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**A HIERARCHICAL ORDERING OF AREA SKILLS BASED ON RULES,
REPRESENTATIONS, AND SHAPES**

The University of Arizona

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A HIERARCHICAL ORDERING OF AREA SKILLS BASED ON
RULES, REPRESENTATIONS, AND SHAPES

By
Adam Schnaps

A Dissertation Submitted to the Faculty of the
DEPARTMENT OF EDUCATIONAL PSYCHOLOGY
In Partial Fulfillment of the Requirements
For the Degree of
DOCTOR OF PHILOSOPHY
In the Graduate College
THE UNIVERSITY OF ARIZONA

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THE UNIVERSITY OF ARIZONA
GRADUATE COLLEGE

As members of the Final Examination Committee, we certify that we have read
the dissertation prepared by Adam Schnaps

entitled A HIERARCHICAL ORDERING OF AREA SKILLS BASED ON RULES,
REPRESENTATIONS, AND SHAPES

and recommend that it be accepted as fulfilling the dissertation requirement
for the Degree of Doctor of Philosophy.

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Final approval and acceptance of this dissertation is contingent upon the
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College.

I hereby certify that I have read this dissertation prepared under my
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TABLE OF CONTENTS

	Page
LIST OF TABLES.....	v
LIST OF ILLUSTRATIONS.....	vii
ABSTRACT.....	viii
I. INTRODUCTION.....	1
II. REVIEW OF LITERATURE.....	4
Area.....	4
Hierarchies.....	8
III. THE PROBLEM.....	10
Hypothesized Hierarchical Foundations.....	10
Hypothesized Task Hierarchy and Associated Rules.....	16
IV. METHOD.....	28
Subjects.....	28
Procedure.....	28
Data Analysis.....	30
V. RESULTS.....	37
VI. DISCUSSION.....	68
APPENDIX A: THE TEST ITEMS.....	77
REFERENCES.....	84

LIST OF TABLES

Table	Page
1. Rules Associated with the Task Items.....	27
2. Subjects by Grade and Sex, and by Grade and Ethnicity.....	29
3. Latent-class Analysis for Hypothesized Area Skills.....	40
4. Preferred Models Observed and Expectency Frequency Counts, Residuals, LR, PRSN, and Latent Class Probabilities for Item Pairs 1,2 and 3,4.....	43
5. Preferred Models Observed and Expectency Frequency Counts, Residuals, LR, PRSN, and Latent Class Probabilities for Item Pairs 1,2 and 5,6.....	44
6. Preferred Models Observed and Expectency Frequency Counts, Residuals, LR, PRSN, and Latent Class Probabilities for Item Pairs 3,4 and 5,6.....	45
7. Preferred Models Observed and Expectency Frequency Counts, Residuals, LR, PRSN, and Latent Class Probabilities for Item Pairs 3,4 and 11,12.....	46
8. Preferred Models Observed and Expectency Frequency Counts, Residuals, LR, PRSN, and Latent Class Probabilities for Item Pairs 3,4 and 13,14.....	47
9. Preferred Models Observed and Expectency Frequency Counts, Residuals, LR, PRSN, and Latent Class Probabilities for Item Pairs 13,14 and 11,12.....	48

LIST OF TABLES--Continued

	Page
10. Preferred Models Observed and Expectency Frequency Counts, Residuals, LR, PRSN, and Latent Class Probabilities for Item Pairs 15,16 and 17,18.....	49
11. Preferred Models Observed and Expectency Frequency Counts, Residuals, LR, PRSN, and Latent Class Probabilities for Item Pairs 15,16 and 11,12.....	50
12. Preferred Models Observed and Expectency Frequency Counts, Residuals, LR, PRSN, and Latent Class Probabilities for Item Pairs 17,18 and 13,14.....	51
13. Number of Correct Responses per Item.....	57
14. Description of Process Codes.....	58
15. Item Numbers Associated with Specific Codes.....	61
16. Process Code Totals for Each Item Pair by Grade.....	62
17. Process Code Totals for Each Item Pair.....	63

LIST OF ILLUSTRATIONS

	Page
Figure	
1. Problem Demonstrating Area Concept.....	6
2. Nonstandard Shaped Region.....	11
3. Region with Grid.....	12
4. Samples of Different Area Shapes.....	13
5. Shearing of a Rectangle.....	15
6. Hierarchy Foundation Flow.....	17
7. Irregular Shaped Region with Grid.....	18
8. Hierarchy of Area Skills Based on Representation and Shape.....	26
9. Rectangle Area Problem.....	35
10. Hypothesized Skill Orderings.....	38

ABSTRACT

A hierarchy of skills in the measurement topic of area was validated on three-hundred and six students between grades six and nine. The hierarchy of skills was based on the rules underlying the individual skills. When a rule for one skill was considered a component of a rule for another skill, then the two skills were hypothesized to be hierarchically ordered. In addition, if a simple rule for a particular skill was replaced by a more complex rule, resulting in a different skill, then these two skills were hypothesized to be hierarchically ordered. The physical representations of the area tasks, as well as the shapes of the area figures were hypothesized as influencing the skill orderings.

The use of Latent-class analysis revealed that seven of the nine skill orderings analyzed were hierarchically ordered based on difficulty level and not prerequisites. The other two skill orderings indicated equiprobable partial mastery classes.

In addition to Latent-class analysis, the incorrect processes used by the students were coded and tabulated. The results revealed that 1) nonstandard shaped area problems were the most difficult for this sample, 2) the most frequent process associated with incorrect responses involved the addition of numbers shown in area problem figures,

3) the second most frequent process involved some form of multiplication, without regard to the area concepts inherent in the task, and 4) students beyond the sixth grade made more errors involving multiplication processes than errors involving addition processes.

The study revealed that the use of rules, representations and shapes as the basis for a hierarchy does appear to have merit. In addition, process analysis revealed that students respond in a large variety of ways when they do not know the correct process for area tasks.

INTRODUCTION

There have been several attempts at studying mathematical concepts at various grade levels. One major focus has been on the examination of basic math skills such as addition and subtraction (Carpenter, Moser, & Romberg, 1982; Riley, Greeno, & Heller, in press). Another area of investigation has been mathematical learning at the high school level (Carry, Lewis, & Bernard, 1980; Greeno, 1978). Nonetheless, numerous gaps in understanding students' learning still exist in many areas of mathematics. One subject of particular importance which has not been investigated is the measurement topic of area.

The investigation of the topic of area is important for a variety of reasons. The concept of area is one of the most commonly used domains of measure in everyday life (Hirstein, Lamb, & Osborne, 1978). For example, the square footage of a house is often a major determinant of the selling price. Also, the amount of paint required for a house is usually dependent on the area of the wall that will be painted. In addition to common use in daily life experiences, the concepts underlying area form the basis for many explanations of other mathematical operations and concepts (Hirstein, Lamb, & Osborne, 1978). For instance, area concepts are often used as a means for introducing students to fractions. In addition, area concepts also help children understand that certain numbers are composites of other numbers (e.g. 24 is composed of 6 by 4, 8 by 3, 12 by 2, and 24 by 1). And finally, the subject matter

of area is included in all school math curricula and spans several grade levels (Heath, 1975; Holt, Rinehart & Winston, 1974; Scott, Foresman & Company, 1975).

Much of the recent work on mathematical skills has focused on the identification of knowledge structures associated with specific mathematical tasks (see for example Carpenter, Moser, & Romberg, 1982). However, attention has also been given to the manner in which knowledge structures change in the course of acquiring sets of interrelated skills (Siegler, in press). One way in which the change in knowledge structures can be represented is through the hierarchical sequencing of skills. A hierarchy of skills is composed of tasks that may be equivalent in their relationship or that may be ordered such that subordinate tasks are prerequisite to superordinate tasks (Gagne, 1977). For example, consider the case of four tasks representing two hierarchically related skills. Assume that two of the tasks are associated with the subordinate skill, and the other two were linked to the superordinate skill. This condition would result in an equivalent relationship between the two tasks associated with the subordinate skill and between the two tasks associated with the superordinate skill. In addition, there would be a prerequisite relationship between the subordinate and superordinate tasks (Bergan, Towstapiat, Cancelli, & Karp, 1982).

In general, the aim of a validated hierarchy of skills is to promote the acquisition of basic skills in education. More specifically, validated hierarchical sequences open the way for the development of assessment tools that reflect changes in student knowledge associated with educational progress (Bergan, 1981; Bergan, Stone, & Feld, in

press). As a result of validated hierarchies, tests may be developed that can be used to individualize the placement of students in an instructional sequence (Glaser & Nitko, 1971; Nitko & Hsu, 1974; Resnick, Wang, & Kaplan, 1973). In addition, the placement of a student in a valid instructional sequence will enable the student to encounter readily attainable goals and at the same time avoid the possibility of having to repeat objectives already mastered (Bergan & Cancelli, 1978). Finally, the use of placement tests may also be used at the end of a sequence of instructions to determine the extent of progress and thereby to establish what should be taught next (Nitko & Hsu, 1974).

A second area in which validated hierarchies can be useful is in instructional design. Bergan, Stone and Feld (in press) note that validated hierarchies can provide a foundation for curriculum design and for the management of instruction that links information about changes in knowledge structures to curriculum and to instructional management decisions. Consequently, a sequence which takes into account prerequisite skills maximizes the likelihood that trainees will have appropriate prerequisite competencies at the time they are needed for superordinate-skill learning. The present study was designed to examine the hierarchical structure of skills in the measurement topic of area.

REVIEW OF LITERATURE

Area

The concept of area can be simply defined as the amount of space covering a region (Woodward & Byrd, 1983). However, the complexities surrounding the calculation of area and the understanding of the area concept are numerous.

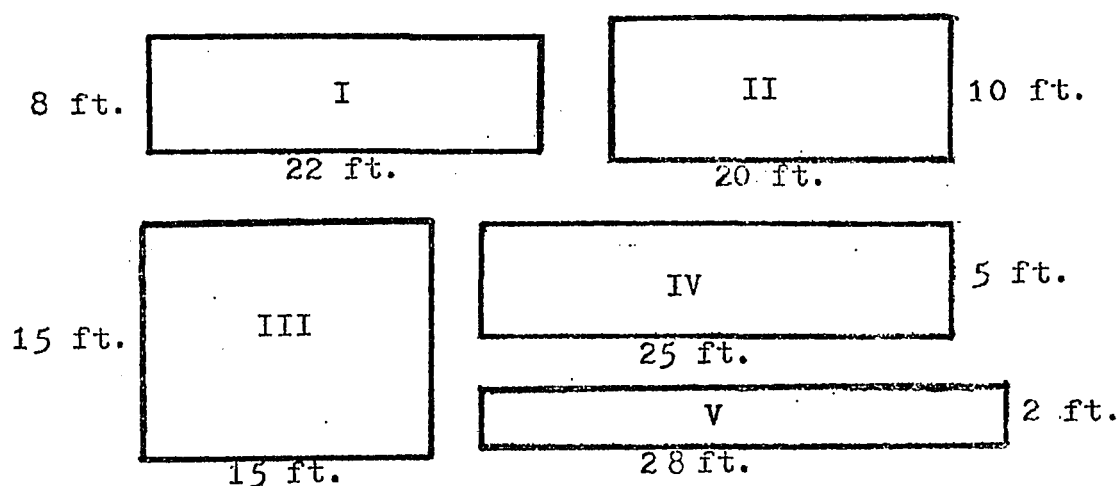
The results of the Second National Assessment in Mathematics (Hirstein, 1981) confirmed that confusion exists among students when it comes to understanding area concepts. This was demonstrated by assessing students' ability to do area problems at junior high school and high school levels (ages 13 and 17). Students in both age groups had difficulty relating the concepts of area to the plane-filling properties of regions. Hirstein (1981) noted that the major difficulty came mostly from the student's misconceptions about area rather than from computational weakness. The students appeared to have difficulty selecting the appropriate procedures needed for calculating area when the area computation required was not a straight forward length times width operation.

Hirstein, Lamb, and Osborne (1978) interviewed 106 children grades three to six after completion of test items on the subject of area. The results of the interviews also revealed that students shared many misconceptions regarding area problems. For instance, a large portion of fifth and sixth graders failed to use partitioning and recombining for area problems of nonstandard shapes, i.e. shapes formed

by a combination of several standard geometric shapes (Hirstein, Lamb, & Osborne, 1978). This suggested to the authors that students did not comprehend the invariance of area upon partitioning and then recombining a figure. In addition, in problems using partial grids, students were seen to count the wrong entities on the edges of figures. Rather than count linear units on the edge of a rectangle, the students counted the indicated grid lines. The author hypothesized that the students did not have a good sense of the linear units that comprised the square units of a rectangle (Hirstein, Lamb, & Osborne, 1978).

Woodward and Byrd (1983) presented an example of a bright seventh grader that believed that rectangles with the same perimeter take up the same amount of space. Although this student had the ability to calculate area, he nevertheless did not understand the concept of area. Woodward and Byrd (1983) revealed that the lack of understanding of area concepts extended upwards to the ranks of college students that were preparing to be elementary school teachers. Although the number of students and their demographics were not indicated, Woodward and Byrd (1983) noted that approximately two-thirds of the students of two class sections at the college level confused the concepts of perimeter and area. The authors had presented the problem shown in figure 1. Approximately two-thirds of junior high school students and college students incorrectly chose the statement indicating that the gardens were the same size. The selection of this statement demonstrated that the question of "the biggest garden" was resolved by a majority of the respondents by focusing on the perimeters of the gardens, rather than their areas. These results led the authors to conclude that

Mr. Young had 60 feet of fencing available to enclose a garden. He wanted the garden to be rectangular in shape. Also, he wanted to have the largest possible garden area. He drew a picture of several possibilities for the garden, each with a perimeter of 60 feet. These drawings are pictured as follows:



Consider Mr. Young's drawings of the garden plots. Check the statement below that he found to be true.

- 1. Garden I is the biggest.
- 2. Garden II is the biggest.
- 3. Garden III is the biggest.
- 4. Garden IV is the biggest.
- 5. Garden V is the biggest.
- 6. The gardens are all the same size.

Figure 1. Problem Demonstrating Area Concept

the subject matter of area was indeed poorly taught (Woodward & Byrd, 1983).

A prime factor in the lack of conceptual understanding of area is that the method of teaching area usually results in students obtaining an algorithmic-computational approach to area that does not enhance the concept of area (Battista, 1982; Olson, 1978). Hirstein (1981) discovered that although many students could relate that the area of a rectangle is the length times the width, they could not determine the area of an irregular shaped figure.

Battista (1982) noted that the consequence of measuring area by substituting numbers into a formula is that students have a vague or false notion about the concept of area. Battista (1982) suggested that after students have been introduced to a square unit to measure the areas of regions, they should then be exposed to finding areas of regions using a variety of nonsquare units. In addition, the relationship between a unit length and the unit area (i.e. the unit area is a square region that has sides of length equal to the unit length) should also be part of the area curriculum.

Several authors (Hirstein, 1981; Hirstein, Lamb, & Osborne, 1978; Olson, 1978) have echoed and supplemented Battista's suggestions as to what to include in math curriculums on the subject of area. However, research providing supportive evidence for these inclusions is lacking. Although research on the topic of area is scarce, the few studies and articles pertaining to this topic suggest that area is a viable and necessary subject for investigation.

Hierarchies

Work on the sequencing of skills has evolved from the introduction of the learning-hierarchy model by Robert Gagne (1962). In a preliminary study, Gagne (1962) attempted to teach seven children how to find formulas for sums of terms in number series. Gagne (1962) suggested that certain prerequisite skills were required in order that students could successfully perform the final goal of finding the formula for the number series. The prerequisite skills were determined by asking the question: "What would the individual have to be able to do in order that he can attain successful performance on this task?". In this way Gagne (1962) derived an ordered set of skills which he termed a hierarchy of knowledge. Having developed the hierarchy, Gagne proceeded to teach skills to the seven children, and observed that the order of skill acquisition resulted in a hierarchy in which none of the students acquired a skill without also acquiring all of the skills that were hypothesized as being subordinate in the hierarchy.

According to White and Gagne (1974), studies since the initial 1962 investigation have provided consistent support for Gagne's theories of learning hierarchies. However, the number of studies validating learning hierarchies has remained small. A major reason for the lack of research in this area has been that the validation procedures used in early hierarchical research to examine the prerequisite-skills assumption were extremely time consuming and therefore may not be applicable for wide scale efforts to validate hierarchies (Bergan & Cancelli, 1978).

The early validation approach consisted of instructional validation in which learners who did not initially possess hierarchical skills were subsequently taught the skills in the hypothesized order. Testing for skill acquisition was conducted during instruction. This procedure does indeed appear laborious when considering the large number of possible hierarchies in the basic skills area.

Another difficulty with the early hierarchical validation studies was that they were marred by methodological flaws (White, 1973). The early research was inadequate in that it failed to provide a statistical test for the prerequisite-skills assumption that took into account errors in measurement (Bergan, 1980; White, 1973).

During recent years a number of attempts have been made to develop procedures to test Gagne's prerequisite skills hypothesis statistically. The psychometric approach is considered more economical in time, personnel cost, and material than the instructional approach since no teaching is involved (Bergan & Stone, in press). This research study utilized the psychometric approach to validate the prerequisite skills hypothesis.

THE PROBLEM

Hypothesized Hierarchical Foundations

Gagne's early research developed hierarchies through the application of the question: "What would the individual have to be able to do in order that he can attain successful performance on this task?" (Gagne, 1962). This approach to the identification of hierarchical structures is subject to problems dealing with the specification of the component elements. Indeed, the early research suffered from imprecise specification of the component elements in the postulated hierarchies (White, 1973). This suggests that the foundation upon which hierarchies are established would benefit from a more precise structuring. Three sources for hierarchical structuring in the area domain were hypothesized in this investigation: rules, representation and shape.

Rules

Recent work on the sequencing of skills has focused on the progressive acquisition of rules representing increasingly higher levels of understanding of mathematical concepts (Bergan, Towstapiat, Cancelli, & Karp, 1982; Siegler, in press). Gagne (1977) suggested that rules can form an ordered relation when one rule can be considered to be a component of another rule. For example, rules for addition can be considered to be component rules for multiplication because addition operations are carried out as part of written multiplication tasks. Insofar as the rules for addition are components of the rules for multi-

plication, it may be hypothesized that problems requiring only the application of addition rules will be subordinate to problems utilizing multiplication rules. In the measurement topic of area, component rules can be seen to be germane to various area tasks. For instance, the calculation of the area of a nonstandard form (figure 2) requires that one be familiar with two sets of rules: 1)



Figure 2. Nonstandard Shaped Region

the rules for calculating the area of a standard geometric form (e.g. a rectangle) and 2) the rules for partitioning and recombining areas. Thus each of the two sets of rules can be considered as a component rule for the superordinate task of calculating the area of a nonstandard form.

A second way in which rules could form an ordered relation is through the replacement of simple rules governing mathematical operations by more complex rules (Bergan, Towstopiat, Cancelli, & Karp, 1982). In this instance, the way in which the individual conceptualizes all or some portion of a task changes. For example, when children first learn the concept of area they usually begin by counting the square units of a grid that is enclosed within a geometrically shaped region (figure 3) to obtain the area (Heath, 1975; Holt, Rinehart, & Winston, 1974; Scott, Foresman & Company, 1975). However, once the familiar

length times width formula is introduced, it is likely to replace the

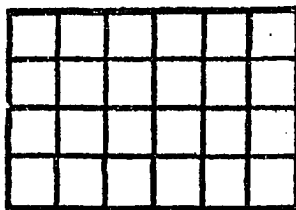


Figure 3. Region with Grid

rule of counting square units to obtain the area. Rule components and rule replacement constituted the major basis for hierarchical ordering of area tasks developed in this study.

In the formulation of ordered relations among area tasks, the sole use of component and replacement rules appear to be insufficient to completely describe the foundations for the hierarchical structure of area tasks. It can be shown that the use of component rules or replacement rules depends on several conceptual structures underlying area math tasks. For instance, the concept of area as defined by the amount of space covering a region can be displayed in several ways (e.g. with and without a square unit grid) leading to varying rules that can be categorized as component or replacement rules. Consequently, in examining the structure of area skills, the task representation must be considered. In addition, consideration should also be given to the shape of the region (nonstandard or standard geometric shape) since the shape also influences the rules for area calculations (see figure 4). The two categories of region representation and the shape of the region are hypothesized structures that influence the rules and that are intended to improve upon the specification of components of the hierarchy.

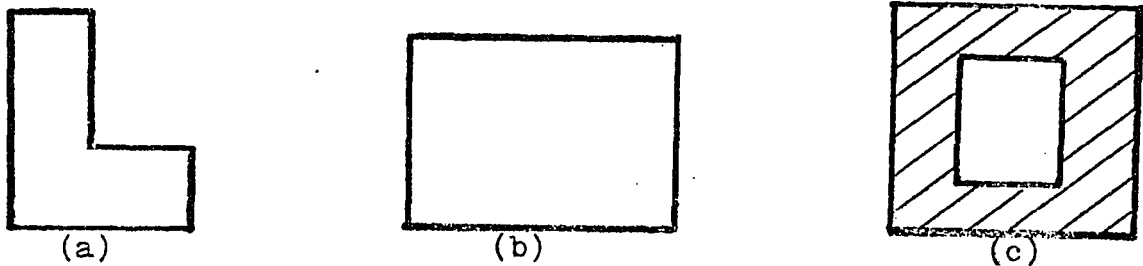


Figure 4. Samples of Different Area Shapes

Representation

The fact that changes exist in the procedures used in problem solving as a result of varying the item representation is particularly important for understanding hierarchical relations among area skills. Newell and Simon (1972) recognized the fact that individuals may represent problems that they are asked to solve in different ways. There are two factors to consider in this regard. The first one is that individuals may solve the same problem in different ways depending on their own internal representations. The second factor is that given two different representations of the same problem, an individual may use different methods for solving the problem. An example of the first factor is the case of a student being presented with the problem of solving for the area of a rectangle that contains a grid of unit squares within its boundaries (figure 3). The student has the option of either counting the number of boxes to obtain the area or of just counting the number of boxes that comprise the length and the width and then multiplying them to get the area. In the latter method, a knowledge of multiplication,

in addition to counting, would be necessary to compute the area. These two options appear to be hierarchically related just by the nature of one method including the other and adding an additional operation (an example of component rules). However, knowing how the student represented this problem internally can only be ascertained by questioning the student.

An example of the second factor involves structuring the tasks such that they are represented in a different manner and thereby increasing the probability that different methods for solving the problems will be used. The computation of the area of a rectangle that is depicted with and without a square unit grid involves different rules for each of the two representations. Although an individual may attempt to internally represent the empty rectangle as one composed of a grid, the more probable representation of the area would be the familiar length times width formula using the rectangle dimensions. This is due to the fact that students are taught to associate area problems of non-grid rectangles with the standard formula (Battista, 1982). The investigation of ordered relations based on task representations was limited to those that fell under the characteristics of the second factor, i.e. differing external representations of the same problem.

Shape

A second source of change in rule-governed behavior of area tasks is the shape of the figure presented. For example, although the general rule may stay the same for area calculations in which the region changes from a rectangle to a triangle or from a triangle to

a parallelogram, the rules for area calculation do change when the figure in question becomes a nonstandard shape (e.g. a combination of several rectangles that together do not form a rectangle). The shape of a region is an intricate part of understanding the concept of area (Spitler, 1982). The question of what happens to the area of a region as its shape is changed is often used in instructional sequences to introduce related geometric shapes (Heath, 1975; Holt, Rinehart, & Winston, 1974; Scott, Foresman & Company, 1975). For example, when a rectangle ABCD (see figure 5) is sheared into a parallelogram

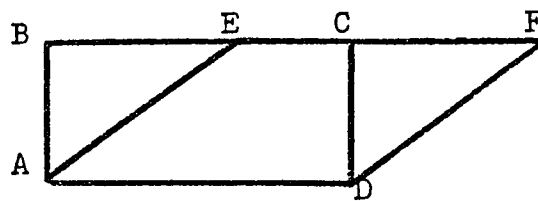


Figure 5. Shearing of a Rectangle

AEFD, the area remains constant leading to the conclusion that given a parallelogram one can find its area by first shearing it into a rectangle and then using the familiar length times width formula (Spitler, 1982). Another example of the use of shapes is the introduction of the area of a right triangle as being equal to one-half the area of a rectangle. In addition to standard shapes, the use of nonstandard shapes as the region in question plays an important part in the understanding of area concepts. The ability to partition and recombine regions is considered a key concept in determining whether an individual comprehends area (Hirstein, Lamb, & Osborne, 1978) Thus, one can see that the shape

of a figure can also be considered as being a possible source of hierarchical ordering of area skills.

The flow of the hierarchical foundation is depicted in figure 6. As indicated in the diagram, the representation of a task and the shape of a figure may lead to the use of either rule replacement or rule components or both. The rules themselves lead to the area skill in question. Finally, the dotted line between shape and representation indicates that the two sources may interact in a variety of combinations. For example, a nonstandard shape may be shown with and without a square unit grid. However, for the purpose of this study the interaction between shape and representation was not investigated.

Hypothesized Task Hierarchy and Associated Rules

The area problems chosen for examination are those that vary according to the two sources described above: item representation and the shape of the region. These two sources cover the comprehension of area conceptually as well as computationally. For example, both item representation and the shape of the region of concern involve tasks that require understanding that the area concept refers to the two dimensional space-filling aspect of measurement. In addition, area problems that involve nonstandard shapes require the ability to partition and recombine regions. Finally, the algorithmic-computational approach is also assessed through area problems concerning traditional geometric shapes.

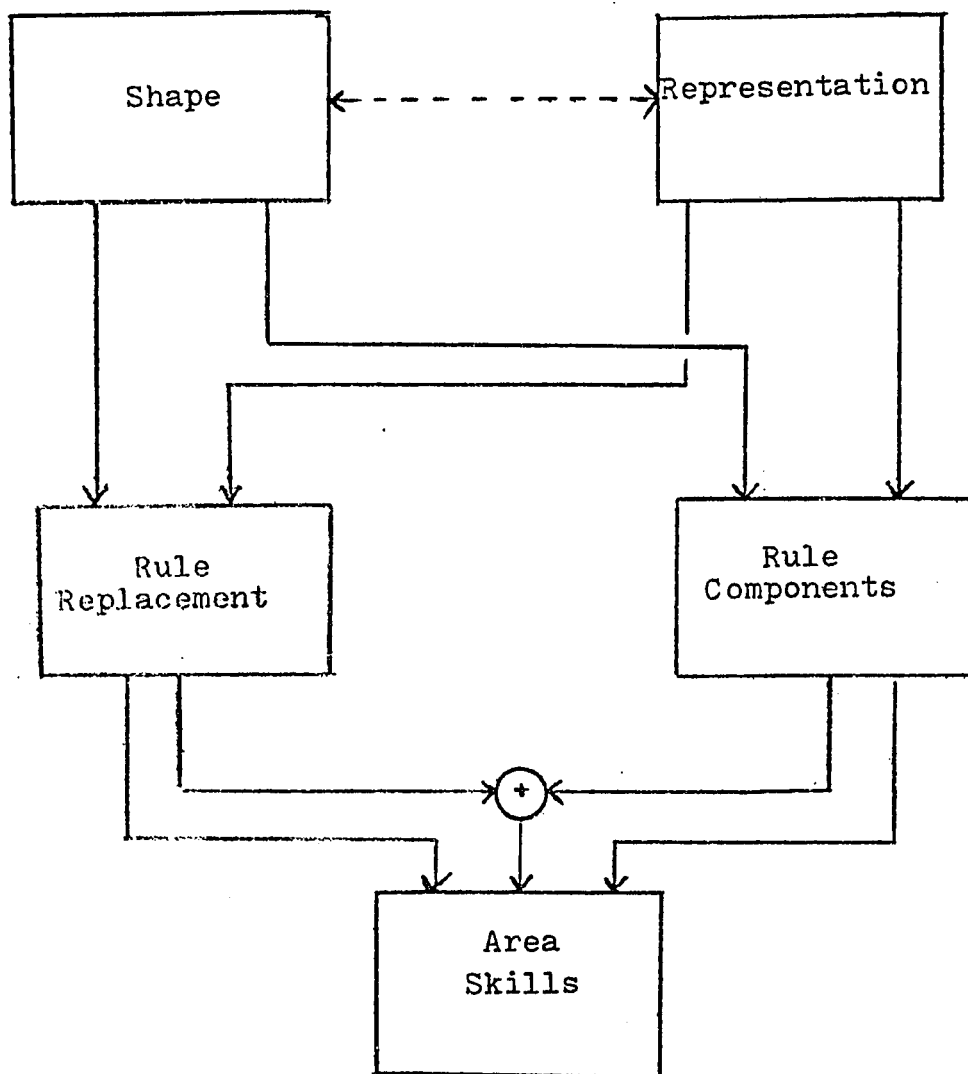


Figure 6. Hierarchy Foundation Flow

Item Representation

Three levels of item representation were hypothesized: figures with a square unit grid, figures with a partial grid and figures without a grid. These three different representations of the same problem (i.e. finding the area of a figure in square units) offer differing methods for solving the problem. The simplest task is seen in the grid representation of a rectangle whereby the individual needs only to count the boxes to obtain the area. Although the more sophisticated student may count the unit squares on the two adjacent sides of a rectangle and then multiply to get the area, students first learn the rule that is conceptually the easiest, i.e. to obtain the area of a rectangle with a square unit grid one counts the square units within the boundaries of the rectangle (Hirstein, Lamb, & Osborne, 1978). To circumvent the possibility of counting sides and then multiplying, the use of an irregular shaped figure was used (see figure 7).

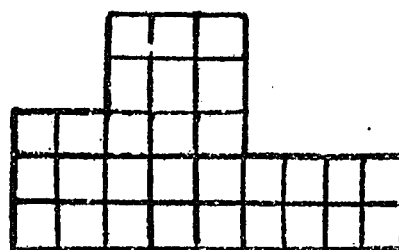


Figure 7. Irregular Shaped Region with Grid

The next representation hypothesized in this skill hierarchy was that of a rectangle without a grid that contains the dimensions on all the sides. The rule associated with this type of problem is the familiar algorithmic-computational approach in which one multiplies

the length times the width to get the area. At this level, the grid has disappeared and the student must be able to make use of rule replacement, whereby the multiplication rule replaces the counting rule due to a change in task representation.

The last level in the representation hierarchy is the partial grid or inferred grid. This representation combines both counting and multiplication for the purpose of area calculation. The rule for area calculation of a rectangle with an inferred grid is as follows: count the unit lengths on two adjacent sides of the rectangle and then multiply the two sides together. In this case component rules are combined to form a superordinate rule for calculating the area of a rectangle with an inferred grid. The possibility may exist that the student may be tempted to draw lines on the diagrams to complete the grid. In order to avoid this possibility, instructions were given that students were not to draw directly on the diagrams.

In a preliminary study that focused only on the percent of a population that correctly answered the problems, Hirstein, et. al. (1978) revealed that the representations of area problems were seen to go from easiest to most difficult in the following order: grid, no grid and partial grid. This was also the hypothesized hierarchy for the representation hierarchy.

Shape of Region

With regards to the shape of the region, there are two main levels: standard shapes and nonstandard shapes. The former can be composed of rectangles, squares, triangles, circles, etc. The latter

is a combination of the standard forms that require partitioning of a figure in order to calculate the areas. A nonstandard shape can be composed of several rectangles, or a rectangle and a triangle, or any combination of standard forms. The two levels of shape can also be further subdivided into more specific hierarchical components.

The standard forms require that either the student be familiar with the formula that is applicable for each unique geometric form or that the student be aware of the relationships among the standard forms (e.g. a right triangle is essentially one-half of a rectangle). In addition, the student must be aware of how the figure's line segments correspond to each unknown in the formula. In some cases the line segments are not obvious and need to be visualized by the student. For example, the formula for the area of a triangle is $\frac{1}{2} B \times H$, where B stands for the base and H refers to the height. In a right triangle there is no need for visualization since the base and height coincide with the two sides that are perpendicular to each other. However, for an acute triangle, the height is a visualized line drawn from any vertex of the triangle perpendicular to the side opposite the vertex, and the base is the side opposite this vertex. Similarly, in an obtuse triangle, the procedure remains the same if the height is drawn from the obtuse angle. However, the height becomes a visualized line outside the triangle when drawn from the tip of the vertex of either acute angle. The base is the length of the side opposite the acute angle that is used. For the purpose of this study all the line segments that require visualization were shown in the task diagrams.

The general rule for standard geometric shapes is as follows: to

find the area of a standard shape, determine the line segments that correspond to formula symbols, substitute the appropriate values into the area formula for that shape and then perform the required operation of multiplication. The subdivisions included in the standard shape hierarchy were the following: a rectangle, a right triangle, and a parallelogram. This hypothesized order is based on instructional sequences found in curriculum texts (Heath, 1975; Winston, 1974; Scott, Foresman & Company, 1975).

As indicated above, nonstandard forms are composed of two or more standard forms that are combined together in such a way that the region in question is no longer a recognizable geometric form. Nonstandard shapes can be subdivided into two categories of problems. The first category includes problems that require the ability to partition and recombine areas (see form 'a' in figure 4). The rule for this type of problem is as follows: To find the area of a region composed of standard shapes, partition the figure into standard shapes, determine the areas of the partitioned shapes using the rule for standard shapes, and then recombine by adding together all the individual areas. This rule is also an example of component rules combined to form a superordinate rule.

The first category of nonstandard shapes focuses on the area of a whole region. In contrast, the second category usually focuses on a section of a region that is contained within the outer boundaries of the figure. The section is usually indicated by a shaded area or by a distinctive pattern (see form 'c' in figure 4). The section in question is a nonstandard shape that is determined by the differencing

of two standard shapes. The rule for the second category of nonstandard shapes is as follows: to find the area of a nonstandard shaped section, find the area of the total region using standard shape rules; find the area of the internal standard shape also using standard shape rules; and then subtract the area of the internal standard shape from the area of the total figure to obtain the area of the desired section. The underlying assumption is that a student that can successfully follow the above rule is aware of two area concepts: 1) the sum of all the areas of the partitioned sections of a figure is equal to the area of the total figure, and 2) the area of a section of a figure is equal to the area of the difference between the area of the total figure and the area of the remaining sections (i.e. the inverse of concept 1)

The hypothesized sequence for the two levels of shape is as follows: computing the area of standard shapes is subordinate to computing the area of nonstandard shapes. This sequence can be attributed to the rule component hypothesis since the rules for standard shapes are components of the rules for nonstandard shapes. The ordering of skills hypothesized within the level of standard shapes, i.e. rectangle, triangle and parallelogram, are based on curriculum sequences given in common mathematic textbooks (Heath, 1975; Holt, Rinehart & Company, 1974; Scott, Foresman & Company, 1975). The use of either component rules or replacement rules is not in effect within the level of standard shapes due to the fact that the general rule for area calculations remains the same for each standard shape. However, the area formulas

and the line segment-symbol correspondences do change for each standard shape, resulting in a hierarchy of standard shapes.

In hypothesizing the ordering among the two nonstandard categories, one encounters the situation in which both component and replacement rules influence the hierarchy. However, in the current analysis the two sources of hierarchy appear to be working in opposition to each other. To further illustrate this point the rules for each category of nonstandard shape are reproduced below:

Category 1: To find the area of a region composed of standard shapes, partition the figure into standard shapes, determine the areas of the partitioned standard shapes using the rules for standard shapes, and then recombine by adding together all the individual areas.

Category 2: To find the area of the nonstandard shaped section, determine the area of the standard shaped regions (the total figure and the inner section) using standard shape rules, and then subtract the area of the internal standard shape from the area of the total figure to obtain the area of the desired region.

In comparing the two categories, several observations can be made: 1) Both rules include computing areas of standard shapes, 2) Category 1 involves the added component of partitioning a region into standard shapes (an example of component rules) and 3) the procedure of adding partitioned shapes in category 1 is replaced by subtracting partitioned shapes in category 2 (an example of replacement rules).

By focusing only on the component rule hypothesis the resulting sequence would place category 2 subordinate to category 1 due to the additional requirement of partitioning. However, by focusing only on the replacement rule hypothesis, the hypothesized sequence is reversed with category 1 being subordinate to category 2. This reversal occurs

because subtracting areas to obtain a nonstandard shaped area is assumed to be superordinate to adding areas to determine the area of a region. Consequently, the replacement rule hypothesis and the component rule hypothesis appear to be exerting contradictory influences on the hierarchical ordering of the two categories. The actual hierarchy that will unfold will depend upon which of the two sources of hierarchy will exert the strongest influence.

The present study hypothesized that in this instance the component rule hypothesis emerges as the stronger influence, resulting in category 2 being subordinate to category 1. Evidence for this sequence is seen in a preliminary study (Hirstein, 1981) of student success on various area tasks. The two nonstandard shape categories were represented by items given on the Second National Assessment in Mathematics (Hirstein, 1981). The results demonstrated that students were more successful on items reflecting category 2 (i.e. calculating the area of a nonstandard section of a figure by differencing areas) than on items representing category 1 (i.e. partitioning of a nonstandard shape and then adding individual areas).

The hierarchy of area skills based on the two sources of representation and shape is displayed in figure 8. The ordering among the skills corresponds to the level of the skill within the diagram. For example, the grid representation is subordinate to the no grid representation, which in turn is subordinate to the partial grid representation. In addition, the assumption that the addition of areas is subordinate to the subtraction of areas was also tested. The items reflecting these two tasks consisted of adding and subtracting areas that have

already been determined. Finally, the presentation of items with partitioned areas that have already been determined can result in a hierarchical relationship with similar items that require the calculation of the partitioned areas. The obvious hierarchy is that items with given information will be subordinate to items that require the calculation of the same information. This relationship was also tested within this study. The rules associated with the task items are presented in table 1. The test is presented in appendix A.

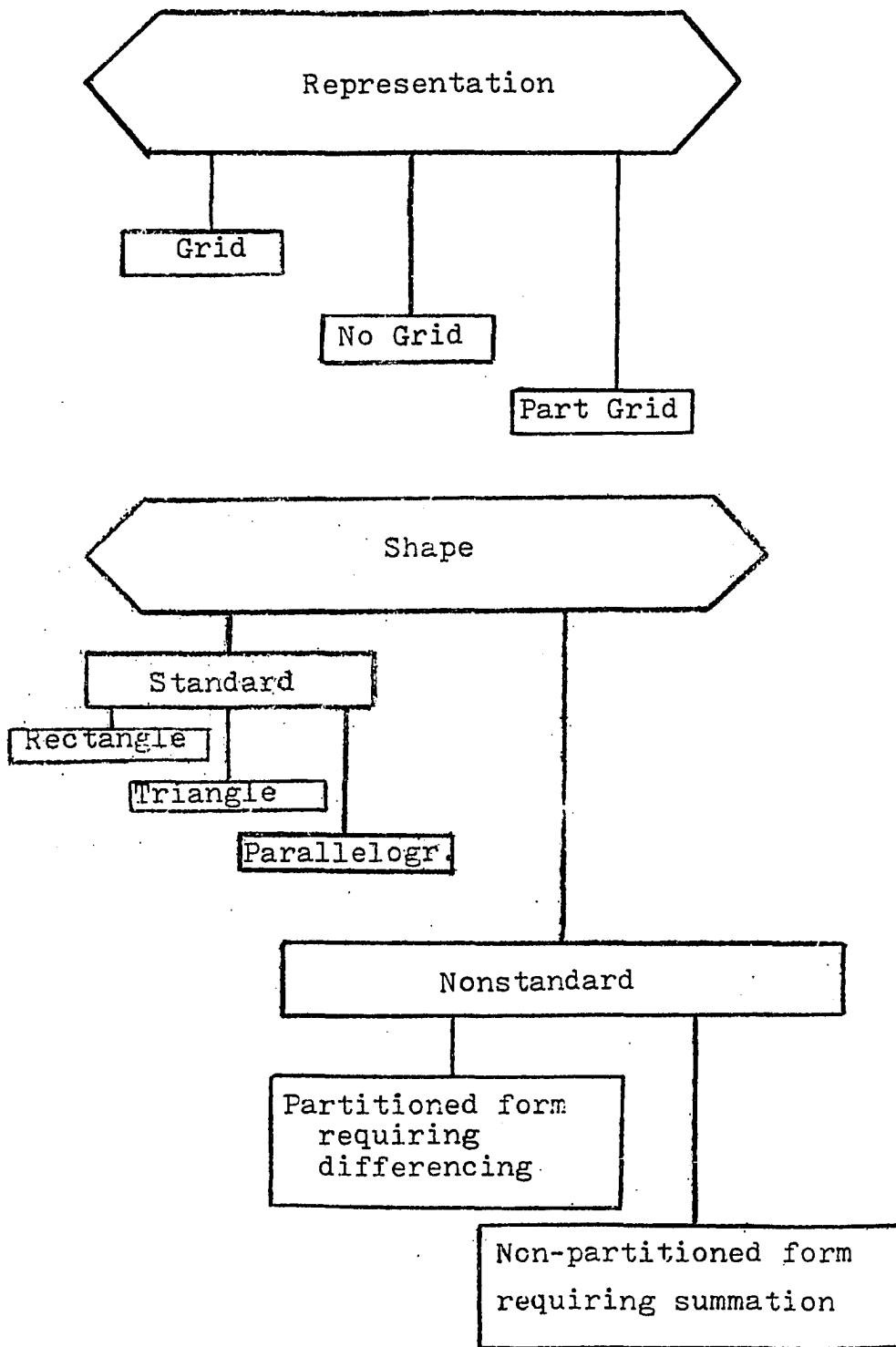


Figure 8.

Hierarchy of Area Skills Based on Representation and Shape

Table 1

Rules Associated with the Task Items

<u>Item Numbers</u>	<u>Rules</u>
1,2	To obtain the area of a figure with a square unit grid one counts the square units within the boundaries of the figure.
3,4,7,8,9,10	To find the area of a standard shape, determine the line segments that correspond to formula symbols, substitute the appropriate values into the area formula for that shape and then perform the required operations (usually multiplication).
5,6	To find the area of a rectangle with an inferred grid, count the unit lengths on two adjacent sides of the rectangle and then multiply the two sides together.
11,12	To find the area of a region composed of standard shapes, partition the figure into standard shapes, determine the areas of the partitioned shapes using the rule for standard shapes, and then recombine by adding together all the individual areas.
13,14	To find the area of a nonstandard shaped section, find the area of the total region using standard shape rules, find the area of the internal standard shape also using standard shape rules, and then subtract the area of the internal standard shape from the area of the total figure.
15,16	To find the area of a nonstandard shaped region, add all the individual standard shaped sections.
17,18	To find the area of a nonstandard shaped section of a region, subtract the area of the standard shaped region from the area of the total region.

METHOD

Subjects

The 18 item area test was administered to 306 students selected from a local public school system in Tucson, Arizona. The subjects ranged in grade level from the 6th grade to the ninth grade. This span of grades was chosen to correspond to age levels at which various area concepts, reflected in the test items, are taught. The grade span is used to demonstrate the progression of skills from non-mastery to mastery, with the possibility of a transition stage in between. These students were selected from a senior high school, a junior high school and two elementary schools. The schools for this study were chosen to approximately reflect minority enrollment within this school district. The breakdown of the subjects by grade, sex and ethnicity is shown in table 2.

Procedure

The test items were administered to the students in the normal course of a classroom period. The classroom teacher distributed the test booklets with the following instructions:

Class, today I am going to give you a quiz on the subject of area. The school is interested in knowing how much you know about area. (PASS OUT SECTION I) The test is in two parts. The first part has 14 questions and the second part has 4 questions. After you have finished part one, I will give you part two.

Table 2

Subjects by grade and sex (in percent of N)

<u>N</u>	<u>Grade</u>	<u>Males</u>	<u>Females</u>
88	6	49%	51%
68	7	50%	50%
70	8	43%	57%
80	9	46%	54%

Subjects by grade and ethnicity (in percent of N)

<u>N</u>	<u>Grade</u>	<u>Anglos</u>	<u>Hispanics</u>	<u>Blacks</u>	<u>Other</u>
88	6	42%	42%	14%	2%
68	7	41%	45%	8%	6%
70	8	56%	38%	4%	2%
80	9	63%	24%	7%	6%

For the first ten questions do not write on or change any of the diagrams. You can do all your work in the open spaces on each page. Put your name and grade at the top of section I and just your name at the top of section II. Remember to write your answers clearly and in the correct space for each question. If you do not know how to do a problem, do not worry about it and just go on to the next one. Do as best as you can. You may begin. (PASS OUT SECTION II AFTER STUDENTS HAVE INDICATED THAT THEY HAVE FINISHED SECTION I AND THEN COLLECT BOTH SECTIONS AT THE END)

The test had been divided into two sections because part two had questions that may have provided clues to items in part one.

Data Analysis

Latent-class analysis

Latent-class analyses (Bergan, 1983; Goodman, 1974) were conducted to examine equivalence and ordered relations among area tasks of interest. Latent-class models attempt to describe the association in a contingency table in terms of one or more latent variables, each of which is comprised of a number of latent classes. For example, it may be useful to explain a relationship among two hierarchically ordered tasks in terms of a latent variable that is composed of the following three latent classes: a mastery class representing mastery of both tasks, a nonmastery class representing failure on both tasks, and a class that reflects passing performance for the subordinate task and failing performance for the superordinate task.

Estimates of expected cell counts are generated for latent-class models using an iterative process (Goodman, 1974). Clifford Clogg (1977) has developed a program which computes maximum likelihood estimates using Goodman's (1974) iterative approach. This program tests latent-

class models by assessing the correspondence between observed cell frequencies and estimates of expected counts using the chi-squared stastic. Low values of the chi-squared stastic indicate models which provide an adequate fit to the data.

Various restrictions can be imposed on latent-class models. For example, it would be possible to restrict the probability of a passing response for items reflecting subordinate task mastery to be equal in the mastery and partial mastery classes. Model variations are produced by altering restrictions, by varying the number of latent variables in a model, and/or by varying the number of latent classes within a latent variable.

Models developed for the present study assume one latent variable reflecting variations in task mastery. The focus of the latent-class analysis was on the determination of ordering between pairs of skills hypothesized to form a hierarchical relationship. Each skill was represented by two items, making it possible to separate variations in responding due to measurement error from variations indicating an ordered relationship. Four latent-class models were used in the analysis of the area skills under investigation.

The first model contained two latent classes, a mastery class and a nonmastery class. This model, which included no restrictions on response probabilities within latent classes, reflects the assumption that all of the items for both skills being examined belong in the same domain (Bergan, 1983; Dayton & Macready, 1976).

The second model included the mastery and nonmastery classes in model 1 and one additional class reflecting partial mastery of the

skills. More specifically, this class assumed mastery of the subordinate skill and nonmastery of the superordinate skill. This assumption was represented by restricting the probability of a passing response for the subordinate skill items to be equal across the mastery and partial mastery classes. In addition, the probability of a failing response for superordinate task items was restricted to be equal across the nonmastery and partial mastery classes. This model reflects the assumption that the skills under examination are prerequisite ordered. Ordering of this kind assumes that mastery of the subordinate skill is necessary to the acquisition of the superordinate skill (Gagne, 1977).

The third model included all of the latent classes and restrictions in model 2 plus an additional latent class reflecting the assumption of superordinate skill mastery accompanied by subordinate skill nonmastery. This assumption was represented by restricting the probability of mastering superordinate skill items to be equal across the last latent class and the mastery class. Likewise, the probability of failing subordinate skill items was restricted to be equal across the nonmastery class and the last latent class. This model assumes that the skills being tested reflect two associated latent variables, each comprised of two latent classes (Goodman, 1974). This assumption can be taken as an indication that the skills are in related subdomains (Bergan & Stone, 1984). In addition, the model assumes that the skills are ordered by difficulty, but that the ordering does not involve prerequisite ness.

The final model was comprised of the same latent classes as those in model 3. However, it imposed a restriction on the latent classes not present in model 3. Model 4 assumes that the skills being

examined are in related subdomains and that the two partial mastery classes are equally probable.

The four models described in the preceding paragraphs are hierarchically related (Goodman, 1974), and it is possible to compare them statistically. The comparison of model 1 with model 2 indicates whether or not the inclusion of the partial mastery class results in a significant improvement in the fit of the model to the data. Instances reflecting significant improvement support the assumption of an ordered relationship between the two skills under examination. The comparison of model 2 with model 3 provides a test for the hypothesis that the two skills being examined are related in a prerequisite fashion. The criterion of prerequisites, which is a hallmark of the concept of hierarchical ordering (see, for example, Gagne, 1977), holds that given suitable allowance for measurement error, there should be no one who has mastered the superordinate skill who is nonmaster of the subordinate skill. Those instances involving a significant improvement in fit for the model containing a latent class reflecting mastery of the superordinate skill accompanied by non mastery of the subordinate skill indicate a state of affairs contrary to the prerequisites hypothesis. On the other hand, in those cases in which there is no improvement in fit, model 2 is preferred over model 3, which supports the assumption that the two skills being examined are prerequisite ordered.

The comparison of model 3 with model 4 tests the assumption that the two partial mastery classes are equally probable. In those cases in which the comparison of 3 and 4 indicates a significant improvement in fit associated with the adoption of model 3, the assumption of equally

probable partial mastery classes is contraindicated. This state of affairs can be taken as an indication that the skills under examination are ordered by difficulty. That is, in those cases in which only one of the skills has been mastered, there will be a higher probability of mastery for one of the skills than for the other.

The models described above were tested to determine their fit to the data using the chi-squared statistic. It is also possible to compare the fit of different models to the data statistically. Statistical comparison is carried out through use of the likelihood ratio statistic (L^2) because the L^2 values can be partitioned exactly (Bishop, Fienberg, & Holland, 1975). Statistical comparisons are conducted by subtracting L^2 values of hierarchical models and their associated degrees of freedom (df). The result is then compared against the chi-square distribution. Models are hierarchical if one contains all the restrictions of the other plus additional constraints. If the resulting L^2 and its associated df is significant, then the additional constraints are important in describing the association in the data. On the other hand, if the difference L^2 is not significant, then the additional restrictions of the one model do not contribute significantly to the fit of the model to the data.

Process analysis

In addition to latent-class analysis, the items were coded and tabulated for the mathematical processes that were used by the students in the calculation of their answers. In coding the process, assumptions were made regarding the technique used by the student to solve the

problem. For example, in the area problem in figure 9, a student that correctly solves this problem (area = 30 sq. units) is assumed to have

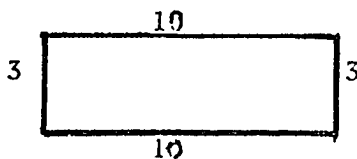


Figure 9. Rectangle Area Problem

used the appropriate area rules. However, a student that solves the problem by summing the sides of the rectangle will produce the answer: area = 26 sq. units. Likewise, a student may multiply all the sides together and come up with an area equal to 900 sq. units.

In order to increase the probability of correctly coding a test item according to process, student responses were coded only if the same process was consistently used across equivalent items. Thus, for example, in order to code the problem in figure 9 as "summing all the sides of the rectangle", a student would have had to solve an equivalent problem in the same manner. In addition, a process had to be used by two or more students before it could be considered codeable. Consequently, even if the investigator was able to determine a process for an item pair, it would not be tabulated unless it had been used by more than one student for that item pair.

In some instances, the process may be interpreted in more than one way. For example, in figure 9 if the student calculated the area to be 26 sq. units, then one interpretation could be that the student added up all the sides of the rectangle. However, another interpretation may be that the student added two adjacent sides and then multiplied

by two. Nevertheless, the final product is still equivalent to obtaining the perimeter of the figure. The processes coded in this study were developed by the investigator based on the individual responses for each of the item pairs.

RESULTS

Latent-class Analysis

Figure 10 illustrates 12 hypothesized skill orderings for the item pairs based on the theory presented in the problem section. The item numbers are presented with the item type. In addition, the item pairs are presented such that the top pair is hypothesized as subordinate to the bottom pair. A total of 9 hypothesized hierarchies were analyzed using latent-class analysis. Three of the hypothesized skill orderings were not suitable for analysis due to the confounding inherent in the items. These three orderings are the item pairs listed under 'Hierarchy based on geometric shape' in figure 10. The perimeters of items 8 and 10 contained numbers that if added together would be equal to the results obtained by using the correct area formulas. Therefore, any analysis involving these two items would not reflect the correct skill mastery.

The results for the latent-class analysis that tests the specific hypothesis generated for the nine item pairs is presented in table 3. This table shows the item numbers examined for each of the pairs of skills studied. Each model tested is represented by the sequence H1 to H4, with the numbers corresponding to the models described in the data analysis section (i.e. H1 refers to model 1). The L^2 values

	<u>Item type</u>	<u>Item #'s</u>
Hierarchy based on representation:	grid	1,2
	no grid	3,4
	grid	1,2
	partial grid	5,6
	no grid	3,4
	partial grid	5,6
Hierarchy based on geometric shape:	rectangle	3,4
	triangle	7,8
	rectangle	3,4
	parallelogram	9,10
	triangle	7,8
	parallelogram	9,10
Hierarchy based on shape forms:	geometric	3,4
	nonstandard	11,12
	geometric	3,4
	nonstandard	13,14
Hierarchy based on level of nonstandard form:	subtracting	3,14
	adding	11,12

Figure 10. Hypothesized Skill Orderings

	<u>Item type</u>	<u>Item #'s</u>
Supporting hypothesis hierarchy for level of nonstandard shaped forms:	adding	15,16
	subtracting	17,18
Hierarchy based on information given for nonstandard shaped problems for addition:	partitioned values given	15,16
	partitioned values not given	11,12
Hierarchy based on information given for nonstandard shaped problems for subtraction:	partitioned values given	17,18
	partitioned values not given	13,14

Figure 10. (Continued)

Table 3

Latent-class Analysis for Hypothesized Area Skills

<u>Item Numbers</u>	<u>Model</u>	<u>L²</u>	<u>DF^{**}</u>	<u>P</u>	<u>Cell Counts</u>			
1,2 3,4	H1	87.58	9	0.00	120	1	0	110
	H2 ^o	23.47	9	0.00	9	0	0	8
	H3*	0.40	7	0.98	8	0	0	11
	H4	37.21	10	0.00	9	0	0	31
1,2 5,6	H1	83.48	6	0.00	126	6	10	89
	H2 ^o	13.31	6	0.02	7	1	2	7
	H3*	4.51	4	0.37	8	1	0	10
	H4	37.19	8	0.00	5	2	5	28
3,4 5,6	H1	133.26	9	0.00	116	5	8	17
	H2 ^o	82.24	8	0.00	1	0	0	0
	H3	0.46	7	0.98	0	0	0	0
	H4*	3.33	8	0.65	29	5	9	117
3,4 11,12	H1	92.44	9	0.00	44	5	18	79
	H2 ^o	18.68	8	0.00	0	0	0	1
	H3*	0.08	7	1.00	0	0	0	0
	H4	61.23	10	0.00	3	1	2	154
3,4 13,14	H1	147.67	9	0.00	56	10	0	80
	H2 ^o	10.21	9	0.07	0	0	0	1
	H3*	0.02	9	1.00	0	0	0	0
	H4	96.18	11	0.00	1	1	0	158
13,14 11,12	H1	119.59	7	0.00	22	4	11	20
	H2 ^o	86.34	6	0.00	3	0	2	6
	H3	2.20	7	0.70	0	0	0	0
	H4*	4.58	8	0.52	22	2	7	208

Table 3 (continued)

<u>Item Numbers</u>	<u>Model</u>	<u>L²</u>	<u>DF</u>	<u>P</u>	<u>Cell Counts</u>			
15,16 17,18	H1	115.67	6	0.00	158	7	4	55
	H2 [°]	49.90	5	0.00	4	0	1	4
	H3*	3.38	4	0.46	7	0	0	11
	H4	14.83	8	0.01	12	1	2	41
15,16 11,12	H1	142.17	6	0.00	40	4	17	163
	H2 [°]	20.12	7	0.00	1	0	1	7
	H3*	2.92	4	0.58	2	1	0	15
	H4	108.35	9	0.00	4	1	2	49
17,18 13,14	H1	203.69	6	0.00	51	8	0	122
	H2 [°]	25.91	8	0.00	0	1	0	7
	H3*	4.39	6	0.35	2	1	0	4
	H4	127.60	10	0.00	4	1	0	106

[°] - Hypothesized model

* - Preferred model

** - The df for the models reflect increases in df due to the convergence of estimated parameters to an extreme value (1.0) during the iteration process (Goodman, 1970).

in the table are the likelihood-ratio chi-squared values. The table also shows the degrees of freedom for each model tested and the associated probability of the L^2 value referred to the chi-squared distribution. Preferred models are indicated by asteriks, and hypothesized models are designated with small o's. Finally, the observed cell counts are also shown. The response patterns reflected in the tables are as follows:

		Task B items				
		B	1	1	2	2
		B'	1	2	1	2
A A'						
Task A items	1 1	F _{ijkl}				
	1 2					
	2 1					
	2 2					

where 1 represents passing and 2 reflects failing and F_{ijkl} indicates the count for cell $ijkl$. Tables 4 to 12 present the observed and expected frequency counts, the residuals, the likelihood ratio statistic (LR) and the chi-squared statistic (PRSN), and the latent class probabilities for the preferred models in each of the 9 latent-class analyses. The results will be presented according to item pairs hypothesized. Reference can be made to figure 10 regarding the item types.

Table 4

Preferred Model Observed and Expectency Frequency Counts,
Residuals, LR, PRSN, and Latent Class Probabilities
for Item pairs 1,2 and 3,4

CELL	OBSERVED	EXPECTED	(STDIZED RESID)			
1	120.	120.35	(-.03)	
2	9.	8.77	(.08)	
3	8.	7.94	(.02)	
4	9.	8.94	(.02)	
5	1.	.82	(.19)	
6	0.	.06	(-.25)	
7	0.	.05	(-.23)	
8	0.	.06	(-.25)	
9	0.	.00	(-.01)	
10	0.	.00	(.00)	
11	0.	.00	(.00)	
12	0.	.00	(.00)	
13	111.	110.83	(.02)	
14	8.	8.17	(-.06)	
15	11.	11.00	(.00)	
16	29.	29.00	(.00)	
FINAL LR		.398	PRSN	.225		
FINAL LATENT CLASS PROBABILITIES						
.4473940	.1104894	.4091184	.0329981			
FINAL CONDITIONAL PROBABILITIES						
LATENT CLASS:		1	2	3	4	
1.	1.	.9324	.1508	.9324	.1508	
1.	2.	.0676	.8492	.0676	.8492	
2.	1.	.9492	.0050	.9492	.0050	
2.	2.	.0508	.9950	.0508	.9950	
3.	1.	.9932	.0000	.0000	.9932	
3.	2.	.0068	1.0000	1.0000	.0068	
4.	1.	1.0000	.0000	.0000	1.0000	
4.	2.	.0000	1.0000	1.0000	.0000	

Table 5

Preferred Model Observed and Expectency Frequency Counts,
Residuals, LR, PRSN, and Latent Class Probabilities
for Item Pairs 1,2 and 5,6

CELL	OBSERVED	EXPECTED	(STDIZED RESID)		
1	126.	126.00	(.00)		
2	7.	7.00	(.00)		
3	8.	8.00	(.00)		
4	5.	5.00	(.00)		
5	6.	6.67	(-.26)		
6	1.	.62	(.48)		
7	1.	.68	(.38)		
8	2.	2.03	(-.02)		
9	10.	11.26	(-.38)		
10	2.	1.06	(.91)		
11	0.	1.17	(-1.08)		
12	5.	3.51	(.80)		
13	90.	88.08	(.21)		
14	7.	8.31	(-.46)		
15	10.	9.15	(.28)		
16	26.	27.46	(-.28)		
FINAL LR	4.508	PRSN	3.622		
FINAL LATENT CLASS PROBABILITIES					
.4549424	.1415291	.3846816	.0188470		
FINAL CONDITIONAL PROBABILITIES					
LATENT CLASS:					
		1	2	3	4
1.	1.	.9521	.1302	.9521	.1302
1.	2.	.0479	.8698	.0479	.8698
2.	1.	.9452	.1256	.9452	.1256
2.	2.	.0548	.8744	.0548	.8744
3.	1.	.9985	.1132	.9985	.1132
3.	2.	.0015	.8868	.8868	.0015
4.	1.	.9999	.0685	.0685	.9999
4.	2.	.0001	.9315	.9315	.0001

Table 6

Preferred Model Observed and Expectency Frequency Counts,
Residuals, LR, PRSN, and Latent Class Probabilities
for Item Pairs 3,4 and 5,6

CELL	OBSERVED	EXPECTED	(STDIZED RESID)			
1	116.	116.22	(-.02)	
2	1.	.80	(.23)	
3	0.	.00	(.00)	
4	29.	28.99	(.00)	
5	5.	4.96	(.02)	
6	0.	.03	(-.18)	
7	0.	.00	(.00)	
8	5.	5.01	(.00)	
9	8.	7.93	(.02)	
10	0.	.05	(-.23)	
11	0.	.00	(.00)	
12	9.	9.01	(.00)	
13	17.	16.89	(.03)	
14	0.	.12	(-.34)	
15	0.	.00	(.00)	
16	116.	115.99	(.00)	
FINAL LR		.458	PRSN		.258	
FINAL LATENT CLASS PROBABILITIES						
		.4202644	.4163325	.0601278	.1032753	
FINAL CONDITIONAL PROBABILITIES						
	LATENT CLASS:		1	2	3	4
1.	1.	.9932	.0000	.9932	.0000	
1.	2.	.0068	1.0000	.0068	1.0000	
2.	1.	1.0000	.0000	1.0000	.0000	
2.	2.	.0000	1.0000	.0000	1.0000	
3.	1.	.9635	.0593	.0593	.9635	
3.	2.	.0365	.9407	.9407	.0365	
4.	1.	.9440	.0327	.0327	.9440	
4.	2.	.0560	.9673	.9673	.0560	

Table 7

Preferred Model Observed and Expectency Frequency Counts,
Residuals, LR, PRSN, and Latent Class Probabilities
for Item Pairs 3,4 and 11,12

CELL	OBSERVED	EXPECTED	(STDIZED RESID)		
1	44.	43.99	(.00)		
2	0.	.02	(-.14)		
3	0.	.00	(.00)		
4	3.	2.99	(.01)		
5	5.	5.00	(.00)		
6	0.	.01	(-.08)		
7	0.	.00	(.00)		
8	1.	.99	(.01)		
9	18.	18.01	(.00)		
10	0.	.01	(-.11)		
11	0.	.00	(.00)		
12	2.	1.98	(.02)		
13	79.	79.00	(.00)		
14	1.	.96	(.04)		
15	0.	.00	(.00)		
16	153.	153.04	(.00)		
FINAL LR	.077	PRSN	.039		
FINAL LATENT CLASS PROBABILITIES					
.2226517	.5076671	.2544725	.0152087		
FINAL CONDITIONAL PROBABILITIES					
LATENT CLASS:		1	2	3	4
1.	1.	1.0000	.0000	1.0000	.0000
1.	2.	.0000	1.0000	.0000	1.0000
2.	1.	1.0000	.0062	1.0000	.0062
2.	2.	.0000	.9938	.0000	.9938
3.	1.	.9042	.0051	.0051	.9042
3.	2.	.0958	.9949	.9949	.0958
4.	1.	.7141	.0044	.0044	.7141
4.	2.	.2859	.9956	.9956	.2859

Table 8

Preferred Model Observed and Expectency Frequency Counts,
Residuals, LR, PRSN, and Latent Class Probabilities
for Item Pairs 3,4 and 13,14

CELL	OBSERVED	EXPECTED	(STDIZED RESID)			
1	56.	56.00	(.00)	
2	0.	.00	(.00)	
3	0.	.00	(.00)	
4	1.	.99	(.01)	
5	10.	10.00	(.00)	
6	0.	.01	(-.07)	
7	0.	.00	(.00)	
8	1.	.99	(.01)	
9	0.	.00	(.00)	
10	0.	.00	(.00)	
11	0.	.00	(.00)	
12	0.	.00	(.00)	
13	80.	80.00	(.00)	
14	1.	.99	(.01)	
15	0.	.00	(.00)	
16	157.	157.01	(.00)	
FINAL LR		.025	PRSN		.005	
FINAL LATENT CLASS PROBABILITIES						
	.2143205	.5190461	.2628037		.0038297	
FINAL CONDITIONAL PROBABILITIES						
LATENT CLASS:		1	2	3	4	
1.	1.	1.0000	.0000	1.0000	.0000	
1.	2.	.0000	1.0000	.0000	1.0000	
2.	1.	1.0000	.0063	1.0000	.0000	
2.	2.	.0000	.9937	.0000	.9937	
3.	1.	.8539	.0000	.0000	.8539	
3.	2.	.1461	1.0000	1.0000	.1461	
4.	1.	1.0000	.0052	.0052	1.0000	
4.	2.	.0000	.9948	.9948	.0000	

Table 9

Preferred Model Observed and Expectancy Frequency Counts,
Residuals, LR, PRSN, and Latent Class Probabilities
for Item Pairs 13,14 and 11,12

CELL	OBSERVED	EXPECTED	(STDIZED RESID)			
1	22.	22.84	(-.18)	
2	3.	4.41	(-.67)	
3	0.	.00	(.00)	
4	22.	17.85	(.98)	
5	4.	2.75	(.75)	
6	0.	.53	(-.73)	
7	0.	.00	(.00)	
8	2.	2.18	(-.12)	
9	11.	9.47	(.50)	
10	2.	1.83	(.13)	
11	0.	.00	(.00)	
12	7.	7.54	(-.20)	
13	20.	24.82	(-.97)	
14	6.	4.79	(.55)	
15	0.	.00	(.00)	
16	207.	206.99	(.00)	
FINAL LR		4.584	PRSN	3.868		
FINAL LATENT CLASS PROBABILITIES						
	.1410538	.6741172	.0924145	.0924145		
FINAL CONDITIONAL PROBABILITIES						
	LATENT CLASS:	1	2	3	4	
1.	1.	.8382	.0000	.8382	.0000	
1.	2.	.1618	1.0000	.1618	1.0000	
2.	1.	1.0000	.0000	1.0000	.0000	
2.	2.	.0000	1.0000	.0000	1.0000	
3.	1.	.8926	.0007	.0007	.8926	
3.	2.	.1074	.9993	.9993	.1074	
4.	1.	.7073	.0002	.0002	.7073	
4.	2.	.2927	.9998	.9998	.2927	

Table 10

Preferred Model Observed and Expectency Frequency Counts,
Residuals, LR, PRSN, and Latent Class Probabilities
for Item Pairs 15,16 and 17,18

CELL	OBSERVED	EXPECTED	(STDIZED RESID)		
1	158.	158.26	(-.02)		
2	4.	3.91	(.04)		
3	7.	6.67	(.13)		
4	12.	12.17	(-.05)		
5	7.	6.72	(.11)		
6	0.	.19	(-.44)		
7	0.	.34	(-.58)		
8	1.	.75	(.29)		
9	4.	4.04	(-.02)		
10	1.	.31	(1.24)		
11	0.	.57	(-.76)		
12	2.	2.08	(-.06)		
13	54.	53.99	(.00)		
14	5.	5.59	(-.25)		
15	11.	10.43	(.18)		
16	40.	40.00	(.00)		
FINAL LR	3.375	PRSN	2.854		
FINAL LATENT CLASS PROBABILITIES					
.5635563	.1896149	.1898477	.0569811		
FINAL CONDITIONAL PROBABILITIES					
LATENT CLASS:		1	2	3	4
1.	1.	.9846	.1857	.9846	.1857
1.	2.	.0154	.8143	.0154	.8143
2.	1.	.9759	.1063	.9759	.1063
2.	2.	.0241	.8937	.0241	.8937
3.	1.	.9613	.0470	.0470	.9613
3.	2.	.0387	.9530	.9530	.0387
4.	1.	.9914	.0065	.0065	.9914
4.	2.	.0086	.9935	.9935	.0086

Table 11

Preferred Model Observed and Expectancy Frequency Counts,
Residuals, LR, PRSN, and Latent Class Probabilities
for Item Pairs 15,16 and 11,12

CELL	OBSERVED	EXPECTED	(STDIZED RESID)			
1	40.	39.99	(.00)	
2	1.	1.39	(-.33)	
3	2.	1.42	(.49)	
4	4.	4.20	(-.10)	
5	4.	4.17	(-.09)	
6	0.	.20	(-.45)	
7	1.	.40	(.95)	
8	1.	1.23	(-.20)	
9	17.	17.02	(.00)	
10	1.	.59	(.53)	
11	0.	.60	(-.78)	
12	2.	1.79	(.16)	
13	162.	161.81	(.01)	
14	8.	7.82	(.07)	
15	15.	15.58	(-.15)	
16	48.	47.78	(.03)	
FINAL LR	2.922	PRSN	2.449			
FINAL LATENT CLASS PROBABILITIES						
.1914091	.2265878	.5543094	.0276938			
FINAL CONDITIONAL PROBABILITIES						
LATENT CLASS:		1	2	3	4	
1.	1.	.9734	.2426	.9734	.2426	
1.	2.	.0266	.7574	.0266	.7574	
2.	1.	.9982	.0670	.9982	.0670	
2.	2.	.0018	.9330	.0018	.9330	
3.	1.	.9991	.0001	.0001	.9991	
3.	2.	.0009	.9999	.9999	.0009	
4.	1.	.7016	.0249	.0249	.7016	
4.	2.	.2984	.9751	.9751	.2984	

Table 12

Preferred Model Observed and Expectency Frequency Counts,
Residuals, LR, PRSN, and Latent Class Probabilities
for Item Pairs 17,18 and 13,14

CELL	OBSERVED	EXPECTED	(STDIZED RESID)			
1	51.	50.98	(.00)	
2	0.	.31	(-.56)	
3	2.	1.97	(.02)	
4	4.	3.74	(.14)	
5	8.	8.72	(-.24)	
6	1.	.14	(2.34)	
7	1.	.34	(1.14)	
8	1.	1.81	(-.60)	
9	0.	.00	(.00)	
10	0.	.00	(.00)	
11	0.	.00	(.00)	
12	0.	.00	(.00)	
13	122.	121.30	(.06)	
14	7.	7.55	(-.20)	
15	4.	4.69	(-.32)	
16	105.	104.45	(.05)	
FINAL LR	4.394	PRSN	7.667			
FINAL LATENT CLASS PROBABILITIES						
.1977506	.3702071	.4171147	.0149276			
FINAL CONDITIONAL PROBABILITIES						
LATENT CLASS:		1	2	3	4	
1.	1.	.9992	.0001	.9992	.0001	
1.	2.	.0008	.9999	.0008	.9999	
2.	1.	.9628	.0666	.9628	.0666	
2.	2.	.0372	.9334	.0372	.9334	
3.	1.	.8759	.0000	.0000	.8759	
3.	2.	.1241	1.0000	1.0000	.1241	
4.	1.	1.0000	.0121	.0121	1.0000	
4.	2.	.0000	.9879	.9879	.0000	

1,2:3,4

The hypothesized order for these pairs of items is that area problems that are represented by a grid (items 1 and 2) are prerequisite related to area problems without a grid (items 3 and 4). The results shown in table 3 indicate that the preferred model for these item pairs is H3. H3 supports the assumption that there is a latent class composed of individuals who are masters of the superordinate task (i.e. area problems without a grid) and at the same time nonmasters of the subordinate task (i.e. area problems with a grid). This state of affairs contradicts the assumption of prerequisite relatedness. Although prerequisite relatedness for these item pairs is not indicated, ordering by difficulty does appear warranted, with the probability of latent class three being higher than the probability of latent class four. In looking at the latent class probabilities in Table 4, one finds that only 3% of the population (probability of latent class 4 = .0325) would be expected to respond in a manner that contradicted the assumption of prerequisite relatedness (i.e. 3% would be successful with skills reflected in items 3 and 4 and not successful with skills reflected in items 1 and 2). This signifies that only 3 out of every 100 students are capable of solving non-grid area problems before they could solve grid area problems.

1,2:5,6

The hypothesized order for these pairs of items is that area problems that are represented by a grid are prerequisite to area problems with a partial grid. The results shown in table 3 indicate that the preferred model for these item pairs is again H3, which contradicts

the assumption of prerequisites. In this instance, the hypothesized sequence was correct, but the ordering was by difficulty rather than by prerequisites. The latent class probabilities in table 5 indicate that approximately 2% (probability of latent class 4 = 0.019) of the population would be expected to respond in a manner that contradicted the assumption of prerequisites. This signifies that only 2 out of 100 students can do partial grid problems before they can do total grid problems.

3,4:5,6

The hypothesized order for this pair of items is that area problems without a grid are prerequisite to area problems with a partial grid. In this instance H4 is the preferred model over H3. Thus, both partial mastery classes are equally probable, signifying that it is just as likely to find students that are masters of 3 and 4 type problems and nonmasters of 5 and 6 type problems as it is to find the reverse, in which students are masters of 5 and 6 type problems and nonmasters of 3 and 4 type problems.

3,4:11,12

The hypothesized sequence for this set of items is that standard shaped area problems are prerequisite to nonstandard shaped area problems. The preferred model based on the latent class analysis is model H3, indicating hierarchical ordering based on item difficulty and not prerequisites. However, table 7 indicates that only 1.5 % (probability of latent class 4 = 0.015) of the population would be expected to respond in a manner that contradicted the assumption of prerequisites.

3,4:13,14

The hypothesized order for this set of items is similar to that of the previous pairs, i.e. standard shaped area problems are prerequisite to nonstandard shaped area problems. Table 3 shows that the preferred model is H3, indicating hierarchical ordering based on item difficulty and not prerequisites. In this case only 0.4% of the population (probability of latent class 4 = 0.0038) would be expected to contradict the prerequisites criterion.

13,14:11,12

In this instance, the hypothesized order was that area problems of nonstandard shapes that required differencing were subordinate to those that required partitioning and recombining. Model H4 is the preferred model for this item pair, with both partial mastery classes being equally probable. This signifies that it is just as likely to find students that are masters of area skills represented by items 13 and 14 and nonmasters of area skills represented by items 11 and 12 as it is to find the reverse, i.e., masters of 11 and 12 type problems and nonmasters of 13 and 14 type problems.

15,16:17,18

These pairs of items tested the hypothesis that combining partitioned areas was subordinate to differencing partitioned areas, assuming that the values of the partitioned areas were already given. The results indicate that H3 is the preferred model, signifying ordering by difficulty and not by prerequisites. Approximately 6% of the population (proba-

bility of latent class 4 = 0.057) would be expected to contradict the prerequisites criterion.

15,16:11,12

The hypothesized order for these pairs of items is that area problems that contained information already calculated (i.e. the area quantities were given) are prerequisitely related to area problems that require the calculations. These item pairs involved nonstandard area problems that required the combining of areas. The results showed that the preferred model was H3, indicating an ordered relationship by difficulty. Approximately 3% of the population (probability of latent class 4 = 0.028) would be expected to violate the prerequisites assumption.

17,18:13,14

These pairs of items are similar to the previous pairs, however, they involve nonstandard area problems that require differencing. Again, the preferred model is H3, indicating ordering by difficulty. Approximately 1.5% of the population (probability of latent class 4 = 0.015) would be expected to contradict the prerequisites assumption.

Process analysis

Table 13 breaks down the number of correct responses by grade level as well as overall totals. Students who had correct responses were assumed to have used correct processes in solving the area tasks. The process codes for incorrect responses are presented in Table 14. In some instances, the student's responses were such that the process was indeterminable, i.e. the investigator could not logically explain

how the student arrived at his(her) answer. For these cases, and for those items that were left blank, a code of '0' was indicated. There were also several cases in which the process was evident for one item, yet the response on the equivalent item was not correct for that particular process. In such cases, both items were coded '0'. The tabulated results reflect only those processes that had been used more than once for any one item pair (i.e. two or more students had to have the same answers corresponding to a coded process for a pair of equivalent items before a code was tabulated). Processes used only once were also coded '0'.

Table 15 presents the codes that were used in conjunction with the item numbers. For example, code G is seen to have been used in items 5,6,9,10,11, and 12. As can be seen in the table, some codes were unique to an item pair. Code '0' was associated with all the items and code 'B' was second in frequency of use by the item pairs.

Table 16 presents the process totals for each item pair by grade level. For instance, in the sixth grade, students' incorrect responses to item pairs 5,6 involved 43 uses of process 'D' (i.e., 43 students counted the partial grid lines on all sides); 21 of code '0' (i.e., 21 students either didn't answer or were not consistent across equivalent items or used uncodeable processes); 2 uses of process 'E' (i.e., 2 students drew in lines or dots and counted square units); and 2 uses of process 'C' (i.e., 2 students multiplied both sets of adjacent sides and added the products).

Table 13

Number of correct responses per item										
<u>Item #:</u> <u>Grade(N)</u>	<u>1</u>	<u>2</u>	<u>3</u>	<u>4</u>	<u>5</u>	<u>6</u>	<u>7</u>	<u>8</u>	<u>9</u>	<u>10</u>
6(88)	66	62	17	18	22	20	4	67	0	3
7(68)	62	60	38	38	45	42	9	34	10	20
8(70)	55	59	36	36	42	39	6	29	8	15
9(80)	67	67	55	55	54	55	14	35	12	14
Total(306)	250	248	146	147	163	156	33	165	30	52
<u>Item #:</u> <u>Grade(N)</u>	<u>11</u>	<u>12</u>	<u>13</u>	<u>14</u>	<u>15</u>	<u>16</u>	<u>17</u>	<u>18</u>		
6(88)	9	8	2	2	67	67	26	28		
7(68)	15	11	9	11	54	48	44	44		
8(70)	14	9	20	26	62	60	58	58		
9(80)	29	25	26	29	59	58	60	59		
Total(306)	67	53	57	68	242	233	188	189		

Table 14

Description of Process Codes

<u>Code</u>	<u>Description</u>
O	Did not answer, or not codeable (either an indeterminable process or a unique process) or not consistent across items.
A	Used multiplication rule without regard to the non-rectangular shape of the figure.
B	Added all the numbers shown in the diagram.
C	Multiplied both sets of adjacent sides and then added the products.
D	Counted the partial grid lines on all sides.
E	Drew in lines or dots and used counting rule.
F	Used $L \times W$ formula for partial grid with L and W equal to the number of grid lines.
G	Added the numbers that make up only the perimeter.
H	Counted the partial grid lines then used procedure in code 'C'.
I	Multiplied all the numbers on the figure.
J	Multiplied two adjacent sides
K	Multiplied two (or three) sides of a figure and added the third (or fourth) side.
L	Multiplied all the numbers and divided by 2.
M	Multiplied two sides and the height.
N	Added two adjacent sides.
P	Same as code 'B' but for the outer rectangle only.
Q	Used the outer area as the answer.

Table 14 (continued)

<u>Code</u>	<u>Description</u>
R	Added numbers written only in the shaded region.
S	Subtracted the added numbers of the inner rectangle from the added numbers of the outer rectangle.
T	Added the calculated areas of the inner and the outer rectangles.
U	Used the inner area as the answer.
V	Added the numbers of the inner rectangle.
W	Counted the spaces between the lines that composed the shady region.
X	Multiplied word problems together.
Y	Added numbers given in word problems.
Z	Divided numbers given in word problems.
1	Applied code 'C' to both the outer and inner rectangles and then subtracted the inner from the outer.

Table 17 presents the process totals across all the grades collapsed. Thus, items 3,4 show that for all the students tested, 110 used code 'B' (i.e. added all the numbers in the diagram); 36 fell under category '0' (i.e. didn't answer or were not consistent across equivalent items or used uncodeable processes); and 16 used code 'C' (i.e. multiplied both sets of adjacent sides and added the results together). As in the latent-class analysis, items 7,8 and 9,10 were left out of the process analysis due to the confounding of items 8 and 10. The numbers reported in tables 15 through 17 will always be based on an overall sample size of 306 students, with 88 at the sixth grade level, 68 at the seventh grade level, 70 at the eighth grade level and 80 at the ninth grade level.

Items 1,2

The process totals listed in table 17 for items 1 and 2 indicate that 56 students either didn't answer one or both of the items, or were inconsistent in their responses across items, or responded in such a way that coding was not possible (code = '0'). Six students were assigned a code 'A' for their responses, indicating that they used the multiplication rule of Length x Width without regard to the non-rectangular shape of the item figures.

The process totals by grade level did not reveal any unusual pattern across the grade levels. However, sixth graders did have 42% of the '0' codes (Table 16).

Table 15

Item Numbers Associated With Specific Codes

<u>Code</u>	<u>Item Numbers</u>
O	all items
A	1,2
B	3,4,5,6,7,8,9,10,11,12,13,14
C	3,4,5,6,9,10,13,14
D	5,6
E	5,6,11,12
F	5,6
G	5,6,9,10,11,12
H	5,6
I	7,8,11,12
J	7,8,9,10,11,12
K	7,8,9,10
L	7,8
M	9,10
N	9,10
P	13,14
Q	13,14,17,18
R	13,14
S	13,14
T	13,14
U	13,14,17,18
V	13,14
W	13,14
X	15,16,17,18
Y	17,18
Z	17,18
1	13,14

Table 16

Process Code Totals for Each Item Pair by Grade

<u>Item #:</u> <u>Grade</u>	<u>1,2</u>	<u>3,4</u>	<u>5,6</u>	<u>11,12</u>	<u>13,14</u>	<u>15,16</u>	<u>17,18</u>	
6	0-23	B-59	D-43	B-56	0-41	0-17	Y-21	
	A-3	0-11	0-21	0-24	B-20	X-4	0-30	
		C-1	E-2		R-9		U-6	
			C-2		P-8		Q-3	
					W-2		X-2	
					S-2			
					Q-2			
					C-1			
					T-1			
	7	0-7	B-17	D-11	0-31	0-38	0-12	0-11
		A-1	0-10	0-10	B-22	P-11	X-8	Y-7
			C-3	B-3	I-2	S-4		U-4
				F-2	J-1	Q-4		X-2
				G-1	C-1			
					T-1			
8		0-14	B-18	0-15	0-35	0-22	0-5	0-5
		A-1	C-9	D-7	B-18	P-9	X-5	Y-4
		0-7	C-4	I-7	Q-7		X-2	
			F-3	G-1	R-4		Z-1	
			E-1		B-2			
			B-1		S-2			
					1-2			
					U-1			
					V-1			
	9	0-12	B-16	0-16	0-25	0-32	0-11	Y-8
A-1		0-6	D-6	B-12	Q-6	X-11	0-6	
		C-3	C-3	I-9	P-4		X-4	
			E-1	G-6	T-3		Z-3	
			F-1	J-3	S-3			
					V-2			
					U-2			
					C-2			

Table 17

Process Code Totals For Each Item Pair							
Item #:	<u>1,2</u>	<u>3,4</u>	<u>5,6</u>	<u>11,12</u>	<u>13,14</u>	<u>15,16</u>	<u>17,18</u>
0-56		B-110	D-67	0-115	0-133	0-45	0-52
A-56		0-36	0-62	B-108	P-32	X-28	Y-40
		C-16	E-4	I-18	B-22		U-10
			C-9	G-8	Q-19		X-10
			F-6	J-4	R-13		Z-4
			B-4		S-11		Q-3
					T-5		
					C-4		
					U-3		
					V-3		
					W-2		
					1-2		

Items 3,4

Thirty-six students were given a code of '0' for items 3 and 4. The most numerous code for items 3 and 4 was a code 'B' indicating that 110 students added up all of the numbers shown in the diagram. In addition, 16 students multiplied both sets of adjacent sides of the rectangle and then added the products together (code = 'C').

The process totals broken up by grade revealed that the frequency of code 'B' decreased dramatically beyond the sixth grade and that there was a corresponding increase in code 'C', although on a smaller scale.

Items 5,6

The codes under items 5 and 6 in table 17 revealed that in addition to code '0' (62 students), there were six different processes that students (i.e. two or more) used in attempting to solve the partial grid problem. The largest number of students (67) used process 'D' which involved just counting the partial grid lines on all four sides of the rectangle. There were seven students that correctly counted the linear units on adjacent sides, and then proceeded to multiply both sets of adjacent sides and add the products (code 'C'). Six students used the correct Length x Width formula for area, but used the number of grid lines in place of the number of linear units on a side (code 'F'). The other codes varied from drawing in dots or lines to complete the grid to just counting the linear units all around the perimeter.

The breakdown of processes by grade level (table 16) showed that 64% of the students that used process 'D' came from the sixth

grade. The rest were closely divided among the upper three grades. The other processes were used infrequently when broken up by grade level.

Items 11,12

One hundred and fifteen students (approximately 38% of the students) were coded '0' for items 11 and 12. Another 108 students (or 35% of the students) added all the numbers shown in the diagram (code 'B'). There were 18 students that multiplied all the numbers shown on the diagram (code 'I'). Eight students were able to determine the values of the sides that lacked numbers and proceeded to calculate the perimeter (code 'G'). And finally, four students multiplied two adjacent sides of the nonstandard figure (code 'J').

In looking at the processes by grade level it is apparent that students in the sixth grade either received a code '0' or a code 'B'. It is not until the seventh grade that the other processes noted above come into play. Code 'G' which involved the determination of unknown values was used mainly in the ninth grade. Multiplying all the numbers in the diagram (code I) was also more evident in the upper two grades.

Items 13,14

Items 13 and 14 seemed to have elicited the most variety of processes among the items analyzed. The largest category coded was the '0' code, indicating that students either didn't respond, or responded inconsistently across both items, or responded with an indeterminable process (133 students). The second largest category (32 students with a code 'P') is similar to previous codes in that the sides of a rectangle

are added together to obtain a perimeter. However, in relation to items 13 and 14, the above process is applied only to the outer rectangle. Code 'B' is the third largest for this item pair (22 students) and it involves the addition of all the numbers shown in the diagram (for both the inner and outer rectangles). There were 19 students that correctly obtained the area of only the outer rectangle (code 'Q'). Thirteen students added the numbers that were only shown in the shaded part of the figure (code 'R') and 11 students subtracted the perimeter of the inner rectangle from the outer rectangle (code 'S'). The rest of the codes involved such processes as adding the areas of the inner and outer rectangles together; multiplying two sets of adjacent sides for the outer rectangle and adding the products; doing the same for the inner rectangle and then subtracting the inner rectangle from the outer rectangle; calculating just the inner area; calculating the perimeter of the inner rectangle, and counting the lines that formed the shaded region.

The breakdown of processes by grade level for items 13 and 14 did not reveal any unusual patterns. It appears that items 13 and 14 elicited varied processes across all grade levels.

Items 15,16

The process totals for items 15 and 16 were confined to two codes: code 'O', with 45 students and code 'X', with 28 students. Code 'X' indicates that the students multiplied the numbers in the word problem instead of adding them. Again, upon visual inspection of the breakdown by grade level, no unusual patterns emerged.

Items 17,18

There were 52 students coded 'O' for items 17 and 18. Forty students incorrectly added the numbers in the word problem (code 'Y') and ten just used the inner area as their answer (code 'U'). There were also ten students that multiplied the numbers in the word problem (code 'X'). Four students divided the numbers in the word problem (code 'Z') and finally, three students used the outer area as their answer (code 'Q').

A breakdown by grade level revealed that the majority of code 'O's and 'Y's were in the sixth grade (58% and 53% respectively). Also there were no code 'Q's beyond the sixth grade.

DISCUSSION

The present study attempted to establish a hierarchy of skills on the topic of area based on rules, representation and shape. The hypothesized skill orderings, shown in figure 10, were analyzed (with the exception of the hierarchies based on geometric shape) through the use of latent-class analysis. In addition, the students' responses were also coded in terms of the processes used to solve the area problems.

The results of the latent-class analysis revealed several skill orderings based on item difficulty. The hypothesized hierarchy based on representation stated that area problems that contained a square unit grid were subordinate to both area problems without a grid and area problems with a partial grid. This hierarchy was supported by the latent-class analysis. However, the ordering was shown to be a function of difficulty and not prerequisites. The latent-class analysis demonstrated that approximately two to three percent of the population was expected to respond in such a way as to contradict the prerequisites criterion for these skills. This suggests that for the skills in question, very few students can do non-grid and partial grid area problems before they can do total grid problems.

The latent-class analysis revealed that prerequisites is contraindicated for the above skills. Future research may involve interviewing those students that responded contrary to the prerequisites ordering for these skills. In this instance, it appears that some of these students had applied a strategy that did not work with the

subordinate tasks but was successful for the superordinate tasks. The process analysis revealed that several students had incorrectly applied the Length x Width formula to the total grid problem. This state of affairs would have resulted in failure for the total grid problem and success on the non-grid problem. Interviewing these students would have revealed whether they were capable of using the counting rule for the total grid problem, but opted for the faster Length x Width method without considering the shape of the figure. Nevertheless, these students were successful with non-grid problems without having to be successful on total grid problems.

In addition to the above skills, the representation hierarchy also hypothesized that area problems without a grid were subordinate to area problems with a partial grid. However, the latent-class analysis did not support this ordering of area skills. In fact, the analysis showed that it was equally probable to find students who have mastered the partial grid problem and not mastered the non-grid problem, as it was to find students who have mastered the non-grid problem and not mastered the partial grid problem. In this instance, a possible explanation for the lack of hierarchical ordering is that students may have internally represented the partial grid problem as a total grid problem. In fact, further evidence of this possibility stems from the observations that despite instructions to the contrary, students had filled in the partial grids, enabling them to use the counting rule in determining the area. One way to reduce the probability of using the counting rule is to increase the number of unit squares such that filling in the diagrams either mentally or on the test itself

would be too complex. This would force the student to combine the counting and Length x Width rules in order come up with the correct answer.

There were several hypothesized skill orderings based on shape. As noted in the results section, the numbers in the figures for items 8 and 10 confounded the underlying skill assumptions related to these items. As a result, the latent-class analysis for the hierarchy based on geometric shape was not performed. However, several implications can be extracted from this confounding. First, it is important to use more than one item in determining a student's mastery of a skill. Without an equivalent item, the confounding may not have been detected. And second, teachers should be cognizant of the fact that the numbers used in area problems may have an effect upon the skills being assessed. As was the case in items 8 and 10, students can correctly answer area problems using processes that have no relation to the skills being assessed (e.g. the perimeter of a geometric shape can be equal to its area).

Latent-class analysis did support the hypothesized skill orderings for hierarchies based on shape forms. The hypothesized sequence for shape forms indicated that area problems that involved a standard geometric shape (i.e. a rectangle) would be subordinate to area problems that involved nonstandard shapes. This sequence was shown to be true for both levels of nonstandard shapes, i.e. shapes that involved partitioning and recombining of areas and shapes that involved differencing of areas.

As in the case of the representation hierarchy, the hierarchical ordering for shape forms was based on difficulty and not on prerequisite-

ness. There existed a class of students that were capable of solving nonstandard area problems, while lacking the ability to solve standard shaped area problems. The percentage of the population that is expected to respond in this way is 1.5% for the skills involving partitioning and recombining areas and 0.4% for the skills involving differencing of areas. A possible explanation for students responding in this manner may be the distancing among items in the test. By the time the student came upon the nonstandard problems (items 11, 12, 13 and 14), he/she may have had a moment of insight into solving nonstandard area problems and yet, failed to return to the earlier standard area problem to correct their responses (items 3 and 4). To test this explanation, future research should place the standard area problems in closer proximity to the nonstandard area problems. In addition, interviewing students would determine how they were able to solve the superordinate problem before the subordinate problem.

The next hypothesized ordering involved levels of nonstandard shapes. In this instance, the latent-class analysis did not support hierarchical ordering among levels of nonstandard shapes. The hypothesis developed for these skills proposed that nonstandard shaped area problems that require differencing of areas are subordinate to those problems that require the partitioning and recombining of areas.

For this pairing of skills, it was also hypothesized that there were two opposing forces that influenced the hierarchical ordering of the two levels of nonstandard shapes: rule components and rule replacement. Based on the literature, rule components were hypothesized to have a stronger influence on the ordering between levels of nonstandard

shapes. As a result, differencing of areas was hypothesized to be subordinate to partitioning and recombining of areas.

In order to demonstrate that rule replacement affected the ordering of the levels of nonstandard shaped area problems, the next set of latent-class analyses focused on the ordering of skills that involved the addition of already partitioned areas and on skills that involved the differencing of already partitioned areas. The effect of including these items was to hold the component rule force constant while attempting to demonstrate the effects of rule replacement.

The results of the latter analysis supported the hypothesis that the addition of already partitioned areas is subordinate to the differencing of already partitioned areas for nonstandard shaped area problems. The ordering was based on difficulty level and not prerequisite-ness. Therefore, the rule replacement force does appear to influence the hierarchical ordering when the rule component force is held constant.

However, the fact that the latent class analysis did not support hierarchical ordering among levels of nonstandard shape seems to suggest that the force of component rules exerted an equal but opposite force to that of rule replacement. This seems to have nullified the effects of rule replacement and resulted in equiprobable mastery classes. To further test this assumption future research may attempt to assess the effects of the component rule force, while holding the replacement rule constant. Nevertheless, the present results indicate that the two types of nonstandard area problems are in related subdomains and that the two partial mastery classes are equally probable. The implication for instruction is that the two skills can be taught at the

same time without having to consider hierarchical ordering.

The final latent class analysis revealed that area problems that provided the student with apriori information, such as the value of partitioned areas, were subordinate to area problems that required calculation of the partitioned areas. This was seen to be true for both the addition and subtraction levels of nonstandard shaped problems. Once again, the ordering was a function of difficulty and not prerequisites. This sequence was expected since problems with given information should be easier than problems without given information. The unexpected result was that there were some students who were successful with area problems that required calculation of partitioned values and were not successful with problems that provided the student with partitioned values. This defiance of logic may have to do with the fact that the test was given in two parts, with the items containing the given information in part II and the problems without the given information in part I. Future research should interview those students that did not fit under the prerequisite model to determine their reasoning processes in these skills.

To sum up the latent class analysis, the bases for the hierarchical ordering of rules, representation and shape seemed to be effective in establishing hierarchical relations for seven of the nine hypothesized item pairs. Although the established hierarchies were seen to be a function of difficulty and not prerequisites, the number of students that contradicted the prerequisites criterion was small. Further research is needed to firmly rule out prerequisites for these item pairs.

The process analysis revealed several interesting points regarding students' approach to area problems. First, the nonstandard shaped items presented themselves as the most difficult items on the test for this sample population. The numbers of correct responses for the four nonstandard shaped items ranged from 53 to 67, or approximately 20% of the sample population ($N = 306$). Although the ninth grade in general performed the best on these items, the percentage of ninth graders that correctly answered these items remained small (from 33% to 36%, depending on the item). These items represent crucial concepts in understanding area (Hirstein, Lamb, & Osborne, 1978). It would appear that the majority of the students sampled do not comprehend the invariance of area upon partitioning and then recombining a figure.

Second, the most frequent processes associated with incorrect responses were those that involved the addition of numbers shown in the item figures. In the case of the partial grid items, the counting of the grid lines around the perimeter was the most frequently used process. These results support Woodward and Byrd's findings (1983) that a great deal of confusion exists between area and perimeter concepts.

The second most frequent processes associated with incorrect responses were those that involved some form of multiplication (e.g. multiplying all the numbers shown in the figures or just multiplying both sets of adjacent sides in the case of a rectangle). It appeared that in these instances students had a vague notion of the concept of area as some mathematical manipulation that required multiplication. However, the processes revealed that the students used multiplication without regard to the area concepts involved in the items.

Based on these results, one may speculate that the majority of errors in the solving of area problems are either those that confuse area with perimeter or those that use multiplication without an appropriate understanding of the plane-filling properties of area. Future research may address this issue by further delineating the type of errors made in area problems.

The partial grid problems demonstrated that students had difficulty comprehending that square units are composed of linear units. Battista's (1982) assertion that students should be exposed to finding areas of regions using a variety of nonsquare units appears warranted in light of the processes used for the partial grid problems. The counting of the partial grid lines, whether used to calculate the perimeter or the area, was the most frequent process encountered for partial grid items. Battista's (1982) approach to instruction could only further increase their understanding of area as a plane-filling concept.

Finally, the patterns of processes across grade levels appears to have revealed a division between the sixth grade and the other three grades with respect to 1) the types of processes used in making errors and 2) the quantity of errors made. After the sixth grade, more errors were made that reflected multiplication processes in comparison to addition processes. In addition, a larger percentage of items that were coded '0' occurred in the sixth grade than in the other grades. A code '0' signifies either no responding, responding inconsistently, or responding with numbers that could not be coded. This was to be expected since students should be learning more as they advance in grades. However, for some of the items the larger percentage was more

noticeable. This may reflect the presentation of new concepts at upper grades beyond the sixth and/or the opportunity to practice and apply skills previously presented but not yet learned.

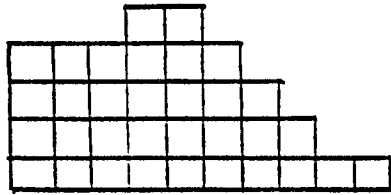
In conclusion, it appears that the use of rules, representations and shapes as a foundation for a hierarchy among area skills does have merit. Further research is still needed to demonstrate that prerequisite-ness ordering can be ruled out in the established hierarchies. In addition, this study demonstrated that a great deal of confusion still remains with respect to the area skills that are learned in the sixth to the ninth grade levels. Further research into the processes used by students in this grade range may shed more light on the instructional approach needed to bridge the gap in understanding the topic of area. The investigation of hierarchical orderings in this study should facilitate the instructional process for area skills.

APPENDIX A

The Test Items

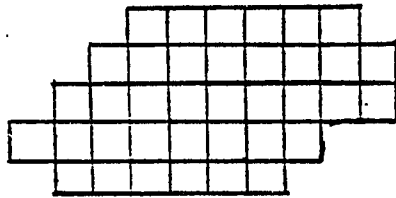
Find the area of the following figures:

1.



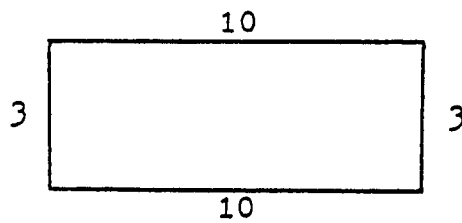
_____ square units

2.



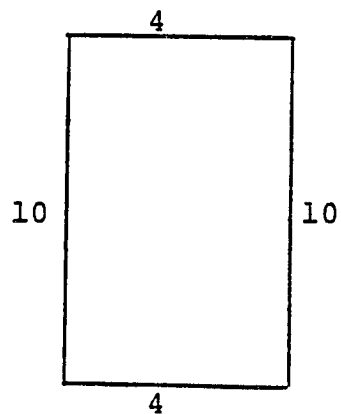
_____ square units

3.



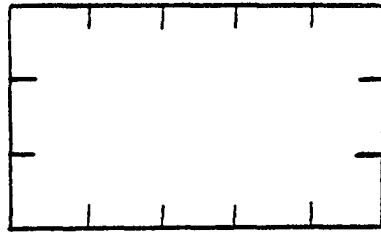
_____ square units

4.



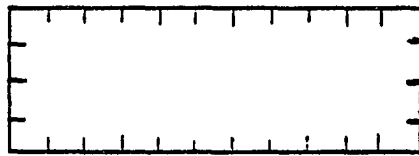
_____ square units

5.



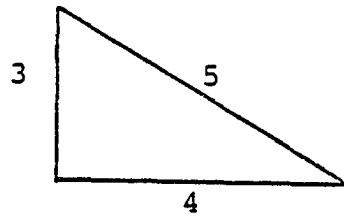
_____ square units

6.



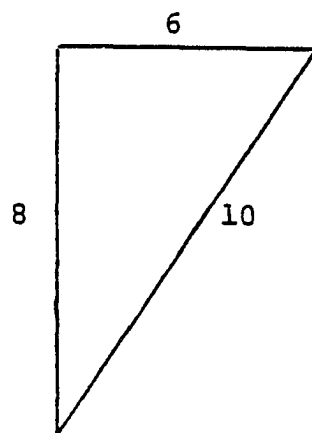
_____ square units

7.



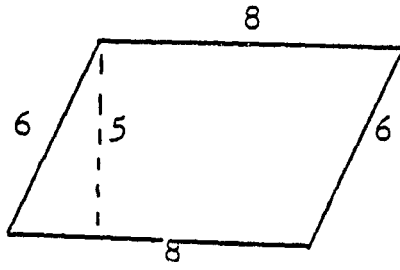
_____ square units

8.



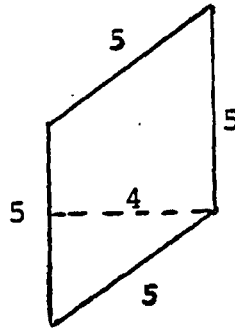
_____ square units

9.



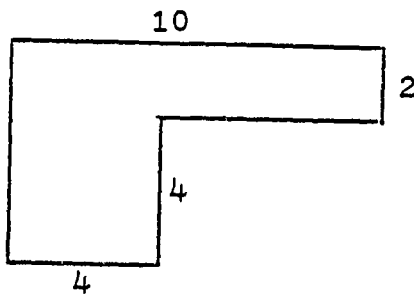
_____ square units

10.



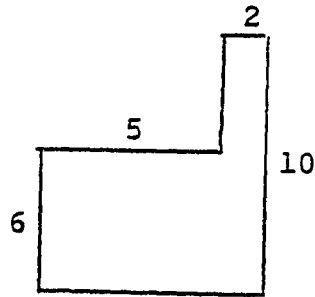
_____ square units

11.



_____ square units

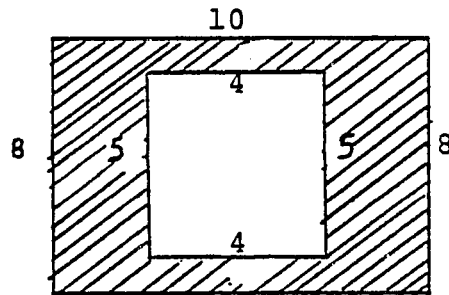
12.



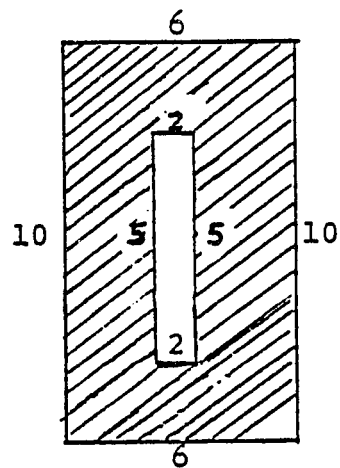
_____ square units

Find the area of the shaded region:

13.



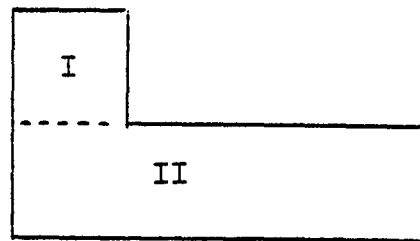
14.



_____ square units

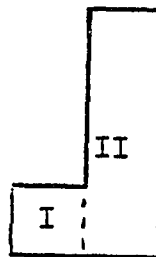
_____ square units

15. Area of section I = 20 square units.
 Area of section II = 60 square units.
 Find the area of the whole figure.



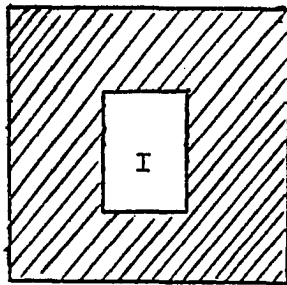
_____ square units

16. Area of section II = 8 square units.
 Area of section I = 20 square units.
 Find the area of the whole figure.



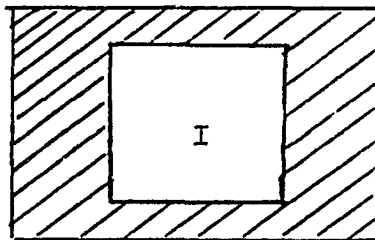
_____ square units

17. Area of the whole figure = 80 square units.
Area of section I = 20 square units.
Find the area of the shaded region.



_____ square units

18. Area of section I = 100 square units.
Area of whole figure = 300 square units.
Find the area of the shaded region.



_____ square units

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