Concrete Buildings, Publication SP-31, American Concrete Institute, Detroit, Mich.

Ruben, H., 1962, "Probability Content of Regions under Spherical Normal Distributions IV," Annals of Mathematical Statistics, Vol. 33, pp. 542-570.

Ruben, H., 1963, "A New Result of the Distribution of Quadratic Forms," Annals of Mathematical Statistics, Vol. 34, pp. 1582-1584.

Shinozuka, M., 1983, "Basic Analysis of Structural Safety," Journal of the Structural Division, ASCE, Vol. 109, No. 3, March, pp. 721-740.

Thoft-Christensen, P., Baker, M. J., 1982, Structural Reliability Theory and Its Applications, Springer-Verlag, N.Y.

Tvedt, L., 1983, "Two Second Order Approximations to the Failure Probability," Det Norske Veritas (Norway) RDIV/20-004-83.

Vannoni, M. G., 1979, "The Hasofer-Lind Safety Index in Probabilistic Design," Master's Report, Aerospace and Mechanical Engineering, The University of Arizona.

Wirsching, P. H., 1981, "Application of Probabilistic Design Theory to High Temperature Low Cycle Fatigue," NASA CR-165488, NASA Lewis Research Center, Cleveland, OH. November.

Wirsching, P. H., and Wu, Y.-T., 1983, " A Review of Modern Approaches to Fatigue Reliability Analysis and Design," ASME Fourth National Congress on Pressure Vessel and Piping Technology, Portland, OR, June, and published in Random Fatigue Life Prediction, ASME, 1983.

Wu, Y.-T., McLain, S.D., Kelly, C. F., and Wirsching, P. H., 1983, "On the Performance of Efficient Algorithms for Computing Structural Reliability," Aero and Mech Engr., The Univ. of Arizona.