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Pickrell, Douglas Murray

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SPIN EXTENSIONS AND MEASURES ON INFINITE

DIMENSIONAL GRASSMANN MANIFOLDS

by

Douglas Murray Pickrell

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As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Douglas Murray Pickrell

entitled Spin Extensions and Measures on Infinite Dimensional Grassmann Manifolds

and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

Geoffrey Martin	4/12/84 Date
Hermann Flaschka	4/12/84 Date
DW McLaughlin	4/12/84 Date
Charles M. Newman	4/12/84 Date
John Palmer	4/12/84 Date

Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

John Palmer
Dissertation Director

4/12/84
Date

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SIGNED: Douglas Murray Pickrell

PREFACE

This paper is organized around the theme of constructing an orbit theory for certain infinite dimensional matrix groups. Although I had previous exposure to the orbit method, my interest in its application to infinite dimensional groups can be traced to a seminar led by John Palmer and Hermann Flaschka in the Spring of 1982.

Since that time I have benefitted frequently from conversations with Hermann Flaschka and Charles Newman. The guidance and insights provided by my adviser and friend John Palmer were indispensable for the completion of this work. I also want to thank him for giving me the opportunity to work side by side with him on a preliminary project.

I have waited ten years for this occasion to thank Elias Toubassi for investing many summer hours in my mathematical education, despite the fact that at that time the Riemann integral was still more my foe than friend.

I hope this work proves worthy of the confidence which my wife, my parents, Elias Toubassi, and John Palmer have expressed in me.

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ABSTRACT

The representation theory of infinite dimensional groups is in its infancy. This paper is an attempt to apply the orbit method to a particular infinite dimensional group, the spin extension of the restricted unitary group.

Our main contribution is in showing that various homogeneous spaces for this group admit measures which can be used to realize the unitary structure for the standard modules.

INTRODUCTION

In the theory of affine Lie algebras, it is commonplace to embed the loop algebra in the algebra of all $\mathbb{Z} \times \mathbb{Z}$ matrices of the form (a_{ij}), where (a_{ij}) has only finitely many nonzero diagonals - a loop of the form $\sum_{-N}^{+N} A_K \zeta^K$ corresponding to the periodic (Toeplitz) matrix

$$(a_{ij}) = \left(\begin{array}{c|c|c} \ddots & \ddots & \ddots \\ A_0 & A_1 & A_2 \\ \hline A_{-1} & A_0 & A_1 \\ \hline A_{-2} & A_{-1} & A_0 \\ \ddots & \ddots & \ddots \end{array} \right) .$$

The affine extension of the loop algebra is then induced by a certain central extension of this Lie algebra of nonperiodic matrices.

This paper primarily concerns applying the orbit method to a Banach Lie group which corresponds to the central extension of the nonperiodic matrices. Some of the results are analogous to those known for affine Lie algebras, as we will point out below. Virtually all of this paper is based in one way or another on the works of David Shale and Graeme Segal.

Here is a description of the contents:

Chapters 1 and 5 concern the restricted unitary group, U_r, and the restricted orthogonal group, O_r, respectively. The group O_r was discovered by Shale: O_r consists of those orthogonal operators which, as automorphisms of the Clifford algebra, are implementable in a certain spin representation. As matrices the elements of O_r have the form $g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$, where b is almost finite dimensional (Hilbert-Schmidt). The extension to which we referred earlier is defined by the spin representation. To understand the representation (and hence the extension), it is useful to view O_r as the product of the subgroups $\left\{\begin{pmatrix} a & \\ & \bar{a} \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : a = 1 + \text{trace class}\right\}$. Reconstructing O_r analytically from this decomposition occupies most of Chapter 5 (that the latter subgroup is naturally a Banach Lie group seems quite amazing to me). Analogous product decompositions are considered for U_r in Chapter 1.

Chapter 2 concerns the affine extension of U_r, and Chapter 6 concerns the extension of O_r defined by the spin representation. These extensions (over the connected components of U_r and O_r, respectively) can be realized in two essentially equivalent ways: (1) as Baer products of groups, or (2) as quotients of semidirect products. The first realization is due to Graeme Segal (although I do not know if he thinks of his model as a Baer product of groups). It has the distinct advantage that it restricts to a model for the affine extension of loops in a natural way. The second is useful for computing the adjoint action, and it also is natural from the viewpoint of an analogy with Mackey's normal subgroup analysis. Since this viewpoint is quite attractive and only implicit in the paper, I would like to briefly explain.

Suppose G is a group, N is a normal subgroup, θ is a G

orbit in the dual of N, L is a fixed representation in θ, and G_L is the stabilizer. For our analogy, modulo a few details, one should think of $G = O(H_{\mathbb{R}})$, $N = O_1(H_{\mathbb{R}}) = \{g \in O(H_{\mathbb{R}}): g = 1 + \text{trace class}\}$, θ the orbit of spin representations, L = fixed spin representation, and $G_L = O_r$, the corresponding restricted orthogonal group. The first problem in normal subgroup analysis is that of extending L from N to G_L. The extent to which this is possible is measured by the so-called Mackey obstruction, F, a central unit circle extension of G_L/N which is characterized by the condition that L does extend to a representation of the Baer product of F and (unit circle) x G_L. If L is realizable via the orbit method, say L corresponds to $f \in \underline{n}^*$ ($\underline{n}$ = Lie algebra of N), then the Mackey obstruction is generally the middle group of the sequence

$$1 \to N_f/N_f^{\perp} \to G_f/N_f^{\perp} \to G_f/N_f \to 1,$$

where N_f is the kernel of the character corresponding to f. This basically follows from the fact that L can be extended to the semidirect product $N \times_t G_f$, for the quotient of the semidirect by the kernel of the extension of L is the Baer product of G_f/N_f and (unit circle) x G, hence the assertion follows by uniqueness of the Mackey obstruction. In terms of our analogy,

$f = \begin{pmatrix} i/2 & \\ & -i/2 \end{pmatrix}$, $G_f = \{\begin{pmatrix} a & \\ & \bar{a} \end{pmatrix} \in O_r\}$, which is isomorphic to a full unitary group $U(W_+)$, and the Mackey obstruction is given by

$$1 \to {}^{U_1(W_+)}/_{SU_1(W_+)} \to F = {}^{U(W_+)}/_{SU_1(W_+)} \to {}^{U(W_+)}/_{U_1(W_+)} \to 1,$$

where $SU_1(W_+) = \{q \ \varepsilon\ U_1(W_+)\colon \det g = 1\}$. The Baer product of $({}^{U}1/_{SU_1}) \times O_r$ and $F = {}^{U}/_{SU_1}$ is basically the realization of Segal. The semidirect product is $O_1(H_{\mathbb{R}}) \times_t U(W_+)$.

Of course, as we mentioned before, this paper concerns principally the orbit method applied to the extensions of U_r and O_r, not the full orthogonal group (although this may be interesting). Hence we have suppressed the viewpoint just expressed. At any rate the reader is forewarned of my preoccupation with semidirect products.

One of the details we overlooked above is that the spin representation for $O_1(H_{\mathbb{R}})$ is projective; spin is a faithful ordinary representation for the double cover of $O_1(H_{\mathbb{R}})$, usually denoted Pin_∞. It also turns out to be necessary to extend spin (by continuity) from O_1 to a subgroup $(O_r)_*$ which we refer to as the predual of O_r (because on the level of Lie algebras, it literally is the predual); the corresponding double cover of $(O_r)_*$ is denoted Pin_*. To show the representation extends we have chosen to use (in a nonessential way) a theorem of my adviser, John Palmer, which is an analogue of the Nelson analytic vector theorem for the spin representation. It asserts that the Lie algebra of O_r can be represented projectively by skew-adjoint operators on a common dense group invariant domain. I have included it because it seems quite satisfying technically, although we have not found any applications for which it is essential.

In Chapter 3 the adjoint action for the affine extension of U_r is computed in two ways: (1) by simply integrating the infinitesimal action and (2) by using the semidirect product formulation.

Chapters 4 and 7 are devoted to the coadjoint action restricted to preduals for the extensions of U_r and O_r, respectively. There is an interesting feature shared with affine algebras: the problem of parameterizing predual coadjoint orbits of U_r is quite intractable, while the corresponding problem for affine predual orbits (those consisting of functionals which do not vanish on the center of the extension) is tractable. In the case of affine algebras the fundamental result of Frenkel and Segal is that each smooth affine orbit contains a constant loop. In our case the fundamental result is that each affine predual orbit contains a diagonal operator.

That the usual Borel-Weil construction can be extended to this infinite dimensional setting to realize the representations corresponding to the integral predual orbits has basically been observed by Robert Boyer and Graeme Segal (Boyer has constructed models for the representations which vanish on the center of the extension, while Segal has constructed the analogue of the basic representation for affine algebras - the general case is just a mixture of the two). The analogue of this construction for affine Lie algebras is apparently soon to be published in a forthcoming book by Graeme Segal and A. Pressley.

The most important part of this paper concerns the unitary structure of the representations above. The idea is contained in the geometric quantization interpretation for the Segal-Bargmann construction of the oscillator representation: view the representation space as holomorphic sections of a line bundle divided by a fixed nonvanishing section, i.e. as holomorphic functions on the base space; now recover the usual inner product via the L^2 inner product relative to Gaussian measure. The key point here is that this works in infinite dimensions, since the polynomials of a finite number of variables are dense. In our case there does not exist a nonvanishing section, but we persist anyway, realizing the representation on a space of meromorphic functions. In graph coordinates these turn out to depend on finitely many variables. The relevant measure is an analogue of Gaussian measure in infinite dimensional flag manifolds.

We have chosen to work this out in the single case of the basic representation. The base space is then a Grassmann manifold, for which the notation is less cumbersome than for a more general flag manifold.

Chapter 8 contains all the relevant differential geometric background.

Chapter 9 contains the relevant measure theory, as well as some additional work on invariant measures in infinite dimensional Grassmann manifolds.

How the invariant measures arise is easy to motivate. Suppose H is an ∞-dimensional ($\mathbb{C}$) Hilbert space, $Gr(n, H)$ the Grassmann manifold of n-dimensional subspaces of H, $\mathcal{L}(\mathbb{C}^n, H)^x$ the set of nonsingular linear maps (so-called admissible bases), and π the projection map $\mathcal{L}(\mathbb{C}^n, H)^x \to Gr(n, H): L \to \text{Range}(L)$. The space $Gr(n, H)$ is a homogeneous space for $U(H)$, and an invariant finitely additive measure ν_F can be obtained in the following manner: the normal distribution n on $\mathcal{L}(\mathbb{C}^n, H)$ is invariant with respect to the action of $U(H)$ by composition; simply put $\nu_F = \pi_*(n)$.

The approach fails for the Grassmann manifold corresponding to the basic representation, for the analogue of the principal bundle of admissible bases above now has infinite dimensional fiber. One way to resolve this difficulty is to observe that ν_F above is actually a cylinder measure in graph coordinates. This approach does generalize. Moreover it suggests how to construct the measures which are germaine to representation theory.

To establish invariance, quasi-invariance etc., it proved useful to use Irving Segal's algebraic integration theory.

In the final Chapter (10) it is established that the correct inner product is yielded by our analogue of Gaussian measure in a Grassmann manifold.

Notation.

For linear sets of operators the subscripts 1 and 2 will denote those which are trace class and Hilbert-Schmidt, respectively. For example $\mathcal{L}_2(H_+, H_-)$ will denote all Hilbert-Schmidt operators from H_+ to H_-, $\underline{u}_1(H)$ will denote the operators in $\underline{u}(H)$ (the space of skew-adjoint operators) which are trace class.

For groups of operators the subscripts 1 and 2 will denote those of the form 1 + trace class and 1 + Hilbert-Schmidt, respectively. For example $U_1(H)$ will denote the unitary operators on H of the fórm 1 + trace class.

Lie groups will be denoted by capital letters, while Lie algebras will be denoted by small underlined letters. For example $O(H)$ will denote the orthogonal group, while $\underline{o}(H)$ will denote the Lie algebra of (bounded) skew-symmetric operators.

CHAPTER 1

THE RESTRICTED UNITARY GROUP

In this section we define the restricted unitary group, U_r, and an important normal subgroup of U_r, $(U_r)_*$, which we think of as the predual. Each of these groups can be viewed as the unitary group of a Banach *-algebra. This provides their analytic structure.

The group U_r can also be viewed as a subgroup of the restricted orthogonal group, O_r, discovered by Shale in [Shale, 1] (we introduce O_r in section 5). In this way the group $(U_r)_0$, the identity component of U_r, can naturally be viewed as the generator of the hidden symmetry in the Kadomtsev-Petviashvili (KP) hierarchy, when this symmetry is formulated in the context of the spin representation (see [Date, 1]). The work of Sato et al on the KP hierarchy was recently clarified in [Segal, 2], where a group denoted GL_{res} is identified as a natural group of symmetries of a space of solutions to the KP hierachy. The group GL_r which we define below is the Hilbert-Schmidt analogue of GL_{res}.

Throughout the next 4 sections, H_+ and H_- will denote ∞-dimensional complex Hilbert spaces, $H = H_+ + H_-$, and $Q_\pm$ are the self-adjoint projections for $H_\pm$, respectively. We will abbreviate $\underline{g\ell}(H)$, GL(H), etc. to $\underline{g\ell}$, GL, etc.

If $\xi \in \underline{g\ell}$ we will often write

$$\xi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where $\alpha(\xi) = Q_+\xi\, Q_+$, etc. If $g \in GL$ we will often write

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a(g) = Q_+\, g\, Q_+$, etc.

Suppose $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

yields the equations

(1.1) $aa^* + bb^* = cc^* + dd^* = a^*a + c^*c = b^*b + d^*d = 1$

(1.2) $ca^* + db^* = a^*b + c^*d = 0.$

(1.3) Definition.

(i) $\underline{g\ell}_r = \{\xi \in \underline{g\ell} : \beta(\xi)$ and $\gamma(\xi)$ are Hilbert-Schmidt$\}$.

(ii) $\underline{u}_r = \underline{u} \cap \underline{g\ell}_r$.

(iii) $GL_r = GL \cap \underline{g\ell}_r$.

(iv) $U_r = U \cap \underline{g\ell}_r$.

(v) $Sym_r = Sym \cap \underline{g\ell}_r$.

(1.4) Definition. For $\xi \in \underline{g\ell}_r$, $\xi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$,

$$|\xi|_r = \left|\begin{pmatrix} \alpha & \\ & \delta \end{pmatrix}\right|_\infty + \left|\begin{pmatrix} & \beta \\ \alpha & \end{pmatrix}\right|_2.$$

(1.5) If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon\, GL_r$, then

$$\begin{pmatrix} a & \\ & d \end{pmatrix} = g - \begin{pmatrix} & b \\ c & \end{pmatrix}.$$

This implies a and d are Fredholm, and index (a) + index (d) = 0

(1.6) Proposition.

(i) $\underline{g\ell}_r$ with the usual adjoint operation and $|\cdot|_r$ is a Banach *-algebra with identity.

(ii) GL_r is an open subset of $\underline{g\ell}_r$, and equipped with the induced complex analytic structure, GL_r is a complex Banach Lie group with Banach Lie algebra $\underline{g\ell}_r$.

(iii) U_r is a Banach Lie subgroup of GL_r with Banach Lie subalgebra $\underline{u}_r$.

(iv) The map $U_r \times Sym_r \to GL_r$ given by

$$(g, \xi) \to g \cdot \exp(\xi)$$

is a diffeomorphism.

(v) The connected components of U_r and GL_r are indexed by the function

$$g \to \text{index } a(g).$$

Proof. (i) Suppose $\xi_j \varepsilon\, \underline{g\ell}_r$ and $A_j = \begin{pmatrix} \alpha_j & \\ & \delta_j \end{pmatrix}$, $B_j = \begin{pmatrix} & \beta_j \\ \alpha_j & \end{pmatrix}$. Then

$$|\xi_1\xi_2|_r = |A_1A_2 + B_1B_2|_\infty + |A_1B_2 + B_1A_2|_2$$

$$\leq |A_1|_\infty|A_2|_\infty + |B_1|_\infty|B_2|_\infty + |A_1|_\infty|B_2|_2 + |B_1|_2|A_2|_\infty$$

$$\leq (|A_1|_\infty + |A_2|_2)(|A_2|_\infty + |B_2|_2),$$

because $|B_j|_\infty \leq |B_j|_2$. (i) is now clear.

(i) and (iii) GL_r is the group of units and U_r is the group of unitaries for the Banach *-algebra with identity $\underline{g\ell}_r$. These are basic examples of Banach Lie groups (see [De la Harpe, 1]).

(iv) follows from polar decomposition.

(v) The map $GL_r \to \mathbb{Z}$ given by $g \to$ index $a(g)$ is a continuous homomorphism, because

$$a(g_1g_2) = a(g_1)a(g_2) + b(g_1)c(g_2),$$

and $b(g_1)c(g_2)$ is compact. It suffices to show the kernel of this map, $(GL_r)_0$, is connected.

Suppose index $a(g) = 0$. GL_r is open in $\underline{g\ell}_r$, and $GL(H_+)$ is dense in Fred (H_+). This implies we can move continuously in GL_r from g to an element $g_1 \in GL_r$ with $a(g_1)$ invertible. We then have

$$g_1 = \begin{pmatrix} 1 & \\ ca^{-1} & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d-ca^{-1}b \end{pmatrix} .$$

The subgroups $\{\begin{pmatrix} 1 & \\ z & 1 \end{pmatrix} \in GL_r\}$ and $\{\begin{pmatrix} a & b \\ o & d \end{pmatrix} \in GL_r\}$ are connected. This implies there is a continuous path in GL_r from g_1 to 1. This proves (v) for GL_r.

To conclude the proof, thanks to (iv), it suffices to verify that index $a(g \exp(\xi))$ = index $a(g)$, where $g \varepsilon U_r$ and $\xi \varepsilon \mathrm{sym}_r$. But $\exp(\xi) > 0$ implies $a(\exp(\xi)) > 0$. Hence index $a(g \exp(\xi))$ = index $a(g)$ + index $\exp(\xi)$ = index $a(g)$.//

The trace functional can be used to identify preduals for the linear spaces in (1.1). We next show these preduals have nice Lie properties.

(1.7) Definition.

(i) $(\underline{g\ell}_r)_* = \{\xi \varepsilon \underline{g\ell}_r\colon a(\xi)$ and $d(\xi)$ and trace class$\}$

(ii) $(\underline{u}_r)_* = \underline{u} \cap (\underline{g\ell}_r)_*$

(iii) $(GL_r)_* = \{1 + \xi \varepsilon GL\colon \xi \varepsilon (\underline{g\ell}_r)_*\}$

(iv) $(U_r)_* = \{1 + \xi \varepsilon U\colon \xi \varepsilon (\underline{g\ell}_r)_*\}$

(v) $(\mathrm{sym}_r)_* = \mathrm{sym} \cap (\underline{g\ell}_r)_*$

(1.8) Definition.

For $\xi + \lambda 1 \varepsilon (\underline{g\ell}_r)^{+\mathbb{C} 1}$, $\xi = \begin{pmatrix} \alpha & \beta \\ \alpha & \delta \end{pmatrix}$, $|\xi + \lambda 1|_* = \left|\begin{pmatrix} \alpha & \\ & \delta \end{pmatrix}\right|_1 + \left|\begin{pmatrix} & \beta \\ \alpha & \end{pmatrix}\right|_2 + |\lambda|$

(1.9) Proposition

(i) $(\underline{g\ell}_r)_*^{+\mathbb{C} 1}$ with the usual adjoint operation and $|\ |_*$ is a Banach *-algebra with identity.

(ii) $\mathbb{C}^* \chi (GL_r)_*$ is an open subset of $(\underline{g\ell}_r)_*^{+} \mathbb{C} 1$, and equipped with the induced complex analytic structure, $(GL_r)_*$ is a complex Banach Lie group with Banach Lie algebra $(\underline{g\ell}_r)_*$.

(iii) $(U_r)_*$ is a Banach Lie subgroup of $(GL_r)_*$ with Banach Lie subalgebra $(\underline{u}_r)_*$.

(iv) The map $(U_r)_* \times (Sym_r)_* \to (GL_r)_*$ given by

$$(g, \xi) \to g \exp(\xi)$$

is a diffeomorphism.

(v) $(GL_r)_*$ and $(U_r)_*$ have the same homotopy type as $GL(\infty, \mathbb{C})$; in particular, both are connected.

(vi) $(GL_r)_*$ and $(U_r)_*$ are normal subgroups of GL_r and U_r, respectively.

Proof. (i) Suppose $\xi_j \in (\underline{g\ell}_r)_*$ and $A_j = \begin{pmatrix} \alpha_j & \\ & \delta_j \end{pmatrix}$, $B_j = \begin{pmatrix} & \beta_j \\ \alpha_j & \end{pmatrix}$. Then

$$|\xi_1\xi_2|_* = |A_1A_2 + B_1B_2|_1 + |A_1B_2 + B_1A_2|_2$$

$$< |A_1|_1|A_2|_\infty + |B_1B_2|_1 + |A_1|_\infty|B_2|_2 + |B_2|_2|A_2|_\infty$$

$$< |A_1|_1|A_2|_1 + |B_1|_2|B_2|_2 + |A_1|_1|B_2|_2 + |B_2|_2|A_2|_1$$

$$= |\xi_1|_*|\xi_2|_*.$$

The nontrivial inequality $|B_1B_2|_1 < |B_1|_2|B_2|_2$ follows easily from a classical s-number inequality due to Weyl (see Corollary 4.1 on page 49 of [Gohberg, 1]). (i) is now clear.

(ii)-(iv) see (1.6).

(v) This follows from a theorem of Geba. For a statement of the theorem, see page 116 of [De la Harpe, 1].

It is very easy to prove $(GL_r)_*$ (and hence $(U_r)_*$) is connected. Suppose $g \in (GL_r)_*$. Since $a(g) = 1 +$ trace class, there exist arbitrarily small finite rank perturbations of $a(\alpha)$ which are invertible. This implies we can move continuously in $(GL_r)_*$ from g to a $g_1 \in (GL_r)_*$ with $a(g_1)$ invertible. The proof is now completed in the same manner as (v) of (1.6).

(vi) By (v) it suffices to check that $g \xi g^{-1}$ is in $(\underline{g\ell}_r)_*$ whenever $\xi \in (\underline{g\ell}_r)_*$ and $g \in GL_r$. This is obvious because $(\underline{g\ell}_r)_*$ is an ideal in $\underline{g\ell}_r$. //

If $g \in (GL_r)_o$, then $a(g)$ is an index zero Fredholm operator, implying $a(g)$ has a polar decomposition. This has important group theoretic consequences, which we now describe.

As a basic reference for notation and basic results concerning semi-direct products of groups, we will use [Varadarajan, 2, section 3.15]. Identify $GL(H_+) \times GL(H_-)$ with the subgroup $\{\begin{pmatrix} a & \\ & d \end{pmatrix} \in GL_r\}$. $GL(H_+) \times GL(H_-)$ acts by conjugation on $(GL_r)_*$. Denote this action by t, and form the semidirect products

$$(U_r)_* \times_t (U(H_+) \times U(H_-)) \quad \text{and} \quad (GL_r)_* \times_t (GL(H_+) \times GL(H_-)).$$

The multiplication is given by

$$(g_1,q_1)(g_2,q_2) = (g_1q_1g_2q_1^{-1},q_1q_2).$$

(1.10) Proposition. The sequences

$$1 \to U_1(H_+) \times U_1(H_-) \xrightarrow{i} (U_r)_* \times_t (U(H_+) \times U(H_-)) \xrightarrow{\pi} (U_r)_o \to 1$$

$$1 \to GL_1(H_+) \times GL_1(H_-) \xrightarrow{i} (GL_r)_* \times_t (GL(H_+) \times GL(H_-)) \xrightarrow{\pi} (GL_r)_o \to 1$$

are exact sequences of Banach Lie groups, where $i(K) = (K^{-1},K)$, and $\pi(g,K) = g \cdot K$.

Proof. Consider the first sequence. We first show π is surjective.

Suppose $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in (U_r)_o$. By (1.5) both a and d are index zero. This implies there are unitary operators q_1 and q_2 satisfying $a = q_1|a|$, $d = q_2|d|$, and

(1.11) $$g = \begin{pmatrix} q_1|a|q_1^{-1} & bq_2^{-1} \\ cq_1^{-1} & q_2|d|q_2^{-1} \end{pmatrix}\begin{pmatrix} q_1 & \\ & q_2 \end{pmatrix}.$$

By (1.1) $\begin{pmatrix} |a| & \\ & |d| \end{pmatrix} = \begin{pmatrix} 1-c^*c & \\ & 1-b^*b \end{pmatrix}^{1/2}$. This implies $|a|$ and $|d|$ are of the form 1 + trace class. If $|a|$ and $|d|$ are invertible, this is obvious: in fact (1.1) shows that

$$g \to \begin{pmatrix} |a| & \\ & |d| \end{pmatrix}$$

is an analytic map into $GL_1(H_+) \times GL_1(H_-)$, because the square root is an analytic map on positive operators in this group. In general $c^*c = \sum \lambda_j P_j$, where each P_j is a one dimensional projection, $0 < \lambda_j \underline{\leq 1}$, and $\sum \lambda_j < \infty$. Then $|a| = 1 - \underline{\Sigma}\,(1 - \sqrt{1 - \lambda_j})P_j$. Also $(1 - \lambda_j)^2 < 1 - \lambda_j$ implies $0 < 1 - \sqrt{1 - \lambda_j} < 1 - (1 - \lambda_j) = \lambda_j$. This shows $|a| = 1 +$ trace class. Similarly, $|d| = 1 +$ trace class. The surjectivity of π now follows from (1.11).

Near $1 \in (U_r)_o$ the polar decomposition of $\begin{pmatrix} a & \\ & d \end{pmatrix}$ uniquely determines the decomposition (1.12). This defines a local cross-section for π. Analyticity of the cross-section follows from (1.1), as we remarked above.//

For purposes of comparison with [Segal, 1], we need to introduce a normal subgroup between $(U_r)_*$ and $(U_r)_o$.

(1.12) Definition.

(i) $\underline{n}^{\mathbb{C}} = \{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \underline{g\ell}_r : \alpha$ is trace class$\}$

(ii) $\underline{n} = \underline{u} \cap \underline{n}^{\mathbb{C}}$

(iii) $N^{\mathbb{C}} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_r : a = 1 +$ trace class$\}$

(iv) $N = U \cap N^{\mathbb{C}}$

(1.13) Definition.

For $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \underline{n}^{\mathbb{C}}$, $\lambda \in \mathbb{C}$, $\left|\begin{pmatrix} \lambda 1 + \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right| = |\lambda| + |\alpha|_1 + |\delta|_\infty + \left|\begin{pmatrix} & \beta \\ \gamma & \end{pmatrix}\right|_2$.

(1.14) Proposition

(i) $\underline{n}^{\mathbb{C}} + \mathbb{C}\,1$ with the usual adjoint operation and $\|\ \|$ is a Banach *-algebra with identity.

(ii) $N^{\mathbb{C}}$ is a Banach Lie subgroup of the group of units of $\underline{n}^{\mathbb{C}} + \mathbb{C}\,1$ with Banach Lie algebra $\underline{n}^{\mathbb{C}}$.

(iii) N is a Banach Lie subgroup of $N^{\mathbb{C}}$ with Banach Lie subalgebra $\underline{n}$.

(iv) N and $N^{\mathbb{C}}$ are connected.

(v) N and $N^{\mathbb{C}}$ are normal subgroups of U_r and GL_r, respectively.

Proof. (i)-(iii) and (v) can be verified in the same manner as the corresponding claims of (1.9).

(iv) slight modification of (1.10) shows that the sequence

$$1 \to U_1(H_+) \xrightarrow{i} (U_r)_* \times_t U(H_-) \xrightarrow{\pi} N \to 1$$

is an exact sequence of Banach Lie groups, where $U(H_\pm)$ are identified with $\{\begin{pmatrix} a & \\ & 1\end{pmatrix} \in U_r\}$ and $\{\begin{pmatrix} 1 & \\ & d\end{pmatrix} \in U_r\}$, respectively. This shows N is connected. The same argument works for $N^{\mathbb{C}}$.//

(1.15) Proposition. The sequences

$$1 \to U_1(H_+) \to N \times_t U(H_+) \to (U_r)_0 \to 1$$

$$1 \to GL_1(H_+) \to N^{\mathbb{C}} \times_t GL(H_+) \to (GL_r)_o \to 1$$

are exact sequences of Banach Lie groups,
where $i(K) = \left(\begin{pmatrix} k^{-1} & \\ & 1 \end{pmatrix}, \begin{pmatrix} k & \\ & 1 \end{pmatrix}\right)$, and $\pi(g,K) = g \cdot \begin{pmatrix} K & \\ & 1 \end{pmatrix}$.

The proof of (1.15) is the same as that of (1.10).

CHAPTER 2

THE AFFINE EXTENSION OF U_r.

If we view the restricted unitary group, U_r, as a subgroup of the restricted orthogonal, O_r, of Shale (see section 5), then the spin representation defines a nontrivial extension of U_r. An abstract model for the corresponding extension of $(GL_r)_0$ is constructed in [Segal, 2, section 3]. In this section we describe the extension for U_r and GL_r, and we relate these extensions to (1.11).

We first discuss the extensions for the connected components, $(GL_r)_0$ and $(U_r)_0$, following [Segal, 2].

(2.1) Definition.

(i) $\widetilde{\widetilde{GL}}_r$ is the group of all pairs (g, q) in $(GL_r)_0 \times GL(H_+)$ satisfying $a(g)q^{-1} - 1 \in L_1$.

(ii) $\widetilde{U}_r$ is the group of all pairs (g,q) in $(U_r)_0 \times U(H_+)$ satisfying $a(g)q^{-1} - 1 \in L_1$.

Identify $U_1(H_+)$ and $GL_1(H_+)$ with $\{1\} \times U_1(H_+)$ and $\{1\} \times GL_1(H_+)$, respectively.

Let π: $\widetilde{\widetilde{GL}}_r \to GL_r$ denote projection onto the first factor. The sequences

$$1 \to U_1(H_+) \to \widetilde{U}_r \overset{\pi}{\to} (U_r)_0 \to 1$$

(2.2)

$$1 \to GL_1(H_+) \to \widetilde{GL}_r \overset{\pi}{\to} (GL_r)_0 \to 1$$

are exact.

(2.3) Definition.

(i) $\hat{U}_r = \tilde{U}_r \,/\, SU_1(H_+)$.

(ii) $\widehat{GL}_r = \widetilde{GL}_r \,/\, SL_1(H_+)$.

Denote the coset containing (g, q) by $[g, q]$.

$\hat{U}_r$ and $\widehat{GL}_r$ are central extensions of $(U_r)_0$ and $(GL_r)_0$ by $\Pi(\cong U_1(H_+) \,/\, SU_1(H_+))$ and $₵^*(\cong GL_1(H_+) \,/\, SL_1(H_+))$, respectively.

Let $\theta^₵ = \{g \in (GL_r)_0 : a(g) \text{ is invertible}\}$,

and $\theta = \theta^₵ \cap U_r$. Analytic cross-sections for the projections $\hat{U}_r \overset{\pi}{\to} U_r$ and $\widehat{GL}_r \overset{\pi}{\to} GL_r$ over θ and $\theta^₵$ are given by

(2.4) $\Gamma : \theta \to \hat{U}_r$, $\Gamma(g) = [g, q(g)]$,

and $\Gamma^₵ : \theta^₵ \to \widehat{GL}_r$, $\Gamma(g) = [g, a(g)]$,

respectively, where $q(g)|a(g)|$ is the unique polar decomposition of $a(g)$. In the coordinates

(2.5) $\pi^{-1}(\theta) \cong \theta \times \Pi$, $\pi^{-1}(\theta^₵) \cong \theta^₵ \times ₵^*$,

multiplication is given by

$$(g_1,\lambda_1) \cdot (g_2,\lambda_2) = (g_1,g_2,\lambda_1\lambda_2\ c(g_1,g_2))$$

and

$$(g_1,\lambda_1) \cdot (g_2,\lambda_2) = (g_1g_2,\lambda_1\lambda_2\ C(g_1,g_2)),$$

respectively, where the multipliers are given by

(2.6) $C(g_1,g_2) = \det a(g_1)a(g_2)a(g_1g_2)^{-1}$, $g_j \in (GL_r)_0$,

and $c(g_1,g_2) = \det q(g_1)q(g_2)q(g_1g_2)^{-1}$

$= \text{phase of } C(g_1,g_2)$, $g_j \in (U_r)_0$.

Now we consider the problem of how to define the extensions over U_r and GL_r. Fix an orthonormal basis $\{e_j : j \in \mathbb{Z}\}$ for H with $\{e_j : j > 0\} \subset H_+$. Define the shift operator S by $S(e_j) = e_{j+1}$. The operator S is in U_r, and index $a(S) = -1$.

(2.7) Lemma. Conjugation by S on $(U_r)_0$ (respectively, $(GL_r)_0$) lifts to an automorphism of $\hat{U}_r$ (respectively, $\hat{GL}_r$) which is the identity on $\Pi \cong U_1(H_+) / SU_1(H_+)$ (respectively, $\mathbb{C}^*$).

Proof. We use the notation in (2.1). For (g,q) in $\tilde{U}_r$, define

$$\sigma(g,q) = (SgS^{-1}, s(q)),$$

where $S(q) = a(S)q\, a(S)^* + e^*_0 \times e_0$. Because $a(S)^* a(S) = 1$, $S(q_1q_2) = S(q_1)S(q_2)$, implying σ is an automorphism of $\tilde{U}_r$. If $(1,q)$ is in $U_1(H_+)$, then $\sigma(1,q) = (1, S(q))$, and $\det S(q) = \det q$. This implies that σ induces an automorphism of $\hat{U}_r$, the quotient of $\tilde{U}_r$ by $SU_1(H)$, which is the identity on Π, the quotient of $U_1(H_+)$ by $SU_1(H_+)$.

This argument also works for GL_r.//

This lemma implies that there is a central extension of U_r

by Π whose identity component is $\hat{U}_r$. The cyclic group generated by S can be identified with $\mathbb{Z}$. If t denotes the action of conjugation by powers of S, then U_r can be identified with $(U_r)_0 \times_t \mathbb{Z}$. The central extension of U_r by Π can be identified with $\hat{U}_r \times_\sigma \mathbb{Z}$.

We now make the connection between the central extension $\hat{U}_r \to (U_r)_0$ and our work on semidirect products in section 1.

The central extension $\hat{U}_r \to (U_r)_0$ is not exact, i.e. $\hat{U}_r$ is not a semidirect product of Π and $(U_r)_0$. The utility of the extension $\check{U}_r \to (U_r)_0$ is that it is exact. In fact the extensions (2.2) are equivalent to the extensions in (1.15). The map inducing the equivalence is given by

(2.8)
$$N \times_t U(H_+) \to \check{U}_r$$

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} q & \\ & 1 \end{pmatrix}\right) \to \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} q & \\ & 1 \end{pmatrix}, q\right)$$

This map also induces an isomorphism

$$N^{\mathbb{C}} \times_t GL(H_+) \to \widetilde{GL}_r \ .$$

Combining this with an analyticity argument as in (1.10) we have

(2.9) Proposition. The sequences

$$1 \to SU_1(H_+) \xrightarrow{i} N \times_t U(H_+) \xrightarrow{\pi} \hat{U}_r \to 1$$

$$1 \to SL_1(H_+) \to N^{\mathbb{C}} \times_t GL(H_+) \to \hat{GL}_r \to 1$$

are exact sequences of Banach Lie groups,
where $i(K) = \left(\begin{pmatrix} K^{-1} & \\ & 1 \end{pmatrix}, \begin{pmatrix} K & \\ & 1 \end{pmatrix}\right)$, and π is given by (2.8).

Finally, we note that the automorphism σ of $\hat{U}_r$ in (2.7) lifts to an automorphism of $N \times_t U(H_+)$. The lift, also denoted by σ, is given by

$$(2.10) \qquad \sigma\left(g, \begin{pmatrix} q & \\ & 1 \end{pmatrix}\right) = \left(SgS^{-1},\ S\begin{pmatrix} q & \\ & 1 \end{pmatrix}S^{-1}\right).$$

It's interesting to note that σ does not lift to $(U_r)_* \times_t (U(H_+) \times U(H_-))$.

CHAPTER 3

THE ADJOINT ACTION OF $\hat{U}_r$.

Let $\hat{\underline{u}}_r$ denote the Lie algebra of $\hat{U}_r$. Because $\hat{U}_r$ is a central extension of U_r by Π, $\hat{\underline{u}}_r$ is a central extension of $\underline{u}_r$ by $i\mathbb{R}$.

The local cross-sections (2.4) of the bundles $\hat{U}_r \to (U_r)_0$ and $\hat{GL}_r \to (GL_r)_0$ determine vector space splittings of $\hat{\underline{u}}_r$, viewed as the tangent space at 1 of $\hat{U}_r \to \hat{GL}_r$. These splittings are identical in the sense that the diagram

$$\begin{tikzcd} \hat{\underline{u}}_r \arrow[r] \arrow[d, "d\Gamma_1"'] & \hat{\underline{g\ell}}_r \arrow[d, "d\Gamma_1^{\mathbb{C}}"] \\ \underline{u}_r \arrow[r] & \underline{g\ell}_r \end{tikzcd}$$

commutes. This is because for $\xi \in \underline{u}_r$, $a(e^{t\xi})$ and $q(e^{t\xi})$ agree up to first order. In the coordinates (2.5) we thus have

$$\hat{\underline{u}}_r \cong \underline{u}_r \times i\,\mathbb{R}\ ,\ \hat{\underline{g\ell}}_r \cong \underline{g\ell}_r \times \mathbb{C}.$$

We claim that in these coordinates the bracket is given by

$$(3.1)\qquad [(\xi_1,is_1),\ (\xi_2,is_2)] = ([\xi_1,\xi_2],\ i\{s_1 + s_2 + w(\xi_1,\xi_2)\}),$$

where the infinitesimal cocycle is given by

$$(3.2)\qquad iw(\xi_1,\xi_2) = \text{trace}\ \{[\alpha(\xi_1),\alpha(\xi_2)] - \alpha([\xi_1,\xi_2])\}.$$

If $\xi_j \in \underline{u}_r$, then

$$iw(\xi_1,\xi_2) = -2i\mathrm{Im}\ \mathrm{trace}\ \beta(\xi_1)\ \gamma\ (\xi_2).$$

To verify (3.2) we must first determine, at least up to second order, the exponential map for the group $\hat{GL}_r$ in the coordinates (2.5).

Suppose $\xi \in \underline{g\ell}_r$. In the coordinates (2.5), $\exp\ (t\xi,\ 0) = (e^{t\xi},\ \phi(t))$, where $\phi(0) = 1$, $\phi'(0) = 0$. The multiplication (2.6) implies

$$\phi(t+s) = \phi(t)\phi(s)\det a(e^{t\xi})a(e^{s\xi})a(e^{t\xi}e^{s\xi})^{-1},$$

$$\phi'(t) = \phi(t)f(t),$$

where $f(t) = \mathrm{trace}\ \{\alpha(\xi) - a(e^{t\xi})^{-1} \frac{d}{dt} a(e^{t\xi})\}$,

$$\phi''(0) = f'(0) = \mathrm{trace}\ \{\alpha(\xi)^2 - \alpha(\xi^2)\}.$$

We thus have

$$\exp(t\xi,0) = (e^{t\xi},\ 1 + \frac{t^2}{2}\ \mathrm{tr}\{\alpha(\xi)^2 - \alpha(\xi^2)\} + 0(t^3))$$

Let $\exp(t\xi_j,\ 0) = (e^{t\xi j},\ \phi_j(t))$. By (2.6)

$$\exp(t\xi_1,\ 0)\ \exp(t\xi_2,\ 0)\ \exp(-t\xi_1,\ 0)\ \exp(-t\xi_2,\ 0)$$

$$= (e^{t\xi_1}e^{t\xi_2}e^{-t\xi_1}e^{-t\xi_2}, \det \{a(e^{t\xi_1}e^{t\xi_2}e^{-t\xi_1}e^{-t\xi_2})^{-1}a(e^{t\xi_1})a(e^{t\xi_2})$$

$$\cdot\, a(e^{-t\xi_1})a(e^{-t\xi_2})\}\ \phi_1(t)\phi_2(t)\phi_1(-t)\phi_2(-t)).$$

We can now simply compute

$$\frac{1}{2}\left(\frac{d}{dt}\right)^2\Big|_{t=0} a(e^{t\xi_1}e^{t\xi_2}e^{-t\xi_1}e^{-t\xi_2})^{-1} = -\alpha([\xi_1,\xi_2]),$$

$$\frac{1}{2}\left(\frac{d}{dt}\right)^2\Big|_{t=0} a(e^{t\xi_1})a(e^{t\xi_2})a(e^{-t\xi_1})a(e^{-t\xi_2})$$

$$= [\alpha(\xi_1),\alpha(\xi_2)] = \alpha(\xi_1^2) - \alpha(\xi_1)^2 + \alpha(\xi_2^2) - \alpha(\xi_2)^2,$$

$$\frac{1}{2}\left(\frac{d}{dt}\right)^2\Big|_{t=0} \phi_1(t)\phi_2(t)\phi_1(-t)\phi_2(-t)$$

$$= \operatorname{trace}\{\alpha(\xi_1)^2 - \alpha(\xi_1^2) + \alpha(\xi_2)^2 - \alpha(\xi_2^2)\},$$

This verifies (3.2).

Let $\Lambda = \begin{pmatrix} i & \\ & -i \end{pmatrix}$. A direct computation and (1.2) shows that

(3.3) $$g^{-1}[\Lambda,g] = 2i\begin{pmatrix} -c^*c & a^*b \\ -d^*c & b^*b \end{pmatrix} = 2i\begin{pmatrix} -c^*c & a^*b \\ b^*a & b^*b \end{pmatrix},$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is unitary.

(3.4) Proposition. The adjoint action of $\hat{U}_r$ on $\underline{u}_r + i\mathbb{R}$ factors through $(U_r)_0$ and is given by

$$g^*(\xi, is) = (g\xi g^{-1}, a(g,\xi) + is),$$

where $a(g,\xi)$ is given by

$$a(g,\xi) = -\text{ trace }\left\{\begin{pmatrix} -c^*c & a^*b \\ b^*a & b^*b \end{pmatrix} \xi \right\}.$$

This formula is also valid for S and σ as in (2.7) (and their powers).

We will give two proofs. In the first we simply integrate the infinitesimal action. In the second we use the semidirect product approach.

Proof 1. Since our formula for the adjoint action must project to the usual adjoint action for $(U_r)_0$, we have

$$g^*(\xi, is) = (g\xi g^{-1}, a(g,\xi) + is),$$

where $a(g,\xi)$ is linear in ξ,

(3.5) $\quad a(g_1g_2;\xi) = a(g_1,g_2\xi\, g_2^{-1}) + a(g_2,\xi),$

and

(3.6) $\quad \frac{d}{ds}\Big|_{s=0} a(e^{s\eta},\xi) = iw(\eta,\xi).$

Substitute $g_1 = e^{s\eta}$ and $g_2 = e^{t\eta}$ into (3.5). By (3.6) we have

$$\frac{d}{ds}\Big|_{s=0} a(e^{(s+t)\eta},\xi) = iw(\eta,e^{t\eta}\xi\, e^{-t\eta}),$$

and

$$(3.7)\qquad a(e^{t\eta},\xi) = i\int_0^t w(\eta, e^{s\eta}\xi\, e^{-s\eta})ds.$$

In particular, if $g = e^{t\eta}$ is in $U(H_+) \times U(H_-)$,

$$(3.8)\qquad a(g,\cdot) = 0.$$

Now assume $\eta = \begin{pmatrix} & \beta \\ -\beta^* & \end{pmatrix}$. Let $e^{s\eta} = \begin{pmatrix} a(s) & b(s) \\ c(s) & d(s)\end{pmatrix}$.

We have

$$(3.9)\qquad \frac{d}{ds}\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} & \beta \\ -\beta^* & \end{pmatrix} = \begin{pmatrix} & \beta \\ -\beta^* & \end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}.$$

Suppose first that $\xi = \begin{pmatrix} \alpha & \\ & \delta\end{pmatrix}$. By (3.2)

$$w(\eta, e^{s\eta}\xi\, e^{-s\eta}) = -2 \operatorname{Im} \operatorname{trace} \beta\, \gamma(e^{s\eta}\xi\, e^{-s\eta})$$

$$(3.10)\qquad = -2 \operatorname{Im} \operatorname{trace} \beta(c\alpha a^* + d\delta b^*)$$

$$= -2 \operatorname{Im} \operatorname{trace} \left\{\begin{pmatrix} a^*\beta c & \\ & b^*\beta d\end{pmatrix}\begin{pmatrix} \alpha & \\ & \delta\end{pmatrix}\right\}.$$

By (3.9)

$$\begin{pmatrix} a^*\beta c & \\ & b^*\beta d\end{pmatrix} = \begin{pmatrix} -\frac{d}{ds}(c^*)c & \\ & b^*\frac{d}{ds}(b)\end{pmatrix}.$$

Moreover $\begin{pmatrix} a^*\beta c & \\ & b^*\beta d\end{pmatrix}$ is self-adjoint: by (1.2) and (3.9)

$$\begin{pmatrix} a^*\beta c & \\ & b^*\beta d\end{pmatrix}^* = \begin{pmatrix} c^*\beta a & \\ & d^*\beta^*b\end{pmatrix} = \begin{pmatrix} c^*d\beta^* & \\ & \beta^*a^*b\end{pmatrix}$$

$$= \begin{pmatrix} -a^*b\beta & \\ & -\beta^*c^*d\end{pmatrix} = \begin{pmatrix} a^*\beta c & \\ & b^*\beta d\end{pmatrix}.$$

This implies

$$\begin{pmatrix} a^*\beta c & \\ & b^*\beta d \end{pmatrix} = \frac{1}{2}\frac{d}{ds}\begin{pmatrix} -c^*c & \\ & b^*b \end{pmatrix}$$

It follows from this, (3.10) and (3.7)

that $a(e^{t\eta},\xi) = -i$ Im trace $\left\{\begin{pmatrix} -c^*c & \\ & b^*b \end{pmatrix}\begin{pmatrix} \alpha & \\ & \delta \end{pmatrix}\right\}$ This agrees with Proposition (3.4) because the trace is pure imaginary.

Now suppose $\xi = \begin{pmatrix} & \beta_1 \\ -\beta_1^* & \end{pmatrix}$. By (3.2)

$$w(\eta, e^{s\eta}\xi\, e^{-s\eta}) = -2 \text{ Im trace } \beta\, \gamma(e^{s\eta}\xi\, e^{-s\eta})$$

$$(3.11) \qquad = -2 \text{ Im trace } \beta(-d^*\beta^*a^* + c\beta b^*)$$

$$= -2 \text{ Im trace } \left\{\begin{pmatrix} & a^*\beta d \\ b^*\beta c & \end{pmatrix}\begin{pmatrix} & \beta_1 \\ -\beta_1^* & \end{pmatrix}\right\} .$$

By (3.9)

$$\begin{pmatrix} & a^*\beta d \\ b^*\beta c & \end{pmatrix} = \begin{pmatrix} & a^*\dfrac{d(b)}{ds} \\ b^*\dfrac{d(a)}{ds} & \end{pmatrix} = \begin{pmatrix} & \beta + \dfrac{d(a^*)}{ds}\, b \\ \dfrac{d(b^*)}{ds}\, a - \beta^* & \end{pmatrix}$$

The trace of $\begin{pmatrix} & \beta \\ -\beta^* & \end{pmatrix}\begin{pmatrix} & \beta_1 \\ -\beta_1^* & \end{pmatrix}$ is real. So (3.11) now shows

$$w(\eta, e^{s\eta}\xi\, e^{-s\eta}) = -\frac{d}{ds} \text{ Im trace } \begin{pmatrix} & a^*b \\ b^*a & \end{pmatrix}\begin{pmatrix} & \beta_1 \\ -\beta_1^* & \end{pmatrix}$$

But this trace is pure imaginary, so by (3.7) we again obtain (3.4).

We have now shown that (3.4) is valid for $g \in (U_r)_0$ of the form $f = \exp\begin{pmatrix} \alpha & \\ & \delta \end{pmatrix}$ or $g = \exp\begin{pmatrix} & \beta \\ -\beta^* & \end{pmatrix}$. By using (3.3) it's easily checked that (3.4) satisfies the equation (3.5). This implies (3.4) is valid for products of group elements of the form $g = \exp\begin{pmatrix}\alpha & \\ & \delta\end{pmatrix}$ or $g = \exp\begin{pmatrix} & \beta \\ -\beta^* & \end{pmatrix}$. This proves (3.4) for $(U_r)_0$, since $(U_r)_0$ is generated by products of this form.

We now consider the shift operator, S, and the automorphism, σ, of the central extension $\hat{U}_r \to (U_r)_0$. Let $S = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$. The operator $\begin{pmatrix}-c^*c & a^*b \\ b^*a & b^*b\end{pmatrix}$ is the self-adjoint projection for $\mathbb{C}e_{-1}$. We must show

$$(3.12) \qquad Ad_\sigma\big((\xi,0)\big) = \big(S\xi S^{-1}, -\langle\xi\, e_{-1}, e_{-1}\rangle\big) .$$

By definition of our identification $\hat{u}_r \cong u_r + i\mathbb{R}$,

$$(\xi,0) \cong \frac{d}{dt}\Big|_{t=0} [e^{t\xi}, q(e^{t\xi})],$$

where $a(e^{t\xi}) = q(e^{t\xi})|a(e^{t\xi})|$ is the unique polar decomposition (t small). Let $q = q(e^{t\xi})$. This implies

$$Ad_\sigma(\xi,0) = \frac{d}{dt}\Big|_{t=0} \sigma\big([e^{t\xi},q]\big)$$

$$= \frac{d}{dt}\Big|_{t=0}[S\ e^{t\xi}S^{-1},\ S(q)]$$

$$= \frac{d}{dt}\Big|_{t=0}[Se^{t\xi}S^{-1}, q(Se^{t\xi}S^{-1})][1,\ q(Se^{t\xi}S^{-1})^{-1}\ S(q)].$$

$$= (S\xi S^{-1},0) + \big(0, \frac{d}{dt}\Big|_{t=0} \det q(Se^{t\xi}S^{-1})^{-1}S(q)\big).$$

The inverse of this determinant is the phase of

$$\lambda(t) = \det a(Se^{t\xi}S^{-1})\ S(a(e^{t\xi}))^{-1}.$$

We will show $\frac{d}{dt}\big|_{t=0} \lambda(t) = \langle \xi\, e_{-1}, e_{-1}\rangle$. Since $\langle \xi\, e_{-1}, e_{-1}\rangle$ is pure imaginary and $\lambda(0) = 1$, this will imply (3.11).

Direct calculation shows

$$
\begin{aligned}
&a(Se^{t\xi}S^{-1})\; S\; (a(e^{t\xi}))^{-1} \\
&= a(S)a(S)^* + b(S)c(e^{t\xi})a(e^{t\xi})^{-1}a(S)^* \\
&+ a(S)b(e^{t\xi})c(S^{-1}) + b(S)d(e^{t\xi})c(S^{-1}).
\end{aligned}
$$

With respect to the ordered basis $\{---, e_n, --, e_1, e_0\}$ for H_+, the matrix of this operator is

$$
\left(\begin{array}{c|c}
1 & Y \\
\hline
X & \langle e^{t\xi}\, e_{-1}, e_{-1}\rangle
\end{array}\right),
$$

where X is a row vector representing $a(S)b(e^{t\xi})c(S^{-1})$ and Y is a column vector representing $b(S)c(e^{t\xi})a(e^{t\xi})^{-1}a(S)^*$.

The determinant of this matrix is

$$
\begin{aligned}
&\langle e^{t\xi}e_{-1}, e_{-1}\rangle - X^t \cdot Y \\
&= \langle e^{t\xi}e_{-1}, e_{-1}\rangle - \text{trace}\,\{a(S)b(e^{t\xi})c(S^{-1})b(S)c(e^{t\xi})a(e^{t\xi})^{-1}a(S)^*\}
\end{aligned}
$$

The trace term is $O(t^2)$ as $t \to 0$ because

$b(e^{t\xi}) = t\,\beta(\xi) + O(t^2)$, $c(e^{t\xi}) = t\gamma(\xi) + O(t^2)$ and

$a(e^{t\xi})^{-1} = 1 + O(t)$ as $t \to 0$. This shows

$$\frac{d}{dt}\Big|_{t=0} \lambda(t) = \langle \xi\, e_{-1},\, e_{-1}\rangle.$$

Because (3.4) satisfies (3.5), the validity of (3.4) for S and σ implies the validity of (3.4) for all powers of S and σ. //

Proof 2. By (2.9) we have exact sequences

$$1 \to SU_1(H_+) \to N \times_t U(H_+) \xrightarrow{\pi} \hat{U}_r \to 1$$

$$0 \to \underline{su}_1(H_+) \to \underline{n} \times_t \underline{u}(H_+) \xrightarrow{d\pi} \underline{u}_r + i\mathbb{R} \to 0.$$

If $\xi \in \underline{n}$, then

$$\pi\,(e^{t\xi}, 1) = [e^{t\xi}, q(e^{t\xi})][1, q(e^{t\xi})^{-1}],$$

where $a(e^{t\xi}) = q(e^{t\xi})\left|a(e^{t\xi})\right|$. This implies

$$\begin{aligned} d\pi(\xi, 0) &= \left(\xi, -\frac{d}{dt}\Big|_{t=0}\det q(e^{t\xi})\right) \\ &= \left(\xi, -\operatorname{trace} \alpha(\xi)\right) \end{aligned}$$

If $\begin{pmatrix}\eta & \\ & 0\end{pmatrix} \in \underline{u}\,(H_+)$, then $\pi\left(\left(1, \begin{pmatrix}e^{t\eta} & \\ & 1\end{pmatrix}\right)\right) = \left[\begin{pmatrix}e^{t\eta} & \\ & 1\end{pmatrix}, e^{t\eta}\right]$ implying

$$d\pi\left(0, \begin{pmatrix}\eta & \\ & 0\end{pmatrix}\right) = \left(\begin{pmatrix}\eta & \\ & 0\end{pmatrix}, 0\right).$$

So we have

$$(3.13)\quad d\pi\left(\xi, \begin{pmatrix}\eta & \\ & 0\end{pmatrix}\right) = \left(\xi + \begin{pmatrix}\eta & \\ & 0\end{pmatrix}, -\text{ trace } \alpha(\xi)\right)$$

The adjoint action for $N \times_t U(H_+)$ is given by

$$(3.14)\quad \begin{aligned} &\mathrm{Ad}_{\left(g_1, \begin{pmatrix}q & \\ & 1\end{pmatrix}\right)}\left(\xi, \begin{pmatrix}\eta & \\ & 0\end{pmatrix}\right) \\ &= \left(g_1\begin{pmatrix}q & \\ & 1\end{pmatrix}\left(\xi+\begin{pmatrix}\eta & \\ & 0\end{pmatrix}\right)\begin{pmatrix}q & \\ & 1\end{pmatrix}^{-1} g_1^{-1} - \begin{pmatrix}q\eta q^{-1} & \\ & 0\end{pmatrix}, \begin{pmatrix}q\eta q^{-1} & \\ & 0\end{pmatrix}\right)\end{aligned}$$

Now suppose $g \in (U_r)_0$ and $\xi = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in \underline{U}_r$. Write $g = g_1 \begin{pmatrix}g & \\ & 1\end{pmatrix}$, where $g_1 \in N$ and $\begin{pmatrix}g & \\ & 1\end{pmatrix} \in U(H_+)$. By (3.13) $\left(\begin{pmatrix}0 & \beta \\ \gamma & \delta\end{pmatrix}, \begin{pmatrix}\alpha & \\ & 0\end{pmatrix}\right)$ is an element in $\underline{n} \times_t \underline{u}(H_+)$ which is mapped to ξ by $d\pi$. By (3.14)

$$\begin{aligned} g^*(\xi,0) &= d\pi\left\{\mathrm{Ad}_{(g_1,(g_1)}, \left(\begin{pmatrix}0 & \beta \\ \gamma & \delta\end{pmatrix}, \begin{pmatrix}\alpha & \\ & 0\end{pmatrix}\right)\right\} \\ &= d\pi\left\{\left(g\ \xi\ g^{-1} - \begin{pmatrix}q\alpha q^{-1} & \\ & 0\end{pmatrix}, \begin{pmatrix}q\alpha q^{-1} & \\ & 0\end{pmatrix}\right)\right\} \\ &= \left(g\ \xi\ g^{-1}, -\text{ trace}\left\{\alpha(g\ \xi\ g^{-1}) - q\alpha q^{-1}\right\}\right) \\ &= \left(g\ \xi\ g^{-1}, -\text{ trace}\left\{a\alpha a^* + b\gamma a^* + a\beta b^* + b\delta b^* - q\alpha q^{-1}\right\}\right). \end{aligned}$$

To see that this agrees with (3.4), we must verify that

$$(3.15)\quad \text{trace } (-c^*c\alpha) = \text{trace } (a\alpha a^* - q\alpha q^{-1})$$

Because $a(g_1) = 1 + \text{trace class}$, we can write $a = q(1 + T)$, where T is trace class. By (1.1), $a^*a = 1 - c^*c = (1 + T)^*(1 + T)$, implying

$$-c^*c = T + T^* + T^*T.$$

We now have

$$\begin{aligned} &\operatorname{trace}\,(a\alpha a^* - q\alpha q^{-1}) \\ &= \operatorname{trace}\,(q(1 + T)\alpha(1 + T)^*q^{-1} - q\alpha q^{-1}) \\ &= \operatorname{trace}\,(T\alpha + \alpha T^* + T\alpha T^*) \\ &= \operatorname{trace}\,(-c^*c\alpha). \end{aligned}$$

This verifies (3.15).

Now consider S and σ. By (2.10) σ lifts to the automorphism of $N \times_t U(H_+)$ given by

$$\sigma\left(g_1, \begin{pmatrix} q & \\ & 1 \end{pmatrix}\right) = \left(Sg_1S^{-1}, S\begin{pmatrix} q & \\ & 1 \end{pmatrix}S^{-1}\right).$$

This implies

$$d\sigma\left(\xi, \begin{pmatrix} \eta & \\ & 0 \end{pmatrix}\right) = \left(S\,\xi\,S^{-1}, S\begin{pmatrix} \eta & \\ & 0 \end{pmatrix}S^{-1}\right),$$

and

$$S^*(\xi, 0) = d\pi\left\{d\sigma\left(\begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} \alpha & \\ & 0 \end{pmatrix}\right)\right\}$$

$$= d\pi\left\{\left(S\begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}S^{-1}, S\begin{pmatrix} \alpha & \\ & 0 \end{pmatrix}S^{-1}\right)\right\}$$

$$= \left(S\,\xi\,S^{-1}, -\operatorname{trace}\,\alpha\left(S\begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}S^{-1}\right)\right)$$

$$= \left(S\ \xi\ S^{-1},\ -\ \mathrm{trace} < \xi\ e_{-1},\ e_{-1} > \right)$$

This agrees with (3.4) by (3.11).//

CHAPTER 4

THE PREDUAL COADJOINT ACTION FOR $\hat{U}_r$.

Let G be a connected Lie group with Lie algebra $\underline{g}$. We will say that a functional $\phi \in \underline{g}^*$ is admissible for G if there is a character X of G_ϕ, the Stabilizer of ϕ in G for the coadjoint representation, having differential $i\phi|_{\underline{g}_\phi}$ (for the groups we shall consider, this definition agrees with the definition of admissible in [Pukanszky, 1, Section 2.1]). If G is simply connected, Kostant discovered an attractive geometric criterion: ϕ is admissible if and only if w_ϕ, the Kirillov symplectic 2-form on $Ad_G^*(\phi)$, defines an integral cohomology class in $H^2(Ad_G^*(\phi), \mathbb{R})$.

According to the orbit method, the irreducible representations of G should correspond to pairs $(Ad_G^*(\phi), X)$, where ϕ and X are as above. The method is a complete success for compact and type I solvable Lie groups and it can be modified to yield a good description of the basic representations for many other classes of finite dimensional groups (see [Pukanszky, 1]). It also seems useful for the classical Banach Lie groups as well (see [Boyer, 1]).

In this section we consider an invariant subspace for the coadjoint action of $\hat{U}_r$, the predual. This approach and the results are similar to those in [Segal, 1, Section 8]. This is hardly surprising, since the extension $\hat{U}_r \to (U_r)_0$ can be used to realize the affine extensions for loop groups (see [Segal, 1, Section 7]). The results in [Boyer, 1] are also relevant.

In this section $(\underline{u}_r)_*$ will simply denote the Banach space - we will not use the Lie structure. Some of what follows can be interpreted in terms of Mackey's normal subgroup analysis, but we will not make this explicit here.

Identify $\hat{\underline{u}}_r$ with $\underline{u}_r^* + i\mathbb{R}$ by using the vector space splitting $\underline{u}_r + i\mathbb{R}$ of $\hat{\underline{u}}_r$ in Section 3. By (3.4) the coadjoint action of $\hat{U}_r$ factors through $(U_r)_0$ and is given by

$$g^*(\phi,\lambda^*) = (\phi^g + \lambda^* \circ a(g^{-1}, \cdot), \lambda^*),$$

where ϕ^g denotes the coadjoint action of U_r. This is also valid for (all powers of) S and σ as in (2.7).

Each λ^* determines an invariant hyperplane.

First consider the hyperplane determined by $\lambda^* = 0$. The action then reduces to the usual coadjoint action for U_r. The predual $(\underline{u}_r)_*$ can be viewed as a subspace of $\underline{u}_r^*$ via trace, and it is invariant under the action.

Fix an orthonormal basis $\{e_j : j \in \mathbb{Z}\}$ for H with $\{e_j : j > 0\} \subset H_+$. We first note that it is not the case that each $(U_r)_0$ orbit in $(\underline{u}_r)_*$ contains a diagonal operator (e.g. the orbit containing $\begin{pmatrix} & \beta \\ -\beta^* & \end{pmatrix}$, where β is not trace class). This precludes a "nice" parameterization of the coadjoint orbits in $(\underline{u}_r)_*$. However, it's easy to see that the admissible functionals in the predual are finite rank and have spectrum contained in $i\mathbb{Z}$. From this it follows that each admissible orbit for $(U_r)_0$ in the predual contains a unique diagonal operator ξ of the form $\xi\, e_j = i\, \mu_j e_j$, where $\mu_j \in \mathbb{Z}$, $0 < \mu_n < \dots < \mu_0$, and $\mu_{-1} < \dots < \mu_{-m} < 0$. In matrix form

$$\xi = \left(\begin{array}{ccc|ccc} i\mu_n & & & & & \\ & \ddots & & & & \\ & & i\mu_0 & & & \\ \hline & & & i\mu_{-1} & & \\ & & & & \ddots & \\ & & & & & i\mu_{-m} \end{array}\right),$$

with respect to the ordered basis $\{\dots,e_n,\dots,e_0,e_{-1},\dots\}$.

Let $n > 0$. Let w_n denote the functional corresponding to

$$\left(\begin{array}{c|c|c} 0 & & \\ \hline & i1_{n\times n} & \\ \hline & & 0 \end{array}\right),$$

w_{-n} the functional corresponding to

$$\left(\begin{array}{c|c|c} 0 & & \\ \hline & -i1_{n\times n} & \\ \hline & & 0 \end{array}\right).$$

Then ξ above can be written uniquely as

(4.0) $\quad \xi = \sum \lambda_j w_j,$

where the λ_j are nonnegative integers.

Now fix $\lambda^*(i\tau) = \tau$. If we project from the hyperplane determined by λ^* to $\underline{u}_r$, then the coadjoint action is carried into the affine action given by

$$(4.1)\qquad g*\phi = \phi^g + a_g,$$

where a_g is the functional

$$(4.2)\qquad a_g(\xi) = (-i)a(g^{-1},\xi) = \text{trace}\left(i\begin{pmatrix}-bb^* & ac^*\\ ca^* & cc^*\end{pmatrix}\xi\right).$$

By (3.3)

$$i\begin{pmatrix}-bb^* & ac^*\\ ca^* & cc^*\end{pmatrix} = \frac{1}{2}(g\,\Lambda\,g^{-1} - \Lambda),$$

and this element is in the predual $(\underline{u}_r)_*$ of $\underline{u}_r$. We will view $(\underline{u}_r)_*$ as a subspace of $\underline{u}_r^*$ via trace.

Fix an orthonormal basis $\{e_j : j \in \mathbb{Z}\}$ for H with $\{e_j : j > 0\} \subset H_+$. This induces a unitary representation of Perm $(\mathbb{Z})$, the permutations of $\mathbb{Z}$.

(4.3) Proposition. (i) The predual $(\underline{u}_r)_*$ is invariant under the affine action (4.1). In the predual the action is given by

$$\begin{aligned} g*\xi &= g\,\xi\,g^{-1} + \frac{1}{2}\left(g\,\Lambda\,g^{-1} - \Lambda\right)\\ &= g\left(\xi + \frac{1}{2}\Lambda\right)g^{-1} - \frac{1}{2}\Lambda \end{aligned}$$

(ii) Each orbit for $(U_r)_0$ in the predual contains a diagonal operator (with respect to $\{e_j\}$).

(iii) Two diagonal operators ξ_1 and ξ_2 are in the same $(U_r)_0$ orbit if and only if there is $g \in \text{Perm}\,(\mathbb{Z}) \cap (U_r)_0$ such that

$$g(\xi_1 + \frac{1}{2}\Lambda)g^{-1} = \xi_2 + \frac{1}{2}\Lambda .$$

(iv) A diagonal operator ξ defines an admissible functional for $\hat{U}_r$ if and only if $(-i)\xi$ is integral.

(v) Each admissible orbit for $\hat{U}_r$ contains a unique ξ of the form

$$\xi = S^k*(\Sigma\, \lambda_j w_j),$$

where the λ_j are nonnegative integers, and the w_j are as in (4.0).

Proof. (i) follows from (4.1) and (4.2).

(ii) Suppose $\xi \in (\underline{u}_r)_*$. The essential spectrum of $\xi + \frac{1}{2}\Lambda$ is $\{\pm \frac{i}{2}\}$ (compact perturbation does not alter essential spectrum). We can list the eigenvalues, repeated according to multiplicity, $i\lambda_j$, $j \in \mathbb{Z}$, with $\lambda_j > 0$ for $j > 0$ and $\lambda_j < 0$ for $j < 0$. Then $\lambda_j \to \pm \frac{1}{2}$ as $j \to \pm \infty$.

The eigenvalues of $(\xi + \frac{1}{2}\Lambda)^2$, repeated according to multiplicity, are $-\lambda^2_j$, $j \in \mathbb{Z}$. We now have

$$(\xi + \frac{1}{2}\Lambda)^2 = -\frac{1}{4} + \text{Hilbert Schmidt},$$

implying

$$\sum_j \left|\frac{1}{4} - \lambda^2{}_j\right|^2 = \sum_j \left|\frac{1}{2} - \lambda_j\right|^2\left|\frac{1}{2} + \lambda_j\right|^2 < \infty .$$

This implies

$$(4.4) \qquad \sum_{j<0}\left|\frac{1}{2} + \lambda_j\right|^2 + \sum_{j>0}\left|\frac{1}{2} - \lambda_j\right|^2 < \infty,$$

because for $j < 0, \left|\frac{1}{2} - \lambda_j\right|^2$ is bounded and always larger than $\frac{1}{4}$, and for $j > 0$, $\left|\frac{1}{2} + \lambda_j\right|^2$ is bounded and always larger than $\frac{1}{4}$.

If we choose a unitary operator g so that

$$g\left(\xi - \frac{1}{2}\Lambda\right)g^{-1}e_j = \lambda_j e_j,\ j \in \mathbb{Z},$$

it follows from (4.4) that

$$g\left(\xi + \frac{1}{2}\Lambda\right)g^{-1} = \frac{1}{2}\Lambda + \text{Hilbert-Schmidt}$$

Multiply this equation on the right by g. It then follows that

$$\frac{1}{2}[g,\Lambda] = (-i)\begin{pmatrix} & b(g) \\ c(g) & \end{pmatrix} = \text{Hilbert-Schmidt},$$

and $g \in U_r$.

If the index of $a(g)$ is n, replace g by $S^n g$ (S is the shift operator of (2.7)). We now have $g \in (U_r)_0$, and $g * \xi$ is still diagonal. This proves (ii).

(iii) ($\Leftarrow$) is clear.

($\Rightarrow$) Suppose $g \in (U_r)_0$ and $g(\xi_1 + \frac{1}{2}\Lambda)g^{-1}$ equals $\xi_2 + \frac{1}{2}\Lambda$. Since $\xi_1 + \frac{1}{2}\Lambda$ and $\xi_2 + \frac{1}{2}\Lambda$ have the same (discrete) spectra and multiplicities, there is a unitary operator g_1 which commutes with $\xi_2 + \frac{1}{2}\Lambda$ and satisfies $g_1 g \in \text{Perm}\,(\mathcal{Z})$. We will show that g_1 is in $(U_r)_0$. This will prove (iii).

The equation $[g_1, \xi_2 + \frac{1}{2}\Lambda] = 0$ implies that g_1 is in U_r.

Since $\beta(\xi_2 + \frac{1}{2}\Lambda) = \gamma(\xi_2 + \frac{1}{2}\Lambda) = 0$, the equation $[g_1, \xi_2 + \frac{1}{2}\Lambda]$ implies $[a(g_1), \xi_2 + \frac{1}{2}\Lambda] = 0$, i.e. the eigenspaces of $\xi_2 + \frac{1}{2}\Lambda$ are invariant under $a(g_1)$.

Now consider the special case when $i/2$ is not an eigenvalue for $\delta(\xi_2 + \frac{1}{2}\Lambda)$. In this case the restrictions of $a(g_1)$ and g_1 to the $i/2$ eigenspace of $a(\xi_2 + \frac{1}{2}\Lambda)$ are identical. In particular the index of $a(g_1)$ restricted to the $i/2$ eigenspace is zero. The other eigenspaces of $\alpha(\xi_2 + \frac{1}{2}\Lambda)$ are always finite dimensional. Hence the index of $a(g_1)$ restricted to any eigenspace is zero. This implies index $a(g_1) = 0$, and $g_1 \in (U_r)_0$.

The general case can be reduced to this special one. For if $i/2$ is an eigenvalue of $\delta(\xi_2 + \frac{1}{2}\Lambda)$, $i/2$ has finite multiplicity. For n sufficiently large, $i/2$ will not be an eigenvalue of $\delta(S^n(\xi_2 + \frac{1}{2}\Lambda)S^{-n})$, and the special case implies $S^n g_1 S^{-n}$ is in $(U_r)_0$. This proves $g_1 \in (U_r)_0$ and (iii).

(iv) By definition ξ corresponds to the real linear functional of $\hat{\underline{u}}_r \cong \underline{u}_r + i\mathbb{R}$ given by

$$\phi(\xi_1, i\tau) = \text{trace}\ \xi\ \xi_1 + \tau.$$

The admissibility of ϕ is invariant under the coadjoint action, and it's easily checked that the integrality condition on the eigenvalues of $(i)\xi$ is invariant under the affine action of U_r in the predual.

Now suppose that $(-i)\xi$ has integer eigenvalues. By applying a finite permutation and the shifting operator, we may assume $\delta(\xi + \frac{1}{2}\Lambda) \equiv -i/2$ and $-i/2$ is not an eigenvalue for $\alpha(\xi + \frac{1}{2}\Lambda)$.

Let $\mu_0 = \frac{1}{2}$, $\mu_1, \ldots, \mu_\ell$ denote the distinct spectral values of $\alpha(\xi + \frac{1}{2}\Lambda)$, $P_0, P_1, \ldots, P_\ell$ the corresponding spectral projections. For $j > 0$, P_j is finite rank. We can identify $U(P_jH_+)$ with

$$\left\{\begin{pmatrix} a & \\ & 1\end{pmatrix} \in U(H_+) \times U(H_-) : a \equiv 1 \text{ on } (P_jH_+)^{\perp}\right\}.$$

The stabilizer of ξ in $(U_r)_0$ is the direct product of $U(H_-)$ and the $U(P_jH_+)$. Because the cross-section Γ of (2.4) is a homomorphism restricted to $U(H_+) \times U(H_-)$, it follows that the stabilizer of ξ in $\hat{U}_r$ can be identified with the direct product

$$(4.5) \qquad \mathop{\times}_j U(P_jH_+) \times U(H_-) \times \Pi,$$

where Π is identified with $U_1(H_+)$ modulo $SU_1(H_+)$ by determinant. Moreover this product decomposition is consistent with the identification of $\hat{\underline{u}}_r$ with $\underline{u}_r + i\mathbb{R}$ where $\mathbb{R}$ is identified with $\underline{u}_1(H_+)$ modulo $\underline{su}_1(H_+)$ by trace. The operator ξ vanishes on P_0H_+ and H_-. On P_jH_+ $\xi = in\,1$, where $n \in \mathbb{Z}$. The functional $i\phi$ restricted to $\underline{u}(P_jH_+)$ is given by

$$i\phi(\eta) = -n \text{ trace } (\eta),$$

and this is the differential of $\det^{-n}$. This shows ϕ is admissible.

Now suppose there is a character, X, for the stabilizer of ξ in $\hat{U}_r$ satisfying $dX = i\phi$. Let $i\lambda$ be a nonzero eigenvalue of $\alpha(\xi)$. If $i(\lambda + \frac{1}{2})$ is an eigenvalue of $\delta(\xi + \frac{1}{2}\Lambda)$, it has finite multiplicity. By applying the shift operator we can suppose $i(\lambda + \frac{1}{2})$ is not an eigenvalue for $\delta(\xi + \frac{1}{2})$.

Let P be the spectral projection for ξ corresponding to $i\lambda$. The stabilizer of ξ in $(U_r)_0$ contains $U(PH)$. As in (4.5), the stabilizer of ξ in $\hat{U}_r$ contains the subgroup

$$U(PH_+) \times \Pi.$$

On the Lie algebra of $U(PH_+)$, $i\phi$ is given by

$$i\phi(\eta) = -\lambda \text{ trace } (\eta).$$

Since determinant generates the character group of $U(PH_+)$, this implies λ must be an integer. This completes the proof of (iv).

(v) We begin by noting that if $n \neq 0$, S^n acts freely on the orbits of $(U_r)_0$. For otherwise there would exist $\xi \in (\underline{u}_r)_*$ and $g \in (U_r)_0$ such that

$$S^n(\xi + \frac{1}{2}\Lambda)\, S^{-n} = g(\xi + \frac{1}{2}\Lambda)g^{-1} .$$

This implies $g^{-1}S^n$ commutes with $\xi + \frac{1}{2}\Lambda$. But in the proof of (iii) we showed that $[g^{-1}S^n, \xi + \frac{1}{2}\Lambda] = 0$ implies $g^{-1}S^n \in (U_r)_0$, a contradiction.

It follows that $S^j * \xi_1$ and $S^k * \xi_2$ are in distinct orbits for $(U_r)_0$ whenever $j \neq k$.

Now suppose ξ has the form $\Sigma\, \lambda_j w_j$. Then $(-i)(\xi + \frac{1}{2}\Lambda)$ has the matrix form

$$(4.6) \qquad M = \left(\begin{array}{ccc|ccc} v_n & & & & & \\ & \ddots & & & & \\ & & v_0 & & & \\ \hline & & & v_{-1} & & \\ & & & & \ddots & \\ & & & & & v_{-m} \end{array}\right),$$

where the v_j are half integers, $\frac{1}{2} < v_n < \dots < v_0$, and $v_{-1} < \dots < v_{-m} < -\frac{1}{2}$. Since the eigenvalues in the upper block are positive and ordered, and the eigenvalues in the lower block are negative and ordered, any permutation, acting by conjugation, which preserves this form actually commutes with $\xi + \frac{1}{2}\Lambda$. Combined with the free action of S, this proves that if ξ_1 and ξ_2 are both of the form $S^k*(\Sigma\ \lambda_j w_j)$, and ξ_1 and ξ_2 are in the same obit for $(U_r)_0$, then $\xi_1 = \xi_2$.

If $(-i)\xi$ is diagonal and integral, then $(-i)\left(\xi + \frac{1}{2}\Lambda\right)$ has the matrix form

$$\begin{array}{c|c|c} 1/2\ I & & \\ \hline & A & \\ \hline & & (-1/2)\ I \end{array} ,$$

where A is diagonal and half integer valued. It's now obvious that we can conjugate by a finite dimensional unitary to arrange that $(-i)\left(\xi + \frac{1}{2}\Lambda\right)$ has the form, $S^k M S^{-k}$, where M is the matrix in (4.6). This completes the proof of (v).//

A few additional comments are in order concerning the preceding proposition (4.3).

First we remark that (ii) and (iii) can be used to show that the coadjoint orbits can be reasonably parameterized.

In (4.3) we considered only the hyperplane corresponding to $s\lambda^*$, where $s = 1$. Parts (i)-(iii) of (4.3) can easily be modified to apply to any hyperplane corresponding to a nonzero s. Thus the coadjoint orbis in any hyperplane, except the hyperplane corresponding to $s = 0$, can be reasonably parameterized.

The hyperplanes containing admissible functionals correspond to integral s. For s equal to a nonzero integer, all of (4.3) is basically valid.

We now want to discuss some aspects of the connection between admissible orbits in the predual and representations.

Again, first consider the hyperspace corresponding to $s\lambda^*$, where $s = 0$. We described a canonical form for admissible functionals in (4.0).

Suppose $\xi = \Sigma \lambda_j w_j$ as in (4.0). In [Boyer, 1] the orbit method is applied to construct an irreducible representation corresponding to ξ for the subgroup $U_2(H)$ of $(U_r)_0$. This construction actually yields a representation for $(U_r)_0$.

This is basically a consequence of

$$(4.7) \qquad U_2(H)/_{U_2(H)_\xi} \cong {}^{(U_r)_0}/_{[(U_r)_0]_\xi} \, .$$

To check (4.7), since $(U_r)_0$ is the product of $U_2(H)$ and $U(H_+) \times U(H_-)$, and $[(U_r)_0]_\xi$ contains $U(S^{\pm k}H_\pm)$ for K sufficiently large, it suffices to check that $U(H_\pm)$ is the product of $U_2(H_\pm)$ and $U(S^{\pm k}H_\pm)$. Let $q \in U(H_+)$, and let $q = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with respect to the decomposition $H_+ = S^k H_+ + \mathrm{span}\,\{e_j : 0 < j < k\}$. The B, C and D are finite rank, and A is Fredholm of index zero as in (1.5). The proof of (1.10) then shows that we can write q as the product of an element from $U(S^k H_+)$ and an element from $U_2(H_+)$.

It follows from (4.7) that

$$(GL_2(H) \times \mathbb{C})/_{GL_2(H)_\xi} \cong ((GL_r)_0 \times \mathbb{C})/_{[(GL_r)_0]_\xi} \, ,$$

where $[(GL_r)_0]_\xi$ is the complexification of $[(U_r)_0]_\xi$, and the other notations are as in [Boyer, 1, Section 2]. This shows that Boyer's construction does yield a representation for $(U_r)_0$.

CHAPTER 5

THE RESTRICTED ORTHOGONAL GROUP

Our goal in sections 5-7 is to carry out the program of sections 1-4 for the spin extension of the restricted orthogonal group. The extension will be considered in section 6.

In this section we define the restricted orthogonal group, 0_r, and a normal subgroup of 0_r, $(0_r)_*$, which we think of as the predual. These groups are naturally Lie groups in the same manner as the unitary groups in Section 1.

The operators in 0_r are distinguished by the fact that they can be realized as automorphisms of the Clifford algebra by unitary operators in a certain Fock representation. This was discovered by Shale and Stinespring in [Shale, 1]. We will review this in the first part of this section.

Our basic references for Clifford algebras and Fock representations will be [Plymen, 1] and [Bratteli and Robinson, 1]. The quadratic form we use to define the Clifford algebra is $\frac{1}{2}$ the form used in [Plymen, 1].

Let H be an ∞-dimensional real Hilbert space, and $W = H^{\mathbb{C}}$. Conjugation in W will be denoted by either $\overline{w}$ or Pw. The complex symmetric extension of the H inner product to W will be denoted by $(\cdot\ ,\ \cdot)$. The complex Hilbert space structure for W induced by the H inner product will be denoted by $\langle \cdot , \cdot \rangle$. These forms are related by

$$\langle \cdot , \cdot \rangle = (\cdot , P \cdot) .$$

Let $C(W,P)$ denote the complex Clifford algebra over H, i.e. $C(W,P)$ is the C^* completion of the $*$-algebra generated by 1 and W subject to the relations

$$w_1 w_2 + w_2 w_1 = (w_1, w_2)1, \; w_j^* = \bar{w}_j,$$

where $w_j \in W$. $C(W,P)$ is a simple C^* algebra, and $O(H)$ acts naturally as a group of automorphisms of $C(W,P)$. The real C^*-algebra generated by 1 and H will be denoted by $C(H)$.

Suppose Λ is an H complex structure. That is, $\Lambda \in O(H)$ and $\Lambda^2 = -1$. Let

$$\langle x, y \rangle_\Lambda = S(x, y) + i\, S(x, \Lambda y),$$

where $x, y \in H$ and S is the H inner product. This form converts (H, Λ) into a complex Hilbert space. The unique state of $C(H)$ such that

$$w_J(xy) = \frac{1}{2} \langle x, y \rangle_\Lambda$$

for x, y ε H is the Fock state relative to Λ. It is known that w_J is pure. The irreducible *-representation of C(H) associated to w_J by the GNS construction is the Λ-Fock representation (see [Plymen, 1, Section 3] for references).

The Λ-Fock representation can be realized in the following way. Let $W_\pm$ denote the $\pm i$ eigenspace of Λ in W. Let $\hat{A}(W_+)$ denote the completion of $A(W_+)$ for which $\{w_{i_1}\Lambda\ldots\Lambda w_{i_k} : i_1 < \ldots < i_k\}$ is an orthonormal basis whenever $\{w_i\}$ is an orthonormal basis for W_+. For w ε W_+, left multiplication by w in $A(W_+)$ has norm $|w|$. Let $\rho(w)$ denote the closure of this operator in $\hat{A}(W_+)$. For w ε W, let $\rho(w) = \rho(\overline{w})^*$, and now extend ρ by linearity to W. Then

$$\rho(w_1)\rho(w_2) + \rho(w_2)\rho(w_1) = (w_1,w_2)1$$

and $\rho(w_j)^* = \rho(\overline{w}_j)$, whenever w_j ε W. Hence ρ extends to a *-representation of C(W,P). The state defined by ρ is w_J (see [Brattelli and Robinson, 1, Section 5.2] for details].

The set of H complex structures coresponds bijectively to the space of maximal $(\cdot,\cdot)$ -isotropic subspaces of W. The correspondence is given by $\Lambda \to W_+ = +i$ eigenspace of Λ. These spaces are homogeneous spaces for O(H). We will let $\underline{\Lambda}$ denote either of these spaces.

Following Shale and Stinespring, define

$$O(H,\Lambda) = \{g \;\varepsilon\; O(H):[g,\Lambda] \text{ is Hilbert-Schmidt}\},$$

where $\Lambda \in \underline{\Lambda}$.

The next proposition follows from results in [Shale, 1] (although it is credited to Manuceau and Verbeure in [Plymen, 1, Section 3]).

(5.1) Proposition. (i) If $\Lambda \in \underline{\Lambda}$, then $g \in O(H)$, as an automorphism of $C(W,P)$, preserves the unitary isomorphism class of the Λ-Fock representation if and only if $g \in O(H,\Lambda)$.

(ii) If $\Lambda_j \in \underline{\Lambda}$ and $g \in O(H)$ satisfies $g\,\Lambda_1 g^{-1} = \Lambda_2$, then

$$g\ O(H,\Lambda_1)g^{-1} = O(H,\Lambda_2).$$

(iii) If we fix $\Lambda \in \underline{\Lambda}$, then the inequivalent Λ-Fock representations are parameterized by the orbits of $O(H,\Lambda)$ in $\underline{\Lambda}$.

Now fix $\Lambda \in \underline{\Lambda}$. Let $W_\pm$ denote the $\pm i$ eigenspace of Λ. With respect to the decomposition $W = W_+ + W_-$, let GL_r, U_r, etc. be defined as in (1.3).

The preceding proposition implies that $O(H,\Lambda) = O(H) \cap U_r$, and up to isomorphism this group is independent of Λ.

In what follows we will consistently identify real operators on H with their complex linear extensions to W. These extensions are characterized by the property that they commute with P. The matrix of an extension has the form $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$, where $\bar{b} = PbP$, $\bar{a} = PaP$.

(5.2) Definition.

(i) $\underline{o}_r^{\mathbb{C}} = \underline{o}(W) \cap \underline{g\ell}_r$

(ii) $\underline{o}_r = \underline{o}(H) \cap \underline{g\ell}_r$

(iii) $O_r^{\mathbb{C}} = O(W) \cap GL_r$

(iv) $O_r = O(H) \cap GL_r$

(5.3) Proposition.

(i) $O_r^{\mathbb{C}}$ is a complex Banach Lie subgroup of GL_r with Banach Lie subalgebra $\underline{o}_r^{\mathbb{C}}$.

(ii) O_r is a banach Lie subgroup of $O_r^{\mathbb{C}}$ and of U_r.

(iii) The map $O_r \times (Sym_r \cap \underline{o}_r^{\mathbb{C}}) \to O_r^{\mathbb{C}}$ given by

$$(g, \xi) \to g \exp (\xi)$$

is a diffeomorphism.

(iv) $O_r \cap (U_r)_0$, and the connected components of O_r are parameterized by the function given by

$$g \to \text{parity of dimension } (\ker a(g)).$$

Proof (i) $\underline{o}_r^{\mathbb{C}}$ is a closed Banach subalgebra of $\underline{g\ell}_r$, and it is complemented in $\underline{g\ell}_r$ by the closed complex subspace of $\underline{g\ell}_r$ consisting of operators which are symmetric with respect to the form $(\cdot,\cdot)$. Part (i) follows from this (see [De La Harpe, 1]).

(ii) Since $\underline{o}_r$ is the real part of $\underline{u}_r$, and $\underline{o}_r^{\mathbb{C}} = \underline{o}_r + (\underline{o}_r^{\mathbb{C}} \cap \mathrm{Sym}_r)$, we can argue as in (i).

(iii) This follows from polar decomposition.

(iv) See [Carey, 2].//

We now introduce the preduals.

(5.4) Definition.

(i) $(\underline{o}_r^{\mathbb{C}})_* = \underline{o}(W) \cap (\underline{g\ell}_r)_*$

(ii) $(\underline{o}_r)_* = \underline{o}(H) \cap (\underline{g\ell}_r)_*$

(iii) $(O_r^{\mathbb{C}})_* = O(W) \cap (GL_r)_*$

(iv) $(O_r)_* = O(H) \cap (GL_r)_*$

(5.5) Proposition.

(i) $(O_r^{\mathbb{C}})_*$ is a complex Banach Lie subgroup of $(GL_r)_*$.

(ii) $(O_r)_*$ is a banach Lie subgroup of $(O_r^{\mathbb{C}})_*$ and of $(U_r)_*$.

(iii) The map $(O_r)_* \times (\underline{O}_r^{\mathbb{C}}) \cap \mathrm{Sym}_r \to (O_r^{\mathbb{C}})_*$ given by

$$(g, \xi) \to g \exp (\xi)$$

is a diffeomorphism.

(iv) $(O_r)_*$ and $(O_r^{\mathbb{C}})_*$ have the same homotopy type as $O(\infty, \mathbb{R})$.

(v) $(O_r)_*$ and $(O_r^{\mathbb{C}})_*$ are normal subgroups of O_r and $O_r^{\mathbb{C}}$, respectively.

Proof. (i)-(iii) can be proved by the same arguments as in (i)-(iii) of (5.3).

(iv) This follows from the theorem of Geba ([De La Harpe, 1, Page 116]), which states that if we fix a basis, then the induced mappings $\underset{n}{\cup} O(n, \mathbb{R}) \to (O_r)_*$ and $\underset{n}{\cup} O(n, \mathbb{C}) \to (O_r^{\mathbb{C}})_*$ are homotopy equivalences.

(v) This follows from (v) of (1.9).

The following proposition is a consequence of (1.10).

(5.6) Proposition. The sequences

$$1 \to U_1(W_+) \to (O_r)_* \times_t U(W_+) \to O_r \to 1$$

$$1 \to GL_1(W_+) \to (O_r^{\mathbb{C}})_* \times_t GL(W_+) \to O_r^{\mathbb{C}} \to 1$$

are exact sequences of Banach Lie groups.

CHAPTER 6

THE SPIN EXTENSION AND $\hat{O}_r$

In finite dimensions the spin representation defines a 2-fold covering of $O(n)$. An analogue in infinite dimensions is that the Λ-spin representation defines a 2-fold covering of $\hat{O}_r$, the extension of O_r induced by $\hat{U}_r \to U_r$. This is proved in (6.11).

We continue with the notation of the previous section. In particular the complex structure Λ is fixed. By (i) of (5.1), there is a projective representation $\tilde{\rho}$ of O_r satisfying

(6.1) $\rho(g(w)) = U\,\rho(w)U^{-1}$,

where $g \in O_r$, $w \in W$, and $U \in U(\tilde{A}(W_+))$ with $\tilde{\rho}(g) = U \cdot (\Pi \cdot 1)$.

(6.2) Definition. Pin is the group of all unitaries U satisfying (6.1) for some $g \in O_r$. Spin is the subgroup of Pin covering SO_r.

Let N denote the number operator in $\hat{A}(W_+)$, i.e. N is the self-adjoint extension of the operator defined on $A(W_+)$ by $N = n$ on $A^n(W_+)$, $n > 0$. Let $D(N)$ denote the domain.

John Palmer has proven the following

(6.3) Proposition. There is a map from $\underline{o_r}$ into skew-adjoint operators on $\hat{A}(W_+)$, $\xi \to d\rho(\xi)$, having the following properties:

(i) if $g = e^{t\xi}$ and $U = e^{td\rho(\xi)}$, then g and U satisfy

(6.1);

(ii) the domain of $d\rho(\xi)$ contains $D(N)$, and $d\rho(\xi)$ is essentially skew-adjoint on $A(W_+)$;

(iii) $d\rho(\xi)$ maps $A(W_+)$ into $D(N)$, and $D(N)$ is invariant under Spin; and

(iv) if we normalize $d\rho(\xi)$ by requiring $\langle d\rho(\xi)1, 1\rangle = 0$, then

$$\langle d\rho(\xi_1)1, d\rho(\xi_2)1\rangle = \frac{1}{2} \text{ trace } \beta(\xi_1) \overline{\beta} (\xi_2),$$

and on $A(W_+)$,

$$[d\rho(\xi_1),d\rho(\xi_2)] = d\rho([\xi_1,\xi_2]) = \frac{1}{2} \text{ trace } ([\alpha(\xi_1),\alpha(\xi_2)] - \alpha([\xi_1,\xi_2])).$$

By (iv) of (5.3), O_r is contained in $(U_r)_0$. Let $\hat{O}_r$ denote the inverse image of O_r in reference to the projection $\hat{U}_r \to (U_r)_0$. Part (iv) of the preceding proposition and (3.2) hint that at least locally Pin is a double covering of $\hat{O}_r$.

Identify $U(W_+)$ with $\{(^a \;_{\bar{a}}) \in O_r\}$. By (1.10) the sequences

$$1 \to SU_1(W_+) \to (O_r)_* \times_t U(W_+) \to \hat{O}_r \to 1$$

(6.4)

$$1 \to U_1(W_+) \to (O_r)_* \times_t U(W_+) \to O_r \to 1$$

are exact sequences of Banach Lie groups.

Because Spin_*, the double covering of $(SO_r)_*$, is connected and simply connected, we can form the semidirect product $\mathrm{Spin}_* \times_t U(W_+)$. Our goal is to show that there is a surjective strong operator continuous homomorphism

$$\rho\colon \mathrm{Spin}_* \times_t U(W_+) \to \mathrm{Spin}.$$

This will then be used to show that Spin is a double covering of $\hat{SO}_r$.

Recall that

$$(\underline{o}_r)_* = \{\begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} \varepsilon\ \underline{o}_r\colon \alpha \text{ is trace class}\}.$$

For $\xi \ \varepsilon\ (\underline{o}_r)_*$, normalize $d\rho(\xi)$ in (6.3) by requiring

$$\langle d\rho(\xi)1,\ 1\rangle = -\frac{1}{2}\,\mathrm{trace}\ \alpha\ .$$

Part (iv) of (6.3) implies that $d\rho$ is a "homomorphism".

It is well known that $d\rho$ restricted to $\underline{o}_1(H)$ exponentiates to a uniform operator norm continuous representation of Pin_∞, the double cover of O_1. This is because O_1 is the subgroup of orthogonal transformations which define inner automorphisms of $C(W,P)$ (see [Shale, 1] or [Plymen, 1, Section 3]).

Our representation ρ on Spin_* is an extension of the spin representation for Spin_1. To check that this extension exists, it suffices to check that we have the right kind of continuity. This is accomplished by lemma (6.6) below, for which we now prepare notation.

Fix an orthonormal basis $\{w_i,\ i > 1\}$ for W_+. The set $\{\bar{w}_i\}$ is an orthonormal basis for W_-, and we have the canonical anticommutation relations

$$\{w_i, w_j\} = \{\bar{w}_i, \bar{w}_j\} = 0,\ \{w_i, \bar{w}_j\} = (w_i, \bar{w}_j) = \delta_{ij}.$$

We identify $O(2n, \mathbb{C})$ with those operators in $O(W)$ which are the identity on the complement of span $\{w_i, \bar{w}_j : i < n\}$. The Lie algebra $\underline{o}(2n, \mathbb{C})$ can be identified with the quadratic elements of $C(W,P)$ generated by $\{w_i, \bar{w}_i : i < n\}$. The precise correspondence is given by

$$(6.5) \qquad \xi \to \frac{1}{2} \Sigma\ \beta_{ij} w_i w_j + \frac{1}{2} \Sigma\ \alpha_{ij} [w_i, w_j] + \frac{1}{2} \Sigma\ \gamma_{ij} \bar{w}_i w_j,$$

where $\xi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \varepsilon\ \underline{o}\ (2n, \mathbb{C})$, and $\beta_{ij} = \langle \beta\ \bar{w}_i, w_j \rangle$, etc. (This is the inverse of the map ψ in [Plymen, 1, Section 2]). The double cover of $SO(2n, \mathbb{R})$, Spin $(2n, \mathbb{R})$, can be identified with $\{\exp(\xi) : \xi\ \varepsilon\ \underline{o}\ (2n, \mathbb{R})\}$, where $\exp(\xi)$ is viewed as an element in the Clifford algebra. For $\xi\ \varepsilon\ \underline{o}\ (2n, \mathbb{R})$, $d\rho(\xi)$ is identical to $\rho(\xi)$, where ξ is viewed as a quadratic element in $C(W,P)$.

Let B and B_1 be normal neighborhoods of 0 in $(\underline{SO_r})_*$ and $\underline{g\ell}_1(W_+)$, respectively. Assume that B is a ball (so that finite rank operators are dense in B), and if $\xi\ \varepsilon\ B$, then $a(e^{\xi})\ \varepsilon\ \exp(B_1)$.

(6.6) Lemma. For ξ in B,

$$e^{d\rho(\xi)}1 = (\det \bar{a})^{1/2} \exp\left(\frac{1}{2} \sum Z_{ij} w_i \wedge w_j\right),$$

where $e^{\xi} = \begin{pmatrix} \underline{a} & \underline{b} \\ \bar{b} & \bar{a} \end{pmatrix}$, $Z = \bar{b}\bar{a}^{-1}$, and the branch of the square root is determined by analytic continuation from 1 in $\exp(\overline{B}_1)$.

Proof. Suppose $\xi \in \underline{O}(2n, \mathbb{R})$. We have

$$\begin{pmatrix} \underline{a} & \underline{b} \\ \bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} 1 & Z \\ & 1 \end{pmatrix} \begin{pmatrix} {a^*}^{-1} & \\ & \bar{a} \end{pmatrix} \begin{pmatrix} 1 & \\ \bar{a}^{-1} b & 1 \end{pmatrix}.$$

If $\eta \in B_1$ and $\exp(\eta) = a$, then by (6.5)

$$e^{d\rho(\xi)}1 = \exp\left(\frac{1}{2} \sum Z_{ij}\rho(w_i)\rho(w_j)\right)\exp\left(\frac{1}{2} \sum -\eta^*_{ij}\rho([w_i, w_j])\right) \cdot 1$$

$$= (\det \bar{a})^{1/2} \exp\left(\frac{1}{2} \sum Z_{ij} w_i \wedge w_i\right).$$

The exponential function is continuous as a map $\hat{A}^2(W_+) \to \hat{A}(W_+)$. Since Z is a continuous function of ξ, and the finite rank operators in B are dense in B, (6.6) is valid for all $\xi \in B$. //

For $\tilde{g} \in (\text{spin}_r)_*$, let g denote the projection of $\tilde{g}$ in $(SO_r)_*$. For $\xi \in (\underline{SO}_r)_*$, let $\exp^{\sim}(\xi)$ denote the exponential of ξ in $(\text{Spin}_r)_*$.

(6.7) Definition. For $\xi \in B$, $\rho(\exp^{\sim}(\xi)) = e^{d\rho(\xi)}$.

(6.8) Proposition. (i) ρ is a local strong operator continuous representation.

(ii) ρ extends uniquely to a strong operator continuous representation of $(\mathrm{Spin}_r)_*$.

Proof. (i) Lemma (6.6) shows that ρ is strongly continuous on 1. If $\tilde{g}_j \in \exp^{\sim}(B)$ and $\tilde{g}_1\tilde{g}_2 = \tilde{g}_3$, then $\rho(\tilde{g}_1)\rho(\tilde{g}_2) = \lambda\, \rho(\tilde{g}_3)$ for some $\lambda \in \Pi$. This implies

$$(6.9) \quad \langle\rho(\tilde{g}_2)1,\ \rho(\tilde{g}_2^{-1})1\rangle = \lambda\langle\rho(\tilde{g}_3)1,\ 1\rangle.$$

If the $\tilde{g}_j$ are in Spin $(2n, \mathbb{R})$, then $\lambda = 1$. By taking limits in (6.9) and using strong continuity of 1, we can conclude that λ is always 1. This shows ρ is a local representation.

To establish continuity, it suffices to establish continuity on the dense set of vectors $C(W,P) \cdot 1$. But because the orthogonal group acts continuously on $C(W, P)$, it suffices to check continuity on the vacuum, 1. As we noted above, this follows from (6.6).

(ii) See [Varadarajan, 1, Section 2.6].

(6.10) Definition. For $(g, q) \in \mathrm{Spin}_* \times_t U(W_+)$,

$$\rho(g, q) = \rho(g)\rho(q),$$

where $\rho(q)$ is the closure of the unique automorphism of the algebra $A(W_+)$ which extends q.

Since the group $SU_1(W_+)$ is simply connected, the inverse image of $SU_1(W_+)$ in $Spin_*$ is a trivial two sheeted covering of $SU_1(W_+)$. Identify $SU_1(W_+)$ with the identity component of this covering.

(6.11) Proposition. (i) ρ is a strong operator continuous representation of $Spin_* \times_t U(W_+)$.

(ii) The sequence

$$1 \to SU_1(W_+) \xrightarrow{i} Spin_* \times_t U(W_+) \to Spin \to 1$$

is an exact sequence, where $i(q) = (q^{-1}, q)$.

(iii) Spin is a double covering of SO_r.

Proof. (i) ρ is actually a homeomorphism of $U(W_+)$ with the strong topology and $\rho(U(w_+))$ with the strong topology. Together with (6.8), this implies that ρ is strong operator continuous.

The intertwining property

$$\rho\left(t_q(\tilde{g})\right) = \rho(q)\rho(\tilde{g})\rho(q)^{-1}$$

is obvious for $q \in U(W_+)$ and $\tilde{g} \in Spin\ (2n, \mathbb{R})$. By continuity it is also valid for $\tilde{g} \in Spin_*$. This implies ρ is a representation.

(ii) To show that ρ is a surjective, it suffices to show $\Pi \cdot 1 \subset \text{Image}\ (\rho)$. If $\tilde{g} \in Spin_*$ and $g = \begin{pmatrix} q & \\ & \bar{q} \end{pmatrix} \in U_1(H_+)$, where g is the orthogonal projection of $\tilde{g}$, then by (6.6)

$$(6.12)\quad \rho(\tilde{g}^{-1},q) = \rho(\tilde{g})^{-1}\rho(q) - (\det q)^{1/2} \cdot 1.$$

This shows ρ is surjective.

If $\rho(\tilde{g},q) = \rho(\tilde{g})\rho(q) = 1$, then $g^{-1} = \begin{pmatrix} q & \\ & \bar{q} \end{pmatrix}$, where $q \in U_1(W_+)$. By (6.12), $q \in SU_1(W_+)$. Since the odd element of the kernel of $\pi:\mathrm{Spin}_* \to (SO_r)_*$ is mapped by ρ to -1, $\tilde{g}$ must be in the identity component of $\pi^{-1}(SU_1(W_+))$. This shows $i(SU_1(W_+))$ is the kernel of ρ.

(iii) Since $\mathrm{Spin}_* \times_t U(W_+)$ is a double cover of $SO_r \times_t U(W_+)$, (ii) and (6.4) imply that the map $\mathrm{Spin} \to \hat{S}o_r$ given by

$$(6.13)\quad \rho(\tilde{g})\rho(q) \to \left[g\begin{pmatrix} q & \\ & \bar{q} \end{pmatrix},\ q\right],$$

where $\tilde{g} \in \mathrm{Spin}_*$, $q \in U(W_+)$ and $\pi(\tilde{g}) = g$, is a well-defined double covering.//

If $[g,\ q] \in \tilde{O}_r$, and g and U satisfy (6.1), then conjugation by U defines an automorphism of Spin which covers conjugation by $[g,\ q]$. This is analogous to Lemma (2.7). This can be used to show that (iii) can be upgraded to state that ρ is a double cover of $\hat{O}_r$.

CHAPTER 7

THE PREDUAL COADJOINT ACTION FOR SPIN

Proposition (6.11) and the closing remark in section 6 imply that the adjoint action of Pin factors through $\hat{0}_r$. The results of section 3 determine the adjoint action of $\hat{0}_r$. We will use these results to classify the admissible coadjoint orbits for Spin in the predual.

Identify $\underline{o}_r + i\mathbb{R}$ with $\hat{\underline{o}}_r$ by using (3.1), and identify $\underline{o}^*_r + (i\mathbb{R})^*$ with $\hat{\underline{o}}^*_r$. The coadjoint action of $\hat{0}_r$ factors through 0_r and is given by

$$g * (\phi, \lambda^*) = (\phi^g + \lambda^* \circ a(g^{-1}, \cdot), \lambda^*),$$

where $a(g^{-1},\xi) = i \operatorname{trace} \left(a_{g^{-1}}\xi\right)$, $a_{g^{-1}} = \frac{1}{2}(g \Lambda g^{-1} - \Lambda)$, and ϕ^g denotes the usual coadjoint action for 0_r.

Each λ^* determines an invariant hyperplane.

First consider the hyperplane determined by $\lambda^* = 0$. The action then reduces to the usual coadjoint action for 0_r. THe predual$(\underline{o}_r)_*$ can be viewed as a subspace of $\underline{o}^*_r$ via $1/2$ trace, and it is invariant under the action.

Fix an orthonormal basis $\{w_j : j \in \mathbb{Z}_+\}$ for W_+. Then $\{\ldots,w_n,\ldots,w_1,\overline{w}_1,\ldots,\overline{w}_n,\ldots\}$ is an ordered orthonormal basis for W. For $n > 1$, let ω_n denote the functional whose matrix is given by

$$\left(\begin{array}{c|c}
0 & i1_{n\times n} \\ \hline
-i1_{n\times n} & 0
\end{array}\right).$$

As in section 3, it is easy to check that each admissible orbit for SO_r in the predual contains a unique element of the form

(7.1) $\quad \xi = \sum^{n} \lambda_j \omega_j,$

where the λ_j are nonnegative integers.

Now fix $\lambda^*(i\tau) = \frac{\tau}{2}$. The action in the λ^* hyperplane is equivalent to the affine action in $\underline{o}^*_r$ given by

(7.2) $\quad g * \phi = \phi^g + \frac{1}{2}\, a(g^{-1}, \cdot).$

We have chosen λ^* to be $\frac{1}{2}$ of the λ^* in section 3 because Spin is a double cover of $\hat{SO}_r$.

View $(\underline{o}_r)_*$, the predual, as a subspace of $\underline{o}^*_r$ via $\frac{1}{2}$ trace.

Fix an orthonormal basis $\{w_j : j > 0\}$ for W_+. Then $\{w_j, \overline{w}_j\}$ is an orthonormal basis for W. This induces a representation of Perm ($\mathbb{Z}$).

(7.3) Proposition (i) $(\underline{o}_r)_*$ is invariant under the affine action (7.2) of $\hat{O}_r$. The action is given by

$$g * \xi = g\left(\xi + \frac{1}{2}\Lambda\right)g^{-1} - \frac{1}{2}\Lambda.$$

(ii) Each orbit for SO_r contains a diagonal operator.

(iii) Two diagonal elements ξ_1 and ξ_2 are in the same SO_r orbit if and only if there is $g \varepsilon \text{Perm}(\mathbb{Z}) \cap SO_r$ such that $g * \xi_1 = \xi_2$.

(iv) A diagonal ξ corresponds to an admissible functional for Spin if and only if $(-i)\xi$ has integral eigenvalues.

(v) Each admissible orbit for Spin contains a unique ξ of the form

$$\xi + R^{\varepsilon} * \sum^{n} \lambda_j \omega_j ,$$

where the λ_j are nonnegative integers, the ω_j are as in (7.1), $\varepsilon = 0$ or 1, and R is the transposition which interchanges w_{n+1} and $\overline{W}_{n+1}$.

Proof. (i) is clear.

(ii) Suppose ξ is in the predual. We have $\xi + \frac{1}{2}\Lambda \varepsilon \underline{o}(H)$, and $\xi + \frac{1}{2}\Lambda$ has discrete specrum. This implies we can find a unitary operator g commuting with P which diagonalizes $\xi + \frac{1}{2}\Lambda$ with respect to $\{w_j, \overline{w}_j\}$.

Let $g(\xi + \frac{1}{2}\Lambda)g^{-1}w_j = i\lambda_j w_j, \lambda_j \varepsilon \mathbb{R}$. Because $g(\xi + \frac{1}{2}\Lambda)g^{-1}$ commutes with P,

$$g(\xi + \frac{1}{2}\Lambda)g^{-1}\,\overline{w}_j = -i\lambda_j \overline{w}_j .$$

Define a relection g_1 by $g_1(w_j) = \overline{w}_j$ if $\lambda_j < 0$ and $g_1(w_j) = w_j$

otherwise. The operator g_1 commutes with P. Replace g by g_1g. We now have a $g \in O(H)$ having the property that the eigenvalues of $\alpha\left(g\left(\xi + \frac{1}{2}\Lambda\right)g^{-1}\right)$ are of the form $i\lambda_j$, $\lambda_j > 0$. We can now repeat the argument in (ii) of (4.3) to conclude that $g \in O_r$.

If g is not in SO_r, we can modify g by a finite odd permutation commuting with P (e.g. switch w_1 and $\overline{w}_1$ and fix the complement) to obtain an appropriate $g \in SO_r$. This proves (ii).

(iii) ($\Leftarrow$) is clear.

($\Rightarrow$) Suppose $g \in SO_r$ and $g\left(\xi_1 + \frac{1}{2}\Lambda\right)g^{-1} = \xi_2 + \frac{1}{2}\Lambda$.

There is a $g_1 \in O(H)$ which commutes with $\xi_2 + \frac{1}{2}\Lambda$ and satisfies $g_1g \in \text{Perm}(\mathcal{H}) \cap O(H)$. As in (iii) of (4.3) $g_1 \in O_r$ and $a(g_1)$ commutes with $\alpha\left(\xi_2 + \frac{1}{2}\Lambda\right)$. If kernel $(a(g_1))$ is even dimensional, then we are done. Suppose otherwise.

Let $\{i\lambda_j\}$ denote the distinct eigenvalues of $\alpha\left(\xi_2 + \frac{1}{2}\Lambda\right)$, and let P_j denote th spectral projection corresponding to $i\lambda_j$.

If $i\lambda_j$ is not an eigenvalue for $\overline{\alpha}\left(\xi_2 + \frac{1}{2}\Lambda\right)$, then g_1 commutes with P_j, implying $a(g_1) = g_1$ on P_jQ. Hence there is an $i\lambda_j$ which is also an eigenvalue for $\overline{\alpha}\left(\xi_2 + \frac{1}{2}\Lambda\right)$.

If $\alpha\left(\xi_2 + \frac{1}{2}\Lambda\right)w_\ell = i\lambda_j w_\ell$, replace g_1 by the produce of g_1 and the transposition which interchanges w_ℓ and $\overline{w}_\ell$. This new g_1 is in SO_r, commutes with $\xi_2 + \frac{1}{2}\Lambda$, and satisfies $g_1g \in \text{Perm}(\mathcal{H}) \cap SO_r$. This proves (iii).

(iv) Let ξ be a diagonal element in the predual, and let ϕ denote the corresponding real functional on $\hat{\underline{o}}_r$.

We first determine the structure of $(SO_r)_\xi$.

If $i\mu$ is a nonzero common eigenvalue of $\alpha(\xi + \frac{1}{2}\Lambda)$ and $\overline{\alpha}(\xi + \frac{1}{2}\Lambda)$, then so is $-i\mu$, and $i\mu \neq -i\mu$. Since $\alpha(\xi + \frac{1}{2}\Lambda)$ and $\overline{\alpha}(\xi + \frac{1}{2}\Lambda)$ have finitely many common eigenvalues, if any, by applying elements in $O_r \bigcap \text{Perm}(\mathbb{Z})$ to switch eigenvalues of $\alpha(\xi + \frac{1}{2}\Lambda)$ and $\overline{\alpha}(\xi + \frac{1}{2}\Lambda)$, we can assume $\alpha(\xi + \frac{1}{2}\Lambda)$ and $\overline{\alpha}(\xi + \frac{1}{2}\Lambda)$ do not have any common nonzero eigenvalues. By applying elements in $U(W_+) \cap \text{Perm}(\mathbb{Z})$, we can assume that zero eigenspace of $\alpha(\xi + \frac{1}{2}\Lambda)$ is the span of $\{w_j : j < k\}$.

Let $i\mu_0 = \frac{i}{2}$, $i\mu_1, \ldots$ be the distinct nonzero points in the spectrum of $\alpha(\xi + \frac{1}{2}\Lambda)$, P_j the corresponding spectral projections.

Identify $SO(2n, \mathbb{R})$ with $\{g \in (SO_r)_* : g \equiv 1 \text{ on } \{w_j, \overline{w}_j : j < k\}\}$. Also view $U(P_jW_+)$ as a subgroup of $U(W_+) \cong \{\begin{pmatrix} q & \\ & \overline{q}\end{pmatrix} \in SO_r\}$ in the same manner.

We now compute stablizers. The stabilizer of ξ in SO_r is the subgroup of elements which commute with $\xi + \frac{1}{2}\Lambda$, i.e.

$$(SO_r)_\xi = \Big(\underset{j>0}{X}\, U(P_jW_+)\Big) \times SO(2k, \mathbb{R}).$$

To describe the stabilizer of ξ in $\hat{SO}_r$, we use the notation in (2.3). Identify $\{[1,q] \in \hat{SO}_r\}$ with Π via determinant. We then have the direct product decomposition

$$(\hat{SO}_r)_\xi = \Big(\underset{j>0}{X}\, \Gamma(U(P_jW_+))\Big) \times \{[g, 1] : g \in SO(2K, \mathbb{R})\} \times \Pi,$$

where Γ is the cross-section in (2.4).

The stabilizer Spin_ξ is the product of the $\rho(U(P_jW_+))$, $\rho(\mathrm{Spin}(2k, \mathbb{R}))$, and $\Pi \cdot 1$. It follows from (6.13) that the projection of Spin_ξ onto $(\hat{SO})_\xi$ is given by

$$(\underset{j>1}{X})\rho(\tilde{g})(\lambda \cdot 1) \to (\underset{j>1}{X} \; \Gamma(g_j))[g,1]\lambda^2,$$

where $q_j \; \varepsilon \; U(P_jW_+)$, $\tilde{g} \; \varepsilon$ Spin $(2k, \mathbb{R})$ and $|\lambda| = 1$. This implies that

$$\mathrm{Spin}_\xi = (\underset{j>0}{X} \; \rho(U(P_jW_+))) \times \rho(\mathrm{Spin}\ (2k, \mathbb{R}))(\Pi \cdot 1),$$

where the intersection of $\rho(\mathrm{Spin}(2k, \mathbb{R}))$ and $\Pi \cdot 1$ is $\{\pm 1\}$.

Now suppose that X is a character of Spin_ξ having differential $i\phi$.

The Lie subalgebra of $\rho(U(P_jW_+))$ in $\hat{\underline{SO}}_r \cong \underline{SO}_r + i\mathbb{R}$ is

$$\underline{u}\ (P_jW_+) \cong \{((^{\eta} \; _{\bar{\eta}}),\ 0) : \eta \; \varepsilon \; \underline{u}\ (P_jW_+)\}.$$

On this subalgebra $i\phi$ is given by

$$\eta \cong ((^{\eta} \; _{\bar{\eta}}),\ 0) \to \frac{i}{2} \text{ trace } \ (\xi \cdot (^{\eta} \; _{\bar{\eta}}))$$

$$= -\lambda_j \text{ trace } (\eta).$$

This implies $\lambda_j \; \varepsilon \; \mathbb{Z}$.

If $K > 1$, then the commutator subgroup of Spin $(2K, \mathbb{R})$ is all of Spin $(2K, \mathbb{R})$. This implies $X(-1) = 0$. But

$$\exp_{\hat{SO}_r}[(0,\ it)] = [1,\ e^{it}]$$
$$\exp_{Spin}[(0,\ it)] = e^{it/2}\cdot 1$$
$$X(e^{it}\cdot 1) = e^{it} \quad (\text{recall } \lambda^*[(0,\ it)] = \tfrac{t}{2}).$$

So we must have $K \leq 1$.

If $K = 1$, then because $SO(2, \mathbb{R}) = U(\mathbb{C}\, w_1)$,

$$\mathrm{Spin}_\xi = \Big(\underset{j>0}{X}\ \rho(U(P_j W_+))\Big) \times \rho(U(\mathbb{C}\, w_1)) \times (\Pi \cdot 1),$$

and the argument above implies λ_1 is integral. So all the λ_j are integral.

The converse follows easily from the explicit form for Spin_ξ,

$$\mathrm{Spin}_\xi = \Big(\underset{j>0}{X}\ \rho(U(P_j W_+))\Big) \times (\Pi \cdot 1).$$

This proves (iv).

(v) Suppose $(-i)\xi$ is integral and diagonal. The matrix of $(-i)(\xi + \frac{1}{2}\Lambda)$ has the form

$$\left(\begin{array}{cccc|cccc} \ddots & & & & & & & \\ & \mu_n & & & & & & \\ & & \ddots & & & & & \\ & & & \mu_1 & & & & \\ \hline & & & & -\mu_1 & & & \\ & & & & & -\mu_2 & & \\ & & & & & & \ddots & \\ & & & & & & & \ddots \end{array}\right),$$

where eventually all $\mu_K = \frac{1}{2}$. We can conjugate by an element from $U(W_+)$ to obtain an equivalent matrix of the form

$$\left(\begin{array}{cc|cc} \frac{1}{2}I & & & \\ & \begin{matrix}\mu_n & \\ & \ddots \\ & & \mu_1\end{matrix} & & \\ \hline & & \begin{matrix}-\mu_1 & \\ & \ddots \\ & & -\mu_n\end{matrix} & \\ & & & (-1/2)I \end{array}\right),$$

where $\frac{1}{2} < |\mu_n| < \ldots < |\mu_1|$.

If an even number of μ_j are negative, we can conjugate by an even number of transpositions to obtain the equivalent matrix

$$\left(\begin{array}{cc|cc} (1/2)I & & & \\ & |\mu_n| \ddots |\mu_1| & & \\ \hline & & -|\mu_1| \ddots -|\mu_n| & \\ & & & (-1/2)I \end{array}\right).$$

This is the matrix of an operator of the form $(-i)(\sum^{n} \lambda_j\ {}_j + \frac{1}{2}\Lambda)$, where the λ_j are nonnegative integers.

If an odd number of μ_j are negative, we can conjugate by an even number of transpositions to obtain the equivalent matrix

$$\left(\begin{array}{cc|cc} (\frac{1}{2})I & & & \\ & \begin{matrix}-1/2 & & \\ & |\mu_n| \ddots & \\ & & |\mu_1|\end{matrix} & & \\ \hline & & \begin{matrix}-|\mu_1| \ddots -|\mu_n| & \\ & 1/2\end{matrix} & \\ & & & (-\frac{1}{2})I \end{array}\right)$$

This is the matrix of an operator of the form $(-i)\left(R * \sum^{n} \lambda_j w_j + \frac{1}{2}\Lambda\right)$.

This shows tht each admissible orbit for Spin contains an operator of the form

$$R^{\varepsilon} * \sum^{n} \lambda_j w_j \quad .$$

Now suppose $\xi = R^{\varepsilon} * \sum^{n} \lambda_j w_j$. Then $(-i)\left(\xi + \frac{1}{2}\Lambda\right)$ has the matrix form

$$\left(\begin{array}{c|c|c|c} (\tfrac{1}{2})I & & & \\ \hline & \nu_n \ddots \nu_1 & & \\ \hline & & -\nu_1 \ddots -\nu_n & \\ \hline & & & (-\tfrac{1}{2})I \end{array}\right),$$

where the ν_j are half integers and $\frac{1}{2} < \nu_n < \ldots < \nu_1$. Any permutation in SO_r, acting by conjugation, which preserves this form actually commutes with $\xi + \frac{1}{2}\Lambda$. This proves (v).//

CHAPTER 8

THE BASIC REPRESENTATION

The next 3 sections of this paper concern a certain measure-theoretic aspect of the representation theory for the group $\hat{U}_r$, which was introduced in Section 2.

In Section 4 we singled out a class of integral coadjoint orbits for the group $\hat{U}_r$, and at the end of Section 4 we indicated that the works of [Boyer, 1] and [Segal, 2] demonstrate that the corresponding representations (via the orbit method) can be realized in Hilbert spaces consisting of holomorphic sections of certain homogeneous hermitian line bundles for the group $\hat{U}_r$. The question of interest to us is whether the global inner product in these representation spaces can be realized by integrating the local inner products of the sections.

Ideally, we would like to find a $\hat{U}_r$-invariant measure on a completion of the base space, $\breve{B}$, so that the global inner product has the form

$$(8.1)\ \langle\sigma_1,\sigma_2\rangle = \int_{\breve{B}} \langle\sigma_1,\sigma_2\breve{\rangle}_{\breve{b}}\, dv(\breve{b}),$$

where $\langle\ ,\ \rangle_b$ denotes the local inner product in the fiber over the point b in the base space, and $\langle\sigma_1,\sigma_2\breve{\rangle}_{\breve{b}}$ denotes a v-measurable extension of the function $\langle\sigma_1,\sigma_2\rangle_b$ to the completion of the base, $\breve{B}$. To extend the function $\langle\sigma_1,\sigma_2\rangle_b$ from the base to $\breve{B}$ in a reasonable way, one would expect that $\langle\sigma_1,\sigma_2\rangle_b$ should be approximable, in some sense, by functions of finitely many variables. This turns out to be possible for the spaces in [Boyer, 1]; unfortunately, a normalization problem precludes the existence of the measure v (see Theorem 2.4 of [Boyer, 1]). For the space in [Segal, 2] the functions $\langle\sigma_1,\sigma_2\rangle_b$ are not approximable. However, it turns out that there is a canonical section σ (the highest weight vector for the representation) having the properties: (1) $\mu \stackrel{\sim}{=} \langle\sigma,\sigma\rangle v$ can be interpreted rigorously as a quasi-invariant cylinder measure, and (2) $\langle\sigma_1,\sigma_2\rangle\langle\sigma,\sigma\rangle^{-1}$ is approximable by functions of finitely many variables. Hence (8.1) should be replaced by the well-defined expression

$$(8.2)\quad \langle\sigma_1,\sigma_2\rangle = \int_{\tilde{B}} \left(\frac{\langle\sigma_1,\sigma_2\rangle_b}{\langle\sigma,\sigma\rangle_b}\right)^{\sim} d\mu \ .$$

In this sense the space in [Segal, 2] can be interpreted as a space of square integrable holomorphic sections.

In the next section, Chapter 9, we will discuss the measure μ, and in Chapter 10 we will consider (8.2). The purpose of this chapter is simply to prepare the way for Chapters 9 and 10. We will review the construction in [Segal, 2] and also some background material. We begin with the finite dimensional analogue of the construction in [Segal, 2].

The mth fundamental representation of $SU(N, \mathbb{C})$ is the representation of $SU(N, \mathbb{C})$ on the space $A^m(\mathbb{C}^N)$, the space of homogeneous tensors of degree m in the alternating algebra. The Borel-Weil theorem says that this representation can also be realized as the action of $SU(N, \mathbb{C})$ on the holomorphic sections of a homogeneous line bundle for $SU(N, \mathbb{C})$. Our first task in this chapter is to make this precise.

Let $\mathbb{C}^N = V = V_+ + V_-$ be an orthogonal decomposition, where $\dim (V_-) = m$. As is our custom, we will write $g \in GL(V)$ as a matrix,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with respect to the decomposition $V = V_+ + V_-$.

Let X be the holomorphic character of $\left\{\begin{pmatrix} a & b \\ o & d \end{pmatrix} \in SL(V)\right\}$ given by

$$X\begin{pmatrix} a & b \\ o & d \end{pmatrix} = \det (d) = \det(a)^{-1} .$$

The group $\left\{\begin{pmatrix} a & b \\ & d \end{pmatrix}\right\}$ has a right holomorphic action on $SL(V) \times \mathbb{C}$ given by

$$(8.3) \quad (g,\lambda) \cdot g_1 = (gg_1, X^{-1}(g_1)\lambda),$$

where $g \in SL(V)$ and $g_1 \in \left\{\begin{pmatrix} a & b \\ o & d \end{pmatrix} \in SL(V)\right\}$. The quotient $SL(V) \times \mathbb{C}/\left\{\begin{pmatrix} a & b \\ & d \end{pmatrix} \in SL(V)\right\}$, denoted by $SL(V) \times_X \mathbb{C}$, is a homogeneous holomorphic line bundle for $SL(V)$.

Since SL(V) is the product of the groups SU(V) and $\{\begin{pmatrix} a & b \\ & d \end{pmatrix} \in SL(V)\}$, SL(V) $\times_X$ ¢ is a homogeneous bundle for SU(V), hence

$$SL(V) \times_X ¢ \cong SU(V) \times_{X_0} ¢ \ ,$$

where X_0 denotes the restriction of X to $\{\begin{pmatrix} a & \\ & d \end{pmatrix} \in SU(V)\}$, and SU(V) $\times_{X_0}$¢ again denotes the quotient SU(V) x ¢$/\{\begin{pmatrix} a & \\ & d \end{pmatrix} \in SU(V)\}$ with respect to the action in (8.3) (see [Wallach, 1, section 5.2]). The point of this is that it's clear there is a SU(V) invariant hermitian structure on SU(V) $\times_{X_0}$ ¢ defined by

$$<[g,\lambda_1],\ [g,\lambda_2]> = \lambda_1 \cdot \bar{\lambda}_2 \ ,$$

where $[g,\lambda_j] \in$ SU(V) $\times_X$¢.

For purposes of comparison with [Segal, 2], we note there is a third, more geometric, realization of this bundle. If $N = n + m$, let Gr(n,V) denote the homogeneous space for SL(V) consisting of all n dimensional subspaces of V. The bundle

$$\mathrm{Det}(V_+,V) = \mathrm{Det} = \coprod_{W \in Gr(n,V)} \Lambda^n(W)$$

is a homogeneous line bundle for SL(V), where

$g \cdot w_1 \Lambda \ldots \Lambda\, w_n = g \cdot w_1\ \Lambda \ldots \Lambda\ g \cdot w_n$, $g \in SL(V)$, and $w_j \in W$. The isotropy subgroup for the fiber over V_+ is $\{\begin{pmatrix} a & b \\ & d \end{pmatrix} \in SL(V)\}$, and the action is given by X^{-1}. This implies that

$$SL(V) \times_X \mathbb{C} \cong \mathrm{Det}^* = \mathrm{Det}^*(V_+,V)$$

where Det* denotes the dual bundle, $\coprod A^n(W)^*$.

The holomorphic sections of $SL(N,\mathbb{C}) \times_X \mathbb{C}$ correspond to holomorphic functions on $SL(N, \mathbb{C})$ which satisfy

$$(8.4) \qquad F(gg_1) = X^{-1}(g_1)F(g),$$

where g_1, $g \in SL(V)$ and $g_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}$. The corresponding section is given by

$$(8.5) \ \sigma_F\left(g \cdot \left\{\begin{pmatrix} a & b \\ & d \end{pmatrix}\right\}\right) = (g,\ F(g)) \cdot \left\{\begin{pmatrix} a & b \\ & d \end{pmatrix}\right\}.$$

The action of $SL(V)$ on sections corresponds to the left regular representation of $SL(V)$ on functions satisfying (8.4).

If F is holomorphic on $SL(V)$, then F is determined by its restriction to $SU(V)$, denoted f. If F also satisfies (8.4), then f satisfies

$$(8.6) \qquad \begin{aligned} & f(gg_1) = X_0^{-1}(g_1)f(g) \\ & \begin{pmatrix} o & \beta \\ o & o \end{pmatrix} \cdot f = 0 \end{aligned}$$

whenever $g,g_1 \in SU(V)$, $g_1 = \begin{pmatrix} a & \\ & d \end{pmatrix}$, and $\begin{pmatrix} o & \beta \\ o & o \end{pmatrix} \in \underline{s\ell}(V)$ is viewed as a left invariant differential operator on $SU(V)$. Conversely, if f is a C^{∞} function on SU(V) satisfying (8.6), then f extends to a holomorphic function on $SL(V) \times_{X_0} \mathbb{C}$ which corresponds to σ_F in (8.5) is given by

$$\sigma_f\left(g \cdot \left\{\begin{pmatrix} a & \\ & d \end{pmatrix}\right\}\right) = \left(g, f(g)\right) \cdot \left\{\begin{pmatrix} a & \\ & d \end{pmatrix}\right\}.$$

We are now prepared to construct the map which intertwines the actions of SL(V) on $A^m(V)$ and holomorphic sections of $SL(V) \times_X \mathbb{C}$.

Let(,) denote the complex bilinear pairing between $A^m(V)$ and its dual, $A^m(V)^*$, and let T denote the dual representation of SL(V) on $A^m(V)^*$.

If $e_{-m},\ldots,e_{-1}$ is an orthonormal basis for V_-, then

$$\begin{pmatrix} a & \\ c & d \end{pmatrix} \cdot e_{-m} \wedge \ldots \wedge e_{-1} = \det(d)\, e_{-m} \wedge \ldots \wedge e_{-1}.$$

If ϕ is the functional which is one on $e_{-m} \wedge \ldots \wedge e_{-1}$ and vanishes on the orthogonal complement, then it follows that

$$(8.7)\quad T\begin{pmatrix} a & b \\ & d \end{pmatrix} \cdot \phi = (\det d)^{-1} \phi\ .$$

Now suppose $V \in A^m(V)$. Let F_V be the holomorphic function on SL(V) defined by

$$(8.8)\quad F_V(g) = (V,\ T(g)\ \phi).$$

The equation (8.7) implies F_V satisfies (8.4), i.e. that F_V defines a section of $SL(V) \times_X \mathbb{C}$. The intertwining property follows directly from the definition of the dual representation (the mapping is actually equivariant for all of $U(V)$ - the multiples of 1 can be checked directly). Since both representations are irreducible, the mapping is an isomorphism (for a proof of irreducibility, see [Wallach, 1].

Let ν denote the unique probability measure on $Gr(n, V)$ which is invariant under the action of $SU(V)$. For each $c > 0$, we can define $SU(V)$ invariant inner product on the space of sections of $SU(V) \times_{X_0} \mathbb{C}$ by

$$(8.9)\qquad \begin{aligned} \langle\sigma_1,\sigma_2\rangle &= c \int \langle\sigma_1(w),\sigma_2(w)\rangle_W d\nu(W) \\ &= c \int (f_1\bar{f}_2)(w)d\nu(w), \end{aligned}$$

where σ_j is the section corresponding to the function f_j.

We wish to determine the unique constant c for which our intertwining map $v \to f_v$ is an isometry, i.e. for which $e_{-m}\Lambda\ldots\Lambda e_{-1}$ will correspond to a section of norm one.

Let $v_1,\ldots,v_d$ be an orthonormal basis for $A^m(V)$, $\sigma_1,\ldots,\sigma_d$ the corresponding sections of $SU(V) \times_{X_0} \mathbb{C}$. If $W = g \cdot V_+$, $g \in SU(V)$, then

$$|\sigma_j(W)|^2 = |(v_j,T(g)\phi)|^2 = |\langle v_j,g \cdot e_{-m}\Lambda\ldots\Lambda e_{-1}\rangle|^2$$

$$\sum_j |\sigma_j(W)|^2 = \sum_j |\langle v_j,\ g \cdot e_{-m}\wedge\ldots\wedge e_{-1}\rangle|^2 = 1$$

$$c = \sum_j c \int |\sigma_j(W)|^2 d\nu(W) = \sum_j |v_j|^2 = d$$

Since $d = \binom{N}{m}$, we have

(8.10) $\langle v_1, v_2\rangle_{A^m(V)} = \binom{N}{m} \int \langle\sigma_1,\ \sigma_2\rangle d\nu$ for any $v_j \in A^m(V)$.

We now turn to the construction in [Segal, 2].

Let $H = H_+ + H_-$, where $H_\pm$ are ∞-dimensional complex Hilbert space, as in chapters 1-4 and [Segal, 2, section 2]. We will consider the Hilbert-Schmidt Grassmannian, rather than the compact Grassmannian defined in [Segal, 2, section 2].

(8.11) Definition. Gr is the set of all closed subspaces W of H satisfying

(i) $Q_+ : W \to H_+$ is Fredholm, and

(ii) $Q_- : W \to H_-$ is Hilbert-Schmidt.

It's verified in [Segal, 2, Lemma 2.2] that GL_r, the restricted general linear group defined in chapter 1, acts on Gr. We now verify that U_r acts transitively on Gr.

Suppose $W \in$ Gr. Part (ii) of (8.11) implies that W_+ and W_- are ∞-dimensional. So we can find $g \in U(H)$ with $g \cdot H_+ = W$.

The diagrams

$$\begin{tikzcd} H_+ \arrow[r, "g"] \arrow[dr, "a"'] & W \arrow[d, "Q_+"] \\ & H_+ \end{tikzcd} \qquad \begin{tikzcd} H_+ \arrow[r, "g"] \arrow[dr, "c"'] & W \arrow[d, "Q_-"] \\ & H_- \end{tikzcd}$$

show that c is Hilbert-Schmidt and a is Fredholm. To show $g \in U_r$, we can assume index $(a) = 0$. We then can find a finite rank operator F such that $a + F$ is invertible. Using (1.1) and (1.2) (which simply says $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is unitary), we have

$$a^*b + c^*d = 0$$

$$\Rightarrow \quad a^*(a + F)(a + F)^{-1}b + c^*d = 0$$

$$\Rightarrow \quad (1 - c^*c + a^*F)(a + F)^{-1}b + c^*d = 0$$

$$\Rightarrow \quad (a + F)^{-1}b \text{ is Hilbert-Schmidt}$$

$$\Rightarrow \quad b \text{ is Hilbert-Schmidt.}$$

Thus $g \in U_r$, and Gr is a homogeneous space for U_r and Gl_r.

Let θ denote the set of $W \in Gr_0$, the H_+ component of Gr, which are transverse to H_-. A holomorphic coordinate is defined on θ by mapping $W \in U$ to the unique $Z \in \mathcal{L}_2(H_+, H_-)$ satisfying $W = \text{graph }(Z)$. The elements in $(GL_r)_0$ act by linear fractional maps in the coordinate $\mathcal{L}_2(H_+,H_-)$:

$$g(W) = \text{graph}\ ((dZ + c)(a + bZ)^{-1}),$$

whenever W and g(W) are transverse. This can be used to show that Gr is a holomorphic space and that GL_r acts holomorphically.

The coordinate map $W \to Z$ can also be interpreted as a cross-section over θ for the map $GL_r \to Gr$ given by

$$g \to g \cdot H_+.$$

This is because $W = \text{graph}(Z) = \begin{pmatrix} 1 & \\ Z & 1 \end{pmatrix} \cdot H_+$. The manifold Gr is thus an analytic quotient of GL_r.

The derivative of 1 of the locally defined mapping

$$\exp\left(\left\{\begin{pmatrix} & \beta \\ \gamma & \end{pmatrix} \varepsilon\ \underline{u}_r\right\}\right) \to \mathcal{L}(H_+,H_-) : g \to \text{graph}(g \cdot H_+)$$

is given by

$$\left\{\begin{pmatrix} & \beta \\ \gamma & \end{pmatrix} \cdot \varepsilon\ \underline{u}_r\right\} \to \mathcal{L}(H_+,H_-) : \begin{pmatrix} & \beta \\ \gamma & \end{pmatrix} \to \gamma.$$

The inverse function theorem implies that Gr is also an analytic quotient of U_r.

The hermitian form on $\left\{\begin{pmatrix} & \beta \\ \gamma & \end{pmatrix} \varepsilon\ \underline{u}_r\right\}$ given by

$$H\left(\begin{pmatrix} & \beta_1 \\ \gamma_1 & \end{pmatrix}, \begin{pmatrix} & \beta_2 \\ \gamma_2 & \end{pmatrix}\right) = \operatorname{tr} \gamma_2{}^*\gamma_1$$

is invariant under the adjoint action of the H_+ stabilizer in $\dot{U}_r$, $\left\{\begin{pmatrix} a & \\ & d\end{pmatrix}\right\}$. This implies that Gr is a hermitian symmetric space for U_r. In particular there is a $(U_r)_0$ invariant metric on each component which induces the original topology (see [Klingenberg, 1, Theorem 1.9.5]).

In [Segal, 2, section 3] the Det* bundle is constructed, and it's shown that $\hat{GL}_r$ (as in (2.3)) acts on Det*. By the preceding paragraph, Det* is a homogeneous bundle for $\hat{GL}_r$ and $\hat{U}_r$. The isotropy subgroup at H_+ is $\left\{[g,q] \in \hat{GL}_r \colon g = \begin{pmatrix} a & b \\ & d\end{pmatrix}\right\}$. The action of this subgroup on the fiber over H_+ is given by

$$(8.12)\quad X([g,q]) = \det q\, a^{-1}.$$

This implies that the maps $\hat{U}_r \times \mathbb{C} \to \text{Det*}$ and $\hat{GL}_r \times \mathbb{C} \to \text{Det*}$ given by

$$([g,q],\lambda) \to (w,\lambda),$$

where w is the admissible basis gq^{-1}, factor to yield isomorphisms

$$\hat{GL}_r \times_X \mathbb{C} \cong \hat{U}_r \times_{X_0} \mathbb{C} \cong \text{Det*},$$

where X_0 denotes the restriction of X.

The usual hermitian structure on $\hat{U}_r \times \mathbb{C}$,

$$\langle([g,q],\lambda_1),\ ([g,q],\lambda_2)\rangle_{[g,q]} = \lambda_1\bar{\lambda}_2,$$

is invariant under the right action of the isotropy subgroup, $\{[g,q] \in \hat{U}_r : g = \begin{pmatrix} a & \\ & d\end{pmatrix}\}$, hence defines a $\hat{U}_r$ invariant hermitian structure on $\hat{U}_r \times_{X_0} \mathbb{C}$. In the geometric terminology of [Segal, 2], this corresponds to

$$\langle(w,\lambda_1),\ (w,\lambda_2)\rangle_W = \lambda_1\bar{\lambda}_2,$$

where w is an isometric admissible basis for $W \in Gr_0$ (note that a subspace in the H_+ component of the compact Grassmannian in [Segal, 2, section 2] will admit an isometric admissible basis if and only if it is in the Hilbert-Schmidt Grassmannian).

Fix an orthonormal basis $\{e_j : j \in \mathbb{Z}\}$ for H with $\{e_j : j > 0\} \subset H_+$. As in [Segal, 2, section 8], let S_0 denote the set of all increasing integral sequences $\{s(i) : i > 0\}$ such that $s(i) = i$ for all but finitely many i. If $s \in S_0$, the complement of s in $\mathbb{Z}$, S^c, can be viewed as a decreasing integral sequence $\{s^c(j) : j < 0\}$ such that $s^c(j) = j$ for all but finitely many j. Define the linear operator U_s by

$$(8.13)\quad U_s(e_j) = \begin{cases} e_{s(j)} & \text{if} \quad j > 0 \\ e_{s^c(j)} & \text{if} \quad j < 0 \end{cases} \quad .$$

It's clear that $U_s^2 = 1$ and $U_s \in (U_r)_0$.

As in [Segal, 2, section 8] we define the Plucker section corresponding to $s \in S_0$ by

(8.14) $\sigma_s(W) = (w, \det (U_s w))$,

where w is an admissible basis for W. Viewed as a section of $\hat{U}_r \times_{X_0} \mathbb{C}$, σ_s corresponds to the function on $\hat{U}_r$ given by

(8.15) $f_s([g,q]) = \det a(U_s g)q^{-1}$.

The section corresponding to the identity sequence is the canonical section σ of [Segal, 2, section 3]. The section σ_s is just the translate of σ by the (Weyl group) element U_s:

$$\sigma_s = [U_s, 1] \cdot \sigma ,$$

where $[g, q] \cdot \sigma$ denotes the usual action of the group element $[g, q]$ on the section σ.

Following [Segal, 2, appendix], we now show that the bundle Det* is essentially a completion of the (direct limit of the) finite dimensional Det* bundles described in the first part of this section.

Let $V = V_+ + V_-$, where $V_\pm$ are n-dimensional subspaces of $H_\pm$, respectively. We view $GL(V)$ embedded in $GL_r(H)$ via the mapping

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \left(\begin{array}{c|c|c|c} 1 & & & \\ \hline & a & b & \\ \hline & c & d & \\ \hline & & & \end{array}\right)$$

where the former matrix is the decomposition of g with respect to $V = V_+ + V_-$, and the latter matrix is the decomposition of g with respect to

$$H = (H_+ - V_+) + V_+ + V_- + (H_- - V_-) .$$

We also view $GL(V)$ as a subgroup of $\hat{GL}_r$ via $g \to [g, 1]$.

The character χ which defines the bundle Det* (see 8.12)) restricts to $\chi(g) = \det a(g)^{-1}$ on $GL(V)$. This yields

$$\begin{tikzcd} GL(V) \times_\chi \mathbb{C} \arrow[r] \arrow[d] & \hat{GL}_r \times_\chi \mathbb{C} \arrow[d] \\ GL(V)/\left\{\begin{pmatrix} a & b \\ & d \end{pmatrix}\right\} \arrow[r] & (GL_r)_0/\left\{\begin{pmatrix} a & b \\ & d \end{pmatrix}\right\} \end{tikzcd}$$

or, in more geometric language,

$$\begin{tikzcd} \mathrm{Det}^*(V_+, V) \arrow[r] \arrow[d] & \mathrm{Det}^* \arrow[d] \\ Gr(V_+, V) \arrow[r] & Gr_0 \end{tikzcd}$$

These embeddings are isometric.

The basic representation of $\hat{U}_r$ is essentially the action of $\hat{U}_r$ on the holomorphic sections of Det*, although to obtain a unitary representation one must restrict this action to the ℓ^2 span of the Plucker sections defined by (8.14). If $s \in S_0$, then U_s defined in (8.13) is in $U(V)$ for some V with $V_+ = \text{span}\,\{e_j : 0 \le j < n\}$ and $V_- = \text{span}\,\{e_j : -n \le j < 0\}$, and hence the Plücker section σ_s restricts to the corresponding Plucker section of $\text{Det}^*(V_+, V)$.

This is used in [Segal, 2, appendix] to describe how the basic representation is equivalent to the action of $\hat{U}_r$ on the zero charge space of a certain spin representation.

In Chapter 10 we will show that the ℓ^2 span of the Plucker sections can also be singled out measure-theoretically as the subspace of holomorphic sections which are square integrable in a certain sense. For the construction of the measure, which is carried out in the next chapter, we will need the following facts which we have established in this chapter:

(8.16) The canonical section $\sigma(= \sigma_s$, where $s \in S_0$ is the identity) corresponds to the function f on $\hat{U}_r$ given by $f([g,q]) = \det a(g)q^{-1}$ (by (8.15)). Hence its local norm is given by

$$\rho(U) = \left|\sigma(U)\right|_U^2 = \det a(g)^* a(g),$$

where $g \cdot H_+ = U \in Gr_0$.

(8.17) Suppose $V = V_+ + V_-$, where $V_\pm \subset H_\pm$ are n-dimensional. The $U(V)$ equivariant isomorphism

$$A^n(V) \to \text{holomorphic sections of } \mathrm{Det}^*(V_+, V) \text{ defined by (8.8)}$$

maps $V = e_{-n} \wedge \dots \wedge e_{-1}$ to the restriction of σ to $\mathrm{Det}((V_+, V)$. For if $g \in SL(V)$,

$$F_V(g) = (g^{-1}e_{-n}\wedge\dots\wedge e_{-1}, \phi) = \det d(g^{-1}) = \det a(g).$$

This last equality is a general fact true for any $g \in SL(\mathbb{C}^n + \mathbb{C}^m)$ of the form $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a is $n \times n$ and d is $m \times m$. For if $*$ denotes the star operator in $A(\mathbb{C}^n + \mathbb{C}^m)$ relative to the standard basis $\{\varepsilon_j\}$, then for $g \in SU(V)$ and $\varepsilon_+ = \varepsilon_1\wedge\dots\wedge\varepsilon_n$, $\varepsilon_- = \varepsilon_{n+1}\wedge\dots\wedge\varepsilon_{n+m}$,

$$\det a(g) = \langle g\, \varepsilon_+, \varepsilon_+\rangle$$

$$= \langle g * \varepsilon_-, * \varepsilon_-\rangle$$

$$= (-1)^{nm}\overline{\langle *g*\, \varepsilon_-, \varepsilon_-\rangle}$$

$$= (-1)^{nm}\overline{\langle (*)^2 g\, \varepsilon_-, \varepsilon_-\rangle}$$

$$= \overline{\langle g\, \varepsilon_-, \varepsilon_-\rangle}$$

$$= \det d(g)^* = \det d(g^{-1}).$$

This is valid for $g \in SL(V)$ by analytic continuation. A little reflection shows that this is properly interpreted as a fact about the spherical functions of contragradient representations.

If $g \in U(V)$, then $g \cdot \sigma$ is also a section of $\mathrm{Det}^*(V_+,V)$, and it corresponds to $g \cdot e_{-n}\Lambda\ldots\Lambda e_{-1}$. Hence the global inner product formula (8.9) implies

$$c \int_{Gr(V_+,V)} \langle g \cdot \sigma,\sigma\rangle dv = \langle g\, e_{-n}\Lambda\ldots\Lambda e_{-1}, e_{-n}\Lambda\ldots\Lambda e_{-1}\rangle = \det d(g)$$

where $c^{-1} = \int_{Gr(V_+,V)} \rho\, dv = \binom{2n}{n}^{-1}$ by (8.10).

CHAPTER 9

MEASURES IN INFINITE DIMENSIONAL GRASSMANN MANIFOLD

In this chapter we construct the cylinder measure μ which will be used to write the global inner product of sections of Det* as an integral of local inner products (see the introduction to the preceding section).

A priori it appears that μ must live on (a completion of) the nonlinear space Gr_0 in (8.11). Fortunately, it turns out that μ is actually supported on (a completion of) a single coordinate patch modelled on $\mathcal{L}_2(H)$, where H is an ∞-dimensional complex Hilbert space. Granted this, μ can be described briefly as follows.

Let $e_1, e_2, \ldots$ be an orthonormal basis for H. We will write operators on H as matrices with respect to this ordered basis. Let $H_n = \text{span}\,\{e_j : j < n\}$, and view $\mathcal{L}(H_n) \hookrightarrow \mathcal{L}_2(H)$ in the obvious manner, i.e. if Z is an $n \times n$ matrix representing an operator on H_n, then

$$Z \to \left(\begin{array}{c|c} Z & \\ \hline & 0 \end{array}\right),$$

where the latter matrix represents a Hilbert-Schmidt operator on H. This embedding is orthogonal with respect to the usual (trace) inner product, and

$$\mathcal{L}(H_n) \hookrightarrow \mathcal{L}(H_N)$$

whenever $n < N$.

The measure μ will be essentially determined by a finitely additive measure, μ_F, living on $\mathcal{L}_2(H)$. In turn the finitely additive measure μ_F is essentially defined by its orthogonal projections onto the spaces $\mathcal{L}(H_n)$. If $P: \mathcal{L}_2(H) \to \mathcal{L}(H_n)$ is orthogonal projection, then $P_* \mu_F = \mu_n$, where

$$d\mu_n(Z) = c_n \det_{\mathbb{C}}(1 + Z^*Z)^{-2n-1} dm_n(Z)$$

and m_n denotes Lebesgue measure.

For this definition to make sense, it must be checked that μ_N projects to μ_n whenever $n < N$. Our proof of this relies on the fact that as a coordinate patch for Gr_0, $\mathcal{L}_2(H)$ is "almost" a homogeneous space for the restricted unitary group.

We have divided this chapter into three parts. In part I we will establish certain facts needed to establish the existence of various measures. In part II we will discuss an invariant measure on Gr_0. This measure does not seem to be of any practical importance, despite the fact its mere existence is quite a surprise. Finally, in part III we will consider the measure μ_F.

PART I

Let $H_\pm$ be ∞-dimensional complex Hilbert spaces, and $H = H_+ + H_-$. As in the previous section, Gr_0 will denote the Grassmannian of all closed subspaces W of H such that

(i) $Q_+: W \to H_+$ is Fredholm of index zero, and

(ii) $Q_-: W \to H_-$ is Hilbert-Schmidt,

where $Q_\pm$ are the self-adjoint projections for $H_\pm$, respectively. Gr_0 is a homogeneous space for the groups $(U_r)_0$ and $(GL_r)_0$ consisting of unitary and invertible operators, respectively, having the matrix form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with respect to $H = H_+ + H_-$, where a is Fredholm of index 0, and b and c are Hilbert-Schmidt.

If V is a closed subspace of H, we let $V_\pm = V \cap H_\pm$ and Q^V, $Q^V_\pm$ denote the self-adjoint projections onto V, $V_\pm$ respectively.

(9.1) Definition. A subspace V of H is admissible if

(i) V is finite dimensional,

(ii) $V = V_+ + V_-$, and

(iii) $\dim(V_+) = \dim(V_-)$.

Suppose V is admissible. We embed $GL(V)$ in $(GL_r)_0$ via the map

(9.2) $\quad g \to Q^V g\, Q^V + (1 - Q^V)$,

i.e.

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \left(\begin{array}{c|c|c|c} 1 & & & \\ \hline & a & b & \\ \hline & c & d & \\ \hline & & & 1 \end{array}\right),$$

where the former matrix is the decomposition of g with respect to $V = V_+ + V_-$, and the latter matrix is the decomposition of g with respect to $H = (H_+ - V_+) + V_+ + V_- + (H_- - V_-)$.

Let $Gr(V_+,V)$ denote the homogeneous space of $U(V)$ and $GL(V)$ consisting of all subspaces of V having the same dimension as V_+. We embed $Gr(V_+, V)$ in Gr_0 via the map

(9.3) $\quad U \rightarrow U + (H_+ - V_+)$,

i.e. if $U = g(H_+)$, $g \in GL(V)$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot V_+ \to \left(\begin{array}{c|c|c|c} 1 & & & \\ \hline & a & b & \\ \hline & c & d & \\ \hline & & & 1 \end{array}\right) \cdot H_+$$

Of fundamental importance is the fact that in graph coordinates the map (9.3) is of a very simple form. The graph coordinate (at H_+) for Gr_0 is the map $\mathcal{L}_2(H_+, H_-) \to Gr_0$ given by

$$Z \to \text{graph } (Z).$$

The graph coordinate at V_+ in $Gr(V_+, V)$ is similarly defined. In these coordinates the embedding $Gr(V_+, V) \to Gr_0$ is given by

(9.4) $Z \to Q_-^V \, Z \, Q_+^V$,

where $Z \in L\,(V_+, V_-)$. If we identify Z with the matrix $\left(\begin{array}{c|c} & \\ \hline Z & \end{array}\right)$ with respect to $V = V_+ + V_-$, then as in (9.2), the mapping $Z \to Q_-^V \, Z \, Q_+^V$ is given by

$$\left(\begin{array}{c|c} & \\ \hline Z & \end{array}\right) \to \left(\begin{array}{c|c|c|c} & & & \\ \cline{2-3} & & & \\ \hline & Z & & \\ \cline{2-3} & & & \end{array}\right).$$

Viewing Z as a matrix in this manner will simplify some later computations.

Although not of great importance here, it's worth noting that the map (9.4) is isometric in two very different senses: it's clearly an isometric embedding of the Hilbert space $\mathcal{L}(V_+, V_-)$ into the Hilbert space $L_2(H_+, H_-)$, while it's also the coordinate expression of the isometric embedding $Gr(U_+, V) \to Gr_0$, where $Gr(V_+, V)$ and Gr_0 are viewed as hermitian symmetric spaces.

Graph coordinates is the linearization of Gr_0 and $Gr(V_+, V)$ we spoke of in the introduction. The sense in which $\mathcal{L}(V_+, V)$ is almost a homogeneous space of $U(V)$ is this: the complement of graphs (over V_+) in $Gr(V_+, V)$ has measure zero. This may require explanation.

Since a homogeneous space of a locally compact group admits just one equivalence class of quasi-invariant measures with respect to absolute continuity, there is an unambiguous notion of measure zero on such a space (see [Varadarajan, 1, page 25]). For homogeneous spaces of Lie groups, measure zero is determined locally by Lebesgue measure in any coordinate system.

The standard cell decomposition of $Gr(V_+, V)$ (see [Griffiths, 1, page 194]) implies that the complement of graphs in $Gr(V_+, V)$ is a finite union of submanifolds, each of which has measure zero. Hence the complement of graphs has measure zero.

An alternate proof is to note that the graphs are the image of $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(V)\colon \det a \neq 0\}$ under the map $GL(V) \to Gr(V_+, V)$ given by $g \to g(V_+)$. This is because $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be written in the form

$$g = \begin{pmatrix} 1 & \\ Z & 1 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ & d_1 \end{pmatrix}$$

if and only if $\det a \neq 0$. The complement of this set of operators in $GL(V)$ has measure zero with respect to Lebesgue measure, which is quasi-invariant for $GL(\dot{V})$. This also implies the complement of graphs in $Gr(V_+, V)$ has measure zero.

We are now prepared to prove the key fact which we will use to show that μ_N projects to μ_n.

Let V be admissible, and let W be admissible or equal to H. Assume that $V \hookrightarrow W$, and let P be the self-adjoint projection from $\mathcal{L}(W_+, W_-)$ to $\mathcal{L}(V_+, V_-)$.

(9.5) Proposition. Suppose $g \in GL(V)$. If U and $g(U)$ are graphs over W_+, then

$$g(P(U)) = P(g(U)),$$

where in the right hand side of this equation we view g as an element in $GL(W)$, using (9.2). If W is admissible, then almost everywhere

$$g \circ P = P \circ g,$$

where we view $P: Gr(W_+, W) \dashrightarrow Gr(V_+, V)$ as a map which is defined almost everywhere.

Proof. Let $U = \text{graph}\ (Z)$, and let $z = P(Z) = Q_-^V Z\, Q_+^V$.

With respect to the decomposition $W = (W_+ - V_+) + V_+ + V_- + (W_- - V_-)$, we have

$$g = \left(\begin{array}{c|c|c|c} 1 & & & \\ \hline & a & b & \\ \hline & c & d & \\ \hline & & & 1 \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right),$$

$$Z = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{0} & 0 \\ \hline * & z & \\ * & * & 0 \end{array}\right) .$$

In graph coordinates g acts by a linear fractional map. We must show

$$(9.6)\quad P\{(C + DZ)(A + BZ)^{-1}\} = (c + dz)(a + bz)^{-1}$$

We first find $(A + BZ)^{-1}$.

$$A + BZ = \left(\begin{array}{c|c|c} 1 & & \\ \hline & a & \\ \hline & & \end{array}\right) + \left(\begin{array}{c|c|c} & & b \\ \hline & & \end{array}\right) \left(\begin{array}{c|c|c} & & \\ \hline * & z & \\ * & * & \end{array}\right)$$

$$= \left(\begin{array}{c|c|c} 0 & 0 & \\ \hline * & a + bZ & \\ \hline & & \end{array}\right)$$

$$(9.7)\qquad (A + BZ)^{-1} = \left(\begin{array}{c|c|c} 1 & & \\ \hline * & (a+bz)^{-1} & \\ \hline & & \end{array}\right) .$$

This now implies

$$C(A+BZ)^{-1} = \left(\begin{array}{cc|c} & & \\ \hline & c & \end{array}\right) \left(\begin{array}{c|c|c} 1 & & \\ * & (a+bz)^{-1} & \\ \hline & & \end{array}\right)$$

$$= \left(\begin{array}{cc|c} & & \\ \hline * & c(a+bz)^{-1} & \end{array}\right), \text{ and}$$

$$DZ(A+BZ)^{-1} = \left(\begin{array}{c|cc} & & \\ \hline & d & \\ & & 1 \end{array}\right) \left(\begin{array}{cc|c} & & \\ \hline * & Z & \\ * & * & \end{array}\right) \left(\begin{array}{c|c|c} 1 & & \\ * & (a+bz)^{-1} & \\ \hline & & \end{array}\right)$$

$$= \left(\begin{array}{cc|c} & & \\ \hline * & dz(a+bz)^{-1} & \\ * & * & \end{array}\right).$$

This proves (9.6).//

Property (iii) of an admissible subspace V, $\dim V_+ = \dim V_-$, is not necessary in the preceding proposition.

This proposition does not seem to say anything about the geometry of $Gr(V_+, V) \hookrightarrow Gr(W_+, W)$, since g is not restricted to be unitary.

PART II

To illustrate the usefulness of (9.5), we will now construct a finitely additive cylinder measure, ν_F, in $\mathcal{L}_2(H_+, H_-)$. This finitely additive measure is of intrinsic interest, although not of practical use in this paper.

If V is admissible, let ν_V denote the unique $U(V)$ invariant probability measure in $Gr(V_+, V)$. The measure ν_V is of course concentrated in the graphs, i.e. $\mathcal{L}(V_+, V_-)$.

(9.8) Proposition. If V and W are admissible, $V \hookrightarrow W$, and P: $\mathcal{L}(W_+, W_-) \to \mathcal{L}(V_+, V_-)$ the self-adjoint projection, then $P_* \ (\nu_W) = \nu_V$.

Proof. $P_* \ (\nu_W)$ is a probability measure on $Gr(V_+, V)$. It suffices to show $P_* \ (\nu_W)$ is a $U(V)$ invariant measure. But this follows immediately from (9.5) and the $U(W)$ invariance of ν_W:

$$g_*(P_*\nu_W) = P_*\nu_W) = P_*(\nu_W),$$

whenever $g \in U(V)$. //

If V is admissible, let P^V denote the self-adjoint projection $\mathcal{L}_2(H_+, H_-) \to \mathcal{L}(V_+, V_-)$.

(9.9) Definition. A subset $E \subset \mathcal{L}_2(H_+, H_-)$ is an admissible cylinder set if

$$E = (P^V)^{-1}(E_V),$$

where V is admissible and E_V is a Borel subset of $\mathcal{L}(V_+, V_-)$.

Because any two admissible subspaces are contained in a third admissible subspace, the admissible cylinder sets form a finite algebra. A standard argument shows that the admissible cylinder sets generate the σ-algebra of all Borel sets in $\mathcal{L}_2(H_+, H_-)$ (see [Kuo, 1, page 75]).

Thanks to (9.8), we can now consistently define a finitely additive measure on the admissible cylinder sets by requiring

(9.10) $P_*^V(\nu_F) = \nu_V$,

for all admissible subspaces B.

(9.11) Lemma. The finitely additive measure ν_F is invariant under the subgroup G of $(U_r)_0$ generated by operators which are either of the form $g = 1 +$ finite rank or of the matrix form $g = \begin{pmatrix} a & \\ & d \end{pmatrix}$.

Proof. Suppose $g = \begin{pmatrix} a & \\ & d \end{pmatrix}$. If V is admissible, then g maps $Gr(V_+, V)$ to $Gr(g(V)_+, g(V))$, and a transport of structure argument implies that $g_*(\nu_V) = \nu_{g(V)}$. Hence

$$P_*^V(g_*\nu_F) = g_*(P_*^{g^{-1}V}\nu_F) = g_*(\nu_{g^{-1}V}) = \nu_V,$$

implying $\nu_F = g_*\nu_F$.

Now suppose $g = 1 +$ finite rank, say $g \in U(V)$, where V is admissible. If $Z \in \mathcal{L}_2(H_+, H_-)$, then $g(\text{graph }(Z))$ is also a graph if and only if $a + bZ$ is invertible by (9.7),

$$\begin{aligned} &\{Z\colon g(\text{graph }(Z)) \text{ is not a graph}\} \\ &= (P^V)^{-1}\{z \in (V_+, V_-)\colon a + bZ \text{ is not invertible}\}, \end{aligned}$$

which is an admisible cylinder set of ν_F measure zero. This together with (9.5) implies that

$$g \circ P^W = P^W \circ g \qquad \text{a.e. } [\nu_F],$$

on $\mathcal{L}_2(H_+, H_-)$, and $P_*^W g_*\nu_F = \nu_W$, whenever W is admissible and contains V. Since any admissible cylinder set is of the form $(P^W)^{-1}(E_W)$, where W is admissible and contains V, it follows that $g_*\nu_F = \nu_F$. This proves (9.11).//

The finite algebra of admissible cylinder sets is not invariant under all of $(U_r)_0$. This is identical to a difficulty one encounters in attempting to prove that the finite Gauss measure on a real Hilbert space is translation quasi-invariant.

One way to resolve this difficulty for Gauss measure is to complete the Hilbert space with respect to a measurable norm. On such a completion the finite Gauss measure can be extended to a σ-finite measure on the σ-algebra of all Borel sets. For example if the Hilbert space is realized as $H^1{}_0[0, 1]$, the space of absolutely continuous functions vanishing at 0 and having square integrable derivative, then an appropriate completion is $C_0[0, 1]$ with the uniform norm. The completion of the Gauss measure is then Wiener measure, and Cameron and Martin explicitly showed that Wiener measure is quasi-invariant under translations by elements in $H^1{}_0[0, 1]$.

It seems very plausible to me that the appropriate completion of Gr_0 is a certain Grassmannian associated to a measurable norm completion of H. However, as of this writing, I have not yet verified this. This conjecture will be discussed further in (9.22) below.

A more efficient approach, perhaps, to the question of $(U_r)_0$ invariance is provided by the algebraic integration theory in [Segal, 1]. In this theory emphasis is placed more on the random variables and less on the measure space.

(9.12) Definition. A function ϕ on $\mathcal{L}_2(H_+, H_-)$ is an admissible tame function if $\phi = \Phi \circ P^V$, where V is an admissible subspace and Φ is a bounded Borel function on $\mathcal{L}(V_+, V_-)$.

For ϕ and V as in (9.12), let

(9.13) $E(\phi) = E_V(\phi) = \int \phi \, d\nu_V$.

By (9.8) $E(\phi)$ does not depend on the choice of V. If $\mathcal{U}$ denotes the algebra of all admissible tame functions, modulo null functions relative to E, then $(\mathcal{V}, E)$ is a (complex) integration algebra as in [Segal, 1]. $(\mathcal{V}, E)$ can also be viewed as the direct limit of integration algebras $(\mathcal{V}_V, E_V)$, where V is admissible, $\mathcal{V}_V$ is all bounded Borel functions on $\mathcal{L}(V_+, V_-)$, and E_V is the integral with respect to ν_V (see [Shale, 2]).

We now repeat the statement in [Shale, 2] of the

(9.14) Segal Representation Theorem. If $(\mathcal{V}, E)$ is an integration algebra, then there is a probability measure space $(\mathcal{M}, \mathcal{N}, \nu)$ and a representation $\phi \to \phi(x)$ of $\mathcal{V}$ as bounded measurable functions (modulo null functions) on M, such that for each ϕ in ν,

$$E(\phi) = \int \phi(x)d\nu(x).$$

Further $\mathcal{M}$ is the smallest σ-ring with respect to which the functions in $\mathcal{V}$ are measurable. Finally the measure algebra of M, that is $\mathcal{M}$ modulo the ideal of null sets, is uniquely determined by $(\mathcal{V}, E)$.

The condition that $\mathcal{M}$ is the smallest σ-ring with respect to which the functions in $\mathcal{V}$ are measurable is equivalent to a number of other conditions: $\mathcal{V}$ is weak* dense in $L^\infty(v)$, $\mathcal{V}$ is dense in $L^P(v)$, $1 \leq P < \infty$, among others (see [Segal, 1, page 433]).

The statement that the finite measure ν_F is invariant under G is equivalent to the assertion that the map $g \to g^*$, where $g^*\phi = \phi \circ g^{-1}$, is a representation of G by automorphisms of $(\mathcal{V}, E)$.

Since $\mathcal{V}$ is dense in $L^P(v)$, each g^* extends to an isometry of $L^P(\nu)$, $1 \leq P < \infty$. Moreover the map

(9.15) $g \to g^* \in \mathrm{Isom}\,(L^P(\nu))$

is a representation of G.

In chapter 1 we show that $(U_r)_0$ has the structure of a Banach Lie group.

(9.16) Lemma. With G in the relative topology, (9.15) is strong operator continuous.

Proof. It suffices to verify strong continuity on a dense set in $L^P(\nu)$. The subset of $\mathcal{V}$ consisting of functions which are of the form $\phi = \Phi \circ P^V$ (as in (9.12)), where Φ is a uniformly continuous function, is dense in $\mathcal{V}$ (and hence in $L^P(\nu)$). Fix a function ϕ in this subset.

Since Gr_0 is a hermitian symmetric space for $(U_r)_0$, there is a global Riemannian metric $d(\cdot,\cdot)$ on Gr_0 which induces the original topology and is invariant under $(U_r)_0$ (see [Klingenberg, 1, Theorem 1.9.5]). We also view $d(\cdot,\cdot)$ as a metric on $L_2(H_+, H_-)$.

Now suppose $\{g_n\}$ is a sequence in G which converges to 1. Given $\varepsilon > 0$, there is a $\delta > 0$ such that $d(Z_1, Z_2) < \delta$ implies that $|\phi(Z_1) - \phi(Z_2)| < \varepsilon$ for all $Z_j \in L_2(H_+, H_-)$. There is an N such that $n > N$ implies that $d(g_n \cdot Z, Z) < \delta$ whenever Z and $g_n \cdot Z$ are in $L_2(H_+, H_-)$. If $n > N$, then

$$\int_M |g_n^*\phi - \phi|^P d\nu = \int_{\mathcal{L}(W_+,\, W_-)} |g_n^*\phi - \phi|^P d\nu_W$$

for some admissible subspace W, since $g_n^*\phi$ and ϕ are admissible. It follows that

$$|g_n^*\phi - \phi|_{L^P} < (\operatorname*{ess\,sup}_{\mathcal{L}(W_+,\, W_-)} |g_n^*\phi - \phi|^P)^{1/P} < \varepsilon.$$

This proves that (9.15) is strong operator continuous.//

This subgroup G is actually dense in $(U_r)_0$. For (1.11) implies tht $(U_r)_0$ is the product of $\{\begin{pmatrix} a & \\ & d\end{pmatrix}\}$ and $U_2(H)$, and it's known that operators of the form 1 + finite rank are dense in $U_2(H)$ (this is clear on the Lie algebra level, and the exponential map is onto $U_2(H)$; see [de la Harpe, 1, pageII.7]). Since the map $g \to g^*$ from G into Isom $(L^P(\nu))$ is strong operator continuous, it follows that this map extends to a strong operator continuous representation of $(U_r)_0$ by isometries of $L^P(\nu)$ (we will use $g \to g^*$ to denote the extension).

(9.17) Lemma. For each $g \in (U_r)_0$, g^* restricts to an automorphism of the integration abgebra $(L^\infty(\nu), \int(\cdot)d\nu)$.

Proof. We know that for $g \in G$, g^* is an automorphism of the integration algebra $(\mathcal{V},E)$. Because multiplication of functions is continuous as a map from $L^2 \times L^2$ to L^1, it follows that

(9.18) $g^*(\phi\Psi) = (g^*\phi)(g^*\Psi)$

for ϕ, $\Psi \in L^2$ and $g \in G$. Strong continuity implies that (9.18) is also valid for $g \in (U_r)_0$. Hence for $\phi \in L^\infty$ and $g \in (U_r)_0$, $g^*\phi$ is also in L^∞ and $|g^*\phi|_\infty \leq |\phi|_\infty$, since

$$\left| \int (g^*\phi)\Psi \, d\nu \right| = \left| \int \phi \, g^{-1*}\Psi d\nu \right| \leq |\phi|_\infty |\Psi|_1$$

whenever $\Psi \in \mathcal{V}$. The fact that $g \to g^*$ is a representation now implies that each g^* actually restricts to an isometry of L^∞. It's trivial to check that the involution $\phi \to \bar{\phi}$ is preserved. This proves (9.17).//

We have proven the following

(9.18) Proposition. Let $(\mathcal{V},E)$ be the integration algebra defined in (9.13) and the ensuing paragraph. Let $(M,\mathcal{M}, \nu)$ be any probability measure space realization of $(\mathcal{V}, E)$ as in the Segal Representation theorem, (9.14). The map $G \to \mathrm{Isom}\,(L^P(\nu))$ in (9.15) extends to a strong operator continuous representation of $(U_r)_0$ by isometries of $L^P(\nu)$, for each $1 \leq P < \infty$. This map also restricts to a representation of $(U_r)_0$ by automorphisms of the integration algebra $(L^\infty(\nu), \int (\cdot)d\nu)$.

We now list a number of remarks concerning ν and related measures.

(9.19) Remark. The integration algebra $(\mathcal{V}, E)$ depends upon the distinguished point $H_+ \in Gr_0$. Given any point $U_+ \in Gr_0$, there is an integration algebra $(\mathcal{V}(U_+), E(U_+))$ associated to the graph coordinate at U_+, constructed in the same manner as $(\mathcal{V}, E)$.

Now view $(\mathcal{V}, E) \to (L^{\infty}(\nu), \int (\cdot)d\nu)$. If $g \in (U_r)_0$ satisfies $g \cdot H_+ = U_+$, then the map $g^*(\mathcal{V}, E) \to (\mathcal{V}(U_+), E(U_+))$ given by $g^*\phi \to \phi \circ g^{-1}$, where $\phi \in V$, is an isomorphism of integration algebras.

It follows that every point in $\{g^*\phi\colon g \in (U_r)_0, \phi \in \mathcal{V}\}$ can be viewed as an admissible tame function in some graph coordinate system, and moreover that $(L^{\infty}(\nu), \int (\cdot)d\nu)$ is a "global" integration algebra which does not depend upon the choice of H_+.

(9.20) Remark. It's clear that we may associate invariant integration algebras to the other components of Gr, using the same graph coordinate technique. The group U_r acts as a group of automorphisms of the direct sum (see (8.11) and (1.3) for the definitions of Gr and U_r, respectively).

(9.21) Remark. Suppose n is a positive integer. The graph coordinate technique yields an invariant integral for the Grassmannian $Gr(n, H)$ consisting of all n-dimensional subspaces of H. This invariant integral is related to one constructed in [Shale, 2], and we will describe this connection below.

To obtain the existence of the integral from our (graph coordinate) point of view, let H_+ be a fixed n-dimensional subspace, and define a subspace V of H to be admissible if

(i) V is finite dimensional, and

(ii) $V = H_+ + V_-$ $(V_- = V \cap H_-$, as before).

Proposition (9.5), the simple fact that linear fractional maps commute with certain orthogonal projections, is valid in this context, and it implies the existence of the integral for admissible tame functions (relative to the graph coordinate system at H_+).

Concerning invariance of the integral, the only changes are for the better. First, it is not necessary to restrict the unitary operators; all of $U(H)$ acts on $Gr(n, H)$. Second, the integration algebra $(\mathfrak{V}(H_+), E(H_+))$ is invariant under all of $U(H)$. This is because all of $U(H)$ is generated by operators which are either of the form 1 + finite rank or of the matrix form $\begin{pmatrix} a & \\ & d \end{pmatrix}$, relative to $H = H_+ + H_-$. Hence the integration algebra $(\mathfrak{V}(H_+), E(H_+))$ does not depend upon the choice of H_+.

In [Shale, 2] it is shown that there is an invariant integral in Isom (H_+, H), the space of isometries of H_+ into H. This space is a $U(H_+)$ principal bundle over $Gr(n, H)$, and Shale's invariant integral pushes down to ν_F, the cylinder measure our technique yielded above.

(9.22) Remark. In the case of $Gr(n, H)$, it is easy to describe an appropriate "completion" which supports a countably additive extension of ν_F. In fact we may choose $Gr(n, B)$, where B is any measurable norm completion of H.

We first note that via graph coordinates $Gr(n, B)$ is a nice analytic manifold which is modelled on $\mathcal{L}(\mathbb{C}^n, B/\mathbb{C}^n)$. Here $\mathbb{C}^n$ may be identified with any n-dimensional subspace of B, for the fact that any finite dimensional subspace of B is complemented easily implies that if $\mathbb{C}^n \cong B_j \hookrightarrow B$, then B/B_1 and B/B_2 are anlytically equivalent via a linear map.

$\mathcal{L}(\mathbb{C}^n, H)$ is a Hilbert space with measurable norm completion $\mathcal{L}(\mathbb{C}^n, B)$. Hence there is a countably additive extension of Gauss measure defined on the Borel σ-algebra of $\mathcal{L}(\mathbb{C}^n, B)$. Let $\mathcal{L}(\mathbb{C}^n, B)'$ denote the regular maps, i.e. those which are 1-1. The complement of this set has measure zero, for it is equal to

(9.23) $\bigcap_N \{L \in \mathcal{L}(\mathbb{C}^n, B) : P_N L \text{ is not 1-1}\}$, where $\{P_N\}$ is a sequence of finite rank projections which converges to 1 strongly, and each of these sets is a cylinder set of measure zero.

It is known that every unitary operator in H can be extended uniquely to a measurable transformation of B which is defined almost everywhere and preserves the extension of Gaussian measure. Hence $U(H)$ acts (almost everywhere) by composition on the space $\mathcal{L}(\mathbb{C}^n, B)$, and this action leaves invariant the subset of regular maps.

We have the commutative diagram

$$(9.24) \qquad \begin{array}{ccc} \mathcal{L}(\mathbb{C}^n, H)' & \longrightarrow & \mathcal{L}(\mathbb{C}^n, B)' \\ \downarrow{\scriptstyle \pi_H} & & \downarrow{\scriptstyle \pi_B} \\ Gr(n, H) & \longrightarrow & Gr(n, B), \end{array}$$

where the $U(H)$ equivariant projections π_H and π_B are given by

$$L \to \text{Range}\ (L).$$

If ν_G is the extension of Gaussian measure to $\mathcal{L}(\mathbb{C}^n, B)'$, then $(\pi_B)_*(\nu_G)$ is a model for the extension of ν_F.

To see this, fix the n-dimensional subspace H_+ of H, and let V be an admissible subspace as in the preceding remark, (9.22), and assume also that Q_-^V extends continuously to B. The projection

$$P: \mathcal{L}(H_+, H) \to \mathcal{L}(H_+, V_-): Z \to Q_-^V Z$$

then extends to an operator

$$P: \mathcal{L}(H_+, B_-) \to \mathcal{L}(H_+, V_-): Z \to Q_-^V Z$$

($B_- = \bar{H}_-$ in B). The complement of graph coordinates $\mathcal{L}(H_+, B) \to Gr(n, B)$ has measure zero relative to $(\pi_B)_*(\nu_G)$, since the π_B-inverse image is

$$\left\{L \in \mathcal{L}(H_+, B): L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \det\ a = 0\right\}$$

(the matrix is with respect to $B = H_+ + B_-$), and this set is seen to have ν_G measure zero by the same argument as in (9.23). It's now easily checked that the following diagram is (almost everywhere) commutative:

$$(9.25)\qquad \begin{array}{ccc} \mathcal{L}(\mathbb{C}^n, B)' & \xrightarrow{Pr} & \mathcal{L}(\mathbb{C}^n, V)' \\ \downarrow{\scriptstyle \pi_B} & & \downarrow{\scriptstyle \pi_V} \\ Gr(n, B) & \xrightarrow{P} & Gr(n, V) \end{array} ,$$

where $Pr: L \to Q^V L$. It follows that the P-projection of $(\pi_B)_*(\nu_G)$ is the invariant probability measure, since the Pr-projection of ν_G is Gaussian measure in $\mathcal{L}(\mathbb{C}^n, V)$, and the π_V-projection of Gaussian measure is the invariant probability measure.

(9.26) Remark. Let $V = V_+ + V_-$, where $\dim (V_+) = n$ and $\dim (V_-) = m$. We will indicate two ways of computing ν_V in the graph coordinate system $\mathcal{L}(V_+, V_-)$, relative to Lebesgue mesure m_V.

The first method is geometric.

Suppose $Z \in \mathcal{L}(V_+, V_-)$, and choose $g \in U(V)$ such that $g(Z) = (dZ + c)(a + bZ)^{-1} = 0$. Then

$$d\nu_V(Z) = \text{constant} \cdot \det_{\mathbb{R}}(dg|_Z)dm_V(Z),$$

where dg denotes the differential. This differential is given by

$$dg|_Z: \mathcal{L}(V_+,V_-) \to \mathcal{L}(V_+,V_-): \beta \to d\beta(bZ + a)^{-1} = d\beta a^*,$$

since $Z = b^*a^{*-1}$ and $(bZ + a)a^* = bb^* + aa^* = 1$. Using (1.1) and (1.2), one can compute that $1 + Z^*Z = (a^*a)^{-1}$ and $1 + ZZ^* = (d^*d)^{-1}$. It follows that

$$\det_{\mathbb{R}} (\beta + d\beta a^*) = \left|\det_{\mathbb{C}}(\beta + d\beta a^*)\right|^2$$

$$= \left|(\det d)^n(\det a^*)^m\right|^2$$

$$= \det (1 + Z^*Z)^{-m-n} .$$

The constant can be computed by the Wirtinger theorem (see [Griffiths, 1, page 171]). The Plucker embedding is the mapping given by

$$Gr(V_+, V) \to \mathbb{P}(A^n(V)) : U \to \mathbb{P}(u_1 \wedge \ldots \wedge u_n),$$

where $U = \text{span}\,\{u_j\}$. Let $e_1,\ldots,e_n$ be an orthonormal basis for V_+. In graph coordinates the mapping is given by

$$\mathcal{L}(V_+,V_-) \to (e_1\wedge\ldots\wedge e_n)^{\perp} : Z \to (e_1 + Ze_1)\wedge\ldots\wedge(e_n + Ze_n),$$

and the differential at V_+ is given by

$$\mathcal{L}(V_+,V_-) \to (e_1\wedge\ldots\wedge e_n)^{\perp} : \beta \to \beta e_1\wedge e_2\wedge\ldots\wedge e_n +\ldots+ e_1\wedge\ldots\wedge\beta e_n .$$

This differential is an isometry with respect to the trace norm on $\mathcal{L}(V_+,V_-)$ and the usual inner product on $A(V)$. Since the Plucker embedding is $U(V)$ equivariant, it now follows from [Griffiths, 1, pgs. 30-31] that

$$\det(1 + Z^*Z)^{-n-m} dm(Z) = ((nm)!)^{-1}(\pi\omega)^{n-m},$$

where ω is the associated (1, 1) - form for the Fubini-Study metric. The degree formula for $Gr(n, V)$ in [Stoll, 1, page 11] now implies that

$$\int \det(1 + Z^*Z)^{-n-m} dm(Z) = \int ((nm)!)^{-1}(\pi\omega)^{nm}$$

$$= ((nm)!)^{-1}\pi^{nm} \text{ degree } (Gr(n, V))$$

$$= \pi^{n \cdot m} \prod_{q=0}^{n-1} \frac{q!}{(n + m - q)!} .$$

Thus for $\phi \in L^1(\nu_V)$,

$$(9.27) \quad \int \phi \, d\nu_V = \pi^{-nm} \prod_{q=0}^{n-1} \frac{(m + n - q)!}{q!} \int \phi(Z) \det(1 + Z^*Z)^{-m-n} dm(Z).$$

The second method of computing ν_V uses the fact that ν_V is the projection of Gaussian measure.

Recall from (9.25) that we have the $U(V)$ equivariant projection

$$\pi_V: \mathcal{L}(\mathbb{C}^n, V)' \to Gr(n, V): L \to \text{Range } (L).$$

In graph coordinates π_V is given by

$$\mathcal{L}(V_+, V)' \to \mathcal{L}(V_+, V_-): L \to Z = \gamma\, \alpha^{-1},$$

where $L = \begin{pmatrix} \alpha & 0 \\ \gamma & 0 \end{pmatrix}$ with respect to $V = V_+ + V_-$.

Thus for $\phi \in L^1(\nu_V)$,

$$\int \phi(Z)d\nu_V(Z) = \pi^{-n(m+n)} \int \phi(\gamma\, \alpha^{-1})e^{-\mathrm{tr}L^*L}dm(L).$$

To express this in terms of Z, we note that $L = \alpha + Z\alpha$ and that the real Jacobian of the mapping $(Z, \alpha) \to \alpha + Z\alpha$ is $\det(\alpha^*\alpha)^m$. It follows that

(9.28)

$$\int\phi(Z)d\nu_V(Z) = \pi^{-n(n+m)}\int\phi(Z)\left\{\int e^{-\frac{1}{2}\mathrm{tr}\alpha^*\alpha}\, e^{-\frac{1}{2}\mathrm{tr}\alpha^*Z^*Z\alpha}\det(\alpha^*\alpha)^m dm(\alpha)\right\}db(Z)$$

$$= \pi^{-n(n+m)}\int\phi(Z)\left\{\int e^{-\mathrm{tr}(1+Z^*Z)\alpha\alpha^*}\det(\alpha^*\alpha)^m dm(\alpha)\right\}dm(Z)$$

$$= \pi^{-n(n+m)}\int\phi(Z)\left\{e^{-\mathrm{tr}\alpha\alpha^*}\det((1+Z^*Z)^{-1}\alpha^*\alpha)^m dm((1+Z^*Z)^{-1/2}\alpha)\right\}dm(Z)$$

$$= \pi^{-n(n+m)}\left\{\int e^{-\mathrm{tr}\alpha^*\alpha}\det(\alpha^*\alpha)^m dm(\alpha)\right\}\int\phi(Z)\det(1+Z^*Z)^{-m-n}dm(Z)\ .$$

We now follow the presentation in [Mehta, 1, sec. 12.1] to compute the "moment" $\pi^{-n^2}\int e^{-\mathrm{tr}\alpha^*\alpha}\det(\alpha^*\alpha)^m dm(\alpha)$. In this calculation α is parameterized by its spectrum $\{z_i\}$ and other auxiliary parameters. Since $\det(\alpha^*\alpha)^m$ does not depend upon these auxiliary parameters, it follows from the discussion in [Mehta, 1, sec. 12.1] that

$$\pi^{-n^2}\int e^{-\mathrm{tr}\,\alpha^*\alpha}\det(\alpha^*\alpha)^m dm(\alpha) = K\int \Pi|z_i|^{2m} e^{-\Sigma|z_i|^2} \prod_{i<j} |z_i - z_j|^2 \Pi dm(z_i)$$

$$= Kn!\int \Pi|z_i|^{2m} e^{-\Sigma|z_i|^2} \left[\det\left(\bar{z}_i^{\,i-1} z_i^{\,j-1}\right)_{1\leq i,j\leq n}\right] \Pi dm(z_i)$$

$$= Kn!\ \det\left(\int |z_i|^{2m}\, \bar{z}_i^{\,i-1} z_i^{\,j-1} e^{-|z_i|^2} dm(z_i)\right)_{1\leq i,j\leq n}$$

$$= Kn!\ \det\left(\delta_{ij}\pi(m+j-1)!\right)_{1\leq i,j\leq n}$$

$$= Kn!\ \prod_{j=1}^{n} \pi(m+j-1)!$$

Here $K^{-1} = K^{-1}{}_c = \pi^n \prod_{j=1}^{n} j!$

Plugging this into (9.28), we again obtain (9.27).

PART III

We now begin the description of the measure μ_F. We will use the notation established in the first part of this chapter (in particular, H_+ is fixed - the measure μ_F depends on the choice of H_+).

Define a function ρ on Gr_0 by

(9.29) $\quad \rho(g \cdot H_+) = \det aa^*$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon (U_r)_0$. If $U = g \cdot H_+$

is a graph, say $U = \text{graph}(Z)$, then (1.1) and (1.2) can be used to show $\rho(U) = \det(1 + Z^*Z)^{-1}$. If U is not a graph (relative to H_+), then $\rho(U) = 0$.

(9.30) Lemma. Suppose U is an admissible subspace and $U \in Gr_0$ is a graph. Then

$$\frac{\rho(g \cdot U)}{\rho(U)} = \frac{\rho(g \cdot P^V \cdot U)}{\rho(P^V \cdot U)}$$

for all $g \in U(V)$.

Proof. Let $g = \left(\begin{array}{c|c|c|c} 1 & & & \\ \hline & a & b & \\ \hline & c & d & \\ \hline & & & 1 \end{array}\right) = \left(\begin{array}{c|c} A(g) & B(g) \\ \hline C(g) & D(g) \end{array}\right)$

as in the proof of (9.5). Choose $h \in (U_r)_0$ so $U = h \cdot H_+ = \text{graph}(Z)$, where $Z = C(h)A(h)^{-1}$. By continuity of ρ, we can supose that Z is finite rank. Then

$$\frac{\rho(g \cdot U)}{\rho(U)} = \frac{\det A(gh)A(gh)^*}{\det A(h)A(h)^*} = \left|\det A(gh)A(h)^{-1}\right|^2,$$

and $A(gh)A(h)^{-1} = A(g) + B(g)C(h)A(h)^{-1} = A(g) + B(g)Z$.

Let $P^V Z = z$, so that $Z = \left(\begin{array}{cc|c} & & \\ \hline * & z & \\ \hline * & * & \end{array}\right)$. Then

$$A(g) + B(g)Z = \left(\begin{array}{c|c|c} 1 & & \\ \hline & a & \\ \hline & & \end{array}\right) + \left(\begin{array}{c|c|c} & b & \\ \hline & & \end{array}\right)\left(\begin{array}{cc|c} & & \\ \hline * & z & \\ \hline * & * & \end{array}\right)$$

$$= \left(\begin{array}{c|c|c} 1 & 0 & \\ \hline * & a + bz & \\ \hline & & \end{array}\right).$$

This implies that

$$\left|\det\ (A(g) + B(g)Z)\right|^2 = \left|\det\ (a(g) + b(g)z)\right|^2.$$

A repeat of the first part of this proof, with h above replaced by an $h \in U(V)$ satisfying $h \cdot H_+ = \text{graph}\ (z)$, shows that the right hand side of this last equation is $\rho(g \cdot \text{graph}\ (z))/\rho(\text{graph}\ (z))$. This proves (9.30).//

Recall that ν_V is the unique probability measure on $Gr(V_+, V)$ which is invariant with respect to the action of $U(V)$. Let μ_V be the probability measure obtained by normalizing $\rho \cdot \nu_V$.

(9.31) Proposition. Let V and W be admissible subspaces, $V \hookrightarrow W$, and $P: \mathcal{L}(W_+, W_-) \to \mathcal{L}(V_+, V_-)$ the orthogonal projection.

Then $P_*(\mu_W) = \mu_V$.

Proof. Supose $g \varepsilon U(V)$. Then from (9.5) and (9.30), it follows that

$$g_*(P_*\mu_W) = P_*(g_*\mu_W) = P_*\{(g^*\rho)\nu_W\}$$

$$= \frac{P^*g^*\rho}{P^*\rho} P_*(\mu_W) = \frac{g^*\rho}{\rho} P_*(\mu_W),$$

i.e. $P_*(\mu_W)$ is a quasi-invariant measure having the same Radon-Nikodym derivative (for every $g \varepsilon U(V)$) as μ_V. Using the fact that $P_*(\mu_W)$ and μ_V are mutually absolutely continuous, this is easily seen to imply (9.31).//

We can now consistently define a finitely additive measure on the admissible cylinder sets by requiring that

$$P_*^V(\mu_F) = \mu_V,$$

for all admissible subspaces V.

Our development of the quasi-invariance properties of μ_F (and its completion) will parallel our development of the invariance properties of ν_F.

First, for $g \varepsilon (U_r)_0$ and $U \varepsilon Gr_0$, define

(9.32) $\qquad \rho(g, U) = \dfrac{\rho(g^{-1}U)}{\rho(U)}$. For $g \varepsilon U(V)$, where V is an admissible subspace, the restriction of $\rho(g, \cdot)$ to $Gr(V_+, V)$ is the

Radon-Nikodym derivative of $g_*(\mu_V)$ relative to μ_V. For g of the matrix form $g = \begin{pmatrix} a & \\ & d \end{pmatrix}$, $\rho(g, \cdot) \equiv 1$.

Since the finitely additive measures ν_F and μ_F have the same (admissible cylinder) sets of measure zero, it's easy to modify the argument in (9.11) to prove

(9.33) Lemma. The finitely additive measure μ_F is quasi-invariant under G. The Radon-Nikodym derivative of $g_*\mu_F$ relative to μ_F is $\rho(g, \cdot)$.

Form the integration algebra $(\mathcal{V}, E) = (\mathcal{V}, E_\mu)$, i.e. $\mathcal{V}$ is the algebra of all admissible tame functions (relative to H_+), and $E = E_\mu$ is the expectation relative to μ_F. Let $(M, \mathcal{M}, \mu)$ be any probability space realiation of $(\mathcal{V}, E)$ as in the Segal representation theorem, (9.14).

Suppose $g \in U(V)$, where V is an admissible subspace. By (9.30), $\rho(g, \cdot) = \rho(g, P^V(\cdot))$ on graph subspaces. This does not quite imply that $\rho(g, \cdot)$ is in $\mathcal{V}$, since $\rho(g, \cdot)$ is not bounded. However, since the projection P^V can clearly be represented as a measurable function on $(M, \mathcal{M}, \mu)$, $\rho(g, \cdot)$ can be represented as a measurable function. This is also the case for any $g \in G$, since we can always write $g = g_1 g_2$, where $g_1 = \begin{pmatrix} a & \\ & d \end{pmatrix}$, $g_2 \in U(V)$, for some admissible V, and $\rho(g, \cdot) = \rho(g_2, \cdot)$.

Let $g \to g^*$ denote the representation of G as automorphisms of $\mathcal{V}$ defined by $g^*\phi = \phi \circ g^{-1}$. In the terminology of [Segal, 1, pages 449-450], it follows from (9.33) that $g \to g^*$ is a representation of G by absolutely continuous transformations of $(\mathfrak{V},E)$, or, equivalently, that $g \to \pi_p(g)$, $1 < p < \infty$, where

(9.34) $$\pi_p(g)(\phi) = g^*\phi\rho(g, \cdot)^{1/p}, \ \phi \in V,$$

gives rise to a representation of G by isometries of $L^P(\mu)$.

(9.35) Lemma. $g \to \pi_p(g)$ is a strong operator continuous representation of G.

Proof. As in (9.16), it suffices to prove strong continuity on $\phi \in \mathfrak{V}$ of the form $\phi = \Phi \bullet P^V$, where Φ is uniformly continuous.

Suppose $g_n \to 1$ in G. For fixed n, there is an admissible W such that

$$\int_M |\pi_\rho(g_n)\phi - \phi|^P d\mu = \int |g_n^*\phi\ \rho(g_n, \cdot)^{\frac{1}{P}} - \phi|^P d\mu_W$$

$$\leq \int |g_n^*\phi|^P|\rho(g_n, \cdot)^{\frac{1}{P}} - 1|^P d\mu_W + \int |g_n^*\phi - \phi|^P d\mu_W$$

$$\leq |\phi|_\infty^P|\pi_p(g_n)\cdot 1-1|_p^P + \operatorname*{ess\,sup}_{(W_+,\ W_-)} |g_n^*\phi - \phi|^P$$

In (9.16) we showed that the second term tends to zero. Hence we can assume $\phi \equiv 1$.

We can also assume $P = 2$. For if this case is established, it then follows that $\rho(g_n, \cdot)^{\frac{1}{P}} \to 1$ almost everywhere $[\mu]$ and that $|\rho(g_n, \cdot)^{1/2}|^2 \to 1$ is a sequence of integrable functions which converge a.e. to an integrable function ($\equiv 2$). Since $|\rho(g_n, \cdot)^{1/p} - 1|^p < \rho(g_n, \cdot) + 1$, we can apply a well-known generalization of the Lebesgue convergence theorem (see [Royden, 1, pg. 89]) to conclude that $\pi_\rho(g_n) \cdot 1 \to 1$ in $L^p(\mu)$.

Our proof in the case $p = 2$ depends upon chapter 1 and the line bundle constructions of the preceding section.

First we factor g_n as in (1.11), $g_n = g'_n \cdot q_n$, where $g'_n = 1 +$ finite rank, $g'_n \to 1$ in $(U_r)_*$ (in particular, $d(g'_n) = 1 + T_n$, where $T_n \to 0$ in trace norm), and g_n is of the form $\left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$. Since $\pi_2(g_n) \cdot 1 = \pi_2(g'_n) \cdot 1$, we can assume $g_n = g'_n$.

If we fix n, then

$$\int |\rho(g_n, \cdot)^{1/2} - 1|^2 d\mu = \int |(\frac{g^*\rho}{\rho})^{1/2} - 1|^2 d\mu_W$$

$$= C_W \int |(g^*_n\rho)^{1/2} - \rho^{1/2}|^2 d\nu_W,$$

where C_W is the normalizing constant for the measure $\rho d\nu_W$, i.e. $C_W = \int \rho d\nu_W$.

It follows from (8.16) that

$$\rho(U) = |\sigma(U)|_U^2$$

where σ denotes the canonical section of the Det* bundle. Hence

$$(g^*\rho)^{1/2}(U) = |\sigma(g^{-1}(U)|_{g^{-1}U}$$

$$= \left|(g \cdot \sigma)(U)\right|_U,$$

where $g \cdot \sigma$ denotes the action of $[g, 1]$ on σ. It follows that

$$\left|(g*_n\rho)^{1/2} - \rho^{1/2}\right|^2 = \left|\, |(g_n \cdot \sigma)(U)| - |\sigma(U)|\,\right|^2$$

$$\leq \left|(g_n \cdot \sigma)(U) - \sigma(U)\right|^2.$$

To show $C_W \int |g_n \cdot \sigma - \sigma| d\nu_W \to 0$, it suffices to show $C_W \int \langle g_n \cdot \sigma, \sigma\rangle d\nu_W \to 1$. But by (8.17) $C_W \int \langle g_n \cdot \sigma, \sigma\rangle d\nu_W = \det d(g_n)$, and this does converge to 1.//

Strong continuity of π_p implies that π_p extends to a strong operator continuous representation of $(U_r)_0$ by isometries of $L^p(\mu)$, $1 \leq p < \infty$.

For all $g \in G$, $\rho(g, \cdot) = \pi_1(g)(1)$ is an $L^1(\mu)$ function. By taking a strong limit we can interpret $\rho(g, \cdot)$ as an element of $L^1(\mu)$ for all $g \in (U_r)_0$. For $\phi \in L^1(\mu)$ and $g \in (U_r)_0$ we can define $g*\phi$ to be the measurable function satisfying

(9.36) $\pi_1(g)(\phi) = g*\phi\ \rho(g, \cdot)$ (a.e.$[\mu]$). Obvious modifications of the proof of (9.17) now yield

(9.37) Lemma. The map from g to the restriction of $g*$ to $L^\infty(\mu)$ defines a representation of $(U_r)_0$ by absolutely continuous transformations of the integration algebra $(L^\infty(\mu), \int (\cdot)d\mu)$.

CHAPTER 10

In this section we will show that the measure μ constructed in the previous section can be used to construct the inner product for the basic representation of $\hat{U}_r$ on holomorphic sections of Det*. The general method applies to many other highest weight representations, but we will confine our discussion here to this single example.

We first expose a certain technical problem. Suppose σ is a holomorphic section of Det*. According to the introduction of Chapter 9, we should define the global L^2 norm of σ by

$$(\sigma,\sigma) = \int \frac{(\sigma,\sigma)_U}{\rho(U)} \, d\mu(U), \tag{10.1}$$

where this may be infinite. The problem with this definition is that the function $\frac{(\sigma,\sigma)_U}{\rho(U)}$ is defined on Gr_0, while the measure lives on a certain "completion", M, of Gr_0. Before we can integrate we must establish that our function can be extended in a reasonable fashion to a measurable function on M.

As of this writing, I do not know whether an arbitrary holomorphic section of Det*, or even one with (σ,σ) bounded, can be reasonably extended.

In the case of Det* we know which sections should be "L^2" - the ℓ^2 linear combinations of Plucker sections. We will show that (10.1) makes good sense for this space of sections.

To accomplish this, we need a slight refinement of (9.30). Recall that the weight function ρ is given by

$$\rho(U) = (\sigma_0, \sigma_0)_U,$$

where σ_0 is the canonical section of Det* corresponding to the function on $\hat{U}_r$ given by $f_0([h,q]) = \det a(h)q^{-1}$. Suppose that $g \in U(V)$, where V is admissible, and identify g with $[g,1] \in \hat{U}_r$.

(10.2) Lemma. The ratio $\dfrac{f_0(g\ \cdot)}{f_0(\cdot)}$ is a function on Gr_0, and as a function of the graph coordinate $Z \in L_2(H_+, H_-)$, it depends only on $z = P^V Z$.

Proof. By the proof of (9.30), if $U = h \cdot H_+ = \text{graph}\ (Z)$, then

$$\frac{f_0(g[h,q])}{f_0([h,q])} = \frac{\det A(gh)q^{-1}}{\det A(h)q^{-1}}$$

$$= \det A(g) + B(g)Z$$

$$= \det(a + bZ).$$

This proves (10.2).//

Fix the orthonormal basis $\{e_j\}$ for H as in the paragraph preceding (9.13). If $s_1, s_2 \in S_0$, then U_{S_1} and U_{S_2} defined by

(9.13) will be in $U(V)$ for some admissible V. By the preceding lemma, the function

$$\frac{(\sigma_{s_1}, \sigma_{s_2})}{\rho} = \left(\frac{f_{s_1}}{f_0}\right)\left(\frac{\overline{f_{s_2}}}{f_0}\right)$$

is a function of $P^V Z$, i.e. it is an admissible tame function. Hence it can be extended to a measurable function on M in a canonical fashion, and we have

$$\int_M \frac{(\sigma_{s_1}, \sigma_{s_2})}{\rho}\, d\mu = \int_{Gr(U_+, V)} \frac{(\sigma_{s_1}, \sigma_{s_2})}{\rho}\, d\mu_V$$

$$= C_V \int (\sigma_{s_1}, \sigma_{s_2}) dv_V$$

$$= \delta_{s_1, s_2} .$$

REFERENCES

Boyer, R. P., Representation Theory of the Hilbert Lie group $U(H)_2$, Duke J. Math., 47 (1980), 325-344.

Brattelli, O. and Robinson, D. W., Operator algebras and quantum statistical mechanics II, Springer, NY-Heidelberg-Berlin, 1981.

Carey, A. L. and O'Brien, D. M., Automorphisms of the infinite dimensional Clifford algebra and the Atiyah-Singer mod 2 index, Topology (1983).

Date et al, Transformation groups for soliton equations-Euclidean Lie algebras and reduction of the KP hierarchy, Publ. RIMS Kyoto University, 18 (1982) 1077-1110.

de la Harpe, P., Classical Banach-Lie algebras and Banach-Lie groups of operators in Hilbert space, Springer Lecture Notes, 285 (1972).

Gohberg, I. C. and Krein, M. G., Introduction to the theory of linear nonselfadjoint operators, AMS translations of Mathematical monographs, Vol. 18 (1969).

Griffiths, P. and Harris, J., Principles of algebraic geometry, Wiley-Interscience, 1978.

Klingenberg, W., Riemannian Geometry, Walter de Gruyter, Berlin-New York, 1982.

Kuo, H., Gaussian measures in Banach spaces, Springer Lecture Notes in Mathematics, 463 (1975).

Mehta, M. L., Random Matrices, Academic Press, 1967.

Plymen, Some recent results on infinite-dimensional spin representations, studies in Algebra and Number Theory, Advances in Mathematics Supplementary Series, Vol. 6 (1979), pg. 159.

_____ and Streater, R. F., A model of the universal covering group $SO(H)_2$, Bull. London Math. Soc. 7 (1975), 283-288.

Pukanszky, Unitary Representations of Lie groups with compact radical and applications, Trans. Amer. Math. Soc., 236 (1978), 1-49.

Royden, H. L., Real Analysis, MacMillan Publishing Co., New York, 1968.

Segal, G. B., Unitary representations of some infinite dimensional groups, Commun. Math. Physics, 80 (1981), 301-362.

_____ and Wilson, G., Loop groups and equations of KdV type, preprint.

Segal, I. E., Algebraic integration theory, Bull. Amer. Math Soc. 71 (1965), 419-489.

Shale, D., and Stinespring, W., Spinor Representations of infinite orthogonal groups, J. Math. Mech. 14 (1965), 315-322.

Shale, D., Invariant integration over the infinite dimensional orthogonal group and related spaces, Trans. Amer. Math Soc. 124 (1966), 148-157.

Stoll, W., Invariant forms on Grassmann manifolds, Annals of Mathematics studies, no. 69, Princeton University Press.

Varadarajan, V. S., Geometry of quantum theory II, Van Nostrand, 1970.

_____, Lie groups, Lie algebras and their representations, Prentice-Hall, 1974.

Wallach, N., Harmonic Analysis on homogeneous spaces, Marcell-Dekker, New York, 1973.