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ASSUMED STRESS FUNCTION
FINITE ELEMENT METHOD

by
Nesrin Sarıgül

A Dissertation Submitted to the Faculty of the
CIVIL ENGINEERING AND ENGINEERING MECHANICS DEPARTMENT
In Partial Fulfillment of the Requirements
For the Degree of
DOCTOR OF PHILOSOPHY
WITH A MAJOR IN ENGINEERING MECHANICS
In the Graduate College
THE UNIVERSITY OF ARIZONA

1984
As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Nesrin Sarigül
entitled Assumed Stress Function

and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

Ralph McCune

Date

Rusu R. Simon

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Date

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Date

Richard Atkinson

Date

Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

Richard Golightly
Dissertation Director

Date
STATEMENT BY AUTHOR

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SIGNED: Nesrin Sarigül
To my beloved parents

Nazmi and Muazzez (Günağ) Sarıgül
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<td>A</td>
<td>Area</td>
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<td>a</td>
<td>Dimension</td>
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<td>D</td>
<td>Plate flexural rigidity</td>
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<tr>
<td>d.o.f.</td>
<td>Degree of freedom</td>
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<td>E</td>
<td>Elastic modulus</td>
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<td>[E]</td>
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<td>[F]</td>
<td>Global flexibility matrix</td>
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<td>[f]</td>
<td>Element flexibility matrix</td>
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<td>Constraint matrix</td>
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<td>L</td>
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<td>lx,ly,lz</td>
<td>Direction cosines</td>
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<td>m</td>
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<td>[N], N</td>
<td>Matrix of shape functions</td>
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<td>n</td>
<td>Number of degrees of freedom</td>
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<tr>
<td>[O],{O}</td>
<td>Null matrix and vector</td>
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<td>p</td>
<td>Number of elements</td>
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<td>Ω</td>
<td>Closed, bounded and simply connected domain</td>
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<td>̅Ω</td>
<td>Closure of the domain Ω, ̅Ω(=Ω∪Ω')</td>
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<td>∂Ω_t, ∂Ω_u</td>
<td>Complementary surface of ∂Ω where tractions and displacements are prescribed, respectively</td>
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<td>Vector of constants in constraint equations</td>
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\( t \)  Thickness.
\( T_x, T_y, T_z \)  Surface (boundary) tractions in \( x, y, z \) directions.
\( U, U^* \)  Strain energy and complementary strain energy.
\( u, v, w \)  Displacement components.
\( V, V^* \)  Potential and complementary potential of applied loads.
\( \text{vol} \)  Volume.
\( d(\text{vol}), dA \)  Elements of volume and area.
\( \{x\}, x, y, z \)  Vector of body forces, and components.
\( x, y, z \)  Cartesian coordinates.
\( [I] \)  Transformation matrix.
\( \{\Delta\} \)  Vector of nodal point displacements.
\( \delta \)  Variational operator; infinitesimal change.
\( \chi \)  General strain vector (normal and shear strain)
\( \{\lambda\}, \lambda_x, \lambda_y, \lambda_z \)  Vector of curvatures in plate bending, and components.
\( \{\lambda\} \)  Vector of Lagrange multipliers.
\( \nu \)  Poisson's ratio.
\( \Pi_c \)  Complementary energy functional.
\( \Pi_p \)  Potential energy functional.
\( \sigma \)  General stress field vector (normal and shear stress).
\( \sigma_x, \sigma_y, \sigma_z \)  Normal stresses.
LIST OF SYMBOLS—Continued

\{ \sigma \} Vector of node point stresses.
\tau_{xy}, \tau_{yz}, \tau_{xz} Shear stresses.
\bar{\phi} Stress function.
\{ \phi \} Vector of node point stress function parameters.
F_x x-direction resultant force per unit thickness.
F_y (-) y-direction resultant force per unit thickness.
M_s counterclockwise direction moment of \( F_x \) and \( F_y \) forces.

ARRAY SYMBOLS

[ ] A rectangular matrix.
\{ \} A column vector.
\[ \] A row vector.
\sim A column or row vector.
ABSTRACT

A finite element formulation, based on assumed stress functions, is developed for the linear elastic analysis of the stresses in two-dimensional elasticity problems, including multiply-connected regions and flat plate bending. For planar analysis the Airy stress function is utilized. The physical significance of the Airy stress function and its normal derivatives are brought out. A new technique is introduced to account for traction type boundary conditions. A family of rectangular finite elements, which enables the direct insertion of stress type boundary conditions, and two higher-order rectangular elements which enable continuous stress variations along the interelement boundaries are constructed using blending function interpolants. In addition, a $C^0$ continuous triangular plate bending element is adapted for use as a plate stretching element. The Southwell stress function is employed for the analysis of flat plates in bending. A computer program is developed to substantiate the proposed methodology. The formulations are evaluated through the comparison of solutions obtained from the
proposed method with classical solutions and solutions obtained from the assumed displacement finite element method. The elements are evaluated by solving the same example problem with different element types. Extensions of the proposed method to account for body forces, initial stresses, material nonlinearities, and shells are briefly discussed. It is demonstrated that the proposed method can directly be integrated with minimal modifications into existing general purpose finite element programs.
CHAPTER 1

INTRODUCTION

1.1. Purpose of Study

In practical applications of the finite element method the displacement based (stiffness or potential energy) formulation has been used exclusively. A simple and clear statement of the formulation inhibited the development of element formulations and applications using the alternative conventional variational principle, the complementary energy principle.

Formulations via complementary energy are valuable on many counts. One aspect is the possibility of establishing upper bounds on certain solution parameters [126]. In addition, with this approach a better approximation is obtained on stress values, which are of major concern to design engineers. It appears that the complementary energy approach is useful in plane stress inelastic analysis [61,128]. In limit analysis, the convenience of the approach is demonstrated in Reference [133]. The inclusion of transverse shear deformation effects can be accomplished easily for plate bending elements if the complementary energy approach is used [107].
In utilizing complementary energy, different approaches are employed. One approach is to form the element flexibilities and to obtain the corresponding element stiffnesses by direct inversion. Subsequently, the analysis is performed as in the direct stiffness method [8]. This approach suffers from the difficulties in obtaining a kinematically stable stiffness model. Another approach is to introduce stress functions for the analysis. This may result in a dual to direct stiffness model. By invoking the duality, general purpose stiffness programs may be rendered applicable to analysis with the use of complementary energy principle. Additionally, the elements formulated for the stiffness formulation may be adapted, with minor alterations, for this approach.

Although several contributions based on the use of the stress functions [9,50,52,61,127-131] have appeared the method is not yet stated with sufficient clarity to attract the finite element practitioner's attention. Another problem encountered is in the handling of traction type boundary conditions.

The purpose of this investigation is to bring clarity to the overall assumed stress function based formulation, to alleviate the difficulties in handling
the traction type boundary conditions by introducing special boundary traction elements, and to develop a new methodology to ease the implementation for applications.

1.2. A Review of the Past Work

The following is a survey of developments of the finite element equilibrium model (i.e., force or stress parameters are the primary unknowns). In the early developments of the finite element method, the desirability of finding bounds to the exact solution led to widespread usage of the force method, together with the displacement method [126]. The force method was used to obtain stresses in such applications as aircraft fuselage and wing analysis, owing to its more accurate stress predictions [48,59]. In addition, the usefulness of flexibility coefficients in vibration analysis is well known [132].

Formulations using the force method suffered from difficulties in selection of the redundant forces, even though many investigators have attempted to automate the selection process [47]. At about the same time, the simple and straightforward direct stiffness method was advanced, which instantly found universal acceptance. The disappearance of the force method from practical applications may be attributed, in part, to
the cumbersome selection of redundant forces, and, in part, to the specific conditions required to be able to obtain bounds on the solution. The research interest in the force method dwindled to a standstill until the interest was revived by the use of stress function concept.

The analogy between the plate stretching and plate bending was advanced in the 1940s [111-114], and an interesting paper describing stress functions for a general continua and containing a bibliography of the stress functions was published by Truesdell [38] in 1960. This concept, however, was not utilized in the finite element context until the paper by Fraeijs de Veubeke and Zienkiewicz [54] appeared in 1967. Following this paper, several investigators adopted stress functions in equilibrium model finite element analysis [9,50,52,61,127-131].

In plate stretching problems Watwood and Hartz [50] gave solutions for linear elastic analysis, while Rybicki and Schmit [128], and Gallagher and Dhalla [61] demonstrated its use for the elasto-plastic analysis. Recently, Harvey [92] utilized the stress function concept and provided a solution to the wedge problem. In this work the compatibility condition was also satisfied in addition to the equilibrium equations. Therefore it
may be appropriate to consider this approach as a Mixed Formulation. Vallabhan and Azene [93] extended the work of Gallagher and Dhall [61] with two additional elements.

One common point in all of the above papers is in the treatment of boundary conditions by construction of constraint conditions which were implemented via the use of Lagrange multipliers. The constraint conditions resulting from the prescribed boundary tractions were expressed as algebraic equations. The total number of equations is increased by use of Lagrange multipliers considerably. There appears to be no rules for constructing the constraint equations nor a rationale in deciding the total number of constraint equations.

Lawther and Kabaila used [20] a single domain approach to solve two dimensional problems.

In plate bending via the application of the principle of the complementary energy Fraeijs de Veubeke [8] derived the element flexibilities and inverted them to obtain element stiffness matrices. Subsequently, the direct stiffness analysis was utilized in this paper. The main drawback of this approach is the difficulty in obtaining a kinematically stable stiffness matrix. Morley [130,131], using quadratic variation of the stress function, and Elias [129], using linear variation
for the stress function, solved the resultant flexibility equations directly. The triangular elements were used in the analysis.

The work of Anderheggen [107] approached the problem by taking both the stresses and displacements as unknowns.

The complementary energy formulation for three dimensional analysis in terms of stress functions was investigated by Charlwood [127]. The stress functions in three dimensions are related to stress values by second derivatives, similar to two dimensional elasticity. In reference [127], the required number of degrees of freedoms for an interelement compatible tetrahedron is given as 246. It may be observed that the reason for the scarce attempts for research in the three dimensional analysis may be attributed to the availability of a comparable tetrahedron element of 12 degrees of freedom in the displacement model [31].

Some extensions of the method are explored by Tabarrok and Sodhi [71] for dynamic analysis, and by Sundararajan [17] for stability analysis.
1.3. Substance of Investigation

In this section an outline of the dissertation is given. Chapter 2 is devoted to the variational principle of the problem, that of minimum complementary energy. Details are given for two-dimensional elasticity and plate bending cases.

The boundary conditions are first examined. The prescribed traction type boundary conditions are given close scrutiny in following section. A new concept is introduced which utilizes the physical significance of stress function. The conditions for multiply-connected regions are described.

Chapter 3 defines the assumed stress function finite element method for two dimensional elasticity. The elements formulated for the two dimensional analysis are detailed. A boundary traction element family is constructed including variable degrees of freedom eight rectangular elements. Two higher-order rectangular elements are also formulated for continuous stresses along the interelement boundaries. A triangular C^0 continuous plate bending element is adopted.

The rectangular elements are constructed using the blending interpolation technique. The section on element formulation, therefore, starts with a brief
outline of basic concepts in the blending function method, followed by detailed developments of the element construction.

The triangular element adapted is based on discrete Kirchhoff concepts for bending elements. The practical aspects of its implementation are discussed at length.

The interchangeability of elements between the plate bending stiffness formulation and plate stretching assumed stress function method is furnished. Certain important new points are brought out.

The assumed stress function finite element method becomes attractive in plate bending problems. Therefore Chapter 4 extends the method for plate bending problems utilizing the Southwell stress functions.

A computer program is developed to perform numerical analyses. This program is outlined in the first section of Chapter 5. The numerical examples are detailed in the following section. Comparisons are made with the classical solution and alternative finite element formulations. The convergence characteristics of the elements are demonstrated by a sequence of gridwork refinements. Computations are carried out on a 32-bit minicomputer, a DG Eclipse MV/10000.
Conclusions pertaining to the current research is presented in Chapter 6. Some possible extensions of the proposed method are given in the last section of Chapter 6.
CHAPTER 2

THE PRINCIPLE OF MINIMUM COMPLEMENTARY ENERGY

2.1. Basic Theory

The principle of minimum complementary energy is utilized for the element flexibility equations. The complementary energy of a structure $\Pi_c$ is given by the sum of the complementary strain energy $U^*$ and the potential of the boundary forces $V^*$ acting through prescribed displacements. Thus,

$$\Pi_c = U^* + V^*$$  \hspace{1cm} (2.1)

For an elastic structure the complementary strain energy is given by

$$U^* = \frac{1}{2} \int_{\Omega} \xi^T [E]^{-1} \xi \, d\Omega$$  \hspace{1cm} (2.2)

where $[E]$ is the matrix of elastic constants, $\xi$ and $\xi = [E]^{-1} \xi$ are the stress and strain fields respectively.
The potential of the boundary forces acting through the prescribed displacements and the potential of the forces $F_i$ acting on the prescribed point displacements $\Delta i$ are,

$$v^* = - \int_{\partial \Omega_u} \bar{u} \times d(\partial \Omega_u) - \sum_{i=1}^{n} \Delta F_i$$  \hspace{1cm} (2.3)$$

Usually there are no prescribed displacements within the body, therefore the integral $\int_{\partial \Omega} \bar{u} \times d\Omega$ is not taken into account in equation (2.3).

The second term in the right side of equation (2.3) is a special case of the first term. Hence, the potential of the forces acting through the prescribed displacements will be expressed as follows:

$$v^* = - \int_{\partial \Omega_u} \bar{u} \times d(\partial \Omega_u)$$  \hspace{1cm} (2.3a)$$

where $\partial \Omega_u$ is the portion of the boundary surface upon which the displacements $\bar{u}$ are prescribed and $\bar{T}$ are the associated tractions (Figure 2.1).
The principle of stationary complementary energy can be stated as: among all the stresses which satisfy the equilibrium equations in the interior of the domain and are in equilibrium with the prescribed tractions on the boundary surface of the domain, those that result in a compatible state of kinematically admissible deformation make the complementary energy assume a
For the stationary value the first variation of the complementary energy functional vanishes. 

\[ \delta \Pi_c = \delta (U^* + V^*) = 0 \]

Using the distributive property of \( \delta \) we get 

\[ \delta \Pi_c = \delta U^* + \delta V^* = 0 \]

or introducing equations (2.2) and (2.3a) into the above equation,

\[ \delta \Pi_c = \int_\Omega \left[ E^1 \delta \sigma \dot{\psi} - \int_{\partial \Omega} \left( \delta \tau \psi \psi \right) \right] \alpha \sigma \, d\Omega = 0 \]  

By examining the second variation of the functional in equation (2.1), we get 

\[ \delta^2 \Pi_c = \delta^2 U^* + \delta^2 V^* = \delta^2 U^* > 0 \]

The inequality prevails because the complementary strain energy is a positive quantity. Thus, the complementary energy attains a minimum value corresponding to a stable equilibrium state.
We seek the state of stress that satisfies the equations of equilibrium, and the prescribed tractions at the boundary

\[ \overline{T} = T \quad \text{on} \quad \partial \Omega_T \quad (2.6) \]

and also minimizes the complementary energy functional \( \Pi_c \):

\[ \delta \Pi_c = \oint_\Omega \left[ \sigma \nabla \delta \epsilon \right] \cdot \delta \epsilon \, d\Omega - \oint_{\partial \Omega_u} \sigma \cdot \delta T \, d(\partial \Omega_u) = 0 \]

An explanation on \( \partial \Omega_u \) and \( \partial \Omega_T \) is necessary at this point. \( \partial \Omega_u \) is the portion of the boundary surface upon which the displacements are prescribed, and \( \partial \Omega_T \) is the portion that the tractions are prescribed.

In addition,

\[ \partial \Omega_T \subseteq \partial \Omega \]

\[ \partial \Omega_u \subseteq \partial \Omega \quad (2.7) \]

meaning that \( \partial \Omega_u \) and \( \partial \Omega_T \) are subsets of the domain \( \partial \Omega \). It is important to note that either \( \partial \Omega_u = \partial \Omega \) or \( \partial \Omega_T = \partial \Omega \) is allowed. The first one corresponds to the case where there are only prescribed displacement type
boundary conditions; the latter case exists when only the prescribed surface traction type boundary conditions are present.

2.1.1. Two-dimensional Elasticity

The complementary energy of a two-dimensional elastic system, for plane stress and plane strain, is (Figure 2.2)

\[ \Pi_c = \frac{1}{2} \int_\Omega \sigma^T \varepsilon \, dA - \int_{S_u} \mathbf{T} \cdot \mathbf{u} \, dS \]  

(2.8)

where \( \sigma \) and \( \varepsilon \) are the stress and strain vectors, \( \mathbf{T} \) and \( \mathbf{u} \) are the tractions and the prescribed displacements, of the form

\[ \varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} T_x \\ T_y \end{bmatrix} \]

**Plane Stress.** The approximation made in thin plates which results with the state of plane stress is that normal and shear stresses \( \sigma_z, \tau_{zx}, \tau_{zy} \) are zero throughout the thickness.
Figure 2.2. Notation in Two-dimensional Elasticity.
The equations of equilibrium, then, take the form

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0
\]

\[
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0
\]

(2.9)

and, tractions on the boundary surface are

\[
\overline{T}_x = l_x \sigma_y + l_y \tau_{xy}
\]

\[
\overline{T}_y = l_y \sigma_y + l_x \tau_{xy} \quad \text{on } \partial \Omega_T
\]

(2.10)

where \(l_x\) and \(l_y\) are the direction cosines of the outer normal to the boundary curve, \(\overline{T}_x\) and \(\overline{T}_y\) are the \(x\) and \(y\)-direction prescribed surface tractions (Figure 2.2).

It should be noted that equations (2.9) can be satisfied identically through introduction of Airy stress function, defined as

\[
\sigma_x = \frac{\partial^2 \Phi}{\partial y^2} \quad \sigma_y = \frac{\partial^2 \Phi}{\partial x^2} \quad \tau_{xy} = \frac{\partial^2 \Phi}{\partial x \partial y}
\]

(2.11)

The prescribed tractions and the stress values are related by the following transformation matrix
The strain component in z-direction is calculated from

\[ \varepsilon_z = \frac{\nu}{1-\nu} (\varepsilon_x + \varepsilon_y) \]

and, the material stiffness matrix [E] and the material flexibility matrix \([E]^{-1}\), for an isotropic material, are

\[
[E] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad [E]^{-1} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}
\]

**Plane Strain.** If the thickness dimension z of a body is large as compared to the dimensions in the xy plane, if it is subjected to a uniform loading and has a uniform cross-section in the z-direction, plane strain assumptions become valid. The strains \( \varepsilon_z, \gamma_{xz}, \gamma_{zy} \)
are zero. The z-direction normal stress is not zero, and is calculated from

$$\sigma_z = \nu (\sigma_x + \sigma_y) \quad (2.14)$$

The equations of equilibrium are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0$$

$$\frac{\partial \sigma_z}{\partial z} = 0 \quad \text{in } \Omega \quad (2.15)$$

and, at the boundary

$$\bar{T}_x = \ell_x \sigma_x + \ell_y \tau_{xy}$$

$$\bar{T}_y = \ell_y \sigma_y + \ell_x \tau_{xy}$$

$$\bar{T}_z = \ell_z \sigma_z$$

on $\partial \Omega_T \quad (2.16)$

The material stiffness and flexibility matrices are

$$[E] = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix}
1 - \nu & \nu & 0 \\
\nu & 1 - \nu & 0 \\
0 & 0 & \frac{1 - 2\nu}{2}
\end{bmatrix}$$

$$[E]^{-1} = \frac{1 - \nu}{E} \begin{bmatrix}
1 & \frac{\nu}{1 - \nu} & 0 \\
-\frac{\nu}{1 - \nu} & 1 & 0 \\
0 & 0 & \frac{2}{1 - \nu}
\end{bmatrix} \quad (2.17)$$
2.2.2. Plate Bending

The geometry and loading of a thin plate of thickness $t$ are given in Figure 2.3. A state of plane stress exists in the plate, and the variation of these stresses across the thickness is assumed to be linear.

The equilibrium equations of a thin plate in bending are

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0$$

$$\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} - Q_y = 0$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \quad (2.18)$$

where $q$ is the transverse distributed load, $Q_x$ and $Q_y$ are the transverse shear loads per unit length in the $x$ and $y$-directions, respectively. $M_x$, $M_y$ and $M_{xy}$ are the direct and twisting moments per unit length. These moment values are obtained by integration of the stress values across the plate thickness. The tractions on the boundary surface are
\[ \bar{M}_n = M_n \]
\[ \bar{M}_nt = M_{nt} \]
\[ \bar{Q}_n = Q_n \]
\[ \text{on } \partial \Omega_T \]

where \( \bar{Q}_n, \bar{M}_n, \) and \( \bar{M}_nt \) are the specified edge line loads, normal and twisting moments.

The normal and twisting moments are related to \( x \) and \( y \) direction moment values by the following expressions

\[ M_n = M_x \cos^2 \alpha + M_y \sin^2 \alpha - 2M_{xy} \sin \alpha \cos \alpha \]
\[ M_{nt} = M_{xy}(\cos^2 \alpha - \sin^2 \alpha) + (M_x - M_y) \sin \alpha \cos \alpha \]
\[ Q_n = Q_x \cos \alpha + Q_y \sin \alpha \]

The Kirchhoff normal force is given by

\[ V_n = Q_n + \frac{\partial M_{nt}}{\partial t} \]

or

\[ V_n = Q_x \cos \alpha + Q_y \sin \alpha - \frac{\partial}{\partial t}[M_{xy}(\cos^2 \alpha - \sin^2 \alpha) + (M_x - M_y) \sin \alpha \cos \alpha] \]

The prescribed normal moment and the equivalent Kirchhoff shear load at the boundary
Figure 2.3. Thin Plate Bending: Loading and geometry.
\[ \bar{M}_n = M_n \quad , \quad \bar{\nu}_n = \nu_n \]

In addition to the conditions above, the concentrated load resulting from the twisting moment \( M \) must be considered. Considering the finite element discretization the sum of such concentrated forces arising from the elements with the common node \( i \) give a resultant force at that node. It is necessary to ensure that the total forces at that node is equal to the applied load, i.e.,

\[ \sum_{j=1}^{e} \bar{R}_i^j = \bar{R}_i \]

where \( e \) is the number of elements joining at node \( i \).

The complementary energy expression is

\[ \Pi_c = \frac{1}{2} \int_{A} [M][E]^{-1}{\{M\}} \, dA - \int_{\Omega_U} q_w \, d\Omega_U \]

\[ - \int_{S_U} (Q_n \bar{w} + M_n \bar{\theta}_n + M_{\tau} \bar{\phi}_{\tau}) \, ds_U \]

(2.19)

where the bars over the quantities denote prescribed values, \( \bar{w} \) and \( \bar{\theta}_n \). \( \bar{\phi}_{\tau} \) are the prescribed transverse and angular displacements, respectively. \( Q_n \)
is edge line load and $M_n$ is the normal moment, and $M_{nt}$ is the twisting moment acting along the prescribed displacements. For isotropic thin plates $[E_f]$ is

$$[E_f] = \frac{E_t^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$ (2.20)

2.2. Boundary Conditions

2.2.1. General Concepts

The prescribed quantities at the boundary of a region, in general, may be in terms of both displacements and tractions. The prescribed displacements are easy to handle since they contribute only to the potential of the applied loads $V^*$. In structural engineering applications, the most commonly encountered prescribed value is zero. This eliminates the term $V^*$ from the complementary energy statement. The traction type boundary conditions require special attention. Therefore, the following section is devoted to these conditions.
2.2.2. A New Concept on Traction Boundary Conditions

Consider the boundary portion $s_1, s_2$ of a simply-connected region subjected to prescribed surface tractions $\overline{T}_x$ and $\overline{T}_y$, (Figure 2.4). In order to examine the stress state of a point on the boundary, say point $P$, consider a triangular differential element $PAB$. Since the triangle is small, body forces and the variation of stresses along the sides can be neglected in the equilibrium equations.

The equilibrium equations of the differential triangle give, for unit thickness

$$\overline{T}_x ds = \sigma_x dy - \tau_{xy} dx$$

$$\overline{T}_y ds = \tau_{xy} dy - \sigma_y dx$$

(2.21)

The body forces are taken as zero. The equilibrium equations and the Airy stress function $\Phi$ are for the plane stress

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

in vol

(2.9)
Figure 2.4. Boundary Surface under Tractions $T_x$ and $T_y$. Two-dimensional Elasticity.
It will be recalled, from equation 2.10 that these equations are identically satisfied if

$$\left\{ \begin{array}{c} \frac{\partial \Phi}{\partial y} \\ \frac{\partial \Phi}{\partial x} \\ -\frac{\partial^2 \Phi}{\partial x \partial y} \end{array} \right\}$$

(2.22)

By substitution of equation (2.22) into equation (2.21) there results:

$$\tau_x ds = \frac{\partial^2 \Phi}{\partial y^2} dy + \frac{\partial^2 \Phi}{\partial x \partial y} dx$$
$$\tau_y ds = -\frac{\partial^2 \Phi}{\partial x \partial y} dy - \frac{\partial^2 \Phi}{\partial x^2} dx$$

(2.21)

Now taking the differentiation of \( \frac{\partial \Phi}{\partial y} \), under the assumption that \( \frac{\partial^2 \Phi}{\partial y^2} \), \( \frac{\partial^2 \Phi}{\partial x \partial y} \) are continuous

$$d\left( \frac{\partial \Phi}{\partial y} \right) = \frac{\partial^2 \Phi}{\partial y^2} dy + \frac{\partial^2 \Phi}{\partial x \partial y} dx$$

(2.23a)

Similarly, the differential of \( \frac{\partial \Phi}{\partial x} \) gives

$$d\left( \frac{\partial \Phi}{\partial x} \right) = \frac{\partial^2 \Phi}{\partial x \partial y} dy + \frac{\partial^2 \Phi}{\partial x^2} dx$$

(2.23b)
Comparison of equation (2.21) with (2.23a-2.23b) results in
\[ d \left( \frac{\partial \Phi}{\partial y} \right) = \tau_x ds \]
\[ d \left( \frac{\partial \Phi}{\partial x} \right) = -\tau_y ds \] (2.24)

Integrating equation (2.24) over the boundary portion \( S_{S_2} \) yields
\[ \int_{S_1}^{S_2} d \left( \frac{\partial \Phi}{\partial y} \right) = \int_{S_1}^{S_2} \tau_x ds \]
\[ \int_{S_1}^{S_2} d \left( \frac{\partial \Phi}{\partial x} \right) = -\int_{S_1}^{S_2} \tau_y ds \]
(2.25)

The differential of \( \Phi \) gives
\[ d \Phi = \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy \] (2.26)

and integrating equation (2.22) in the \( S_1 S_2 \) interval
\[ \Phi \bigg|_{S_1}^{S_2} = \int_{S_1}^{S_2} \left[ \frac{\partial \Phi}{\partial x} \right] dx + \int_{S_1}^{S_2} \left[ \frac{\partial \Phi}{\partial y} \right] dy + C_3 \] (2.27)

Integration by parts yields
\[ \int_{S_1}^{S_2} \left[ \frac{\partial \Phi}{\partial x} \right] dx = (x \frac{\partial \Phi}{\partial x})_{S_2} - (x \frac{\partial \Phi}{\partial x})_{S_1} + \int_{S_1}^{S_2} x \tau y ds \] (2.28)
Substitution of equations (2.28) and (2.29) into equation (2.27) gives

\[ 
\int_{\tilde{s}_1}^{\tilde{s}_2} \left[ \frac{\partial \phi}{\partial y} \right] dy = (y \frac{\partial \phi}{\partial y})_{\tilde{s}_2} - (y \frac{\partial \phi}{\partial y})_{\tilde{s}_1} - \int_{\tilde{s}_1}^{\tilde{s}_2} y \tilde{T} dx ds 
\]

(2.29)

Hence, three equations are obtained (equations 2.25 and 2.30), which relate the \( \frac{\partial \phi}{\partial x} \) and \( \frac{\partial \phi}{\partial y} \) to \( \tilde{T}_x \) and \( \tilde{T}_y \).

Examination of these equations shows that \( \frac{\partial \phi}{\partial y} \) and \( \frac{\partial \phi}{\partial x} \) are the \( y \) and \( x \) components of the resultant force on the \( S_1, S_2 \) part of the boundary. In order to simplify them consider the case when \( s_1 = 0 \) and \( s_2 = s \).

Then,

\[ 
\left( \frac{\partial \phi}{\partial y} \right)_s = \int_0^s \tilde{T}_x ds + C_1 \quad (2.31a) 
\]

\[ 
\left( \frac{\partial \phi}{\partial x} \right)_s = - \int_0^s \tilde{T}_y ds + C_2 \quad (2.31b) 
\]
\[(\Phi)_s = \int_0^s \left( \frac{\partial \Phi}{\partial x} \right) dx + \int_0^s \left( \frac{\partial \Phi}{\partial y} \right) dy + C_3 \] (2.31c)

The three integration constants \((C_1, C_2, C_3)\) must be assigned their values prior to the solution in order to avoid a singular flexibility matrix. For simply-connected regions zero values can be assigned to the integration constants \((C_1 = C_2 = C_3 = 0)\), since there is no contribution of these to the stress values. In this case

\[\frac{\partial \Phi}{\partial y} \bigg|_{s_1}^{s_2} = \int_{s_1}^{s_2} T_x ds \] (2.32a)

\[\frac{\partial \Phi}{\partial x} \bigg|_{s_1}^{s_2} = -\int_{s_1}^{s_2} T_y ds \] (2.32b)

\[\Phi \bigg|_{s_1}^{s_2} = -\int_{s_1}^{s_2} [(x_{s_2} - x_{s_1}) - x] T_y ds + \int_{s_1}^{s_2} [(y_{s_2} - y_{s_1}) - y] T_x ds \] (2.32c)

Using basic statics to compute \(x\) and \(y\) direction components of resultant forces, for unit thickness, acting in the \(s_1, s_2\) interval and the moments, in counterclockwise direction, of these forces about \(s_2\), we get
\[ \begin{align*}
\text{Fx} &= \int_{s_1}^{s_2} T_x \, ds \quad (2.33a) \\
\text{Fy} &= \int_{s_1}^{s_2} T_y \, ds \quad (2.33b) \\
\text{Ms} &= \int_{s_1}^{s_2} [(y_{s_2} - y_{s_1}) + y] T_x \, ds - \int_{s_1}^{s_2} [(x_{s_2} - x_{s_1}) - x] T_y \, ds \quad (2.33c)
\end{align*} \]

By comparison of equations (2.32) and (2.33), we conclude

\[ \begin{align*}
\frac{\partial \Phi}{\partial y} \bigg|_{s_1}^{s_2} &= \text{Fx} \quad (2.35a) \\
\frac{\partial \Phi}{\partial x} \bigg|_{s_1}^{s_2} &= -\text{Fy} \quad (2.35b) \\
\Phi \bigg|_{s_1}^{s_2} &= \text{Ms} \quad (2.35c)
\end{align*} \]

In Figure 2.5, the physical significance of the Airy stress function is given for simply-connected regions. It may be noted that if \( \Phi, \frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y} \) are known the stress field is uniquely defined. Equations (2.35) are considered in the solution of the system of equations, which is obtained by writing the first variation of the complementary energy functional. These conditions are imposed directly into the equations, details of which are given in Chapter 3.
Further, by using the differential of $\frac{\partial \phi}{\partial n}$, it is concluded that two conditions $\phi$ and $\frac{\partial \phi}{\partial n}$ (either $\frac{\partial \phi}{\partial x}$ or $\frac{\partial \phi}{\partial y}$) are enough for the solution. Hence, the linearly independent constants are identified.

The differential of $\frac{\partial \phi}{\partial n}$ gives (n is the outer normal to the boundary curve)

$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial x} \frac{dx}{dn} + \frac{\partial \phi}{\partial y} \frac{dy}{dn} \quad (2.36)$$

On examination of the above equation, we obtain

$$\frac{\partial \phi}{\partial n} \bigg|_{s_1} = -F_y \sin \alpha - F_x \cos \alpha$$

$$= - (F \cos \alpha + F \sin \alpha) = -F_t$$

where $F_t$ is the resultant force in the tangential direction per unit thickness. Hence,

$$\frac{\partial \phi}{\partial n} \bigg|_{s_1} = -F_t \quad (2.37)$$

This verifies that if $\phi$ and $\frac{\partial \phi}{\partial n}$ are known the stress field is uniquely defined.
Figure 2.5. Physical significance of Airy Stress Function in Simply-connected region.
Multiply-connected Regions. The integration constants in equations (2.31) can not be assigned arbitrarily in multiply-connected regions (Figure 2.6). There are three constants of integration for each closed boundary. These constants may be assigned arbitrarily for only one of the closed boundaries. Denoting the closed boundaries by \( B_1, B_2, \ldots, B_m \), the required number of constants \( c \) is

\[
c = 3(m-1) \quad (2.38)
\]

Figure 2.6. Multiply-connected Region.

where \( m \) is the number of closed boundaries in the region.
The compatibility condition in plane stress, in terms of Airy stress function \( \Phi \), is given by the biharmonic equation

\[
\frac{\partial^4 \Phi}{\partial x^4} + 2 \frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = 0
\]  \hspace{1cm} (2.39)

or using the Laplacian operator

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\]  \hspace{1cm} (2.40)

we get

\[
\nabla^2 \nabla^2 \Phi = \nabla^4 \Phi = 0
\]

For multiply-connected regions the compatibility conditions are alone not enough to ensure the single valuedness of stresses and strains. It is also necessary to invoke the conditions on displacements and rotations for every closed boundary,

\[
\oint \partial u = 0
\]

\[
\oint \partial v = 0
\]

\[
\oint \partial w = 0
\]  \hspace{1cm} (2.41)
where \( u, v \) are the displacements in the \( x \) and \( y \)-direction and \( \omega \) is the rotation about the \( z \)-direction.

It may be noted that the additional conditions on \( \bar{Q} \) are given by Mitchell [111] by assuming the strains to be continuous and the rotations and the displacements to be single valued. Mindlin [99] gave these conditions for prescribed discontinuities in rotations and displacements. Three conditions (equations 2.41) are used to obtain the constants of integration, on each closed boundary.

2.2.3. Lagrange Constraints

The prescribed boundary stresses are examined from a different point of view than that described in section 2.2. It may be noted that Lagrange constraints were utilized in References [61,93]. Consider a portion of boundary \( x_i x_j \), where stresses \( \bar{\sigma}_y \) and \( \bar{\tau}_{xy} \) are prescribed, (Figure 2.7).

By integration of equations (2.11), which are the relations between the Airy stress function and the stress values, over \( x_i x_j \) interval and by evaluation of the integrations at the limits there results

\[
\int_{x_i}^{x_j} \bar{\sigma}_y \, dx = \frac{\partial \Phi}{\partial x} \bigg|_j - \frac{\partial \Phi}{\partial x} \bigg|_i \quad (2.42a)
\]
Figure 2.7. Planar element under prescribed stresses.

\[
\int \int_{x_i}^{x_j} \tau_{xy} \, dx = \phi_j - \phi_i - x_j \frac{\partial \phi}{\partial x_i} + x_i \frac{\partial \phi}{\partial x_j} \quad (2.42b)
\]

\[
\int_{x_i}^{x_j} \bar{\sigma}_{xy} \, dx = \frac{\partial \phi}{\partial y_i} - \frac{\partial \phi}{\partial y_j} \quad (2.42c)
\]

Now, consider an element with degrees of freedoms at the corner points only, and with prescribed
boundary stresses $\sigma_x$, $\sigma_y$ and $\tau_{xy}$ along its edges, (Figure 2.7). The above equations then take the following forms for different edges of the element,

side 1-2

$$\int_0^a \sigma_y \, dx = \phi_x - \phi_x$$

$$\int_0^a \sigma_y \, dx = \phi_x - \phi_1 - a \phi_x$$

$$\int_0^a \tau_{xy} \, dx = \phi_y - \phi_y$$  \hspace{1cm} (2.43)

side 2-3

$$\int_0^b \sigma_x \, dy = \phi_y - \phi_y$$

$$\int_0^b \sigma_x \, dy = \phi_y - \phi_y - b \phi_y$$

$$\int_0^b \tau_{xy} \, dy = \phi_x - \phi_x$$  \hspace{1cm} (2.44)

side 4-3

$$\int_0^a \sigma_y \, dx = \phi_x - \phi_x$$
\[
\int \int_{x} \bar{t}_{y} \, dx = \phi_{3} - \phi_{4} - a \phi_{x4}
\]

\[
\int_{0}^{a} \tau_{xy} \, dx = \phi_{y4} - \phi_{y3}
\]  \hspace{1cm} (2.45)

Side 1-4

\[
\int_{0}^{b} \bar{t}_{x} \, dy = \phi_{y4} - \phi_{y1}
\]

\[
\int_{0}^{b} \bar{t}_{x} \, dy = \phi_{4} - \phi_{1} - b \phi_{y1}
\]

\[
\int_{0}^{b} \tau_{xy} \, dy = \phi_{x1} - \phi_{x4}
\]  \hspace{1cm} (2.46)

On examination of equations (2.43) and (2.45) there results, using matrix notation,

for edges in the x-direction (1-2 and 4-3)

\[
\begin{bmatrix}
-1 & -a & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
\phi_{i} \\
\phi_{x_i} \\
\phi_{y_i} \\
\phi_{j} \\
\phi_{x_j} \\
\phi_{y_j}
\end{bmatrix}
= 
\begin{bmatrix}
\int_{0}^{a} \bar{t}_{x} \, dx \\
\int_{0}^{a} \tau_{xy} \, dx
\end{bmatrix}
or

\[
[C]_a \{ \phi \}_x = \{ s \}_x
\]  

(2.47)

where \( i \) and \( j \) denote the nodal points, and \( a \) is the size of the element in the \( x \)-direction.

For the \( y \)-direction, \( b \) will be replaced by \( a \), and \( \bar{\phi}_y \) will be replaced by \( \bar{\phi}_x \) and

\[
[C]_b \{ \phi \}_y = \{ s \}_y
\]

(2.48)

where \([C]_a\) and \([C]_b\) are the "boundary line stiffnesses".

The edges of an element which is at the boundary is referred to as "boundary line".

For the structure the equations above may be written as

\[
[C] \{ \phi \} = \{ s \}
\]

(2.49)

It may be noted that there are three times as many equations as boundary lines.
These conditions must be taken into account in the process of minimizing the complementary energy. One of the methods is the Lagrange multipliers technique.

There are additional constraints to be considered if the element has higher order derivatives as degrees of freedom. These are directly assigned to the nodal values by using the relations between the Airy stress function and the stress values, equations (2.11).
CHAPTER 3

ASSUMED STRESS FUNCTION FINITE ELEMENT METHOD;
PLANE STRESS AND PLANE STRAIN

3.1. General Concepts

The stress function concept is now utilized. The
equilibrium equations (2.9) are automatically satisfied
by the introduction of a stress function, which in the
planar case (equation 2.11) is the "Airy Stress
Function", $\Phi$, defined by

$$\begin{align*}
\sigma_x &= \frac{\partial^2 \Phi}{\partial y^2} \\
\sigma_y &= \frac{\partial^2 \Phi}{\partial x^2} \\
\gamma_{xy} &= -\frac{\partial^2 \Phi}{\partial x \partial y}
\end{align*}$$

(3.1)

or, in vector form

$$\begin{pmatrix}
\sigma_x \\
\sigma_y \\
\gamma_{xy}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial^2 \Phi}{\partial y^2} \\
\frac{\partial^2 \Phi}{\partial x^2} \\
-\frac{\partial^2 \Phi}{\partial x \partial y}
\end{pmatrix} \Phi \quad \text{or} \quad \mathbf{\tau} = \mathcal{L} \Phi$$

(3.2)

where $\mathcal{L}$ is the differentiation vector operator defined by
From equation (2.8) with use of equation (3.3), we get

\[ L = \begin{pmatrix} \frac{\partial^2}{\partial y^2} \\ \frac{\partial^2}{\partial x^2} \\ -\frac{\partial^2}{\partial x \partial y} \end{pmatrix} \quad \text{and} \quad L^T = L \begin{pmatrix} \frac{\partial^2}{\partial y^2} \\ \frac{\partial^2}{\partial x^2} \\ -\frac{\partial^2}{\partial x \partial y} \end{pmatrix} \]

(3.3)

By substitution of equation (3.2) into equation (2.11), there results

\[ \pi_c = \frac{1}{2} \int_\text{vol} L^T \Phi \left[ E \right]^{-1} L \Phi \, \text{d} \text{vol} - \int_\text{su} \left( \frac{\partial}{\partial \xi} \right) \text{d} S_u \]

(3.4)

Thus equation (3.4) becomes

\[ \left[ \Gamma_{\tau\sigma} \right] \Phi \]

(3.5)

Thus equation (3.4) becomes

\[ \pi_c = \frac{1}{2} \int_\text{vol} L^T \Phi \left[ E \right]^{-1} L \Phi \, \text{d} \text{vol} - \int_\text{su} \left[ \Gamma_{\tau\sigma} \right] L \Phi \, \text{d} S_u \]

(3.6)

The complementary energy of an element \( \pi_c^e \) is given by
where $A$ is the area of the element, $t$ is thickness, and $C_u$ is the boundary curve of the element on which the displacements are prescribed. It may be noted that $C_T$ is the boundary curve on which the tractions are prescribed.

The finite element discretization is based on stress function field $\mathbf{\Phi}$ within the element expressed as

$$\mathbf{\Phi} \simeq [N] \{\phi\} \quad (3.8)$$

where $[N]$ is the shape functions vector and $\{\phi\}$ is the vector of nodal values of stress function parameters.

If we define the matrix $[G]$ as

$$[G] = [L] [N]$$

we have

$$\mathbf{\Sigma} = [G] \{\phi\} \quad (3.9)$$
By substituting equations (3.8) and (3.9) into equation (3.7) we get

\[ \Pi^e_c = \frac{1}{2} \int_{A} [G]^T \Phi \Phi^T [E]^{-1} [G] \{ \phi \} \tau dA - \int_{C_u} \{ \Gamma \}^T [G]^T \Phi \Phi^T \{ U \} \tau dCu \]

or

\[ \Pi^e_c = \frac{L_{\phi_1}}{2} \left[ f \right] \{ \phi \} - L_{\phi_1} \{ \bar{A} \} \]

(3.10)

(3.11)

(3.12)

(3.13)

where \([f]\) is the element "generalized" flexibility matrix. The term "generalized" is introduced since the \(\{ \phi \} \) vector contains a combination of stress function values and their derivatives. \(\{ \bar{A} \} \) is the element prescribed displacement vector.

The complementary energy in discrete form is

\[ \Pi_c = \sum_{i=1}^{P} \Pi^e_{c_i} \]

(3.14)
where \( p \) is the total number of elements.

Equation (3.14) is valid if the conditions on the field variable (\( \Phi \)) is met. These conditions require the continuity of \( \Phi \) and \( \Phi_n \) at the interelement boundaries for the unique evaluation of the complementary energy functional. The continuity of the normal derivative is also required since equation (3.6) includes the second derivatives of the stress function. Therefore, \( C^1 \) continuous elements need to be used in order to preserve the upper bound to complementary strain energy.

The complementary energy of the structure, using equation (3.14) is

\[
\Pi_c = \sum_{i=1}^{p} \frac{L \Phi_{i,j}}{2} [f_i] \{\phi_i\} - L \phi_{j} \{\bar{z}\}
\]

or

\[
\Pi_c = \frac{L \phi_{j}}{2} [F] \{\phi\} - L \phi_{j} \{\bar{z}\} \tag{3.15}
\]

where \([F]\) is the "global generalized flexibility matrix", and \( \{\phi\} \) is the vector of stress function parameters for the structure.
Equation (2.6) must be taken into account in the solution process. The conventional way of dealing with these conditions is to write a set of constraint conditions of the form as given in the last section of Chapter 2,

\[ [C][\phi] = \{s\} \]

(2.49)

These conditions can be enforced by use of Lagrange multipliers. The augmented complementary energy functional \( \Pi_{c_A} \) then becomes

\[
\Pi_{c_A} = \frac{1}{2} [F] \{ \phi \} - [\phi] [A] + \lambda [C] \{ \phi \} - \lambda [s] \]

The differentiation with respect to variables \( \{\phi\} \) and \( \{\lambda\} \) gives

\[
\frac{\partial \Pi_{c_A}}{\partial \{\phi\}} = 0 \quad \frac{\partial \Pi_{c_A}}{\partial \{\lambda\}} = 0
\]

which results in the following equations

\[
[F] \{\phi\} - \{A\} - \lambda [C] = 0
\]
After rearranging the terms, we get

\[
\begin{bmatrix}
F & C \\
C & 0
\end{bmatrix}
\begin{bmatrix}
0 \\
\lambda
\end{bmatrix} = 
\begin{bmatrix}
\Delta \\
\kappa
\end{bmatrix} \tag{3.17}
\]

This approach is classified as "a Mixed Solution" in this report, since the unknown vector contains both the stress function parameters and the Lagrange multipliers which, because they are conjugate to the stress function parameters, are "displacement" parameters.

It may be noted that there are several drawbacks in the Mixed Solution procedure. The number of equations to be solved increases considerably. There are no guidelines either for writing the constraint equations or for deciding the total number of constraint equations.

In this study an approach is proposed which keeps the formulation consistent throughout the analysis, resulting in only the stress function parameters as unknowns. This is done by identifying the
derivatives of the stress functions on the boundary with the prescribed tractions. The mathematical basis of this development is given in section 2.2.2 of Chapter 2. The new approach, "Assumed Stress Function Finite Element Method" (ASFFEM), utilizes the complementary energy functional without augmenting it. Therefore, in general, it gives

$$\frac{\partial \mathcal{K}_c}{\partial \{\phi\}} = 0$$

or

$$[F]\{\phi\} = \{\tilde{\alpha}\}$$

(3.18)

with

$$\{\phi\} = \begin{bmatrix} \phi_u \\ \phi_t \end{bmatrix} \quad \{\tilde{\alpha}\} = \begin{bmatrix} \tilde{\alpha} \\ \tilde{\alpha}_t \end{bmatrix}$$

$$[F] = \begin{bmatrix} F_{uu} & F_{ut} \\ F_{tu} & F_{tt} \end{bmatrix}$$

where \{\phi_t\} are the stress function parameters at the nodes where the tractions are prescribed. Eliminating \{\phi_t\} from equation (3.19) we get
\[ [Fuu]\{\phi u\} = - [Fut]\{\phi t\} + \{\bar{\alpha}\} \quad (3.20) \]

or denoting \{t\} as

\[ \{t\} = - [Fut]\{\phi t\} \]

we obtain

\[ [Fuu]\{\phi u\} = \{\bar{\alpha}\} + \{t\} \]

In most practical applications pertaining to structural analysis, the prescribed displacements are zero. When this is the case, equation (3.20) becomes

\[ [Fuu]\{\phi u\} = \{t\} \quad (3.21) \]

From this avenue it seems possible to obtain generalized displacements directly at the joints where the surface tractions are prescribed. It is observed from the numerical studies that the generalized displacements are zero wherever the actual displacements are zero.
3.2. Formulation of Elements

The stress function field for a finite element in plate stretching must be continuous in both function and its normal derivatives (C\(^1\) continuity) across the element boundaries, for a valid complementary energy formulation. This requirement may be observed from the complementary energy functional, equation (3.4), which involves the second derivatives of Airy stress function.

On the other hand, by definition of Airy stress function, continuous stresses across the interelement boundaries are obtained by requiring the second derivatives of the stress function be continuous. With this formulation the number of degrees of freedom increase considerably. This approach is utilized to construct higher order rectangular elements.

Another approach is to construct elements to satisfy the stress type boundary conditions exactly. These elements require to have continuous second derivatives only along the boundaries where the stresses are prescribed. With this formulation elements may be obtained with an optimal number of degrees of freedom. A family of boundary traction element is constructed utilizing this approach.
The elements formulated in the present investigation are illustrated in Figures 3.1 A, B, C. There are 11 C' continuous rectangular elements, and 1 C° continuous triangular element which may be categorized as;

1. C' continuous elements; namely,
   i) Boundary Traction Element family, (BTE),
      consisting of nine rectangular elements eight of which are variable degrees of freedom.
   ii) Higher-order Rectangular Elements, (HRE),
      consisting of two rectangular elements with 24 degrees of freedom


The "boundary traction element" family (Figure 3.1A) is constructed by using the blending interpolation technique. These elements are termed boundary traction elements since they enable satisfaction of the traction boundary conditions exactly. With this family, any rectangular region under arbitrary variation of tractions may be modeled. It may be observed from the Figure 3.1A that at the interior nodes, denoted by i, there are four nodal d.o.f. (\(\phi\), \(\phi_x\), \(\phi_y\), \(\phi_{xy}\)). The nodes that are on the edges j, k and l, have six (\(\phi\), \(\phi_x\), \(\phi_y\),
Typical degrees of freedom

at node j
\[ L \phi J = L \phi \phi_x \phi_y \phi_{xy} \phi_{xx} \phi_{yy} \]

at node i
\[ L \phi J = L \phi \phi_x \phi_y \phi_{xy} \]

at node k
\[ L \phi J = L \phi \phi_x \phi_y \phi_{xy} \phi_{xx} \]

at node l
\[ L \phi J = L \phi \phi_x \phi_y \phi_{xy} \phi_{yy} \]

Figure 3.1A. Boundary Traction Element Family and Degrees of Freedom.
Figure 3.1B. Higher-order Rectangular Elements.

Figure 3.1C. 9 Degrees of Freedom Triangular Element.
\( \phi_{xy}, \phi_{xx}, \phi_{yy} \) five \( (\phi, \phi_{x}, \phi_{y}, \phi_{xy}, \phi_{xx}) \), and five \( (\phi, \phi_{x}, \phi_{y}, \phi_{xy}, \phi_{yy}) \), respectively.

The higher-order rectangular elements (element type 10 and 12 in Table 3.1) are formulated using the quintic and cubic blended interpolants, respectively. These elements have six d.o.f. \( (\phi, \phi_{x}, \phi_{y}, \phi_{xy}, \phi_{xx}, \phi_{yy}) \) per node.

The 9 d.o.f. triangular element is the \( C^0 \) continuous Discrete Kirchhoff Theory (DKT) bending element [10]. It is adapted here as a plane stress element to be employed in the assumed stress function finite element method. The element has three d.o.f. \( (\phi, \phi_{x}, \phi_{y}) \) in each node.

Numerical results are given for several examples, using each one of the elements developed, in Chapter 5.

3.2.1. Concept of Blending Function Interpolants

The method was first advanced by Coons [134] in 1964 for computer aided design applications. Further developments of the method were given by Gordon [44], and by Gordon and Hall [36]. The blending function methods find many potential application areas. These include representation of free surfaces (e.g. ship hulls, fuselages) in computer aided design; for data
smoothing and for generation of functions via the interpolation of curves or surfaces in numerical analysis; in approximate solution of variational problems, etc. Further extensions and applications of the method are given in References [32,34,35,84,95,124].

The blending interpolation technique is utilized to construct the rectangular elements shown in Figures 3.1A and 3.1B. In order to demonstrate the concept the constant thickness biunit square \([-1,-1] \times [1,1]\) is studied, Figure (3.2). We will assume that a \(C^1\) continuity is desired. \(C^1\) continuity is the continuity of the normal derivative \(\tilde{\phi}_n\) of the function.
Table 3.1. Element Type Numbers and Total Degrees of Freedoms.

<table>
<thead>
<tr>
<th>Element</th>
<th>Type</th>
<th>Node</th>
<th>Total d.o.f</th>
<th>Remarks</th>
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<tr>
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<td>12</td>
<td>6</td>
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</tr>
</tbody>
</table>
In order to demonstrate the concept consider a rectangular element shown in Figure (3.2). The exact values of the function and its normal derivatives along the boundary lines of the square are denoted by $\Phi(-1,s)$, $\Phi_r(-1,s)$, $\Phi(l,s)$, $\Phi_r(l,s)$, $\Phi(r,-1)$, $\Phi_s(r,-1)$, $\Phi(r,1)$, and $\Phi_s(r,1)$. An approximation polynomial $\Phi(r,s)$ is sought such that it will satisfy the boundary values of $\Phi$ exactly. That is,

$$\Phi(r,1) \quad \Phi_s(r,1)$$
$$\Phi(-1,s) \quad \Phi_r(-1,s)$$
$$\Phi(l,s) \quad \Phi(r,l)$$
$$\Phi(r,-1) \quad \Phi_s(r,-1)$$

Figure 3.2. Boundary values of $\Phi(r,s)$ and $\Phi(r,s)$.

for the sides 2-3 and 4-1 (r-direction):

$$\Phi(r,s)\bigg|_{r=1} = \Phi(1,s) \quad \Phi_r(r,s)\bigg|_{r=1} = \Phi_r(1,s)$$

$$\Phi(r,s)\bigg|_{r=-1} = \Phi(-1,s) \quad \Phi_r(r,s)\bigg|_{r=-1} = \Phi_r(-1,s)$$
and similarly, for the sides 3-4 and 1-2 (s-direction):

\[
\begin{align*}
\phi'(r,s)|_{s=1} &= \phi'(r,1) & \phi'(r,s)|_{s=1} &= \phi'(r,1) \\
\phi'(r,s)|_{s=-1} &= \phi'(r,-1) & \phi'(r,s)|_{s=-1} &= \phi'(r,-1)
\end{align*}
\]

where subscripts denote the respective differentiation with respect to indicated variables. It may be noted that the functions \( \phi(1,s) \), \( \phi(r,1) \), \( \phi(-1,s) \), \( \phi(-1,1) \), \( \phi(r,-1) \), \( \phi(s,-1) \), \( \phi'(r,-1) \), and \( \phi'(s,1) \) are the so-called "interpolation functions".

First considering only the r-direction, we need to find a polynomial to interpolate the end values of \( \phi'(r,s) \) at \( r=-1 \) and \( r=1 \), as given in equations (3.22a), namely \( \phi(-1,s) \), \( \phi(-1,1) \), \( \phi(1,s) \), and \( \phi(1,1) \). The four conditions to be satisfied define a cubic polynomial uniquely. The suitable polynomial in this case results in the two point cubic Hermitian polynomial given by, in the \(-1\leq r\leq 1\) interval,

\[
\begin{align*}
C_1 &= \frac{1}{4} (2 -3r + r^3) & \text{node 1 (r=-1)} \\
C_1r &= \frac{1}{4} a(1-r-r^2 + r^3) & \text{node 2 (r= 1)} \\
C_2 &= \frac{1}{4} (2+3r-r^3) \\
C_2r &= \frac{1}{4} a(-1 -r + r^2 + r^3)
\end{align*}
\] (3.23)
The following operator interpolates the boundary values exactly,

\[ Pr[\tilde{\phi}(r,s)] = \tilde{\phi}(-l,s)C_1(r) + \tilde{\phi}(1,s)C_2(r) + \tilde{\phi}(-l,s)C_3(r) + \tilde{\phi}(1,s)C_4(r) \]

(3.24)

where, \( \tilde{\phi}'s \) are the "interpolation functions", \( C 's \) are the "blending functions" and \( Pr \) is the "projector operator" [36]. The blending functions may be chosen from a variety of functions, such as Lagrange polynomials, splines, trigonometric functions, Hermitian polynomials etc. In the present research the Hermitian polynomials are selected as the blending functions, in order to avoid mid-side nodes.

Similarly for the \( s \)-direction, in order to interpolate \( \tilde{\phi}(r,l), \tilde{\phi}_s(r,l) \) and \( \tilde{\phi}(r,-l), \tilde{\phi}_s(r,-l) \), we define the operator \( Ps \) as

\[ Ps[\tilde{\phi}(r,s)] = \tilde{\phi}(r,-l)C_1(s) + \tilde{\phi}(r,l)C_2(s) + \tilde{\phi}_s(r,-l)C_3(s) + \tilde{\phi}_s(r,l)C_4(s) \]

(3.25)

The operator \( Ps \) and \( \frac{\partial Pr}{\partial n} \) interpolate the function \( \tilde{\phi} \) and its normal derivatives \( \tilde{\phi}_n \) along the opposite sides of the square (i.e., \( s=-l, s=l \)) by taking the exact
boundary values of $\bar{\Phi}$ and $\bar{\Phi}n$.

In order to find a polynomial to interpolate the boundary values of $\bar{\Phi}$ and $\bar{\Phi}n$ at four sides of the element, a combination of these two operators $Pr$ and $Ps$ is required. Making use of the tensor product $PrPs[\bar{\Phi}(r,s)]$ and using cubic Hermitian polynomials as blending functions, we get

$$PrPs[\bar{\Phi}(r,s)] =$$

$$C_1(r)[\bar{\Phi}(-1,-1)C_1(s) + \bar{\Phi}(-1,1)C_2(s) + \bar{\Phi}s(-1,-1)C_1 s(s) + \bar{\Phi}s(-1,1)C_2 s(s)] +$$

$$C_2(r)[\bar{\Phi}(1,-1)C_1(s) + \bar{\Phi}(1,1)C_2(s) + \bar{\Phi}s(1,-1)C_1 s(s) + \bar{\Phi}s(1,1)C_2 s(s)] +$$

$$C_1 r(r)[\bar{\Phi}r(-1,-1)C_1(s) + \bar{\Phi}r(-1,1)C_2(s) + \bar{\Phi}sr(-1,-1)C_1 s(s) + \bar{\Phi}sr(-1,1)C_2 s(s)] +$$

$$C_2 r(r)[\bar{\Phi}r(1,-1)C_1(s) + \bar{\Phi}r(1,1)C_2(s) + \bar{\Phi}sr(1,-1)C_1 s(s) + \bar{\Phi}sr(1,1)C_2 s(s)]$$

(3.26)

where

$$\bar{\Phi}(-1,-1) = \phi_1$$

$$\bar{\Phi}r(-1,-1) = \phi_1 r$$

$$\cdot$$

$$\cdot$$
The operator PrPs interpolates only the common points of the two operators given in equations (3.24) and (3.25). These common points are the corner points of the square with the values of \( \phi, \phi_r, \phi_s, \phi_{rs} \). It may be noted that \( \phi_{rs} \), the cross derivative appears as a nodal variable, which mathematically assures the commutative property of the tensor product; i.e,

\[
PrPs = PsPr
\]

(3.27)

On the other hand, it accounts to the satisfaction of the condition of moment equilibrium in the absence of body couples.

\[
Trs = Tsr
\]

(3.28)

The combination operator PrPs is not sufficient to interpolate the boundary values exactly, since it only interpolates the corner values \((\phi(-1,-1), \phi(-1,1), \phi_s(-1,-1) \) and \( \phi_s(-1,1) \) etc.). Therefore, the effects of Pr and Ps also need to be included into the formulation. This may be achieved by using the Boolean sum [36].

\[
\Phi(r,s) = (Pr + Ps)\phi(r,s)
\]

\[
= Pr\phi(r,s) + Ps\phi(r,s) - PrPs\phi(r,s)
\]

(3.29)
It may be noted that at the boundary of the element $\partial A$, the Boolean sum (i.e., $P[\bar{\phi}(r,s)]$) takes the exact values of related interpolation functions. The boolean sum results

$$P[\bar{\phi}(r,s)] = \bar{\phi}(r,s)$$  

(3.30)

where $P$ is the "blended interpolating operator" and $\bar{\phi} = P[\bar{\phi}(r,s)]$ is the so-called "blended (weighted) interpolant of $\bar{\phi}$".

As indicated above the Hermitian polynomials are utilized as blending functions in this research. Hermitian polynomials are, by definition, two point $(2k+1)$th order polynomials. The following expression is deduced by using the properties of these polynomials. It may be noted that this expression is valid for any order of Hermitian polynomials, e.g. if $k=1$ (to interpolate the first derivative) cubic Hermitian polynomials are obtained.

$$\frac{d^n H_n(r_m)}{d r^j} = \delta_{ij} \delta_{nm} \quad \text{for} \quad n = 1,2$$

$$j = 0, k$$

$$m = 1,2$$

$$i = 0, k$$

(3.31)
where $\delta_{ij}$, $\delta_{nm}$ are the Kronecker delta, defined by

$$
\delta_{ij} = \begin{cases} 
1 & \text{for } i=j \\
0 & \text{for } i \neq j
\end{cases}
= \begin{cases} 
1 & \text{for } n=m \\
0 & \text{for } n \neq m
\end{cases}
$$

Subscript $n$ denotes the nodes with $r = -1$ and $r = 1$. $H$ denotes a Hermitian polynomial in general, and cubic and quintic Hermitian polynomials are denoted by replacing $H$ by $C$ or $Q$, respectively. Figures A.1-A.2 (Appendix A) illustrate the cubic (first order, $k=1$), and quintic (second order, $k=2$) Hermitian polynomials. The projector operator given by equation (3.24) may be written, in concise form, using the summation notation as follows,

$$
Pr[\hat{Q}(r,s)] = \sum_{n=1}^{2} \sum_{i=0}^{k} H_n(r) \hat{Q}(r_n,s)
$$

(3.32)

where

$n$ : denotes the nodes $n=1$ ($r=-1$), and $n=2$ ($r=1$).

$k$ : The order of the derivative that need to be interpolated

$i$ : denotes the shape function at node $n$ corresponding to $i$th derivative; $i=0$ implies that the function itself is continuous
e.g.,

\[ H_1 = H_1^{(0)} \] shape function at node 1

\[ H_2 = H_2^{(1)} \] shape function at node 2 corresponding to the first derivative.

\[ H_{2r} = H_2^{(2)} \] shape function at node 2 corresponding to second derivative.

In order to have the C' continuity, the function and its first order derivatives are required to be continuous; hence \( k \), in the Hermitian polynomial definition, is 1. In general, \( k \) is one order less than the maximum order of the derivative, which appears in the functional.

Similarly, in summation notation the equation in the s-direction may be written

\[ P_s[\overline{\phi}(r,s)] = \sum_{n=1}^{2} \sum_{i=0}^{k} H_n(s) \overline{\phi}(r,s_n) \] (3.33)

and the tensor product takes the following form

\[ P_{rPs}[\overline{\phi}(r,s)] = \sum_{n=1}^{2} \sum_{m=1}^{2} \sum_{i=0}^{k} \sum_{j=0}^{k} H_n H_m(s) \overline{\phi}^{(i)}(r_n, s_m) \] (3.34)
Finally, in short form, the Boolean sum (equation 3.29) may be written as as follows,

\[ P[\Phi(r,s)] = \sum_{n=1}^{2} \sum_{i=0}^{k} H_n(r)\Phi(r_n,s) + \sum_{n=1}^{2} \sum_{i=0}^{k} H_n(s)\Phi(r,s_n) \]

\[ - \sum_{n=1}^{2} \sum_{m=1}^{2} \sum_{i=0}^{k} \sum_{j=0}^{k} H_n(r)H_m(s)\Phi(r,s) \]

(3.35)

In the present development two types of Hermitian polynomials are utilized as blending functions. These are the cubic and quintic polynomials. The Boolean sum obtained using these polynomials are given in Tables 3.2 - 3.3.
Table 3.2. Boolean Sum for Cubic Hermitian Blending Functions on biunit square \([-1,1] \times [-1,1]\).

<table>
<thead>
<tr>
<th>(P[r(s)])</th>
<th>Boolean Sum ( (P = Pr \oplus Ps = Pr + Ps - PrPs))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Pr[r(s)])</td>
<td>(\Phi(-1,s)C_1(r) + \Phi(1,s)C_2(r) + \Phi(-1,s)C_3(s)) + (\Phi(1,s)C_4(s))</td>
</tr>
<tr>
<td>(Ps[r(s)])</td>
<td>(\Phi(r,-1)C_1(s) + \Phi(r,1)C_2(s) + \Phi(r,-1)C_3(s)) + (\Phi(r,1)C_4(s))</td>
</tr>
<tr>
<td>(PrPs(r,s))</td>
<td>(\Phi_1[C_1(r)C_1(s)] + \Phi_1[r[C_1(r)C_1(s)]] + \Phi_1[s[C_1(r)C_1(s)]] + \Phi_1[r[s[C_1(r)C_1(s)]]]) + (\Phi_2[C_2(r)C_2(s)] + \Phi_2[r[C_2(r)C_2(s)]] + \Phi_2[s[C_2(r)C_2(s)]] + \Phi_2[r[s[C_2(r)C_2(s)]]]) + (\Phi_3[C_3(r)C_3(s)] + \Phi_3[r[C_3(r)C_3(s)]] + \Phi_3[s[C_3(r)C_3(s)]] + \Phi_3[r[s[C_3(r)C_3(s)]]]) + (\Phi_4[C_4(r)C_4(s)] + \Phi_4[r[C_4(r)C_4(s)]] + \Phi_4[s[C_4(r)C_4(s)]] + \Phi_4[r[s[C_4(r)C_4(s)]]])</td>
</tr>
</tbody>
</table>
Table 3.3. Boolean Sum for Quintic Hermitian Blending Functions on biunit square \([-1,1] \times [-1,1]\).

\[
P[\Phi (r,s)] = \Phi (-1,s)Q_1(r) + \Phi (1,s)Q_2(r) + \Phi (-1,s)Q_1r(r) + \Phi (1,s)Q_1r(s) + \Phi (-1,s)Q_1rr(r) + \Phi (1,s)Q_1rr(s)
\]

\[
P[\Phi (r,s)] = \Phi (r,-1)Q_1(s) + \Phi (r,1)Q_2(s) + \Phi (r,-1)Q_1s(s) + \Phi (r,1)Q_2s(s)
\]

\[
Pr[\Phi (r,s)] = \Phi \left[ Q_1(r)Q_1(s) \right] + \Phi \left[ Q_1r(r)Q_1s(s) \right]
\]

\[
Ps[\Phi (r,s)] = \Phi \left[ Q_2(r)Q_1(s) \right] + \Phi \left[ Q_2r(r)Q_1s(s) \right]
\]

\[
PrPs[\Phi (r,s)] = \Phi \left[ Q_1(r)Q_2(s) \right] + \Phi \left[ Q_1r(r)Q_2s(s) \right]
\]
3.2.2. Construction of Rectangles

The elements need to preserve the \( C^1 \) continuity conditions to ensure an upper bound to the strain energy. The \( C^1 \) continuity is the continuity of the function, in this case \( \bar{\varphi} \), and its normal derivative, \( \bar{\Phi}_n \). An approximation to \( \bar{\varphi} \) may be given in terms of nodal degrees of freedoms as

\[
\bar{\varphi} = [N]l\{\phi\}_e
\]

(3.36)

where \([N]\) is the vector of shape functions, and \( \{\phi\}_e \) is the vector of nodal degrees of freedoms for the element. The geometry of the rectangles is given in Figure 3.3. The shape functions \([N]\) are given in terms of the nondimensional coordinates \((r,s)\) in the square region \([-1,1] \times [-1,1]\). In the computations, the derivatives and the area integral are expressed in terms of nondimensional coordinates as follows;

\[
\frac{1}{ax} = \frac{1}{a} \frac{a}{ar}
\]

\[
\frac{1}{ay} = \frac{1}{b} \frac{a}{as}
\]

and

\[
\int_{A} \bar{\phi}(x,y) \, dx \, dy = 4ab \int_{-1}^{+1} \int_{-1}^{-1} \bar{\phi}(r,s) \, dr \, ds
\]

(3.37)
In order to construct a rectangular element with $C'$ continuity, at least 4 d.o.f. (i.e., $\phi$, $\phi_r$, $\phi_s$, $\phi_{rs}$) are required (equation 3.26). Therefore, the simplest $C'$ continuous rectangular element will have 16 d.o.f. (Figure 3.4). The details of the construction of this element follows.

The cubic blending functions are utilized in the element formulation. Considering that the $\bar{\phi}(r,-1)$ will interpolate the nodal values $\phi_i$, $\phi_2$, $\phi_r$, $\phi_{ir}$ and $\bar{\phi}s(r,-1)$ will interpolate the nodal values $\phi_is$, $\phi_{irs}$, $\phi_zs$, $\phi_{zrs}$ etc., at least cubic polynomials are required as interpolation functions. The interpolation functions are uniquely determined by four nodal values. The vector of nodal values (degrees of freedom) is given by

Figure 3.3. Geometry of Rectangles.
The interpolation functions for 16 d.o.f element are given as

\[ \mathbf{\phi} = [ \phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi_6 \phi_7 \phi_8] \]

(3.38)

Figure 3.4. Boundary Traction Element Type 5.

side 1-2
\[ \bar{\Phi}(r,-l) = \phi_1\xi_1(r) + \phi_2\xi_2(r) + \phi_3\xi_3(r) + \phi_4\xi_4(r) + \phi_5\xi_5(r) + \phi_6\xi_6(r) + \phi_7\xi_7(r) + \phi_8\xi_8(r) \]
\[ \bar{\Phi}_s(r,-l) = \phi_1\xi_1(r) + \phi_2\xi_2(r) + \phi_3\xi_3(r) + \phi_4\xi_4(r) + \phi_5\xi_5(r) + \phi_6\xi_6(r) + \phi_7\xi_7(r) + \phi_8\xi_8(r) \]

side 4-3
\[ \Phi(r,1) = \phi_1\xi_1(r) + \phi_2\xi_2(r) + \phi_3\xi_3(r) + \phi_4\xi_4(r) + \phi_5\xi_5(r) + \phi_6\xi_6(r) + \phi_7\xi_7(r) + \phi_8\xi_8(r) \]
\[ \Phi_s(r,1) = \phi_1\xi_1(r) + \phi_2\xi_2(r) + \phi_3\xi_3(r) + \phi_4\xi_4(r) + \phi_5\xi_5(r) + \phi_6\xi_6(r) + \phi_7\xi_7(r) + \phi_8\xi_8(r) \]

side 1-4
\[ \bar{\Phi}(-l,s) = \phi_1\xi_1(s) + \phi_2\xi_2(s) + \phi_3\xi_3(s) + \phi_4\xi_4(s) + \phi_5\xi_5(s) + \phi_6\xi_6(s) + \phi_7\xi_7(s) + \phi_8\xi_8(s) \]
\[ \bar{\Phi}_r(-l,s) = \phi_1\xi_1(s) + \phi_2\xi_2(s) + \phi_3\xi_3(s) + \phi_4\xi_4(s) + \phi_5\xi_5(s) + \phi_6\xi_6(s) + \phi_7\xi_7(s) + \phi_8\xi_8(s) \]
\[ \phi_{i}(l,s) = \phi_{1}C_{1}(s) + \phi_{2}C_{2}(s) + \phi_{2}SC_{1}(s) + \phi_{3}SC_{2}(s) \]
\[ \phi_{r}(l,s) = \phi_{2}rC_{1}(s) + \phi_{3}rC_{2}(s) + \phi_{2}rsC_{1}(s) + \phi_{3}rsC_{2}(s) \]

Substitution of the above values into the Boolean sum expression for cubic Hermitian Blending Functions (Table 3.2) yields

\[ \Phi = \]
\[ C_{1}(r)C_{1}(s)\phi_{1} + C_{1}(r)C_{1}(s)\phi_{1}r + C_{1}(r)C_{1}(s)\phi_{1}s + C_{1}(r)C_{1}(s)\phi_{1}rs + \]
\[ C_{2}(r)C_{1}(s)\phi_{2} + C_{2}(r)C_{1}(s)\phi_{2}r + C_{2}(r)C_{1}(s)\phi_{2}s + C_{2}(r)C_{1}(s)\phi_{2}rs + \]
\[ C_{2}(r)C_{2}(s)\phi_{3} + C_{2}(r)C_{2}(s)\phi_{3}r + C_{2}(r)C_{2}(s)\phi_{3}s + C_{2}(r)C_{2}(s)\phi_{3}rs + \]
\[ C_{1}(r)C_{2}(s)\phi_{4} + C_{1}(r)C_{2}(s)\phi_{4}r + C_{1}(r)C_{2}(s)\phi_{4}s + C_{1}(r)C_{2}(s)\phi_{4}rs + \]

or, by denoting the shape functions as

\[ N_{i} = C_{1}(r)C_{1}(s) \ldots N_{i} = C_{1}(r)C_{1}(s) \ldots N_{i} = C_{1}(r)C_{1}(s) \]

and by using the equation (3.38), the stress function \( \Phi \) becomes

\[ \Phi = [N]\{\phi\}^{e} \]
The element is complete in cubic terms, and it preserves all the monomials [45] of the form

\[ x^i y^j \quad i, j \leq 3 \]  

(3.41)

It is interesting to note that with the above selection of the blending functions and interpolation functions, using the procedure described, the bicubic Hermitian blending element is obtained [106].

**Boundary Traction Elements Family.** The Airy stress function \( \Phi \) is related to normal stresses by the second derivatives (equation 3.1). In order to impose the stress type boundary conditions directly, the second derivatives of stress function need to be taken as nodal variables. It means that a \( C^2 \) continuity is required at the boundaries, wherever the stresses are prescribed.

The boundary traction elements family is constructed to account for the prescribed stress type boundary conditions with a minimal nodal d.o.f. [39]. Hence, the cubic blending functions are utilized in constructing these variable d.o.f element family. The interpolation functions at the boundary are selected as quintic Hermitian polynomials to preserve \( C^2 \) continuity. At the boundaries common with the adjacent elements, cubic interpolation functions are retained in order to keep
compatibility of the elements with the 16 d.o.f element. At the interior nodes the degrees of freedoms are $\phi$, $\phi_x$, $\phi_y$, $\phi_{xy}$.

In order to illustrate the development for boundary traction elements family element type 1 is exemplified. The interpolation functions of this element are taken as:

**side 1-2**

$$
\begin{align*}
\Phi(r, -1) &= \phi_1C_1(r) + \phi_2C_2(r) + \phi_1rC_1r(r) + \phi_2rC_2r(r) \\
\Phi_s(r, -1) &= \phi_1sC_1(r) + \phi_2sC_2(r) + \phi_1rsC_1r(r) + \phi_2rsC_2r(r)
\end{align*}
$$

**side 4-3**

$$
\begin{align*}
\Phi(r, 1) &= \phi_4Q_1(r) + \phi_3Q_2(r) + \phi_4Q_1r(r) + \phi_3Q_2r(r) \\
&\quad + \phi_4rQ_1rr(r) + \phi_3rQ_2rr(r) \\
\Phi_s(r, 1) &= \phi_4sC_1(r) + \phi_3sC_2(r) + \phi_4rsC_1r(r) + \phi_3rsC_2r(r)
\end{align*}
$$

**side 1-4**

$$
\begin{align*}
\Phi(-1, s) &= \phi_1Q_1(s) + \phi_4Q_2(s) + \phi_1sQ_1s(s) + \phi_4sQ_2s(s) \\
&\quad + \phi_1ssQ_1ss(s) + \phi_4ssQ_2ss(s) \\
\Phi_r(-1, s) &= \phi_1rC_1(s) + \phi_4rC_2(s) + \phi_1rsC_1s(s) + \phi_4rsC_2s(s)
\end{align*}
$$

**side 2-3**

$$
\begin{align*}
\Phi(1, s) &= \phi_2C_1(s) + \phi_3C_2(s) + \phi_2sC_1s(s) + \phi_3sC_2s(s) \\
\Phi_s(1, s) &= \phi_2sC_1(s) + \phi_3rC_2(s) + \phi_2rsC_1s(s) + \phi_3rsC_2s(s) \\
&\quad (3.42)
\end{align*}
$$
By substitution of the interpolation functions given in equation (3.42) into the Boolean sum expression for cubic blending functions (Table 3.2), the element shape functions are obtained and the final expression is given in Table B.1, (Appendix B). This element has 20 degrees of freedom, and it is \( C^1 \) continuous. It interpolates the normal stress \( \sigma_n \) values along the boundaries up to a cubic variation exactly. The shear stress \( \tau \) is interpolated up to a quadratic variation exactly.

It may be noted that, construction of three elements; i.e., element types- 1, 2, 5,- is adequate, as the other elements in this family may be obtained by proper transformations. However, in the present research all the elements are constructed for the reasons of efficient computer implementation and economy in execution time. The shape functions for all types of this family are given in Tables B.1-9.

**Higher-order Rectangular Elements.** Two rectangular elements with 24 d.o.f are also constructed in this research (element types 10 and 12). The element type 10 is constructed using the quintic blending functions, (Table 3.3) and quintic interpolation functions, (Figure A.2). The terms corresponding to nodal variables \( \phi_{rrs}, \phi_{ssr} \) and \( \dot{\phi}_{rrss} \) are neglected in
the development of the element formulations. It may be noted that the element passes the required patch test [125].

The other element, type 12, is constructed by using the cubic blending functions. The quintic and cubic functions are selected as interpolation functions. The procedure for the formulations of these elements follow the same lines as the boundary traction elements. The derivations are carried out as described, and the final form of the shape functions of these elements are given in Tables B.10 and B.12 (Appendix B).

It may be noted that these elements represent the boundary conditions exactly when the normal stresses vary cubically. In addition, these elements may be used together with a higher order triangular element given in Reference [120] to discretize irregular regions.

3.2.3. $C^0$ Continuous Triangular Element

The element is adapted from plate bending displacement model [10]. The nodal degrees of freedoms are the stress function $\phi$, and its derivatives $\phi_x$, $\phi_y$ (Figure 3.1B). The variation of $\phi$ along the boundary is cubic, and the variation of the derivatives are quadratic. Therefore, the normal stress variation is linear along the boundaries.
It is observed that the element does not enforce the equality of shear stresses. Hence, it may be utilized to solve problems if the body couples are present.

3.3. Interchangeability of Elements

The compatibility equations in two-dimensional elasticity, written in terms of the Airy stress function, yields the biharmonic homogeneous equation, under the assumption that there are no body forces and no initial strains. The equilibrium equations in plate bending written in terms of transverse deformation also gives the biharmonic equation for zero distributed loads, \([111]\). It is due to this analogy that the elements constructed for displacement model plate bending analysis is transferable to the equilibrium model plate stretching analysis. Due account must be taken in material property matrix and curvature_displacement/stress-stress_function relations.

3.4. Discussion on "support" conditions

The system of algebraic equations obtained from either using the Lagrange multipliers technique, (equation 3.17), or by using the present ASFFEEM, (equation 3.18), cannot be solved, because the determinant of the coefficient matrix is zero. This is
analogous to a situation that is encountered in the displacement method, the solution of which is obtained by suppressing the rigid body modes. In the assumed stress function finite element formulation, these correspond to three conditions which give a reference level for the values of \( \phi, \phi_x, \) and \( \phi_y \). These values can be assigned arbitrarily in simply-connected regions, since they do not effect the stresses. It is also verified numerically that these three values may be assigned arbitrarily at any point. For convenience, the values for \( \phi, \phi_x, \) and \( \phi_y \) are assigned to zero in this research. It may be recalled from Chapter 2 that the Airy stress function \( \Phi \) and its derivatives are the moment and forces per unit thickness.
**DISPLACEMENT MODEL**  
Plate Bending

**EQUILIBRIUM MODEL**  
Plate Stretching

1. Field Variable

transverse displacement  
\[ w \]

Airy stress function  
\[ \phi \]

2. Material Properties

material stiffness matrix
\[
\begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1-\nu}{2}
\end{bmatrix}
\]

material flexibility matrix
\[
\begin{bmatrix}
1 & -\nu & 0 \\
-\nu & 1 & 0 \\
0 & 0 & 2(1+\nu)
\end{bmatrix}
\]

3. Curvature-displacement/ stress-stress function

\[
\kappa = \begin{bmatrix}
-\frac{\partial^2 w}{\partial x^2} \\
-\frac{\partial^2 w}{\partial y^2} \\
-2 \frac{\partial^2 w}{\partial x \partial y}
\end{bmatrix}
\]

\[
\varphi = \begin{bmatrix}
\frac{\partial^2 \phi}{\partial x^2} \\
\frac{\partial^2 \phi}{\partial y^2} \\
-\frac{\partial^2 \phi}{\partial x \partial y}
\end{bmatrix}
\]

\[
\kappa_x \longleftrightarrow (-) \quad \sigma_y
\]

\[
\kappa_y \longleftrightarrow (-) \quad \sigma_x
\]

\[
\kappa_{xy} \longleftrightarrow (-) \quad \tau_{xy}
\]

**Figure 3.5** Analogy between Plate Bending/Plate Stretching.
In order to detail the implementation of "support" conditions, various boundary conditions are examined and corresponding values of $\phi$, $\phi_x$, $\phi_y$ are given. Consider a hypothetical structure as shown in Figure 3.6 which includes both static and kinematic type boundary conditions.

![Figure 3.6 Static and kinematic boundary conditions.](image)

In order to impose the boundary conditions at any node on the boundary, first, a point on the boundary (node 1) is selected. The boundary conditions at any
other node is prescribed in an orderly manner, traversing in counterclockwise direction from the selected node, along the boundary line.

**Free Boundary (1-2 portion of boundary).** For any node on this portion of the boundary, the values of \( \phi \), \( \phi_x \), and \( \phi_y \) remain unchanged. That is, the values at node 2 will be the same as at node 1, which is chosen as zero in simply-connected regions.

**Prescribed Traction.** For nodes on this boundary portion, the values of \( \phi \), \( \phi_x \), and \( \phi_y \) are calculated using the equations derived in Chapter 2, (equation 2.18).

**Concentrated Forces.** In calculating the resultant forces and moment per unit thickness (i.e., \( \phi_x \), \( \phi_y \), \( \phi \)) the effect of the concentrated force is also to be considered.

**Prescribed Displacements.** On this portion of the boundary, displacements are prescribed. Owing to the nature of the complementary energy formulation the stress parameters are not prescribed. (These are accounted for in the complementary energy functional.)

**Symmetry Conditions.** Along the symmetry axis the shear stress is assigned a zero value. In addition, it may be observed that the first derivative of the stress...
function in the normal direction, \( \phi_n \), is constant. It may be noted that at the center of symmetry, the Airy stress function becomes zero.

Antisymmetry Conditions. In this case, the normal stress is zero along the antisymmetry axis. The first derivative of the Airy stress function in tangential direction is found to be a constant.
4.1. General Concepts

The equations of equilibrium of a thin plate, equation (2.18), are satisfied if the moment and shearing forces are obtained from the Southwell stress functions $\Phi_u$ and $\Phi_v$ [23]. These relations may be expressed as

\[
\begin{align*}
M_x &= -\left(\frac{\partial \Phi_v}{\partial y} + \Omega\right) \\
M_y &= -\left(\frac{\partial \Phi_u}{\partial x} + \Omega\right) \\
M_{xy} &= \frac{1}{2} \left(\frac{\partial \Phi_u}{\partial y} + \frac{\partial \Phi_v}{\partial x}\right) \\
Q_x &= \frac{1}{2} \frac{\partial}{\partial y} \left(\frac{\partial \Phi_u}{\partial y} - \frac{\partial \Phi_v}{\partial x}\right) - \frac{\partial \Omega_1}{\partial x} \\
Q_y &= \frac{1}{2} \frac{\partial}{\partial x} \left(\frac{\partial \Phi_v}{\partial x} - \frac{\partial \Phi_u}{\partial y}\right) - \frac{\partial \Omega_2}{\partial y}
\end{align*}
\]

(4.1)
in which and are parameters and related to the intensity of applied load \( q \) by

\[
q = \frac{\partial^2 \mathcal{L}_1}{\partial x^2} + \frac{\partial^2 \mathcal{L}_2}{\partial y^2}
\]  

(4.2)

Considering that the prescribed displacements are in general zero, only the complementary strain energy is examined.

\[
\Pi_c = U^* = \frac{1}{2} \int_\mathcal{A} \{M\}^T [E_f]^{-1} \{M\} \, d\mathcal{A}
\]  

(4.3)

where \( \{M\} \) is the vector of normal moments \( (M_x, M_y) \), and twisting moment \( (M_{xy}) \) per unit length, obtained by integration of direct stresses across the thickness, and \( [E_f] \) is the material flexibility matrix as given by equation (2.20).

Defining a differentiation operator \( R \) as,

\[
R = \begin{bmatrix}
0 & -\frac{\partial}{\partial y} \\
-\frac{\partial}{\partial x} & 0 \\
\frac{1}{2} \frac{\partial}{\partial x} & \frac{1}{2} \frac{\partial}{\partial y}
\end{bmatrix}
\]
then, the vector of moments

\[
\{M\} = \begin{bmatrix}
0 & -\frac{3}{\partial y} \\
-\frac{3}{\partial x} & 0 \\
\frac{1}{2} \frac{3}{\partial x} & \frac{1}{2} \frac{3}{\partial y}
\end{bmatrix}
\begin{bmatrix}
\{\tilde{\phi}_u\} \\
\{\tilde{\phi}_v\}
\end{bmatrix}
+ \begin{bmatrix}
-\mathcal{R}_1 \\
-\mathcal{R}_2 \\
0
\end{bmatrix}
\]

or

\[
\{M\} = R \tilde{\mathcal{R}} + \mathcal{R}
\]

(4.4)

where

\[
\mathcal{R} = \begin{bmatrix}
-\mathcal{R}_1 \\
-\mathcal{R}_2 \\
0
\end{bmatrix}
\]

and

\[
\tilde{\mathcal{R}} = \begin{bmatrix}
\{\tilde{\phi}_u\} \\
\{\tilde{\phi}_v\}
\end{bmatrix}
\]

By substitution of equation (4.4) into the equation (4.3), we obtain

\[
\mathcal{K}_c = \frac{1}{2} \int_A (R \tilde{\mathcal{R}} + \mathcal{R})^T \mathcal{E} \mathcal{E}^T (R \tilde{\mathcal{R}} + \mathcal{R}) dA.
\]
or

\[
\mathcal{K}_c = \frac{1}{2} \int_A (R \mathbf{\bar{\phi}})^T [E_f]^{-1} (R \mathbf{\bar{\phi}}) \, dA
\]

\[
+ \int_A (R \mathbf{\bar{\phi}}) [E_f]^{-1} \mathbf{\bar{\eta}} \, dA
\]

\[
+ \frac{1}{2} \int_A \mathbf{\bar{\eta}} [E_f]^{-1} \mathbf{\bar{\eta}} \, dA
\]

(4.5)

When the first variation of the complementary energy is performed, the last integral on the right side of equation (4.5) vanishes and the second integral results in a constant term. Therefore, the last term is excluded from further consideration.

The complementary energy of an element is, then,

\[
\mathcal{K}_c = \frac{1}{2} \int_A \{R \mathbf{\bar{\phi}}\}^T [E_f]^{-1} \{R \mathbf{\bar{\phi}}\} \, dA + \int_A \mathbf{\bar{\eta}} [E_f]^{-1} \mathbf{\bar{\eta}} \, dA
\]

(4.6)

Using the finite element approximation to stress functions \( \mathbf{\bar{u}} \) and \( \mathbf{\bar{v}} \) as

\[
\mathbf{\bar{u}} = \{N\} \{\phi_u\}, \quad \mathbf{\bar{v}} = \{N\} \{\phi_v\}
\]

(4.7)
Figure 4.1. Rectangular plate element in bending.

or

\( \bar{\phi} = [N] \phi \) \hspace{1cm} (4.8)

where \( \{\phi_u\} \) and \( \{\phi_v\} \) are the stress function parameters which include the values of the stress function and its derivatives, and \( [N] \) are the shape functions.

By substituting equation (4.8) into equation (4.3) one obtains

\( M = [R][N] \phi + \bar{N} \)
or by defining matrix $[D]$ as

$$ [D] = R \{N\} \quad (4.9) $$

the stress resultant–stress function relations take the following form

$$ M = [D]\phi + \rho \quad (4.10) $$

Substitution of equation (4.10) into the complementary energy expression yields

$$ \mathcal{K} \mathcal{C} = \frac{L}{2} \int_{A} [D]^T [E_\gamma^{-1}] [D] \{\phi\} dA + L \phi \int_{A} [D]^T [E_\gamma^{-1}] \{R\} dA $$

or

$$ \mathcal{K} \mathcal{C} = \frac{L}{2} \{f\} \{\phi\} + \{\phi\} \{c\} \quad (4.11) $$

with

$$ [f] = \{ \int_{A} [D]^T [E_\gamma^{-1}] [D] dA \} $$

$$ \{c\} = \{ \int_{A} [D]^T [E_\gamma^{-1}] \{\phi\} dA \} \quad (4.12) $$

where $[f]$ is the "generalized" flexibility matrix, and $\{c\}$ is a vector of constants.
The complementary energy of the structure is obtained by summation of the element complementary energies.

\[ \Lambda_c = \sum_{e=1}^{p} \Lambda_c^e \]  \hspace{1cm} (4.13)

where \( p \) is the total number of elements. Or,

\[ \Pi_c = \sum_{i=1}^{p} \frac{L_{\phi; i}}{2} [f_i] \{\phi_i\} + L\phi_i \{c_i\} \]

\[ \Pi_c = \frac{L_{\phi; i}}{2} [F] \{\phi_i\} + L\phi_i \{c_i\} \]  \hspace{1cm} (4.14)

where \([F]\) is the global flexibility matrix for the structure and the vector \( \phi_i \) contains all the degrees of freedoms of the structure.

The principle of stationary complementary energy is then invoked together with the prescribed boundary tractions. These conditions may be handled in a similar
manner given for planar case (Chapter 3). The boundary conditions in terms of Southwell stress functions are given in the following section.

4.2. Boundary Conditions

In order to examine the boundary conditions, bending moments $M_n$, twisting moments $M_{nt}$ and shear forces $Q_n$ acting along the boundary are expressed in terms of $x$ and $y$ direction stress resultants [130-131].

Equations (2.20) may be written, considering the stress functions, in the form

$$M_n = -\sin \alpha \frac{\partial \Phi_u}{\partial t} + \cos \alpha \frac{\partial \Phi_v}{\partial t} + \Omega_1 \cos \alpha - \Omega_2 \cos \alpha$$

$$M_{nt} = \frac{1}{2} \left( \cos \alpha \frac{\partial \Phi_u}{\partial t} - \sin \alpha \frac{\partial \Phi_u}{\partial n} + \sin \alpha \frac{\partial \Phi_v}{\partial t} + \cos \alpha \frac{\partial \Phi_v}{\partial n} \right)$$

$$- (\Omega_1 - \Omega_2) \sin \alpha \cos \alpha$$

$$Q_n = -\frac{1}{2} \left[ \cos \alpha \frac{\partial \Phi_u}{\partial t} + \sin \alpha \frac{\partial \Phi_u}{\partial n} + \sin \alpha \frac{\partial \Phi_v}{\partial t} + \cos \alpha \frac{\partial \Phi_v}{\partial n} \right]$$

$$- \cos \alpha \frac{\partial \Omega_1}{\partial x} - \sin \alpha \frac{\partial \Omega_2}{\partial y}$$

(4.17)
Figure 4.2. Boundary conditions in bending.
The Kirchhoff normal force is given by

\[ V_n = -\frac{\omega}{\partial t} \left\{ \cos \alpha \frac{\partial \Phi_u}{\partial t} + \sin \alpha \frac{\partial \Phi_u}{\partial t} - (\omega_1 - \omega_2) \sin \alpha \cos \alpha \right\} \]

\[ - \cos \alpha \frac{\partial \omega}{\partial x} - \sin \frac{\partial \omega}{\partial y} \]

(4.18)

The prescribed boundary values must satisfy the relations

\[ \overline{M}_n = M_n = -\sin \alpha \frac{\partial \Phi_u}{\partial t} + \cos \alpha \frac{\partial \Phi_v}{\partial t} \]

\[ \overline{V}_n = V_n = -\frac{\omega}{\partial t} \left[ \cos \alpha \frac{\partial \Phi_u}{\partial t} + \sin \alpha \frac{\partial \Phi_v}{\partial t} \right] \]

The twisting moment in equation (4.17) gives a concentrated force \( R \) at vertices of elements. This force acts in the same direction as the applied load \( q \).

\[ \sum_{i=1}^{e} R_i = \overline{R}_i \]

(4.19)
It is necessary to ensure that the total force resulting from that is balanced by the prescribed concentrated load at the corresponding boundary node.

4.3. Interchangeability of elements

By use of Southwell stress functions, plate bending problems can be solved with only \( C^0 \) continuity requirements. Thus, considering the duality between plate stretching displacement model and plate bending equilibrium model, the elements formulated for displacement model are transferable to the equilibrium model (Figure 4.3). It may be noted that due consideration must be given to modification of the strain-displacement relations.
DISPLACEMENT MODEL
Plate Stretching

EQUILIBRIUM MODEL
Plate Bending

1. Field Variables

\[ \begin{align*}
\text{displacements} & \quad \text{Southwell stress functions} \\
u & \quad \phi_u \\
v & \quad \phi_v
\end{align*} \]

2. Material Properties matrix

\[ [E] = \frac{E}{1-v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \]

\[ [E_\text{f}]^{-1} = \frac{12(1-v^2)}{E t^3} \begin{bmatrix} 1 & -v & 0 \\ v & 1 & 0 \\ 0 & 0 & 2(1+v) \end{bmatrix} \]

3. Strain-displacement/moment-stress function

\[ \varepsilon = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} \]

\[ \begin{cases} \\ \varepsilon_x & \rightarrow (-) \\ \varepsilon_y & \rightarrow (-) \\ \varepsilon_{xy} & \rightarrow 1/2 \end{cases} \]

\[ \begin{cases} \\ \sigma_x & \rightarrow (-) \\ \sigma_y & \rightarrow (-) \\ \sigma_{xy} & \rightarrow 1/2 \end{cases} \]

\[ \begin{cases} \\ M_x & \rightarrow 1/2 \sigma_{xy} \end{cases} \]

\[ \begin{cases} \\ M_y & \rightarrow 1/2 \sigma_{xy} \end{cases} \]

Figure 4.3. Plate Stretching / Plate Bending.
CHAPTER 5

NUMERICAL EXPERIMENTS

5.1. Computer Implementation of the Analysis

A general purpose finite element computer program was developed to facilitate analysis using either the potential energy or the complementary energy approach. The user has the option of selecting the method to include the set of constraints by utilizing the Lagrange multipliers or by direct substitution into the system of algebraic equations.

The elements developed in Chapter 3 (Table 3.1), are incorporated into this program. These form a library of elements which includes the boundary traction element family, higher order rectangular elements, and a $C^0$ continuous triangular element.

Numerical integration is employed in the computation of the element generalized flexibilities. Due to the different order of polynomials utilized in the element formulations, different numbers of integration points are required. There is a subroutine built into the system which contains the Gaussian quadrature data up to six point integration. The element flexibilities select suitable data from this subroutine.
depending on the type of the elements. A three point integration scheme is adopted for the triangular element.

The block diagram, depicted in Figure (5.1), describes the overall logic of the computer program. The major subroutines and their features are outlined briefly for completeness.
Figure 5.1 Block diagram for Dual Analysis.
FILE: This subroutine requests the names of the input and output files from the user interactively, and creates automatically the output file.

DATA: The input data is read from the disk file for the problem: The title of the problem, solution method (i.e., flag to indicate either complementary or potential energy approach), and constraints (i.e., Lagrange multipliers or direct substitution), geometric data, material properties and the loading together with the boundary conditions. Additional data pertaining to constraints is to be supplied, if the Lagrange multipliers method is selected.

STIFF: The element stiffness or generalized flexibilities are computed depending on the desired element type.

LOAD: Calculates the element loads (applicable for displacement model only)

ASSEM: Assembles the element stiffness/generalized flexibility and loads into the global arrays.
LAGSTF: Calculates the generalized flexibilities at the boundaries.

LAGLOD: Calculates the boundary integrals.

LAGASM: Assembles the stiffnesses of "boundary line elements".

5.2. Numerical Examples

The elements constructed have been evaluated by using the patch test and by comparisons of results with other solution methods.

5.2.1. Patch Tests for Equilibrium Model Elements

The purpose of the patch tests in the present analysis is twofold; firstly, to check the correctness of the element formulations, and secondly to evaluate the performance of the element. The description of the applicable patch tests are outlined.

**Constant Shear.** A constant shear stress is applied along the boundaries of a rectangle (Figure 5.2). The test is performed using four elements. The values of $\phi$, $\phi x$, $\phi y$, and $\phi xy$ are prescribed along the boundaries. The patch test requires that the stress values at any point within the region correspond to a constant shear situation (normal stresses being zero).
Figure 5.2. Patch Test. Constant Shear.

\[ \Phi = -\tau_{xy} \]

\[ \Phi_x = -\tau_0 y \]
\[ \Phi_y = -\tau_0 x \]
\[ \Phi_{xy} = -\tau_0 \]
\[ \sigma_x = \Phi_{yy} = 0 \]
\[ \sigma_y = \Phi_{xx} = 0 \]
\[ \tau_{xy} = -\Phi_{xy} = \tau_0 \]

\[ E = 10^7 \]
\[ \nu = 0.2 \]
\[ t = 1.0 \]
\[ L = 2 \]
\[ \tau_0 = 8 \]
Constant Normal Stress. The normal stress distribution along the edges of the region is shown in the Figure 5.3. The test requires that $\sigma_x$ and $\sigma_y$ be constant throughout the region with $\tau_{xy}$ being zero.

Constant Normal and Shear Stress. The stress distribution is prescribed to be constant along the edges as shown in Figure 5.4. The test, in this case, requires that the values of $\sigma_x$, $\sigma_y$, and $\tau_{xy}$ at any interior point result in constant values.

Patch Test Results. The elements are tested making use of the computer program developed. The patch tests described are utilized for the rectangular elements constructed. It is observed that the elements pass the patch tests. It may be concluded, based on the tests performed, that the element formulations are correctly implemented and that the convergence of the results are assured in any given application.

5.2.2. Square Plate Subjected to Parabolically Varying Edge Stresses.

This problem is selected because of the availability of a solution based on the principle of least work [135] which utilizes a polynomial representation for stress functions.
Figure 5.3. Patch Test. Constant normal stress.

\[ \Phi = \frac{1}{2}(x^2 + y^2) \]

\[ \frac{\partial \Phi}{\partial x} = x \]

\[ \frac{\partial \Phi}{\partial y} = y \]

\[ \frac{\partial^2 \Phi}{\partial x \partial y} = 0 \]

\[ \frac{\partial^2 \Phi}{\partial x^2} = 1 \]

\[ \frac{\partial^2 \Phi}{\partial y^2} = 1 \]
Figure 5.4. Patch Test. Constant normal and shear stress.

\[ \Phi = 1 + 2x + 3y + 4xy + 5x^2 + 6y^2 \]

\[ \frac{\partial^2 \Phi}{\partial x \partial y} = 4 \]

\[ \Phi_{xx} = 10 \]

\[ \Phi_{yy} = 12 \]
Figure 5.5. Square plate subjected to parabolically distributed edge loads. Geometry-loading.

Figure 5.6. One quarter of the plate for analysis and boundary conditions.
The material properties are assumed as follows:

- Poisson's Ratio $v = 0.3$
- Plate Thickness $t = 0.1\text{mm}$
- Plate Dimensions $L = 400\text{mm}$
- Modulus of Elasticity $E = 70000\text{MPa}$

One quarter of the structure is analyzed, owing to the double symmetry. Each of the element type in the boundary traction element family is used (Figure 5.5). The purpose of this endeavor is to demonstrate the usefulness of the additional degrees of freedom at the boundaries. It is observed from the tests, in general, that the elements with additional degrees of freedom perform better (Table 5.1). In particular, the element type 7 results in better stress predictions. This may be attributed to the element having appropriate degrees of freedom to interpolate the prescribed traction in this problem.
Table 5.1 Stress values using boundary traction element family

<table>
<thead>
<tr>
<th>NODE</th>
<th>STRESS</th>
<th>BOUNDARY TRACTION ELEMENT FAMILY</th>
<th>LEAST WORK</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_y$</td>
<td>1  2  3  4  5  6  7  8  9</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.0</td>
<td>5.00 5.00 5.00 5.00 5.00 5.00 0.0 0.0 0.0</td>
<td>0.0 0.0 0.0</td>
</tr>
<tr>
<td></td>
<td>$\sigma_x$</td>
<td>0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\tau_{xy}$</td>
<td>0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5.00</td>
<td>35.0 35.0 35.0 35.0 35.0 35.0 30.0 30.0 30.0</td>
<td>30.0 30.0 30.0</td>
</tr>
<tr>
<td></td>
<td>$\sigma_y$</td>
<td>7.74 7.74 12.85 7.63 7.63 12.76 12.51 7.37 7.37</td>
<td>12.51</td>
</tr>
<tr>
<td></td>
<td>$\sigma_x$</td>
<td>7.74 7.74 12.85 7.63 7.63 12.76 12.51 7.37 7.37</td>
<td>12.51</td>
</tr>
<tr>
<td></td>
<td>$\tau_{xy}$</td>
<td>0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>24.64</td>
<td>24.64 24.89 27.37 27.37 27.55 27.62 27.62 27.81 25.86</td>
<td>25.86</td>
</tr>
<tr>
<td></td>
<td>$\sigma_y$</td>
<td>-7.74 -7.74 -4.58 -7.63 -7.63 -4.45 -4.11 -7.37 -4.11</td>
<td>-4.14</td>
</tr>
<tr>
<td></td>
<td>$\sigma_x$</td>
<td>-7.74 -7.74 -4.58 -7.63 -7.63 -4.45 -4.11 -7.37 -4.11</td>
<td>-4.14</td>
</tr>
<tr>
<td></td>
<td>$\tau_{xy}$</td>
<td>0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\sigma_y$</td>
<td>0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\sigma_x$</td>
<td>0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\tau_{xy}$</td>
<td>0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0</td>
<td></td>
</tr>
</tbody>
</table>

Diagram shows a square with nodes labeled 1, 2, 3, 4.
The rectangular plate, comprising the entire model, is analyzed by making use of different combinations of the element types from the library. The results are depicted in Table 5.2, which show the stress values at the center. On examination of the results, we may infer that the higher order rectangular element predicts better stresses for the mesh size considered.
Table 5.2. Stress values at center.

<table>
<thead>
<tr>
<th>STRESS</th>
<th>GRID (1)</th>
<th>GRID (2)</th>
<th>GRID (3)</th>
<th>GRID (4)</th>
<th>GRID (5)</th>
<th>GRID (6)</th>
<th>LEAST WORK</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_x$</td>
<td>27.62</td>
<td>27.37</td>
<td>27.62</td>
<td>25.40</td>
<td>25.80</td>
<td>27.37</td>
<td>25.86</td>
</tr>
<tr>
<td>$\Sigma_{xy}$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
The stress values are improved owing to the higher order completeness of the polynomials, at the expense of an increase in the total number of unknowns. Figures (5.8) and (5.9).

A. Gridworks for boundary traction element family.

B. Gridworks for rectangular elements and triangular elements.

Figure 5.7 Gridworks for analysis
In order to evaluate the performance, the problem is analyzed using different element types and mesh sizes. The gridworks for the analysis are given in Figure 5.7. The element mesh sizes considered are 4, 16, and 64 respectively. The stress values at the center of the model are summarized for different element types in Table (5.3). The plot of the stresses at the center versus the gridwork refinement is given in Figure (5.10).
Figure 5.8. Comparison of solution along x-axis. Element type BTE 5.
Figure 5.9. Comparison of solution along x-axis. Element type BTE 10.
Table 5.3. Comparison of stress values at center using different element types.

<table>
<thead>
<tr>
<th>Number of Elemts.</th>
<th>16 dof BTE 5</th>
<th>24 dof BTE 10</th>
<th>BTE Family</th>
<th>9 dof ΔDRT</th>
<th>Ref. [135]</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>27.369</td>
<td>25.408</td>
<td>27.623</td>
<td>20.000</td>
<td>25.856</td>
</tr>
<tr>
<td></td>
<td>-7.631</td>
<td>-4.592</td>
<td>-7.376</td>
<td>0.0</td>
<td>-4.143</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
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<tr>
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<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>64</td>
<td>26.01</td>
<td>25.76</td>
<td>25.91</td>
<td>25.52</td>
<td>25.856</td>
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<td></td>
<td>-4.31</td>
<td>-4.242</td>
<td>-4.22</td>
<td>-3.85</td>
<td>-4.143</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.23</td>
<td>0.0</td>
</tr>
</tbody>
</table>

\[ \sigma = \sigma_0 \left(1 - 4 \left(\frac{a}{L}\right)^2\right) \]
The higher order rectangles give better accuracy, even with a coarse mesh. The boundary traction elements family give very close results with a mesh consisting of 16 d.o.f. elements. We might have expected that the boundary traction elements perform better than the 16 d.o.f. elements mesh. On the contrary, it is observed that the results obtained are very close. This may be attributed to the simple nature of the traction considered. We may expect superior performance of the boundary traction elements family for an arbitrary distribution of prescribed tractions, for which the elements are developed.

The same example problem is utilized to validate the proposed method of handling the prescribed traction type boundary conditions. The input data is modified to reflect the change in the switch to flag this mode of analysis. The results are obtained using both the Lagrange Multipliers and the proposed method, Table (5.4). The results given by both the approaches are observed to be identical and further are in complete agreement with the reported values in Reference [61], which adopted Lagrange multipliers for the solution procedure.
### Table 5.4 Stress values at center using 16 d.o.f. Boundary Traction Element, BTE 5.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>( \sigma_y ) 25.86</td>
<td>27.36</td>
<td>27.36</td>
<td>27.36</td>
</tr>
<tr>
<td></td>
<td>( \sigma_x ) -4.14</td>
<td>-</td>
<td>-7.63</td>
<td>-7.63</td>
</tr>
<tr>
<td></td>
<td>( \tau_{xy} ) 0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>16</td>
<td>( \sigma_y ) 25.86</td>
<td>26.61</td>
<td>26.61</td>
<td>26.61</td>
</tr>
<tr>
<td></td>
<td>( \sigma_x ) -4.14</td>
<td>-</td>
<td>-4.63</td>
<td>-4.63</td>
</tr>
<tr>
<td></td>
<td>( \tau_{xy} ) 0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
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<td>26.01</td>
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<tr>
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<tr>
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#### 5.2.3. Cantilever Beam Under End Load.

Figure 5.11 illustrates the next example problem considered, a cantilever beam of unit thickness subjected to an end load \( P \). The end load is distributed as a parabolically varying shear stresses. The geometry and the material data is detailed in the Figure 5.11.

The stress distributions at cross-sections 0.25L, 0.50L, 0.75L distance from the built-in end are given in Figure 5.12. Comparison solutions are taken from the beam theory and from Ref[20]. In solution 16 d.o.f. rectangular elements are utilized. The solution obtained using 16 elements gridwork coincides with the beam theory solution.
Figure 5.10. Numerical comparison: Assumed stress function finite element method, assumed displacement, analytical solution.
Figure 5.11 Cantilever beam under parabolically distributed end load.

\[ \tau_{xy} = -1.5(1-4y^2) \]
\( \sigma_x \) and \( \tau_{xy} \) distribution (16 d.o.f. element) BTE: 5

Figure 5.12 Normal and shear stress distribution at 0.25L, 0.50L, and 0.75L.
5.2.4. Cantilever Beam Under End Shear, and End Moment

Figure 5.13 illustrates a cantilever beam under end moment, and end shear. The results are given in terms of the distribution of normal stress $\sigma_x$ along $y=h/4$ axis for the end shear. The comparison solutions are obtained from beam theory and the finite element displacement model*. For displacement model solution constant strain quadrilateral and linear strain quadrilateral are utilized. The equilibrium model analysis is performed using 24 d.o.f. rectangular element.

The stresses predicted by ASFFEM are in close agreement with the beam theory solution. The Constant Strain Quadrilateral (CSQ) gives stresses with an error of sixtyseven percent (67%), while the Linear Strain Quadrilateral (LSQ) improves the stresses to within thirtyone per cent (31%). Figure (5.14).

The results obtained from the end moment loading case are given as the distribution of normal stress at $x=L/2$ cross-section. The percentage of error in stresses is about 0.06 for ASFFEM, while the displacement model LSQ and CSQ result in 2.5 and 64 respectively. Figure (5.15).

* The displacement model solutions are obtained by utilizing GIFTS program. [104].
Figure 5.13. Cantilever Beam. (a) end shear, (b) end moment
Figure 5.14. Normal stress variation at $y=h/4$. 

- Built-in end
- Beam theory solution
- Assumed displacement, Linear strain quadrilateral
- Assumed stress function, Higher-order rectangle
- Assumed displacement, Constant strain quadrilateral

Distance from built-in end, (in.)
Figure 5.15 Normal stress distribution. End moment.
5.2.5 Cantilever wedge under uniformly distributed load

In order to illustrate the versatility of the proposed ASFFEM, a cantilever wedge under uniformly distributed load is considered, (Figure 5.16). The wedge problem has mixed type boundary conditions (prescribed displacements and tractions).
Figure 5.16 Uniformly distributed loaded wedge.
The variation of normal stress at the built-in end is depicted in Figure (5.17). A close agreement is obtained with the analytical solution [135]. The finite element solution given in Reference [92] is also shown. Since the prescribed displacements are zero at the built-in end, no contribution is obtained to the complementary energy functional (i.e., $V^*$) due to these displacements. Hence, it may be noted that there are no constraints applied at the built-in edge. This reference calculates the stresses by removing the tip element and prescribing the exact stresses. However, in the present approach this is not required.

The shear stress distribution at the built-in end is shown in Figure (5.18). The shear stress variation given by the present approach is observed to be closer to the analytical solution than that given in Reference [92].
Figure 5.17 Normal stress values at built-in end.
Figure 5.18 Shear stress values at built-in end.
5.2.6. Rectangular plate with multiple cutouts

In order to demonstrate performance in multiply-connected regions, a plate with two cutouts under in-plane loading is analyzed (Figure 5.19). The analytical solution for this problem is given by Savin [67]. Figure 5.20 depicts the principal minimum stress distribution, and it is seen to be in agreement with the values given by Savin.

5.2.7. Square plate with a rectangular cutout

A square plate with a rectangular cutout under tension is selected as another application, the data and loading of which is given in Figure 5.21. The major stress variation at the critical section are given and a stress concentration factor of 2.23 is obtained.
Figure 5.19 Rectangular plate with multiple cutouts.
Figure 5.20 Distribution of minimum principal stresses.
Figure 5.21 Square plate with a rectangular cutout.
5.2.8 Square plate with a circular cutout

A square plate with a circular cutout is analyzed using the $C^0$ continuous triangular elements. The geometry and loading conditions are given in Figure (5.22), and the gridwork in Figure (5.23). Symmetry conditions are invoked to reduce the size of the problem. The ratio of normal stress to the applied load per unit thickness is plotted against the distance from the center of the hole.

A ratio of 2.2 is obtained at the edge of the cutout by using the ASFFEM, while the displacement method by using the constant strain triangle yields a value of 1.78.
\[ \nu = 0.3, \quad E = 1 \times 10^4 \text{k/in}^2, \quad t = 0.1'' \]

Figure 5.22 Square plate with a circular cutout at center
Figure 5.23 Gridwork for analysis.
Figure 5.24 The variation of $\sigma_{yt}/p$ ratio along $x$-axis.
6.1. Concluding Remarks

The generalized displacement-force relations for the boundary traction element family and for two higher order rectangular elements have been formulated. In order to obtain elements which interpolate the prescribed tractions exactly, the continuity of the higher order derivatives is required along the edges on their boundary. This necessitates the usage of more variables on the edges, which is accomplished by utilizing the blending interpolation technique. Also, a $C^0$ continuous plate bending triangular element has been adapted to be used as a plate stretching element.

A computer program was developed for the verification of the overall method and element formulations. An element library was incorporated in the program with an option for the user to select any of the elements for the specific problem at hand.

The elements have been evaluated through performance of analyses, patch tests, and comparison of results with those obtained from the finite element displacement model, and from classical solutions. The
results obtained are in agreement with the comparison solutions.

The mesh generated with the variable degrees of freedom elements at the boundary and 16 degrees of freedom element at the interior results in better stress predictions than using only 16 degrees of freedom elements throughout the region. This may be attributed to the values of the degrees of freedoms along the edges being assigned the prescribed tractions exactly.

A high degree of agreement is attained with relatively coarse grids using the 24 degrees of freedom rectangular elements. This can be ascribed to both the completeness of polynomial up to quartic terms and to the additional degrees of freedoms introduced. It may also be noted that the number of unknowns are increased by the use of these elements.

All of the rectangular elements formulated may be used as plate bending elements with the finite element displacement model. In the process of element testing the displacement model is used to check the element formulation.

A different approach is proposed herein to account for the prescribed traction boundary conditions. The stress function and the normal derivatives are assigned their values directly on the boundary nodes.
The proposed approach eliminates the need for the additional constraint equations resulting from the prescribed tractions. It may also be noted that the number of equations to be solved remains unchanged.

The importance of obtaining continuous and more accurate stress values was emphasized in the introduction. The displacement method has proved to be inadequate for stress predictions in several applications. The need for better stress predictions, in a way, formed the motivation for this research. The application areas of the present method is broad; such as, aerospace, civil, mechanical and nuclear structural components.

The plate bending analysis with the displacement method requires a $C^1$ continuity. Difficulties are encountered in constructing $C^1$ conformable elements. The present formulation, which requires only a $C^0$ continuity for bending elements is well suited in such applications.

The procedures to extend the formulations to account for inelastic analysis, shells, and axisymmetric solids are briefly outlined. It may be noted that although the potential exists for extensions, a substantial research effort needs to be devoted to integrate these formulations into the analysis.
Some limitations of the method do exist in obtaining the displacements. The following observations can be made in this regard. In this formulation, the primary unknowns are the stress function parameters, the conjugate variables of which are the generalized displacements. Three approaches may be pursued in order to obtain an estimate of the displacements. The displacements at the nodes where the tractions are prescribed may be obtained directly, analogous to computing the support reactions in the conventional finite element displacement model. It is observed that wherever the displacements are zero, the corresponding generalized displacements are also zero. It may be concluded on examination of generalized displacements that the actual displacements may be obtained from them by a suitable transformation.

The second scheme is to obtain the displacements through the use of the unit load method. One other scheme is to perform direct integration using the strain-displacement relations, yielding relative displacements at the nodes.

An important outcome of the present research is the ease with which a design analyst familiar with the finite element displacement method can use the present method effectively. It may be mentioned that, the
proposed method is such that the user need only input
the node numbers lying on the boundaries in addition to
the usual input data required for the finite element
displacement analysis - namely, relevant geometry,
material properties, loads and boundary conditions. The
program is designed to calculate automatically the
required integrals in preparation for the solution.

6.2. Some Future Developments

Extensions of the present formulation to account
for body forces, initial stresses and material
nonlinearities are possible, and are briefly discussed.

Body forces. In the presence of body forces, the
equilibrium equations and stress-stress function
relations are

for constant body forces \((X = Xc, Y = Yc)\)

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + Xc = 0
\]

\[
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + Yc = 0
\]

and

\[
\sigma_x = \frac{\partial^2 \Phi}{\partial y^2} - Xc x
\]

\[
\sigma_y = \frac{\partial^2 \Phi}{\partial x^2} - Yc y
\]

\[
\tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}
\]
for the body forces derived from a potential, $V$

\[
(x = \frac{\partial V}{\partial x}, \quad y = -\frac{\partial V}{\partial y})
\]

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + x &= 0 \\
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + y &= 0
\end{align*}
\]

and

\[
\begin{align*}
\sigma_x &= \frac{\partial^2 \phi}{\partial y^2} + V \\
\sigma_y &= \frac{\partial^2 \phi}{\partial x^2} + V \\
\tau_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y}
\end{align*}
\]

It may be noted that in the presence of body forces, which are derived from a potential function, the compatibility conditions are different for plane stress and plane strain cases. Additionally, due account must be taken of the body forces in the expression for the complementary energy.

**Initial Strain.** The constitutive relationships, when initial strains are present, are given by

\[
\{ \sigma \} = [ E ] \{ \varepsilon \} - [ E ] \{ \varepsilon \}^i
\]
Substituting this expression in to the complementary energy functional, we may obtain the initial stress vector.

**Material Nonlinearities.** The application of complimentary energy formulation in material nonlinearities is well established [53, 61, 93, 133]. By combining the present formulation with different flow rules and with different solution procedures [5, 64, 76, 77, 78], an efficient method may be obtained for inelastic analysis.

**Shells.** By the use of Airy Stress function, it is possible to compute the displacements in cylindrical shells, under the action of membrane stresses [62]. This may be achieved by expressing the stress resultants in terms of Airy Stress function. The equations of equilibrium are satisfied directly.

**Geometric Modelling/Analysis.** The present formulation is based partly on the Blending interpolation technique. The evidence presented by numerical experiments suggest a promising potential for the application to practical problems in stress predictions. Several investigators have utilized the blending interpolation technique for geometric/solid modelling [124]. It may be beneficial to integrate the
analysis capabilities developed here and the geometric modelling into a common data-base for practical applications (CAD/CAM) of ASFFEM.
APPENDIX A

HERMITIAN POLYNOMIALS
\[ C_1 = \frac{1}{4} (2 - 3r + r^3) \]
\[ C_{1r} = \frac{1}{4a} (1 - r - r^2 + r^3) \quad -1 \leq r \leq 1 \]
\[ C_2 = \frac{1}{4} (2 + 3r - r^3) \quad r = \frac{x}{a} \]
\[ C_{2r} = \frac{1}{4a} (-1 - r + r^2 + r^3) \]

Figure A.1  The cubic Hermitian polynomials.
for $-1 \leq r \leq 1$ and $r = x/a$

$Q_1(r) = \frac{1}{16} \left(-3r^5 + 10r^3 - 15r + 8\right)$

$Q_{1r}(r) = \frac{1}{16a} \left(-3r^5 + r^4 + 10r^3 - 6r^2 - 7r + 5\right)$

$Q_{1rr}(r) = \frac{1}{16a^2} \left(-r^5 + r^4 + 2r^3 - 2r^2 - r + 1\right)$

$Q_2(r) = \frac{1}{16} \left(3r^5 - 10r^3 + 5r + 8\right)$

$Q_{2r}(r) = \frac{1}{16a} \left(-3r^5 - r^4 + 10r^3 + 6r^2 - 7r - 5\right)$

$Q_{2rr}(r) = \frac{1}{16a^2} \left(r^3 + r^4 - 2r^3 - 2r^2 + r + 10\right)$

Figure A.2 The quintic Hermitian polynomials.
APPENDIX B

ELEMENT LIBRARY AND SHAPE FUNCTIONS
Table B.1 Element Library.

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Table B.2 Shape Functions. Boundary Traction Element BTE 1.

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<td>$C_2r(r)C_1(s)$</td>
<td>$C_2r(r)C_2(s)$</td>
<td>$C_1r(r)C_2(s)$</td>
<td></td>
</tr>
<tr>
<td>$\phi_s$</td>
<td>$C_1(r)C_1s(s)$</td>
<td>$C_2(r)Q_1s(s)$</td>
<td>$C_2(r)Q_2s(s)$</td>
<td>$C_1(r)C_2s(s)$</td>
<td></td>
</tr>
<tr>
<td>$\phi_{rs}$</td>
<td>$C_1(r)C_1s(s)$</td>
<td>$C_2r(r)C_1s(s)$</td>
<td>$C_2r(r)C_2s(s)$</td>
<td>$C_1r(r)C_2s(s)$</td>
<td></td>
</tr>
<tr>
<td>$\phi_{ss}$</td>
<td>-</td>
<td>$C_2(r)Q_1ss(s)$</td>
<td>$C_2(r)Q_2ss(s)$</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>
Table B.8 Shape Functions. Boundary Traction Element BTE 7.

<table>
<thead>
<tr>
<th>d.o.f.</th>
<th>NODE</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi )</td>
<td>( C_1(r)Q_1(s) + Q_2(r)C_1(s) - C_1(r)C_1(s) )</td>
<td>( Q_2(r)C_1(s) )</td>
<td>( C_2(r)C_2(s) )</td>
<td>( C_1(r)Q_2(s) )</td>
<td></td>
</tr>
<tr>
<td>( \phi r )</td>
<td>( Q_1(r)C_1(s) )</td>
<td>( Q_2(r)C_1(s) )</td>
<td>( C_2(r)C_2(s) )</td>
<td>( C_1(r)C_2(s) )</td>
<td></td>
</tr>
<tr>
<td>( \phi s )</td>
<td>( C_1(r)Q_1(s) )</td>
<td>( C_2(r)C_1(s) )</td>
<td>( C_2(r)C_2(s) )</td>
<td>( C_1(r)Q_2(s) )</td>
<td></td>
</tr>
<tr>
<td>( \phi s r )</td>
<td>( C_1(r)C_1(s) )</td>
<td>( C_2(r)C_1(s) )</td>
<td>( C_2(r)C_2(s) )</td>
<td>( C_1(r)C_2(s) )</td>
<td></td>
</tr>
<tr>
<td>( \phi r r )</td>
<td>( Q_1(r)C_1(s) )</td>
<td>( Q_2(r)C_1(s) )</td>
<td>( _ )</td>
<td>( _ )</td>
<td></td>
</tr>
<tr>
<td>( \phi s s )</td>
<td>( C_1(r)Q_1(s) )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( C_1(r)Q_2(s) )</td>
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</table>
Table B.9 Shape Functions. Boundary Traction Element BTE 8.

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<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>$Q_1(r)Q_1(s)$</td>
<td>$Q_2(r)C_1(s)$</td>
<td>$C_2(r)C_2(s)$</td>
<td>$C_1(r)C_2(s)$</td>
<td></td>
</tr>
<tr>
<td>$\phi_r$</td>
<td>$Q_1r(r)C_1(s)$</td>
<td>$Q_2r(r)C_1(s)$</td>
<td>$C_2r(r)C_2(s)$</td>
<td>$C_1r(r)C_2(s)$</td>
<td></td>
</tr>
<tr>
<td>$\phi_s$</td>
<td>$C_1(r)C_1s(s)$</td>
<td>$C_2(r)C_1s(s)$</td>
<td>$C_2(r)C_2s(s)$</td>
<td>$C_1(r)C_2s(s)$</td>
<td></td>
</tr>
<tr>
<td>$\phi_r$s</td>
<td>$C_1r(r)C_1s(s)$</td>
<td>$C_2r(r)C_1s(s)$</td>
<td>$C_2r(r)C_2s(s)$</td>
<td>$C_1r(r)C_2s(s)$</td>
<td></td>
</tr>
<tr>
<td>$\phi_{rr}$</td>
<td>$Q_1rr(r)C_1(s)$</td>
<td>$Q_2rr(r)C_1(s)$</td>
<td>-</td>
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Table B.10 Shape Functions. Boundary Traction Element BTE 9.

<table>
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<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>ϕ</td>
<td></td>
<td>Q₁(r)C₁(s)</td>
<td>+Q₂(r)C₁(s)</td>
<td>C₂(r)Q₂(s)</td>
<td>C₁(r)C₂(s)</td>
</tr>
<tr>
<td>ϕr</td>
<td></td>
<td>Q₁r(r)C₁(s)</td>
<td>Q₂r(r)C₁(s)</td>
<td>C₁r(r)C₂(s)</td>
<td>C₁r(r)C₂(s)</td>
</tr>
<tr>
<td>ϕs</td>
<td></td>
<td>C₁(r)C₁s(s)</td>
<td>C₂(r)C₁s(s)</td>
<td>C₂(r)Q₂s(s)</td>
<td>C₁(r)C₂s(s)</td>
</tr>
<tr>
<td>ϕrs</td>
<td></td>
<td>C₁r(r)C₁s(s)</td>
<td>C₂r(r)C₁s(s)</td>
<td>C₁r(r)Q₂s(s)</td>
<td>C₁r(r)C₂s(s)</td>
</tr>
<tr>
<td>ϕrr</td>
<td></td>
<td>Q₁rr(r)C₁(s)</td>
<td>Q₂rr(r)C₁(s)</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>ϕss</td>
<td></td>
<td>-</td>
<td>C₂(r)Q₁ss(s)</td>
<td>C₂(r)Q₁ss(s)</td>
<td>-</td>
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</table>
Table B.11 Shape Functions. Boundary Traction Element \textbf{HRE} 10.

<table>
<thead>
<tr>
<th>d.o.f.</th>
<th>NODE 1</th>
<th>NODE 2</th>
<th>NODE 3</th>
<th>NODE 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>$Q_1(r)Q_1(s)$</td>
<td>$Q_2(r)Q_1(s)$</td>
<td>$Q_2(r)Q_2(s)$</td>
<td>$Q_1(r)Q_2(s)$</td>
</tr>
<tr>
<td>$\phi_r$</td>
<td>$Q_1(r)Q_1(s)$</td>
<td>$Q_2(r)Q_1(s)$</td>
<td>$Q_2(r)Q_2(s)$</td>
<td>$Q_1(r)Q_2(s)$</td>
</tr>
<tr>
<td>$\phi_s$</td>
<td>$Q_1(r)Q_1(s)$</td>
<td>$Q_2(r)Q_1(s)$</td>
<td>$Q_2(r)Q_2(s)$</td>
<td>$Q_1(r)Q_2(s)$</td>
</tr>
<tr>
<td>$\phi_{rs}$</td>
<td>$Q_1(r)Q_1(s)$</td>
<td>$Q_2(r)Q_1(s)$</td>
<td>$Q_2(r)Q_2(s)$</td>
<td>$Q_1(r)Q_2(s)$</td>
</tr>
<tr>
<td>$\phi_{rr}$</td>
<td>$Q_1(r)Q_1(s)$</td>
<td>$Q_2(r)Q_1(s)$</td>
<td>$Q_2(r)Q_2(s)$</td>
<td>$Q_1(r)Q_2(s)$</td>
</tr>
<tr>
<td>$\phi_{ss}$</td>
<td>$Q_1(r)Q_1(s)$</td>
<td>$Q_2(r)Q_1(s)$</td>
<td>$Q_2(r)Q_2(s)$</td>
<td>$Q_1(r)Q_2(s)$</td>
</tr>
</tbody>
</table>
Table B.11 Shape Functions for \( C^0 \) continuous Triangular Element DKT.

| Shape Functions, [10] | \[G\] = \( \frac{1}{2A} \) 
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( N = 2(1-\xi-\eta)(1/2-\xi-\eta) )</td>
<td>[ \begin{bmatrix} -x_{21}H_{11}^T -x_{12}H_{12}^T \ y_{31}H_{11}^T + y_{12}H_{12}^T \ -x_{31}H_{11}^T - x_{12}H_{12}^T + y_{12}H_{11}^T - y_{31}H_{12}^T \end{bmatrix} ]</td>
</tr>
<tr>
<td>( N = \xi(2\xi-1) )</td>
<td></td>
</tr>
<tr>
<td>( N = \eta(2\eta-1) )</td>
<td></td>
</tr>
<tr>
<td>( N = 4\xi\eta )</td>
<td></td>
</tr>
<tr>
<td>( N = 4\eta(1-\xi-\eta) )</td>
<td></td>
</tr>
<tr>
<td>( N = 4\xi(1-\xi-\eta) )</td>
<td></td>
</tr>
</tbody>
</table>

The flexibility matrix

\[
[F] = 2A \int \int \begin{bmatrix} [G] \end{bmatrix}^T \begin{bmatrix} [E] \end{bmatrix}^{-1} [G] \, d\xi \, d\eta
\]
Table B.12 Shape Functions. Boundary Traction Element HRE 12.

<table>
<thead>
<tr>
<th>d.o.f.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>$C_1(r)Q_1(s)$</td>
<td>$C_2(r)Q_1(s)$</td>
<td>$C_2(r)Q_2(s)$</td>
<td>$C_1(r)Q_2(s)$</td>
</tr>
<tr>
<td>$\phi_r$</td>
<td>$Q_1Q_1(r)C_1(s)$</td>
<td>$Q_2Q_1(r)C_1(s)$</td>
<td>$Q_2Q_1(r)C_2(s)$</td>
<td>$Q_1Q_1(r)C_2(s)$</td>
</tr>
<tr>
<td>$\phi_s$</td>
<td>$C_1(r)Q_1(s)$</td>
<td>$C_2(r)Q_1(s)$</td>
<td>$C_2(r)Q_2(s)$</td>
<td>$C_1(r)Q_2(s)$</td>
</tr>
<tr>
<td>$\phi_{rs}$</td>
<td>$C_1(r)C_1(s)$</td>
<td>$C_2(r)C_1(s)$</td>
<td>$C_2(r)C_2(s)$</td>
<td>$C_1(r)C_2(s)$</td>
</tr>
<tr>
<td>$\phi_{rr}$</td>
<td>$Q_1Q_1(r)C_1(s)$</td>
<td>$Q_2Q_1(r)C_1(s)$</td>
<td>$Q_2Q_1(r)C_2(s)$</td>
<td>$Q_1Q_1(r)C_2(s)$</td>
</tr>
<tr>
<td>$\phi_{ss}$</td>
<td>$C_1(r)Q_1(s)$</td>
<td>$C_2(r)Q_1(s)$</td>
<td>$C_2(r)Q_2(s)$</td>
<td>$C_1(r)Q_2(s)$</td>
</tr>
</tbody>
</table>
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