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SOME ASPECTS OF VORTEX LINE RECONNECTION

by

Arie Dagan

A Dissertation Submitted to the Faculty of the
DEPARTMENT OF MATHEMATICS
In Partial Fulfillment of the Requirements
For the Degree of
DOCTOR OF PHILOSOPHY
WITH A MAJOR IN PROGRAM IN APPLIED MATH
In the Graduate College
THE UNIVERSITY OF ARIZONA

1986
As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Arie Dagan entitled Some Aspects of Vortex Line Reconnection and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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Turbulence has long been believed to be associated with the behavior of vorticity. Ever since experiments showed clearly the presence of vortex structures in turbulent flow, concentrated efforts have tried to identify the important dynamics of three-dimensional vortex flow. In particular, conjectures abound about the importance of vortex stretching and vortex line reconnection. Numerical experiments based on ad hoc assumptions on the nature of the cores of vortex filaments have shown interesting behavior. In some cases, it has been argued that singularities develop in finite time and in other cases that the filament exhibits fractal dimensions. These inviscid calculations also show that filaments of opposite signed vorticity tend to pair up and that the local flow is two-dimensional.

Consequently, we have begun a study that clarifies the behavior of a pair of counter-rotating vortices in the presence of an external strain flow that would be induced by the presence of vorticity well away from the local two-dimensional plane. So far, the results are quite interesting and depend on the nature of the strain flow. We always assume that the horizontal component of the strain pushes the filaments together. It is the other two components that then affect the results. Without any strain along the axes of the filaments, the vortex cores are pulled into parallel elliptical shapes. Eventually, the cores are so deformed that they become unstable in the same way a
parallel shear flow would and the vortex structures disrupt. This phenomenon will be missed by filament codes that assume the cores remain circular.

On the other hand, a strain component along the filaments increases the vorticity but keeps the core structure mostly circular. As the cores approach one another, viscous effects overcome the increase in vorticity due to stretching and the cores dissipate away.
CHAPTER 1

INTRODUCTION

1.1 On the Reconnection of Vortex Line Filaments

Recent concern over the hazard presented by trailing vortices produced by large aircraft, has stimulated research into flow with vortex filaments in free motion. Since these vortices can be strong and persistent enough to pose a safety hazard to other aircraft, it is clearly desirable to be able to predict the structure, position and persistence of such vortices, as well as to understand the mechanisms by which vortex wakes are dissipated. In a homogeneous fluid, trailing vortices undergo a natural sinusoidal instability that eventually causes them to touch, and then break into a series of ring-like structures. This process destroys the initial wake structure more rapidly than viscous or turbulent decay of the individual filaments. The character and the potential hazard of the residual wake structure after completion of the sinusoidal instability is still not understood.

In order to assess the hazard presented by aircraft trailing vortices, it is necessary to predict the structure, motion and persistence of these flows in the real atmosphere. Such a study is beyond the present capabilities, as our understanding of the problem is based on solutions to more idealized problems. For example, the generation of trailing vortices by finite wings is well understood.
(Batchelor, [1983]). The spanwise distribution of the aerodynamic lift, $L$, is related to the flight speed, $U$, fluid density, $\rho$, and the circulation in each spanwise cross-section $c$, as $L(y) = U(y) \cdot \rho \cdot c$. Stokes' theorem requires that a trailing vortex sheet of strength, $\gamma$, be created, as the circulation changes along the span. This vortex sheet will roll up, and the wake structure will be established within a few spans downstream of the wing. For lightly loaded wings at high Reynolds numbers, roll up takes place gradually enough to justify a local two-dimensional model of the wake, and yet rapidly enough to assume an inviscid flow. It appears that the vortex sheet from each half of the wing rolls up into a single vortex, and the far field behavior is that of two counter-rotating vortex filaments.

If the aircraft wake could be represented by a two-dimensional vortex pair, the discussion of the motion in the atmosphere might be straightforward. However, the vortex pair is unstable to three-dimensional disturbances, so that it soon loses its two-dimensional character. Nevertheless, for studies of vortex wake behavior, the three-dimensional structure has been ignored in the hope of making the problem tractable.

In an ideal homogeneous fluid, it is well known that a vortex pair will descend with a constant speed at a constant spacing. The descent of a vortex pair in stratified atmosphere poses a problem that has not been completely solved, at least judging by the disagreement of the existing theories on whether vortex spacing increases or decreases (Lissaman et al., [1973]; Tulin & Schwartz, [1971]).
A vortex pair is also affected by atmospheric wind. To a first approximation, the vortex cores are convected by the local wind speed. However, if the wind is nonuniform, the vortex motion may be affected by the wind shear (Lissaman et al., [1973]).

The dissipation of the aircraft wake occurs through a sequence of events, a longitudinal, sinusoidal perturbation of the initial vortex pair develops, the amplitude grows until the vortices touch, and then linking occurs, which transforms the pair into a series of vortex rings that dissipate rapidly. Scorer and Davenport [1970] have proposed atmospheric stratification as a cause of this instability. Since their study is based on a two-dimensional model of the wake, three-dimensional effects on the vortices are not included.

The first quantitative analysis of the three-dimensional instability of counter-rotating vortex filaments in an ideal homogeneous fluid was done by Crow [1970]. A longitudinal instability results from the mutually induced velocity of a sinusoidally perturbed pair. A single, sinusoidally perturbed vortex filament will rotate about its own axis, as a consequence of its self-induced velocity. The longitudinal instability of the vortex pair occurs when the velocity field induced by the other filament annuls the self-induced rotation. The filament vortices then kink in planes placed symmetrically at an approximate angle of 45 degrees. The instability is characterized by both a symmetric and antisymmetric mode. However, only the symmetric one was found to be dominant, which agrees with the experimental observation.
Crow used a simple model for the vortex wake. He assumed the vortex filaments were merely lines, that is, had no core structure. This assumption presents difficulties in evaluating the Biot-Savart integral for the fluid velocity at the line vortex. The self-induced velocity is undefined because of a singularity in the integration. The "cut-off" method restores some meaning to the approximation by including a simple representation for the filament core when determining the self-induced velocity. Consequently, the integral may be "cut-off" a distance \( d \), proportional to the core size, on either side of the singularity. The constant of proportionality was chosen to agree with Kelvin's solution for waves on a core with constant vorticity. Further studies by Widnall et al. [1971] found an analytic expression for the cut-off distance that is valid when the core is essentially circular and its radius is smaller than the radius of curvature at the filament. They used matched asymptotic expansions to obtain their results, which agree very well with the results of Saffman [1971], and with the theory of Ting et al. [1967].

Moore [1972] has extended Crow's results by using a numerical method to study the growth of the longitudinal waves of finite amplitude. In order to find the non-linear shape of the vortex filaments, the dynamical equations were integrated in time, where the velocity field was obtained by the "Biot-Savart" integral. The results showed only minor differences to the linear theory (Crow, [1970]). In particular, the filaments touched in finite time, at which point a
complicated process occurs during which the vortex lines reconnect to form ring-like structures.

The process of vortex line reconnection, when the trailing vortices touch, and then reform as a vortex ring, is not understood (Saffman & Baker, [1979]). It is obvious that the reconnection phenomena is a three-dimensional, viscous effect. In the neighborhood of the reconnection, the length scale of the spacing between the vortices is of the same order as the viscous core of each vortex. Hence, it would seem that the main mechanism for reconnection is the rapid viscous annihilation of the longitudinal vorticity. The remaining small, non-longitudinal components establish a new reconnected vortex line which forms ring-like structures. The precise details of this process are one of the main areas of the present study.

The most natural equations to use in the study of vortex reconnection are the vorticity-velocity equations.

\[ \frac{\partial w_i}{\partial t} + u_j \frac{\partial w_i}{\partial x_j} = w_j \frac{\partial u_j}{\partial x_j} + v \frac{\partial^2 w_i}{\partial x_j^2} \]  

\[ w_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j} \]  

\[ \frac{\partial u_j}{\partial x_j} = 0. \]

The first equation is known as the vorticity equation and is obtained by elimination of the pressure from "Navier-Stokes" equations. The second
term in equation (1-1a) is known as the advection or convection term. The first term in the right-hand side is the rigid body rotation and elongation of vortex elements. The last term in the vorticity equation is the dissipation term, which results from viscous effects; \( \nu \) is the kinematic viscosity. Equation (1-1b) relates the velocity field to the vorticity, and last, equation (1-1c) is the conservation of mass for incompressible flow.

The vorticity equation as it appears in equation (1-1a) is written in Eulerian form, but an alternate form, the Lagrangian form, is often used to understand the basic properties of vortex flows. The Lagrangian form of the vorticity equation describes the dynamic evolution of vortex elements as they move with the fluid flow. Hence, it is easy to see that vorticity is increased by the stretching term while the dissipation term smears out the vorticity. This term may play a crucial role in vortex reconnection because the dissipation term is able to balance the gain of the vorticity strength that comes from the extension of the vortex line (Batchelor, [1983]) (See Chapter 5).

For our specific study of vortex line reconnection, we choose \( x_1, x_2 \) and \( x_3 \), as in Figure 1.1. Essentially, \( x_3 \) lies in the longitudinal direction and \( x_1, x_2 \) lie in the cross sectional plane. In order to approximate the physical process, we assume that the length scale \( b \) in \( x_1, x_2 \) corresponds to the viscous length but along the longitudinal direction we assume that the length scale is the same as the length of the most unstable wave, which was found by Crow [1970] to be about 5 times the initial spacing between the vortices. It is easy to see that
Figure 1.1 Shape of unstable modes after transients have died away (A-antisymmetric mode, S-symmetric mode).
the term \( \partial \partial x_3 \) is of order \( (b/L) \), which is small. The two cross-sectional components of the vorticity, \( w_1 \) and \( w_2 \), are of the same order, i.e., \( O(b/L) \). Therefore, to a first approximation, equation (1-1) reduces to the 2-D equation for a counter-rotating pair of vortices. We now assume that in the reconnection region

\[
w_3 = w_3^0 + \frac{b}{L} w_3^1
\]

\[
w_j = \frac{b}{L} w_j^1 \quad j = 1, 2
\]

\[
\frac{\partial \psi}{\partial x} = \frac{b}{L} \frac{\partial \psi}{\partial z}
\]

The zeroth order approximation;

\[
\frac{\partial w_j^0}{\partial t} + u_1^0 \frac{\partial w_j^0}{\partial x_1} + u_2^0 \frac{\partial w_j^0}{\partial x_2} = \nu \left[ \frac{\partial^2 w_j^0}{\partial x_1^2} + \frac{\partial^2 w_j^0}{\partial x_2^2} \right] \quad (1.2a)
\]

\[
\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} = w_j^0 \quad (1.2b)
\]

\[
u_2 = - \frac{\partial \psi}{\partial x_1} \quad (1.2c)
\]

\[
u_1 = \frac{\partial \psi}{\partial x_2} \quad (1.2d)
\]
Note that the streamfunction $\psi$ has been introduced; $\psi$ satisfies the stream function equation (1.2b) and the velocity components are then calculated by (1.2c) and (1.2d).

The first order $O(b/L)$ in the perturbation expansion satisfies

$$\frac{aw^j}{\partial t} + u^0 \frac{aw^j}{\partial x^j} = w^j + v \frac{a^2w^j}{\partial x^j} \quad i = 1, 2, \quad j = 1, 2 \quad (1.3a)$$

The velocity components $u^0_j$ or the stream function must satisfy conditions at the boundaries of the reconnection region that match the outer flow. The outer flow is specified by the motion of the vortex filaments, which is given by the Biot-Savart integral,

$$u_k = \sum \frac{3}{4\pi} \frac{a^3w^j}{\partial x^j} \int \frac{w^i \, dV'}{|x-x'|}$$

In the case where $x$ is far away from the vortex cores, this integral may be approximated by an integral along line vortices at the centroids of the filaments. Hence,

$$\dot{u} = -\frac{\Gamma}{4\pi} \sum \left( \frac{\dot{x}_1 \cdot \vec{d}x_1}{|x-x_1|^3} + \frac{\Gamma}{4\pi} \left( \frac{\dot{x}_2 \cdot \vec{d}x_2}{|x-x_2|^3} \right) \right) \quad (1-4)$$

where $\Gamma$ is the circulation of each point along the vortex filaments and $a \times b$ represents a vector cross-product. The behavior of the outer solution (equation 1-4) in the neighborhood of the reconnection region, reduces to that of two-dimensional counter-rotating vortices, plus additional terms that are due to the shape of the vortex filaments as they undergo the longitudinal instability (Widnall, Bliss and Zalay,
10

[1971]; Crow, [1970]). The additional terms are known to give a strain flow.

The Biot-Savart law is a valid description of the velocity field induced by a thin vortex filament, if a proper account is taken of the flow within and near the vortex core (Widnall, Bliss and Zalay, [1971]). However, this analysis was carried out for a single filament vortex. For the reconnection problem, it is questionable whether this approach is valid. Moreover, Batchelor [1983] pointed out that the mathematical notion of a line vortex is of a limited direct value in problems involving development in time because the core may deform significantly. In the light of the above complications, we have decided to model the strain flow rather than use equation (1-4). Actually, there have been no studies of the viscous merging of counter-rotating vortices in two dimensions and so, in that sense, our studies add further insight into vortex flows in a more fundamental way.

There have been some studies in vortex reconnection. The first numerical computation on 3-D vortex reconnection appeared in 1984 (Chamberlain and Weston, [1984]). They studied the collision of a system of ring vortices. The computational domain is a rectangular box with 33 x 33 x 33 nodal points. In order to define the numerical boundary condition, the asymptotic behavior of equation 1-4 is used. The results are convincing, but their numerical scheme is costly because of the way the numerical boundary condition is evaluated. Their method requires $O(N^5)$ computer operations, where, in the next chapter, we see that in our spectral representation, the numerical
boundary condition appears in a simplified form. However, the 3-D numerical computation of vortex line reconnection is not feasible yet, due to numerical difficulties that occur when the vortex line intersects the boundary of the computational domain. Some suggestions to overcome this difficulty can be found in Lin, Ting and Weston [1985], but they have not yet been implemented.

Due to the difficulties experienced in three-dimensional flow, we have decided to study the simpler equations (1-2), and to look for a computational method that will treat the computational boundary condition efficiently. Details are given in Chapters 2 and 3. Since reconnection is associated with a strain flow, we use different straining flows to explore the various possibilities. In Chapter 4, we use the simple planar strain, $\psi = A x_1 x_2$, while in Chapter 5 a radial strain is assumed with stretching in the $x_3$ direction; this case is the one Siggia [1985] believes is relevant. Because vorticity is stretched, special care must be taken numerically to avoid instabilities; details are given in Chapter 5.
CHAPTER 2

ASYMPTOTIC SOLUTION FOR THE VISCOUS INTERACTION OF COUNTER-ROTATING VORTICES

2.1 Derivation of the Equations

As a first approximation to the reconnection of two vortex lines, the 3-D equations can be reduced to the 2-D viscous merge of counter-rotating vortices. This can be done whenever the ratio between the radius of the viscous core to the length of the most unstable wave is small enough. Physically there are two mechanisms, vorticity transport and dissipation, which spreads the vorticity.

For the solitary vortex in an infinite domain, the analytical solution is a vorticity distribution with a Gaussian profile (White, [1974]). The sensitivity of the solitary vortex to the initial condition has been checked by Ting [1971]. The Gaussian profile was taken in previous research on the viscous merge of vortices as an initial condition (Robert et al., [1976]; Hecht et al., [1981]). In this chapter, we will examine the sensitivity of the solution to the Gaussian vorticity distribution as an initial condition. Since the vorticity is initially concentrated in a narrow region, the initial conditions may cause substantial numerical problems, due to the sharp gradient. An asymptotic solution for a short time period is given in this chapter, which clarifies the question of the importance of the initial conditions.
The basic assumption in this research is that initially the ratio between the vortex core size to the vortex spacing is very small or \[ \frac{vt'}{(2H_d)^2} \ll 1 \] where \( H_d \) is the initial distance between the centroid of the vortices. Hence, there is no viscous interaction between the two vortices and each one of them is under the influence of the induced velocity of its neighbor. In other words, this is a singular perturbation problem, where each one of the vortices is under the influence of the far field solution of its neighbor.

The vorticity equation can be written in polar coordinates, where the origin is located at the vorticity maximum of the positive vortex. The angle \( \theta \) is measured from the radius \( r \) to the \( x \)-axis.

The vorticity and the stream function equations, written in polar coordinates, are

\[
\frac{\partial \psi'}{\partial t'} + v_r \frac{\partial \psi'}{\partial r'} + v_\theta \frac{\partial \psi'}{\partial \theta'} = -\nu \frac{\partial^2 \psi'}{\partial \theta'^2}
\]

\[
\frac{\partial^2 \psi'}{\partial r'^2} + \frac{1}{r'} \frac{\partial \psi'}{\partial r'} = 0
\]

where all primed quantities are dimensional. In order to write the equations in dimensionless form, let

\[
r' = br
\]

\[
v_\theta = \frac{r'}{2\pi b} v_\theta
\]

\[
v_r = \frac{r'}{2\pi b} v_r
\]
\[ w' = \frac{r}{2\pi b^2} w \]
\[ t' = T_0 t \]
\[ \psi' = \psi_0 \psi \]

where:
\[ Re = \frac{r}{2\pi \nu} \]
\[ \psi_0 = \frac{r}{2\pi} \]
\[ T_0 = \frac{2t_0}{Re} \]
\[ b^2 = 2\nu t_0 \]

and \( \Gamma \) is the magnitude of the initial circulation around each vortex.

The parameter \( t_0 \) is introduced to relate a time scale to the initial viscous length scale. Hence, the equations can be written in dimensionless form;

\[ \frac{\partial w}{\partial t} + v_r \frac{\partial w}{\partial r} + \frac{v_\theta}{r} \frac{\partial w}{\partial \theta} = \frac{1}{Re} \psi^2 w \]
\[ \psi^2 \psi = w \]
\[ v_\theta = \frac{\partial \psi}{\partial r} \]
\[ v_r = -\frac{1}{r} \frac{\partial \psi}{\partial \theta} \]

The vorticity equation can also be written as:

\[ 2 \frac{\partial w}{\partial t} + Re \left\{ v_r \frac{\partial w}{\partial r} + \frac{v_\theta}{r} \frac{\partial w}{\partial \theta} \right\} = \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \frac{\partial w}{\partial r} \right\} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \]  
(2-1)
In this representation, the time \( T = 2t/Re \) or \( T = t/t_0 \). The expressions for the stream function and the velocity are the same as before, i.e.:

\[ \nabla^2 \psi = w \]

\[ v_r = -\frac{1}{r} \frac{\partial \psi}{\partial \theta} \]

\[ v_\theta = \frac{\partial \psi}{\partial r} \]

As long as the viscous core of each vortex is smaller than the vortex spacing, we can regard the induced velocity coming from the negative vorticity as that due to a point vortex. Hence, the asymptotic behavior of the stream function equation is:

\[ \nabla^2 \psi = 0 \]

Under these assumptions, we write the velocity field as follows:

\[
\begin{align*}
  v_\theta &= v_{\theta,0} + v^{(1)}_\theta + v_{\theta,1} \\
  v_r &= v^{(1)}_r + v_{r,1} \\
  w &= w_0 + w_1
\end{align*}
\]

where \( v_{\theta,0} \) is the tangential velocity obtained from the solution of the solitary vortex in an infinite domain, and \( v^{(1)}_\theta \) is the induced velocity from the negative vortex. The symbols \( (i), (0) \) have a similar meaning for the radial velocity, too. \( (v_{\theta,1}, v_{r,1}) \) are the perturbed velocities in the neighborhood of the positive one.

We assume that \( w_0 = \exp(r^2/2T)/T \) and \( v_{\theta,0} \) is the velocity obtained from this vorticity distribution.
Substituting equation (2-3) into the vorticity equation yields:

\[
2 \frac{\partial w_1}{\partial t} + \text{Re} \left\{ (v_r^{(I)} + v_r^{(I)}) \left[ \frac{\partial w_0}{\partial r} + \frac{\partial w_1}{\partial r} \right] + \frac{v_\theta^{(I)} + v_\theta^{(I)}}{r} \frac{\partial w_1}{\partial \theta} \right\} \\
= \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial w_1}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 w_1}{\partial \theta^2}
\]  

(2-4)

Assuming \((v_r^{(I)}, v_\theta^{(I)}, v_\theta^{(I)}, v_r^{(I)}, w_1)\) are of small order, the above equation can be written as follows:

\[
2 \frac{\partial w_1}{\partial t} + \text{Re} \left\{ (v_r^{(I)} + v_r^{(I)}) \left[ \frac{\partial w_0}{\partial r} + \frac{v_\theta^{(I)} + v_\theta^{(I)}}{r} \frac{\partial w_1}{\partial \theta} \right] \right\} \\
+ \text{Re} \left\{ (v_r^{(I)} + v_r^{(I)}) \left[ \frac{\partial w_1}{\partial r} + \frac{v_\theta^{(I)} + v_\theta^{(I)}}{r} \frac{\partial w_1}{\partial \theta} \right] \right\} \\
= \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial w_1}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 w_1}{\partial \theta^2}
\]  

(2-5)

and the second expression describing the advection of vorticity will be assumed smaller. This equation can be written in similarity variables \((T, \frac{r}{\sqrt{T}})\) as follows:

\[
2 \frac{\partial w_1}{\partial T} = -\frac{\eta}{T} \frac{\partial w_1}{\partial \eta} + \text{Re} \left\{ \left[ -\frac{1}{n} \frac{\partial \psi_{1,1}}{\partial \eta} + \frac{C_1}{(2a)^2} \right] \frac{\partial w_0}{\partial \eta} + \frac{q(n)}{n^2} \frac{\partial w_1}{\partial \eta} \right\} \\
+ \text{Re} \left\{ \left[ \frac{C_1}{(2a)^2} - \frac{\partial \psi_{1,1}}{\partial \eta} \right] \frac{\partial w_1}{\partial \eta} + \frac{1}{n} \left[ \frac{C_2}{(2a)^2} + \frac{\partial \psi_{1,1}}{\partial \eta} \right] \frac{\partial w_1}{\partial \eta} \right\} \\
= \frac{1}{T} \left\{ \frac{1}{n} \frac{\partial}{\partial \eta} \left( n \frac{\partial w_1}{\partial \eta} \right) + \frac{1}{n^2} \frac{\partial^2 w_1}{\partial \eta^2} \right\}
\]  

(2-6)

Where \(\eta = \frac{r}{\sqrt{T}}\). The induced velocities have been written as:
\[ v_r^I = \sqrt{\frac{T}{2\pi}} \, C_1(t, n, \theta) \]  
\[ v_\theta^I = \sqrt{\frac{T}{2\pi}} \, C_2(t, n, \theta) \]  

and the other quantities are defined by:

\[ w_{0,0} = e^{-n^2/2} \]
\[ q(n) = 1 - e^{-n^2/2} \]
\[ w_0 = \frac{w_{0,0}}{T} \]
\[ \psi_1 = \sqrt{\frac{I}{2a}} \, \psi_{1,1} \]
\[ w_1 = \frac{w(T, n, \theta)_{1,1}}{(2a)^2} \]
\[ a = \frac{H_d}{2\nu t_0} = \frac{H_d}{b} \]

We define \( \tau = \frac{I}{(2\pi)^2} \), or in dimensional form \( \tau = 2\nu t/(2H_d)^2 \), as the ratio between the radius of the viscous core to the spacing between the vortices. Hence, the vorticity equation can be written as:

\[ 2\tau \frac{\partial w_{1,1}}{\partial \tau} - n \frac{\partial w_{1,1}}{\partial n} + Re \left\{ \left[ C_1 - \frac{3\psi_{1,1}}{n^2} \right] \frac{\partial w_{0,0}}{\partial n} + \frac{q(n)}{n^2} \frac{\partial w_{1,1}}{\partial \theta} \right\} \]
\[ + Re \tau \left\{ \left[ C_2 + \frac{3\psi_{1,1}}{n^2} \right] \frac{\partial w_{1,1}}{\partial n} + \frac{1}{n} \left\{ C_1 + \frac{3\psi_{1,1}}{n^2} \right\} \frac{\partial w_{1,1}}{\partial \theta} \right\} \]
\[ = \frac{1}{n} \frac{\partial}{\partial n} \left( n \frac{\partial w_{1,1}}{\partial n} \right) + \frac{1}{n^2} \frac{\partial^2 w_{1,1}}{\partial \theta^2} \]
\[ \frac{1}{n} \frac{\partial}{\partial n} \left( n \frac{\partial \psi_{1,1}}{\partial n} \right) + \frac{1}{n^2} \frac{\partial^2 \psi_{1,1}}{\partial \theta^2} = w_{1,1} \]  

where:
\( \psi_1 = \tau \psi_{1,1}(\tau, n, \theta) \)

Let us assume that the particular solution for the vorticity can be written as:

\[
\psi_{1,1} = \sum_{\kappa=0}^{\infty} \frac{\tau^{\kappa/2}}{\kappa!} \sum_{n=-\infty}^{\infty} A_{\kappa,n} e^{in\theta}
\]

\[
\psi_{1,1} = \sum_{\kappa=0}^{\infty} \frac{\tau^{\kappa/2}}{\kappa!} \sum_{n=-\infty}^{\infty} \rho_{\kappa,n} e^{in\theta}
\]

where

\[
A_{\kappa,n} = f(n) \]
\[
\rho_{\kappa,n} = f(n)
\]

The homogeneous solution will be discussed in the next section, and we will show that it is damped away. Because we seek real solutions, \( A_{\kappa,-n} = A_{\kappa,n}^*; \rho_{\kappa,-n} = \rho_{\kappa,n}^* \) where (*) denotes the complex conjugate. In this representation, the value of \( \psi_{1,1} \) can be found as:

\[
\frac{1}{n} \frac{\partial}{\partial n} \left( n \frac{\partial \rho_{\kappa,n}}{\partial n} \right) - \frac{n^2}{n^2} \rho_{\kappa,n} = A_{\kappa,n}
\]

(2-11)

For convenience, we shall use \( \frac{\partial}{\partial n} \) rather than \( \frac{d}{d\eta} \), even though \( \rho_{\kappa,n}(n) \) and \( A_{\kappa,n}(n) \) are only functions of \( n \). Since we expect the vorticity to decay faster than any power, we take \( A_{\kappa,n} \) for the asymptotic result to be zero. The asymptotic solution to this equation is:

\[
\rho_{\kappa,n} = \frac{\beta_{\kappa,n}}{n|n|}
\]

(2-12)
where $\beta_{\kappa,n}$ is a constant. Hence, the asymptotic representation of the stream function can be written as:

$$
\psi_{1,1}\big|_{n=\infty} = \sum_{\kappa=0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{\beta_{\kappa,n}}{n|n|} e^{i\theta} \left( \sum_{\kappa=0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{\beta_{\kappa,n}}{n|n|} e^{i\theta} \right)
$$

$$
= 2\text{Re} \left\{ \sum_{\kappa=0}^{\infty} \tau^{\kappa/2} \sum_{n=0}^{\infty} \frac{\beta_{\kappa,n}}{n|n|} e^{i\theta} \right\} \left( \sum_{\kappa=0}^{\infty} \sum_{n=0}^{\infty} \frac{\beta_{\kappa,n}}{n^2} \right) \right\}
$$

$$
(2-13)
$$

where $z = ne^{i\theta}$. But the last expression may be written in complex form,

$$
\tilde{w}_{1,1} \equiv \tilde{\psi}_{1,1} + i\tilde{\phi}_{1,1} = 2 \sum_{\kappa=0}^{\infty} \sum_{n=0}^{\infty} \frac{\beta_{\kappa,n}}{z^n}
$$

$$
(2-14)
$$

where $\phi_{1,1}$ is the harmonic conjugate potential function. Hence:

$$
\phi_1 = -i\tilde{w}_{1,1} = \phi_{1,1} + i(-\psi_{1,1}) = -2 \sum_{\kappa=0}^{\infty} \sum_{n=0}^{\infty} \frac{i\beta_{\kappa,n}}{z^n}
$$

$$
(2-15)
$$

Note that the definition of the stream function used in this research differs from the conventional one by a negative sign (see the derivation of the velocity from the stream function).

To complete the construction of the solution, we must add a contribution that assures that there is no flow through the line of symmetry. Let $\phi_1^+ = \phi_1$ be the solution associated with the positive vortex. To this we add the mirror image $\phi_1^-$, which we assume was the form
\[
\phi^- = -2i \sum_{k=0}^{\infty} \tau^{k/2} \sum_{n=0}^{\infty} \frac{\beta_{k,n} e^{-i\pi n}}{n! e^{-i\eta n}}
\]

Thus, we require

\[
\text{Im} \{ \phi^- + \phi^+ \} = -2 \text{Im} \sum_{k=0}^{\infty} \tau^{k/2} \sum_{n=0}^{\infty} \left\{ \frac{i\beta_{k,n} e^{-i\pi n}}{n! e^{-i\eta n}} + \frac{i\beta_{k,n} e^{i\pi n}}{n! e^{-i\eta n}} \right\} = 0 \quad (2-16)
\]

Or:

\[
\beta_{k,n} = -\beta_{k,n} e^{i\pi}
\]

(2-17)

The last expression guarantees that equation (2-16) has only a real part. Hence, the asymptotic behavior of the induced complex potential of the negative vortex \( \phi^- \) can be written in the neighborhood of the positive one as:

\[
\phi^- = -\frac{1}{\tau} \ln \left[ z + \frac{1}{\sqrt{\tau}} \right] + 2\tau \sum_{\kappa=0}^{\infty} \tau^{1/2} \sum_{n=-\infty}^{\infty} \frac{i\beta_{k,n} e^{i\pi n}}{z + \frac{1}{\sqrt{\tau}}^n} \quad (2-18)
\]

where the induced velocity in the stationary coordinates can be written as:

\[
(v_r - i v_\theta) e^{i\theta} = \frac{1}{2a \tau^{1/2}} \frac{dw}{dz} =
\]

\[
= \frac{i}{2a} \left\{ \frac{1}{z + \frac{1}{\sqrt{\tau}^2}} - 2 \sum_{\kappa=0}^{\infty} \tau^{(k+1)/2} \sum_{n=0}^{\infty} \beta_{k,n} e^{i\pi n} \frac{n}{[1 + \tau^{1/2}z^{n+1}]} \right\} \quad (2-19)
\]
The induced velocity at $z = 0$ must be subtracted in order to express the induced velocity in the moving coordinates, i.e.:

$$v_r^{(I)} - iv_\theta^{(I)} =$$

$$= \frac{i}{2a} e^{i\theta} \left\{ \frac{1}{1 + \tau^{1/2} z} - 1 - 2 \sum_{\kappa=0}^{\infty} \frac{\tau(\kappa+1)/2}{\tau(n+1)/2} \beta_{\kappa,n} \ e^{i\pi n} \right\}$$

$$\cdot \left\{ \frac{1}{(1 + \tau^{1/2} z)^{n+1}} \right\}$$

(2-20)

It can be seen that the last term $m$ in the brackets $\{ \}$ is of order $O(\tau^2)$, since

$$\sum_{\kappa} \tau^{(\kappa+1)/2} \sum_{n} \tau^{(n+1)/2} \beta_{\kappa,n} \ e^{i\pi n} \left[ \frac{1}{[1 + \tau^{1/2} z]^{n+1}} - 1 \right]$$

$$= \sum_{\kappa} \tau^{(\kappa+1)/2} \sum_{n} \tau^{(n+1)/2} \beta_{\kappa,n} \ e^{i\pi n} \left\{ (n+1) \tau^{1/2} z + ... \right\}$$

$$= O(\tau^2)$$

(2-21)

We have decided to work up to $O(\tau^{3/2})$, which means that the induced velocity is:

$$v_r^{(I)} - iv_\theta^{(I)} = \frac{i}{2a} e^{i\theta} \left\{ \frac{1}{1 + \tau^{1/2} z} - 1 \right\} = \frac{i}{2a} e^{i\theta} \sum_{\kappa=1}^{2} \tau^{\kappa/2} (-z)^{\kappa} + O(\tau^{3/2})$$

$$= \frac{i}{2a} \tau^{1/2} \sum_{\kappa=1}^{2} \tau^{(\kappa-1)/2} (-n)^{\kappa} e^{i(\kappa+1)} + O(\tau^{3/2})$$

(2-22)

The velocity components are,
From (2-7),

\[ C_1(T,n,e) = \sum_{K=1}^{K(l-1)/2} (-n)^K \sin(K+1)e + O(T) \]

\[ C_2(T,n,e) = \sum_{K=1}^{K(l-1)/2} (-n)^K \cos(K+1)e + O(T) \]

One point that we have to be aware of is that for the expansion of \(1/(1+\tau^{1/2}z)\) to be valid, the ratio \(\tau^{1/2}z\) \(\ll 1\). For small time \(\tau \ll 1\), we can expect that the vorticity will decay given this condition.

\(C_1, C_2\) can be expressed in the following form:

\[ C_1 = i \sum_{K=0}^{\infty} \tau^{K/2} \sum_{n=-\infty}^{\infty} C_{1,K,n} e^{i n \theta} \]

\[ C_2 = \sum_{K=0}^{\infty} \tau^{K/2} \sum_{n=-\infty}^{\infty} C_{2,K,n} e^{i n \theta} \]

\[ C_{1,K,n} = \frac{1}{2} \delta_{K,n} |n|^{-2} (-n)^{n|-1} \text{sign}(n) \]

\[ C_{2,K,n} = -\frac{1}{2} \delta_{K,n} |n|^{-2} (-n)^{n|-1} \]

where \(\delta_{i,j} = 1\) for \(i = j\), and \(\delta_{i,j} = 0\) for \(i \neq j\). Hence, by inserting equation (2-10) into equation (2-9), the following expression can be obtained:
\[ \sum_{\kappa} \left\{ \kappa A_{\kappa,n} - n \frac{\partial A_{\kappa,n}}{\partial n} + i \text{Re} \left\{ \left[ C_{1,\kappa,n} - n \frac{\rho_{\kappa,n}}{n} \right] \frac{\partial w_{0,0}}{\partial n} \right\} \right\} \tau^{|n|} + \frac{q(n)}{n^2} n A_{\kappa,n} \right\} \tau^{\kappa/2} e^{i n \theta} \]

\[ = \sum_{\kappa} \sum_{n} \left\{ \kappa A_{\kappa,n} - n \frac{\partial A_{\kappa,n}}{\partial n} \right\} \tau^{\kappa/2} e^{i n \theta} \]

where \(f_{\kappa,n}\) is obtained from:

\[ \sum_{\kappa} \sum_{n} f_{\kappa,n} e^{i n \theta} \tau^{\kappa/2} = \text{Re} \left\{ \left[ C_{1} - \frac{\partial \psi_{1,1}}{\partial n} \right] \frac{\partial w_{1}}{\partial n} \right\} \]

\[ + \frac{1}{n} \left\{ C_{2} + \frac{\partial \psi_{1,1}}{\partial n} \right\} \frac{\partial w_{1}}{\partial n} \]

\[ = \text{Re} \left\{ \sum_{\kappa} \sum_{n} \sum_{\kappa_{1}} \sum_{n_{1}} \left\{ C_{1,\kappa,n} - n \frac{\rho_{\kappa,n}}{n} \right\} \frac{\partial A_{\kappa_{1},n_{1}}}{\partial n} \right\} \tau^{(\kappa+\kappa_{1}+2)/2} e^{i(n+n_{1}) \theta} \]

For each \(\kappa\) and for each \(n\), equation (2.26) can be written

\[ -\kappa A_{\kappa,n} + n \frac{\partial A_{\kappa,n}}{\partial n} - i \text{Re} \left\{ \frac{q(n)}{n^2} n A_{\kappa,n} - n \frac{\rho_{\kappa,n}}{n} \frac{\partial w_{0,0}}{\partial n} \right\} \]

\[ + \frac{1}{n} \frac{\partial}{\partial n} \left\{ n \frac{\partial A_{\kappa,n}}{\partial n} \right\} - \frac{n^2}{n^2} A_{\kappa,n} \]

\[ = f_{\kappa,n} + i \text{Re} C_{1,\kappa,n} \frac{\partial w_{0,0}}{\partial n} \]

In order to evaluate \(f_{\kappa,n}\) in equation (2.27), define:
\[ \lambda = \kappa + \kappa_1 + 2 \quad \quad 0 \leq \lambda \leq \infty \]
\[ m = n + n_1 \quad \quad -\infty < m < \infty \]

Then:

\[ \sum_{\lambda} \sum_{m} f_{\lambda, m} e^{im\theta} e^{\lambda \theta / 2} = \]

\[ = i \text{Re} \sum_{\lambda=2}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{\kappa=0}^{\infty} \sum_{n=\infty}^{\infty} \left\{ C_{1,\kappa,n} - n \frac{\rho_{\kappa,n}}{n} \right\} \frac{\partial A_{\lambda-\kappa-2,m-n}}{\partial \eta} \]

\[ + \frac{1}{n} \left\{ C_{2,\kappa,n} + \frac{\partial \rho_{\kappa,n}}{\partial \eta} \right\} (m-n) A_{\lambda-\kappa-2,m-n} \}

(2-29)

For \( \lambda > 2 \),

\[ f_{\lambda, m} = i \text{Re} \sum_{\kappa=0}^{\infty} \sum_{n=-\infty}^{\infty} \left\{ C_{1,\kappa,n} - n \frac{\rho_{\kappa,n}}{n} \right\} \frac{\partial A_{\lambda-\kappa-2,m-n}}{\partial \eta} \]

\[ + \frac{1}{n} \left\{ C_{2,\kappa,n} + \frac{\partial \rho_{\kappa,n}}{\partial \eta} \right\} (m-n) A_{\lambda-\kappa-2,m-n} \}

For \( \lambda = 0, 1 \), then \( f_{\lambda, m} = 0 \). For \( \lambda = 2 \), then \( \kappa = 0 \), hence:

\[ f_{2, m} = i \text{Re} \sum_{n=-\infty}^{\infty} \left\{ \left\{ C_{1,0,n} - n \frac{\rho_{0,n}}{n} \right\} \frac{\partial A_{0,m-n}}{\partial \eta} + \frac{1}{n} \left\{ C_{2,0,n} + \frac{\partial \rho_{0,n}}{\partial \eta} \right\} (m-n) A_{0,m-n} \right\} \]

From (2-25), \( C_{1,0,n} \) and \( C_{2,0,n} \) have finite values only when \( n = \pm 2 \).

Thus, equations (2.28) and (2.25) which are two equations for \( A_{\kappa,n} \) and \( \rho_{\kappa,n} \), have a nonvanishing forcing term only when \( n = \pm 2 \). So, \( \rho_{\kappa,n} \) and \( A_{\kappa,n} \) have non-zero values only when \( n = \pm 2 \). Finally, \( f_{2, m} \) has non-zero values when \( n = \pm 2 \) and \( m-n = \pm 2 \); hence:
\[ f_{2,4} = i \text{ Re} \left\{ \left[ \frac{\text{sgn}(n)}{2} (-n) \left| n \right|^{-1} - n \frac{\rho_0 n}{\eta} \right] \frac{\partial A_0}{\partial n} - n \frac{\rho_0 n}{\eta} \right\} \]

\[ + \frac{1}{n} \left\{ - \frac{1}{2} (-n) \left| n \right|^{-1} + \frac{3\rho_0 n}{\partial n} \right\} (m-n) A_{0,m-n} \bigg|_{m=4}^{n=2} \]

\[ f_{2,4} = i \text{ Re} \left\{ \left[ \frac{n}{2} + 2 \frac{\rho_0,2}{n} \right] \frac{\partial A_0,2}{\partial n} + \frac{1}{n} \left\{ \frac{n}{2} + \frac{3\rho_0,2}{\partial n} \right\} 2 A_{0,2} \right\} \quad (2-30) \]

and for \( m = 0, n = \pm 2: \)

\[ f_{2,0} = i \text{ Re} \left\{ \left[ \frac{n}{2} + 2 \frac{\rho_0,2}{n} \right] \frac{\partial A_0,2}{\partial n} + \frac{1}{n} \left\{ \frac{n}{2} + \frac{3\rho_0,2}{\partial n} \right\} (-2) A_{0,-2} \right\} \]

\[ + \left\{ \frac{n}{2} + 2 \frac{\rho_0,2}{n} \right\} \frac{\partial A_0,2}{\partial n} + \frac{1}{n} \left\{ \frac{n}{2} + \frac{3\rho_0,2}{\partial n} \right\} 2 A_{0,2} \right\} \text{ Re} \quad (2-31) \]

This may be written as

\[ f_{2,0} = 2 \text{ Im} \left\{ \left[ \frac{n}{2} + 2 \frac{\rho_0,2}{n} \right] \frac{\partial A_0,2}{\partial n} + \frac{2}{n} \left\{ \frac{n}{2} + \frac{3\rho_0,2}{\partial n} \right\} A_{0,2}^* \right\} \text{ Re} \quad (2-32) \]

where we have used \( A_{\kappa,-n} = A_{\kappa,n}^* \) (see (2.10)).

Hence, equations (2.11) and (2.26) can be written as

\[ -\kappa A_{\kappa,n} + n \frac{\partial A_{\kappa,n}}{\partial n} = - i \text{ Re} \left\{ \frac{q(n)}{n^2} n A_{\kappa,n} - n \frac{\rho_0 n}{\eta} \frac{\partial A_{0,0}}{\partial n} \right\} \]

\[ + \frac{1}{n} \frac{\partial}{\partial n} \left\{ n \frac{\partial A_{\kappa,n}}{\partial n} \right\} - \frac{n^2}{n^2} A_{\kappa,n} \]

\[ = f_{\kappa,n} + i \text{ Re} \left\{ \frac{3\kappa,0}{\eta} \frac{\partial A_{0,0}}{\partial n} \right\} \quad (2-33) \]
\[ \frac{1}{n} \frac{\partial}{\partial n} \left[ n \frac{\partial \rho_{\kappa,n}}{\partial n} \right] = \frac{n^2}{n^2} \rho_{\kappa,n} = A_{\kappa,n} , \]

where

\[ f_{\kappa,n} = \begin{cases} 0 & 0 \leq \kappa < 2 \\ 1 \text{ Re} \left\{ \left[ \frac{n}{2} + 2 \frac{\rho_{0,2}}{n} \frac{\partial A_{0,2}}{\partial n} + \frac{1}{n} \left( \frac{n}{2} + \frac{3\rho_{0,2}}{\partial n} \right) A_{0,2} \right] \right\} & \kappa = 2, n = 4 \end{cases} (2-34) \]

\[ 2 \text{ Im} \left\{ \left[ \frac{n}{2} + 2 \frac{\rho_{0,2}}{n} \frac{\partial A_{0,2}}{\partial n} + \frac{2}{n} \left( \frac{n}{2} + \frac{3\rho_{0,2}}{\partial n} \right) A_{0,2} \right] \right\} \text{ Re} \kappa = 2, n = 0 \]

and

\[ q(n) = 1 - e^{-n^2/2}, \quad w_{0,0} = e^{-n^2/2} \]

These results can be expressed more simply by using the expansions value to \( O(\tau^{3/2}) \)

\[ \psi_{1,1} = \rho_2 e^{2i\theta} + \tau^{1/2} \rho_3 e^{3i\theta} + \tau (\rho_4 e^{i4\theta} + \rho_0) \]

\[ w_1 = A_2 e^{2i\theta} + \tau^{1/2} A_3 e^{3i\theta} + \tau (A_4 e^{i4\theta} + A_0) \]  

(2-35)

The equations (2-33) can now be written as
\[-\kappa A_{\kappa+2} + n \frac{\partial A_{\kappa+2}}{\partial n} - i \text{Re} \ (\kappa+2) \left\{ \frac{q(n)}{n^2} A_{\kappa+2} - \frac{\rho_{\kappa+2}}{n} \frac{\partial w_{0,0}}{\partial n} \right\} \]

\[+ \frac{1}{n} \frac{\partial}{\partial n} \left( n \frac{\partial A_{\kappa+2}}{\partial n} \right) - \frac{(\kappa+2)^2}{n^2} A_{\kappa+2} \]

\[= f_\kappa + \frac{i}{2} \text{Re} \ (-n)^{\kappa+1} \frac{\partial w_{0,0}}{\partial n} \]

\[\frac{1}{n} \frac{\partial}{\partial n} \left( n \frac{\partial \rho_{\kappa+2}}{\partial n} \right) - \frac{(\kappa+2)^2}{n^2} \rho_{\kappa+2} = A_{\kappa+2} \]

\[f_\kappa = \begin{cases} 
0 & 0 \leq \kappa < 2 \\
2 = i \text{Re} \left\{ \left( -\left( \frac{n}{2} + 2 \frac{\rho_2}{n} \right) \frac{\partial A_2}{\partial n} + \frac{1}{n} \left( \frac{n}{2} + 2 \frac{\partial A_2}{\partial n} \right) \right) 2A_2 \right\} \end{cases} \quad (2-36)\]

where we are only interested in \(0 \leq \kappa \leq 2\). Finally,

\[2A_0 = n \frac{\partial A_0}{\partial n} - \frac{1}{n} \frac{\partial}{\partial n} \left( n \frac{\partial A_0}{\partial n} \right) = -2 \text{Im} \left\{ \left( \frac{n}{2} + 2 \frac{\rho_2}{n} \right) \frac{\partial A_0^*}{\partial n} \right\} \text{Re} \]

\[+ \frac{2}{n} \left( \frac{n}{2} + \frac{\partial^2}{\partial n} \right) A_0^* \]

These equations describe the particular solution for the influence of the negative vortex on the positive one. Next, we consider the homogeneous solution so that the correct initial conditions may be satisfied.
2.2 The Homogeneous Solution

In the previous section, we found the particular solution for 2-D counter-rotating vortices. However, we must take the initial conditions into account. We assume that at \( \tau = \tau_0 \), the vorticity is a Gaussian distribution. This assumption seems reasonable for a numerical computation of this problem, where in the limit \( \tau_0 = 0 \) and the vorticity distribution is a delta function. Since, for a short time, we can regard these vortices as independent of each other, we expect the vorticity to be mainly distributed as a Gaussian profile.

The general solution for a short time can be written

\[
\psi_{1,1} = \sum_{\kappa=0}^{\infty} \tau^{\kappa/2} \sum_{n=-\infty}^{\infty} (A^P_{\kappa,n} + A^H_{\kappa,n}) e^{i n \theta}
\]

where

\[
\begin{align*}
A^P_{\kappa,n} &= f(n) \\
p^P_{\kappa,n} &= f(\tau,n) \\
A^H_{\kappa,n} &= f(n) \\
p^H_{\kappa,n} &= f(\tau,n)
\end{align*}
\]

but
where \( A^p_{k, n}, \rho^p_{k, n} \) are the particular solution described in the previous section and \( A^H_{k, n}, \rho^H_{k, n} \) are the homogeneous solutions. Since at \( \tau = \tau_0 \), \( w_1 \) and \( \psi_{11} = 0 \), then

\[
\begin{align*}
A^H_{k, n} &= -A^p_{k, n} \\
\rho^H_{k, n} &= -\rho^p_{k, n}
\end{align*}
\]

at \( \tau = \tau_0 \)

The boundary conditions are the same as those for the particular solution, namely

\( \rho^H_{k, n} \) and \( A^H_{k, n} \) + 0 as \( n \to \infty \)

and

\( \rho^H_{k, n} = A^H_{k, n} = 0 \) at \( \eta = 0 \).

The homogeneous part must satisfy

\[
-2 \tau \frac{A^H_{k, n}}{\rho^H_{k, n}} - \kappa A^H_{k, n} + \frac{3 \rho^H_{k, n} A^H_{k, n}}{\rho^H_{k, n}} = \text{in Re} \left\{ \frac{1-e^{-\eta^2/2}}{\eta^2} A^H_{k, n} + e^{-\eta^2/2} \rho^H_{k, n} \right\}
\]

\[
+ \frac{1}{\eta} \frac{3 \rho^H_{k, n} A^H_{k, n}}{\rho^H_{k, n}} \right\} - \frac{n^2}{\eta^2} A^H_{k, n} = 0
\]

(2-37)

The solution of this system can be written as a linear combination of the eigenfunctions
\[
\begin{bmatrix}
A_{\kappa,n}^H \\
\rho_{\kappa,n}^H
\end{bmatrix} = \sum_{\alpha} \left( \frac{i}{\tau_0} \right)^{\alpha} \begin{bmatrix}
C_{\alpha} \phi_{\alpha}(n) \\
\mu_{\alpha} C_{\alpha} B_{\alpha}(n)
\end{bmatrix}
\]

Note that each \( \alpha \) will depend on \( \kappa \) and \( n \). The equations for the eigenfunctions are

\[-2\alpha_1 \phi_{\alpha} + n \frac{\partial \phi_{\alpha}}{\partial n} - \text{Re} \left\{ \frac{1 - e^{-n^2/2}}{n^2} \phi_{\alpha} + e^{-n^2/2} B_{\alpha} \mu_{\alpha} \right\} + \frac{1}{n} \frac{\partial}{\partial n} \left( n \frac{\partial \phi_{\alpha}}{\partial n} \right) - \frac{n^2}{n^2} \phi_{\alpha} = 0.\]

(2-38)

\[
\frac{1}{n} \frac{\partial}{\partial n} \left( n \frac{\partial B_{\alpha}}{\partial n} \right) - \frac{n^2}{n^2} B_{\alpha} = \frac{\phi_{\alpha}}{\mu_{\alpha}}
\]

where \( \alpha_1 = \alpha + \frac{\kappa}{2} \).

Multiplying the first equation in (2.38) by \( n e^{n^2/2} \phi_{\alpha}^* \) (\( \phi_{\alpha}^* \) is the complex conjugate), adding it to the complex conjugate of the resulting equation, and then integrating, one obtains

\[
-4 \alpha_1 (r) \left[ \int_0^{\infty} n e^{n^2/2} \phi_{\alpha} \cdot \phi_{\alpha}^* d n + \int_0^{\infty} n^2 e^{n^2/2} \frac{\partial}{\partial n} (\phi_{\alpha} \cdot \phi_{\alpha}^*) d n \right]
\]

\[
- \text{in Re} \left[ \int_0^{\infty} (\mu_{\alpha} B_{\alpha} \phi_{\alpha}^* - \mu_{\alpha}^* B_{\alpha}^* \phi_{\alpha}) n d n \right]
\]
\[
+ \int_{0}^{\infty} \left[ \frac{\Phi_{n}}{n} \frac{\partial}{\partial n} \left( n \frac{\partial \Phi_{n}}{\partial n} \right) + \frac{\Phi_{n}}{n} \frac{\partial}{\partial n} \left( n \frac{\partial \Phi_{n}}{\partial n} \right) \right] e^{-\eta^2/2} \, d\eta
\]

\[- n^2 \int_{0}^{\infty} \frac{\Phi_{n} \cdot \Phi_{n}^{*}}{n} e^{-\eta^2/2} \, d\eta = 0 \quad (2-39)\]

where \(\alpha_{1}^{(r)}\) is the real part of \(\alpha_{1}\).

The third term in equation (2-39) can be shown to be zero, i.e.,

\[
\int_{0}^{\infty} \mu_{\alpha} B_{\alpha} \Phi_{n} \, d\eta = \mu_{\alpha} B_{\alpha} \int_{0}^{\infty} \left( \eta \frac{\partial B_{\alpha}^{*}}{\partial \eta} \right) - \frac{n^2}{n} B_{\alpha}^{*} \, d\eta
\]

\[
= - \mu_{\alpha} B_{\alpha} \int_{0}^{\infty} \left( \eta \frac{\partial B_{\alpha}^{*}}{\partial \eta} + \frac{n^2}{n} B_{\alpha} \cdot B_{\alpha}^{*} \right) \, d\eta
\]

Note that integration by parts has been used and that

\[
\frac{\partial B_{\alpha}}{\partial \eta} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty
\]

since we require the stream function to decay at \(\infty\). Since the integral is real,

\[
\int_{0}^{\infty} \left( \mu_{\alpha} B_{\alpha} \cdot \Phi_{n}^{*} - \mu_{\alpha} B_{\alpha}^{*} \Phi_{n} \right) \, d\eta = 0.
\]

which establishes the result. Hence, after the integration by parts of the fourth term of (2-39), some terms cancel and one obtains
-4a_1(r) \int_0^\infty n \, e^{\eta^2/2} \phi_\alpha \cdot \phi^*_\alpha \, d\eta = 2 \int_0^\infty \left[ n \frac{\partial \phi_\alpha}{\partial \eta} \cdot \frac{\partial \phi^*_\alpha}{\partial \eta} \right] e^{\eta^2/2} \, d\eta
+ 2n^2 \int \frac{\phi_\alpha \cdot \phi^*_\alpha}{n} e^{\eta^2/2} \, d\eta
\tag{2-40}

Clearly, for a non-trivial solution of $$\phi_\alpha$$ to exist, we must have $$a_1^{(r)} < 0$$ since the right-hand side of (2-40) must be positive. Consequently,

$$A_{\kappa,n} = \sum_\alpha \left( \frac{\tau_0}{\tau} \right)^{\left( \frac{\kappa}{2} - a_1 \right)} C_\alpha \, \phi_\alpha + A^p_{\kappa,n} \tag{2-41}$$

$$\sum_\alpha C_\alpha \, \phi_\alpha(n) = -A^p_{\kappa,n}.$$ Note that $$\phi_\alpha(n)$$ can be specified independently of $$\tau_0$$ and so $$C_\alpha$$ is independent of $$\tau_0$$. Actually, the appropriate choice of an inner product in which the eigenfunctions are orthogonal was beyond our ability. However, the key observation is that for small $$\tau_0$$, so that $$\frac{\tau}{\tau_0} \gg 1$$ but $$\tau \ll 1$$, $$A_{\kappa,n}$$ tends to $$A^p_{\kappa,n}$$. In particular, if $$\tau_0 = 0$$, the particular solution is the only solution to the problem.

We can also show that the homogeneous solution to (2-37) is unique. If we keep $$\tau \frac{\partial A}{\partial \tau}$$ rather than $$a_1^{(r)} \phi_\alpha$$, and let $$\psi$$ be the difference between two solutions of (2-37) which both satisfy the same boundary and initial conditions, then we find,

$$\tau \frac{\partial}{\partial \tau} \int_0^\infty n \, e^{\eta^2/2} (\psi \cdot \psi^*) \, d\eta = -2 \int_0^\infty \left[ n \, \frac{\partial \psi}{\partial \eta} \cdot \frac{\partial \psi^*}{\partial \eta} \right] e^{\eta^2/2} \, d\eta
-2n^2 \int_0^\infty \frac{\psi \cdot \psi^*}{n} e^{\eta^2/2} \, d\eta
\tag{2-42}$$
in the same way that (2-40) is derived.

Since the right-hand side of (2-42) is negative, the equation states that a positive quantity must decay. However, the initial value is zero and hence \( \psi = 0 \) is the only possibility. Hence, the solution to (2-37) is unique.

2.3 Spacing of Vortices and Descent Velocity

One of the most controversial points in studies of counter-rotating vortices in stratified environments is the spacing between the two vortices. It is well known that in ideal fluids the two vortices will descend with a constant speed and a constant spacing (Widnall, [1975]). This analysis and subsequent numerical solutions show that even in unstratified fluids, the spacing between the vortices is not constant but grows. As will be seen later, the motion of the spacing between vortices is a result of viscous effects.

The motion of the origin of coordinates located at the maximum of the positive vortex can be written as

\[
\frac{1}{(2a)^2} \frac{d\xi^*}{dt} = \text{Re} \left( v_r - iv_\theta \right) e^{i\theta} = \frac{\text{Re}}{2} \cdot \frac{1}{2a} \frac{1}{\tau^{1/2}} \frac{dw}{dz} \bigg|_{z=0^+} \tag{2-43}
\]

\[ \xi^* = x - iY \]

where \( \frac{dw}{dz} \) is given by equation (2-19), thus

\[
u - iv = \frac{i}{2a} \left\{ 1 - 2 \sum_{\kappa=0}^{\infty} \tau^{(\kappa+1)/2} \sum_{n=0}^{\infty} \frac{\tau^{(n+1)/2}}{n!} b_{\kappa,n} e^{in\pi n} \right\} \tag{2-44} \]
Finally, the motion is described by

\[
\frac{dx^*}{d\tau} = i \Re a \left\{ 1 - 2\tau \sum_{\kappa=0}^{\infty} \sum_{n=0}^{\kappa} \beta_{\kappa-n,n} e^{i\pi n} \right\}
\]

or

\[
x^* = i \Re a \left\{ \tau - 4 \sum_{\kappa=0}^{\infty} \sum_{n=0}^{\kappa} \beta_{\kappa-n,n} e^{i\pi n} \right\}
\]

Specifically, the descent distance is

\[
Y = -\Re a \left\{ \tau - 4 \sum_{\kappa=0}^{\infty} \sum_{n=0}^{\kappa} \beta_{\kappa-n,n} e^{i\pi n} \right\} \Re a
\]

and the vortex spacing is

\[
x = -4 \Re a \sum_{\kappa=0}^{\infty} \sum_{n=0}^{\kappa} \Im(\beta_{\kappa-n,n})(-1)^n
\]

Since \( \beta_{\kappa-n,n} \sim \delta_{\kappa-n,n-2} \) or \( \kappa = 2n-2, \) i.e., \( \kappa \) is even, we find

\[
x = 2 \Re a \sum_{\kappa=1}^{3} \frac{\tau^{\kappa+2}}{\kappa+2} \Im(\beta_{\kappa-1,\kappa+1})(-1)^{\kappa+1}(\kappa+1)
\]

Explicitly

\[
x = 2 \Re a \left\{ \frac{2}{3} \Im(\beta_{0,2}) - \frac{3}{4} \tau \Im(\beta_{1,3}) + \frac{4}{5} \tau^2 \Im(\beta_{2,3}) \right\}
\]

where \( \beta_{\kappa,n} \) is the asymptotic behavior which can be derived from the following expressions
Our subsequent calculations show that \( \text{Im}(\beta_{0,2}) > 0 \) and so the spacing between the vortices increases as a function of the ratio between the radius of the viscous core, and the initial spacing between the vortices. A particular case occurs when it is zero, i.e., in ideal fluids, the spacing remains constant and the vortex pair descends with a constant spacing between them (Widnall, [1975]).

2.4 Numerical Scheme and Numerical Boundary Conditions

The set of equations (2-36) was solved numerically, basically because it is easier to solve these linear O.D.E.'s numerically rather than to find an analytic solution.

We pointed out before that the similarity solution allowed us to solve an O.D.E., as opposed to a P.D.E. This is advantageous because it will save computer time for the short time evolution of the vorticity, compared with the time required for the spectral solution.

Before describing the numerical method used to solve (2-36), we first discuss the numerical boundary conditions. The numerical boundary conditions at the edge of the computational domain are,

\[
\rho_\kappa = \frac{\text{const}}{\eta^\kappa} \quad (2-51)
\]
\[ \frac{\partial \rho_{\kappa}}{\partial n} + \frac{k}{n} \rho_{\kappa} = 0, \quad \kappa = 2, 3, 4 \]

where \( \rho_0 = 0 \) (essentially equal to a constant, which was chosen as 0). This numerical boundary condition will be discussed in the next chapter.

The numerical boundary condition at \( n = 0 \) was chosen according to the physical behavior of the vorticity and the stream function. The quantities \( \psi, w \) are considered in the two-dimensional equations as scalars. Hence, in order to avoid isotropic behavior of both quantities, then \( A_\kappa = 0, \rho_\kappa = 0 \) for \( n = 0 \), and \( \kappa = 1, 2, 3, 4 \). For \( \kappa = 0 \) \( A_\kappa \) and \( \rho_\kappa \) can be regarded as smooth functions, due to the radial symmetry. Hence,

\[
\begin{align*}
\frac{3A_{\kappa}}{3n} &= 0 \\
& \quad n + 0, \quad \kappa = 0 \\
\frac{3\rho_{\kappa}}{3n} &= 0
\end{align*}
\]

An approximation of equation (2-36) by finite differences is

\[
-\kappa A_{\kappa+2,j} + \frac{1}{2} \left( A_{\kappa+2,j+1} - A_{\kappa+2,j-1} \right) - (\kappa+2) \left\{ A_{\kappa+2,j} Q_j + \rho_{\kappa+2,j} D_j \right\}
\]

\[
+ \frac{1}{n^2} \left\{ (1 + \frac{1}{2j}) A_{\kappa+2,j+1} - \left[ 2 + \frac{(\kappa+2)^2}{j^2} \right] A_{\kappa+2,j} + (1 - \frac{1}{2j}) A_{\kappa+2,j-1} \right\}
\]

\[
= f_{\kappa,j} + (-n_j)^{\kappa+2} D_j / 2
\]

and
\( (j + \frac{1}{2}) \rho_{k+2,j+1} - \left(2j + \frac{(k+2)^2}{j} \right) \rho_{k+2,j} + (j - \frac{1}{2}) \rho_{k+2,j-1} = h^2j A_{k+2,j} \) \hspace{1cm} (2-53)

where

\[ \eta_j = jh \]
\[ Q_j = i \text{Re} \frac{q(n)}{\eta_j^2} \]
\[ D_j = i \text{Re} e^{-\eta_j^2/2} \]

\[ f_{k,j} = \begin{cases} 0 & 0 \leq \kappa < 2 \\ f_{k,j} = i \text{Re} \left\{ \frac{\eta_j}{2} + 2 \frac{\rho_{2,j}}{\eta_j} \frac{A_{2,j+1} - A_{2,j-1}}{2h} \right. \\ \left. + \frac{1}{\eta_j} \left[ \frac{\eta_j}{2} + \frac{(\rho_{2,j+1} - \rho_{2,j-1})}{2h} \right] A_{2,j} \right\} \end{cases} \]

This set can be rearranged in the following form:

\[ a_{f,j} A_{k+2,j-1} + a_{g,j} A_{k+2,j} + a_{h,j} A_{k+2,j+1} + b_{f,j} \rho_{k+2,j} = f_j \] \hspace{1cm} (2-54)

\[ \lambda_{1,j+1} \rho_{k+2,j-1} + \lambda_{2,j+1} \rho_{k+2,j} + \lambda_{3,j+1} \rho_{k+2,j+1} = A_{k+2,j} \] \hspace{1cm} (2-55)

where the coefficients are obtained by the collection of terms that refer to each variable. In addition, we add the following approximations to the boundary conditions (2-51), (2-52),

\[ a_{1,1} = 0 \]
\[ a_{3,N} = 0 \]

and

\[ \rho_{k+2,N+1} - \rho_{k+2,N-1} + \frac{2k}{j} \rho_{k+2,N} = 0 \]

Substituting equation (2-55) into (2-54) leads to a pentadiagonal matrix.
which was solved with the Thomas algorithm (Roache, [1971]).

2.5 The Perturbation Results

The perturbation equations (2-35) were solved numerically as described above (equations 2-54, 2-55). The number of node points along the \( n \) direction is 1000, and the chosen computational length (\( n_{\text{max}} \)) is 20 (0 \( \leq n \leq n_{\text{max}} \)). The sensitivity of the results was checked by comparing the results for different mesh sizes, and different computational lengths. As can be seen in Figures 2.1 through 2.6, the computed results reach their asymptotic values even for a shorter computational domain length (\( n_{\text{max}} \)), i.e., \( n_{\text{max}} = 6 \).

The moment invariant \( I_2 \) was used to check accuracy. Now,

\[
I_2 = \int_A w(x) \, dA = \int_A w(a + r\cos\theta) \, dA = \int A_0 \, r \, dr \, d\theta
\]

The last term can be shown to be zero. We write the last equation of equation (2-36) in its conservative form, i.e.,

\[
2A_0 - n \frac{\partial A_0}{\partial n} - \frac{1}{n} \frac{\partial}{\partial n} \left( n \frac{\partial A_0}{\partial n} \right) = -2 \Re \left( \frac{1}{2n} \frac{\partial}{\partial n} (A_2^* \cdot n^2) + \frac{2}{n} \frac{\partial}{\partial n} (p_2 A_2^*) \right)
\]

By multiplying the last equation by \( n \) and integrating it by parts, it can be shown that the last term in the moment invariant is zero. Thus, the only contribution to \( I_2 \) is \( \Gamma a \). We found that \( I_2 \) was constant to \( O(\tau^{3/2}) \) for our calculations.

We will compare the asymptotic solution to the numerical solution in the next chapter. Further details on this phenomena will be given.
Figure 2.1 Perturbation of the vorticity for Re = 1.
Figure 2.2 Stream function perturbation for $Re = 1$. 
Figure 2.3 Perturbation of the vorticity for Re = 10.
Figure 2.4 Stream function perturbation for Re = 10.
Figure 2.5 Perturbation of the vorticity for Re = 100.
Figure 2.6 Stream function perturbation for Re = 100.

\[ a_{1,\kappa} + ia_{2,\kappa} = (\rho_{\kappa}) \cdot n^\kappa \]
3.1 The Coordinate System and the Choice of the Numerical Scheme

In the previous chapter, many questions appeared about the sensitivity of the solution to the initial conditions, and the spacing between the vortices that increased in time. These questions are to be checked numerically in this chapter.

Initially, the vorticity is concentrated in a narrow region. Physically, this means a sharp gradient in the vorticity distribution, in the region where the vorticity is concentrated, and as a result, fine resolution in this region is required. It is thus important to devise a proper coordinate system in order to resolve efficiently the problem numerically.

In order to solve the problem numerically, three types of coordinate systems will be considered here.

A - Planar coordinates (Figure 3.1)

B - A polar coordinate system in which the origin is located on the maximum of the positive vortex (Figure 3.2)

C - A polar coordinate system in which the origin is located on the line of symmetry (Figure 3.3).

There is little advantage to using the planar coordinates. As will be seen later, this is due to the numerical boundary condition
Figure 3.1 Planar coordinates.
Figure 3.2 Polar coordinate system in which the origin is located on the maximum of the positive vortex.
Figure 3.3 A polar coordinate system in which the origin is located on the line of symmetry.
that is required along the edge of the computational domain. In order to understand this, let us briefly consider the vorticity equation and the stream function equation, i.e.,

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} = \nu \nabla^2 w$$

(3-1)

$$\nabla^2 \psi = w$$

where $w$ is the vorticity and $\psi$ is the stream function.

To compute a numerical solution to these equations, a boundary condition along the edge of the computational domain is required. Since the stream function equation is elliptic and the vorticity equation is parabolic, boundary values for both of them must be specified. It is natural to require that $w = 0$ on the boundary, because we expect the vorticity to decay exponentially. As for the stream function, we assume that it behaves as any arbitrary constant as the radial distance from the vortex core goes to infinity. However, the choice $\psi = 0$ along the boundary (this constant was chosen as zero) causes considerable error because the stream function decays more slowly than the vorticity. Physically the condition $\psi = 0$ on the computational boundary generates a flow field like that shown in Figure 3.1. In this case, each boundary generates a mirror image which induces velocity that causes a sizeable error unless the boundary is far away. A better approximation is to assume that the stream function along the numerical boundary is that of a dipole, i.e.,
\[ \psi = \beta \frac{\sin \theta}{R} + O\left( \frac{1}{R^2} \right) \]

or

\[ \psi \sim \beta \frac{x}{x^2 + y^2} \]

where \( \beta \) is a constant, to be determined.

Since the error in the approximate boundary condition is of \( O(1/R^2) \), where \( R \) is the radial distance from the origin to the nearest point of the computational domain, reasonable accuracy can be obtained for moderate values of \( R \). Of course, increased accuracy can be obtained if necessary by including higher order poles in (3-2). The effect of the reflected computational domain will be discussed further in Chapter 4.

Spectral techniques are the most efficient in solving Poisson's equation for a given level of accuracy. As a consequence, it is natural to transform this problem to a radial coordinate system where one coordinate is periodic and then to apply the spectral method. The main advantage of the spectral method is its exponential rate of convergence to the exact solution when the solution is \( C^\infty \). However, at first glance, it seems that the spectral method, when applied to the vorticity equation, may require more computation than the finite difference method, in order to evaluate the nonlinear advection term. However, the use of pseudo-spectral techniques to evaluate nonlinear terms takes about as much effort as finite difference methods and the accuracy is much better. Thus, the number of computer operations required to evaluate the advection term for a given accuracy is much
less for spectral methods since less terms are required. Moreover, the elliptic term in the vorticity equation is linear, which means that in the Fourier representation the dissipation terms can be handled implicitly rather than explicitly, giving improved stability requirements on the time step. Finally, the application of the boundary condition (3-2) must be considered. Previous research has used the general solution of the stream function equation in integral form (Robert et al., [1976]), but this method required substantial computational work and became almost impractical. Fortunately, the Fourier representation for the stream function accommodates the numerical boundary condition in a natural way. Details are given in the next section.

In order to apply the spectral method, we have to consider cases B and C, where both are in polar coordinates. Case B is inferior compared to case C, even though in the first glance it seems promising, in the sense that the general solution can be written in a fashion similar to the solution in Chapter 2. In other words, the advection term can be treated semi-implicitly in the spectral method. As a result, it is possible to reduce the C.F.L. type errors, and to increase the time step. Even though that case B is the best choice in the sense of fine resolution of the physically sharp gradients, the implementation of the boundary condition along the line of symmetry is difficult. It is possible to solve the whole problem (including the negative vortex), but due to the coarse resolution near the negative vorticity compared to the fine resolution near the positive one, the
error may accumulate along the line of symmetry. Consequently, for long time evolution, the boundary condition \( w = 0 \) would not be fulfilled. Moreover, case B requires twice as many spectral modes as case C.

As a result of the limitation in the first two cases, we have decided to work with the coordinate system proposed in case C. Note than an odd representation in the Fourier series is enough to describe the flow field because of symmetry.

### 3.2 The Spectral Method

The vorticity and the stream function can be represented as

\[
\psi = \sum_{n=0}^{\infty} \psi_n(t,r) \sin(n\theta)
\]

\[
w = \sum_{n=0}^{\infty} w_n(t,r) \sin(n\theta)
\]

where \( \theta \) is the azimuthal angle measured from the negative \( y \)-axis (Figure 3.3). Upon substitution, equation (3-1) becomes

\[
\sum_{n} \left\{ \frac{\partial w_n}{\partial t} + F_n - \frac{1}{Re} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial w_n}{\partial r} \right) - \frac{n^2}{r^2} w_n \right] \right\} \sin(n\theta) = 0
\]

\[
\sum_{n} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial \psi_n}{\partial r} \right] - \frac{n^2}{r^2} \psi_n - w_n \right\} \sin(n\theta) = 0.
\]

where

\[
\left[ v_r \frac{\partial w}{\partial r} + \frac{v_\theta}{r} \frac{\partial w}{\partial \theta} \right] = \sum_{n} F_n \sin(n\theta)
\]
The orthogonality of the sine functions means that for each $n$

$$\frac{\partial w_n}{\partial t} + F_n - \frac{1}{Re} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial w_n}{\partial r} \right) - \frac{n^2}{r^2} w_n \right) = 0.$$  

and

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi_n}{\partial r} \right) - \frac{n^2}{r^2} \psi_n = w_n \quad (3-6)$$

The computation of $F_n$ is difficult in general but by restricting it to a finite range, pseudo-spectral techniques may be used.

To describe the pseudo-spectral method, we introduce a finite difference approach in $r$ and a central difference in time for the operator $\partial_t$. Hence, the vorticity equation is written

$$\frac{wn_{n+1}^{j+1} - wn_{n+1}^{j-1}}{2 \cdot DT} + F_{\kappa,j} = \frac{1}{Re} \left( \frac{1}{r_{j}} \delta_r (r_{j} \delta r) - \frac{n^2}{r_{j}^2} \right) \frac{wn_{n+1}^{j+1} + wn_{n+1}^{j-1}}{2} \quad (3-7)$$

$$\frac{1}{r_{j}} \delta_r \left( r_{j} \delta r \psi_{\kappa,j} \right) - \frac{n^2}{r_{j}^2} \psi_{\kappa,j} = w_{\kappa,j} \quad (3-8)$$

where $DT$ is the time step and

$$h \delta r f_{j} = f_{j + (1/2)} - f_{j - (1/2)} \quad (3-9)$$

$$r_{j} = j \cdot h$$

The vorticity and the velocity can be evaluated at $\kappa$ as:
\[ w^K_{n,j} = \frac{1}{r_j} \delta_r (r_j \delta_r \psi^K_{n,j}) - \frac{n^2}{r_j^3} \psi^K_{n,j} \]

\[ v^K_{r,m,j} = -\frac{1}{r_j} \sum_{n=1}^{N-1} n \psi^K_{n,j} \cos \frac{mn}{N} \]

\[ v^K_{\theta,m,j} = \sum_{n=1}^{N-1} \delta_r \psi^K_{n,j} \sin \frac{mn}{N} \]

\[ \frac{\partial w}{\partial r} |^K_{m,j} = \sum_{n=1}^{N-1} \delta_r w^K_{n,j} \sin \frac{mn}{N} \]

\[ \frac{\partial w}{\partial \theta} |^K_{m,j} = \sum_{n=1}^{N-1} n w^K_{n,j} \cos \frac{mn}{N} \]

All of these sums can be computed for each \((j)\) with the aid of the Fast Fourier Transform in \(O(N \log(N))\) computer operations.

Hence, for each \((j)\), the nonlinear term can be evaluated directly and then expressed in terms of a Fourier series,

\[ \{ v_r \frac{\partial w}{\partial r} + \frac{v_\theta}{r} \frac{\partial w}{\partial \theta} \}^K_{m,j} = \sum_n F^K_{n,j} \sin \frac{mn}{N} \]

and \(F^K_{n,j}\) is obtained by a Fast Fourier Transform.

This method is different from the method used by Orszag and Patera [1983] in that they computed the nonlinear terms for each time step and predicted them at \(\kappa + 1/2\), using the Adams Bashforth algorithm.
This scheme is the "Leap Frog" method for the inviscid part of the vorticity equation, and Crank-Nicolson for the dissipation part. The leap frog method is known to be conditionally stable for the hyperbolic equations. Hence, in this method, it is possible to devise a conditionally stable numerical scheme, due to the fact that the inviscid part of the vorticity equation is conditionally stable and the dissipation terms are unconditionally stable. For more details about the numerical stability, see Chapter 5.

The nonlinear term has been calculated at each time level ($\kappa$). It was found numerically that the choice of time step size for stable calculations is sensitive to the number of modes. For 16 modes and mesh size $h = .1$, the time step that was used was $DT = .01$, where for 32 modes and the same mesh size, the time step $DT = .005$ was found to give stable results.

This method has second order accuracy in time and in space. Second order accuracy in time is important in order to avoid error terms that simulate viscous effects. To avoid artificial viscosity, which is proportional to the time step, it would be necessary to severely limit the time step. The second order accuracy in space might generate a dispersion wave due to the spatial truncation error in the advection term. The "Leap Frog" scheme can be written in its differential form

$$
\frac{\partial w}{\partial t} + \nu_r \frac{\partial w}{\partial r} + \frac{v_0}{r} \frac{\partial w}{\partial \theta} + O(h^2) \frac{\partial^2 w}{\partial r^2} + O(h^2) \left[ \frac{\partial^2 w}{\partial r^2} , \frac{\partial w}{\partial r} \ldots \right] = \frac{1}{Re} \nu^2 w \quad (3-12)
$$

where terms of $O(t^2h^3)$ are neglected.
The linearized version of equation (3-12) permits dispersive waves in the absence of viscous dissipation and hence develop numerical dispersion waves in the case where $1/\text{Re} \ll O(h^2)$.

Finally, we describe how the boundary conditions are satisfied. The boundary conditions are satisfied along the line of symmetry due to the odd representation of the Fourier series, i.e.:

$$\psi(t,r,0) = 0$$

$$\psi(t,r,\pi) = 0$$

$$w(t,r,0) = 0$$

$$w(t,r,\pi) = 0$$

The other boundary conditions are applied at $r = R$ (R is the length of the computational domain). There $w = 0$ for all $n$. As was mentioned before, the stream function equation requires a different numerical boundary condition. If we examine equation (3-4), we can see that as $(r)$ goes to infinity, the stream function behaves asymptotically as a harmonic function (i.e., $w = 0$). Thus, its asymptotic behavior can be written as:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi_n}{\partial r} \right) - \frac{n^2}{r^2} \psi_n = 0$$

and the solution we require is
\[ \psi_n = \frac{\text{const}}{r^n} \]  

(3-15)

Essentially, the last expression describes the multi-pole expansion of the vortical behavior in the far field. We can take advantage of this representation of the stream function in order to specify the numerical boundary condition at \( r = R \) in a similar fashion to that done by Fasel [1976].

\[ \frac{\partial \psi_n}{\partial r} + \frac{n}{r} \psi_n = 0. \]  

(3-16)

Numerically,

\[ \delta_r \psi_n^j + \frac{n}{r^j} \psi_{n,j} = 0. \]  

(3-17)

Additional numerical boundary conditions are required at \( r = 0 \). At this point, as well as in the rest of the computational domain, we cannot expect non-isotropic behavior of the stream function or the vorticity. Hence, the only possibility is:

\[ [\psi] = 0. \text{ at } r = 0 \]  

(3-18)

This completes the description of the numerical method, except for the specification of the initial condition, which we return to in the next section.
3.3 The Initial Condition

The initial condition used in this research is a superposition of two Gaussian profiles where each one corresponds to the solution of a single viscous vortex in an infinite domain, i.e.,

\[ w = \left\{ \exp\left[-\frac{r_1^2}{2t_{in}}\right] - \exp\left[-\frac{r_2^2}{2t_{in}}\right]\right\} / t_{in} \]  \hspace{1cm} (3-19)

where

- \( r_1 \): radial distance from the location of the positive vortex
- \( r_2 \): radial distance from the location of the negative vortex
- \( t_{in} \): initial time.

As \( t_{in} \rightarrow 0 \), the initial condition becomes two delta functions. The delta function is the obvious choice if initial details in the vortex structure are unimportant. Since we are interested in whether the initial structure affects the viscous interaction, we choose \( t_{in} > 0 \). Also, it is impossible to start the numerical code from a delta function distribution. Since the initial interaction between the vortices is weak, it is acceptable to represent the initial vorticity distribution, as a Gaussian profile. This representation is realistic when compared to experimental results (Hecht, [1981]). For our studies, \( t_{in} \) was chosen as small as possible provided there was enough resolution in the spectral scheme to resolve the vorticity distribution.
For two-dimensional motion, it is well known that there are several invariants which are determined by the initial flow. In particular the vertical impulse is given by

\[ I_2 = \int_A w x \, da = \text{const.} \tag{3-20} \]

In the spectral representation, this integral is:

\[ I_2 = \left[ \int_0^\infty \int_0^\pi r^2 \sin \theta \sum \omega_n \sin(n\theta) \, d\theta \, dr \right] = \frac{\pi}{2} \left[ \int_0^\infty r^2 \omega_1 \, dr \right] \]

\[ = \frac{\pi}{2} \left[ \int_0^\infty r \frac{\partial}{\partial r} \left( r \frac{\partial \psi_1}{\partial r} \right) \, dr \right] - \int_0^\infty \psi_1 \, dr = -\pi C_1. \tag{3-21} \]

where

\[ \psi_1 = \frac{C_1}{r} \quad \text{as} \quad r \to \infty \]

This invariant provides one useful way to monitor the accuracy of the calculations.

### 3.4 Numerical Results

The vorticity equation was solved spectrally in the \( \theta \) direction, and with a finite difference approximation in the radial direction for Reynolds numbers (1, 10, 100, 1000). The initial value was taken as \( t_{in} = .25 \). The results were computed with between 100 and 200 nodal points in the radial direction, and 16 modes in the \( \theta \) direction. There is a small difference between the computed results at each Reynolds number (see Figures 3.4 through 3.18). For a low Reynolds number (Re = 1), the vorticity contours seem to have symmetric shapes, but for
Figure 3.4 Contour of constant vorticity $\text{Re} = 1, \ T = .26$ and $\ T = 1.41$. 
Figure 3.5 Contour of constant vorticity $Re = 1$, $T = 3.47$ and $T = 7.43$. 
Figure 3.6 Contour of constant vorticity $Re = 1$, $T = 14.97$ and $T = 29.39$. 

PARAMETERS $Re = 1$, $\alpha = 0.00$, $\text{NUM of PTS}(\theta) = 100.15$, $\Delta T = 0.01$

$T = 14.97$

$T = 29.39$
Figure 3.7 Contour of constant vorticity Re = 10, T = .25 and T = 1.85.
Figure 3.8 Contour of constant vorticity $Re = 10$, $T = 3.45$ and $T = 5.05$. 
Figure 3.9 Contour of constant vorticity $Re = 10$, $T = 6.65$ and $T = 8.25$. 
Figure 3.10 Contour of constant vorticity $Re = 10$ and $T = 16.33$. 
Figure 3.11 Contour of constant vorticity $Re = 100$, $T = .25$ and $T = .46$. 
Figure 3.12 Contour of constant vorticity $Re = 100$, $T = .68$ and $T = .92$. 
Figure 3.13 Contour of constant vorticity $Re = 100$, $T = 1.19$ and $T = 1.49$. 
Figure 3.14 Contour of constant vorticity $Re = 100$, $T = 1.81$ and $T = 2.18$. 
Figure 3.15 Contour of constant vorticity $Re = 100$, $T = 2.59$ and $T = 3.05$. 
Figure 3.16 Contour of constant vorticity Re = 1000, T = .25 and T = .43.
Figure 3.17 Contour of constant vorticity $Re = 1000$, $T = .61$ and $T = .79$. 
Figure 3.18 Contour of constant vorticity Re = 1000, T = .97 and T = 1.15.
Re = 10, the contours are asymmetric (Figure 3.7) even after a long time. Similarity in the contour shape is again found when the Reynolds number is increased (Re = 100 and 1000). This orientation of the vorticity contours at Re = 10 is due to the initial structure of the vortex.

A necessary condition for the accuracy of the results is the constancy of the invariant (3-21). In most of the numerical computations, the moment invariant was unchanged up to the fourth digit. The initial value is easily calculated analytically as 4\pi.

Asymptotically, the stream function can be regarded as a dipole representation, i.e.,

\[ \phi + i(-\psi) = \frac{\Gamma}{2\pi i} \left\{ \ln[z-a] - \ln[z+a] \right\} = -\frac{\Gamma}{2\pi i} \frac{2a}{z} + O(\frac{(a^2)}{z}) \]

\[ = -\frac{\Gamma}{2\pi i} \frac{2a}{r} e^{-i(\theta - \frac{\pi}{2})} \]

The term \((-\psi)\) is due to the definition of the stream function in this work, and \(a\) is the initial location of the vorticity. Since the initial condition implies \(\Gamma = 2\pi\), it follows that

\[ \psi_1 = -2a \]

or

\[ I_2 = 2\pi a \]

The initial distribution of the vorticity has \(a = 2\). Hence, the moment invariant is \(I_2 = 4\pi\). This value gave a good indication of the resolution in both the \(r\) and \(\theta\) directions. After a long time, the
vorticity dissipates and reaches the computational boundary. The computational boundary condition \( w = 0 \) is therefore not accurate any more. At this time, the invariant changed by 10\% and the computation was stopped. We believe that before this time the results are reliable. The computational domain was chosen with the length \( R_{\text{max}} = 10 \) divided into 100 spatial intervals in the \( r \) direction and 16 modes in the \( \theta \) direction. For 32 mode points, the solution did not change significantly. For the results in this section, it was found satisfactory to use a fixed length for the computational domain. For long time evolution, we shall show, in Chapter 5, that an asymptotic expression describes the results well.

The sensitivity of the results to the initial conditions was checked (see Figures 3.19 through 3.22). The computations were done for \( \text{Re} = 100 \), 16 modes, and 100 spatial points in the \( r \) direction. In one case, the initial value was set with \( t_{\text{in}} = .5 \) and in the other \( t_{\text{in}} = .25 \). It appears from the numerical computation that the vorticity contours oscillate along the \( y \)-axis. Numerically, this oscillation appeared to damp out after \( t = 2t_{\text{in}} \). It is caused by the difference between the "true" initial vorticity distribution, (i.e., the asymptotic result obtained in the previous chapter), and the Gaussian profile, used as the initial condition (Figure 3.19). This difference can be considered as a disturbance of the "true" solution, which will oscillate until it damps away. Even for the initial condition at \( t_{\text{in}} = .25 \), there is a period of time where the results oscillate until the disturbance is damped. But as the initial time is small, the initial disturbance is
Figure 3.19 Contour of constant vorticity \( \text{Re} = 100, T = .5 \) and \( T = .54 \).
Figure 3.20 Contour of constant vorticity $Re = 100$, $T = .58$ and $T = .62$. 
Figure 3.21 Contour of constant vorticity $Re = 100$, $T = .66$ and $T = .7$. 
Figure 3.22 Contour of constant vorticity $Re = 100$, $T = .74$ and $T = .76$. 
also small. The results computed starting at the initial time \( t_{in} = 0.25 \) have reached a stage where perturbations of the initial condition have no effect on the results. We will elaborate upon these comments in this section.

The comparison between the asymptotic solution and the spectral solution is shown in Figure 3.23 for Reynolds number 10 and the same nondimensional time as was used in this chapter, i.e., \( T = t/t_0 = 1.25 \). For higher Reynolds number, the comparison is good, but only for short time evolution. Figure 3.24 shows comparison between the computed results for Reynolds number 100, and for \( T = 0.65 \), and \( T = 0.85 \), which is in good agreement compared with the spectral solution. It can be seen from Figure 3.24 that for \( T = 0.85 \), there is small deviation between the asymptotic results and the numerical results which were computed in the spectral method. This deviation increases as the Reynolds number increases, but the asymptotic results still agree very well with the numerical results with small evolution time. The reason for this is that there are not enough terms in the perturbation expansion to resolve the problem for higher Reynolds number.

The typical computer time that was required to solve the perturbation solution is about 0.5 seconds on the Cray X-MP 48, with a fully vectorized code. The computation of the spectral solution on the same machine for semi-vectorized code (the F.F.T. was not vectorized) required 500 seconds. The spectral computation was done with 16 modes in the \( \theta \) direction, and 100 points in the \( r \) direction, with time step \( DT = 0.01 \) for 6000 time steps. Moreover, because of the difficulty in
Figure 3.23 Comparison between the asymptotic result to the spectral result for Re = 10, T = 1.25.
Figure 3.24 Comparison between the asymptotic result to the spectral result for $Re = 10$, $T = .65$ and $T = .85$. 
resolution for small $t_{in}$, the perturbation solution is superior to the spectral solution for early times. Since the perturbation solution agrees very well with the spectral solution, and requires less computer time, it seems worthwhile to carry out the perturbation expansion for higher order terms.

In this chapter it was pointed out that there is a slight difference between the computed results for the values of Reynolds numbers 1, 10 and 100. This can be seen from Figures 3.4 and 3.18. The asymptotic solution does verify that such differences exist (Figures 3.23, 3.24). For $Re = 10$ (Figure 3.23), the vorticity contours have a small inclination compared to a higher Reynolds number, i.e., $Re = 100$ (Figure 3.24). In order to examine the perturbation solution, let us write the perturbation of the vorticity up to the second order, i.e.,

$$w(\eta) = \frac{e^{-\eta^2/2}}{1} - 2|A_2| \cos(2\theta + \arg(A_2))$$

It can be seen from Figure 2.1, 2.3 and 2.5 that $\arg(A_2) = \frac{\pi}{2}$, $\frac{\pi}{4}$, 0 for the values of Reynolds number 1, 10, and 100. Hence, the vorticity contour may have an inclination of $\frac{\pi}{4}$ for $Re = 1$, $\frac{\pi}{8}$ for $Re = 10$, and 0 for $Re = 100$. For Reynolds numbers 10 and 100, the computed results agree very well with this analysis. For $Re = 1$, no such orientation has been observed (Figure 3.4). Since the magnitude of the term $(A_2)$ is small compared with the same term for $Re = 10$ and 100, the perturbation does not affect the solution very much when $Re = 1$. 
It was found in this research that the spacing between the vortices increased as a function of time. This phenomenon is a result of the higher perturbation term of the vorticity in which their far field solutions behave as a multi-pole, and the orientation of the multi-poles is a result of viscous effects. As a result, there is a small velocity which advects the vorticity away from the y-axis. Figure 3.25 shows the spacing between the vortices for $Re = 10$ as a function of time while Figures 3.26 and 3.27 show the comparison between the vortex spacing for $Re = 10$ and 100 as obtained from the spectral solution and the asymptotic solution (equation 2.49). The agreement is good up to about $T = 1.5$, but beyond these values the asymptotic results diverge from the spectral solution. Thus, more terms are required in the perturbation expansion. It is worthwhile to note that the centroid of the vorticity is advected outward from the y-axis. It is easy to verify that this phenomenon is a result of the viscous effect.

The decay of the maximum of the vorticity vs. time is shown in Figure 3.28. This result was computed from the spectral code. There is a small difference between the results for $Re = 10$ and 100 as is shown in this figure. This difference does verify the computations of Robert [1976] who pointed out that there is a small difference between the maximum vorticity decay of the solitary vortex and the counter-rotating vortices. At least from the perturbation expansion which was done in the previous chapter, it appears that this difference is of order $O(\tau)$, (equation 2-35).
Figure 3.25 Spacing between maximum of vorticity vs. time for Re = 10.
Figure 3.26 Spacing between maximum of vortices vs. time for Re = 10.
Figure 3.27 Spacing between maximum of vortices vs. time for Re = 100.
Figure 3.28 Maximum vorticity of counter-rotating vortices (varying time step) Re = 1, 10, 100.
The descent velocity for \( Re = 1, 10 \) and 100 is shown in Figure 3.29. There is a small difference between \( Re = 100 \) and \( Re = 1 \) and 10 where the last two seem to have an asymptotic behavior, while \( Re = 100 \) is different from the other two. The descent velocity is nondimensional as was derived in Chapter 2. It appears that for a very short time the descent velocity oscillates as shown in Figure 3.30, where the computation was made for \( Re = 100 \). This region is magnified as shown in Figure 3.31. The sensitivity of this region was checked numerically by increasing the number of points in the \( r \) direction from 100 and 200 and reducing the time step. However, the oscillations are exactly the same. As a second possibility, the numerical initial condition for the "Leap Frog" method was changed, i.e., \( w^{1}_{n,j} = w^{0}_{n,j} \) rather than computing it from \( w^{1}_{n,j} = 2D(-F_{n,j} + 0(\frac{1}{Re})) \), but the same phenomenon was observed. Due to the consistency of these numerical results, it is reasonable to believe that this is a physical phenomena. Note that a similar oscillation occurs in the spacing between vortices (Figure 3.27). Since the descent velocity is computed in the spectral code at the point of maximum vorticity, it is clear that if the location of the maximum vorticity oscillates, then the descent velocity will oscillate too. The initial behavior is characterized by small oscillations around a vertical axis which goes through the maximum vorticity, as well as horizontal oscillation in the vortex spacing (Figure 3.27). These oscillations are a result of the Gaussian initial condition which was used in the numerical code. This conclusion was verified by using the initial condition as computed by the asymptotic
Figure 3.29 Descent velocity of counter-rotating vortices (varying time step) $Re = 1, 10, 100$. 
Figure 3.30  Descent velocity vs. time for Re = 100.
Figure 3.31 Descent velocity vs. time for Re = 100.
solution (equation 2-35) for an initial time $t_{in} = .26$. The new spectral solution now does not show any kind of oscillation (Figure 3.32). Any initial disturbances on the asymptotic solution having zero circulation oscillate around the vortex until they are damped away.

In light of what we did in Section 2.2, it can be understood that the oscillation in the vorticity contours, and in the vortex spacing, is a result of the homogeneous part of the perturbation solution. From the numerical result obtained by the spectral code, this part of the solution decays rapidly. It was estimated in this chapter that the decay time is about $t = 2t_{in}$. Hence, the solution for the counter-rotating vortices behaves for a long time period as the particular part of the perturbation expansion.
Figure 3.32  Descent velocity vs. time for Re = 100, (asymptotic initial condition).
CHAPTER 4

COUNTER-ROTATING VORTICES UNDER A POTENTIAL STRAINING VELOCITY

In this chapter, we consider the influence of an externally imposed strain flow that pushes the two vortices together but stretches them in a direction parallel to the plane of symmetry and perpendicular to vortex filaments. In general, the strength of the strain flow would be determined by the farfield vorticity. In this study, the strain field will be assumed to be given by $\psi = Ax$ where $A$ is a positive constant.

Consequently, the vorticity equation can be written as:

$$\frac{\partial \omega}{\partial t} + (-Ax + u) \frac{\partial \omega}{\partial x} + (Ay + v) \frac{\partial \omega}{\partial y} = \frac{1}{Re} \nabla^2 \omega$$

$$\nabla^2 \psi = \omega$$

$$u = - \frac{\partial \psi}{\partial y}$$

$$v = \frac{\partial \psi}{\partial x}$$

The presence of the strain flow strongly affects the choice of a computational domain, since the vorticity soon lies stretched in a narrow region adjacent to the plane of symmetry.
If an attempt is made to solve this problem in the polar coordinates, it fails to give a good solution because of the lack of resolution in the $\theta$ direction, and a numerical error with a wave-like pattern appears because of the lack of higher modes in the spectral representation. Moreover, this solution, due to the advection term in the $y$-direction, soon reaches the computational boundary, and the numerical method fails to provide accurate solutions. Thus, it is imperative to use a different computational mesh in order to capture the resolution of the vorticity and to guarantee that the computational boundary will not affect the calculations.

4.1 The Computational Domain

There are several possible choices for a computational domain. For example, elliptic coordinates or some conformal mapping may be used. The advantage in this technique lies in the fact that the transformation remains orthogonal, which means that the Laplacian, as well as the boundary conditions, transforms into a simple form. Elliptic coordinates can be written as:

\[ z = \zeta + a^2 / \zeta \]

where $\zeta$ is the transform plane $\zeta = \xi + i\eta$ and $z$ is the physical plane $z = x + iy$. Under this transformation, the vorticity equations can be written in a similar form to equations (4-1), but the term $\left( \frac{\partial w}{\partial \zeta} \right)$ as well as the vorticity term in the stream function equations are multiplied by the Jacobian of the transformation. In the new coordinates, the
computational domain is a circle. Since the far field behaves as \( z = \zeta \), the asymptotical representation of the stream function can be easily determined in terms of the new coordinates. In particular, it can be specified at the circular boundary as a numerical boundary condition. The main disadvantage in this technique is that the coordinate lines cluster near the singularity of the transformation, \( z = (2a, -2a) \), where the Jacobian is zero. That means that the scheme will have a severe restriction on the time step. The second disadvantage is that the presence of the Jacobian makes the spectral solution more expensive. Thus, this method has not been used in this research.

Due to the concentration of vorticity along the y-axis, it seems that a grid which moves with the vorticity is more suitable for ensuring good resolution, or in other words, an adaptable grid which adapts itself at each time step to the new position of the vorticity. The main idea is to track the coordinates with the transport of the vorticity. However, the problem of devising a suitable grid is complicated and is a new field in computational fluids. It is worthwhile to mention that solving the above problem in a stationary grid and updating the result whenever the vorticity is advected requires high order interpolation, otherwise the solution may contain an error \( (h^n/Dt) \) where \( h \) is the spatial interval, \( Dt \) is the time step and \( n \) is the order of interpolation. As was pointed out by Roache [1971], the second order interpolation contains an additional dissipative term proportional to \( h^2/Dt \). In order to devise a coordinate system which
adapts itself to the location of the vorticity each time step, we look at the Lagrangian formulation, i.e.:

\[
\frac{d x}{d t} = -A x + u
\]

\[
\frac{d y}{d t} = A y + v
\]

(4-2)

Unfortunately, the values of \( u \) and \( v \) are not given explicitly. Moreover, the solution of this O.D.E. system (4-2) will generate a curved coordinate system, which will make the set of equations (4-1) more complicated, and more expensive to use. However, one can anticipate that the vortex structure will track with the strain flow and this conjecture is confirmed in the next section.

\[
\xi = \frac{x}{\lambda_1(t)}
\]

(4-3)

\[
n = y \cdot \lambda_1(t)
\]

\[
\lambda_1(t) = \exp \left[ - \int_0^t A(\tau) \, d\tau \right]
\]

Essentially, \( \xi \), \( n \) are the characteristics of the far field solution of Euler equations. Under this transformation the vorticity equation (4-1) can be written as:
\[ \frac{\partial w}{\partial t} + \lambda_1^2 \left( u \frac{\partial w}{\partial \xi} + v \frac{\partial w}{\partial n} \right) = \frac{1}{\text{Re} \: \lambda_1^2} \left[ \frac{\partial^2 w}{\partial \xi^2} + \lambda_1^4 \frac{\partial^2 w}{\partial n^2} \right] \]

\[ \frac{\partial^2 \psi}{\partial \xi^2} + \lambda_1^4 \frac{\partial^2 \psi}{\partial n^2} = w \]  

(4-4)

\[ u = -\frac{\partial \psi}{\partial n} \]

\[ v = \frac{\partial \psi}{\partial \xi} \]

where

\[ \psi_p = \lambda_1^2 \psi \]

\[ u_p = \lambda_1^3 u \]  

(4-4a)

\[ v_p = \lambda_1 v \]

where the above scalings have been introduced in the anticipation that the new quantities will be 0(1). All the quantities with (p) are dimensional.

In this transformation, the physical equations have been normalized. The initial conditions in the computational domain remain the same as in the physical plane and the boundary conditions are the same as in Chapter 2, i.e., \((\psi, w) = 0\) as \(\xi, n \to \infty\) and along \(\xi = 0\). As was mentioned before, the stream function does not decay as fast as the vorticity. As a result, a proper numerical boundary condition is
required on the boundary of the computational plane in order to ensure that reflections will not cause a problem. The same approach which was used before, i.e., taking the asymptotic solution as a boundary condition, is not feasible in the transform plane because the dipole behavior of the asymptotic boundary condition in the radial coordinates appears to be more complicated than in the physical plane. Moreover, the transformation of equation (4-4) to radial coordinates will give additional mixed terms in the dissipation expression multiplied by \( \sin(\theta) \) or \( \cos(\theta) \), and as a result the spectral method will lose its simplicity, and may be more expensive. Hence, the advantage of using it in the physical plane appears to be lost in this case.

4.2 Asymptotic Analysis

Due to the straining velocity, we may expect that the initial vorticity distribution will lose its radial nature and advect with the local velocity. In the case where the strength of the straining field is large enough, it may elongate along the \( y \)-axis and will have a 2-D structure rather than a radial structure, as it had in the previous chapters. The local velocity can be taken as a superposition of the straining velocity and the induced velocity at any point, equation (4-2). The induced horizontal velocity for such an elongation appears to be negative in the upper part of the vorticity and positive in the lower part of the vorticity.

Hence, the upper part of the vorticity will advect towards the \( y \)-axis, while the lower part will advect outward from the \( y \)-axis. In this case, the upper part will remain along the \( y \)-axis mainly because
the induced horizontal velocity decays as the material point moves towards the y-axis, where as in the lower part the induced horizontal velocity increases as a function of the distance from the y-axis. In this analysis, we have not included yet the effect of the straining velocity, where the straining velocity in the upper part appears to be in the same direction as the induced velocity (for a positive vorticity), but in the lower is in the opposite direction to the induced velocity. In that region, we can expect that the vorticity will be advected until the induced velocity will be balanced by the straining velocity. However, this situation has not been found in this research.

The phenomenon can be regarded as a sequence of pictures at each time step. Equation (4-4) can be written with $\lambda_1$ frozen at $\lambda_{1,0} = \lambda_1(t_0)$ when $t = t_0$ where $t_0$ is the picture time sequence,

$$\frac{aw}{at} + \lambda_{1,0}^2 \left( u \frac{aw}{\xi} + v \frac{aw}{\eta} \right) = \frac{1}{Re} \frac{1}{\lambda_{1,0}^2} \left[\frac{\partial^2 w}{\partial \xi^2} + \lambda_{1,0} \frac{\partial^2 w}{\partial \eta^2}\right] + O(t-t_0)$$

$$\frac{\partial^2 \psi}{\partial \xi^2} + \lambda_{1,0}^4 \frac{\partial^2 \psi}{\partial \eta^2} = w + O(t-t_0)$$

(4-5)

$$u = - \frac{\partial \psi}{\partial \eta}$$

$$v = \frac{\partial \psi}{\partial \xi}$$

It is possible to observe the advection of the lower part of the vorticity due to the induced velocity, where the straining velocity
stretches and compresses the $y$ and $x$ coordinates, respectively. That is to say, at each time step we may regard the phenomena as a slender initial vorticity distribution, where the lower part is advected by the induced velocity outward from the $y$ axis. This mechanism has to be updated by including the stretching in the $y$ direction and the compression of the coordinate in the $x$ direction.

It can be expected that the dissipation term will smooth the results. This means that this term will come into effect when $\text{Re} \lambda_1^2 = O(1)$. But for high Reynolds number, and for a short time period, the main phenomena is characterized by the advection terms. Hence, we may regard the Euler equation as an approximate solution with slender vorticity distribution as the initial condition, and under zero straining velocity. The governing equations are:

$$\frac{\partial w}{\partial t} + \lambda_0^2 \left( u \frac{\partial w}{\partial \xi} + v \frac{\partial w}{\partial \eta} \right) = 0$$

$$\frac{\partial^2 \psi}{\partial \xi^2} + \lambda_0^2 \frac{\partial^2 \psi}{\partial \eta^2} = w$$

$$u = - \frac{\partial \psi}{\partial \eta}$$

$$v = \frac{\partial \psi}{\partial \xi}$$

where $\lambda_0^2$, the slenderness ratio of the vorticity, is the ratio of the width of the vorticity distribution to its height.
If we restrict our solution to a small time interval in the neighborhood of the initial condition, the above equation can be written as

$$\frac{Dw}{Dt} = 0$$

$$\frac{de}{dt} = \lambda \phi u(t_0^+t, \xi, n)$$  \hspace{1cm} (4-7)

$$\frac{dn}{dt} = \lambda \phi v(t_0^+t, \xi, n)$$

The last equation is identical to (4-6) but is written in the Lagrangian system, which describes the characteristics along which the quantity w remains constant. Let

$$\xi = \xi_0 + \xi_1$$

$$\eta = \eta_0 + \eta_2$$  \hspace{1cm} (4-8)

$$\frac{d\xi_1}{dt} = \lambda \phi \exp[t \partial t_0 + \epsilon_1 \partial \xi_0 + \epsilon_2 \partial \eta_0] u(t_0, \xi_0, \eta_0)$$

$$= \lambda \phi (1 + \epsilon_1 \partial \xi_0 + \epsilon_2 \partial \eta_0 + t \partial t_0 + \cdots) u(t_0, \xi_0, \eta_0)$$

$$\frac{d\epsilon_2}{dt} = \lambda \phi \exp[t \partial t_0 + \epsilon_1 \partial \xi_0 + \epsilon_2 \partial \eta_0] v(t_0, \xi_0, \eta_0)$$

$$= \lambda \phi (1 + \epsilon_1 \partial \xi_0 + \epsilon_2 \partial \eta_0 + t \partial t_0 + \cdots) v(t_0, \xi_0, \eta_0)$$

where (4-8) can be expressed as
where $\xi_0, \eta_0$ are the initial locations of each material point and $u^0(\xi, \eta), \nu^0(\xi, \eta)$ are the components at $t = t_0$. Hence, if we denote by $\alpha, \beta$ the characteristics, then

$$
\xi = \alpha + \lambda_0^2 \frac{u^0(\alpha, \beta)t}{2} \left[ \frac{1}{\lambda_0^2} \frac{\partial u}{\partial \xi} + u \frac{\partial u}{\partial \xi} + \nu \frac{\partial u}{\partial \eta} \right]_{\xi_0, \eta_0}^0 + \cdots
$$

$$
\eta = \beta + \lambda_0^2 \frac{v^0(\alpha, \beta)t}{2} \left[ \frac{1}{\lambda_0^2} \frac{\partial v}{\partial \xi} + u \frac{\partial v}{\partial \xi} + \nu \frac{\partial v}{\partial \eta} \right]_{\alpha, \beta}^0 + \cdots
$$

and the explicit form of the characteristic is

$$
\alpha = \xi - \lambda_0^2 t \frac{u^0(\xi, \eta)}{2} \left[ \frac{1}{\lambda_0^2} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial \xi} + \nu \frac{\partial u}{\partial \eta} \right]_{\xi, \eta}^0 + \cdots
$$

$$
\beta = \eta - \lambda_0^2 t \frac{v^0(\xi, \eta)}{2} \left[ \frac{1}{\lambda_0^2} \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial \xi} + \nu \frac{\partial v}{\partial \eta} \right]_{\xi, \eta}^0 + \cdots
$$

The expression for the vorticity in terms of the characteristics satisfies Euler's equation up to $O(\lambda_0^9 t^3)$. We do not expect characteristics of the same family to intersect, since that will lead to the appearance of "shock waves" which are impossible under the assumption of incompressible flow. Hence, we can use the characteristic to generate a new grid, since under the last assumption
there is no singularity of the transformation from \((\xi, n) + (\alpha, \beta)\). Hence

\[
\eta = \rho \left[ 1 - \lambda \frac{\partial t}{\partial \rho} \right] \eta = \rho - \lambda \frac{\partial t}{\partial \rho} \eta = O(\lambda \frac{\partial t}{\partial \rho}) 
\]

The stream function equation can be written

\[
\left\{ \left[ 1 - 2 \lambda \frac{\partial t}{\partial \rho} \right] \partial_t \psi - \left[ \lambda \frac{\partial t}{\partial \rho} \frac{\partial^2 \psi}{\partial \rho^2} \right] \right\} = w(\alpha, \beta) 
\]

and, \(\psi\) can be expressed as

\[
\psi = \psi_0 + \lambda \psi_1 + \cdots 
\]

where

\[
\frac{\partial^2 \psi_0}{\partial \rho^2} = w(\alpha, \beta) 
\]

and

\[
\frac{\partial^2 \psi_1}{\partial \rho^2} = \left\{ 2 \frac{\partial \psi_0}{\partial \rho} \frac{\partial^2 \psi_0}{\partial \rho^2} + \frac{\partial^2 \psi_0}{\partial \rho^2} \frac{\partial^2 \psi_0}{\partial \rho \partial \beta} + \frac{\partial^2 \psi_0}{\partial \rho^2} \frac{\partial \psi_0}{\partial \beta} \right\}
\]

It can be seen from the last expression that \(\frac{1}{\lambda^2} \frac{\partial u}{\partial t} = O(1)\). Equation (4-14) was solved in this research with a Gaussian profile as an initial condition. Hence, the velocity and the vorticity can be written as...
\[ v^0 = \frac{3\psi}{3\xi} = 2e^{-\beta_1^2} \left[ \text{erf}(\alpha_1) - \text{erf}(\alpha_2) \right] / \sqrt{2\tau_0} \]

\[ u^0 = -\frac{3\psi}{3n} = 4\beta_1 e^{-\beta_1^2} \left[ \alpha_1 \text{erf}(\alpha_1) - \alpha_2 \text{erf}(\alpha_2) \right] + \frac{1}{2} (e^{-\alpha_1^2} - e^{-\alpha_2^2}) / \sqrt{2\tau_0} \]

\[ w = (e^{-\alpha_1^2} - e^{-\alpha_2^2}) e^{-\beta_1^2} / t_0 \]

where

\[ \alpha_1 = \frac{\alpha - a}{\sqrt{2\tau_0}}, \]

\[ \alpha_2 = \frac{\alpha + a}{\sqrt{2\tau_0}}, \]

\[ \beta_1 = \frac{\beta}{\sqrt{2\tau_0}}, \]

Finally,

\[ \xi = \alpha + \lambda_0^0 u^0 t \]

\[ n = \beta + \lambda_0^0 v^0 t \]

These expressions are illustrated graphically in Figures 4.1 and 4.2, where Figure 4.1 describes the initial value of vorticity as it appears in the physical plane and in the computational plane for the slender initial vorticity distribution of ratio \( \lambda_0 = 0.2 \) at \( t = 0 \) and \( t = 1 \), \( t \) is the nondimensional time expressed as: \( t = t' / (\frac{2\pi \delta_x^2}{\lambda_0^0 \Gamma}) \), and \( \delta_x \) is the
Figure 4.1 Approximate solution for counter-rotating vortices under zero straining velocity, $t = 0$. 
Figure 4.2 Approximate solution for counter-rotating vortices under zero straining velocity, $t = 1$. 
dimensional distance from the y coordinate up to the maximum vorticity, \( \Gamma \) is the circulation of the positive vortices). In the computational domain \((x, y)\) are the physical coordinates stretched by the factor \( \left( \frac{1}{\lambda_0}, \lambda_0 \right) \) for \( y \) and \( x \), respectively. The vorticity distribution appears in the computational domain, as is shown in Figures 4.1 and 4.2, where the lower part of the vorticity is advected by the induced velocity outwards from the y-axis, while the upper part remains elongated along the y-axis, Figure 4.2. It is difficult to predict by this approximate solution the behavior of the physical phenomena for the long time evolution, but it gives a qualitative understanding for the short period.

Since the vorticity is elongated along the y-axis, the stability of the upper portion as well as the lower one has to be examined. It is known that an infinite strip of vorticity under zero straining field might induce instability, and moreover under certain conditions the infinite strip of vorticity in a weak straining velocity might exhibit instabilities [Moore, 1977]. This question will be reviewed in the next section.

### 4.3 Instability of 2-D Vorticity Distribution under a Potential Straining Velocity

Moore and Griffith Jones [1974] have shown that if a vortex sheet is stretched fast enough, the sheet is stable to small 2-D disturbances. The same effect was found by Dagan [1975] for the Rayleigh-Taylor instability when the interface between two fluids of
different densities was being stretched. In a former paper, Moore [1976] has shown that it is possible for the disturbances on the distant part of the vortex sheet to control what happens locally, if the disturbances on the distant part grow more rapidly.

The same phenomena can happen when the vorticity is stretched along the y-axis, loses its radial nature and becomes a strip of vorticity. In that case, the question of instability is similar to the previous research, but more complicated since the phenomenon here is a finite vortex sheet in the y direction rather than an infinite strip. It is worthwhile to mention that the approach presented here is qualitative rather than exact.

Let assume a finite vorticity strip as shown in Figure 4.1, where \( \lambda \) will denote the characteristic length scale in the x direction and \( 1/\lambda \) the characteristic length in the y direction. The vorticity equation can be written as

\[
\frac{\partial \omega}{\partial t} + \frac{(u_0 + u)}{\lambda} \frac{\partial \omega}{\partial (x/\lambda)} + \lambda (\nu_0 + \nu) \frac{\partial \omega}{\partial (\lambda y)} = \frac{1}{Re\lambda^2} \left[ \frac{\partial^2 \omega}{\partial (x/\lambda)^2} + \lambda^4 \frac{\partial^2 \omega}{\partial (\lambda y)^2} \right]
\]

\[
\frac{\psi}{\lambda^2} \left[ \frac{\partial^2 \psi}{\partial (x/\lambda)^2} + \lambda^4 \frac{\partial^2 \psi}{\partial (\lambda y)^2} \right] = \omega
\]

where \( \psi \) is a measure of the stream function \( \psi_p \). \( u_0, \nu_0 \) are the straining velocity, i.e.,

\[
u_0 = -Ax
\]
\[ v_0 = Ay \]

In this transformation, we have scaled the variable, which means

\[
\left\{ \begin{align*}
\frac{\partial}{\partial (x/\lambda)} & = O(1) \\
\frac{\partial}{\partial (\lambda y)} &
\end{align*} \right. \]

Because we expect the vorticity to be of \( O(1) \)

\[ \bar{\psi} = O(\lambda^2) \]

and the induced velocity components are:

\[ u = -\frac{\partial \psi}{\partial y} = -\bar{\psi} \lambda \frac{\partial \psi}{\partial (y\lambda)} = O(\lambda^3) \]

\[ v = -\frac{\partial \psi}{\partial x} = \frac{\bar{\psi}}{\lambda} \frac{\partial \psi}{\partial (x/\lambda)} = O(\lambda) \]

The straining velocity is

\[ u_0 = -Ax = -A\lambda \left( \frac{x}{\lambda} \right) = O(A\lambda) \]

\[ v_0 = Ay = \frac{A}{\lambda} (y\lambda) = O(A/\lambda) \]

which means that as long as \( A \) is larger than \( O(\lambda^2) \), the straining velocity appears to be dominant and the first order approximation can be written as
This equation can be reduced by using the characteristic transformation (4-3) to the following equation

\[
\frac{\partial w}{\partial t} + u_0 \frac{\partial w}{\partial x} + v_0 \frac{\partial w}{\partial y} = \frac{1}{Re} \frac{\partial^2 w}{\partial x^2}
\]

where

\[
\xi = \frac{x}{\lambda_1(t)}
\]

and \( \lambda_1(t) \) is of the same length scale as was used in equation (4-3), i.e.,

\[
\lambda_1(t) = \exp \left[ - \int_0^t A(\tau) \, d\tau \right]
\]

Hence, the derivation of equation (4-4) is fully legitimate whenever the vorticity under consideration appears to be slender. Under this transformation equation (4-3) is obtained.

In order to solve equation (4-3), let:

\[
w = w_0(t, \xi) + w_1(t, \xi, \eta)
\]

\[
\psi = \psi_0(t, \xi) + \psi_1(t, \xi, \eta)
\]

where the zero subscript stands for the base solution and (1) stands for the perturbation. Hence, the base solution is:
\[
\frac{\partial \psi_0}{\partial t} = \frac{1}{\operatorname{Re} \lambda_1^2} \frac{\partial^2 \psi_0}{\partial x^2}
\]

\[
\frac{\partial^2 \psi_0}{\partial x^2} = \psi_0
\]

Essentially, the base solution can be independent of \( n \).

In order to check the stability of the above scheme, we assume that we are looking for the modes where

\[
\frac{\partial}{\partial \eta} = 0 \left( \frac{1}{\lambda^2} \right)
\]

In that sense, we restrict our model to high mode disturbances where the base solution can be regarded locally as independent of \( n \), while \( \psi_1 \) and \( w_1 \) vary rapidly. So

\[
\frac{\partial}{\partial x} (\psi_0 + w_1) - \lambda_1^2 \left( \frac{\partial \psi_0}{\partial \eta} + \frac{\partial \psi_1}{\partial \eta} \right) \left( \frac{\partial \psi_0}{\partial \xi} + \frac{\partial \psi_1}{\partial \xi} \right)
\]

\[
+ \lambda_1^2 \left( \frac{\partial \psi_0}{\partial \xi} + \frac{\partial \psi_1}{\partial \xi} \right) \left( \frac{\partial \psi_0}{\partial \eta} + \frac{\partial \psi_1}{\partial \eta} \right)
\]

\[
= \frac{1}{\operatorname{Re} \lambda_1^2} \left( \frac{\partial^2 \psi_0}{\partial \xi^2} + \lambda_1^4 \frac{\partial^2 \psi_0}{\partial \eta^2} \right) (\psi_0 + w_1)
\]

This equation can be written as
In order to answer the question of stability, let the variables be written in the form

\[
\begin{align*}
\psi_1 & = \psi_\theta(\xi, t) \\
w_1 & = w_\theta(\xi, t)
\end{align*}
\]

where \( \theta \) is the wave number for the perturbation. Upon substitution,

\[
\frac{\partial w_\theta}{\partial t} - \left[ i \theta \psi_\theta \frac{\partial w_\theta}{\partial \xi} - \frac{\partial^2 \psi_\theta}{\partial \xi^2} i \theta w_\theta \right] \lambda_1^2
\]

\[
= \frac{1}{\text{Re} \lambda_1} \left\{ \frac{\partial^2 w_\theta}{\partial \xi^2} - \lambda_1^4 \psi_\theta w_\theta \right\} \quad (4-17)
\]

\[
\frac{\partial^2 \psi_\theta}{\partial \xi^2} - \lambda_1^4 \psi_\theta = w_\theta
\]

When the base flow is not a function of time and \( \lambda_1 = \text{constant} \), we can assume that

\[
\psi_\theta = \phi(\xi) e^{i\theta t}
\]

\[
w_\theta = \dot{\phi}(\xi) e^{i\theta t} \quad (4-18)
\]

Equation (4-17) then becomes
\[ [ic (a_x^2 - \theta_\lambda^2) + (u_0(a_x^2 - \theta_\lambda^2) - u_0') \theta_\lambda] \dot{\psi} \]

\[ = \frac{1}{Re \lambda_1^2} \left\{ a_x^2 - (\theta_\lambda^2) \right\}^2 \dot{\psi} \]

where \( \theta_\lambda = \lambda_1^2 \theta \).

This is the Orr-Sommerfeld equation for \( (Re \lambda^2) \) and \( \theta_\lambda \). As is well known, one condition for this equation to be unstable is the existence of an inflection point in the mean velocity. In this problem, one inflection point exists where \( \frac{3w_0}{\delta^2} = 0 \), or the required condition in this case is a point of maximum vorticity. Without accounting for the stretching, the flow is clearly unstable.

We now return to Equation (4-17) where \( \lambda_1 \) is given by (4-3). We look at the stability of the modes whose variation in time is much larger than the changes induced by the strain flow. Let

\[ t = \bar{t} + t_1. \]

Then

\[ \theta_\lambda = \bar{\theta}_\lambda + O(t_1) \]

and

\[ (\psi_0, w_0) = (\bar{\psi}_0, \bar{w}_0) + O(t_1) \]

To lowest order (4-17) have the form of (4-18), where the time derivative is \( t_1 \) rather than \( t \). The perturbation solution can be written as
\[ \psi_\theta = \hat{\psi}(\xi, \xi) e^{i \xi(\xi)t} \]

where we now regard all the constants as slowly varying in \( \xi \). In Figure 4.3, a typical stability diagram for (4-18) is given. Initially, there is a band of modes \( \theta \xi^s \) that lie in the instability region provided \( \text{Re} \lambda_1^2 \) is above a critical value. Since \( \theta \xi \) is a function of time \( \theta \xi = 0 \lambda_1^2 \), a new set of wave lengths will be excited as time advances. Amplification of the modes will move to those wave lengths which were previously damped. Hence, there is uncertainty whether the new excited waves will be amplified significantly, because \( \lambda_1 \) decreases, and as a result these modes \( (\theta \xi) \) are shifted to the neighborhood of \( \theta \xi = 0 \). In this region (Figure 4.3), \( \text{Re}(i \xi) < 0 \), and these modes are no longer excited. Moreover, the equivalent Reynolds number \( (\text{Re} \lambda_1^2) \) decreases and as a result the Reynolds number is shifted toward \( (\text{Re} \lambda_1^2)_{\text{crit}} \). When \( \lambda_1^2 < (\text{Re} \lambda_1^2)_{\text{crit}} \), no mode will be amplified and the solution will be damped completely. As a conclusion, it can be said that for a small straining velocity some modes will be amplified until \( \text{Re} \lambda_1^2 \) reaches the critical value. A straining velocity of \( O(1) \) may not exhibit instability at all since the modes have not been sufficiently amplified. For \( \lambda_1 = \text{constant} \) (no straining velocity), the solution will show the same instability as the classical Orr-Sommerfeld equation.

If the Reynolds number is large enough, then the Orr-Sommerfeld equation reduces to the classical Rayleigh equation. If \( (\theta \xi) \) goes asymptotically to zero, the modes which initially were amplified asymptotically as \( t \) goes to infinity will have the same value
Figure 4.3 Schematic description of stability diagram.
corresponding to $\theta_{\lambda} = 0$. Hence, instability can exist if for $\theta_{\lambda} = 0$, $\text{Real}(\text{i}c) > 0$. Essentially this question is more hypothetical rather than relevant, because actually the phenomena is a viscous one and as a result after a finite time, the dissipation term will be dominant (for the validity of equation, $\lambda_1 << 1$).

The base solution was given analytically, i.e.:

$$w_0 = \frac{c_1}{\sqrt{\tau}} \left\{ \exp \left[ - \frac{(\xi - \xi_0)^2}{4\tau} \right] - \exp \left[ - \frac{(\xi + \xi_0)^2}{4\tau} \right] \right\}$$

$$\frac{\partial w_0}{\partial \xi} = \frac{c_1}{\sqrt{\tau}} \left\{ \frac{\xi - \xi_0}{2\tau} \exp \left[ - \frac{(\xi - \xi_0)^2}{4\tau} \right] - \frac{\xi + \xi_0}{2\tau} \exp \left[ - \frac{(\xi + \xi_0)^2}{4\tau} \right] \right\}$$

where

$$\frac{d\tau}{dt} = \frac{1}{\text{Re} \lambda_1}$$

The method of finding the eigenvalues, when $\lambda_1$ and $\tau$ are constant, is a laborious $O(N^3)$ computer operation, but is straightforward and leads to Figure 4.3. As an alternative, equation (4-17) can be solved as an initial problem where the main interest in this approach is in those modes which are unstable. The initial condition can be chosen arbitrarily. The same method can be applied to the Orr-Sommerfeld equation where disturbances can be expressed by a linear combination of the eigenfunctions. Those terms which correspond to positive $\text{Real}(\text{i}c)$, will get excited. For long time evolution, the
solution will behave asymptotically as though the mode corresponding to
maximum \( \text{Real}(i\nu) \) had been chosen initially.

A finite difference approximation was used to solve the
equation with a uniform mesh size, even though an equal mesh size is
not optimal for this kind of problem due to the singular behavior of
equation (4-17). (The highest derivative is multiplied by a small
number \( 1/\text{Re} \lambda_1^2 \), [Drazin, 1981].)

The vorticity equation is approximated by

\[
\frac{w_j^{\nu+1} - w_j^\nu}{\Delta t} + i\theta \lambda \left[ \frac{\mu(\partial \psi_j)}{\partial \xi_j} \ n w_j - \mu \psi_j \ \frac{\mu(\partial w_j)}{\partial \xi_j} \right] \\
= \frac{1}{\text{Re} \lambda_1^2} \left( \delta_j^3 \ n w_j^3 - \theta^2 \ h^2 \ \mu w_j^3 \right) / h^2
\]

and the streamfunction is approximated by

\[
\delta_j^3 \ \mu \psi_j^3 - \theta^2 \ h^2 \ \mu \psi_j^3 = \mu w_j^3 \ h^2
\]

where the difference operator \( \mu \) is defined by

\[
\mu w_j^\nu = \left\{ \ n w_j^{\nu+1} + w_j^\nu \right\} \frac{1}{2}
\]

and \( \delta_j^3 \ \psi_j = \psi_{j+1} - 2\psi_j + \psi_{j-1} \). \( \theta \) subscript present in (4-17) has been
dropped and the superscript \( \kappa \) indicates the time level. The subscript \( j \)
indicates the spatial point along the \( \xi \) coordinate. The above equations
are linear equations for the unknowns at the \( (\nu+1) \)th time level. They
have the following form;
$$a_{1,j} w_{j-1}^k + a_{2,j} w_j^k + a_{3,j} w_{j+1}^k + \beta_j \psi_j^{k+1} = f_j^k$$

and

$$\psi_j^{k+1} - (2 + \theta_j^2 h^2) \psi_j^k + \psi_{j-1}^k = w_j^{k+1} h^2$$

where $a_{1,j}$, $a_{2,j}$, $a_{3,j}$, $\beta_j$ are the coefficients that can be obtained from the previous equations and $f_j^k$ contains known values of $w_j^k$ and $\psi_j^k$. The boundary conditions used here are the exponential decay of the disturbance modes and will be discussed later on in this chapter. The number of grid points along the $\xi$ direction is 900, where the initial condition was chosen as

$$\psi = \xi e^{-\left(\xi - 0.5\right)^2}$$

which satisfies the boundary condition at $\xi = 0$ and $\xi = \infty$. The result has been computed for Reynolds numbers of 10,000 and 1000 and for values of $A = .05$ and .01. There is a good agreement between the analysis in this chapter with the computed results, where for a large straining velocity the solution seems to propagate towards the short waves (see Figures 4.4-4.7). For $Re = 1000$, $A = .05$, it can be seen that the solution is damped (see Figure 4.7). That is due to the decrease in $\lambda_1$, which means that the initial modes which were originally excited are no longer in the region of instability.

Care must be taken to get reliable results. Previous results with less points in the $\xi$ direction show that growing modes appear periodically in the $\theta$ direction, but by increasing the resolution, this phenomenon disappears. The difference between the computed result for
Figure 4.4 Amplification of the disturbance wave number as a function of time, $\text{Re} = 10000$, $A = .05$. 
Figure 4.5 Amplification of the disturbance wave number as a function of time, $Re = 10000$, $A = .01$. 
Figure 4.6 Amplification of the disturbance wave number as a function of time, Re = 1000, A = .01.
Figure 4.7 Amplification of the disturbance wave number as a function of time, $Re = 1000$, $A = .05$. 
the unstable mode is of the order of square of the mesh size. Finally, the solution was compared at 200, and 400 spatial points, and was basically the same.

4.4 Numerical Scheme

Due to the fact that the spectral representation of set (4-4) in polar coordinates requires many more azimuthal modes and hence much computer work, it seems that the best way to solve those equations is in cartesian coordinates, in a rectangular computational plane. As was explained before, the main limitation in this approach is the proper boundary condition for the stream function along the computational domain. The second point is the best choice of the numerical scheme, since the problem is not periodic any longer. One possibility is to formulate the equations in a Finite Difference Approximation (F.D.A.), where less computer work is required in order to evaluate the advection term in the vorticity equation. The main disadvantage is that the solution of the stream function equations may require the use of a Fast Poisson Solver, or at least a factor of $N^2 \ln(N)$ in computer operations. Moreover, in F.D.A., if the dissipation terms are taken explicitly, there is a stability condition on the choice of time step. So, it seems worthwhile to use the spectral representation for this problem, where $(\psi, w)$ is expressed by a Fourier series along the $n$ direction, and F.D.A. in the $\xi$ direction. Thus, we write

$$\begin{bmatrix} \psi \\ w \end{bmatrix} = \sum_{n=1}^{\infty} \begin{bmatrix} \psi_n \\ w_n \end{bmatrix} \sin \frac{n \pi x}{L}$$

(4-19)

Similar to the spectral representation in the radial coordinate, the set (4-4) can be written:
The finite difference approximation is now applied. At each point \( (j) \) in the \( \xi \) direction and for each time step \( (\kappa) \), equations (4-20) are represented by

\[
\frac{w_{n,j}^{\kappa+1} - w_{n,j}^{\kappa-1}}{2\Delta t} + \left[ \lambda_1^2 F \right]_{n,j}^{\kappa} = \frac{1}{\text{Re} \lambda_1^2} \left[ \frac{\delta^2}{\xi^2} - \lambda_1^4 \left( \frac{n\pi}{L} \right)^2 \right] w_{n,j}^{*}
\]

where

\[
w_{n,j}^{*} = \{ w_{n,j}^{\kappa+1} + w_{n,j}^{\kappa-1} \} \frac{1}{2}, \text{ and}
\]

\[
\sum_n F_{n,j} \sin \frac{n\pi n}{L} = (u \frac{\partial w}{\partial \xi} + v \frac{\partial w}{\partial n})_j
\]

\[
\left[ \frac{\delta^2}{\xi^2} - \{ \lambda_1^4 \left( \frac{n\pi}{L} \right)^2 \} \right] \psi_{n,j}^{\kappa} = w_{n,j}^{\kappa}.
\]

(4-21)
Note that the velocity may be determined by

\[ u_j^K = \sum_{n=1}^{N-1} \frac{(n\pi)}{L} \psi_{n,j}^K \cos \frac{n\pi n}{L} \]

\[ v_j^K = \frac{1}{n} \sum_{n=1}^{N-1} \delta_\xi \psi_{n,j}^K \sin \frac{n\pi n}{L} \]

Note that the sums have been truncated. The Fourier coefficients may be obtained via the Fast Fourier Transform. The nonlinear term has been evaluated similarly to the spectral method in polar coordinates, where the quantities \( \frac{\partial w}{\partial \xi} \), \( u, v, \frac{\partial w}{\partial \eta} \) were transformed from Fourier space to the physical space, the advection term evaluated and they transformed back to Fourier space. The computation of the advection term required \( 5NM \ln N \) computer operations, while the F.D.A. required \( cMN \) where \( c \) depends on the order of accuracy of the F.D.A. and \( N,M \) are the number of points in the \( \xi, \eta \) direction respectively. Note that, unlike in the previous chapter, the computational domain is fixed and must be shifted downwards periodically to avoid the vorticity reaching the computational boundary.

### 4.5 Boundary Condition and Initial Condition

The boundary conditions in the \( \eta \) direction are determined automatically by equation (4-19), i.e., \( (\psi, w) = 0 \) along \( \eta = (0,L) \), and as a result, the computational domain is under the influence of the induced velocity coming from the images above and below it. Previous work (Robert, [1976]) used the general solution of "Poisson's" equation,
but this method required \((NM)^2\) computer operations and as a result it was not feasible. In order to estimate the induced velocity, consider the streamfunction that results from the periodic vorticity distribution (see Figure 4.8);

\[
\psi_p = \frac{1}{4\pi} \int \int w(x_1, y_1) \ln \left[ \frac{y-y_1}{(x-x_1)^2 + (y-y_1)^2} \right] \, dx_1 \, dy_1 \quad (4-22)
\]

Quantities subscripted with \(p\) are physical (not in dimensionless form, see 4-4a). The integration is over the full plane. The velocity components that are induced by the nonphysical images can be written

\[
u_p = \frac{\partial \psi_p}{\partial y} = -\frac{1}{2\pi} \int \int w(x_1, y_1) \frac{y-y_1}{(x-x_1)^2 + (y-y_1)^2} \, dx_1 \, dy_1 \quad (4-23)
\]

and \(\Omega\) is the area \(y > \kappa\) and \(y < 0\). The above equation can be written in compact form

\[
u_p = \frac{1}{2\pi i} \int \int w(x_1, y_1) \frac{dx_1 \, dy_1}{z-z_1}
\]

where \(z = x + iy\) and \(z_1 = x_1 + iy_1\). The induced velocity is:

\[
u_p = \frac{1}{2\pi i} \sum_{\kappa=-\infty}^{\infty} \int_{\kappa \neq 0}^{(\kappa+1)\kappa} \int \int w(x_1, y_1) \frac{dx_1 \, dy_1}{z-z_1} \quad (4-25)
\]

where \(\kappa\) is the length of the computational domain in the \(n\) direction. By a suitable change in the integration variable, we may write
Figure 4.8 Schematic description of the computational domain and its images.
\[ u_p - iv_p = \frac{1}{2\pi i} \sum_{\kappa \neq 0}^{\infty} (-1)^{\kappa} \int_{-d_0}^{x_0} \frac{w(x_1,y_1) \, dx_1dy_1}{z_{1,-1\kappa z} - z_1} \]

We now center our coordinates on the centroid of the right hand vortex which we define as \( z_c = (x_c, d_0) \). Thus,

\[ u_p - iv_p = \frac{1}{2\pi i} \sum_{\kappa \neq 0}^{\infty} (-1)^{\kappa} \int_{-d_0}^{x_0} \frac{w(x_1,y_1) \, dx_1dy_1}{z'_{1,-1\kappa z} - z_1} \]

where \( z' = z - z_c \). Since \( z_1 \ll z'_{1,-1\kappa z} \), we may expand this expression as follows

\[ u_p - iv_p = \frac{1}{2\pi i} \sum_{\kappa \neq 0}^{\infty} (-1)^{\kappa} \int_{-d_0}^{x_0} \frac{w(x_1,y_1) \, dx_1dy_1}{z'_{1,-1\kappa z} - z_1} \]

Since \( \int_{-\infty}^{\infty} w(x_1,y_1) \, dx_1 = 0 \), the dominant contribution is

\[ u_p - iv_p = \frac{1}{2\pi i} \sum_{\kappa \neq 0}^{\infty} (-1)^{\kappa} \int_{-d_0}^{x_0} \frac{w(x_1,y_1) \, dx_1dy_1}{z'_{1,-1\kappa z}^2} \]

\[ = \frac{\Gamma}{\pi i} \sum_{\kappa \neq 0}^{\infty} (-1)^{\kappa} \frac{x_c}{(z'_{1,-1\kappa z})^2} \]

where we use the definition of the centroid,

\[ 2\pi x_c = \int_{-d_0}^{x_0} \int_{-\infty}^{\infty} w(x_1,y_1) \, dx_1dy_1. \]

Note that for small values of \( z' \), that is the field point is close to the centroid, (4-27) can be approximated by

\[ u_p - iv_p = -\frac{\Gamma x_c}{\pi i} \sum_{\kappa \neq 0}^{\infty} (-1)^{\kappa} \frac{2z'}{1\kappa z} \left[ 1 + \frac{2z'}{1\kappa z} \right] \]
or in components

$$u_p = + \frac{2\Gamma}{\pi L^2} \sum_{\kappa \neq 0} \frac{(-1)^\kappa}{\kappa^2}$$

$$v_p = - \frac{\Gamma}{\pi L^2} \sum_{\kappa \neq 0} \frac{(-1)^\kappa}{\kappa^2}$$

These formula may be expressed in the transformed variables as

$$u = 2\lambda_1^2 \frac{\Gamma}{\pi L^2} \xi C \xi \sum_{\kappa \neq 0} \frac{(-1)^\kappa}{\kappa^2}$$

$$v = -\lambda_1^2 \frac{\Gamma}{\pi L^2} \sum_{\kappa \neq 0} \frac{(-1)^\kappa}{\kappa^2}$$

where $x_C = \lambda_1 \xi C$.

Hence, the induced velocity is in this case of $O(\lambda_1^2)$ and decays as $L^3$ for the velocity component in the $\xi$ direction, and as $L^2$ for the velocity component in the $\eta$ direction. Hence, the contribution of the images can be neglected when $L$ is large enough. For the case where $\lambda_1$ is $O(1)$, there is an effect from the images, unless a large computational domain is chosen. Fortunately, this is not the case for this research, since $\lambda_1$ is small.
The numerical boundary condition at $\xi = \xi_{boundary}$ can be evaluated in a similar fashion to that described in the previous chapter. The asymptotic behavior of the stream function is given by equation (4-20) or

$$\frac{\partial^2 \psi_n}{\partial \xi^2} - \lambda_1^2 \left( \frac{n \pi}{L} \right)^2 \psi_n = 0 \quad \text{as} \quad \xi \to \infty$$  \hspace{1cm} (4-31)

The last equation has the solution:

$$\psi_n = C_n \exp\left[-\lambda_1^2 \frac{n \pi}{L} \xi\right]$$  \hspace{1cm} (4-32)

which means that for small values of $\lambda_1^2 \left( \frac{n \pi}{L} \right)$, $\psi_n$ does not decay rapidly enough. The second solution of (4-31) cannot exist because it does not decay to zero as $\xi \to \infty$. By the differentiation of (4-32), one gets

$$\frac{\partial \psi_n}{\partial \xi} + \lambda_1^2 \left( \frac{n \pi}{L} \right) \psi_n = 0$$  \hspace{1cm} (4-33)

The last expression was taken as the numerical boundary condition, i.e

$$\psi_{n,N+1} - \psi_{n,N-1} + 2 \lambda_1^2 \left( \frac{n \pi}{L} \right) h \psi_{n,N} = 0$$ \hspace{1cm} (4-34)

where $N$ is the point on the computation boundary in the $j$ direction (i.e., $\xi$ direction). Equation (4-23) is analogous to the same approach used by Fasel [1976].

The initial condition was taken as a Gaussian profile

$$w = \left\{ \exp\left[-\frac{(\xi-a)^2}{2t_{in}} - \frac{n^2}{2t_{in}}\right] - \exp\left[-\frac{(\xi+a)^2}{2t_{in}} - \frac{n^2}{2t_{in}}\right]\right\} / t_{in}$$ \hspace{1cm} (4-35)
Here $t_{in}$ is a time parameter determining the initial spread of the vorticity and $a$ is the distance of the maximum vorticity to the $n$ axis. This approximation is a close one to the experimental measurement of the vorticity (Hecht, [1981]). It is relevant to mention that previous work used constant vorticity in a closed region (Moore and Saffman, [1971]). However, this approach is just a crude approximation. In this research, (4-35) is preferred because this initial condition is $C_{in}$, and as a result, the Fourier series won't yield "Gibbs" phenomena, as in the case of nonsmooth initial conditions.

4.6 Numerical Analysis and Results

Equation (4-20) was solved in a fashion similar to the spectral solution in polar coordinates (Chapter 2), where the $\theta$ direction in the previous code is replaced by the coordinate $\frac{\pi n}{L}$, and $L$ is the length of the computational domain in the $n$ direction. The hybrid scheme (4-21), i.e., "Crank-Nicolson" for the dissipation and the "leap frog" for the advection term, was solved numerically with time step varying between 0.01 and 0.05. Stable results can be obtained even for larger time steps, but due to the rapid change of the physical phenomena these time steps were used and found satisfactory for these calculations.

In order to check the sensitivity of the proposed model, the numerical result computed in this model was compared to the spectral solution in polar coordinates where the straining velocity was included by taking
\[ \psi = \psi_1 + \psi_S \]

where \( \psi_1 \) is the stream function as was described in Chapter 3, i.e.:

\[ \psi_1 = \sum_{n=1}^{N-1} \psi_n \sin(n\theta) \]

and \( \psi_S \) is the stream function due to the straining velocity:

\[ \psi_S = -A \frac{r^2}{2} \sin(2\theta) \]

For this comparison, \( A \) was chosen as .1 in order to avoid the rapid advection of the vorticity towards the computational boundary. The results of the two calculations show good agreement (Figures 4.9 through 4.18), where the computations in the adaptive cartesian coordinates are shown in Figures 4.14 through 4.18 and in the radial coordinates in Figures 4.9 through 4.13. The same solution is shown in the computational domain \((\xi, \eta)\) coordinates, Figures 4.19 through 4.28. The number of modes in the computational domain was 128, number of spatial points in the \( \xi \) direction was 150, and the length of the computational domain in the \( \eta \) direction was chosen as 12.8, and in the \( \xi \) direction as 15. As for the calculation in the polar coordinates, the number of modes is 16, and the number of spatial points in the \( r \) direction is 200, where the computational domain length was chosen as \( R_{\text{max}} = 10 \). The physical phenomena does not differ from the previous result obtained in Chapter 3 for the short time evolution for this particular straining coefficient \( A = .1 \), where, in this case, like in the previous one, the oscillations of the vorticity contours is due to the
Figure 4.9 Contours of constant vorticity of counter-rotating vortices under a potential straining velocity $Re = 1000$, $T = .25$ and $T = .2520$. 
Figure 4.10 Contours of constant vorticity of counter-rotating vortices under a potential straining velocity $Re = 1000$, $T = .2540$ and $T = .2560$. 
Figure 4.11 Contours of constant vorticity of counter-rotating vortices under a potential straining velocity Re = 1000, T = .2580 and T = .2600.
Figure 4.12 Contours of constant vorticity of counter-rotating vortices under potential straining velocity Re = 1000, T = .2620 and T = .2640.
Figure 4.13 Contours of constant vorticity of counter-rotating vortices under a potential straining velocity Re = 1000, T = 0.2660 and T = 0.2680.
Figure 4.14 Contours of constant vorticity (cartesian coordinate $\psi = Axy$) $Re = 1000$, $T = .25$ and $T = .252$. 

STRAINING VELOCITY $Re=1000$, $A=.1$ (CARTESIAN COORDINATE $PSI=AXY$) 

$T = 0.250$ 

$V = \frac{.3972 \times 10^1}{.2565 \times 10^-3}$ 

$V = \frac{.2975 \times 10^1}{.1945 \times 10^-3}$ 

$V = \frac{.9925 \times 10^-1}{.0000 \times 10^-3}$ 

$X \ 0.0000 \times 10^-3 \ 0.1945 \times 10^-3 \ 0.3075 \times 10^-3$ 

$Y \ 0.3075 \times 10^-3 \ 0.1945 \times 10^-3 \ 0.0000 \times 10^-3$ 

$T = 0.252$ 

$V = \frac{.3972 \times 10^1}{.2565 \times 10^-3}$ 

$V = \frac{.2975 \times 10^1}{.1945 \times 10^-3}$ 

$V = \frac{.9925 \times 10^-1}{.0000 \times 10^-3}$ 

$X \ 0.0000 \times 10^-3 \ 0.1945 \times 10^-3 \ 0.3075 \times 10^-3$ 

$Y \ 0.3075 \times 10^-3 \ 0.1945 \times 10^-3 \ 0.0000 \times 10^-3$
Figure 4.15 Contours of constant vorticity (cartesian coordinate $\psi = Axy$) $Re = 1000$, $T = .254$ and $T = .256$. 
Figure 4.16 Contours of constant vorticity (cartesian coordinate \( \psi = Axy \)) \( Re = 1000 \), \( T = .258 \) and \( T = .260 \).
Figure 4.17 Contours of constant vorticity (cartesian coordinate $\psi = Axy$) $Re = 1000$, $T = .262$ and $T = 2.64$. 
Figure 4.18 Contours of constant vorticity (cartesian coordinate $\psi = Axy$) Re = 1000, T = .266 and T = .268.
Figure 4.19 Contours of constant vorticity in the computational domain $A = 0.1$, $Re = 10000$, $T = 0.25$. 
Figure 4.20 Contours of constant vorticity in the computational domain $A = .1$, $Re = 10000$, $T = .2502$. 
Figure 4.21 Contours of constant vorticity in the computational domain $A = .1$, $Re = 10000$, $T = .2504$. 
Figure 4.22  Contours of constant vorticity in the computational domain $A = .1$, $Re = 10000$, $T = .2506$. 
Figure 4.23 Contours of constant vorticity in the computational domain $A = .1$, $Re = 10000$, $T = .2508$. 
Figure 4.24  Contours of constant vorticity in the computational domain $A = .1$, $Re = 10000$, $T = .2510$. 
Figure 4.25 Contours of constant vorticity in the computational domain $A = 0.1$, $Re = 10000$, $T = 0.2512$. 
Figure 4.26 Contours of constant vorticity in the computational domain $A = 0.1$, $Re = 10000$, $T = 0.2514$. 
Figure 4.27 Contours of constant vorticity in the computational domain $A = .1$, $Re = 10000$, $T = .2516$. 
Figure 4.28  Contours of constant vorticity in the computational domain $A = .1, \text{Re} = 10000, T = .2518$. 
initial condition which was taken as a Gaussian profile. However, after a long time (Figures 4.13 and 4.18), a sharp front appears in the vorticity contour. At first sight this phenomena seems related to numerical error, but it occurs in both programs. Unfortunately, neither program is able to compute the result any further due to the formation of a sharp gradient in the vorticity, and it is difficult to predict the behavior of this sharp front.

From the numerical results, Figures 4.18 and 4.28, it appears that the height of the computational domain is too large. As a result, a distorted shape for the vorticity is obtained in the computational domain, which means that the n coordinates are stretched faster than the actual advection of the vorticity. In the physical plane (Figure 4.18) the same phenomenon can be seen as a lack of resolution and as a result, the vorticity contour is less smooth than the solution in polar coordinate (Figure 4.13). Moreover, Figures 4.20 and 4.28 show the appearance of short waves with the same wave length as the mesh size in the n direction. These waves are a result of lack of resolution of the vorticity in the computational domain. Clearly, the computational box needs to be shorter so that the vorticity may be better resolved without an excess number of mesh points. These difficulties appear more severely in the next problem.

Calculations were also performed for the motion of the counter-rotating vortices under zero straining velocity with the same initial condition as in Chapter 3 (see Figure 4.29). In this case, the slender ratio $\lambda_0 = 1$ and consequently the solution is affected by the
Figure 4.29 Contours of constant vorticity (cartesian coordinates)
A = 0, Re = 10, T = 1.85.
induced velocity from the computational images. It can be seen that the solution is similar to the corresponding one in Chapter 3 (Figure 3.7) where the maximum of the vorticity in this case is .524, compared to the previous one of .522. That is to say, by a proper choice of computational length size (L), it is possible to control the effect of the induced velocity of the computational images.

In order to check the sensitivity of the results, two computations have been done for an initial vorticity distribution with slenderness ratio of \( \lambda_0 = 0.04 \) under zero straining velocity, and Reynolds number of 10000 with a computational domain of length of 10 in the \( \xi \) direction and 6.4 in the \( \eta \) direction. The number of spatial points in the first case was 100 along the \( \xi \) direction and 64 modes in the \( \eta \) direction (Figures 4.30 through 4.35). In the second case, the number of points in the \( \xi \) direction was 150 and the number of modes 128 (Figures 4.36 through 4.41). Both results agree very well on the large-scale structure. There are fewer waves in the \( \eta \) direction in the second case (Figure 4.41) than in the first (Figure 4.36). The main reason for this improvement is the finer resolution in the second case. The sensitivity of the results to the boundary condition has been checked by repeating the last computation with twice the length of the computational plane in the \( \eta \) direction. As can be verified from the computed results, there is no change in the physical phenomena (Figures 4.42 through 4.47).

In summary, the calculation of the results in the computational plane for the above problem (an initial vorticity distribution with
Figure 4.30  Contours of constant vorticity (computational domain)
Re = 10000, A = 0, N_x = 100, N_y = 64, T = .25.
Figure 4.31 Contours of constant vorticity (computational domain)
Re = 10000, A = 0, N_x = 100, N_y = 64, T = .2502.
Figure 4.32 Contours of constant vorticity (computational domain)
$R = 10000, A = 0, N_x = 100, N_y = 64, T = .2504.$
Figure 4.33 Contours of constant vorticity (computational domain)
$R = 10000$, $A = 0$, $N_x = 100$, $N_y = 64$, $T = .2506$. 
Figure 4.34 Contours of constant vorticity (computational domain)
Re = 10000, A = 0, N_x = 100, N_y = 64, T = .2508.
Figure 4.35  Contours of constant vorticity (computational domain)
Re = 10000, A = 0, N_x = 100, N_y = 64, T = .2510.
Figure 4.36 Contours of constant vorticity (computational domain)
Re = 10000, A = 0, N_x = 150, N_y = 128, T = .2500.
Figure 4.37 Contours of constant vorticity (computational domain)
Re = 10000, A = 0, N_x = 150, N_y = 128, T = .2502.
Figure 4.38 Contours of constant vorticity (computational domain)
Re = 10000, A = 0, N_x = 150, N_y = 128, T = .2504.
Figure 4.39 Contours of constant vorticity (computational domain)
Re = 10000, A = 0, N_x = 150, N_y = 128, T = .2506.
Figure 4.40. Contours of constant vorticity (computational domain)
Re = 10000, A = 0, \( N_x = 150 \), \( N_y = 128 \), \( T = .2508 \).
Figure 4.41 Contours of constant vorticity (computational domain)
Re = 10000, A = 0, N_x = 150, N_y = 128, T = .2510.
Figure 4.42 Contours of constant vorticity (sensitivity in the computational domain $Re = 10000$, $A = 0$, $N_x = 150$, $N_y = 128$, $T = .2500$.}
Figure 4.43 Contours of constant vorticity (sensitivity results in the computational domain) $\text{Re} = 10000$, $A = 0$, $N_x = 150$, $N_y = 128$, $T = .2502$. 
Figure 4.44 Contours of constant vorticity (sensitivity results in the computational domain) $Re = 10000$, $\Delta = 0$, $N_x = 150$, $N_y = 128$, $T = .2504$. 
Figure 4.45 Contours of constant vorticity (sensitivity results in the computational domain) $Re = 10000$, $A = 0$, $N_x = 150$, $N_y = 128$, $T = .2506$. 
Figure 4.46 Contours of constant vorticity (sensitivity results in the computational domain) $Re = 10000$, $A = 0$, $N_x = 150$, $N_y = 128$, $T = .2508$. 
Figure 4.47  Contours of constant vorticity (sensitivity results in the computational domain) \( \text{Re} = 10000, A = 0, N_x = 150, N_y = 128, T = .2510 \).
slenderness ratio of 0.04 and under zero straining velocity) shows that the vorticity in the computational plane is contracted to a narrow band (Figure 4.41) and as a result a wave pattern appears in the η direction indicating lack of resolution.

The results of an initial vorticity distribution with slenderness ratio of 0.04 and zero straining velocity have been plotted in the physical plane, (Figures 4.48 through 4.58), where the first case is for Reynolds number 1000 and the second case is for 10000. As was pointed out by Moore and Saffman [1971], a constant vorticity distribution of slenderness ratios of λ<1/2.9 is stable to disturbances. Despite the fact that this analysis was obtained for inviscid flow, constant vorticity in an elliptical region, and hyperbolic straining velocity, it seems reasonable to look for instability for this ratio or lower. Hence, an initial vorticity distribution was chosen of slenderness ratio 0.04 as a test case. This computation was done with zero straining velocity and as a result the slenderness ratio was kept fixed. It can be seen that for a short time the results are in good agreement with the approximate solution obtained previously (Figures 4.3 and 4.4), where the induced velocity advected the lower part of the vorticity away from the y axis. The full nonlinear development shows a more complicated behavior with the lower part rolling-up into a concentrated vortex core (Figures 4.51 through 4.53), with a small tail of vorticity, (which was originally the lower edge of the vorticity), wrapping around the concentrated vortex. This concentrated vortex behaves as a single vortex by inducing a strong velocity on the upper
Figure 4.48 Counter-rotating vortices under zero straining velocity
Re = 1001, T = .25 and T = .2520.
Figure 4.49  Counter-rotating vortices under zero straining velocity
Re = 1001, T = .254 and T = .256.
Figure 4.50  Counter-rotating vortices under zero straining velocity
R = 1001, T = .258 and T = .2600.
Figure 4.51 Counter-rotating vortices under zero straining velocity
Re = 1001, T = .262 and T = .264.
Figure 4.52 Counter-rotating vortices under zero straining velocity
Re = 1001, T = .266 and T = .268.
Figure 4.53 Counter-rotating vortices under zero straining velocity
Re = 1001, T = .270.
Counter-rotating vortices under zero straining velocity

Figure 4.54 R = 10000, T = .2500 and T = .2504.
Counter-rotating vortices under zero straining velocity
$T = 0.2508$ and $T = 0.2512$.

Figure 4.55 Counter-rotating vortices under zero straining velocity
$Re = 10000$, $T = 0.2508$ and $T = 0.2512$. 
Figure 4.56  Counter-rotating vortices under zero straining velocity
Re = 10000, T = .2516 and T = .2520.
Figure 4.57  Counter-rotating vortices under zero straining velocity  
Re = 10000, T = .2524 and T = .2526.
Counter-rotating vortices under zero straining velocity
Re = 10000, T = .2532 and T = .2536.

Figure 4.58 Counter-rotating vortices under zero straining velocity
Re = 10000, T = .2532 and T = .2536.
portion of the vorticity (Figures 4.56, 4.57, 4.52 and 4.53). As a result of this advection velocity, the upper part seems to stretch further along the y direction. In the second stage, the upper part seems to separate from the vortex core, where a narrow neck appears between the elongated vortex and the vortex core (Figures 4.52, 4.53 and 4.56). It is difficult to predict the nature of the separation point, i.e., the narrow neck, due to the fact that when the neck becomes narrow and there is not enough resolution in the x direction to resolve that region. Moreover, the tail which previously wraps around the vortex core appears to separate from the vortex and move to the region of the narrow neck (Figures 4.57 and 4.58). It appears that for a large Reynolds number, the solution is no longer smooth (Figure 4.58). The first possibility is that there is insufficient resolution in the region of interest. Alternately, it is possible that for a large Reynolds number there is an additional physical process not presently understood that leads to instability. However, the limitations of grid size and computer time prevent us from determining the reason. Similarly, it is difficult to give a detailed description of the separation of the vorticity tail. It is obvious from the last computation that it does separate, but it is not clear if the nonsmoothness of the tail after separation is due to numerical instability or some physical process. If separation does occur between the vortex core and the upper portion of the vorticity, it is reasonable to believe that the vortex core will descend faster than the upper portion, because its circulation is larger than the upper portion.
In this case, the advection velocity which was induced by the concentrated vortex will decay and as a result the upper portion will repeat the same phenomena. We may be able to see in that case structures of small vortices traveling along the y axis where the biggest descended faster than the others. Essentially this process illustrates the principle that an initially unstable structure tends to evolve into a stable structure. The initially unstable vorticity distribution breaks into a concentrated vortex with an estimated slenderness ratio 1, which is stable.

Figure 4.59 shows the evolution of counter-rotating vortices initially having a slenderness ratio of 1, under the influence of a strong straining velocity (A = 1). In this case, the vorticity stretched quickly along the y axis. The behavior is similar to the previous calculations where in this case the lower portion advects outward from the y axis. Further computation shows similar results to the initial slender vorticity under zero straining velocity, until the dissipation terms became dominant and affects the solution by smearing it. However, this behavior is hardly seen in the physical plane due to the small scale in the x direction. The adaptive grid used here is adequate for this problem and resolved it fairly well (Figures 4.60 and 4.65).

In conclusion, due to the lack of resolution in the computational plane, the spectral method fails to fully resolve the behavior of counter-rotating vortices in the straining flow that stretched the vortices in the y direction. Short waves develop in the n direction, which are initially very weak, about 1% of the maximum
Figure 4.59 Counter-rotating vortices under straining velocity
Re = 10000, A = 1, T = .2500 and T = .2504.
Figure 4.60 Counter-rotating vortices under straining velocity (computational domain) Re = 10000, A = 1, T = .2500.
Figure 4.61 Counter-rotating vortices under straining velocity (computational domain) $Re = 10000$, $A = 1$, $T = .2502$. 
Figure 4.62 Counter-rotating vortices under a straining velocity (computational domain) $Re = 10000$, $A = 1$, $T = .2504$. 
Figure 4.63 Counter-rotating vortices under a straining velocity (computational domain) $Re = 10000$, $A = 1$, $T = .2506$. 
Figure 4.64 Counter-rotating vortices under a straining velocity (computational domain) $Re = 10000$, $A = 1$, $T = .2508$. 
Figure 4.65 Counter-rotating vortices under a straining velocity (computational domain) $Re = 10000$, $A = 1$, $T = 0.2510$. 
vorticity, but later on they grow and affect the behavior of the numerical results. For example, a negative vorticity appears in the computational plane, and as a result it interacts with the positive one. This problem can be resolved by increasing the number of modes, or alternatively by the generation of a better grid to resolve the problem. The proposed model is believed adequate enough to see the main physical phenomena and to understand it.
CHAPTER 5

COUNTER-ROTATING VORTICES UNDER RADIAL STRAINING VELOCITY

5.1 Formulation of the Equations

In the previous chapter we discussed the two-dimensional motion of a pair of counter-rotating vortices under a potential straining velocity, \( \psi = Axy \), where \( A \) is some constant. As discussed in Chapter 1, it is difficult to determine the vertical straining velocity using the "Biot Savart" law. This is because that the usual approximations to the induced velocity are true far away from the viscous core of the vortex, and the flow inside the viscous core of the vortex is more difficult to determine (Widnall, [1975]).

Since the straining velocity can exhibit a longitudinal velocity (Widnall, [1975]), we have decided to look at the simple case of a radial straining velocity. Together with the results of the previous chapter, this case represents the two extreme possibilities for the strain flow. Specially, the radial velocity component of the strain flow is \(-\alpha r\) while the axial component is \(2\alpha z\) (Batchelor, [1983]).

The vorticity equations in polar coordinates, where the origin is located on the symmetric line between the vortices, and in the presence of a radial straining velocity, can be written as

\[
\frac{\partial \omega}{\partial t} + (v_r - \alpha r) \frac{\partial \omega}{\partial r} + \frac{v_\theta}{r} \frac{\partial \omega}{\partial \theta} = 2\alpha \omega + \frac{1}{Re} \nu \frac{\partial^2 \omega}{\partial \theta^2} \tag{5-1}
\]
The vorticity is assumed to point only in the $z$ direction and is independent of $z$.

A stream function may be introduced as follows

$$
\psi_v = -\frac{1}{r} \frac{\partial \psi}{\partial \theta}
$$

$$
\psi_\theta = \frac{\partial \psi}{\partial r}
$$

and so $\psi^2 = w$. The above equations describe two different processes. The first one is the radial advection (straining velocity $-ar$, and the velocity $v_r$, $v_\theta$ due to the vortex flow), and the longitudinal transport corresponding to the longitudinal velocity ($2az$). The main difference between this vorticity equation and the ones considered in the previous chapters is the longitudinal stretching term ($2a$). In the absence of the dissipation term, this equation can be written in the Lagrangian form as

$$
\frac{dw}{dt} = 2aw,
$$

which may be integrated as

$$
w = C_1 e^{2at}
$$

That means that any material elements which have a given vorticity will be transported by this flow field and their vorticity strength will grow exponentially. Moreover, since the vorticity is advected by the straining velocity, we expect that the vorticity distribution will be concentrated as a function of time in a narrow region around the origin.
The main difficulty in solving these equations is the transport mechanism which advects the vorticity to a narrow region around the origin. As a result, the grid which was proposed in the previous chapter is no longer capable of resolving this problem. The second difficulty is the exponential growth of the vorticity. This will cause numerical instability since the nonlinear term grows as a function of the vorticity growth. As a result, it will violate the C.F.L. condition for fixed time steps. In order to devise a proper adaptable grid for this problem, let us use one similar to that of Chapter 4. As a result, the vorticity advects towards the origin, and \( v_r \) becomes small. This means that asymptotically the vorticity equation behaves as

\[
\frac{3w}{at} - ar \frac{3w}{ar} = 0 \tag{5-5}
\]

where equation (5-5) is the Euler equation. For the moment, the dissipation is neglected.

Equation (5-5) may be solved by the method of characteristics; \( w \) is a constant along the curves

\[
\frac{dr}{dt} = -ar \quad r = C_3 e^{-at} \tag{5-6}
\]

Hence, the characteristics in \((r,t)\) plane are:

\[
\xi = r/e^{-at} \tag{5-7}
\]

Let the vorticity and the stream function now depend on the new variables. Hence, we have that
The vorticity equation as well as the stream function equation can be written in the new variable as

\[ \frac{\partial w}{\partial t} + \psi^* \frac{\partial w}{\partial \xi} + \frac{\partial \psi}{\partial \theta} \frac{\partial w}{\partial \theta} = 2\omega w + \frac{e^{2at}}{Re} \left( \frac{1}{\xi} \frac{\partial}{\partial \xi} \left( \xi \frac{\partial w}{\partial \xi} \right) + \frac{1}{\xi^2} \frac{\partial^2 w}{\partial \theta^2} \right) \]

\[ \nabla^2 \psi^* = w \]  

\[ \psi^* = - \frac{1}{\xi} \frac{\partial \psi^*}{\partial \theta} \]

\[ \psi^* = \frac{\partial \psi^*}{\partial \xi} \]

This set of equations is much more convenient to handle because the equations are written in a stretched coordinate, which is capable of resolving the vorticity distribution. It is important to note that as the vorticity is advected towards the center, dissipation effects will become important.

As we have pointed out, the vorticity equation has an exponential type of growth. As a result, we have to scale time step in order not to violate the C.F.L. condition. Hence, let us write the vorticity and the new time scale as

\[ w = w(t, \xi) \]

\[ \psi = e^{-2at} \psi(t, \xi) \]
\[ w = e^{2a\tau} w^* (\tau, \xi, \theta) \]
\[ \psi = \psi (\tau, \xi, \theta) \]  
\[ \tau = \frac{e^{2a\tau} - 1}{2a} \]  

The vorticity equation now becomes

\[ \frac{\partial w^*}{\partial \tau} + v_r^* \frac{\partial w^*}{\partial \xi} + v_\theta^* \frac{\partial w^*}{\partial \theta} = \frac{1}{Re} \left( \frac{1}{\xi} \frac{\partial}{\partial \xi} \left( \xi \frac{\partial w^*}{\partial \xi} \right) + \frac{1}{\xi^2} \frac{\partial^2 w^*}{\partial \theta^2} \right) \]

\[ \nabla^2 \psi = w^* \]  
\[ v_r^* = - \frac{1}{\xi} \frac{\partial \psi}{\partial \theta} \]

\[ v_\theta^* = \frac{\partial \psi}{\partial \xi} \]

Essentially, the original reason for using the vorticity representation described in equation (5-10), was to find a new variable \( w^* \) that does not grow in time, and to scale the time step according to maximum amplification of the nonlinear term. The vorticity was then chosen as it is in equation (5-10) because of the vorticity behavior.

The main advantage to equation (5-11) is that it is equivalent to the 2-D form describing counter-rotating vortices with no straining velocity (see Chapter 2 and 3). Hence, the solution of the 2-D problem is applicable here. In other words, counter-rotating vortices in a constant radial strain are similar to 2-D counter-rotating vortices with no straining velocity. It is worthwhile to mention here that even
though the analysis that was carried out in this chapter was for a
time-independent straining velocity, it can easily be extended to a
radial straining velocity which is a function of time.

Since the physical time $t$ is related logarithmically to $\tau$, it
becomes expensive to solve equation (5-11) numerically for a long time
by using a fixed time-step in $\tau$. However, vorticity grows to some
maximum and then decays due to dissipative effects. Hence, when the
vorticity is dissipating there is no longer any reason to use small time
steps. In order to reduce the number of computer operations we took
advantage of the fact that the vorticity dissipates and its maximum
decays in time by decreasing the time step appropriately.

In order to choose the right dynamical time step, let us
rewrite the vorticity equation (5-11) as

$$\frac{dT}{dt} \frac{\partial w^*}{\partial T} + v_r^* \frac{\partial w^*}{\partial \xi} + v_\theta^* \frac{\partial w^*}{\partial \theta} = \frac{1}{\text{Re}} \left\{ \frac{1}{\xi} \frac{\partial}{\partial \xi} \left( \xi \frac{\partial w^*}{\partial \xi} \right) + \frac{1}{\xi^2} \frac{\partial^2 w^*}{\partial \theta^2} \right\} \quad (5-12)$$

where here we use a new computational time scale ($T$). Equation (5-12)
can be approximated by finite differences in the $r$ direction and in time
as follows:

$$\frac{d\xi}{d\tau} \left( \frac{w_{j+1} - w_j}{\Delta \xi} \right) + v_r^* \frac{w_{j+1} - w_{j-1}}{2h} + \frac{v_\xi^*}{\xi_j} \frac{\partial w_j^*}{\partial \theta}$$

$$= \frac{1}{\text{Re} \xi_j} \left\{ \frac{1}{\xi_j} \left( \xi_j + (1/2) \right) \mu(w_{j+1} - w_j - \xi_j + (1/2) \right) - \xi_j + (1/2) \right) \mu[w_{j+1} - w_{j-1}] + \frac{h^2}{\xi_j} \frac{\mu(\partial^2 w_j)}{\partial \theta^2} \right\} \quad (5-13)$$
where \(\mu w_j^\kappa = \frac{1}{2} (w_j^{\kappa+1} + w_j^\kappa)\). The stream function equation can be approximated by

\[
\frac{1}{h^2 \epsilon_j} \left[ \xi_{j+(1/2)} (\psi_{j+1}^\kappa - \psi_j^\kappa) - \xi_{j+(1/2)} (\psi_j^\kappa - \psi_{j-1}^\kappa) \right] + \frac{1}{\epsilon_j} \frac{\partial^2 \psi_j^\kappa}{\partial \theta^2} = w_j^\kappa
\]

\[
v_{\kappa,j} = - \frac{1}{\epsilon_j} \frac{\partial \psi_j^\kappa}{\partial \theta}
\]

\[
v_{\theta,j} = \frac{\psi_{j+1}^\kappa - \psi_{j-1}^\kappa}{2h}
\]

For convenience, the (*) symbol was omitted and the subscript \(j\) refers to the location in \(r\) while the superscript \(\kappa\) indicates the time level.

Before solving these equations, a linear stability analysis is done to verify that the method may give reliable results. The error can be written as a difference between the numerical results and the exact solution or

\[
w_j^{\kappa+1} = w_j^0 + e_j^{\kappa+1}
\]

\[
\psi_j^{\kappa+1} = \psi_j^0 + \delta_j^{\kappa+1}
\]

Hence, the velocity field can be written

\[
v_{\kappa,j} = - \frac{1}{\epsilon_j} \frac{\partial \psi_j^0}{\partial \theta} - \frac{1}{\epsilon_j} \frac{\partial \delta_j^\kappa}{\partial \theta}
\]

\[
v_{\theta,j} = \frac{\partial \psi_j^0}{\partial \xi} \bigg|_j + \frac{\delta_j^{\kappa+1} - \delta_j^{\kappa-1}}{2h}
\]
Let us substitute this relation into the difference approximation of the vorticity equation. Neglecting terms of order \((h\delta, h\epsilon, \epsilon\delta)\), we obtain:

\[
\frac{dT'}{d\tau} \left( \frac{e_j^{K+1}-e_j^{K-1}}{2\Delta r} \right) + v_{ij} \frac{\epsilon_j^{K+1}-\epsilon_j^{K-1}}{2h} + \frac{\theta_{ij} \epsilon_j^K \epsilon_j^K}{\epsilon_j^K} + \frac{\theta_{ij} \epsilon_j^K \epsilon_j^K}{\epsilon_j^K} \left( \frac{\delta_j^{K+1} - \delta_j^{K-1}}{2h} \right)
\]

\[
= \frac{1}{Re} \left( (e_j^{K+1} - 2e_j^{K} + e_j^{K-1}) + \frac{h}{\epsilon_j^K} (e_j^{K+1} - e_j^{K-1}) + \frac{h^2}{\epsilon_j^K} \epsilon_j^K \right) \quad (5-17)
\]

and

\[
\delta_j^{K+1} - 2\delta_j^{K} + \delta_j^{K-1} + \frac{1}{2\epsilon_j^K h} (\delta_j^{K+1} - \delta_j^{K-1}) + \frac{1}{\epsilon_j^K} \epsilon_j^K \epsilon_j^K = \epsilon_j^K \quad (5-17a)
\]

If we regard the wave length of the error as smaller than the characterized length scale of the mean flow, we regard the mean flow as a constant at each point. Thus (5-17) is a linear equation with constant coefficients and so we assume that the error can be represented as

\[
\begin{bmatrix}
\epsilon_j^K(	heta) \\
\delta_j^K(	heta)
\end{bmatrix} = \sum_m \sum_n \begin{bmatrix}
A_{n,m}^K \\
B_{n,m}^K
\end{bmatrix} e^{in\theta} e^{imjh} \quad (5-18)
\]

By substituting (5-18) into (5-17), the following equation is obtained
\[
\frac{dT}{d\tau} \left( \frac{\lambda_{\omega}^{m-1} - \lambda_{\omega}^{m-1}}{2\Delta T} \right) + i \frac{v_r^0}{n} A_{\omega}^{m} \sin(mh) + \frac{v_\theta^0}{\xi} \sin(A_{\omega}^{m})
\]

\[
= \frac{1}{Re} \{ \text{dissipation} \} \quad (5-19)
\]

where \( \frac{dT}{d\tau} \), \( v_r^0 \), \( v_\theta^0 \) are the frozen values (constants). The numerical behavior of the dissipation term is to stabilize the equation. Hence, if a stable choice for the time step is made when the dissipation is neglected, we can anticipate that the full method will be stable. The linear equation (5-19) may be solved by letting \( A_{\omega}^m = \lambda^m \). Thus,

\[
\left(\frac{dT}{d\tau}\right) \left(\frac{\lambda - \frac{1}{\lambda}}{2\Delta T}\right) + i\beta = 0
\]

where

\[
\beta = \frac{v_r^0}{n} \sin(mh) + \frac{v_\theta^0}{\xi} n \quad (5-20)
\]

For stability, we require that \( |\lambda| < 1 \). The roots of (5-20) are easily determined and so we require

\[
\bar{\beta} \Delta T < \frac{dT}{d\tau}
\]

where

\[
\bar{\beta} = \max \left| \frac{v_r^0}{n} \sin(mh) + \frac{v_\theta^0}{\xi} \right|
\]
Suppose that $\tilde{\beta}_0 = \tilde{\beta}(t=0)$ is such that the stability constant is satisfied and $\frac{dT}{d\tau}$ is initially 1, then the choice

$$\frac{dT}{d\tau} = \frac{\beta}{\beta_0}$$

ensures that the stability condition is always satisfied. In practice, this method was conditionally stable even for equation (5-9), where the stretching term was approximated by the "Crank Nicolson" method.

5.2 Asymptotic Behavior of Counter-Rotating Vortices

Under Radial Straining Velocity

As we have noted, there are difficulties in obtaining the long time evolution of the solution, since many time steps are required. Moreover, the vorticity distribution spreads towards the computational boundaries, and the solution is no longer reliable (see Chapter 3). In order to overcome these difficulties, rather than look for a dynamical computational domain, we can assume that equation (5-11) behaves asymptotically as the heat equation, i.e.,

$$\frac{\partial w^*}{\partial \tau} = \frac{1}{Re} \nabla^2 w^* \quad (5-21)$$

The advection term is considered asymptotically small compared to the linear term. The solution of equation (5-21) that satisfies the boundary conditions is
\[ w^* = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(\sigma, s) e^{-\left(\sigma^2 + s^2\right) \tau} \frac{e^{i\alpha y^* + i\alpha x^*}}{Re} \, ds \, d\sigma \quad (5-22) \]

where

\[ A(\sigma, s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(0, x^*, y^*) e^{-i\alpha x^*} e^{-i\alpha y^*} \, dx^* \, dy^* \]
\[ = -2i \int_{0}^{\infty} \int_{-\infty}^{\infty} w(0, x^*, y^*) e^{-i\alpha y^*} \sin sx^* \, dy^* \, dx^* \quad (5-23) \]

where we have used the oddness of the vorticity distribution. For convenience, \( x^*, y^* \) are the stretched Cartesian coordinates, that correspond to the polar coordinates of equation (5-7). Let

\[ \sqrt{\frac{\tau}{Re}} \left[ \sigma - \frac{iRe}{2\tau} \, y^* \right] = \lambda \]
\[ \sqrt{\frac{\tau}{Re}} \left[ s - \frac{iRe}{2\tau} \, x^* \right] = \rho \quad (5-24) \]

\[ w^* = \frac{1}{4\pi^2} e^{-\frac{Re^2}{4\tau}} \frac{Re}{\tau} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(\sigma, s) e^{-\left(\lambda^2 + \rho^2\right)} \, d\lambda \, d\rho \]

where \( r \) is defined by (5-7). For large \( \tau \), we approximate this integral by
From equation (5-23), the following relations can be obtained

\[ A(0,0) = 0 \]

\[ \frac{\partial A}{\partial \sigma} \bigg|_0 = 0. \]

\[ \frac{\partial A}{\partial \sigma} = -2i \int_{-\infty}^{\infty} x^* w(0,x^*,y^*) \, dx^* \, dy^* = -2i \, I_2 \]

\[ \frac{\partial^2 A}{\partial \sigma^2} \bigg|_0 = 0 \]

\[ \frac{\partial^2 A}{\partial \sigma^2} \bigg|_0 = 0 \]

where \( I_2 \) is the hydrodynamic force that was derived in Chapter 2.

Hence, the asymptotic behavior of equation (5-22) is

\[ w^* = \frac{Re^2 I_2}{4\pi^2} \, x^* \, e^{-\frac{Re}{4\pi} \, r^2} \]  

(5-26)
where \( r^2 = x^2 + y^2 \). The maximum of the vorticity in the transformed plane is

\[
\omega_{\max}^* = \frac{\text{Re}^2 I_2}{4\pi \tau^2} \left\{ \frac{2\tau}{\text{Re} \cdot 2.71828} \right\}^{1/2}
\]  

(5-27)

But we have \( \omega_{\max} = e^{2\alpha t} \omega_{\max}^* \), \( \tau = \frac{e^{2\alpha t}}{2\alpha} \) and \( I_2 = 4\pi \). Hence, the maximum of the vorticity in the physical plane is:

\[
\omega_{\max} = \left\{ \frac{16 \text{Re}^3 \alpha^3}{2.71828} \right\}^{1/2} e^{-\alpha t}
\]  

(5-28)

For counter-rotating vortices in an external radial straining velocity there is no steady state as can be seen from equation (5-28). For the long time evolution, the maximum vorticity decays as \( e^{-\alpha t} \). When \( A(0,0) \neq 0 \), as in a single stretched viscous vortex (Batchelor, [1983]), or for two co-rotating vortices, a steady state solution can exist.

Comparisons between the numerical results and the asymptotic results shows good agreement for the values of the straining coefficient of \( \alpha = .25, .5, \) and \( \text{Re} = 10, 40, \) and 100. Most of the results were computed by equation (5-9) and have been compared to the numerical solution of equation (5-11). In the next section we give results in detail.
5.3 Analysis of the Results

The computed results are shown in Figures 5.1 through 5.5, where Figure 5.1 shows the comparison between the numerical computation of equation 5-9 and the numerical calculation of equation 5-11. Equation 5-9 was computed with variable time steps, but equations 5-11 was computed with a fixed time step size, as was done in Chapter 3. The agreement between the computed results of the two different schemes is good. Even though the vorticity strength increased, no significant deviation as a result of numerical error appears. It was found that it is less expensive to use equation 5-11 for short time evolution, but for a long time evolution, equation 5-9 is better. In both cases, it is necessary to vary the time step size in order to ensure stability, and to save computer time. All the remaining results that appear here refer to the computation of equation 5-9 with a variable time step.

The comparison between the asymptotic solution of equation 5-28 and the numerical solution is good, as shown in Figures 5.2 through 5.5. Due to the dissipation of the vorticity, the numerical computation was stopped whenever the vorticity reached the computational boundary. The numerical computation could be carried further by increasing the computational domain. Due to the good agreement between the numerical solution and the asymptotic solution, it is preferrable to represent the long time evolution of the solution by the asymptotic solution.

The sensitivity of the results was checked by increasing the number of grid points in the radial direction from 100 to 200, and
Figure 5.1 Maximum vorticity for longitudinal stretching (varying time step, logarithmic time scale, and linear time scale) $Re = 100, \alpha = .5$
Figure 5.2 Maximum vorticity for longitudinal stretching (varying time step) $Re = 100, \alpha = .5$. 
Figure 5.3 Maximum vorticity for longitudinal stretching (varying time step) Re = 10, $\alpha = .5$. 

Numerical solution ($\Delta t = .1$, $N_r = 100$, $N_\theta = 16$.)

Numerical solution ($\Delta t = .05$, $v_r = 200$, $N_\theta = 16$)

Asymptotic solution
Figure 5.4 Maximum vorticity for longitudinal stretching (varying time step) Re = 100, α = .25.
Figure 5.5 Maximum vorticity for longitudinal stretching (varying time step) $Re = 40$, $\alpha = .25$. 
reducing the initial time step to half of its original value. The same results were obtained, as can be seen in Figure 5.3.

The computation time on the "VAX" 11/750 takes 22 CPU hours for 100 grid points in the radial direction and 16 modes in the \( \theta \) direction. The initial time step that was used is \( \Delta T = .01 \), with 20,000 time steps. The use of variable time steps proved numerically to be reliable and stable. No instabilities due to the violation of the C.F.L. condition have been encountered in the numerical computation, even for large Reynolds number (the maximum of the vorticity is proportional to Reynolds number). For a large Reynolds number, the numerical computation used smaller time steps. Hence, more computer time was required to solve this problem.

In summary, we describe the physical phenomenon involved in this problem. Due to the stretching mechanism, the strength of the vorticity increases. At the same time, the cross-sectional area of the vorticity distribution decreases until the length scale in the radial direction is small enough for the dissipation terms to be important. The dissipation terms then balance the effect of the stretching. Hence, it can be understood from the results obtained that for large Reynolds number the length scale is smaller than for small Reynolds number. The strength of the vorticity will increase up to some maximum which is proportional to the Reynolds number. Then, due to the effect of the dissipation, it will decrease. The maximum of the vorticity reaches a maximum value in time which appears to be linearly proportional to \( \alpha \text{Re} \).
The flows that have been considered in this thesis may be important in the understanding of the onset of turbulence. Recently it was shown that the "Euler" equations can exhibit a singularity in finite time, where the solution breaks down in particular points in the flow field. The assumption today is that dissipation effects prevent the singularities from forming and instead cores of concentrated vorticity appear at these particular points. It is worthwhile to mention that the appearance of singularities in finite time is due to the stretching mechanism. The subsequent behavior of these concentrated cores of vorticity remain one of the most important problems in fluid mechanics.
REFERENCES


