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RELIABILITY ANALYSIS OF SERIES STRUCTURAL SYSTEMS

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RELIABILITY ANALYSIS OF SERIES STRUCTURAL SYSTEMS

by

Lidvin Kjerengtroen

A Dissertation Submitted to the Faculty of the

DEPARTMENT OF AEROSPACE AND MECHANICAL ENGINEERING

In Partial Fulfillment of the Requirements
For the Degree of

DOCTOR OF PHILOSOPHY
WITH A MAJOR IN MECHANICAL ENGINEERING

In the Graduate College

THE UNIVERSITY OF ARIZONA

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entitled Reliability Analysis of Series Structural System

and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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Yih-Tsuen Wu Date 1/25/85

P H Wirsching Date 1/25/85

Dimitri Kececioglu Date 1/25/85

Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

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SIGNED: Lidvin Kjerengtroen

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NOMENCLATURE

A	Area
a	Constant relating element stresses
B	Random variable to account for uncertainty in stress modelling
$COV(\cdot)$	Covariance of
C_R	Coefficient of variation of the strength variable
C_{R_i}	Coefficient of variation of the strength variable, element i
C_S	Coefficient of variation of the stress variable
D	Fatigue damage
D_{NB}	Fatigue damage for a narrow band process
E	Percent error
$E_j(\cdot)$	Event of
$E(\cdot)$	Expected value of
F	Cumulative distribution function (cdf); subscript indicates variable or distribution type
f	Probability density function (pdf); subscript indicates variable
$g(\underset{\sim}{x})$	Failure or limit state function of $\underset{\sim}{X}$
$g'(\underset{\sim}{x})$	Reduced failure of limit state function of $\underset{\sim}{X}$
$\vec{i}$	Unit directional vector
K	Emperical constant in S-N curve
Ln	Lognormal distribution
m	Negative reciprocal slope of S-N curve
N	Cycles to failure
N_T	Total number of fatigue stress cycles driving life T
N_o	Design life

n	Number of elements
$\vec{n}$	Normal to the plane at the design point
$P(\cdot)$	Probability of
P	Correlation matrix
p_i	Element probability of failure
p_s	System probability of failure
p_s^*	Approximate system probability of failure
Q_i	Load in element i
R	Strength
R_i	Strength of element i
R_s	System Reliability
$\vec{r}$	Position vector vector to any point in the limit state plane
S	Stress, in the S-N curve equation S denotes stress range
S_i	Stress in element i
T	Time to failure
t	Integration variable
u	Reduced variable
V	Stress random variable
$\underset{\sim}{x}$	A vector of design variables
α	Weibull shape parameter or ratio of element to system probability of failure
β	Weibull scale parameter
β_i	Element safety index
β_s	System or generalized safety index
β_{o1}	Element target safety index
β_{oS}	System target safety index

γ	Parameter defining load gradient
Δ	Damage at failure or difference
δ	Ratio of exact to system probability of failure
λ	Constant to accounts for load dependency or a constant to relate equivalent narrow band and wide band fatigue damage
μ_i	Mean value of element i
π	Product
ρ	Correlation coefficient between to random variables (subscript indicates which variables)
ρ_e	Equivalent normal correlation coefficient
σ	Standard deviation (subscript indicate variable)
σ_N	Equivalent normal standard deviation
Σ	Sum
ϕ	Normal probability density function
Φ	Normal probability distribution function
Φ^{-1}	Inverse normal probability distribution function
$\cap$	Intersection
$\mid$	Conditioning
~	Median value if above variable and vector if below vector
$\cdot$	Dot or inner product
*	Denotes variable at design point

ABSTRACT

Reliability analysis of series structural systems with emphasis on problems typical for metal fatigue is addressed. Specific goals include the following: (1) Given the distribution of strength of the components and the distribution of external loads on the system what is the probability of failure of the system? (2) Given the target safety index for the system, what would be the target safety index for the components.

Exact solutions in the analysis of series structural systems only exists for some special problems. Some of these special problems are investigated. In particular some special cases of the problem of unequal element reliabilities are considered and some interesting observations are made. Numerical integration is in general required even when an exact solution exists. A correction or adjustment factor is developed for an important class of problems. This factor makes it possible to relate element and system probabilities of failure without numerical integration.

However in most cases no exact solution to the structural series system problem exists. Approximations by for instance bounds on the probability of failure or Monte Carlo simulation has been the only way of approximating solutions. These two methods are generally not good approximation schemes since they are either too crude or too expensive. In this dissertation an approximation scheme for analysis of series systems where no exact solution exists is developed. The method only requires a simple numerical integration if the component safety index and the correlation coefficient between failure modes is known. Numerous examples are used to verify the method against known exact results and excellent estimates are obtained. Application by practical examples is also given.

In the appendix the problem of convergence of fatigue life distributions is also summarized.

CHAPTER 1

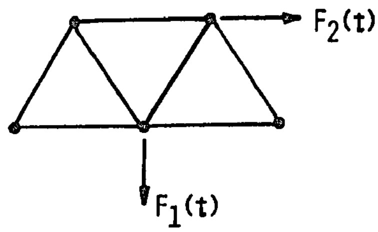
INTRODUCTION

1.1 Description of System Reliability

A structural or mechanical system is defined as a collection of components. A series system is defined as one which fails if <u>any</u> component fails. Examples of structural series systems are provided in Fig. 1.1. In general, the components do not have equal strength and the load in each component may be different. The statically determinate structure of Fig. 1.1a is an example of such a system.

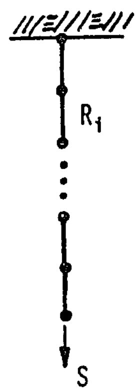
In both conventional allowable stress design (ASD) and in reliability based design, analysis is often aimed at the selection of individual components. The individual elements, however, are almost always a part of a structural assemblage in which there is an interaction between the system and the individual components. Consequently, it is reasonable to believe that, in theory, a design based on system considerations can be more reliable and/or more cost effective than one based on component ASD. In this regard, it is necessary to define how the overall system reliability relates to individual element reliabilities. For a series system, the system reliability is always less than or equal to the individual element reliabilities. At the other extreme is a redundant system, which is a system that may survive failure of one or more of it's components. Redundancy in a system may be of the "active" or the "standby" type (Ang, (1984)). In the case of active

(a)



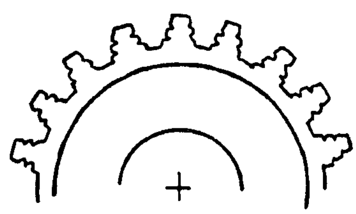
Resistance distribution and loads in each member could be different.

(b)



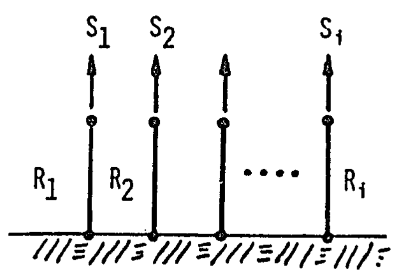
Load in each member is the same, but resistance distribution could be different for each.

(c)



Turbine disk. Failure occurs if crack initiates at any root; similar to Fig. 1b.

(d)



A series system where loads and resistance are independent. (failure if any member fails)

Fig. 1.1 Examples of Series Systems

redundancies, all the components of a system are participating (e.g., carrying or sharing loads). For systems with standby redundancies some of the redundant components are inactive and become activated only when some of the active components have failed.

For the analysis of series systems, it may be unsafe to use ASD on individual elements, unless there is perfect correlation between the failure functions of the individual elements. If the element failure functions or limit state equations are positively correlated it is always conservative to assume that the element reliabilities are independent.

The general goal of this dissertation is, to study series system reliability with special emphasis on fatigue. It is in particular the goal to develop an approximation method for analysis of complicated series systems for which there exists no exact solution.

1.2 Previous Work

First attempts to apply probabilistic and statistical concepts in structural analysis date back to the beginning of this century with publications by Forsell (1924) and Mayer (1926). In spite of these early works, the subject did not receive much attention before the 1940's. However, after World War II, publications in structural reliability have steadily increased through the years; numerous specialty conferences on structural safety and probabilistic design have been held worldwide. Prot (1936, 1948, 1949, 1950, 1951) published several papers in which his main interest was the statistical distribution of stresses. He defined ways of splitting up the stress distribution due to different effects and gave rules for combining them to determine probabilities of failure. Kjellman (1940) and Wastlund (1940) defended the probabilistic approach to structural design and introduced some basic concepts.

Freudenthal, who is widely regarded as the father of modern probabilistic design, wrote numerous papers on structural reliability (1947, 1954, 1961, 1966). His publications of 1947 and 1954 were landmark papers and laid the foundation for the development and implementation of structural reliability.

Torroja and Paez (1949, 1951, 1952) considered different causes of variability and attempted to quantify their distributions. They derived relations between traditional safety factors and probabilities of failure. Early work on distributions for loads and stresses was carried out by Johnson (1953) and Borges (1952, 1954).

From the early 1950's and up to about 1970, much of the research was done in the United States. In addition to Freudenthal, the most active investigators have been, Shinozuka (1966), Benjamin (1968, 1970), Cornell (1967), Ang (1973, 1979, 1981, 1982), Moses (1967, 1974) and Kececioglu (1964). But during this period Pugsley from Great Britain (1951, 1955, 1966) also made important contributions. Haugen (1968, 1980) also deserves to be mentioned with the publication of his two books and other important work.

However in the past 10 to 15 years, most of the pioneering work in structural reliability has been done in Europe. Although Hasofer and Lind (1974) solved the invariance problem in the definition of the safety index. Rackwitz and Fiessler (1977) introduced an algorithm where non-normally distributed variables at the design point is transformed into a set of normally distributed variables. Their method allows rapid numerical estimation of the probability of failure for non-normal and non-linear limit states. It is impossible to summarize all the work that has been done in the past decade. However, the work of Ditlevsen (1973, 1977, 1981) and Lind (1974) deserves to be mentioned. In some regards they have been central figures in the evolution of probabilistic design in much the same manner as Freudenthal was in the early period. Other important contributions has been made by Wirsching (1975, 1977, 1980), Thoft-Christensen (1980, 1982) as well as Galambos and Ravindra (1978) and many others.

The focus of this dissertation is on reliability consideration of series structural systems. Much of the fundamental theory has been presented in the literature by Freudenthal (1947), Ang (1968, 1979), Moses (1970, 1974) and Gorman (1979). Garson (1980) investigated the importance of correlated failure modes in weakest link systems and concluded that they are of minor importance if the correlation coefficient between element failure functions is less than approximately 0.7 and the probability of failure less than about 10^{-3}. Grigoriu and Turkstra (1979) and Thoft-Christensen and Dalsgard Sorensen (1982) looked at the importance of correlated component resistances in weakest-link systems. Grigoriu (1983) has developed methods to approximate system reliabilities when the loads and the strengths are non-normally distributed. Finally Ditlevsen (1984) has developed equivalent correlation coefficients, based on first and second order

Taylor expansions, when the element failure functions are unequally correlated.

Very little work and effort has been put into practical applications of existing series system theory. It is a goal of this dissertation to use and modify the existing basic theory so that practical problems can be solved in a simple and accurate manner. Solution of practical problems has often been prohibited for two reasons, (a) the existing theory is restricted by many assumptions and (b) the analysis would be too complicated for practicing engineers.

Approximation techniques are developed so that simple design procedures with good accuracy results. In particular it is necessary to develop a method so that the system reliability can be obtained directly from knowledge of the components reliabilities and parameters.

In recent years there has been a considerable effort to develop upper and lower bounds on the system reliability. Early attempts were made by Cornell (1967) and Ang and Amin (1968). However, these proved to be too general for practical purposes. Ove Ditlevsen (1979) introduced a new set of narrow bounds on system probability of failure. These bounds take into consideration the possible correlation between failure functions of any two elements and also between failure modes. When the correlation coefficient increases, even these bounds become quite wide. A discussion and illustration of "Ditlevsen-bounds" can be found in Ang and Tang (1984) and Thoft-Christensen and Baker (1982).

Finally it should be mentioned that considerable effort has been put into application of reliability methods to calibration of existing and development of new design codes using partial safety factors. This is a "semi-probabilistic" approach to design and is undoubtedly the area where probabilistic methods has real practical application. However, this subject is not included in the scope of this dissertation.

1.3 Some Basic Definitions and Concepts

In this section some fundamental definitions and concepts in the theory of probabilistic design and structural reliability analysis will be presented. A structural/mechanical system, and in particular, a series system was defined in Section 1.1.

The reliability of a system or a component is the probability that s system/component survives a mission or satisfies some predefined requirements. According to Kececioglu (1981) these requirements are:

- Operating specifications without failure must be met
- The reliability must be related to age and operating time
- Environment, application and load history must be specified
- The confidence level on the reliability should be known.

In probabilistic design theory all these requirements in the definition of reliability may not be fulfilled and for this reason probabilities should often be considered "notional".

Correlation and dependency play an important role in reliability analysis of systems. It is therefore appropriate to explain some basic ideas regarding correlation. Consider two jointly distributed random variables X and Y. For example suppose variables X and Y represent the fatigue life of a welded specimen and the notch radius at the weld toe respectively. It may be found that a small notch radius is associated with a short fatigue life. This points to a statistical relationship between the two variables. It is then of interest to know how frequent can certain values of the two variables be observed together. "Correlation theory" deals with such questions of dependency. Two random variables are said to be independent if any measurable information on one random variable provides no measurable information on the other random variable.

The correlation coefficient, ρ, is a dimensionless indicator of how closely the regression of one variable on an other follows a linear relationship,

$$\rho_{xy} = \frac{\text{cov }[x,y]}{\sigma_x \sigma_y} \quad -1 \leq \rho \leq 1$$

where cov is the covariance between X and Y and σx and σy is the standard deviation of X and Y respectively. The correlation coefficient can often be used to establish that two random variables X and Y are dependent. Indeed if X and Y are independent then $\rho_{xy} = 0$. Thus if $\rho_{xy} \neq 0$ it must be the case that X and Y are dependent. On the other hand if $\rho_{xy} = 0$ it cannot in general be asserted that X and Y are independent. There is just no linear dependency. Hence uncorrelated random variables are not necessarily independent.

Failure mode is another fundamental concept that needs to be understood and defined when working with reliability analysis of components and systems. At the component level, a failure mode is any possible mechanism under which the component may fail. Typical component failure modes are in stability (buckling), fracture, excessive yield, fatigue, creep, wear and so on. In this dissertation fatigue, will be the only failure mode considered at the component level. Fundamental failure modes may also occur at the system level. Moreover, it should be recognized that the reliability of a multicomponent system is essentially a problem involving multiple failure modes, Ang and Tang (1984).

Failure of different components or sets of components constitute distinct and different modes of failure of the system. The consideration of multiple modes of failure is fundamental to the problem of system reliability, see Fig. 1.2.

Finally it is appropriate to define the failure function or the limit state function of a component. Given a set of basic variables $\underline{X} = (X_1, ---, X_n)$ assume that they are chosen so that a failure surface corresponding to a given failure mode can be defined in the n-dimensional basic variable space. A failure surface divides the basic variable space into two regions, a failed region Ω_F and a safe region Ω_S. The failed region contains all combinations of $\underline{X}$ that would result in failure. It is convenient to describe the failure surface by an equation of the form,

$$g_j\ (\underline{X}) = g_j\ (X_1, ---, X_n)\ ;\ j = 1, 2, ---, k \tag{1.2}$$

where g_j is the failure function of the jth failure mode and there are k failure modes all together. Individual failure events are then defined as,

$$E_j = g_j\ (\underline{X}) < 0 \tag{1.3}$$

and the probability of failure of event j,

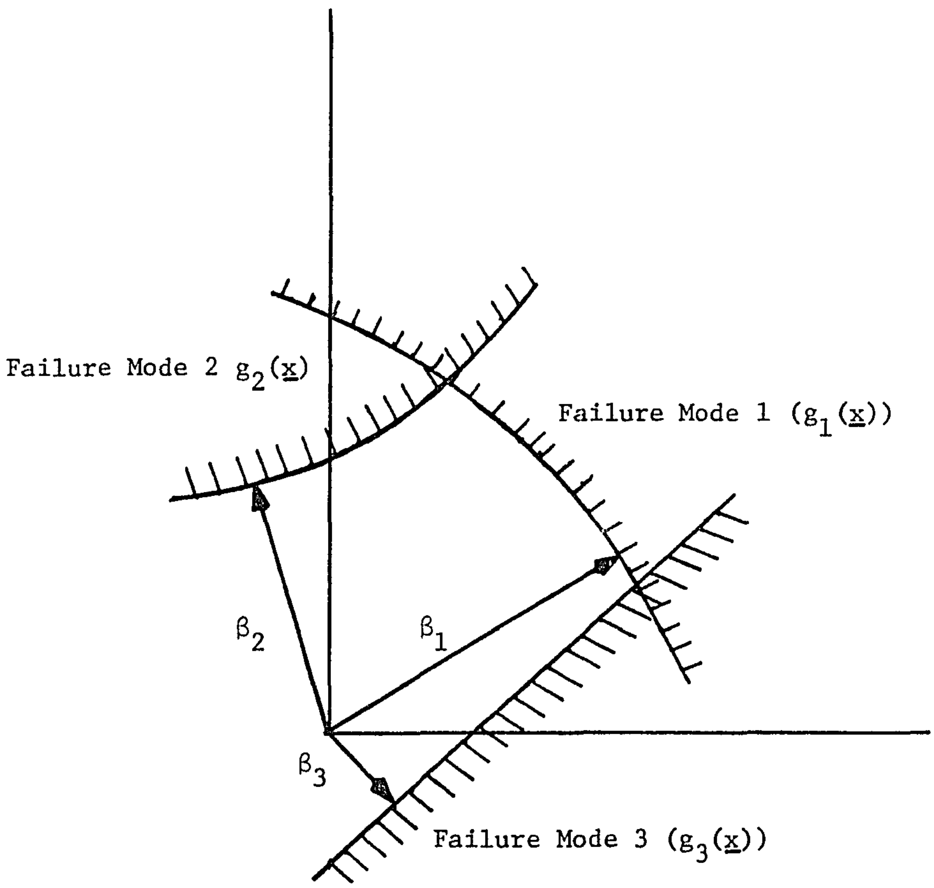


Fig. 1.2 Multiple Modes of Failure, Ang (1984) (g_i is the failure function of element (s) i)

$$P_j = P[g_j(x)] < 0 \tag{1.4}$$

Figure 1.2 shows a failure surface for multiple modes of failure in the case of three variables. β_j is defined as the safety index and is the minimum distance to failure surface j. By the Hasofer-Lind (1974) "minimum-distance" method,

$$\beta_j = \min \sqrt{\sum_{i=1}^{k} u_i^2} \tag{1.5}$$

where

$$u_i = \frac{x_i - \mu_i}{\sigma_i} \tag{1.6}$$

is the reduced variable with μ_i and σ_i denoting mean and standard deviation of X_i respectively.

1.4. The Scope and Outline of the Present Study

The main goal of this study is to develop the technology to describe the behavior of series systems with regard to practical design and safety check problems. It is in particular the goal to develop an approximate method for reliability analysis of series systems when no exact solution exists. In chapter 3 a simple approximation method is developed and presented. It is shown by numerous examples that the method works well. It was in particular of interest to analyze systems were fatigue is the main failure mode at element level. However, most of the theory and basic principles developed herein are generic in that they apply to other failure modes as well as combinations of failure modes.

In chapter 2 analysis of some special cases for which there is an exact solution is summarized and investigated. Given the "tools" from chapters 2 and 3 specific goals include the following:

1. Given the distribution of strength of an element what is the distribution of strength of the system?

2. Given the distribution of strength of the components and the loads on the system what is the probability of failure of the system?

3. Given the target safety index of the system what is the target safety index for the components?

For design and safety check purposes, element reliabilities have to be obtained from the limit state function by the use of some appropriate probabilistic design method (e.g., the log-normal format or the Rackwitz-Fiessler algorithm). In chapters 5, limit state functions are defined for the SN curve fatigue model. Applications are demonstrated by use of practical examples. The importance of some basic design variables are also investigated.

A study of the convergence of the approximate distribution of minimum is included in Appendix A. In Appendix B, a comparison study of the distribution of cycles to failure from several data sources are summarized.

CHAPTER 2

AN EXACT METHOD FOR ANALYSIS OF SERIES SYSTEMS

2.1 Introduction

The fundamental theory of series system analysis has been developed and is well documented in the literature, e.g., Ang and Tang (1984), Thoft-Christensen and Baker (1983), Grigoriu and Turkstra (1978). In this chapter the basic theory will be summarized and extensions to the theory are presented.

As discussed in Chapter 1, there are two basic methods that can be used for analysis of series systems having dependent component failure functions. Chapter 2 and 3 are devoted to a detailed development of these two methods. It is the general goal to develop procedures for practical engineering applications. In this chapter, exact methods are presented for some special cases of series systems. In general it will be assumed that the element strengths and the common (system) load distribution are known.

An example of the most general type of weakest link or series system is shown in Fig. 2.1. In this system all element internal loads are dependent. The external loads Q_i may as well be correlated with different distribution, moreover, the various elements may have different strength distributions.

The event of survival of element i is defined as,

$$E_i = [S_i < R_i] \tag{2.1}$$

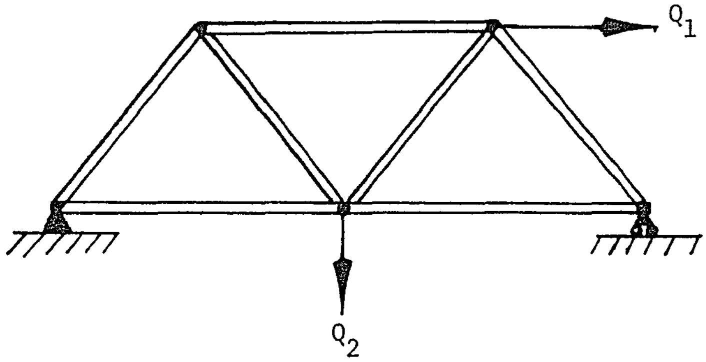


Fig. 2.1 Simple Pin Jointed Truss for Which the Resistance (R_i) and Stress (S_i) in Each Member Could be Different (Note; Q_1 and Q_2 may be correlated)

where S_i and R_i are random variables denoting the stress and strength respectively of the ith member. Let R_s be the system reliability (probability of non-failure). From elementary probability theory,

$$R_s = P\ [(S_1<R_1) \cap (S_2<R_2) \cap \ldots \cap (S_n<R_n)] \tag{2.2}$$

Expanding the probability expression,

$$R_s = P(E_1)P(E_2|\,E_1)P(E_3|\,E_1 \cap E_2) \cdots$$
$$\cdots\ P[E_n|\,E_1\ E_2 \cap \ldots \cap\ E_{n-1}] \tag{2.3}$$

For example the event $(E_3\,|E_2 \cap E_1)$ means that components 3 operates under the condition that both 1 and 2 are operating. In practice, these conditional probabilities are often very difficult if not impossible to assess. In general they have to be obtained experimentally. However, if the correlation coefficients between the element strengths are known, a transformation to independent coordinates can be made (Leporati (1979)). Reliability analysis of the more general cases of series systems are often impractical or impossible to solve by the direct method of this chapter and will be treated in chapter 3.

The types of series or "chain-like" systems addressed in this chapter is shown in Fig. 2.2. Two important general assumptions are made; (a) the stress, S, common to all elements is independent of R_i, and (b) all R_i are independent and identically distributed. The general theory will first be discussed and outlined. Some special cases of practical interest will then be investigated and illustrated by examples. Finally the concept of an empirical adjustment or correction factor, λ, is introduced to describe dependency in failure events, E_i. This factor

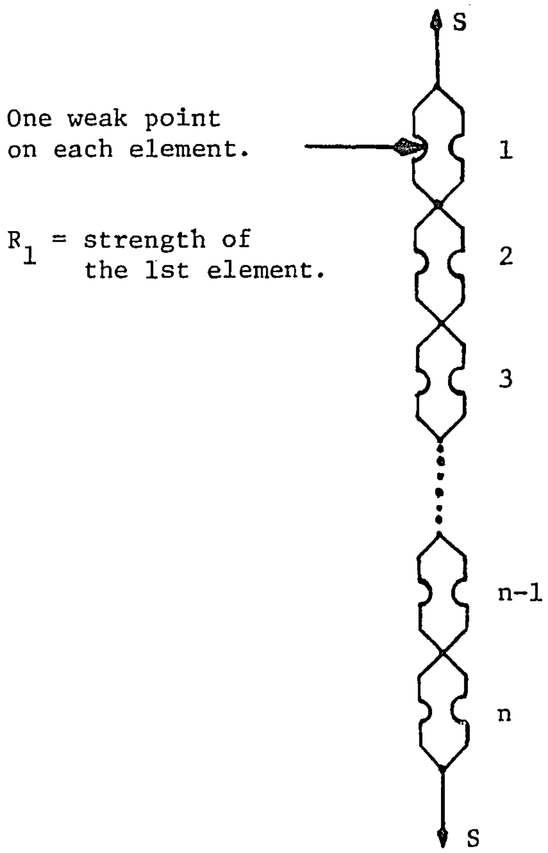


Fig. 2.2 Chain-Type Structure System Where All Components have the Same Load

enables the engineer or designer to directly relate system and component reliabilities without numerical integration.

2.2 The General Series System With Unequal Element Reliabilities

Consider a system where both load and strength may vary from element to element. It is assumed that all elements have identical stress distributions and that the load between any elements is known by a functional relationship,

$$S_i = a_i S \tag{2.4}$$

where a_i is a known constant. Combining Eqs. 2.2 and 2.4

$$R_s = P\left[\left(S < \frac{R_1}{a_1}\right) \cap \left(S < \frac{R_2}{a_2}\right) \cap \ldots \cap \left(S < \frac{R_n}{a_n}\right)\right] \tag{2.5}$$

If R_i is a random variable, then $\frac{R_i}{a_i}$ is also a random variable, defined as "modified strength". Let,

$$\rho_i = \frac{R_i}{a_i} \tag{2.6}$$

where the distribution of R_i is known. The distribution function of ρ_i is,

$$F_{\rho_i}(x) = P\left[\rho_i < x\right] = P\left[\frac{R_i}{a_i} < x\right] = P\left[R_i < a_i x\right]$$

or

$$F_{\rho_i}(x) = F_{R_i}(a_i x) \tag{2.7}$$

Equation 2.7 modifies the initial strength distribution to account for a_i. For example, if the Weibull is the initial strength distribution it is easy to show that the scale parameter will change from β to β/a_i. Hence the component strength distribution function will be,

$$F_{\rho_i}(r) = 1 - \exp\left[-\left(\frac{r}{\beta/a_i}\right)^{\alpha}\right] \tag{2.8}$$

When the lognormal distribution is the initial distribution, the location parameter $\mu_R = \ln\tilde{R}$ changes to $\ln\left(\frac{\tilde{R}}{a_i}\right)$ and,

$$F_{\rho_i}(r) = \Phi\left[\frac{\ln\left(\frac{a_i r}{\tilde{R}}\right)}{\sigma_{\ln R}}\right] \tag{2.9}$$

Φ is the standard normal distribution function and $\tilde{R}$ and $\sigma_{\ln R}$ are the parameters.

Assuming that the element strengths are independent, elementary extreme value theory is employed to derive the distribution function of R, the strength of the system equals the strength of the weakest element, (Bury, (1975)).

$$F_R(r) = 1 - \prod_{i=1}^{n} [1-F_{\rho i}(r)]$$

$$= 1 - \prod_{i=1}^{n} [1-F_{Ri}(a_i r)] \tag{2.10}$$

The system can now be considered to be a "super-element" with strength $F_R(r)$. The probability of failure of a single element is known from fundamentals to be (Haugen (1980)),

$$p_i = \int_0^{\infty} F_R(r)\, f_s(r)\,dr \tag{2.11}$$

Then for the "super-element" or the series system,

$$P_s = \int_0^\infty [1-\prod_{i=1}^{n} (1-F_{R_i}(a_i r))] f_s(r)dr \qquad (2.12)$$

In summary, for the special case of proportional stresses, the modified strength distribution concept produces an equivalent reliability model in which the stress in each element is the same, but the strength distribution differs. Thus each component has different p_i.

2.3 System Having A Linear Stress Gradient

It is of interest to examine the behavior of systems with unequal element reliabilities. Unfortunately only special cases can be solved by Eq. 2.12. As an example, the system in Fig. 2.3 having a linear stress gradient may have some practical application, e.g., the tendon of a tension leg drilling and production platform. The linear stress variation defined in Fig. 2.3 can be discretized as follows,

$$a_i = a_1 - [a_1 - a_n] \frac{i-1}{n-1} \quad , i = 1_1 ----,n \qquad (2.13)$$

where a_1 may be chosen as unity. Introducing Eq. 2.13 into Eq. 2.12 system probability of failure P_S can be obtained by numerical integration.

In order to interpret the results of the integration of Eq. 2.12, a parameter $\gamma = \frac{a_n}{a_1}$ is introduced. γ defines the load gradient, this is the ratio of the magnitude of the ratio of the lowest to the highest stressed element. So $\gamma \leq 1$, with high stress gradients defined by small γ. It is also convenient to define $\alpha = p_1/p_s$, which is the ratio of weakest element to system reliability.

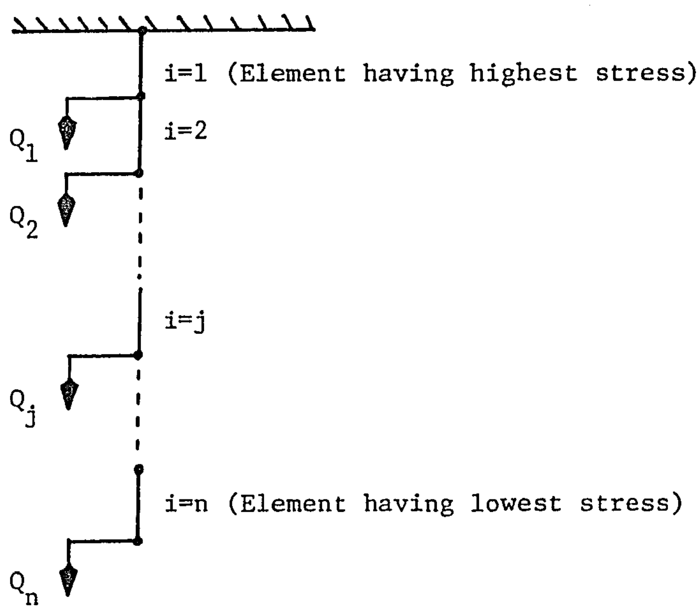


Fig. 2.3 Series "chain-like" System Having a Linear Stress Variation (Loads, Q_i, applied so that there is a linear variation of stress)

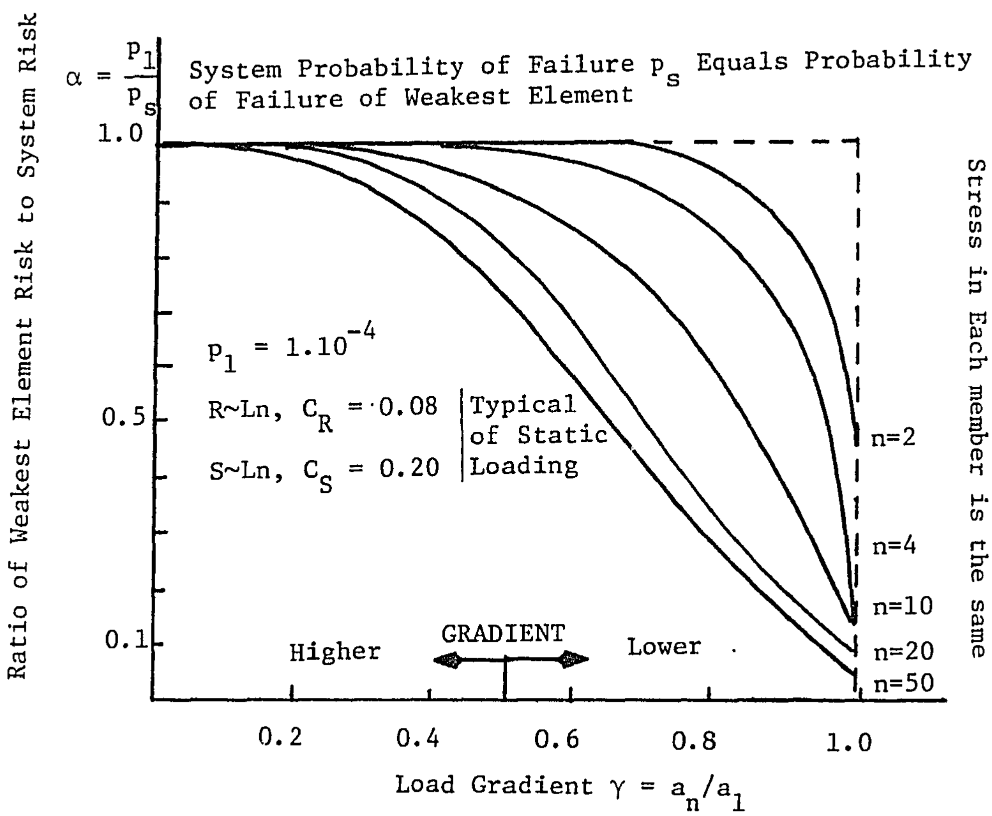


Fig. 2.4 Ratio of Weakest Element Risk to System Risk vs. Load Gradient

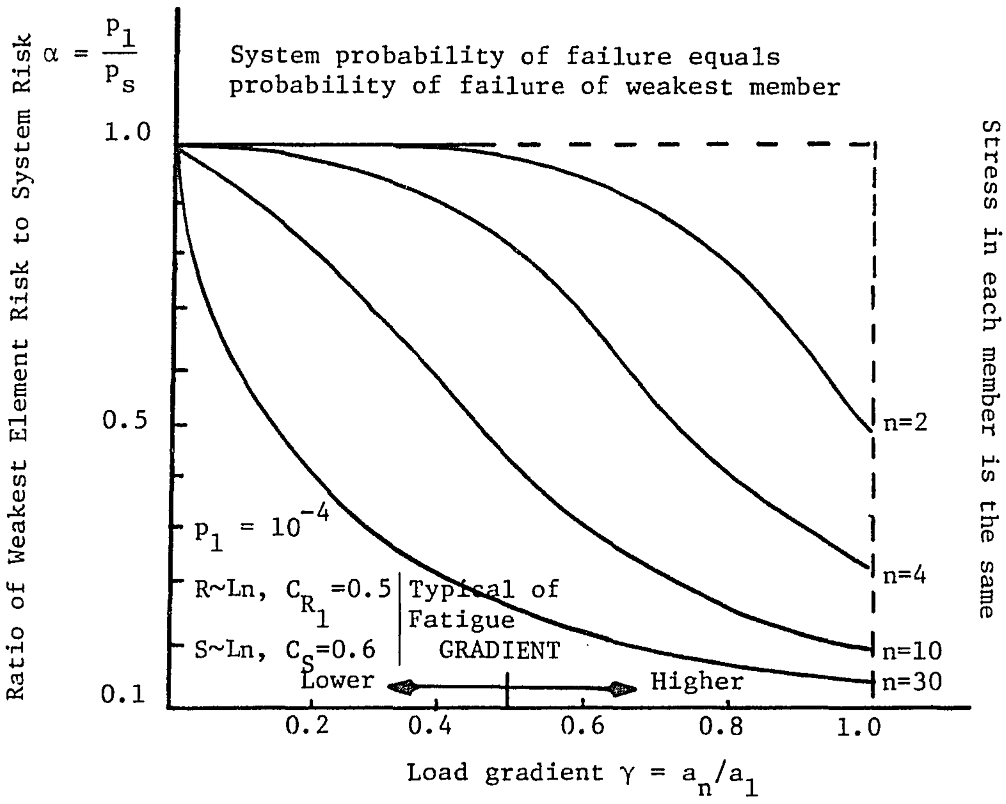


Fig. 2.5 Ratio of Weakest Element Risk to System Risk vs. Load Gradient

2.4. Example: The Relation Between The Weakest And The System Probability of Failure

It is expected that the element with the highest probability of failure will most influence p_s and therefore govern the design of the system. In this example it is the goal to see if this holds true.

Both the component strength R_i, and the load, S, were assumed to be lognormally distributed, p_1 can therefore be obtained from the lognormal format, (Wirsching (1979))

$$\beta_1 = \frac{\ln(\frac{\tilde{R}_1}{\tilde{S}})}{\sqrt{\ln(1+C_{R_i}^2)(1+C_S^2)}} \tag{2.14}$$

β is the safety index (standard normal fractile), $\tilde{R}_i$ and $\tilde{S}$ are median strength and stress respectively and C_{R1} and C_S are the coefficient of variation of strength and stress. The element probability of failure is then obtained from,

$$p_1 = \Phi[-\beta_1] \tag{2.15}$$

where Φ denotes the standard normal distribution function.

Unfortunately the calculations of Eq. 2.12 are very tedious. For this reason only two cases were considered, one in which the coefficients of variation are considered typical to be a static design problem; the other typical of a fatigue problem where COV's are expected to be high. In the "static" case, Fig. 2.4 C_{Ri} was 8% and C_S was 20%, the "dynamic" problem had a C_{Ri} and C_S of 50% and 60% respectively.

In Figs. 2.4 and 2.5 the ratio of $\alpha = p_1/p_s$ is plotted against $\gamma = a_n/a_1$. The median values of stress and strength was adjusted so that the probability of failure of the weakest element was 10^{-4}. It is seen from both Fig. 2.4 and 2.5 that the weakest element generally does not always dominate the system performance. This is in particular true for increased system size and relatively small load gradients, i.e., large γ. In this example the weakest component had a probability of failure, p_1, of 10^{-4}. However, it was observed that these curves did not change much when p_1 was changed to 10^{-5} and 10^{-6}.

Example: System Probability of Failure vs. Load Gradient

It is reasonable to expect that small changes in the load gradient, for a system as shown in Fig. 2.3, may significantly influence the system probability of failure. This expectation is confirmed by examination of Figs. 2.6 and 2.7. For example, if the load gradient in Fig. 2.7 is changed from 0.9 and 0.8 almost an order of magnitude in p_s results. For higher load gradients this sensitivity is even more severe. When C_{Ri} and C_S is large as in fatigue this sensitivity is not quite as distinct. However, it is clear that conservative design will result if the design is based on the weakest element and even a small load gradient is neglected.

2.5. Relation Between Element and System Reliability For the Special Case of Equal Component Reliabilities (Kjerentroen and Wirsching(1984))

A particular important application of Eq. 2.12 is when all the component strengths R_i are independent and identically distributed, and there is a common load $S_i=S$. The goal of the analysis is to relate component risk to system risk.

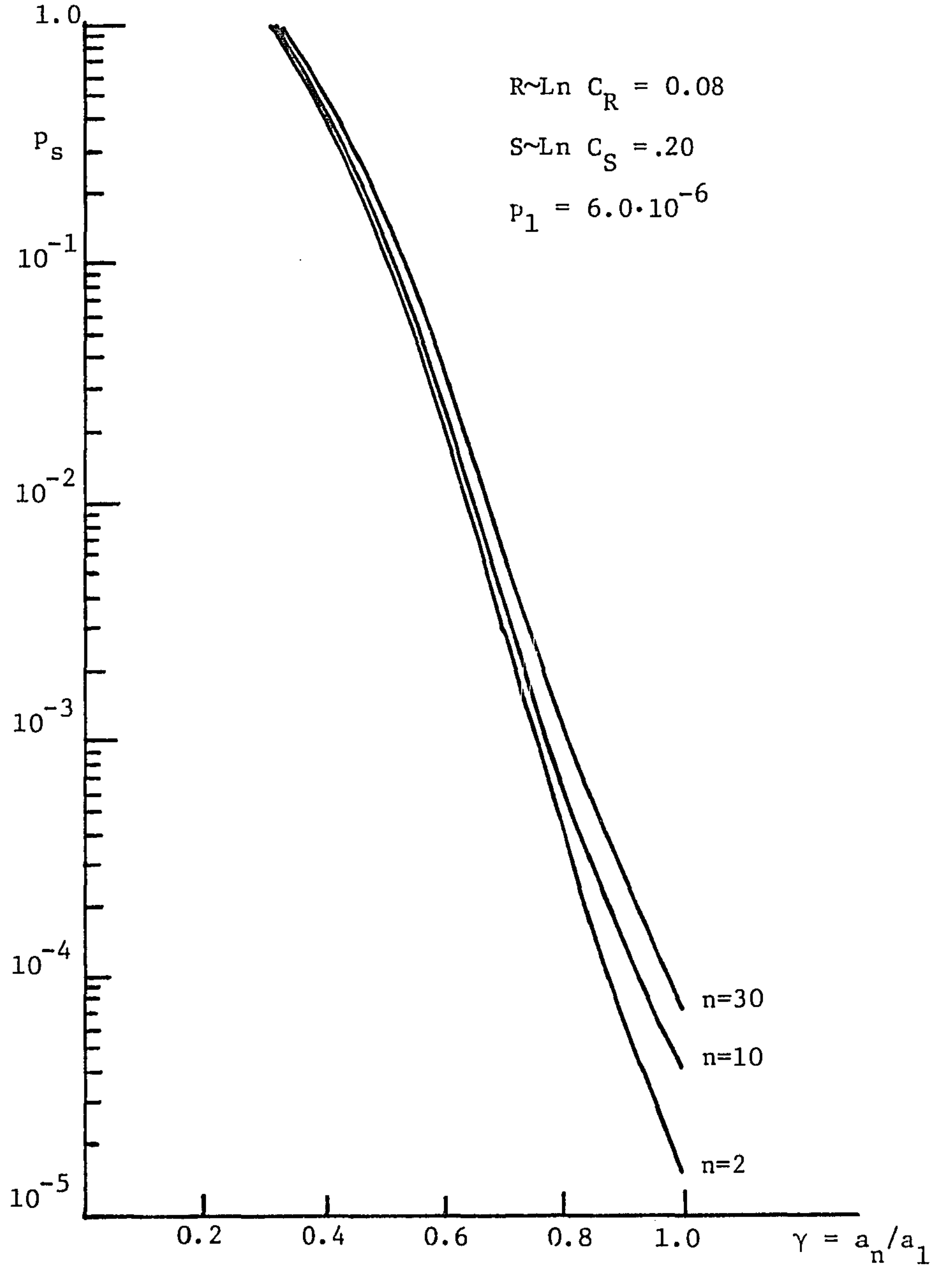


Fig. 2.6 System Probability of Failure for the "Static Case"

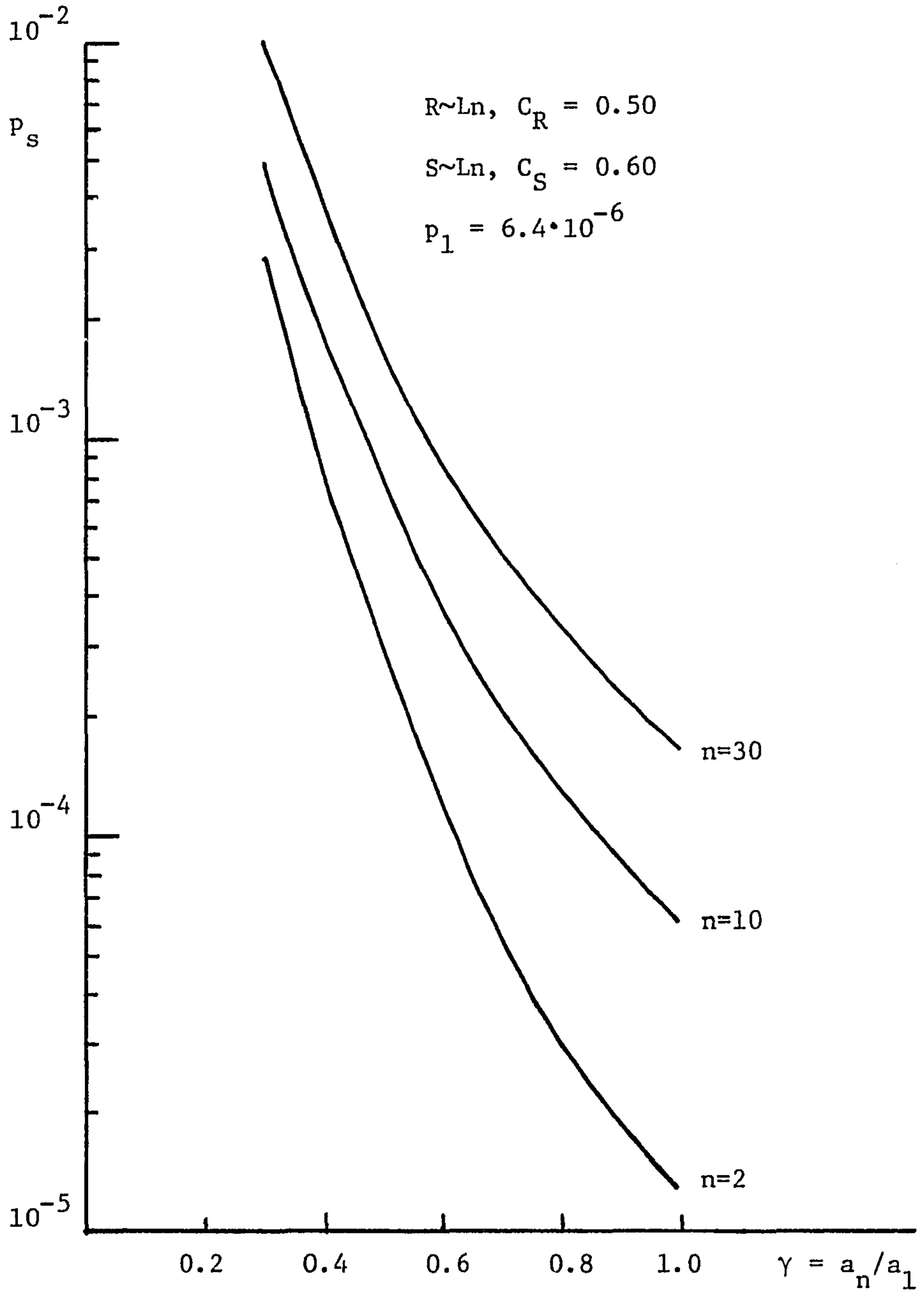


Fig. 2.7 System Probability of Failure vs. Load Gradient for the "Dynamic" Case

If the events of failure, $E_i = [g_i(R_i,S) \leq 0]$, of each member in a series system were mutually independent, the system probability of failure, could be formulated in terms of component probabilities of failure p_i. From basic probability considerations,

$$p_S^* = 1-(1-p_1)^n \tag{2.16}$$

where p_i is the probability of failure of all the identical elements. The star subscript indicates an approximation to p_s. If p_l is small, it is easily shown that

$$p_S^* \cong n\ p_1$$

However the events of failure are not independent because the same random variables S appears in each component failure event. This dependency can be accounted for by numerical integration of Eq. 2.12. If $a_i = 1$ and all R_i are independent and identically distributed, Eq. 2.12, the exact form of p_s,

$$p_S = \int_0^\infty [1-(1-F_{Ri}(r))^n]f_s(r)dr \tag{2.17}$$

Nevertheless Eq. 2.16 is an upper bound on system probability of failure, assuming that the element failure functions are positively correlated (Ang (1984)) which they will be for the chain like sysems under consideration. The topic of bounds on system reliability is covered in detail in Ang (1984).

Numerical integration of Eq. 2.17 is both tedious and impractical for engineering purposes. It is therefore the goal to produce a general expression for p_S, Eq. 2.17 can be modified by introducing an adjustment or correction factor, λ, which accounts for stress dependency.

$$p_S = \lambda p_S^* = \lambda[1-\prod_{i=1}^{n} (1-p_i)] \tag{2.18}$$

where $0 \leq \lambda \leq 1$. λ can be established by computing the ratio of exact to the approximate probability of failure,

$$\lambda = p_S/p_S^* \tag{2.19}$$

where p_S is given by Eq. 2.16 and p_S by Eq. 2.17. Numerical integration is required for p_S. In general, λ will be a function of four parameters.

$$\lambda = \lambda \; [C_{Ri}, C_S, p_i, n] \tag{2.20}$$

where C_{Ri} is the coefficient of variation of the strength variable, C_S is the coefficient of variation of the stress variable, n is the number of elements and p_i is the element probability of failure. λ depends upon the statistical distribution of R_i and S. Because of the number of factors which influenced λ, it would be difficult to produce a general or closed form λ.

The percent error, E, by making the independence assumption function for Eq. 2.22 is related to λ by, (assuming that the exact system risk is p_s)

$$E = \frac{100 \; (1-\lambda)}{\lambda} \tag{2.21}$$

2.6 Example: Lognormal Stress and Lognormal and Weibull Strength

The purpose of this example is to derive charts for λ as a function of stress and strength parameters that are typical for fatigue problems. The elements are assumed to be identical and p_S is obtained from Eq. 2.17 and the approximate form from Eq. 2.16.

The lognormal is assumed to be the distribution of load, S. The lognormal, commonly used for design variables, seems to provide a reasonable description of a wide variety of physical phenomena including loads. The Weibull and lognormal were both considered for strength distributions. λ was computed for C_S and C_{Ri} ranging from 0.10 to 0.60, values considered to be representative for both static failures and fatigue. Element probabilities of $p_1 = 10^{-4}$ and 10^{-5} were used. The number of elements in the system ranges from n = 1 to 100.

Computed values of λ are presented in Fig. 2.8 through 2.13. The values of λ provided in the Figs. 2.8 through 2.13 define, for the distributions and parameters considered, the relationship between system and component risk.

$$p_S = \lambda\ [1-(1-p_1)^n] \tag{2.22}$$

Or, if $p_1 < 10^{-4}$ and $n < 100$, the error is less than 0.5% using the approximation,

$$p_S = \lambda\ np_1 \tag{2.23}$$

In general it will be necessary to interpolate the values of λ on the charts demonstrated below.

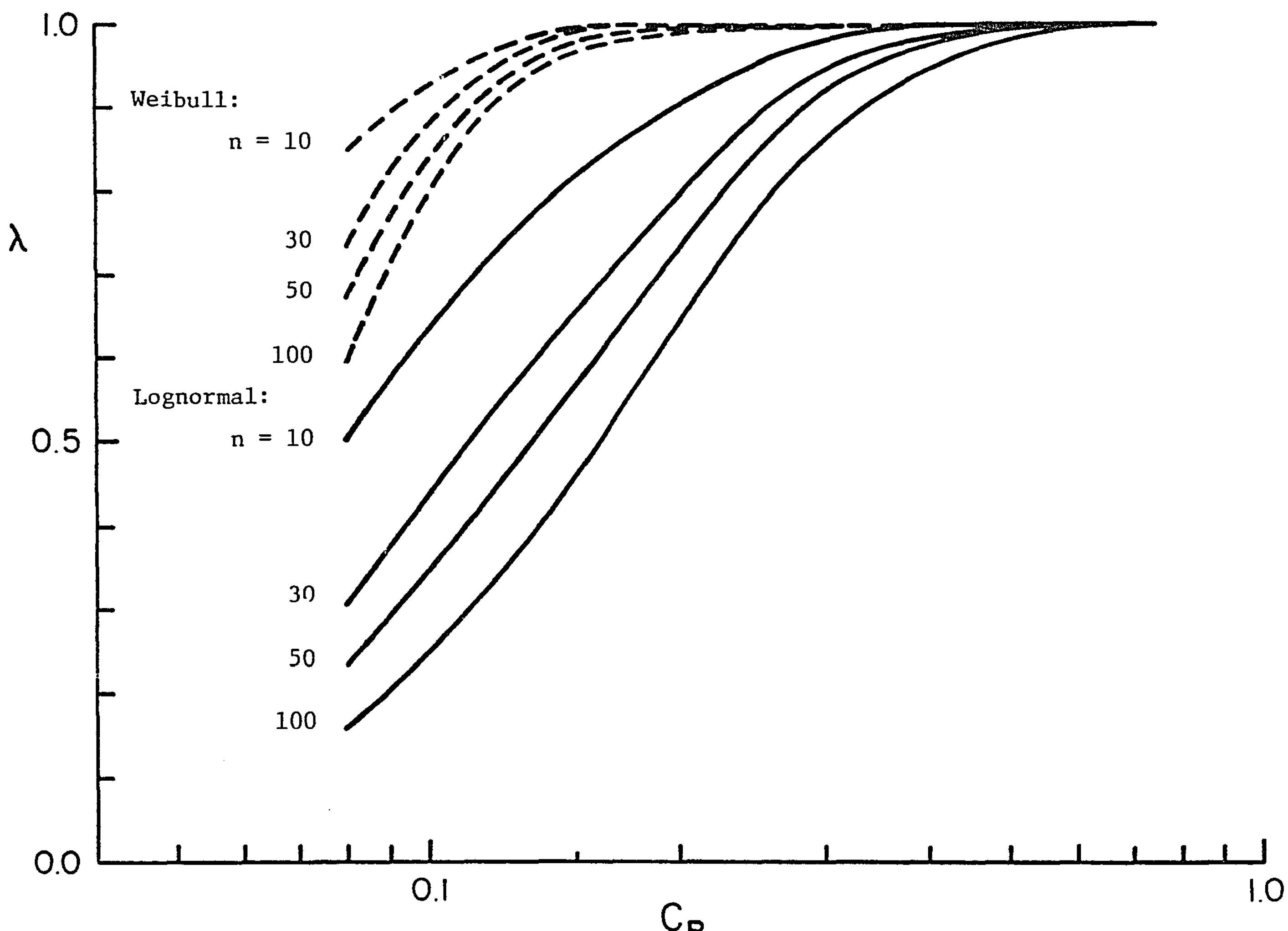


Fig. 2.8 λ vs. C_{R_i} for $C_S = 0.2$ and $p_1 = 10^{-4}$; S is Lognormal

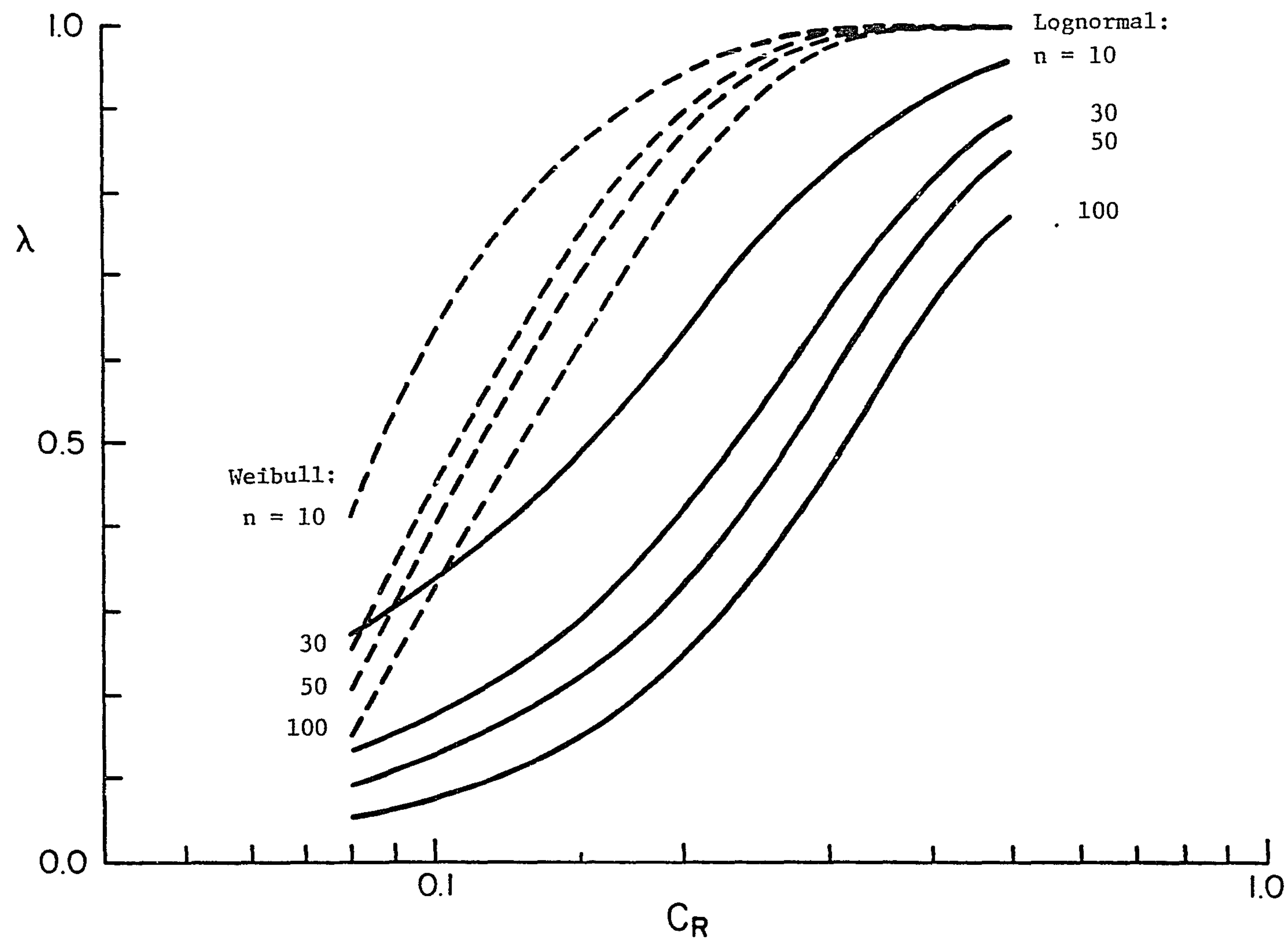


Fig. 2.9 $-\lambda$ vs. C_{R_i} for $C_S = 0.4$ and $p_1 = 10^{-4}$; S is Lognormal

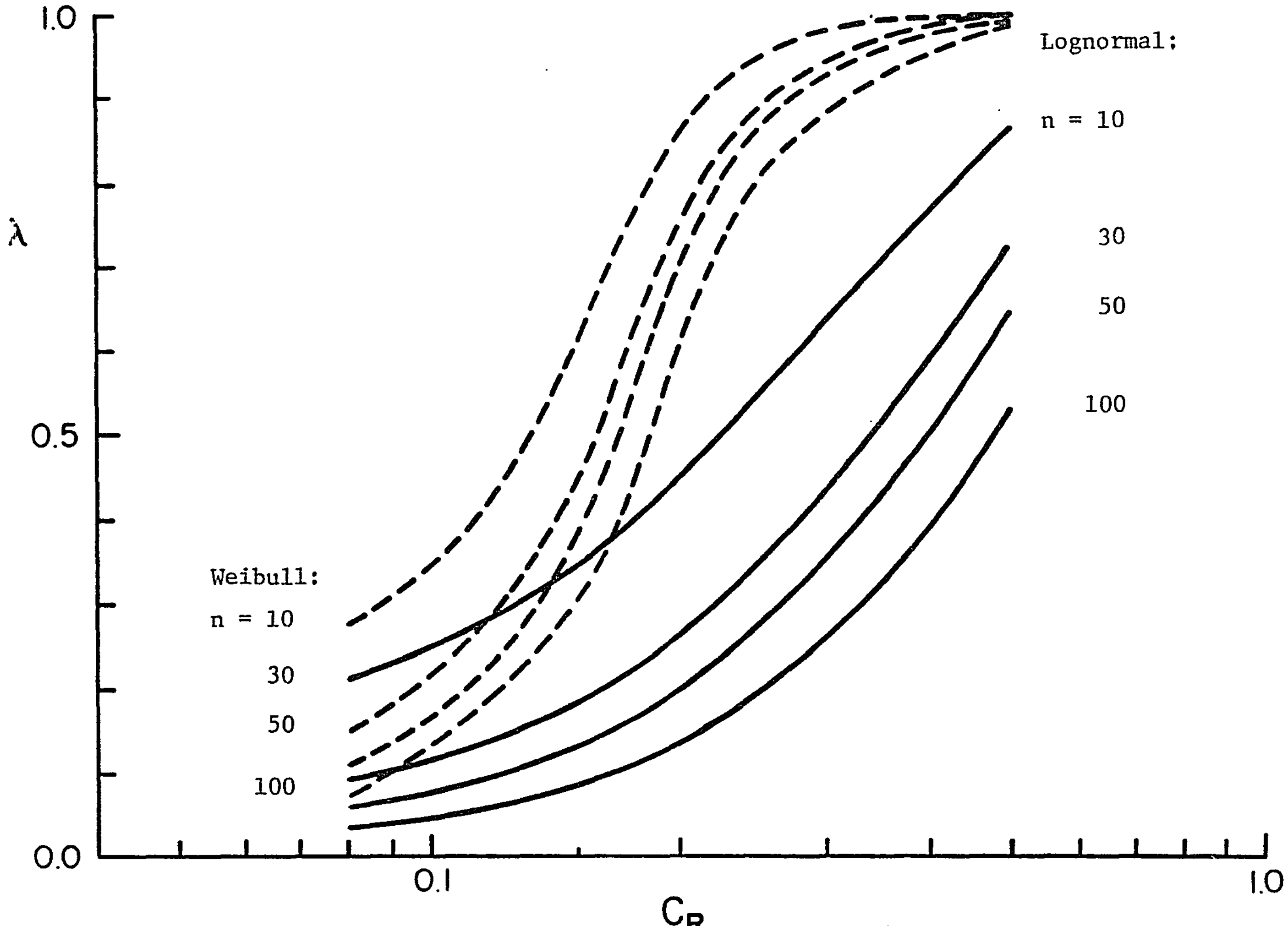


Fig. 2.10 λ vs. C_{R_i} for $C_S = 0.6$ and $p_1 = 10^{-4}$; S is Lognormal

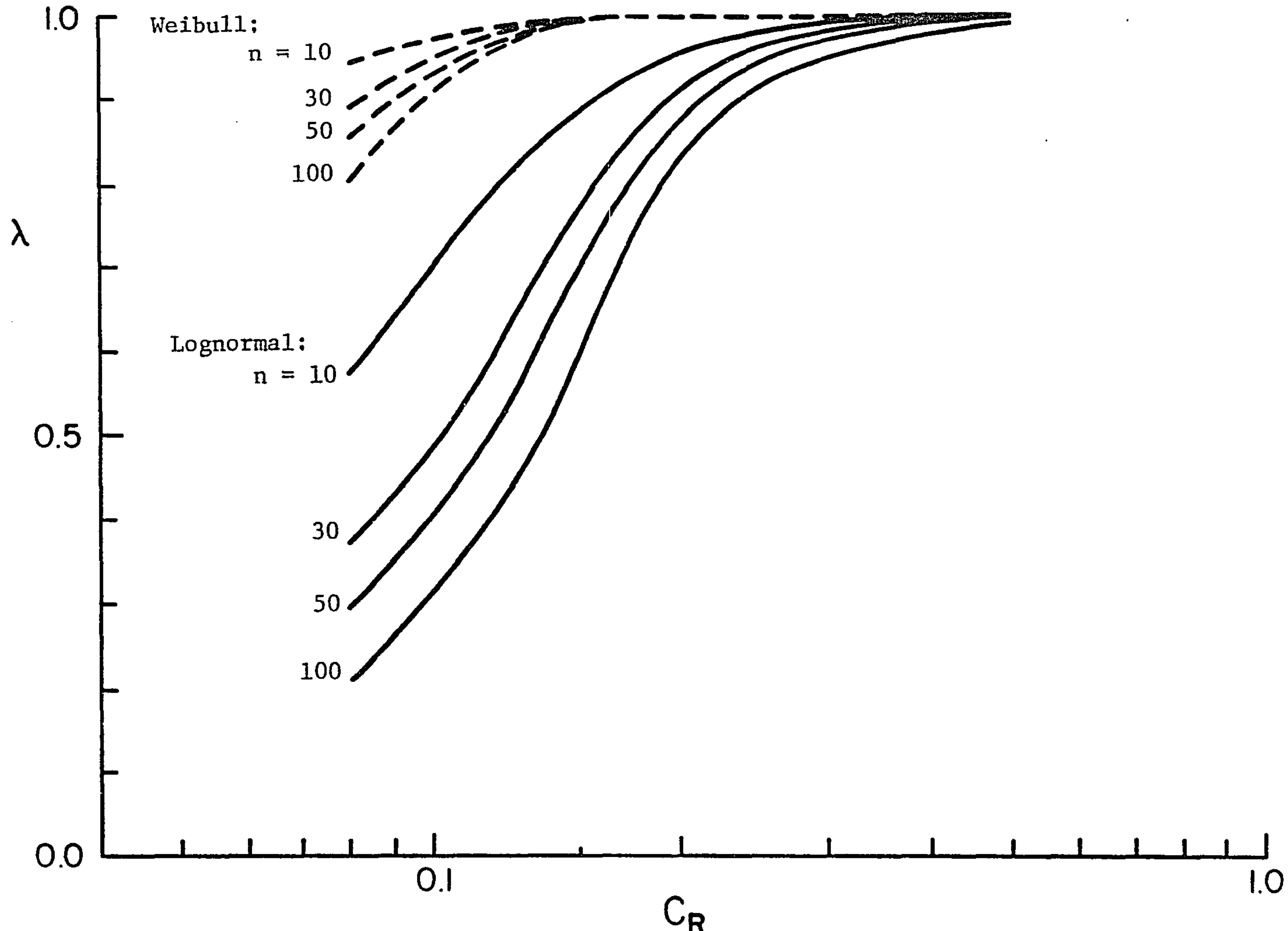


Fig. 2.11 $-\lambda$ vs. C_{R_i} for $C_S = 0.2$ and $p_1 = 10^{-5}$; S is Lognormal

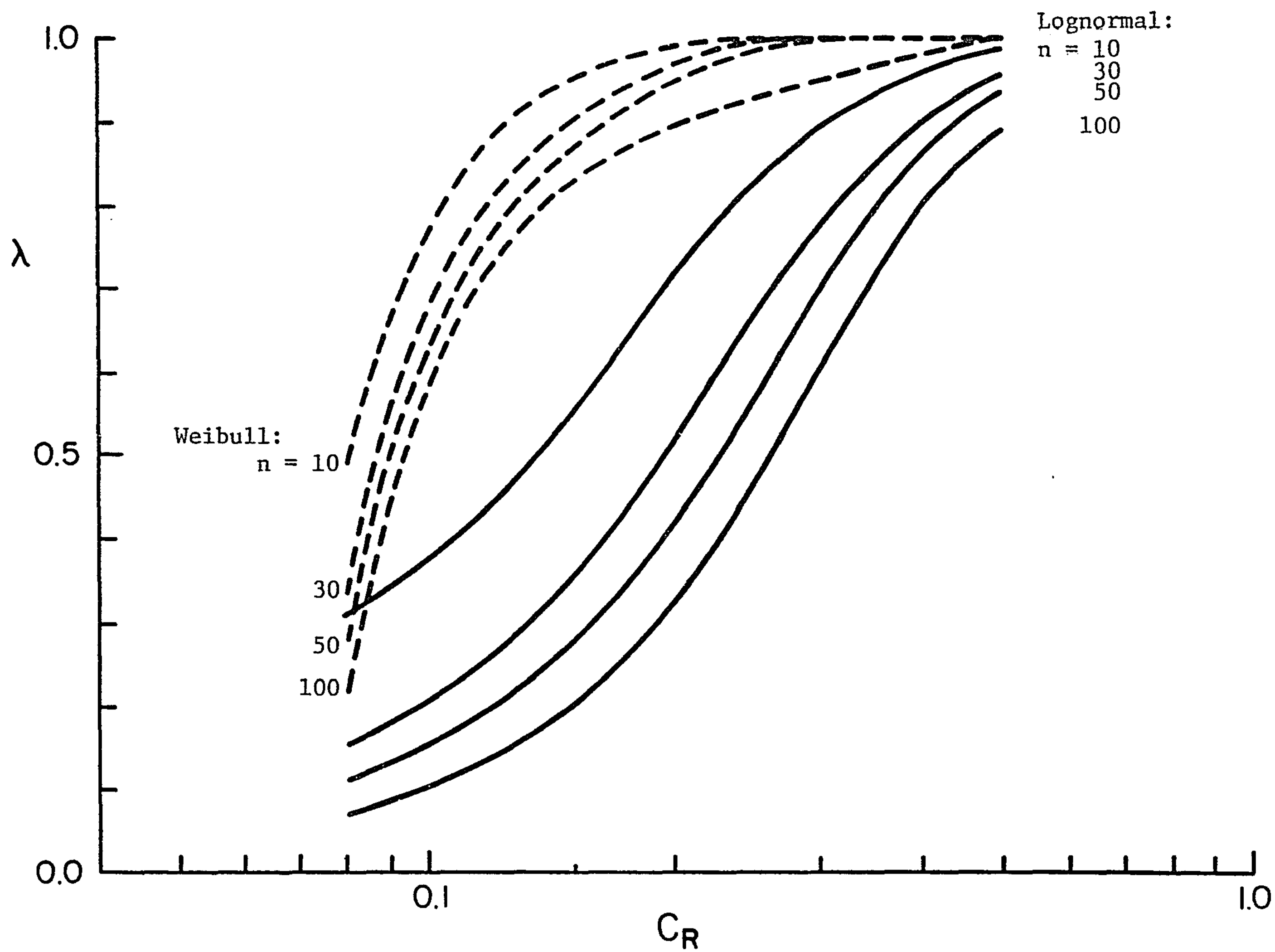


Fig. 2.12 λ vs. C_{R_i} for $C_S = 0.4$ and $p_1 = 10^{-5}$; S is Lognormal

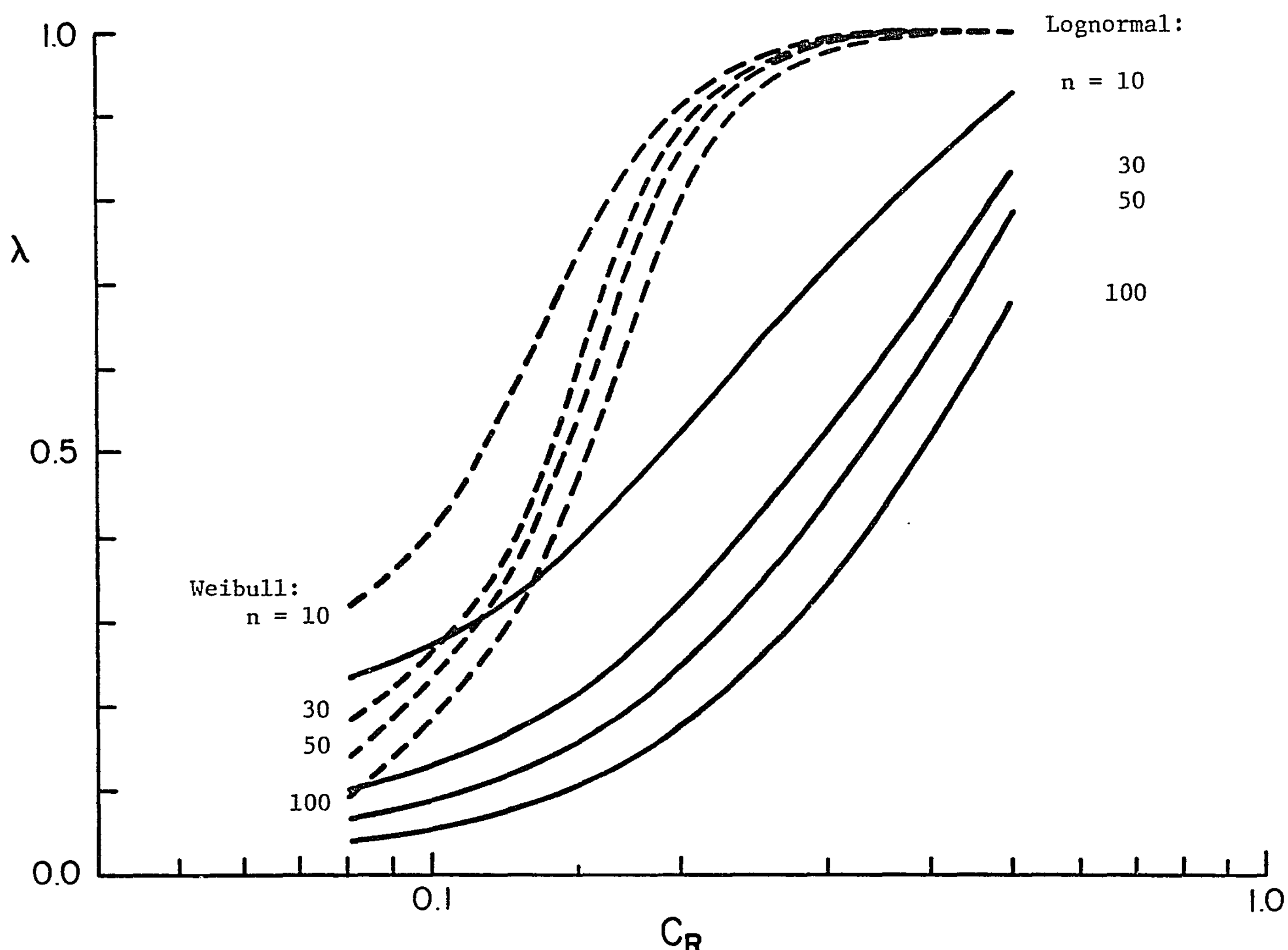


Fig. 2.13 λ vs. C_{R_i} for $C_S = 0.6$ and $p_1 = 10^{-5}$; S is Lognormal

Example

The purpose of this example is to illustrate how Figs. 2.8 through 2.13 can be applied. The example also demonstrates the importance of including load dependency in series system analysis. A target probability of failure for the system is given as p_{os}. A target safety index β_{01}, for each element is selected to meet this system. Both component strength, R_i and stress S, are assumed to be lognormally distributed. Given C_{Ri}, C_S, and the system size, n, the element probability of failure can be obtained by interpolation from Figs. 2.8 through 2.13. In this case, the lognormal format can be used. The expression for the safety index β_1 is,

$$\beta_1 = \frac{\ell n(\tilde{R}_1/\tilde{S})}{\sigma} \tag{2.24}$$

where

$$\sigma = [\ell n(1 + C_S{}^2)(1 + C_{R1}^2)]^{1/2} \tag{2.25}$$

where $\tilde{R}_1$ and $\tilde{S}$ are the median values of the element strength and the stress respectively. β_1 is related to the probability of failure by

$$p_1 = \Phi\ (-\beta_1)$$

where Φ is the standard normal distribution function. The element survives if, $p_1 < p_{o1}$. Equation 2.24 can now be solved for $\tilde{R}$,

$$\tilde{R}_1 = \tilde{S}\ \exp\ (\beta_1\sigma) \tag{2.26}$$

Table 2.1 summarizes the cases considered. The comparison between R_a (approximate median strength) and R_e (exact median strength) provides an

Table 2.1 Comparison Between Exact and Approximate Median Elemental Strength Requirements for a Given System Probability of Failure. (The subscripts e and a denote exact and approximate respectively. The approximate is computed using $\lambda=1$)

C_R	C_S	λ	R_e	R_a	% overweight when R_a is used
0.1	0.20	0.27	2.41S	2.57S	6.6
0.15	0.20	0.56	2.79S	2.88S	3.2
0.10	0.40	0.07	4.22S	5.46S	29.4
0.50	0.40	0.91	13.29S	13.46S	1.3
0.40	0.60	0.50	16.01S	17.81S	11.2
0.40	0.40	0.80	9.12S	9.39S	3.0

Target $p_{oS} = 10^{-3}$; Number of Elements, n = 100 illustration of the importance of the load dependency effect on design. R_a is the case when load dependency is neglected or $\lambda = 1$. For example, consider the first line. Note that the resistance is directly proportional to weight for a tension member. This means that neglecting load dependency in some series systems, or more generally the assumption of independent element failure events, is conservative. Therefore when weight is of concern or when material costs is of importance the design can be improved by including the effects of dependent element failure functions.

2.7 Summary

The general theory for calculations of exact series systems reliabilities has been outlined. It was shown that for a system with unequal element reliabilities even small variations in the linear element reliability gradient may strongly influence the system performance.

Charts for a coefficient, λ, that relates element to system reliability were developed. It was shown that this λ was strongly a function of distribution type as well C_R, C_S and n.

CHAPTER 3

AN APPROXIMATE METHOD FOR ANALYSIS OF SERIES SYSTEMS

3.1 Introduction

An exact method for analysis of a special class of series systems was investigated in Chapter 2. A special form relating component and system reliabilities was presented in Eq. 2.12. However for a general series system (e.g., Fig. 3.1), reliability analysis using Eq. 2.12 may be difficult if not impossible. The goal of this chapter is to develop and investigate an approximate method for general series system analysis.

Consider a series system for which each element has a linear failure function of the form,

$$g_i(\underset{\sim}{x}) = a_o + \sum_{K=1}^{n} a_K X_K \tag{3.1}$$

where x_K is a normally distributed random variable and a_o and a_K are constants. From reproductive properties of the normal probability distribution, g_i is also normally distributed (Bury, 1975).

It is assumed that g_i and g_j are correlated. The correlation coefficient is defined as

$$\rho_{g_i g_j} = \frac{COV(g_i g_j)}{\sigma_{g_i} \sigma_{g_j}} \tag{3.2}$$

In some cases, e.g. fatigue, it is necessary to make a log transformation $h_i = ln_{g_i}$. This would be the case where g_i is a multiplicative function, by lognormal variates. Then the ρ of Eq. 3.2 relates to h_i and h_j.

R_i = strength of ith member

S_i = stress in ith member

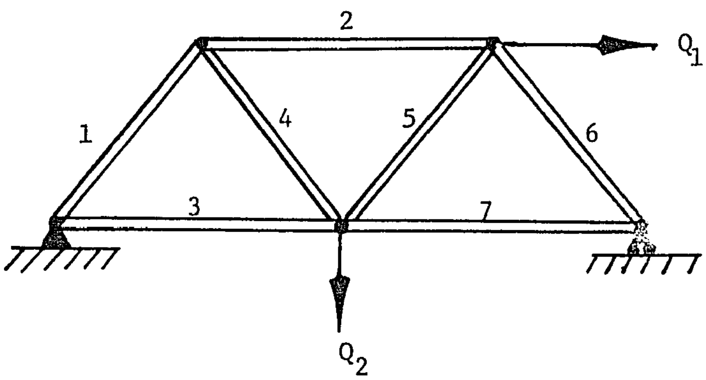


Fig. 3.1 Example of a Series System; Simple Pin Jointed Truss for Which the Resistance (R_i) and Stress (S_i) in Each Member have Different Magnitudes and Distributions but are Correlated (NOTE: Q_1 and Q_2 may be correlated and each may have different distributions)

Cov $(g_i g_j)$ is the covariance of g_i and g_j and defined as,

$$\mathrm{cov}(g_i g_j) = E[(g_i - E(g_i))(g_j - E(g_j))] \tag{3.3}$$

Let μ_{g_i} and σ_{g_i} represent the mean and standard deviation respectively of g_i. Then if the event of survival of element i is defined by,

$$E_i = (g_i > 0) \tag{3.4}$$

The reliability of the system R_S is,

$$R_s = P[E_1 \cap E_2 \cap \ldots \cap E_n] \tag{3.5}$$

Solution of Eq. 3.5 requires evaluation of the n-th order joint normal density function, (from the definition of the joint density function)

$$R_S = \int_0^\infty \int_0^\infty \cdots \int_0^\infty f_{g_1 g_2 \ldots g_n} \; dg_1 dg_2 \ldots dg_n \tag{3.6}$$

An alternate form using the standard normal can be written. Using the standard normal transformation,

$$g_i' = \frac{g_i - \mu_{g_i}}{\sigma_{g_i}} \tag{3.7}$$

Then $f_{g_1' g_2' \ldots g_n'}$ is a joint standard normal density function; the system reliability is,

$$R_S = \int_{-\beta_1}^\infty \int_{-\beta_2}^\infty \cdots \int_{-\beta_n}^\infty f_{(g_1' g_2' \ldots g_n')} \; d_{g_1'} \, d_{g_2'} \ldots d_{g_n'} \tag{3.8}$$

where $\beta_i = \mu_{g_i} / \sigma_{g_i}$ and is called the "standard normal fractile" or more commonly the "safety index". It should also be recognized that if the ith element has the probability of failure p_i then,

$$\beta_i = -\Phi^{-1}(p_i) \tag{3.9}$$

Furthermore $\beta_s = -\Phi^{-1}(p_s)$ is called the system (or generalized) reliability index. The multiple integral in Eq. 3.8 may be evaluated by numerical integration using the method developed by Stuart (1958), Gupta (1963) and others. From here on it will be referred to as the "G-S equation". The deviation of this equation is found in Johnson (1972). Using the probability of system failure $p_s = 1-R_S$ the G-S equation is,

$$p_s = 1 - \int_{-\infty}^{\infty} \prod_{i=1}^{n} [\Phi(\frac{\beta_i + \sqrt{\rho}\, t}{\sqrt{1-\rho}})]\ \phi(t)dt \tag{3.10}$$

where Φ and ϕ denote the standard normal distribution and density function respectively. Furthermore $\rho_{ij}=\rho$ is the equicorrelation coefficient in the multinormal integral for all i=j . To summarize, the G-S equation (3.10) is exact if the following assumptions are fulfilled, a) the basic variables are all normally distributed, b) the element failure functions g_i must be linear in the basic variables so that g_i is normal and, c) the correlation coefficient, $\rho_{g_i g_j}$, between any two element failure functions must be the same. These assumptions are restrictive and few practical applications of the G-S equation.

A major contribution of this study has been to show that the G-S equation can be used to estimate the probability of a series system failure p_s when the failure functions g_i are non normal. It is suggested that good estimates of p_s will result if the following two conditions are met; a) A good estimate of the safety index β_i (the equivalent normal fractile) is obtained. b) An equivalent normal correlation coefficient can be derived.

The safety index β_i can be approximated by the Rackwitz-Fiessler (R-F) algorithm or any similar fast probability integration method. An equivalent normal correlation coefficient ρ_e based on the transformed equivalent normal failure function will be derived. It is shown that excellent estimates of series system probability of failure may be obtained by the G-S equation even when the failure functions are highly non normal.

Several examples are used to demonstrate the general use and application of the G-S equation. The suggested approximation scheme is also verified and demonstrated by several examples.

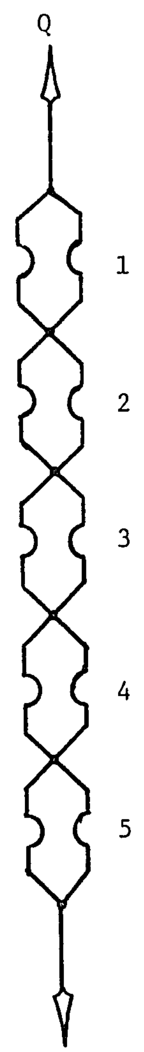
3.2 Example: Examination of the Correlation Coefficient

The goal of this example is to show how the exact correlation coefficient between linear failure functions is derived.

Consider the weakest link system (a "chain" having an axial load) shown in Fig. 3.2 The load in each member is the same, but the resistance could have a different distribution for each element. Here the failure function can be written as

$$g_i = R_i - S \tag{3.11}$$

Defined so that component failure corresponds to $g_i \leq 0$. Both R_i and S are assumed to be normally distributed random variables.



R_i = strength of ith element

Fig. 3.2 A Chain Like System where all the Elements are Subject to the Common Load Q

The covariance can be derived from the definition by applying fundamental rules of expectation (Bury (1975))

$$\begin{aligned}\mathrm{cov}[g_i g_j] &= E[g_i - E(g)_i)][g_j - E(g_j)] \\ &= E[g_i g_j] - E(g_i)\, E(g_j) \\ &= E[R_i R_j] - E[R_i]\, E[R_j] - [E(R_i S) - E(R_i)\, E(S)] \\ &\qquad - [E(R_j S) - E(R_j)\, E(S)] + E(S^2) - E(S)^2 \\ &= \rho_{R_i R_j} \sigma_{R_i} \sigma_{R_j} - \rho_{R_i S} \sigma_{R_i} \sigma_S - \rho_{R_j S} \sigma_{R_j} \sigma_S + \sigma_S^2 \end{aligned} \tag{3.12}$$

where $\rho_{R_i R_j}$ is the correlation coefficient between the strengths of element i and element j, and $\rho_{R_i S}$ is the correlation coefficient between the strength of element i and the load S. Notice that the covariance is equal to the variance when i=j, the variance of g_i is

$$\sigma_{g_i}^2 = \sigma_{R_i}^2 + \sigma_S^2 - 2\, \rho_{R_i S}\, \sigma_{R_i}\, \sigma_S \tag{3.13}$$

Then from Eqs. 3.2, and 3.12 and 3.13 the correlation coefficient between g_i and g_j becomes,

$$\rho_{ij} = \frac{\sigma_S^2 + \rho_{R_i R_j} \sigma_{R_i} \sigma_{R_j} - \rho_{R_i S} \sigma_{R_i} \sigma_S - \rho_{R_i R_j} \sigma_{R_i R_j}}{(\sigma_{R_i}^2 + \sigma_S^2 - 2\rho_{R_i S} \sigma_{R_i} \sigma_S)(\sigma_{R_j} + \sigma_S^2 - 2\rho_{R_i S} \sigma_{R_j} \sigma_S)]^{-0.5}} \tag{3.14}$$

Now assume (a) stress and strength are independent, and (b) the strength of elements i and j are independent. Then $\rho_{R_i S}$ and $\rho_{R_i R_j}$ are both zero. Hence Eq. 3.17 reduces to,

$$\rho_{ij} = \frac{\sigma_S^2}{\sigma_{R_i} \sigma_{R_j} + \sigma_S^2} \tag{3.15}$$

In Eq. 3.15, the only source of correlation is the common element load S.

For the special case when all the elements have the same reliability index, β_1, and $\rho_{ij}=\rho$ Eq. 3.8 becomes,

$$p_s = 1 - \int_{-\infty}^{\infty} [\Phi(\frac{\beta_1 + \sqrt{\rho}\, t}{\sqrt{1-\rho}})]^n \phi(t)dt \tag{3.16}$$

In Figs. 3.3 and 3.4 the influence of β_1 and ρ on the ratio P_s/p_1 is shown for the case when all the elements in the system have the same safety index, β_1, n=10 for Fig. 3.3 and n=100 for Fig. 3.4. It is shown that the system and the element probability of failure approach each other as ρ goes to 1 which is the case of highly correlated functions. This would be the case where the strengths R_i have a correlation of one or when the variance in the load is much larger than the variance of strength. On the other hand as ρ decreases and the element reliability index β_1 increases, the approximation of,

$$p_s = 1 - [1 - p_1]^n$$

gets increasingly good.

3.3. Example: Analysis of a* Simple Series System

This simple example demonstrates the computation of p_s. Consider again the chain like series system of Fig. 3.2. The system consists of five elements all subject to the common load Q. The system fails if the load in any one element exceed the yield strength. All elements are assumed to have identical strength distributions. The failure function for all the elements is then,

$$g_1(Q,R) = R_1 - Q \tag{3.17}$$

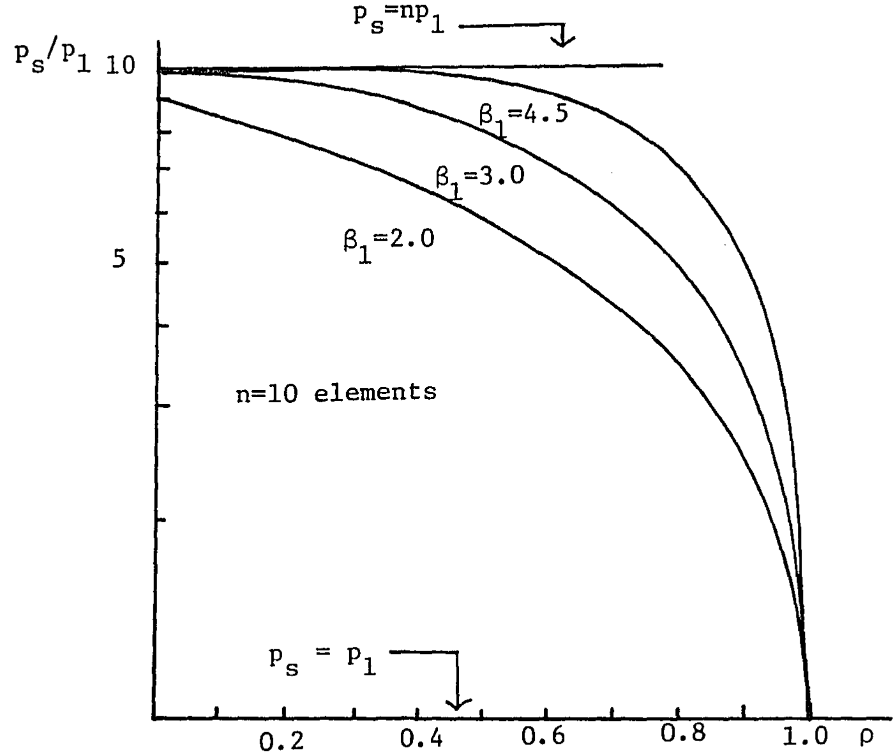


Fig. 3.3 The Influence of ρ and β_1 on the ratio p_s/p_1 for a system of n=10 elements

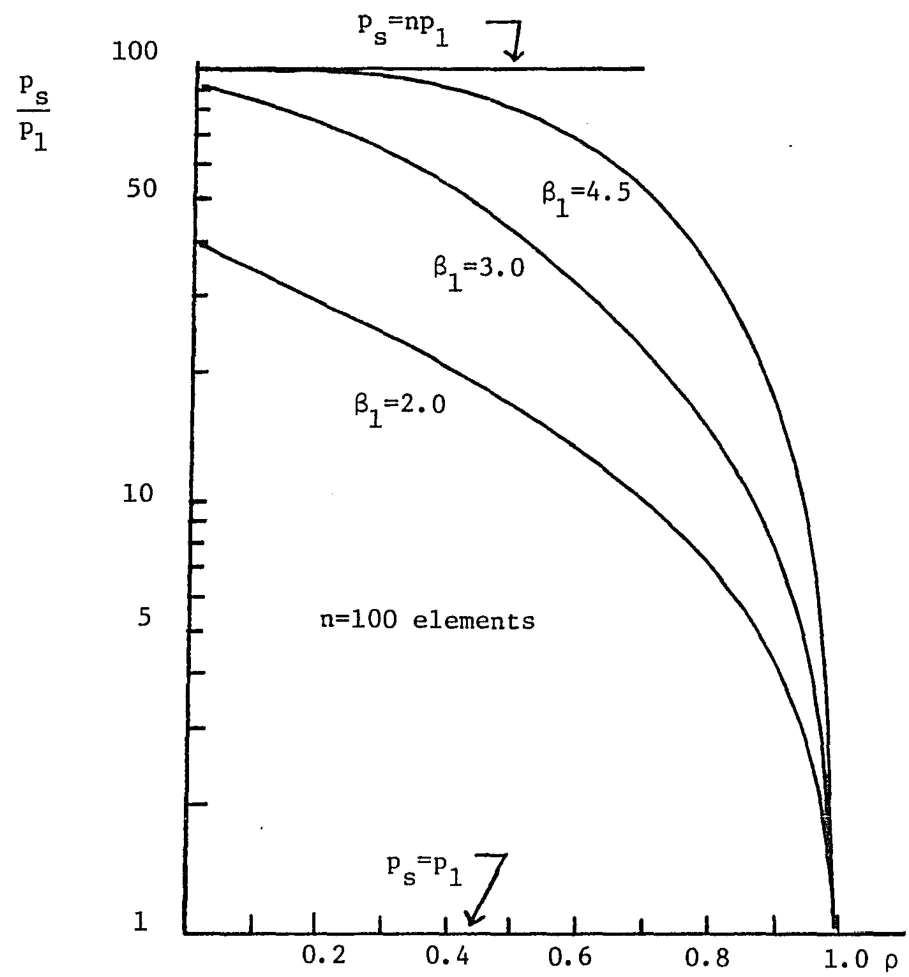


Fig. 3.4 The Influence of ρ and β_1 on the ratio p_s/p_1 for a System of n=100 elements

and the event of failure of any element is defined by $g_1(Q,R)<0$. Both R_1 and Q are normally distributed random variables, $R_1 \sim N(10,2)$ and $Q \sim N(1.06,1.0)$. The values within the parenthesis denotes mean and standard deviation respectively. It is assumed that failure by yield is the only possible failure mode. Because R_1 and Q are normally distributed $g_1(Q,R)$ is also normally distributed. The element safety index can be found from fundamentals (Haugen (1980)).

$$\beta_1 = \frac{\mu_g}{\sigma_g} = \frac{\mu_{R_i} - \mu_s}{\sqrt{\sigma_R^2 + \sigma_s^2}} = \frac{10-1.06}{\sqrt{2^2+1^2}} = 4.00 \tag{3.18}$$

The probability of failure of a component

$$p_1 = \Phi(-\beta_1) = 3.17\ 10^{-5} \tag{3.19}$$

Because the five elements are subjected to the same load the failure events are correlated. If R_1 and Q are statistically independent the correlation coefficient between the element failure events is found from Eq. 3.15 assuming $\sigma_{R_i} = \sigma_{R_j}$

$$\rho = \frac{\sigma_s^2}{\sigma_R^2+\sigma_s^2} = 0.200 \tag{3.20}$$

Substitution of 3.18 and 3.20 into the G-S equation and for n=5 elements gives after numerical integration a system probability of failure of 1.52 E-4. The approximation from $p_s = n\ p_1$ gives $p_s = 1.59E-4$.

3.4 Examples: Unequal Element Risk

Analysis of series chain-like system with unequal element risks was considered in some detail in Chapter 2. It is demonstrated how the unequal element problem can be solved by this method as well. The main conclusions made in Chapter 2, regarding this problem will be reaffirmed through two examples.

A linear variation in the reliability index, β_i, was assumed and estimated by,

$$\beta_i = \beta_1 + [\beta_n - \beta_1] \frac{i-1}{n-1} \qquad , \quad i = 1,2,\ldots,n \tag{3.21}$$

β_1 and β_n is the reliability index of the weakest and the strongest element respectively.

Example

The goal of this exercise is to examine how a linear variation in the element reliability index influences system risk. The weakest element is chosen to have a reliability index of 3.0. The "strongest" element has a reliability index varying from 3.0 (equal elements) to 4.5. It is seen from Fig. 3.6 that unequal element reliabilities strongly influence the system risk. As the correlation coefficient between the element failure functions increases, the importance of unequal elements becomes less important. This is to be expected because for $\rho = 1$ the system probability of failure equals that of the weakest element. In other words the system becomes "deterministic." It is also not surprising to see that as β_n increases the weakest

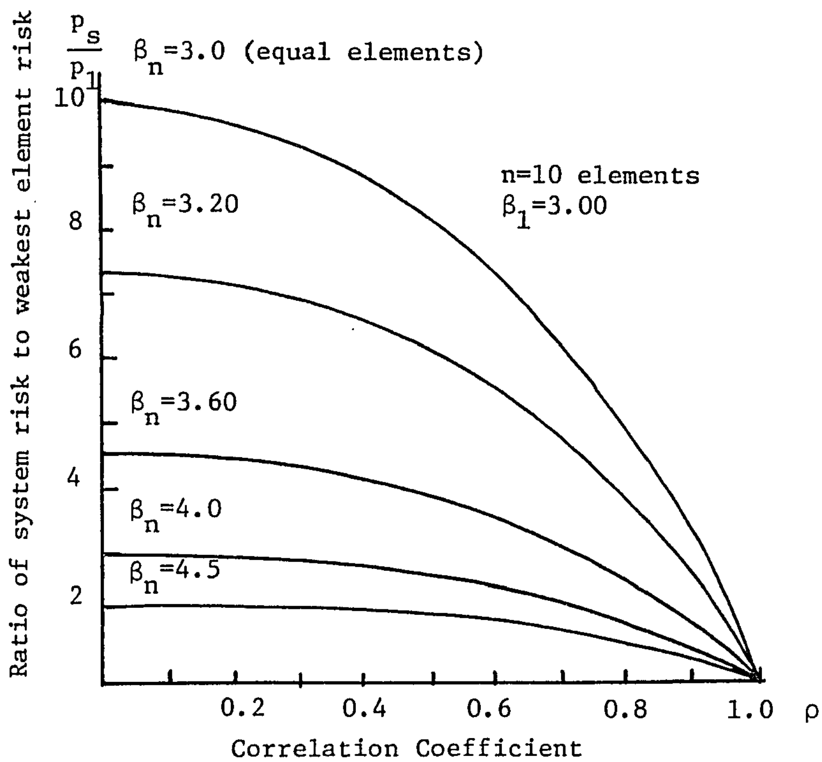


Fig. 3.5 The Influence of β_n/β_1 on System Risk

element more and more dominates the system performance. The same observations were made in Chapter 2.

Example

In this example the goal is to examine how the reliability index of the weakest and strongest element, β_1 and β_n, relate to each other for a given target reliability assigned to the system p_{so}. This will give information on how much the weakest element influences the system. The system target reliability is chosen $p_{so} = 10^{-3}$ corresponding to a system target safety index p_{so} of 3.09. Systems of 10, 30, and 100 elements are investigated. The correlation coefficient was predetermined to be 0.300. In Fig. 3.7 β_1 is plotted against β_n when the system safety index β_s is fixed at 3.09. It is seen that the different between β_1 and β_n is large. As ρ increases β_1 is expected to play a more important role.

3.5 Series Systems With Non Normal Failure Functions; An Approximation to the General Series Reliability Problem

In the G-S equation ρ is the equicorrelation coefficient between normally distributed variates and β_1 is the normal safety index. However, the basic random variables will in general not be normally distributed and the distribution of g_i may not even be known. The failure function g_i may be non normal because a) some or all of the basic variables x_i in Eq. 3.1 are non normal and/or b) $g_i(\underset{\sim}{x})$ is non linear.

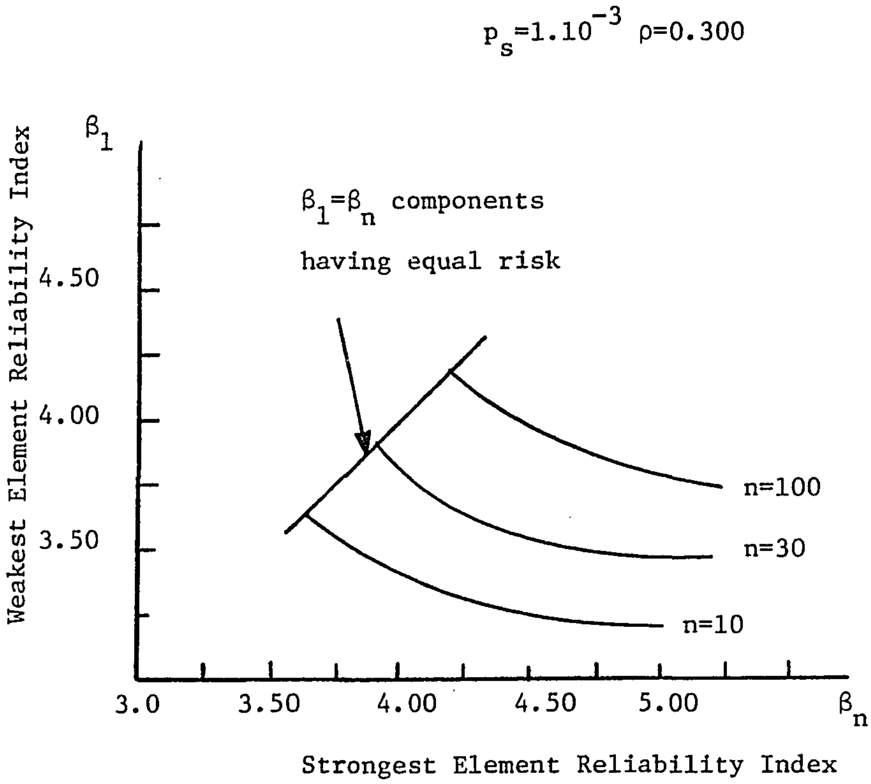


Fig. 3.6 Relation between the Reliability Index of the Weakest and the Strongest Element when the System Reliability Index is Fixed

The overall goal is to show that the G-S equation may provide excellent estimates of p_s, even though the failure functions are non normal, if a) an equivalent normal correlation coefficient ρ_e can be derived, and b) if a good estimate of the equivalent normal safety index β_1 can be made.

The above scheme will be based on some fast probability integration method such as the R-F algorithm. These techniques has in recent years received much attention and are well documented in the general literature such as Leporati (1979), Thoft-Christensen and Baker (1982) and Ang and Tang (1984). The fundamental idea behind the R-F algorithm will therefore only be reviewed in Appendix C. By use of the R-F algorithm for example, an equivalent linear normal failure function of $g(\underset{\sim}{u})$ may be obtained. Where $\underset{\sim}{u}$ is a vector of the transformed basic variables in the normal probability space. From the design point $g(\underset{\sim}{u}^*)$ the equivalent normal safety index β_1 is obtained as the minimum distance from origo to the failure surface (see Appendix C). The correlation coefficient may be derived from $g(\underset{\sim}{u})$ by the use of fundamentals.

3.6 Some Comments and Observations Regarding the Equivalent Normal Safety Index

It was suggested above that the R-F algorithm can be used in the estimation of β_1. However it should be kept in mind that this algorithm is only approximate for non-linear failure functions in the reduced variable space. The accuracy of these approximation techniques not only depends upon the linearity of the limit state function around the design point but also on the shape of the probability distributions of the basic variables. More sophisticated methods have been developed to account for non-linearities in the limit state function in the normal space and may be used if necessary. For instance the Wu-algorithm (Wu (1984)) is known to give more accurate estimates of β_1. It is important to keep in mind that β_1 is the normal fractile. If $g(\underset{\sim}{x})$ is non normal β_1 is estimated by an equivalent normal fractile such as β_{R-F} or β_{Wu}. It is known that p_1 the element probability of failure is sensitive to the accuracy of β_1 good estimates of β_1 are therefore important. The next question which arises is, how does an error in β_1 influence p_s? This question will be addressed by the following example.

<u>Influence on System Uncertainty From Uncertainty or Error in</u>

If β_1 is approximated by the R-F algorithm ($\beta_1 \tilde{=} \beta_{R-F}$) there may be some error in β_1 at the element level. In this example it is then the goal to see how this may influence p_s. The G-S equation is therefore solved for β_1=4.0 and 4.05 while ρ and n are fixed.

In Table 3.1 it is shown that even though probability of failure is very sensitive to changes in β_1, an error in β_1 made on the element level does not magnify with increased system size. It is seen that a change in β_1 from 4.0 to 4.05 (1.3%) results in a 23.7% error in element probability of failure, however for a system of 30 or 100 elements this error remains approximately the same. This observation is very encouraging since it suggests that as far as β_1 is concerned p_s is very insensitive to

Table 3.1 Importance of Uncertainty in β on p_s

	ρ = 0.200			ρ = 0.600			ρ = 0.900		
n	β_1=4.0	β_1=4.05	Δ%	β_1=4.0	β_1=4.05	Δ%	β_1=4.0	β_1=4.05	Δ%
1	$3.168.10^{-5}$	$2.562.10^{-5}$	23.7	$3.168.10^{-5}$	$2.561.10^{-5}$	23.7	$3.168.10^{-5}$	$2.561.10^{-5}$	23.7
5	$1.882.10^{-4}$	$1.279.10^{-4}$	23.7	$1.486.10^{-4}$	$1.206.10^{-4}$	23.2	$9.595.10^{-5}$	$7.821.10^{-5}$	22.7
10	$3.159.10^{-4}$	$2.555.10^{-4}$	23.6	$2.802.10^{-4}$	$2.280.10^{-4}$	22.9	$1.428.10^{-4}$	$1.169.10^{-4}$	22.2
30	$9.417.10^{-4}$	$7.624.10^{-4}$	23.5	$7.206.10^{-4}$	$5.903.10^{-4}$	22.1	$2.477.10^{-4}$	$2.039.10^{-4}$	21.5
100	$3.076.10^{-3}$	$2.497.10^{-3}$	23.2	$1.833.10^{-3}$	$1.516.10^{-3}$	20.9	$3.113.10^{-4}$	3.433.1.10-^{4}	20.8

uncertainly in β_1 or to put it in other words; an error in the estimate of p_1 or β_1 does not appear to magnify as the system size increases. This observation was also confirmed by several other examples not included herein. However this does not mean that good estimates of β_1 is unimportant because a large error in p_1 at the element level will of course also result in poor estimates of p_s. It only suggests that if good techniques are available in the estimation of element reliabilities system reliabilities can also be estimated with good accuracy. Furthermore, it appears from numerous examples that there is little combined effect in ρ and β. Or p_s appears to be sensitive to n mainly through ρ.

3.7 Derivation of an Equivalent Correlation Coefficient

In this section, an equivalent normal correlation coefficient ρ_e between the equivalent normal failure functions $g_i(\underset{\sim}{u})$ will be derived. The equivalent failure function is defined as the failure function in the space where the basic variables is transformed to reduced equivalent normal variables as in Fig. 3.8. It is expected that such a correlation coefficient together with a good estimate of β_1 will result in excellent estimates of p_s by the use of the G-S equation when the element failure functions $g(\underset{\sim}{x})$ are non normal random random variables.

Consider the 2-dimensional transformed linear failure function $g(\underset{\sim}{u})$ in Fig. 3.8. If this figure is extended to higher dimensions $g(\underset{\sim}{u})$ is a plane through the normal space and β is a line perpendicular to this plane. It is now the goal by the use of vector calculus to derive an

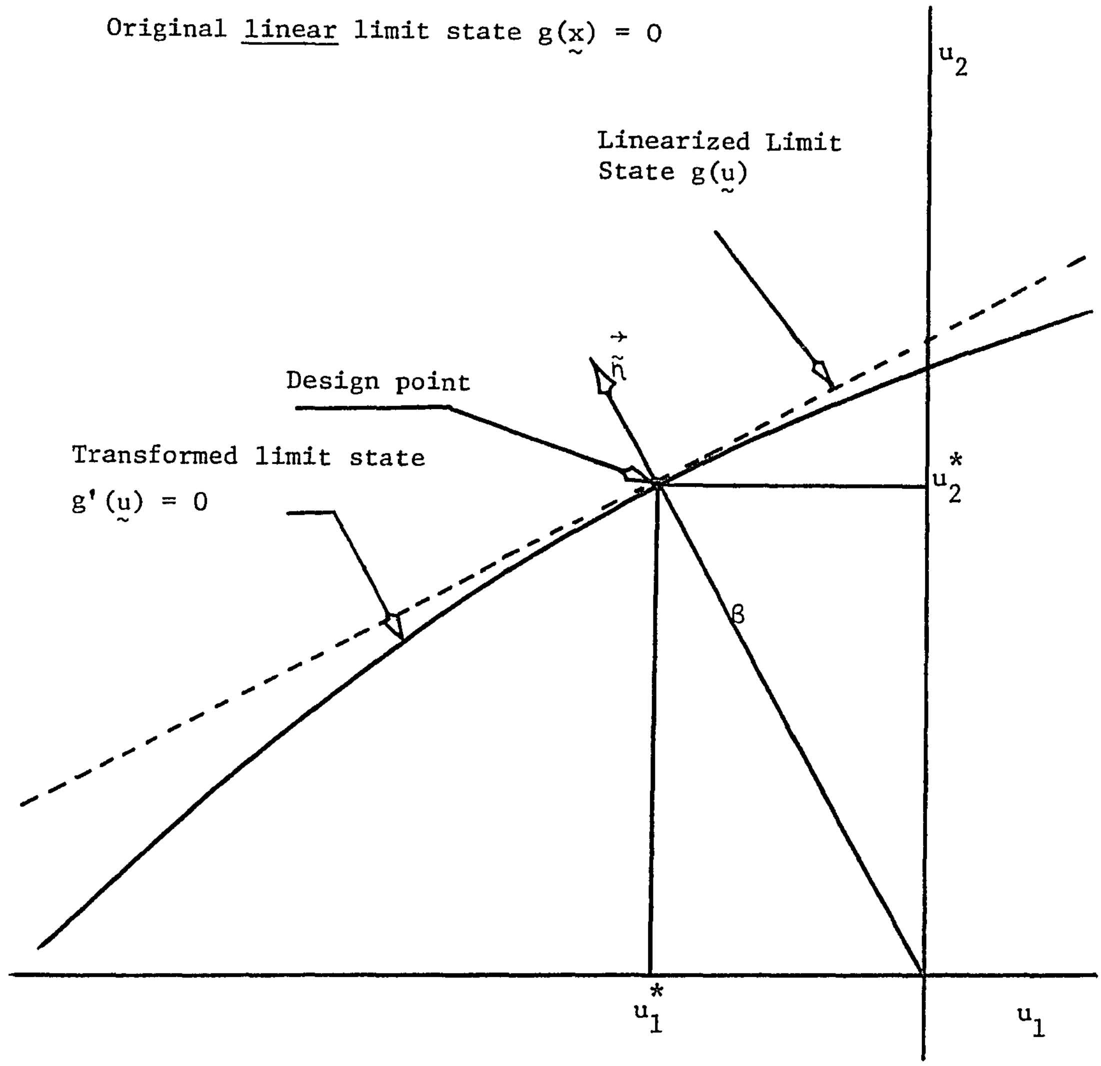


Fig. 3.7 Transformation of a Limit State

expression for $g(\underset{\sim}{u})$ in terms of u_i^*, u_i and β, where u_i^* is the reduced variable at the design point. The design point is the reduced variables at the shortest distance from origo to the failure surface which is defined by β (see Fig. 3.8).

From calculus $g(\underset{\sim}{u}) = 0$ can be expressed as (Kregszig (1979))

$$a_1 u_1 + a_2 u_2 + \ldots\ldots + a_n u_n = \beta \tag{3.22}$$

where a_1 through a_n are some unknown constants. At the design point the normal to this plane is

$$\vec{n} = \frac{u_1^* \vec{i}_1 + u_2^* \vec{i}_2 + \ldots + u_n^* \vec{i}_n}{\sqrt{u_1^{*2} + u_2^{*2} + \ldots + u_n^{*2}}} \tag{3.23}$$

where $\vec{i}_j$ is the unit directional vector. Furthermore recognize that the design point $\underset{\sim}{u}^*$ and β are related

$$u_1^{*2} + u_2^{*2} + \ldots + u_n^{*2} = \beta^2 \tag{3.24}$$

Substitute 3.24 into 3.23,

$$\vec{n} = \frac{u_1^{*2} \vec{i}_1 + \ldots + u_n^{*2} \vec{i}_n}{\beta} = \frac{\sum_{j=1}^{n} u_j^{*2} \vec{i}_j}{\beta} \tag{3.25}$$

A position vector to any point in the plane may be expressed as,

$$\vec{r} = u_1 \vec{i}_1 + u_2 \vec{i}_2 + \ldots + u_n \vec{i}_n = \sum_{j=1}^{n} u_j \vec{i}_j \tag{3.26}$$

Note that,

$$\vec{n} \cdot \vec{r} = \beta \tag{3.27}$$

where $\cdot$ denote the dot product. Substituting 3.25 and 3.26 into 3.27, it

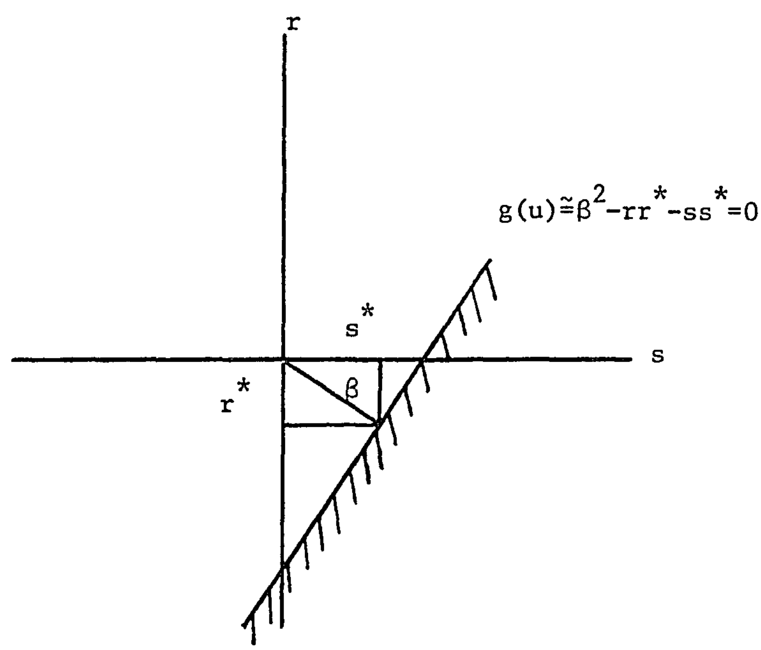


Fig. 3.8 The Limit State g(R,S)=R-S Linearized in the Reduced Normal Space

follows that,

$$\frac{\sum_{j=1}^{n} u_j^* \vec{i}_j}{\beta} \cdot u_j \quad \vec{i}_j = \beta \tag{3.28}$$

Then after taking the dot product and rearranging,

$$\sum_{j=1}^{n} u_j^* u_j = \beta^2 \tag{3.29}$$

the linear failure function $g_i(\underset{\sim}{u})$ in the reduced space is then,

$$g_i(\underset{\sim}{u}) = \beta^2 - u_1^* u_1 - u_2^* u_2 - \ldots - u_n^* u_n \tag{3.30}$$

Then from expectations (Bury (1975))

$$\text{cov}(g_i(\underset{\sim}{u}) g_j(\underset{\sim}{u})) = E[(g_i - E(g_i))(g_j - E(g_j))] \tag{3.31}$$

and after algebra,

$$\text{cov}(g_i(\underset{\sim}{u}) g_j(\underset{\sim}{u})) = \sum_{j=1}^{n} u_i^{*2} \sigma_{u_i}^2 + \sum_{i \neq j}^{k} \sum \text{cov}\ (u_i u_j) \tag{3.32}$$

also recognize that

$$\text{cov}(u_i u_j) = \rho_{u_i u_j} \sigma_{u_i} \sigma_{u_j} \tag{3.33}$$

and that

$$\sigma_{u_i} = 1 \tag{3.34}$$

Substitute 3.38 and 3.39 into 3.37,

$$\text{cov}(g_1(\underset{\sim}{u}) g_j(u)) = \sum_{j=1}^{n} u_i^{*2} + \sum_{i \neq j}^{k} \sum \rho_{u_i u_j} \tag{3.35}$$

Similarly if i=j the covariance becomes the variance

$$g^2_{g_i}(\underset{\sim}{u}) = \sum_{j=1}^{n} u_i^{*2}$$

$$= \beta_1^2 \qquad (3.36)$$

Notice that k in Eq. 3.35 is less than or equal to n in Eq. 3.36. This physically means that the reduced variables 1 through k occur in both g_i and g_j and for this reason $g_i(\underset{\sim}{u})$ and $g_j(\underset{\sim}{u})$ becomes correlated. Then from Eq. 3.2, 3.35 and 3.36,

$$\rho_{g_i(\underset{\sim}{u})g_j(\underset{\sim}{u})} \equiv \rho_e = \frac{\sum\limits_{i=j}^{n} (u_i^*)^2 + \sum\limits_{i \neq j}^{k} \sum \rho ij}{\beta_1^2} \qquad (3.37)$$

where now ρ_e will be referred to as an "equivalent normal correlation coefficient." In most practical examples $\rho_{ij}=0$ and the equivalent correlation coefficient is,

$$\rho_e = \frac{\sum\limits_{i=1}^{n} u_i^{*2}}{\beta_1^2} \qquad (3.38)$$

Example

In this example it is the goal to shown how ρ_e is found for a simple linear failure function. Consider the following failure function,

$$g_i(R,S) = R_i - S \qquad (3.39)$$

R_i is the element strength and s is the stress common for all elements. Both R_i and s are assumed to be non normally distributed and independent. The linearized failure function in the reduced normal space may look like the one shown in Fig. 3.9. From Eq. 3.30 the reduced failure function is given by,

$$g_i(r,s) = \beta^2 - r_i r_i^* + s\,s^* \tag{3.40}$$

where r_i and s are the reduced equivalent normal strength and stress respectively. The correlation coefficient between the reduced failure functions is then from Eq. 3.37.

$$\rho_e = \frac{u_s^{*2}}{\beta^2} \tag{3.41}$$

3.8 Example: Check With Some Known Exact Cases

In this example it will be shown that excellent estimates of p_s can be obtained when the failure function is given by Eq. 3.35 and the basic variables are non normal. The R-F algorithm was used to estimate the safety index as well as ρ_e. Exact element and system probability of failure was obtained by the use of the method described in Chapter 2. Several combinations of strength and stress distributions were used and the results are shown in Tables 3.2 and 3.3. Table 3.3 contains examples where ρ_e is close to 1 where in particular good estimates of the correlation coefficient is necessary. It is seen in that two approximate results are given in each case. One is based on the safety index obtained by the R-F algorithm β_{R-F} and the other is based on the equivalent β_e which can be found by

$$\beta_e = -\Phi^{-1}(P_1) \tag{3.42}$$

where p_1 is the exact element probability of failure and obtained by the

Table 3.2 Approximation of System Probability of Failure by G-S Equation

$\mu_R=60$, $\sigma_R=10$, $\mu_S=10$, $\sigma_S=5$ in all cases

$g(\underset{\sim}{x}) = R-S$

Case A: (R) = Weibull (S) = Extreme value $\rho_e=0.449$

No. of elements	Exact	Approximate based on $\beta_e=3.687$	% Error	Approximate based on $\beta_{R-F}=3.7276$	% Error
1	1.135 E-4	1.135 E-4	0.0	9.666 E-3	17.4
10	1.031 E-3	1.065 E-3	3.3	0.180 E-3	13.2
100	7.589 E-3	7.920 E-3	4.4	6.862 E-3	10.7

Case B: (R) = Weibull (S) = Weibull $\rho_e=0.251$

No. of elements	Exact	Approximate based on $\beta_e=3.881$	% Error	Approximate based on $\beta_{R-F}=3.869$	% Error
1	5.091 E-5	5.202 E-5	2.2	5.474 E-5	7.5
10	5.059 E-4	5.165 E-4	2.1	5.433 E-4	7.4
100	4.789 E-3	4.865 E-5	1.6	5.110 E-3	6.7

Table 3.2--Continued

Case C: (R) = Weibull	(S) = Lognormal ρ_e=0.6105				
No. of elements	Exact	Approximate based on β_e=3.533	% Error	Approximate based on β_{R-F}=3.614	% Error
1	2.053 E-4	2.054 E-4	0.1	1.510 E-4	35.9
10	1.637 E-3	1.669 E-5	1.9	1.245 E-3	31.5
100	9.157 E-3	9.052 E-3	1.2	6.972 E-3	31.3

Case D: (R) = Weibull	(S) = Extreme value ρ_e=0.8185				
No. of elements	Exact	Approximate based on β_e=4.157	% Error	Approximate based on β_{R-F}=4.149	% Error
1	1.609 E-5	1.612 E-5	0.2	1.671 E-5	3.8
10	1.305 E-4	1.320 E-4	1.2	1.369 E-4	4.9
100	4.126 E-4	4.246 E-4	2.9	4.389 E-4	6.4

Table 3.3 Approximation of System Probability of Failure by G-S Equation

$\mu_R=60,\ \sigma_R=1.2,\ \mu_S=10,\ \sigma_S=10$

$g(\underset{\sim}{x}) = R-S$

Case A: (R) = Lognormal (S) = Extreme value $\rho_e=0.9983$

No. of elements	Exact	Approximate based on $\beta_e=3.112$	% Error	Approximate based on $\beta_{R-F}=3.112$	% Error
1	9.295 E-4	9.291 E-4	0.1	9.291 E-3	0.1
10	1.165 E-3	1.143 E-3	1.9	1.143 E-3	1.9
100	1.303 E-3	1.303 E-3	3.1	1.303 E-3	3.1

Case B: (R) = Weibull (S) = Lognormal $\rho_e=0.9991$

No. of elements	Exact	Approximate based on $\beta_e=2.568$	% Error	Approximate based on $\beta_{R-F}=2.571$	% Error
1	5.112 E-3	5.114 E-3	0.0	5.064 E-3	1.0
10	5.752 E-3	5.825 E-3	1.3	5.769 E-3	0.3
100	6.448 E-3	6.321 E-3	2.0	6.261 E-3	3.0

Table 3.3--Continued

Case C:	(R) = Weibull		(S) = Lognormal		ρ_e=0.9864
No. of elements	Exact	Approximate based on β_e=4.965	% Error	Approximate based on β_{R-F}=4.964	% Error
1	3.439 E-7	3.435 E-7	0.1	3.453 E-7	0.4
10	7.706 E-7	7.679 E-7	0.4	7.718 E-7	0.2
100	1.302 E-6	1.311 E-6	0.7	1.319 E-6	1.2

Case D:	(R) = Normal		(S) = Extreme value		ρ_e=9980
No. of elements	Exact	Approximate based on β_e=3.112	% Error	Approximate based on β_{R-F}=3.112	% Error
1	9.295 E-4	9.279 E-4	0.2	9.279 E-4	0.2
10	1.169 E-3	1.161 E-3	0.7	1.161 E-3	0.7
100	1.354 E-3	1.338 E-3	1.2	1.338 E-3	1.2

methods of chapter 2. It is seen that for the cases when β_{R-F} is a relatively poor estimate a considerable better estimate of p_s may be obtained using β_e. This of course suggests as mentioned earlier that a more advanced method than the R-F algorithm may be used to estimate β. On the other hand it is expected that the estimate of ρ_e by the R-F algorithm is sufficiently accurate. The reason for this is that ρ is relatively insensitive to distributions.

It is seen that in all cases excellent estimates of p_s results even when $g(\tilde{x})$ is highly non normal. Systems of 10 and 100 elements are shown and it is in particular encouraging that the error of the estimates does not strongly magnify as the system size increases. This fact suggests that the estimates of ρ_e is sufficiently good and that the main source of error is due to nonlinearity in the reduced failure function.

Example: Extension to Non Linear Failure Function

The simple strength-stress relationship is extended to the case where the stress is a non-linear function,

$$g(\underset{\sim}{x}) = R-V^3 \tag{3.43}$$

where V^3 is identified as the "stress" function, S, the distribution of which is derived and described in Appendix D. Four different cases are run and the results are shown in Table 3.4. Again the results are shown to be excellent.

Table 3.4 Approximation of System Reliability by G-S Equation for Non Linear Failure Functions

$g(\underset{\sim}{x}) = R-V^3$

$\mu_R=80,\ \sigma_R=3.0,\ \mu_V=2.1304,\ \sigma_V=0.4694$

Case A (R) = Weibull (S) = Extreme value $\rho_e=0.9979$

No. of elements	Exact	Approximate based on $\beta_e=2.9733$	% Error	Approximate based on $\beta_{R-F}=2.9813$	% Error
1	1.475 E-3	1.473 E-3	0.1	1.435 E-3	2.8
10	1.907 E-3	1.840 E-3	3.6	1.793 E-3	6.3
100	2.451 E-3	2.118 E-3	15.7	2.065 E-3	18.7

$g(\underset{\sim}{x}) = R-V^3$

$\mu_R=40,\ \sigma_R=6.0,\ \mu_V=3.1257,\ \sigma_V=0.4799$

Case B (R) = Normal (S) = Lognormal $\rho_e=0.6204$

No. of elements	Exact	Approximate based on $\beta_e=3.9672$	% Error	Approximate based on $\beta_{R-F}=4.0250$	% Error
1	3.636 E-5	3.634 E-5	0.1	2.849 E-5	27.6
10	3.0460 E-4	3.147 E-4	3.3	2.483 E-4	22.7
100	2.023 E-4	1.963 E-3	3.1	1.579 E-3	28.1

Table 3.4--Continued

$$g(\underset{\sim}{x}) = R-V^3$$

Case C (R) = Normal (S) = Extreme value ρ_e=0.9987

μ_R=80, σ_R=2.0, μ_V=2.1440, σ_V=0.3109

No. of elements	Exact	Approximate based on β_e=3.7915	% Error	Approximate based on β_{R-F}=3.7977	% Error
1	7.490 E-4	7.502 E-5	0.2	7.302 E-5	6.5
10	9.636 E-5	9.293 E-5	3.7	9.048 E-4	11.7
100	1.233 E-4	1.064 E-4	15.9	1.037 E-4	18.9

$$g(\underset{\sim}{x}) = R-V^3$$

Case D (R) = Extreme value (S) = Weibull ρ_e=0.8577

μ_R=20, σ_R=1.2, μ_V=2.1507, σ_V=0.1860

No. of elements	Exact	Approximate based on β_e=4.443	% Error	Approximate based on β_{R-F}=4.402	% Error
1	4.432 E-6	4.436 E-6	0.1	5.368 E-5	20.9
10	2.748 E-5	2.662 E-5	3.2	3.204 E-5	16.6
100	9.828 E-5	1.031 E-4	4.9	1.228 E-4	24.9

Several other cases which are not included were run as well and case A and C in Table 3.4 were the worst results that was found.

3.9 Example: Application to a General Failure Function

The goal of this example is to demonstrate the principals of this approximate method for series system analysis to a general element failure function where no exact solution exists. Consider the following element failure function,

$$g_i = \frac{A_i^2 + Y_i^{0.5}}{x_i} - \frac{1}{4}\,(Z^2R^{1/3}+D^2) \tag{3.43}$$

where A,Y,X,Z, R and D are all random variables their distribution and parameters are shown in Table 3.5. All the random variables are mutually independent. The goal is to check chain like series system consisting of 100 identical elements which mean that all the elements are governed by the failure function in Eq. 3.43. The terms $1/4\ (Z^2R^{1/3}+D^2)$ is termed the stress part and is common for all elements. And $(A_i^2 + Y_i^{0.5})/X_i$ is the strength term it is independent and identical for all elements. the system is considered safe if $p_s < 5$ E-5. By use of the R-F algorithm $\beta_{R-F} = 4.6192$ and the reduced design variables are listed in Table 3.5. The equivalent correlation coefficient ρ_e is found by Eq. 3.37

$$\rho_e = \frac{u_z^{*2}+u_R^{*2}+u_D^{*2}}{\beta^2} = \frac{(4.17344)^2+(0.42969)^2+(0.89844)^2}{(4.61922)^2} = 0.8628 \tag{3.44}$$

The system probability is found by numerical integration of the G-S equation.

Table 3.5 Distribution, Mean Value, Standard Division and Design Point for the Element Failure Function

Variable	Distribution	Mean	Standard Deviation	Design Point
A_i	Normal	1.5	0.2	-1.500
Y_i	Weibull	4	0.3	-0.2375
X_i	Weibull	1	0.1	0.79003
Z	Extreme value	1	0.2	4.17344
R	Extreme value	2.5	0.4	0.42969
D	Normal	1.5	0.3	0.89844

$g_i=(A^2+Y^{0.5})/X - \frac{1}{4} Z^2R^{1/3}D^2$, $\beta_{R-F}=4.61922$, $p_1=1.926$ E-6

$$p_s = 4.671\text{ E-5} < 5 \cdot \text{E-5 (safe)} \tag{3.45}$$

or

$$\beta_s = -\phi^{-1}(p_s) \cong 3.91 \tag{3.46}$$

so the system is safe. If independence had been assumed $p=1-[1-p_1]^n$ = 1.926 E-4 which is greater than 5E-5. This means that the system is safe if dependency is considered and not safe if independent failure events are assumed.

3.10. A Simple Factor to Account for Correlation Between Failure Functions

In this section simple charts will be developed so that numerical integration of the G-S equation in many cases may be eliminated. It was shown in Chapter 2 that the system probability of failure could be expressed on the following form,

$$p_s = \delta\, p_s^* = \delta[1-\prod_{i=1}^{n} (1-p_i)] \tag{3.47}$$

where p_s is the system probability of failure if the element failure events were independent. δ is a factor which accounts for correlation between failure events, and $\delta=\delta(\rho_{g_i g_j}, \beta_1, n)$. In Figs. 3.10 through 3.14 it is assumed that all the elements are identical. The use of Fig. 3.10 through 3.14 should be self explanatory. If β_1, ρ, and n are known the system probability of failure can be estimated without numerical integration. It may of course be necessary to interpolate if n or β_1 differs from the discrete values in the figures.

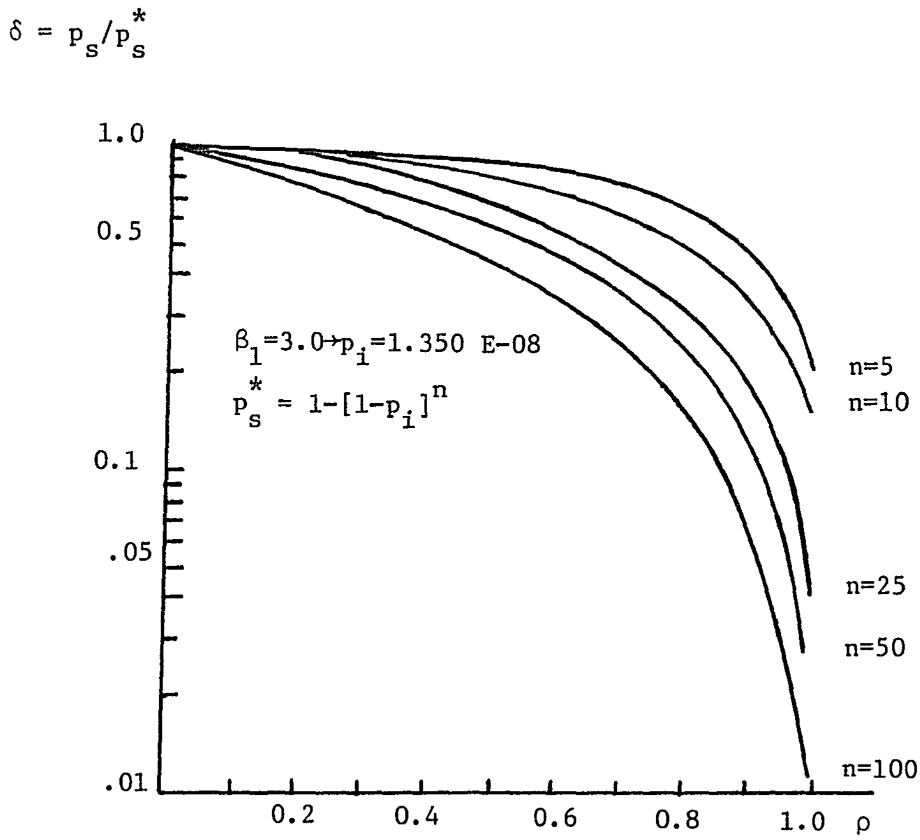


Fig. 3.9 The Ratio of p_s/p_s^* vs ρ(the correlation coefficient between element failure modes) at β_1=3.0

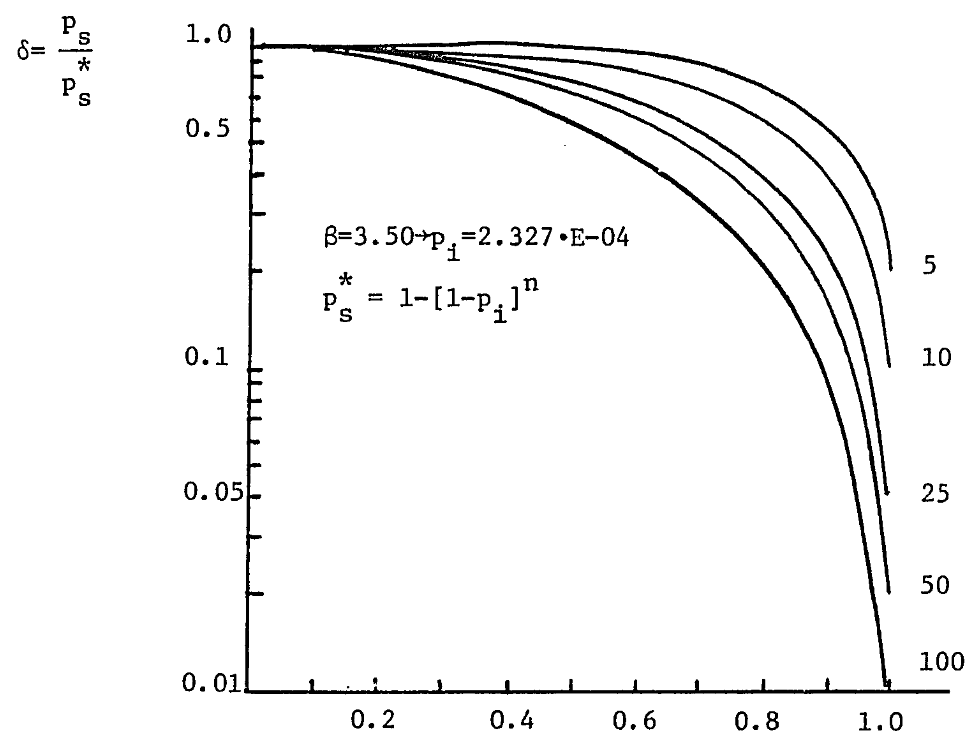


Fig. 3.10 The Ratio of p_s/p_s^* vs ρ at β_1=3.50

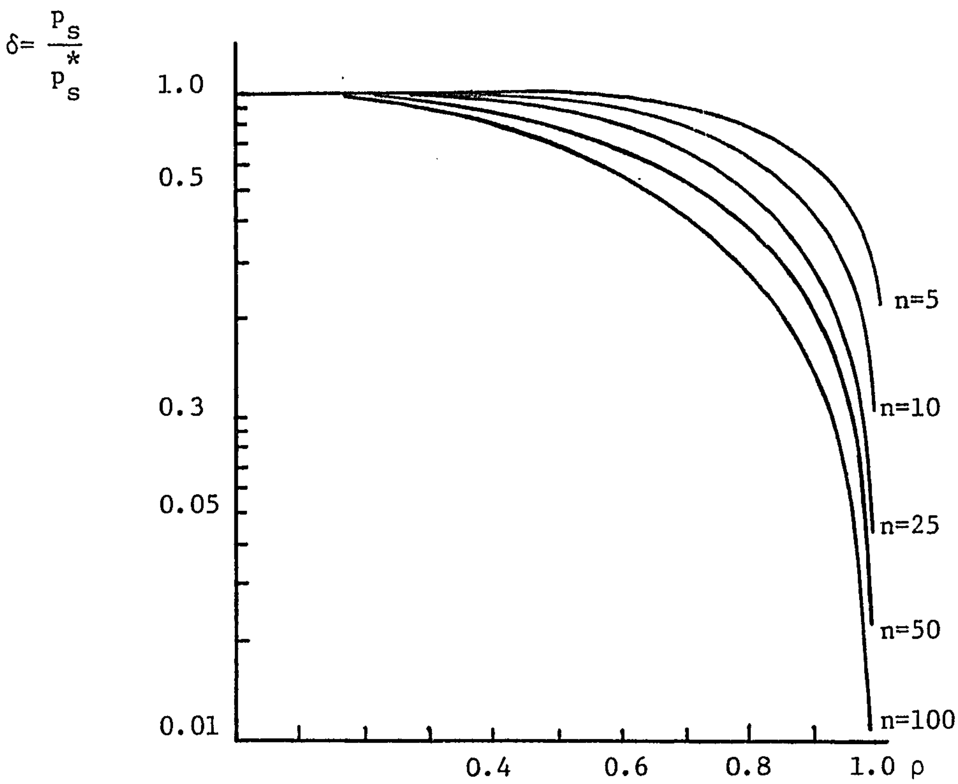


Fig. 3.11 The Ratio of p_s/p_s^* vs ρ at β_1=4.00

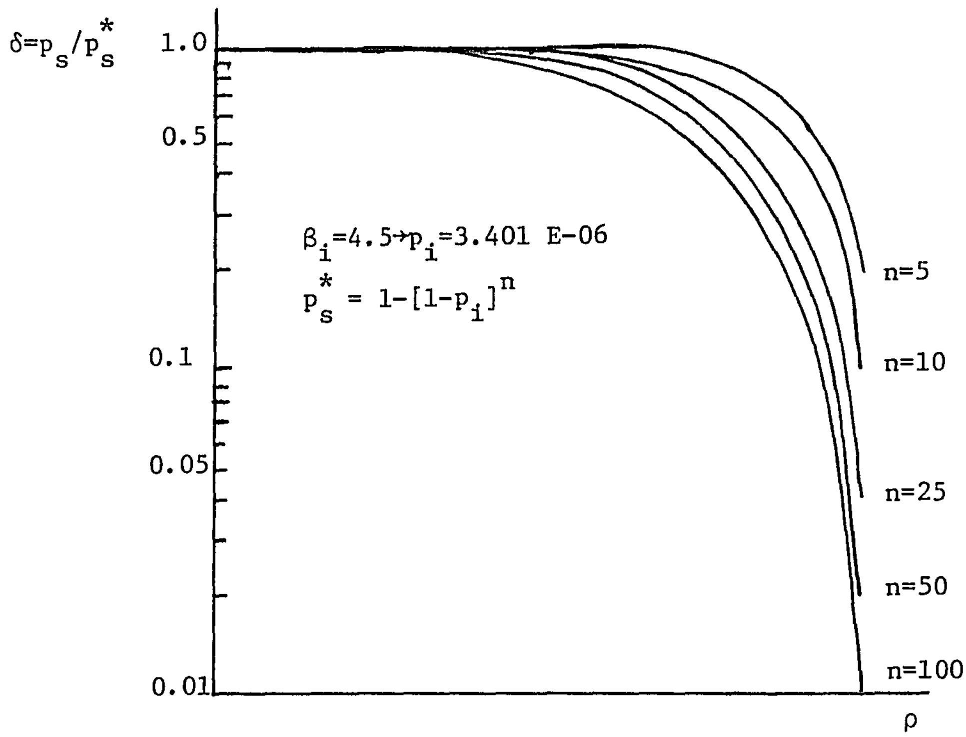


Fig. 3.12 The Ratio of p/p_s^* vs ρ at β_1=4.50

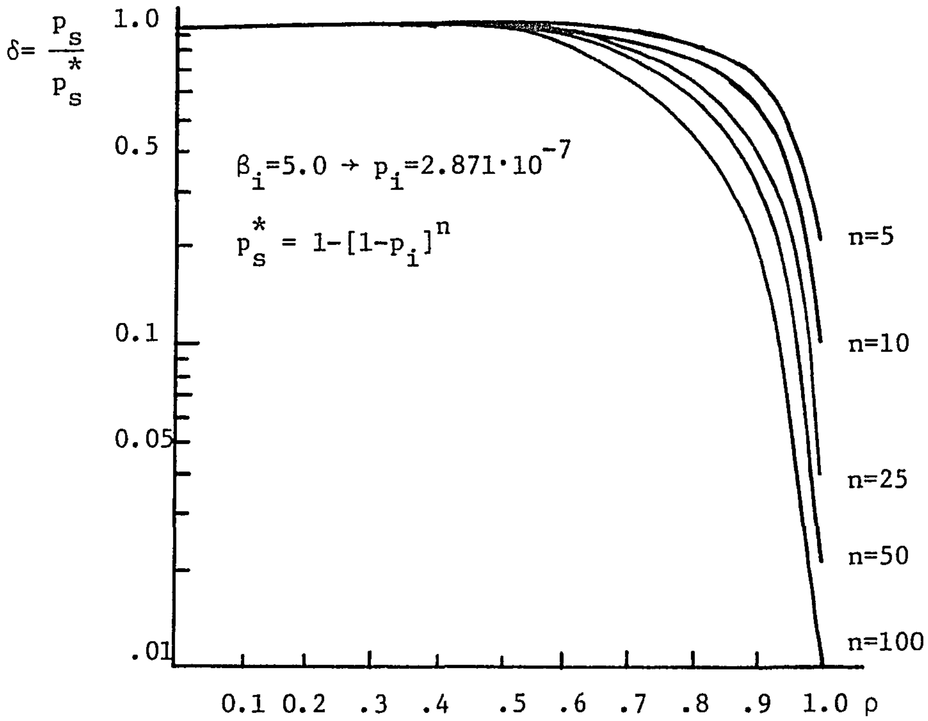


Fig. 3.13 The Ratio of p_s/p_s^* vs ρ at β_1=5.0

Summary and Conclusions

A special form of the multinormal integral has been used for calculation of series system reliabilities. It has been demonstrated that good estimates of system probabilities of failure results even when the basic variable are non normal. It is particularly interesting to notice that good results are obtained for highly correlated failure functions. This is significant since any bounds on the system reliability breaks down when the correlation coefficient approaches unity. This new method for series system analysis is therefore a significant new tool which greatly simplifies the analysis. Furthermore problems that before had no closed formed solution can now easily be solved.

CHAPTER 4

SERIES SYSTEMS WITH UNEQUALLY CORRELATED FAILURE FUNCTIONS

4.1 Introduction

In chapter 3 an "extended" G-S (Gupta-Stuart) equation was introduced for analysis of series systems. A key assumption for the use of this equation is that the correlation coefficient ρ_{ij} between any two element failure functions g_i and g_j, $i \neq j$ is the same. The failure functions were said to be "equicorrelated" but in general ρ_{ij} varies within the system. For example if $\sigma_{Ri} \neq \sigma_{Rj}$ the G-S equation no longer applies. However, the approximation scheme that was developed in chapter 3 proved to be a very powerful tool in the analysis of series systems. Therefore an attempt will be made in this chapter to extend the use this method in the more general case when the assumption of equicorrelated is violated.

Thoft-Christensen and Dalsgard Sorensen (1982) and Ditlevsen (1984) has developed the concept of an equivalent correlation coefficient which can be used together with the G-S equation. In this chapter the concept of the equivalent correlation coefficient will be reviewed and summarized. This will be followed by an example of the analysis of a general series system with unequally correlated elements.

4.2 The Equivalent Correlation Coefficient (Ditlevsen (1984))

The G-S equation as described in chapter 3 is a solution to a special case of the n-th order standard normal distribution function. A key assumption in this equation is that all the joint standard normal variables are equally correlated.

In general the n-dimensional standard normal distribution function can be written as

$$\Phi(\underset{\sim}{\beta};P) = \Phi(\beta_1,\beta_2,\ldots,\beta_n;P) \tag{4.1}$$

where $\underset{\sim}{\beta}$ is a set of n standard normal variables with zero mean and unit variance and with a correlation matrix $P=[\rho_{ij}]$. The total differential of $\Phi(\underset{\sim}{\beta};p)$ with respect to ρ_{ij} at the equicorrelation can be calculated,

$$d\Phi = \sum_{i<j} \frac{\partial \Phi}{\partial \rho_{ij}} (\rho_{ij}-\rho) \tag{4.2}$$

where ρ is the point of equicorrelation.

The goal of the analysis is to find ρ in terms of ρ_{ij}, or a equicorrelation matrix of the form

$$[P] = [\rho + (1-\rho)\delta_{ij}] \tag{4.3}$$

so that $d\Phi$is zero; δ_{ij} is the Kroneker delta. The above may be achieved by the following identity, (Ditlevsen (1982))

$$\frac{\partial \Phi}{\partial \rho_{ij}} = \frac{\partial^2 \Phi}{\partial \beta_i \partial \beta_j} \qquad i=j \tag{4.4}$$

Substitute Eqs. 3.10 and 4.4 into 4.2. It can be shown that at the equicorrelation point,

$$d\Phi = \frac{1}{1-\rho} \int_{-\infty}^{\infty} \phi(u) \prod_{i=1}^{n} \Phi\left(\frac{\beta_K+\sqrt{\rho}\,+}{\sqrt{1-\rho}}\right)[\Sigma a_i(t)a_j(t)(\rho_{ij}-\rho)]dt \qquad (4.5)$$

in which

$$a_i(t) = \phi\left(\frac{\beta_j+\sqrt{\rho}\ t}{\sqrt{1-\rho}}\right) / \left(\Phi\ \frac{\beta_j+\sqrt{\rho}\ t}{\sqrt{1-\rho}}\right) \qquad (4.6)$$

the scope is then to solve for $\rho_{ij}-\rho$ so that $d\phi=0$. In the special case of equal component safety indices, $\beta_i-\beta_j$; $i\neq j$, $\beta_1=---=\beta_n$ the differential is,

$$d\Phi = \frac{1}{1-\rho} \int_{-\infty}^{\infty} [\phi(t)\phi(\frac{\beta+\sqrt{\rho}\ t}{\sqrt{1-\rho}})^2\Phi(\frac{\beta+\sqrt{\rho}\ t}{\sqrt{1-\rho}})^{n-2}dt] \sum_{i<j} (\rho_{ij}-\rho) \qquad (4.7)$$

Thus if $\sum_{i<j} (\rho_{ij}-\rho) = 0$ i.e., if ρ is selected as the simple average

$$\rho = \frac{2}{n(n-1)} \sum_{i<j} \rho_{ij} \qquad (4.8)$$

it follows that $d\Phi=0$, i.e., the equicorrelation is a stationary point of $\Phi(\underset{\sim}{\beta},\underset{\sim}{\rho})$ with respect to such variations of P for which the restriction of Eq. 4.8 is satisfied. In case $\beta_i=\beta_j,\beta_n$ are not all equal it follows from Eq. 4.5 that there is no simple solution to the problem of determining so that $d\Phi=0$ except for $\rho=0$. In any case, in order to use the G-S equation it is wise to choose ρ so that $d\Phi$ becomes approximately zero.

Finding this value of ρ by simple iteration procedure, the first order approximation to $\Phi(\underset{\sim}{\beta_j},\underset{\sim}{P})$ is calculated by

$$\Phi(\underset{\sim}{\beta},P) \cong \Phi[\underset{\sim}{\beta};[\rho+(1-\rho)\delta_{ij}]] + d\Phi \qquad (4.9)$$

where $d\Phi$ is zero. This form implies that the G-S equation can be used. Ditlevsen (1984) also suggested a 2nd order approximation

$$\Phi(\underset{\sim}{\beta}:P) \cong \Phi[\underset{\sim}{\beta};[\rho+(1-\rho)\delta_{ij}]]+d\Phi+\frac{1}{2}d^2\Phi \tag{4.10}$$

This iterative procedure to find the equivalent ρ now becomes much more complicated and is not included herein. Remarkably good results may be obtained by the 2nd order approximation; however, for practical engineering purposes it may be too complicated.

Based on extensive Monte Carlo simulation Thoft-Christensen and Dalsgard Sorensen (1982) suggests that the equivalent correlation coefficient gives results which are close to the upper bimodal bounds on p_s. Bimodal bounds or "Ditlevsen" bounds are narrow bounds on p_s which accounts for correlated failure functions.

4.3 .Reliability Analysis of a Statically Determinate Truss

It is the goal of this example to illustrate how a more general series system can be analyzed by the G-S equation. The statically determinate truss in Fig. 4.1 is used for illustration. It is assumed that the external load, F, is a random variable which is normally distributed. The element stress S_i are proportional to F,

$$S_i = a_i F \tag{4.11}$$

The element stresses will then also be normally distributed. Furthermore the mean and the standard deviation of S_i are proportional to the mean and standard deviation of F by the same factor. The individual element strengths are assumed to have identical distributions.

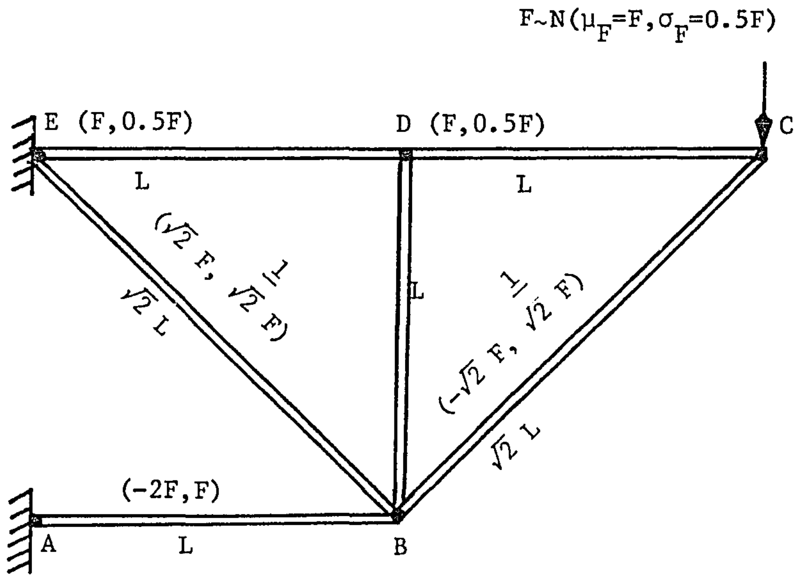


Fig. 4.1 A Weakest Link System with Unequal Element Loads (μ_{Ri}=5F, σ_{Ri}=F) (Mean and Standard Deviation of Stress in the parenthesis on each members)

The first step is to establish an element failure function for the axially loaded members,

$$g_i = R_i - S_i \tag{4.12}$$

where R_i and S_i are the element strength and stress respectively; failure occurs when $g_i \leq 0$. In the next step, complete σ_{gi} and cov (g_i, g_j). From fundamentals of probability theory, (e.g., see sec 3.2)

$$\sigma^2_{g_i} = \sigma^2_{R_i} + \sigma^2_{S_i} - 2\rho_{R_iS_i}\sigma_{R_i}\sigma_{S_i} \tag{4.13}$$

and

$$\text{cov}\ (g_i, g_j) = \rho_{S_iS_j}\sigma_{S_i} + \rho_{R_iR_j}\sigma_{R_i}\sigma_{R_j} - 2\rho_{R_iS_i}\sigma_{R_i}\sigma_{S_i} \tag{4.14}$$

In linear-elastic structural mechanics, stress and strength are often assumed independent so that $\rho_{R_iS_i}=0$. It is also common to assume that the individual element strengths are independent of each other. If so $\rho_{R_iR_j}=0$. For the simple limit state function in Eq. 4.12 it has been shown that dependent strengths has minor influence on the system reliability (Grigoriu and Turkstra (1979) and Thoft-Christensen and Dalsgard Sorensen (1982)). By invoking the above assumptions the correlation coefficient between any two elements becomes

$$\rho_{g_ig_j} = \frac{\rho_{S_iS_j}\ \sigma_{S_i}\sigma_{S_j}}{\sqrt{(\sigma^2_{R_i} + \sigma^2_{S_i})(\sigma^2_{R_j} + \sigma^2_{S_j})}} \tag{4.15}$$

In this problem the individual element loads are correlated. The next step is to find, $\rho_{S_iS_j}$, the correlation coefficient between element <u>stresses</u> in terms of the statistics of F.

$$\rho_{S_iS_j} = \frac{a_i a_j \operatorname{cov}(F_i, F_j)}{(a_i \sigma_{F_i})(a_j \sigma_{F_j})} \tag{4.16}$$

$$= \frac{\operatorname{cov}(F_i F_j)}{\sigma_{F_i} \sigma_{F_j}} \tag{4.17}$$

Now

$$\operatorname{cov}(F_i F_j) = E[F_i F_j] - E[F_i]E[F_j] \tag{4.18}$$

However F_i and F_j are both related to the external load F by the constants b_i and b_j respectively, it then follows,

$$\rho_{S_iS_j} = \frac{b_i b_j \sigma_F^2}{(b_i \sigma_F)(b_j \sigma_F)} = 1 \tag{4.19}$$

and

$$\operatorname{cov}[F_i F_j] = b_i b_j \sigma_F \tag{4.20}$$

It is important to recognize that if there are more than one external load, $\rho_{S_iS_j}$ may possibly not be 1. In fact $\rho_{S_iS_j}=1$ if and only the external loads are independent of each other. If $\rho_{S_iS_j} \neq 1$, a solution exists only if the degree of dependency between the external loads are known.

For this example Eq. 4.15 then becomes,

$$\rho_{g_i g_j} = \frac{\sigma_{S_i}\sigma_{S_j}}{[\sigma_R^4 + \sigma_R^2\,[\sigma_{S_i}^2 + \sigma_{S_j}^2] + \sigma_{S_i}^2\sigma_{S_j}^2]^{0.5}} \qquad (4.21)$$

The correlation coefficients are then calculated from the element statistics in Fig. 4.1. The results are shown in Table 4.1. If each member is assumed to have a unit cross sectional area, the constants a_i are shown in Table 4.2. It is necessary to calculate an equivalent correlation coefficient (Ditlevsen (1984))

$$\rho = \frac{2}{n(n-1)} \sum_{i<j} \rho_{g_i g_j} = 0.120 \qquad (4.22)$$

The subject of an equivalent correlation coefficient was covered in the previous section. The element safety indecies are not expected to be equal so an iterative procedure suggested by Ditlevsen (1984) should have been used. However this is a cumbersome process requiring numerical integration for each iterative step. Because ρ is low and the system reliability is not affected even by a significant error in $\rho_{g_i g_j}$, a refinement in the calculation for ρ is not considered to be justified.

Finally the element safety indecies can be evaluated from the normal format (Thoft-Christensen and Baker (1982))

$$\beta_i = \frac{\mu_{R_i} - \mu_{S_i}}{\sqrt{\sigma_{R_i}^2 + \sigma_S^2}} \qquad (4.23)$$

The respective values are shown in Table 4.2. If the failure function had been non-normal, the method from chapter 3 would have to be employed in obtaining ρ_i.

Table 4.1 Correlation Coefficient for Truss Example

Elements	$\rho g_i g_j$
AB - BC	0.2887
AB - BD	0
AB - BE	0.2887
AB - CD	0.1581
AB - DE	0.2887
BC - BD	0
BC - BE	0.1667
BC - CD	0.0913
BC - BE	0
BD - CD	0
BD - DE	0
BE - CD	0.0913
CD - DE	0.0913
BC - DE	0.1667
BC - DE	0.1667

Table 4.2 Element Safety Indicies for Truss Example and the Constant a_i for a Unit Cross-Sectional Area

Element	β_i	p_i	a_i
AB	4.950	3.712.E-7	-2F
BC	5.237	8.177.E-8	$-\sqrt{2}$ F
BD	5.000	2.871.E-7	0
BE	2.928	1.706.E-3	$\sqrt{2}$ F
CD	3.578	1.732.E-4	F
DE	2.928	1.706.E-3	F

By numerical integration of the G-S equation system probability of failure is 2.05. E-3 (β_S=2.86). This is greater than the weakest element safety index as it should be.

4.4 Summary

The concept of an equivalent correlation coefficient has been summarized and reviewed. When a series system has unequally correlated elements this equivalent correlation coefficient can be used in connection with the G-S equation. According to Ditlevsen (1984) and Thoft-Christensen and Dalsgard Sorensen (1982) good estimates of p_s will result. From Monte Carlo simulation Thoft Christensen has observed that the use of an equivalent correlation coefficient gives results close to the upper bimodal bounds on p_s.

If the failure functions g_i are non normally distributed the method of chapter 3 can be used to find $\rho_{g_i g_j}=\rho_e$.

CHAPTER 5

FATIGUE IN SERIES SYSTEMS: THE CHARACTERISTIC S-N CURVE

5.1 Introduction

In Chapters 2, 3 and 4, methods for series system analysis was reviewed and developed. In this chapter, it is the goal to apply this theory to systems in which fatigue is the governing failure mode. The characteristic S-N curve approach is investigated. Other fatigue models like the fracture mechanics model can often be reduced to the S-N form as well. The elementary theory of fatigue is well documented in the literature (Collins (1981), Fuchs and Stephens (1980), Broek (1984) and Wirsching (1983)) and is therefore only summarized in this study. Examples are used to illustrate the use of the methods from Chapters 2 and 3.

The development of reliability models or failure functions at the component level is to a great extent based on Wirsching's earlier work. The reliability model in this case is based on a project sponsored by the American Petroleum Institute (API). This was a four year project and its primary goal was to investigate the fatigue design process in welded joints of steel offshore structures (Wirsching (1983, 1984)). This theory is quite general and can be applied to other areas of fatigue design as well.

5.2 Reasons for Reliability Analysis in Fatigue (Wirsching and Wu (1983))

Metal fatigue under stress cycling is a principal mode of failure in mechanical and structural components. But fatigue design factors are subject to considerable uncertainty. For example, enormous scatter is observed in cycles to failure fatigue test data with coefficients of variation ranging typically from 20 to 70%.

Furthermore, scatter exists in operating environment data, and uncertainties are present in the models used to predict stresses. Therefore life predictions, which rely on the fatigue models and the data for such models, are also subject to uncertainty. It is suggested that the appropriate mathematical model to describe fatigue design factors is a probabilistic one rather than a deterministic one. Uncertainty in environments and in fatigue resistance imply uncertainty in fatigue life predictions. This uncertainty can be analyzed rationally only using probability theory.

A reliability approach to fatigue using probabilistic design theory has at least the promise if not the guarantee of producing better engineered design, i.e., components which are more safe, reliable, and cost effective, relative to a deterministic approach. Typically conventional design procedures tend to be conservative and produce inconsistent levels of risk in components of a system. The pay off for an improved design criteria, components, would be a savings in cost, weight, and/or improvement in reliability.

5.3 The S-N Relationship

The theory in this section is based on fatigue analysis of welded joints in offshore structures. This is in part based on the API project mentioned above. The basic principles are general; extensions to other applications should follow without difficulty.

Fatigue strength of an element may be described by a S-N curve. A S-N curve is typically obtained from testing of several similar specimens or models. It is typical for a SN relation that, at first, the stress which can be applied repeatedly rapidly decreases as the numbers of cycles increases but the curve then flattens out and a comparatively small decrease in stress brings about a large increase in the number of cycles which can be withstood, see Fig. 5.1. For plain specimens of ferrous material, it has been found that an endurance limit exists at about 2-5 million cycles.

However, there is doubt as to whether an endurance limit exists in specimens containing stress concentrations. Time and costs generally prohibits testing beyond about $2 \cdot 10^6$ cycles and it may be only of academic interest as to whether a endurance limit exists at say 10^7 cycles since most structures has a finite design life.

There are two principal problems in the development of fatigue reliability formulas. The first is how to predict fatigue failures under such irregular stress histories as shown in Fig. 4.2. The method employed invokes Miner's linear damage accumulation rule to define fatigue damage. The second problem is that of specifying uncertainties

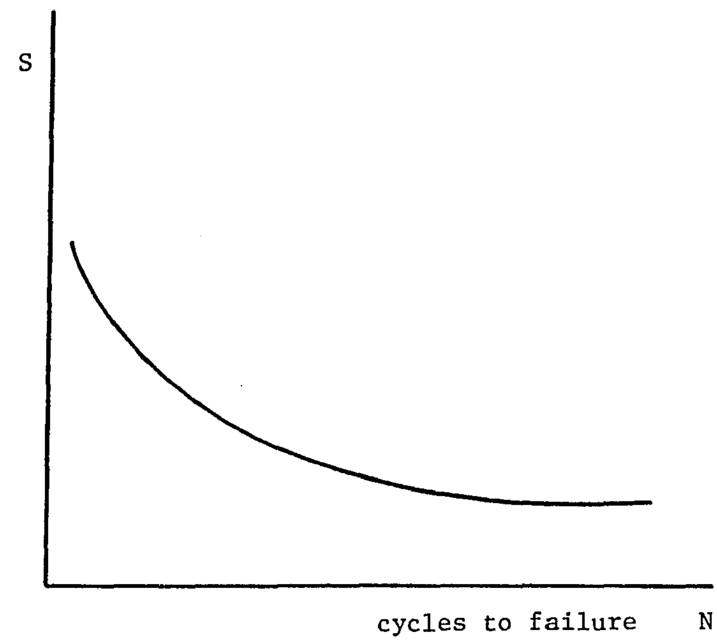


Fig. 5.1 Typical SN Curve (Stress-Life Relation) on a Linear Scale

associated with each fatigue design factor and then of performing reliability analyses.

A fundamental assumption is that the curve which characterizes fatigue behavior under constant amplitude loading is of the form

$$NS^m = K \tag{5.1}$$

m and K are empirical constants. Equation 5.1 defines the number of cycles to failure N as a function of stress range S. It is assumed that Eq. 5.1 applies for all S>0. Therefore no endurance limit exists in the model.

5.4 Miner's Rule to Define Damage

Assume that an irregular load history is applied as blocks of constant amplitude stress, i.e. stress level S_i applied for n_i cycles followed by stress level S_{i+1} applied for n_{i+1} cycles, etc. Miner's rule states that the fractional damage for the i^{th} block is n_i/N_i where N_i is the cycles to failure of the material at stress level S_i. Total damage D is the sum of fractional damages. Thus for k blocks of constant amplitude stress,

$$D = \sum_{i=1}^{k} n_i/N_i \tag{5.2}$$

Now consider an irregular stress process characterized by a sequence of stress ranges S_i; $i = 1, N_T$ each S_i in general of different magnitude; and N_T is the total number of cycles. Miner's rule states that the "damage" due to one cycle (the i^{th} cycle) is

$$D_i = 1/N(S_i) = S_i^m/K \tag{5.3}$$

The total damage is obtained by summing the damage of each stress cycle

$$D = (1/K) \sum_{i=1}^{N_T} S_i^m \tag{5.4}$$

For large N_T, the following approximation can be made

$$E(S^m) \cong \frac{1}{N_T} \sum_{i=1}^{N_T} S_i^m \tag{5.5}$$

$E(S^m)$ is the "expected", "mean", or "average" value of S^m, and S is now defined more generally as a random variable denoting fatigue stress range. Assuming equality, Eq. 5.4 becomes

$$D = (N_T/K)E(S^m) \tag{5.6}$$

Miner's rule is easy to apply in practice and is used in most design codes all over the world. But because a simple model is used to describe such a complicated phenomena, fatigue predictions by Miner's rule are often inaccurate. Descriptions of the performance of Miner's rule are provided in Wirsching and Light (1979) and Wirsching (1980).

It is at this point important to distinguish between actual stress range in the member, denoted by S_a, and the estimated stress range denoted by S. It is assumed that $S_a = B \cdot S$, in which B is a random variable that quantifies modeling error. B plays a key role in the equation for fatigue because (a) it may have relatively large variability and (b) its influence is magnified by the presence of m as an exponent. Uncertainties in stress arise from, manufacturing, assembly, load data, transfer-functions and so on.

If the average frequency of stress cycles is defined as,

$$f_o = \frac{N_T}{T} \tag{5.7}$$

the expression for fatigue damage can then be written as

$$D = \frac{TB^m\Omega}{K} \tag{5.8}$$

in which $\Omega = f_o E(S^m)$. Several methods are currently used to evaluate Ω and a summary of some of these can be found in Wirsching (1983 and 1984). The only method that will briefly be discussed here is the spectral method (Maddox and Wildenstein (1975), Wirsching (1983)).

It is assumed that Eq. 5.8 is valid for a non-stationary as well as a stationary process. For the special case when the process is stationary, gaussian and narrow band it can be shown that (Wirsching and Light (1979))

$$\Omega = (2\sqrt{2}\,\sigma)^m \; \Gamma\left(\frac{m}{2} + 1\right) \tag{5.9}$$

where σ is the root mean square value of the process and $\Gamma(\cdot)$ is the gamma function. This is the form when K is based on range. If K is based on amplitude, then the term contains and reads $(\sqrt{2}\;\sigma)^m$.

Fatigue stress cycles are easily identified in a narrow band process; an example of this is given in Fig. 5.2b. If the process is wide band, it is not immediately obvious how to measure fatigue stress cycles to be used with Miners rule. The rainflow algorithm (Dowling (1972)) is known to be the best method for counting of cycles in a wide band process. Wirsching and Light (1980) proposed that damage under a

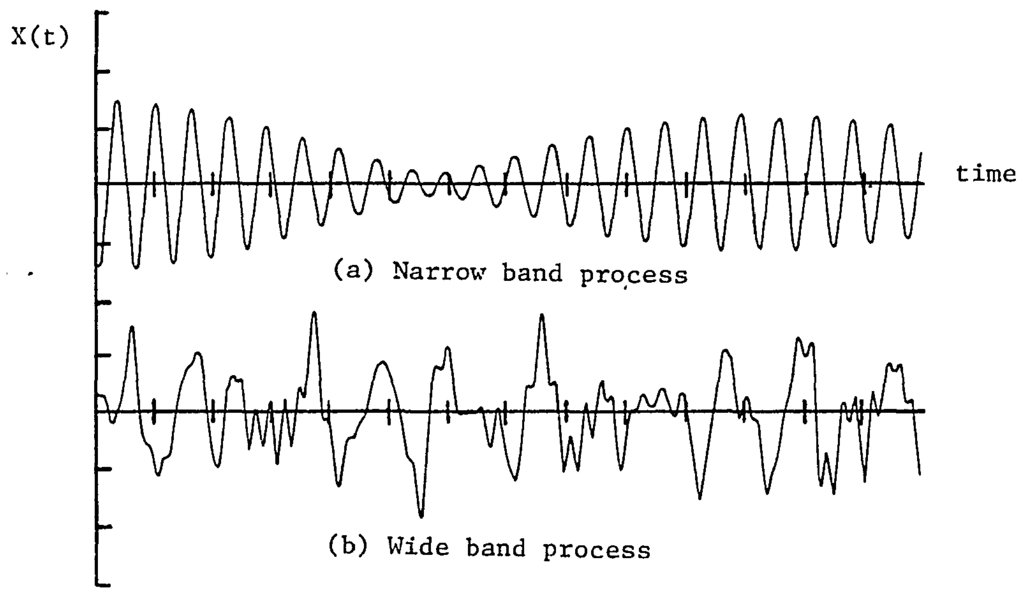


Fig. 5.2 Samples of Random Processes (Each Process has Same RMS and Expected Rate of Zero Crossings with Positive Slope) (Wirsching and Light (1980))

a wide band stationary process can be written as

$$D = \lambda D_{NB} \tag{5.10}$$

where D_{NB} denotes damage under a narrow band process and λ is a correction factor for an equivalent narrow band formula. They used simulation based on rainflow counting to obtain an empirical form for λ.

5.5 Reliability Analysis of a Component (Wirsching (1984))

Significant uncertainty exists in the factors of the fatigue damage expressions. Assuming that each uncertainty can be quantified, an expression for reliability can be derived.

Miner's rule states that failure under irregular stress ranges occurs when fatigue damage $D \geq 1$. But random fatigue experimental results have suggested that it is appropriate to describe fatigue failure more generally as $D \geq \Delta$, in which Δ is a random variable denoting damage at failure, which quantifies modeling error associated with Miner's rule. Uncertainties in fatigue strength, as evidenced by scatter in S-N data, are accounted for by considering K to be a random variable (with m = constant). Inaccuracies in fatigue stress estimation are described by the random variable B.

Let T denote time to fatigue failure. Letting $D = \Delta$, the basic damage expression of Eq. 5.8 can be expressed in terms of time to failure,

$$T = \frac{\Delta K}{(B^m \Omega)} \tag{5.11}$$

Because Δ, K, and B are random variables, T also is a random variable. Define the intended service life of the structure as T_s. Then the probability of fatigue failure of a joint is

$$p_f = P(T \leq T_s) \tag{5.12}$$

It may in many cases be reasonable to assume that Δ, K and B are all lognormally distributed. If this is the case, the lognormal format can be used, and an exact expression for probability of failure results. A detailed summary of the lognormal distribution is provided in Wirsching (1983). Employing mathematical properties of lognormal variables, an expression for p_f can be derived as

$$p_f = \Phi(-\beta)$$

in which $\Phi(\cdot)$ = the standard normal distribution function; and β is defined as the safety index.

$$\beta = \frac{\ell n(\frac{\tilde{T}}{T_s})}{\sigma_{\ell nT}} \tag{5.13}$$

$\tilde{T}$ = the median value of T and is equal to

$$\tilde{T} = \frac{\tilde{\Delta}\tilde{K}}{(B^m\Omega)} \tag{5.14}$$

in which the tildes indicate median values. Also, the standard deviation of ln T is

$$\sigma_{\ell nT} = [\ell n(1+C_\Delta^2)(1+C_K^2)(1+C_B^2)^{m^2}]^{1/2} \tag{5.15}$$

in which the C's denote the coefficients of variation, COV, of each variable. It should be emphasized that, because of the assumptions made, computed values of p_f do not necessarily provide actuarial estimates. Computed values of p_f, often called "notional" probabilities of failure, are useful in providing relative measures of safety.

The reasons for using the lognormal format are (1) A relatively simple closed form and exact expression for p_f results; (2) it has been demonstrated that the lognormal is a valid model for a wide variety of structural design variables; and (3) specifically, the lognormal has been shown to provide a good fit to data on both Δ and K.

5.6 Example of Series System Reliability Analysis

In this section the methods from Chapter 2 will be used to demonstrate system reliability analysis. It will in particular be shown how the approximate correction factor λ, which accounts for load dependency in a chain like system, can be obtained from the derived charts at the end of Chapter 2. The goal of the example is to find the component risk level so that a given system target risk of p_{os} = 1.35 E-3 is met.

An example of a series fatigue problem is a tendon of a tension leg drilling and production platform. The tendon can be considered to be a chain of n identical elements. Oscillatory stresses produced from wave loading are superimposed on a mean stress. It is assumed that the joints are welded and that in the first approximation, mean stress will be ignored. In this example, there are n=10 joints. Following the reliability format in Section 5.5 and if $n=f_oT$ and $\Omega=f_oE(S^m)$ Eq. 5.11 can be written as,

$$N_i = \Delta K_i / B_i^m S_e^m \tag{5.16}$$

N_i is cycles to failure and S_e is Miner's stress equal to $[E(S^m)]^{1/m}$, and assumed to be constant.

If the design life is given as N_o, the event of component failure is $(N_i<N_o)$ and the probability of failure is

$$P_i = P(N_i<N_o) \tag{5.17}$$

Substituting Eq. 5.16 and rearranging terms,

$$P_i = P(K_i < \frac{N_o B^m S_e^m}{\Delta}) \tag{5.18}$$

The term on the left side of the inequality is the "strength" term having a different value for each member. The term on the right is the "stress" term. Each stress variable relates to the system as a whole; failure occurs if this term gets too large. It is assumed that K, B, and Δ have lognormal distributions so that both stress and strength terms will also be lognormal.

In this example it is given that C_{K_I} (and thus C_{R_I}) = 0.50. It is also given that C_S = 0.70. Just for reference, if values C_B, C_Δ, and m are known, C_S can be computed as (Wirsching (1983))

$$C_S = [(1+C_\Delta^2)(1+C_B^2)^{m^2}-1]^{1/2} \tag{5.19}$$

Then it can be shown that the component safety index is,

$$\beta_i = \frac{\ell n(\tilde{\Delta}\tilde{K}/B^m \tilde{S}_e^m N_o}{\sigma} \tag{5.20}$$

where

$$\sigma = \{\ell n[(1+C_\Delta^2)(1+C_K^2)(1+C_B^2)^{m^2}]\}^{0.5} \tag{5.21}$$

"Safety" is achieved when $\beta > \beta_{o1}$ where β_{o1} is the component target safety index. It is the goal of this exercise to demonstrate how to compute β_{o1} given the target β_{oS} for the system.

The design requirement for a tendon is specified in terms of a target safety index $\beta_{oS} = 3.00$. The goal of this analysis is to specify a target safety index β_{o1} for a single component, i.e., a welded joint. This design requirement is computed using the following steps.

1. A $\beta = 3.00$ translates into a notional probability of system failure by

$$p_{oS} = \Phi(-\beta) = 1.35\ E\text{-}3 \tag{5.22}$$

where Φ is the standard normal distribution function. The problem is to relate p_{oS} to the notional component probability of failure.

2. To find $1-\lambda$ for $C_S = 0.70$, extrapolation of Figs. 2.8 through 2.13 is required. This is done in Fig. 5.3 for both $p_1 = 1\cdot E\text{-}4$ and $1\cdot E\text{-}5$. The solid points shown are obtained directly from Figs. 2.8 through 2.13. Extrapolated curves shown produce $\lambda = 0.72$ for $1\cdot E\text{-}4$ and 0.88 for $1\cdot E\text{-}5$.

3. Using $p_S = \lambda n p_1$, values of p_S are computed as $p_S = 7.2\cdot E\text{-}4$ corresponding to $p_1 = 1\cdot E\text{-}4$ and $p_S = 8.8\ E\text{-}5$ corresponding to $p_1 = E\text{-}5$. These two points are plotted as the solid points on Fig. 5.4.

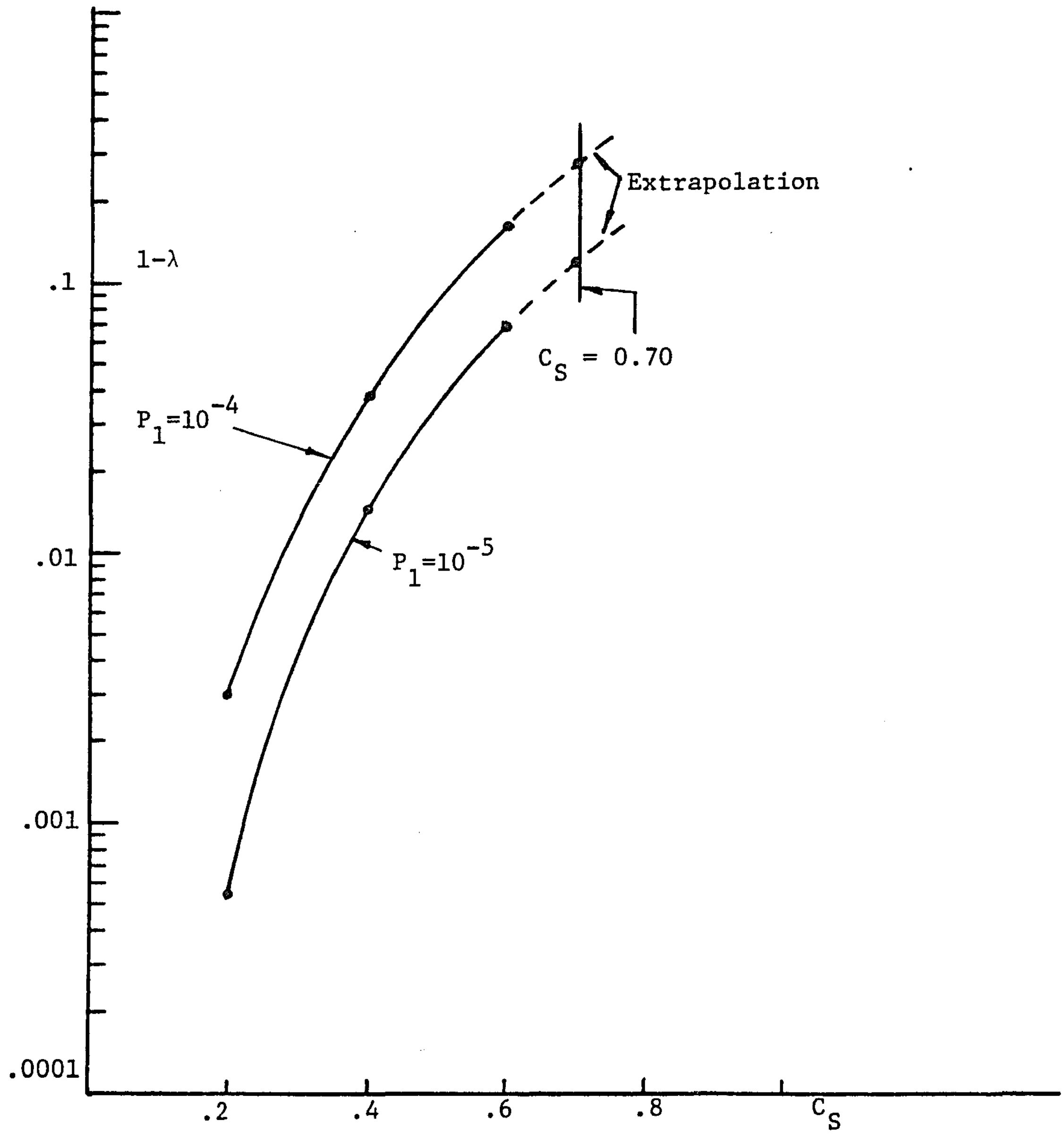


Fig. 5.3 1-λ versus C_S for R_i Lognormal and C_{Ri}=0.50

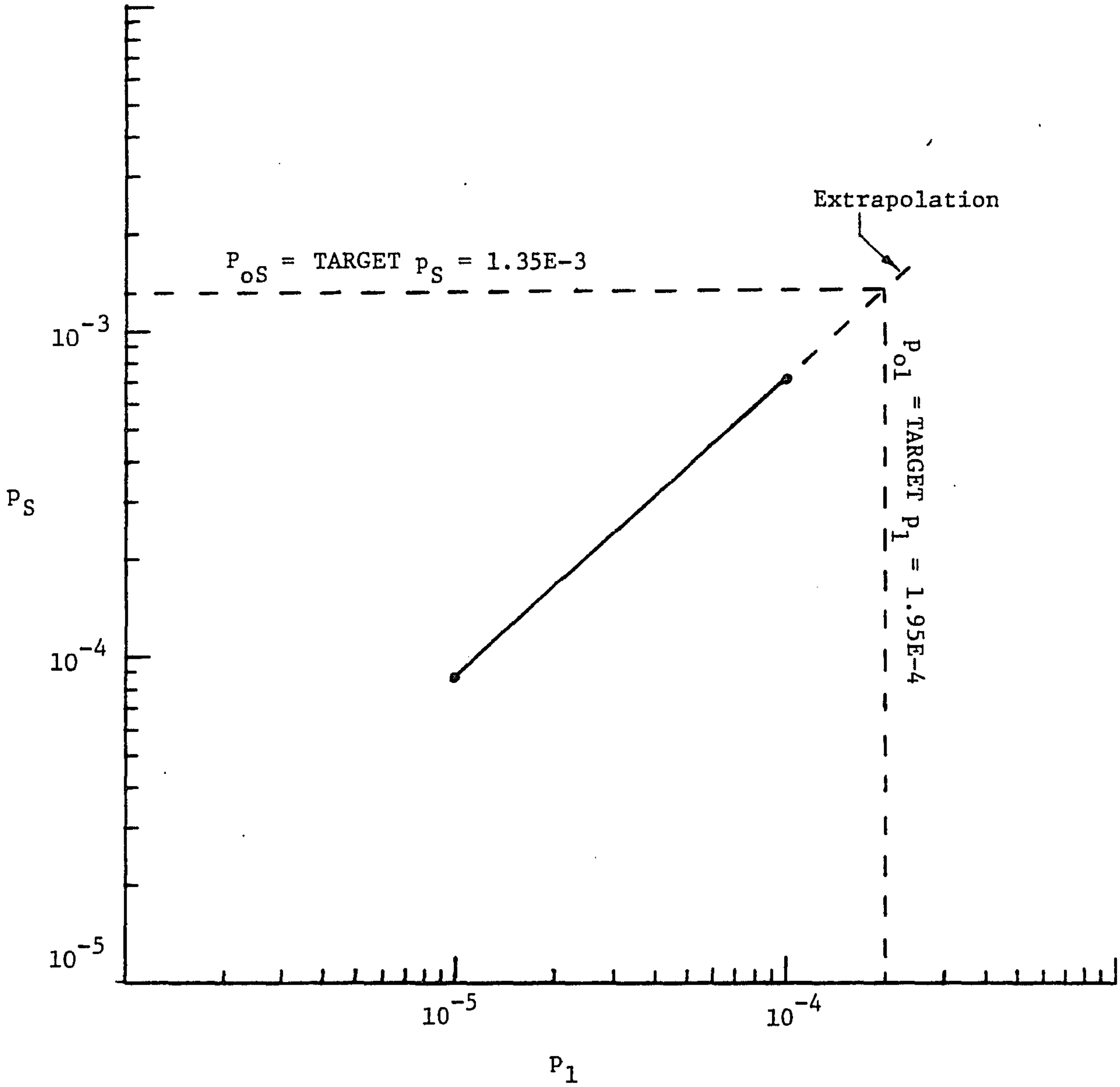


Fig. 5.4 p_S versus p_1 for Example

4. It is assumed that the p_S-p_l relationship is linear. This would be the case if $\lambda=1$, but it is not expected that the picture is significantly distorted by the λ's of this problem. The relationship between p_S and p_l is therefore established, and an extrapolation provides a target component risk of $p_{ol}=1.95$ E-4 corresponding to $p_{oS}=1.35$ E-3.

5. The target safety index is computed as

$$\beta_{ol} = \Phi^{-1}(1-p_{ol}) = 3.55 \tag{5.23}$$

It is interesting to note that using the approximate form, $p_{oS}=1-(1-p_{ol})^n$ leads to a value of $\beta_{ol}=3.64$. Use of this approximate form implies a weight penalty of about 9.5% for a tension member.

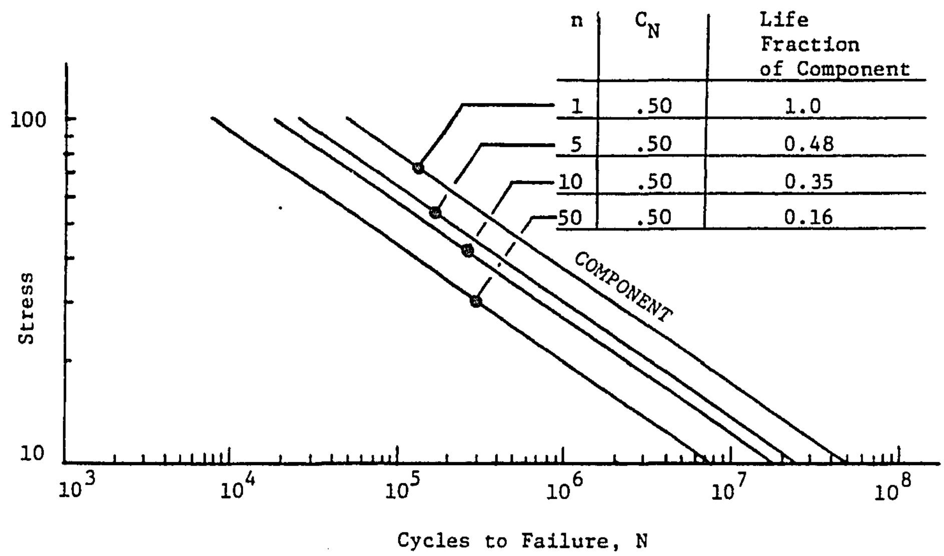
5.7 Comparison of "System" SN Curves: An Example

It is of interest to consider fatigue strengths of a series system of identical components for which the component cycles to failure distribution is either Weibull or lognormal. Consider as an example, component fatigue strength defined by the S-N curve $NS^m=K$ where m and K are empirical constants. Assume that m=3.0, and the component coefficient of variation of cycles to failure is, $C_N=0.50$. These values are typical of welded joint fatigue. In this context C_N is now interpreted as C_{Ri}.

The system median S-N curves are shown in Fig. 5.5. The curves are derived from the distribution of minimum, Bury (1976) and Appendix B.

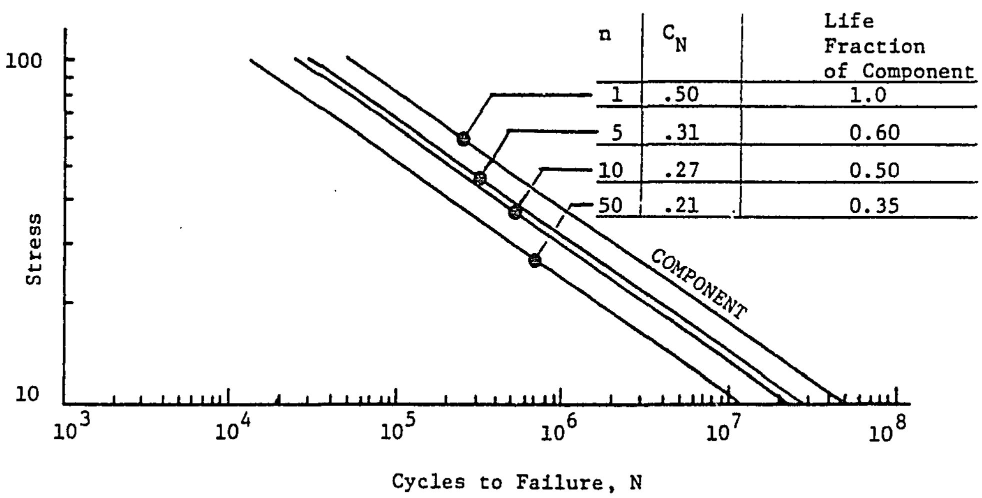
Clearly, computed system performance depends strongly on the model used for initial distribution. This difference in system behavior

(a) Component has <u>Weibull</u> cycles to failure distribution



n	C_N	Life Fraction of Component
1	.50	1.0
5	.50	0.48
10	.50	0.35
50	.50	0.16

(b) Component has <u>lognormal</u> cycles to failure distribution



n	C_N	Life Fraction of Component
1	.50	1.0
5	.31	0.60
10	.27	0.50
50	.21	0.35

Fig. 5.5 Examples of Median S-N Curves for Series Systems

can in part be explained by comparing the system coefficient of variation, COV, for the Weibull and lognormal as initial distributions. From extreme value theory it is straight forward to estimate C_N (Bury (1976)). A dramatic difference is demonstrated in Fig. 5.6. Both the Weibull and lognormal have been proposed for cycles to failure. It is out of the scope of this study to address the issue of statistical modeling, however, a comparison test by Littleton (1984) is summarized in Appendix B. In another test (Wirsching (1983)), the lognormal was found to be a landslide winner over the Weibull as a preferred model for cycles to failure of welded tubular joint fatigue data. In another extensive study of welded joints by Engesvik (1981) the lognormal was the best fit distribution for cycles to failure.

5.8 System Analysis by the G-S Equation

In Chapter 3 a new and alternate method for series system analysis was investigated. The method is exact for linear, equally correlated failure functions with normally distributed basic variables. It was shown that good approximations could often be made if one or all of the above assumptions were violated.

If the element life is based on Eq. 5.16 and this equation is linearized by taking the logarithm of both sides, the correlation coefficient becomes, (see sec. 3.1 and 3.2)

$$\rho_{\ell nN_i \ell nN_j} = \frac{m^2\sigma^2_{\ell n\beta}+\sigma^2_{\ell n\Delta}+\rho_{\ell nK_i \ell nK_j}\sigma^2_{\ell nK}}{m^2\sigma^2_{\ell n\beta}+\sigma^2_{\ell n\Delta}+\sigma^2_{\ell nK}} \tag{5.24}$$

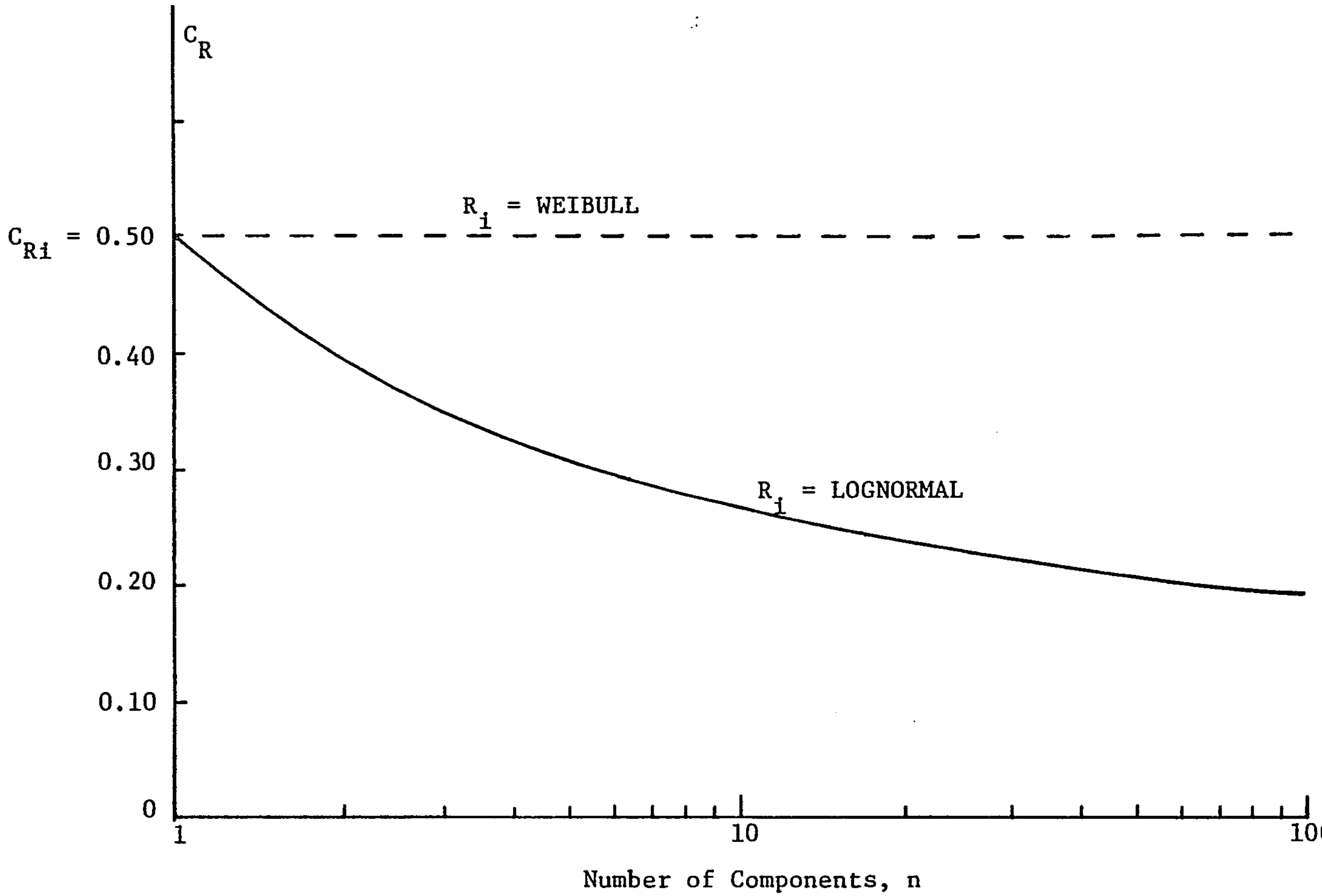


Fig. 5.6 System Strength Coefficient of Variation for Component Weibull and Lognormal Distributions

$\rho_{\ell nN_i \ell nN_j}$ is the correlation coefficient between the element "strengths". In design this is almost always assumed to be zero (see discussion in sec 5.10). In Eq. 5.23 it is assumed that the mean and the standard deviation of the basic variables are identical for all the elements. If all the basic variables are lognormally distributed Eq. 5.23 becomes

$$\rho_{\ell nN_i \ell nN_j} = \frac{\ell n[1+C_B^2]^{m^2}[1+C_\Delta^2][1+C_K^2]^{\rho_{\ell nK_i \ell nK_j}}}{\ell n[1+C_B^2]^{m^2}[1+C_\Delta^2][1+C_K^2]} \qquad (5.25)$$

it should be recognized that $\ln N_i$ is normally distributed; exact system reliability follows from the G-S Equation and 5.23.

Applications of Eqs. 5.23 and 5.24 to system analysis by the G-S equation is trivial and will not be covered in any further detail. However, it is of some interest to observe from Eq. 5.24 that C_B, the COV of stress uncertainty can dominate $\rho_{\ell nN_i \ell nN_j}$. The companion question, "how large would the correlation coefficient be in practice?"

Example: Calculations of System Reliabilities

The goal of this example is to demonstrate how the concepts of an "equivalent" correlation coefficient can be used in estimates of the system probability of failure. A 50 component chain-like series system is to be investigated. All the components are identically distributed, the design variables with statistics and distributions are given in Table 5.2. In particular note that K_i and K_j are correlated.

From Eq. 5.18 the event of component failure is given by,

$$P_1 = P(K_1 < \frac{N_o B^m S_e^m}{\Delta}) \qquad (5.18)$$

Then by the use of the R-F algorithm and the information given in Table 5.1 the element safety index is found to be; β_1=3.3582 or p_1=3.9235 E-04. From the R-F algorithm the reduced variables are found to be,

$$K^* = -1.94607$$

$$B^* = 2.53125$$

$$\Delta^* = -1.04043$$

Then from the methods of chapter 3 the system probability can be obtained by the G-S Equation. The equivalent correlation coefficient ρ_e is obtained from Eq. 3.37

$$\rho_e = \frac{B^{*2} + \Delta^{*2} + \rho^2_{K_iK_j}}{\beta^2} \qquad (3.37)$$

or

$$\rho_e = \frac{(2.53125)^2 + (1.04043)^2 + 0.40^2}{(3.35817)^2} = 0.6996$$

Then after substitution of β_1 and ρ_e into the G-S Equation the system probability of failure is found to be; p_s=7.8046 E-3. If the "strength" correlation $\rho_{K_iK_j}$ = 0.40 had been assumed to be zero, p_s = 8.6962 E-3 or an 11.4% difference. Furthermore the simple upper and lower bounds on p_s can be found by assuming independent on perfectly correlated failure events respectively;

$$p_{s\ upper} = 1-(1-p_1)^{50} = 1.943\ E-2 \qquad (5.26)$$

Table 5.1 Design Variables for Calculations of Component and System Probabilities of Failure

Variable	Mean/Median	Coefficient of Variation	Distribution
B	0.80	0.40	Weibull
Δ	1.0	0.20	Normal
K	5.25•E 10	0.70	Lognormal
m	3.0	-	Deterministic
$\rho_{K_iK_j}$	0.40	-	Deterministic
S_e	9 Ksi	-	Deterministic
N_o	5•E 6 cycles	-	Deterministic

$$p_{s\ lower} = p_1 = 3.924\ E\text{-}4 \tag{5.27}$$

or

$$p_{s\ lower} = 3.924\ E\text{-}4 < p_s = 7.805\ E\text{-}4 < p_{s\ upper} = 1.943\ E\text{-}3 \tag{5.28}$$

It is seen that a significant error result if either independent or perfectly correlated failure events are assumed.

5.9 Solution of a Design Problem by the G-S Equation

Example:

The goal of this example is to solve the problem from section 5.6 by use of the G-S equation. In section 5.6 the methods of Chapter 2 was used. The goal of the example is to find the component risk level so that a given system target risk of p_{oS}=1.35 E-3 (or β_{oS}=3.0) is met. A 10 element chain like series system was investigated. The element failure functions were defined by Eq. 5.16 and 5.17. It was shown that the element probability of failure is,

$$p_1 = P(K_1 < \frac{N_o B^m S_e^m}{\Delta}) \tag{5.18}$$

All the terms in Eq. 5.18 is defined in sections 5.4 through 5.6. In section 5.6 C_s=0.70 and C_K=0.50 then from Eqs. 3.15, 5.19, 5.21 and 5.23 $\rho_{\ln N_i \ln N_j}$ = 0.6412. The G-S equation is then solved for β_1=3.2; 3.5 and 3.8. In Fig. 5.6 the system probability of failure is then plotted

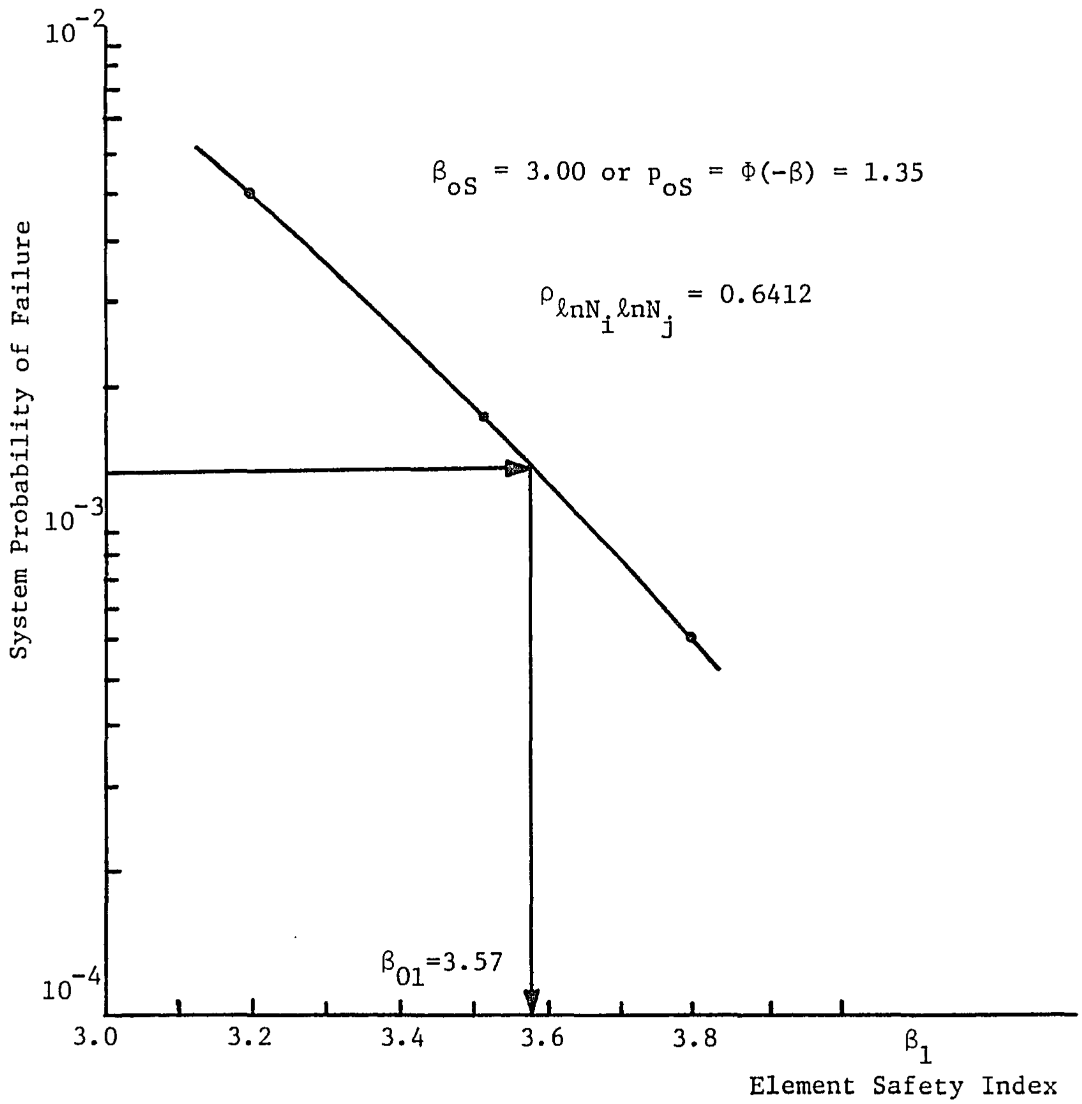


Fig. 5.7 Element Reliability Index β_1 vs. System Probability of Failure

against β_1 and for p_{OS}=1.35 E-3 β_{o1} is found to be 3.57. This is close to 3.55 which is the result from section 5.6. Note the solution by the G-S equation is considerably easier.

5.10 The Importance of Correlated Strength

Example:

Several investigators have in the past few years been concerned with the problem of correlated strength, (Grigoriu and Turkstra (1978), Thoft-Christensen and Dalsgard Sorensen (1982), and Garson (1980)). It has been suggested that the degree of correlation between member resistances is an essential parameter of safety, except for some series systems which might have a few overdesigned components.

Correlation between element strengths defined by ρ_{RiRj} may occur when the material comes from the same plant and the same sample. It would be difficult (and expensive) to determine the value of the strength correlation coefficients in most applications. However, it is of interest to see its relative importance, because if correlated strengths significantly alters, p_S, it may at least serve as a "hidden" factor of safety in the case of a series system.

The importance of correlated strengths in the SN curve equation may be determined from Eqs. 5.24 and the G-S equation. In Fig. 5.7 one illustration of this is made. It is seen that when $\rho_{\ell nK_i \ell nK_j}=1$. The system probability of failure equals the component probability of failure. It is clear that even a low degree of strength dependency may significantly influence the strength dependency. From Chapter 3, Figs.

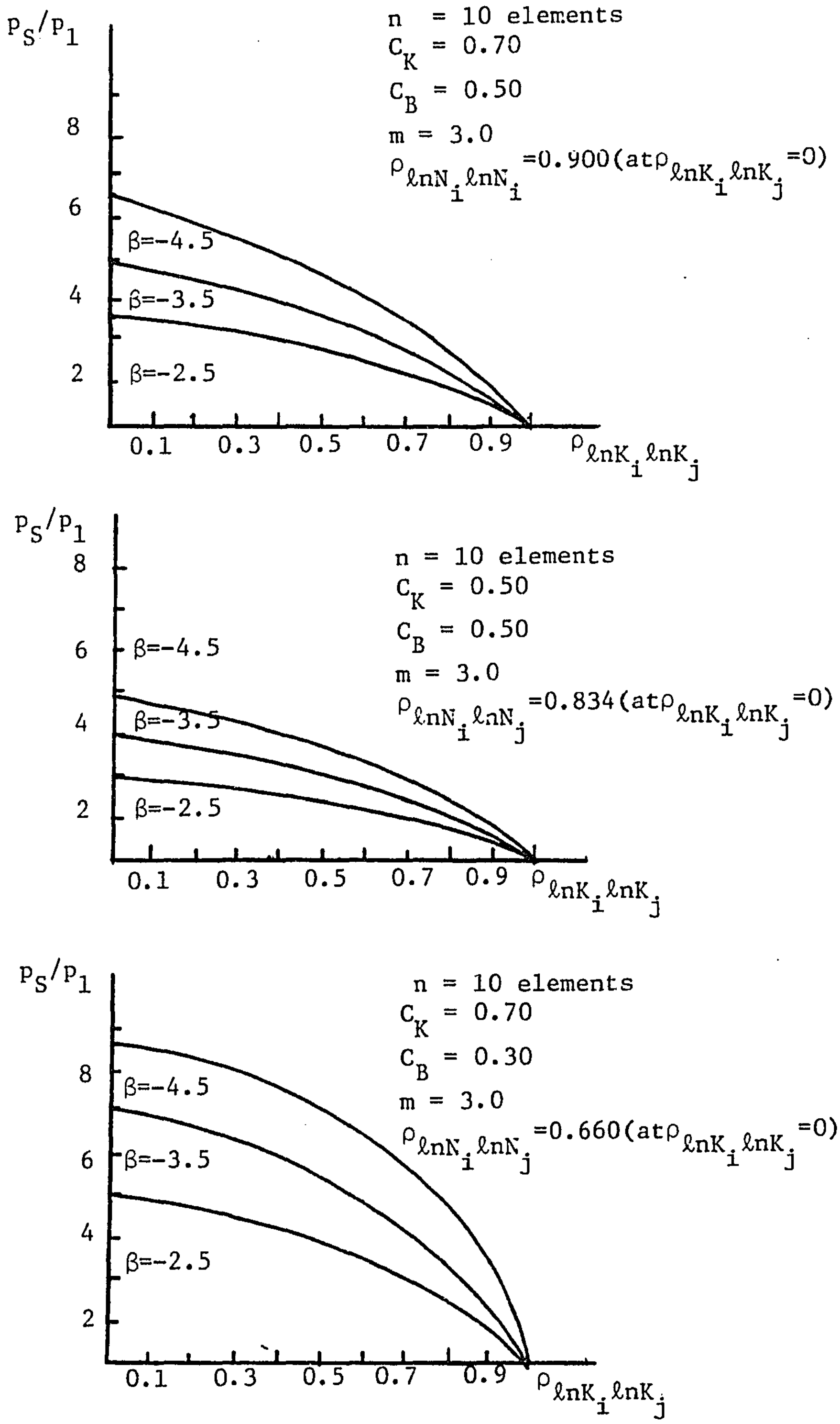


Fig. 5.8 Ratio of System to Element Probability of Failure vs. the Degree of Strength Correlation (C_Δ=0 in all cases)

3.3 and 3.4 it can be concluded that even small degrees of strength correlation may be significantly when, a) $\rho_{\ell nK_i \ell nK_j}$ approaches 1, b) the element reliability is high, say β_1 larger than 4.0, and c) as the system size increases.

5.11 Summary

Basic principles concerning the characteristic SN curve approach to fatigue has been outlined. An element reliability format for the SN curve equation has also been summarized and discussed. It was demonstrated that the underlaying distribution for cycles to failure is of importance and that significant differences in the design may result, depending on which distribution is used for cycles to failure. Examples were used to demonstrate how system reliability analysis may be carried out when the characteristic SN curve is the element failure mode. Finally it was demonstrated that correlated element strengths may significantly influence the system performance. This may even be true when the degree of correlation is small.

CHAPTER 6

SUMMARY AND CONCLUSIONS

Reliability analysis of series structural systems with emphasis on problems typical for mental fatigue was addressed. Specific goals included the following: (1) Given the distribution of strength of the components and distribution of external loads on the system what is the probability of failure of the system? (2) Given the target safety index for the system, what would be the target safety index for the components.

Exact solutions in the analysis of series structural systems only exists for some special problems. Some of these special problems were investigated. In particular some special cases of the problem of unequal element reliabilities were considered. It was concluded that for systems with a linear variation in element reliabilities even a small gradient significally influenced the system performance. From examples it was shown that (1) The weakest element will in general not dominate the system performance. Which implies that the reliability of all the components must be included in analysis of the system reliability. (2) Even a small gradient in element reliabilities may significally influence the system reliability. For a system where the weight of the system itself may cause a load gradient in the longitudinal direction this means that a nonconservative design may result if this is neglected.

Numerical integration is in general required even when an exact solution exists. A correction or adjustment factor is developed for an important class of problems. This factor makes it possible to relate element and system probabilities of failure without numerical integration. This factor was shown to be a function of system size n, element probability of failure p_i, the coefficient of variation of element strength, C_{R_i}, and element stress C_S, and finally the probability distribution of strength and stress. It was in particular of interest to observe that this factor strongly depended upon the distributions of strength. The two common fatigue strength distributions Weibull and the lognormal was used. It was demonstrated that the issue of distributional choice is indeed important and future research on this topic is necessary.

The main goal of this study was to develop a method for analysis of series systems when no exact solution exists. Chapter 3 is devoted to this problem and an approximation scheme based on a special form of the multinormal integral was developed. This method requires an approximation of the equivalent normal fractile or more commonly known as the safety index for each element. The method also requires estimates of the correlation coefficient between element failure functions. The equation which is used for estimates of the system probability of failure is termed the G-S equation after Gupta (1959) and Stuart (1964) which developed it. The G-S equation yields exact results when the element failure functions are normally distributed with equally correlated elements. When these requirements are violated estimates of the safety index and the correlation coefficient is necessary. There are several possible methods which may be used in the estimates of the safety index. Some of these are the Rackwitz-Fiessler algorithm or a more advanced fast probability integration method such as the Wu-algorithm (Wu (1984)). Other reasonable methods includes for instance Monte Carlo simulation even though that should be used as a last resort. For estimates of the correlation coefficient a new method was developed. This method is based on equivalent normal variates and the safety index, excellent estimates of the equivalent correlation coefficient results. It is well known that element probabilities of failure is sensitive to uncertainties in the safety index. It is therefore necessary to make sure that good estimates is obtained. However it was encouraging to notice that an error in the element safety index does not magnify with the size of the system.

Numerous examples are used to verify the method and the results are shown to be excellent. Systems with very complicated element failure functions can now be solved. This is a major breakthrough in analysis of series systems. It is in particular interesting to notice that good estimates of the system probability of failure can be obtained when the element failure functions are highly correlated. This is encouraging since ven narrow bimodal bounds break down in this region and highly correlated failure functions are common in for instance fatigue analysis.

Chapter 4 is devoted to the case when one of the underlaying assumptions for the use of the G-S equation is violated. That is the case of unequally correlated failure functions. Ditlevsen (1984) has developed the concept of an equivalent correlation coefficient. This method is summarized and discussed in this chapter.

Chapter 5 is devoted to applications of fatigue. Several examples are used to illustrate basic and advanced principles. Special problems such as the development of "system" SN curves was addressed. It was again shown that the problem of distributional fill is important. Another problem of interest is the case when the element strengths are correalted. It is shown that when the element failure function are highly correlated even low degrees of correlated strengths may significantly influence the system probability of failure. This is an other area where future research is needed.

Summarized in appendix A is a study of the statistical distribution of the minimum of a sample size n independent and identically distributed lognormal variates.

The goal of the study was to, (a) examine the properties of the statistical distribution of the minimum of a random sample size n taken from a lognormal distribution, and (b) quantify the differences between the exact distribution of minima and the asymptotic distributions of minima which may be considered as approximate for large n. It can be shown (Bury (1975)) that the Weibull is the limiting distribution of minima as the same size $n \to \infty$. This distribution is also sometimes referred to as the type III asymptotic distribution of minima.

It was shown that tail area properties in general has slow convergence and the approximation form of the distribution of minima should be used with caution. On the other hand reasonable approximations can be for median and mean values this is in particular true for low coefficients of variations.

Summarized in appendix B is a comparison study of distributions for cycles to failure. For high cycle fatigue it was shown that for the cases investigated the lognormal distribution generally provided a better fit to the data than the Weibull distribution.

APPENDIX A

THE DISTRIBUTION OF MINIMA OF A SAMPLE OF LOGNORMAL VARIATES

A.1 Introduction

Summarized in this appendix is a study of the statistical distribution of the minimum of a sample of size n independent and identically distributed lognormal variates. This work was performed by J. J. Paul Gagne (1983). The author served as an "informal advisor" because it was an important component of this dissertation.

The goal of the study was to, (a) examine the properties of the statistical distribution of the minimum of a random sample of size n taken from a lognormal distribution, and (b) quantify the differences between the exact distributions of minima and the asymptotic distributions of minima which may be considered as approximate forlarge n. It can be shown (Bury (1975)) that the Weibull is the limiting distribution of minima as the sample size $n \to \infty$. This distribution is also sometimes referred to as the type III asymptotic distribution of minima.

The exact distribution of minima is usually difficult to work. Thus the approximate asymptotic distribution may be preferred provided the sample is large enough or some correction factor for small n can be obtained. Gagne's study presents a method where the Weibull approximation of the minimum of a sample from the lognormal distribution can be used and adjusted to obtain the exact distribution properties.

These properties are the median, mean, standard deviation, coefficient of variation, and selected left tail probabilities and quantiles.

When the Weibull is the initial distribution the type III distribution of minima is exact. Thus the Weibull is reproductive with respect to its own minimum.

A.2 A Brief Summary of Theory

In engineering applications it is a common problem to be concerned about the largest or smallest in a sample. For example, when the operating lifetime of a system component is investigated, n specimens may be put on a test: the shortest lived component will give rise to the first measurement $X_{(1)}$, the next longer lived component will be measured next $x_{(2)}$, and so forth. Such an "ordered sample" satisfies the inequalities,

$$X_{(1)} \leq X_{(2)} \leq \cdot \cdot \cdot \cdot \leq X_{(r)} \leq \cdot \cdot \cdot \cdot \leq X_{(n)} \qquad (A.1)$$

The order statistic $x_{(r)}$ is clearly a random variable associated with a statistical distribution.

Application of order statistics to engineering problems are numerous. For example, in this study the weakest link will determine the fatigue life of a series structure.

From elementary probability considerations it is straightforward to show that the distribution function of the smallest of a sample is given by, (Bury (1984)),

$$F_{X(1)}(\chi) = 1 - \prod_{i=1}^{n} [1-F_X(\chi)] \tag{A.2}$$

or for identical initial distributions,

$$F_{X(1)}(\chi) = 1-[1-F_X(\chi)]^n \tag{A.3}$$

From the derivative of the distribution function the probability density function is,

$$f_{X(1)}(\chi) = n\ [1-F_X(\chi)]^{n-1} f_X(\chi) \tag{A.4}$$

The r^{th} moment can be obtained as follows,

$$E[X^r_{(1)}] = \int_{\text{all } x} X^r f_{X(1)}(\chi)\, d\chi \quad \text{where } r = 1, 2, \ldots \tag{A.5}$$

The variance of $x_{(1)}$ is,

$$\sigma^2_{X(1)} = V(X_{(1)}) = E[X^2_{(1)}]-E[X_{(1)}]^2 \tag{A.6}$$

where $E[x_{(1)}]$ and $E[x_{(1)}]$ are the mean and the mean square values of $x_{(1)}$ respectively. An other important measure of dispersion of $x_{(1)}$ is its coefficient of variation,

$$C_{X(1)} = \frac{\sigma_{X(1)}}{\mu_{X(1)}} \tag{A.7}$$

Finally the median value of the minimum is,

$$\tilde{X}_{(1)} = F_X^{-1}[1-(0.5)^{1/n}] \tag{A.8}$$

Note that all these functions and characteristics of $x_{(1)}$ are all dependent on the initial distribution and the sample size n.

A.3 Relation Between the Initial Distribution and Approximate Distribution of Minima

The Weibull distribution or the type III asymptotic distribution of the smallest extreme values is given by the following distribution function,

$$F_W(X) = 1\text{-}\exp[-(\frac{X}{\beta})^{\alpha}]$$

where σ and β are the shape and scale parameters respectively. This distribution is the asymptotic distribution of the smallest extreme value of a sample taken from an initial distribution bounded to the left.

The Weibull moments are defined as,

$$\mu_X = E(X) = \beta\, \Gamma(1 + \frac{1}{\alpha}) \tag{A.10}$$

$$\sigma_X^2 = V(X) = \beta^2[\Gamma(1 + \frac{2}{\alpha}) - \Gamma^2(1 + \frac{1}{\alpha})] \tag{A.11}$$

$$C_X = \frac{\sigma_X}{E(X)} = \left[\frac{\Gamma(1 + \frac{2}{\alpha})}{\Gamma^2(1 + \frac{1}{\alpha})} - 1\right]^{1/2} \tag{A.12}$$

where $\Gamma(\cdot)$ is the gamma function.

When the Weibull distribution is used as an asymptotic distribution of minima its parameters must be approximated from the parameters of the initial distribution (Bury (1984), Cramer (1951)). With th lognormal distribution as the initial distribution these relations are,

$$\alpha = \frac{H}{\sigma_Y} \tag{A.13}$$

$$\beta = \exp[\mu_Y - G\ \sigma_Y] \tag{A.14}$$

where

$$H = \sqrt{2 \ln n} \tag{A.15}$$

$$G = \frac{2 \ln n - (0.50) \ln(4\pi \ln n}{H} \tag{A.16}$$

and σ_y and μ_y are the standard normal parameters.

In the following sections it will be shown that the convergence between the exact and the approximate distributions of minima is slow. Hence if the Weibull approximation is to be used correction factors must be applied.

A.4 Outline of the General Procedure

The general goal was to study the properties of the distribution of the minimum of a sample of size n from a lognormal distribution. A specific goal was to quantify the differences between the exact distribution of minima and the approximating Weibull asymptotic distribution. The basic approach to achieving these goals is outlined as follows:

(1) Select a set of different lognormal distribution parameters in combination with different sample sizes taken from these distributions. Values of $\tilde{X} = 1$, and $C_X = 0.10$ to 1.00 were chosen.

(2) The sample sizes n = 2, 5, 10, 20, 50, 500, 1,000, 5,000 were chosen.

(3) For a given initial distribution and sample size, (defined by n and C_X) calculate both the exact and asymptotic distributions of the smallest extreme and its moments using the Table 1 formulas.

(4) Plot the asymptotic and exact CDFs of the minimum value on Weibull probability paper to provide a visual comparison.

(5) Quantify the relative difference between the exact and asymptotic results for the mean, median, coefficient of variation, and selected probabilities of $X_{(1)}$, for the matrix of initial distribution parameters and sample sizes.

These steps will be discussed in detail in the rest of this chapter.

A.5 Initial Distribution Parameters/Sample Sizes

The initial lognormal distributions are specified by their coefficient of variation, C_X. As was mentioned before, without loss of generality, all initial distribution models are the special case of the reduced lognormal where $\tilde{X} = 1$, thus $\mu_y = 0$. The initial median, $\tilde{X}$, is directly proportional to the minimum value quantiles, moments, and the Weibull scale parameters, β.

The selected C_X are 0.1, 0.2, 0.3, 0.5, 0.7 and 1.0. These values are representative of what is commonly encountered in design. The sample sizes taken from each initial distribution are 2, 5, 10, 20, 50, 100, 500, 1000 and 5000. These magnitudes of n were selected to observe

Table A.1 Summary of Exact and Asymptotic (Weibull) Formulas for Distribution of $X_{(1)}$

Property of $X_{(1)}$	Exact Distribution	Approximating Asymptotic Distribution
C.D.F.	$F_{X(1)}(x) = 1 - \left[1 - \Phi\left(\frac{\ln x - \mu_Y}{\sigma_Y}\right)\right]^n$ where Φ = standard normal CDF μ_Y = initial lognormal scale parameter σ_Y = initial lognormal shape parameter	$F_W(x) = 1 - \exp\left[-(\frac{x}{\beta})^\alpha\right]$ where $\alpha = H/\sigma_Y$ $\beta = \exp[\mu_Y - G\sigma_Y]$ $H = \sqrt{2 \ln n}$ $G = \frac{H^2 - (0.50)\ln(4\pi \ln n)}{H}$
Mean	$E(X^k_{(1)}) = \int_0^\infty x^k f_{X(1)}(x)\,dx$ where k = 1 (mean), 2 (mean square)	$E(X_{(1)}) = \beta\Gamma(1 + \frac{1}{\alpha})$
Variance	$V(X_{(1)} = E(X^2_{(1)}) - [E(X_{(1)})]^2$	$V(X_{(1)}) = \beta^2[\Gamma(1 + \frac{2}{\alpha}) - \Gamma^2(1 + \frac{1}{\alpha})]$
Standard Deviation	$\sigma_{X(1)} = \sqrt{V(X_{(1)})}$	$\sigma_{X(1)} = \sqrt{V(X_{(1)})}$

Table A.1--continued

Property of $X_{(1)}$	Exact Distribution	Approximating Asymptotic Distribution
Median	$\tilde{X}_{(1)} = F_{LN}^{-1}[1 - (0.5)^{1/n}]$ where F_{LN}^{-1} = inverse lognormal CDF	$\tilde{X}_{(1)} = \beta[\ln 2]^{1/\alpha}$
Quantile	$X_{(1)q} = F_{LN}^{-1}[1 - (1 - q)^{1/n}]$ where $q = P(X_{(1)} \leq x)$	$X_{(1)q} = \beta[\ln(\frac{1}{1-q})]^{1/\alpha}$ where q is the same as for the exact case
COV	$C_{X(1)} = \frac{\sqrt{V(X_{(1)})}}{E(X_{(1)})}$	$C_{X(1)} = \left[\frac{\Gamma(1 + \frac{2}{\alpha})}{\Gamma^2(1 + \frac{1}{\alpha})} - 1\right]^{1/2}$

and measure the small and large sample size effects on the convergence of the exact minima distributions to their respective limiting asymptotic distributions.

To measure the relative degree of convergence as a function of initial C_X and n,

$$R_q = R_q(C_X, n) = \frac{X_{(1)qW}}{X_{(1)qE}} \tag{A.16}$$

where

$q = P(X_1 \le x)$, a specific probability/quantile order.

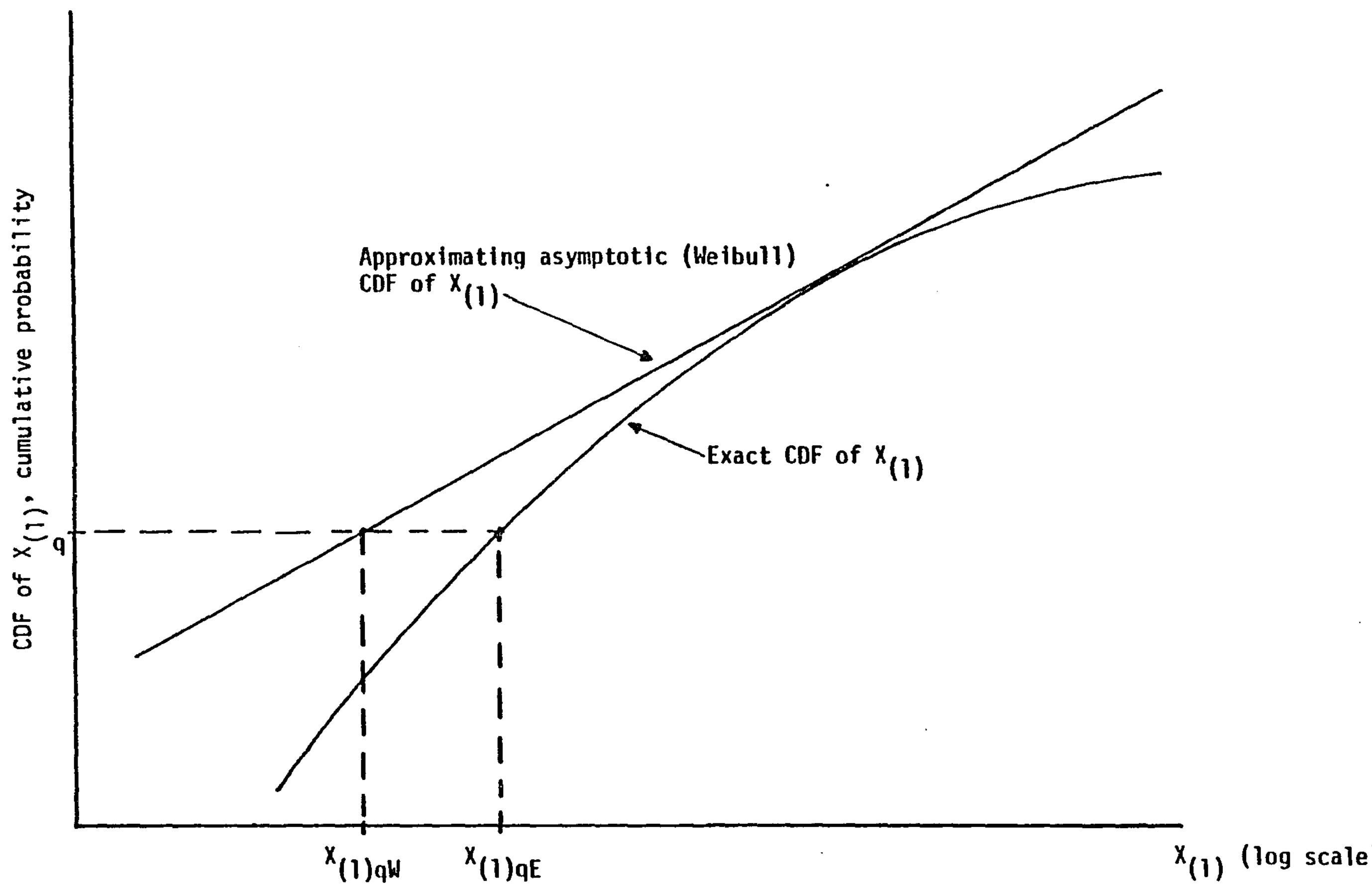
$X_{(1)qW}$ = Weibull minimum quantile, where $X_{(1)qW}$ is the root of $F_W(x) = q$.

$X_{(1)qE}$ = Exact minimum quantile, where $X_{(1)qE}$ is the root of $F_{X(1)}(x) = q$.

The quantiles $X_{(1)qW}$ and $X_{(1)qE}$ are illustrated in Fig. A.1 on a Weibull probability plot.

Such ratios were calculated for all runs and for $q = 10^{-2}$, 10^{-3}, 10^{-4} and 10^{-5}. Because the Weibull approximation is usually employed in engineering practice only for "strength" variables for which the left tail is important, only the smaller probability levels were considered in this study. The ratios were then plotted as functions of C_X and n.

Ratios of the Weibull approximation to the exact were also plotted as functions of C_X and n for the median, mean, standard deviation and coefficient of variation. These are denoted as $R_{\tilde{X}}$, R_μ, R_σ and R_C respectively.



Minimum of a sample of size n taken from a lognormal population

Fig. A.1 Definition of the asymptotic and exact minimum quantiles ($X_{(1)qW}$ and $X_{(1)qE}$, respectively), shown on Weibull probability paper

A.6 Results

The results presented in this appendix do not cover all of Gagne's work, however the most significant results and trends are included.

Comparisons of the exact and asymptotic quantities for the mean, median, standard deviation and coefficient of variation of $X_{(1)}$ are plotted as functions of C_X and n in Figs. A.2, A.3, A.4 and A.5 respectively. The plot of the ratios of the median, $R_{\tilde{X}}$, in Fig. A.2 indicates that the exact and asymptotic values of $\tilde{X}_{(1)}$ are quite close (note ordinate scale) and fairly stable as n increases. This agrees with the closest approach between $F_{X(1)}(x)$ and $F_W(x)$ occurring in the vicinity of $\tilde{X}_{(1)}$ in the Weibull plots. Note how the maximum difference in $\tilde{X}_{(1)}$ (minimum $R_{\tilde{X}}$) occurs at n = 10, for all C_X, as $R_{\tilde{X}}$ first diverges and then converges as n increases.

The ratio of mean values, R_μ, varies somewhat more in Fig. A.3. For C_X = 0.1, 0.2 and 0.3, R_μ consistently converges to one as n increases. However, for $C_X \geq 0.5$, the diverging/converging phenomena, observed for $R_{\tilde{X}}$, occurs for R_μ also. Here the maximum difference between exact and asymptotic (minimum R_μ) occurs at n = 4-5, apparently increasing slightly as C_X increases.

The plots of R_σ and R_C in Figs. A.4 and A.5 exhibit quite a different situation. Whereas $R_{\tilde{X}}$ and R_μ are always less than one, R_σ and R_C for a given C_X shows a minimum ratio value point which occurs at decreasing values of n as $C_{\tilde{X}}$ increases. The plot of R_C is greatly influenced by R_σ.

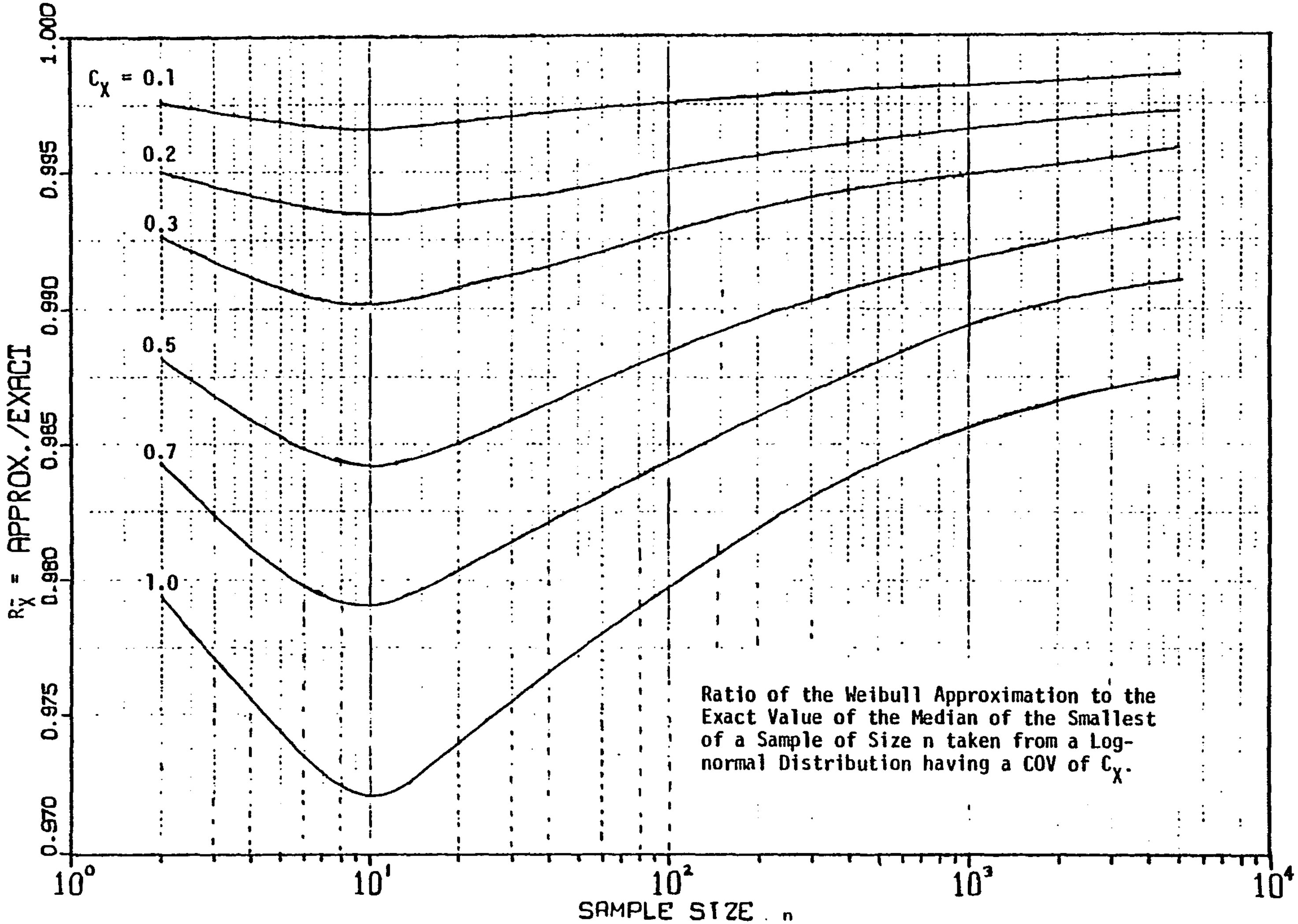


Fig. A.2 Median Ratio Plot

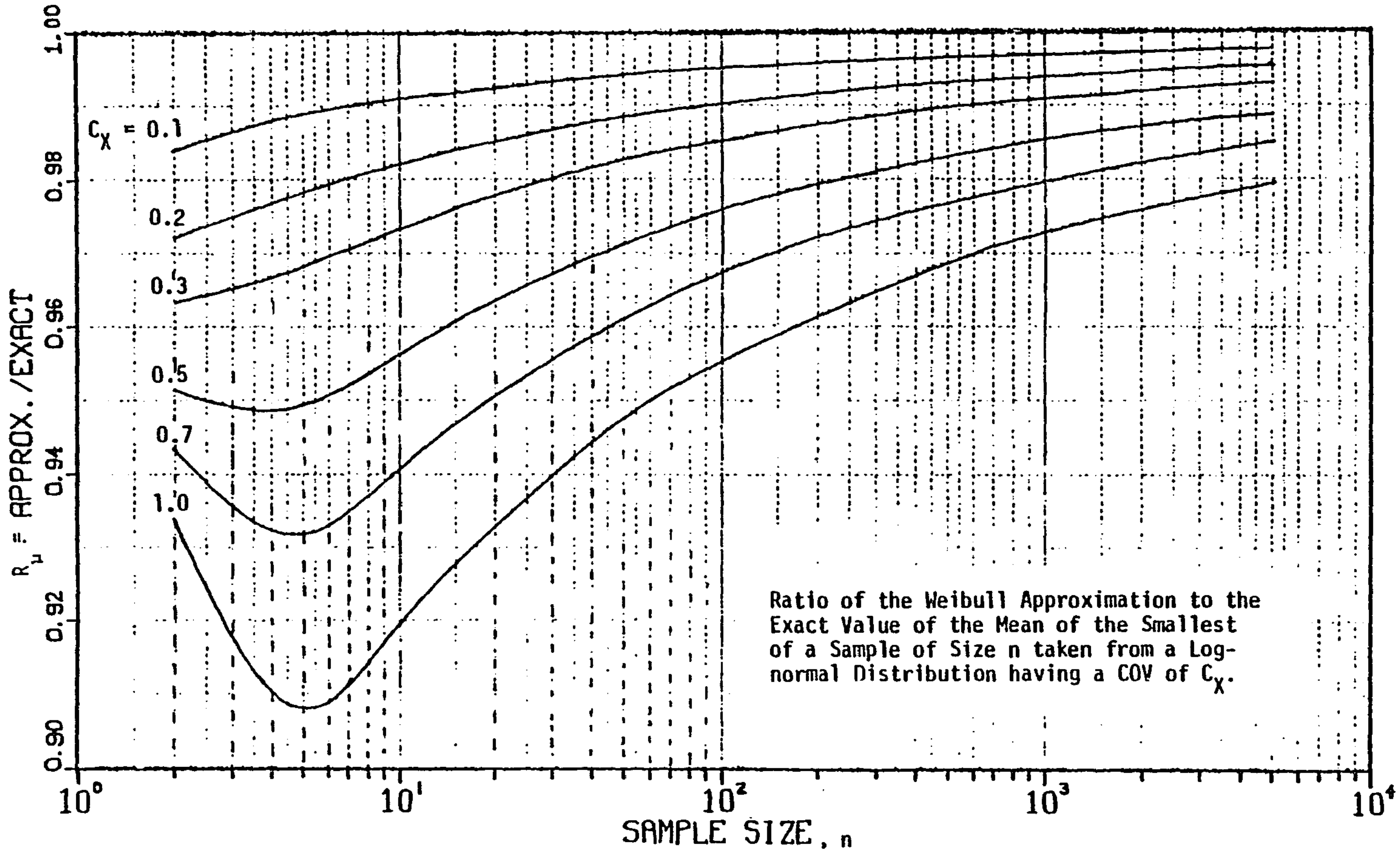


Fig. A.3 Mean Value Ratio Plot

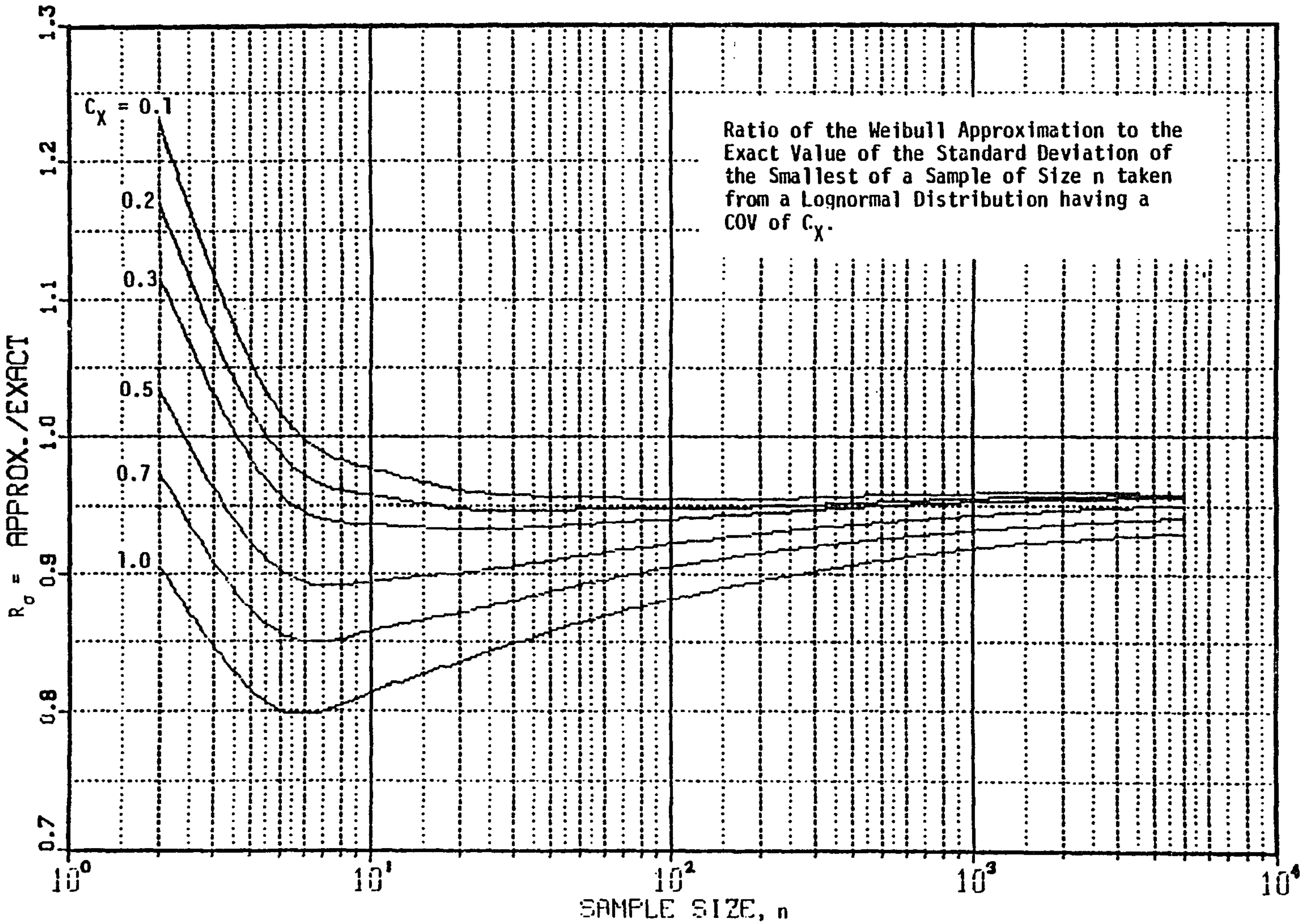


Fig. A.4 Standard Deviation Ratio Plot

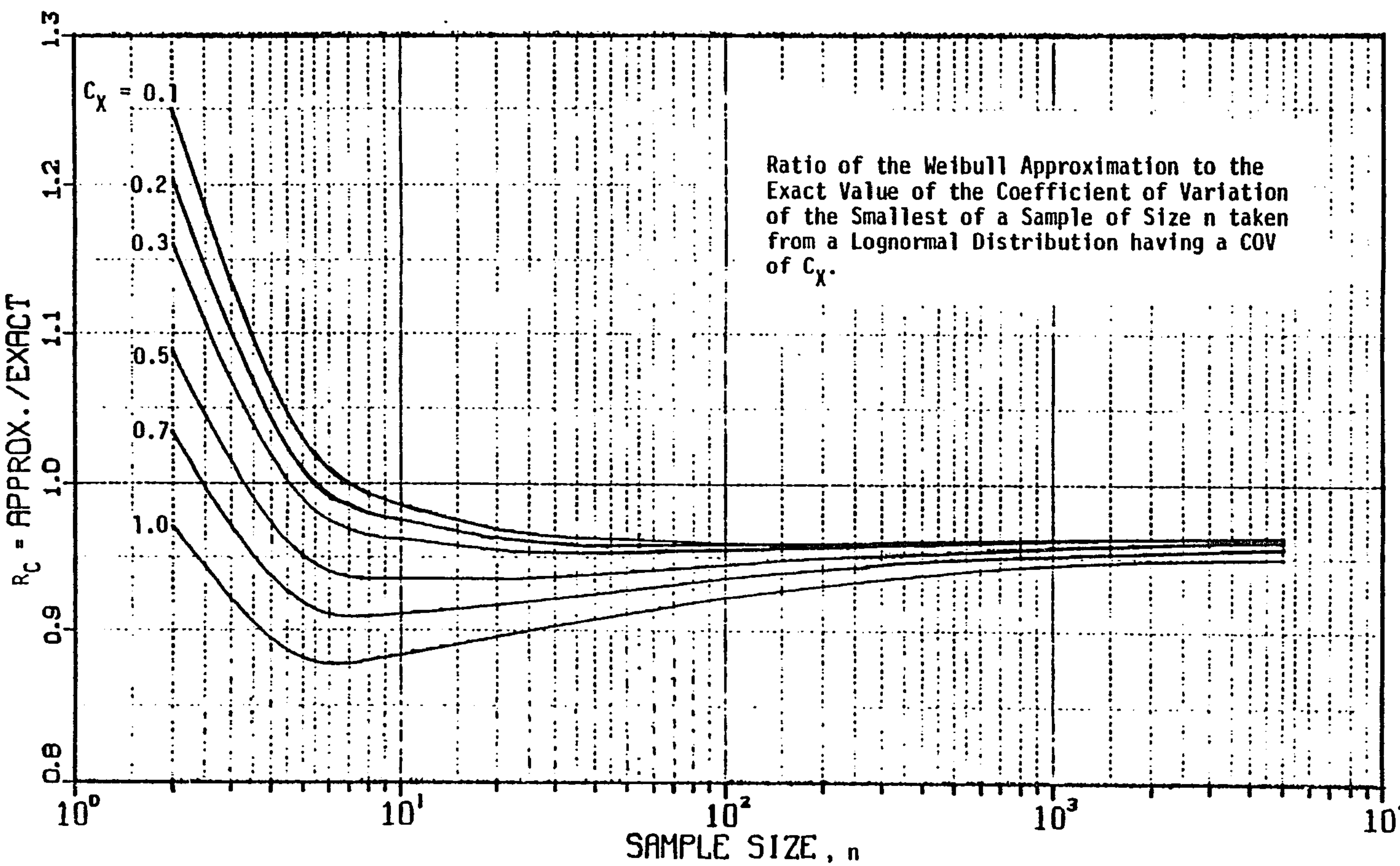


Fig. A.5 Coefficient of Variation Ratio Plot

The best comparisons between the Weibull approximation and the exact solutions relate to quantile and probability estimates. The method of comparison, described in Sec. , again uses the ratios of approximate to exact as used above for the moments.

The exact and asymptotic quantiles defined graphically in Fig. A.1, can also be defined numerically as:

$$\text{(exact)} \qquad X_{(1)qE} = F_{LN}^{-1}\left[1-(1-q)^{1/n}\right] \tag{A.17}$$

$$\text{(Weibull)} \quad X_{(1)qW} = \beta\left[\ln\left(\frac{1}{1-q}\right)\right]^{1/\alpha} \tag{A.18}$$

The quantiles for the left-tail probabilities $q = 10^{-2}$, 10^{-3}, 10^{-4} and 10^{-5} were thus found and the R_q ratios were obtained from Eq. 16.

The R_q is plotted as a function of the initial $C_{\tilde{X}}$ and n in Figs. A6 through A8. The differences in the left tails of the exact and asymptotically approximate distributions are observed. The asymptotic approximation gets better a the initial C_X decreases, n increases and q approaches 0.5.

The R_q plots can also be used as a practical design tool. The Weibull parameters (α, β) can be quickly calculated from $(\tilde{X}, C_X)$ and n. Specifying q, the approximate quantile $X_{(1)qW}$ can be obtained from Eq. 35. By entering the R_q plots in Figs. A.6 through A.8 with the initial C_X, n, and q, the appropriate R_q factor may be interpolated. The final step is to divide the approximate quantile by R_q to obtain the exact quantile of $X_{(1)}$. This method is valid for $10^{-5}<q<10^{-2}$.

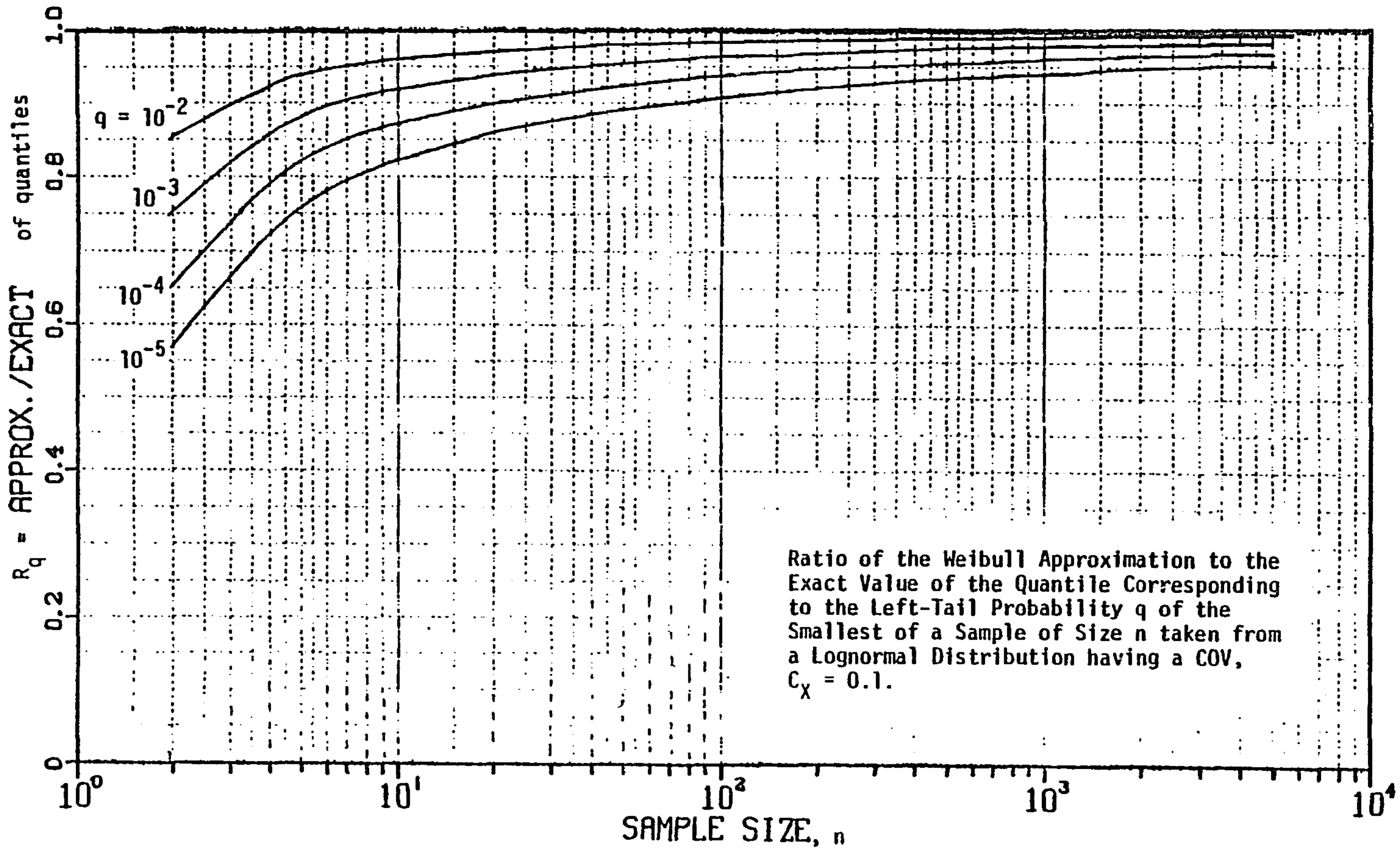


Fig. A.6 Left Tail Convergence Behavior

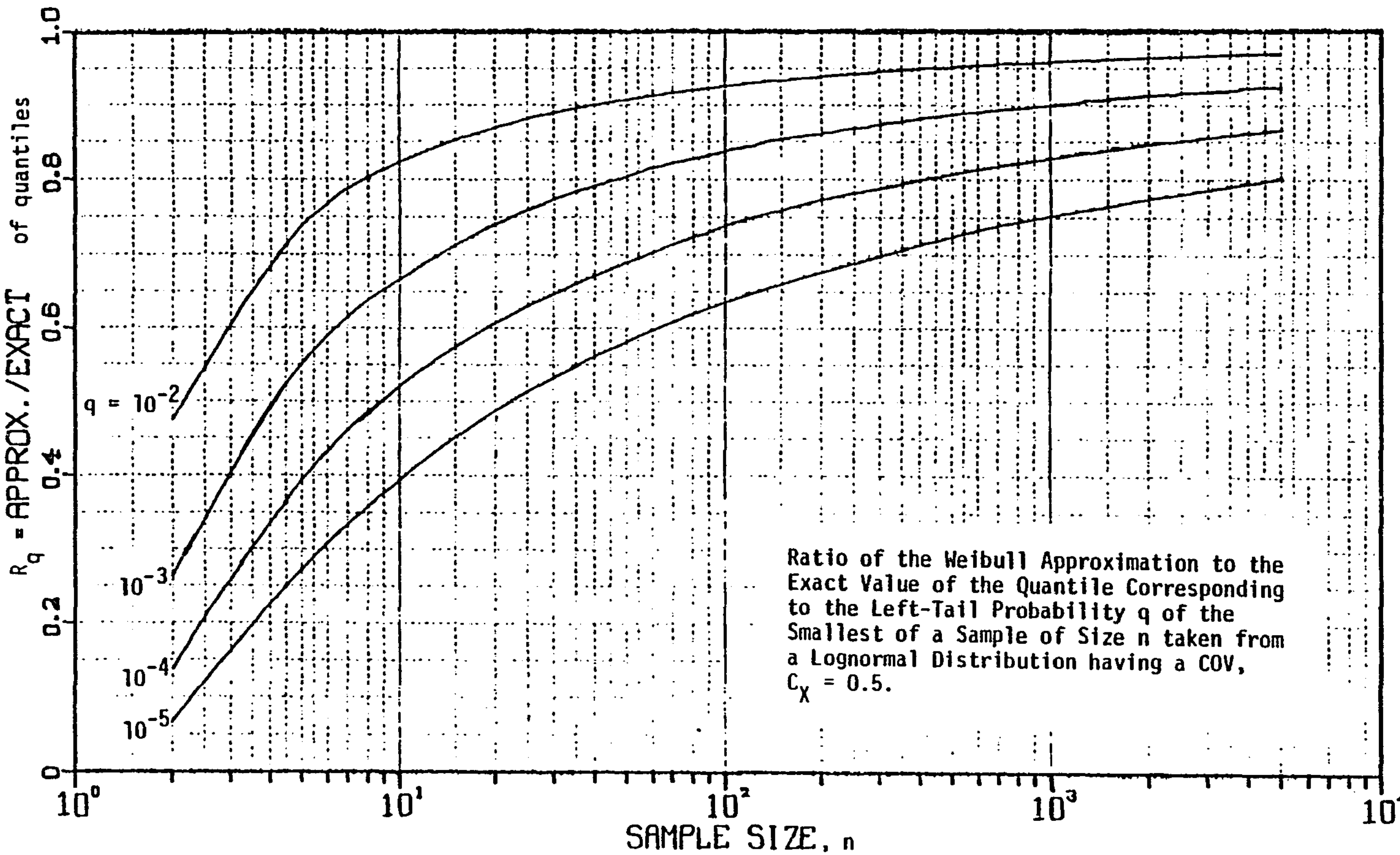


Fig. A.7 Left Tail Convergence Behavior

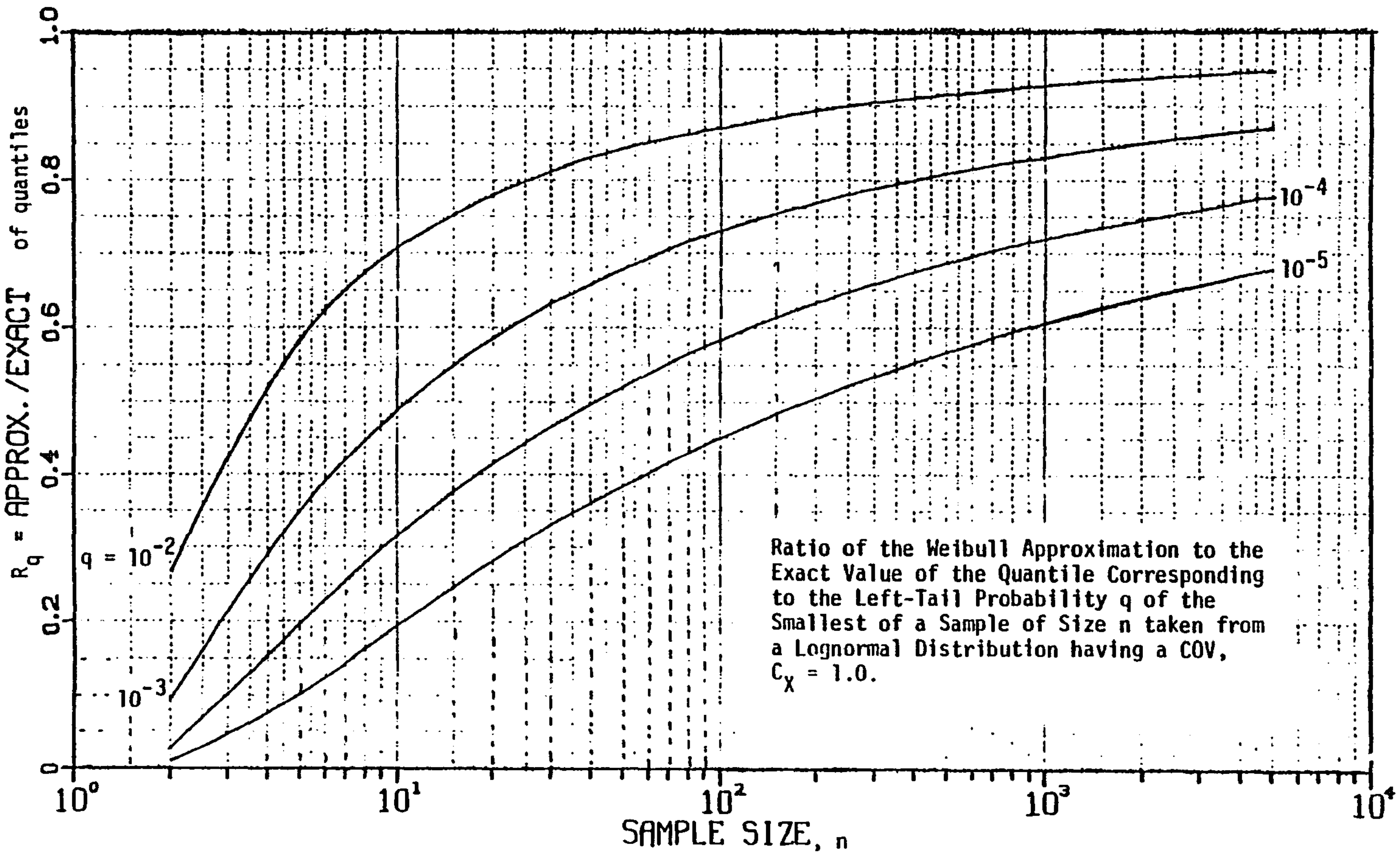


Fig. A.8 Left Tail Convergence

A.7 Summary

Convergence of the approximate distribution of minima has been investigated. it is shown that tail area properties in general has slow convergence and the approximate form of the distribution of minima should be used with caution. On the other hand reasonable approximations can be for median and mean values this is in particular true for low coefficients of variations.

APPENDIX B

COMPARISON OF LIFE DISTRIBUTIONS FOR SOME FATIGUE DATA

Summarized in this appendix is a comparison study of distributions for cycles to failure. The work was performed by Paul Littleton (1984). Littletons masters report came as a necessary offspring from this dissertation and the main results are therefore summarized in this appendix.

One of the most difficult issues in mechanical reliability analysis is the assignment of appropriate statistical distributions to the various random design factors. Small sample sizes or complete lack of data are typical problems. In fatigue reliability analysis the Weibull and the lognormal distribution are the most common distributions used for cycles to failure. The Weibull distribution has a "stronger" left tail and consequently yields conservative results compared to the lognormal distribution. It is demonstrated in both chapters 2 and 5 that the "choice" of life distribution may significantly influence the design results. This difference increases as the probability of failure decreases.

A comparison study of some available fatigue data sets has been performed. A summary is provided in Table B.1. The W-statistic, based upon a form of the Cramer-Von Mises statistic, was used in these comparison tests (Wirsching and Carlson (1977)).

Table B.1 Comparison Between the Weibull and the Lognormal Distribution from Various Data Sources

Data Source	Elastic Strain Range		C_N	Plastic Strain Range	C_N
	Sample Size	Best Fit Distribution		Best Fit Distribution	
4340 st 118 ksi	20	WEI	0.31	---	---
4340 st 118 ksi	20	LN	0.28	---	---
4340 st 55 ksi	20	LN	0.74	---	---
1045 st 55 ksi	20	WEI	0.39	---	---
1045 st 67 ksi	20	LN	0.24	---	---
1045 st 67 ksi	20	LN	0.26	---	---
COPPER 22.5 ksi	20	WEI	0.09	---	---
STEEL 60 ksi	12	WEI	0.42	---	---
STEEL 53 ksi	12	WEI	0.14	---	---
STEEL 46.7 ksi	12	LN	0.26	---	---
STEEL 41.2 ksi	12	LN	0.10	---	---
STEEL 36.4 ksi	12	WEI	0.33	---	---
STEEL 32.1 ksi	12	WEI	0.54	---	---
RQC 100 LAB A	17	LN	0.54	WEI	0.50
RQC 100 LAB B	10	LN	1.23	LN	0.19
RQC 100 LAB C	10	WEI	0.33	WEI	0.34
RQC 100 LAB D	10	LN	0.72	LN	0.22
SAE 950 ST	24	WEI	1.23	WEI	0.41
1008 ST	51	LN	0.51	LN	0.42
1015 ST	31	LN	0.75	LN	0.46
4340 ST	14	LN	0.67	WEI	0.38
4130 ST	22	LN	1.36	WEI	0.50
1005 ST	15	LN	1.32	WEI	0.44
RQC 100	16	LN	0.57	WEI	0.24
ASTM 514	12	LN	1.41	WEI	0.60
SAE 980	14	LN	0.26	LN	0.76
SAE 950	16	LN	1.94	LN	0.58
WASPALOY B, T = 1000F	44	LN	0.55	LN*	0.34
TUBULAR JOINTS BASIC CURVE	42	LN	0.63	---	---
INCOLOY 800, 70F	11	LN	0.47	WEI	0.31
INCOLOY 800, 1100F	9	LN	1.11	LN	0.20
INCOLOY 800, 1400F	9	LN	0.59	LN	0.36
MANTEN	9	LN	0.74	WEI	0.38
A36	9	LN	0.54	WEI	0.25
AISI 4340	10	LN	0.41	LN	0.22

*n = 31

Both high and low cycle fatigue data sets were considered. With some exceptions, the lognormal distribution proved to be the best fit for high cycle data for most cases. Based on this study it seems reasonable to use the lognormal distribution as a distribution for cycles to failure in high cycle fatigue when no other information is available. This point of view is supported by other investigations like Engesvik (1981) and Wirsching (1983). The low cycle, plastic strain range, data does not seem to show a preference toward any particular distribution. It is expected that much larger data sets are needed.

It is recognized that the distribution fit problem in fatigue reliability analysis is somewhat controversial. It might be argued that a distribution should preferably be picked on physical grounds. However fatigue is an extremely complicated phenomenon where empirical relations are still used to describe cycles to failure. In the near future, it is not expected that the physical mechanisms concerning metal fatigue will be understood thereby providing a theoretical basis for a particular choice.

As a final note it is often argued that the Weibull distribution should be used in fatigue analysis rather than the lognormal since it has an increasing hazard function or instantaneous failure rate. However as shown in Fig. B.1 this is true only for coefficients of variations, C_X, less than 1. It is also well known that the coefficient of variation in fatigue analysis may very well be larger than 1. This is in particular true in analysis of welded joints.

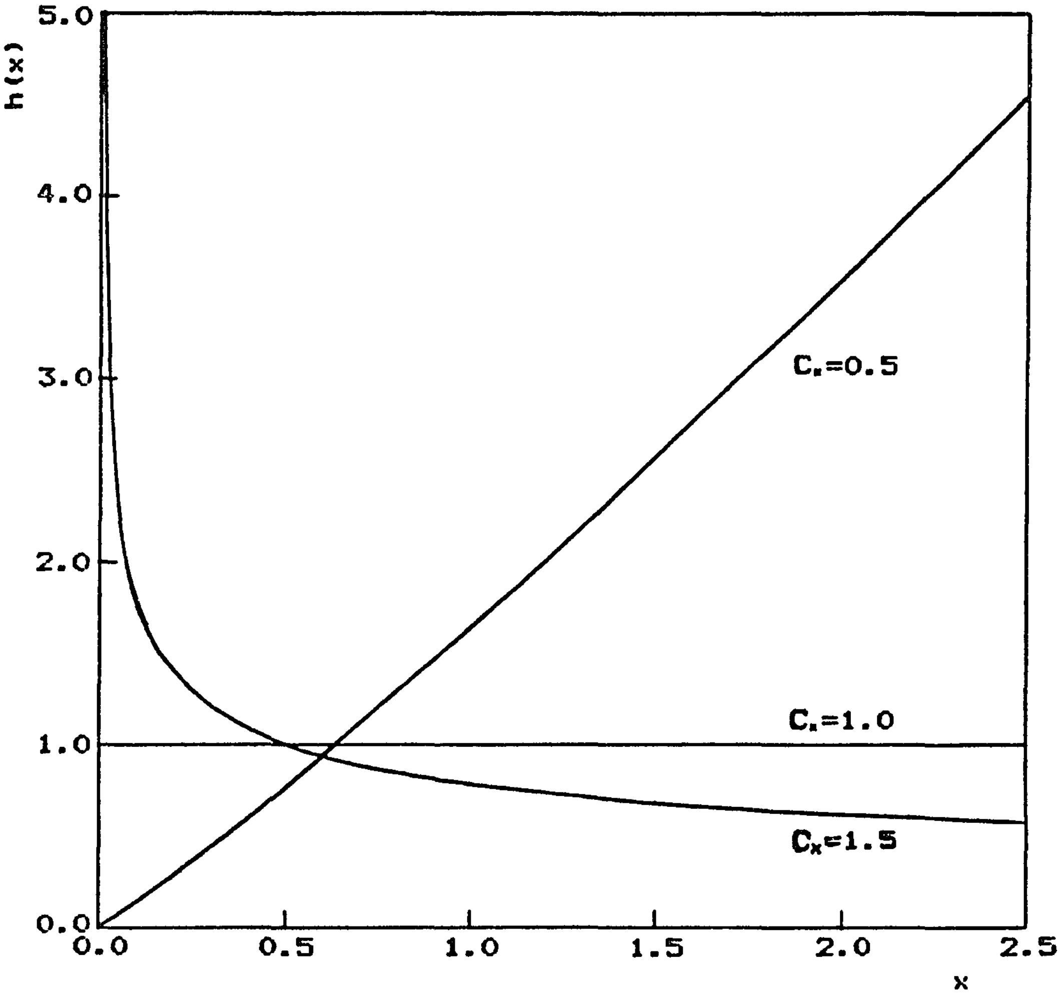


Fig. B.1 Weibull Hazard Function

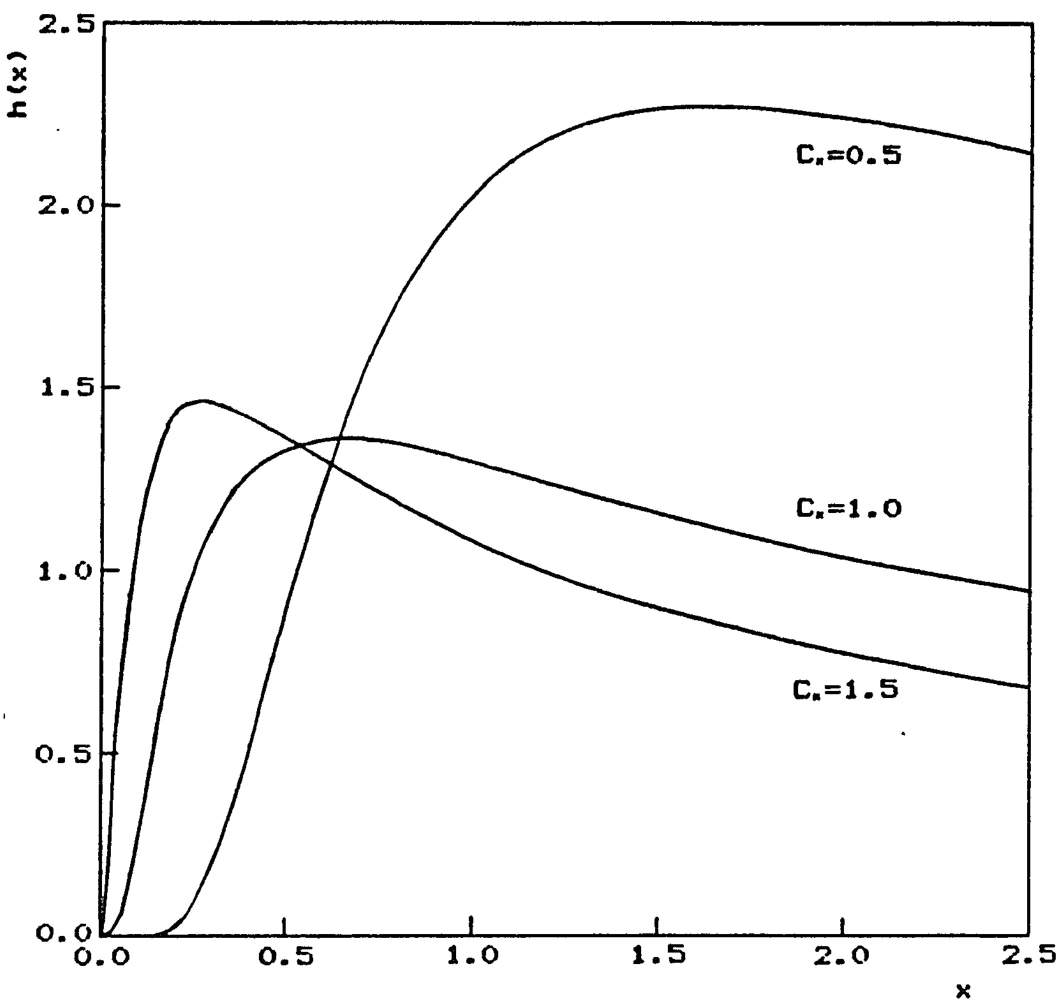


Fig. B.2 Lognormal Hazard Function

APPENDIX C

THE RACKWITZ-FIESSLER (R-F) ALGORITHM

The algorithm proposed by Rackwitz and Fiessler (1978) has been extensively described in recent literature (Leporati (1980)).

Thoft-Christensen and Baker (1982). The procedure for calculating the R-F algorithm safety index can be summarized as follows:

1. Define each design factor, X_i (i=1, n) and its corresponding probability distribution F_i and f_i denotes to cdf and pdf of X_i respectively.

2. Define reduced variables

$$u_i = \frac{X_i - \mu_i}{\sigma_i} \quad i = 1, n \tag{C.1}$$

where (μ_i, σ_i) = mean and standard deviation of X_i respectively.

3. Define the limit state in reduced variables

$$g_1(\underset{\sim}{u}) = 0 \tag{C.2}$$

where $\underset{\sim}{u} = (u_1, u_2, \ldots . u_n)$.

4. Make an initial estimate of the safety index

$$\beta = \min \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2} \tag{C.3}$$

subject to $g_1(\underset{\sim}{u}) = 0$.

5. Calculate the corresponding design point, x*

$$x_i^* = u_i^* \sigma_i + \mu_i \quad i = 1, n \tag{C.4}$$

6. Calculate the mean and standard deviation of the equivalent normal distributions for each non-normal variable

$$\sigma_{Ni} = \frac{\phi[\Phi^{-1}(F_i(x_i^*))]}{f_i(x_i^*)} \tag{C.5}$$

$$\mu_{Ni} = x_i^* - \Phi^{-1}[F_i(x_i^*)]\sigma_{Ni}$$

where ϕ = standard normal pdf and Φ = standard normal cdf

7. Define the new reduced variables

$$u_i' = \frac{X_i - \mu_{Ni}}{\sigma_{Ni}} \tag{C.6}$$

8. Calculate a new estimate of the safety index

$$\beta_1 = \min \sqrt{(u_1')^2 + \ldots + (u_N')^2} \tag{C.7}$$

subject to $g_1(\underset{\sim}{u}') = 0$.

9. Repeat steps 6 through 10 until the difference

$$|\beta_N - \beta_{N-1}| \le t \tag{C.8}$$

where t is the "error". In this study, the value t = 0.001 was used.

10. The probability of failure is calculated using $\hat{\beta} = \beta_N$

$$p_f = \Phi(-\hat{\beta}) \tag{C.9}$$

APPENDIX D

THE FULL DISTRIBUTIONAL APPROACH FOR DERIVED DISTRIBUTIONS

In order to formulate simple limit state equations for evaluation by numerical integration, consider the following development for a derived distribution.

Let $Y = X^a$, $a > 0$. To find the cdf of Y:

$$F_Y(y) = P[Y \leq y] = P[X^a \leq y] = P[X \leq y^{1/a}] \tag{D.1}$$

Therefore,

$$F_Y(y) = F_X(y^{1/a}) \tag{D.2}$$

or, in words, the cdf of Y evaluated at Y is equal to the cdf of X evaluated at $y^{1/a}$.

As an example, let

$$g(\underset{\sim}{U}) = g(R,V) = R - V^2 \tag{D.3}$$

where V is Weibull distributed with cdf

$$F_V(v) = 1 - \exp\left[-\left(\frac{v}{\beta}\right)^\alpha\right] \tag{D.4}$$

and S is equal to V^2. Hence, the cdf of S is

$$F_S(s) = 1 - \exp\left[-\left(\frac{s^{1/2}}{\beta}\right)^\alpha\right] \tag{D.5}$$

The evaluation of Eq. D.3 by the full distributional approach given the relationship in Eq. D.2 necessitates a form of Eq. 2.10:

$$p_f = 1 - \int_0^{\infty} f_R(r) F_S(r) \, dr \tag{D.6}$$

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