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CONSTITUTIVE MODELLING FOR ANISOTROPIC HARDENING
BEHAVIOR WITH APPLICATIONS TO COHESIONLESS SOILS

by

Sujithan Somasundaram

A Dissertation Submitted to the Faculty of the
DEPARTMENT OF CIVIL ENGINEERING AND ENGINEERING MECHANICS
In Partial Fulfillment of the Requirements
For the Degree of
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WITH A MAJOR IN CIVIL ENGINEERING

In the Graduate College
THE UNIVERSITY OF ARIZONA

1986
As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Sujithan Somasundaram entitled "Constitutive Modelling for Anisotropic Hardening Behavior with Applications to Cohesionless Soils" and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

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A constitutive model based on rate-independent elastoplasticity concepts is developed to simulate the behavior of geologic materials under arbitrary three-dimensional stress paths, stress reversals and cyclic loading. The model accounts for the various factors such as friction, stress path, stress history, induced anisotropy and initial anisotropy that influence the behavior of geologic materials. A hierarchical approach is adapted whereby models of progressively increasing sophistication are developed from a basic isotropic-hardening associative model. The influence of the above factors is captured by modifying the basic model for anisotropic (kinematic) hardening and deviation from normality (nonassociateness).

Both anisotropic hardening and deviation from normality are incorporated by introducing into the formulation a second order tensor whose evolution is governed by the level of induced anisotropy in the material. In the stress-space this formulation may be interpreted as a translating potential surface \( Q \) that moves in a fixed field of isotropic yield surfaces. The location of the translating surface in the stress-space, at any stage of the deformation, is given by the 'induced anisotropy' tensor.

A measure to represent the level of induced anisotropy in the material is defined. The validity of this representation is investigated
based on a series of special stress path tests in the cubical triaxial device on samples of Leighton Buzzard sand.

The significant parameters of the model are defined and determined for three sands based on results of conventional laboratory test results. The model is verified with respect to laboratory multiaxial test data under various paths of loading, unloading, reloading and cyclic loading.
CHAPTER 1

INTRODUCTION

Background

Solution of complex soil-structure interaction problems such as dams, power plants and offshore structures subjected to earthquakes, explosive shock or wave action require the use of advanced computational methods. A critical aspect of such numerical methods is the appropriate simulation of the constitutive response of geologic materials. The highly complex nonlinear inelastic characteristics of geologic media are very difficult to simulate. This is especially so if the loading involves multidimensional stress paths and their reversal, cyclic loading and rotations of principal stresses. When modeling the behavior of geologic materials, it is also necessary to account for the influence of factors such as inherent and induced anisotropy, state of stress and strain, stress path, deformation history, volume changes under shear and fluid in the pores.

Development of constitutive relations capable of simulating these characteristics, within the framework of continuum mechanics, has been the focus of a number of researchers over the last several years. This has led to significant progress in the field of constitutive modeling and the parallel development of more sophisticated and powerful computational techniques.
This study employs the phenomenological approach to constitutive modeling by using rate-independent elastoplasticity concepts. The phenomenological approach is based on the experimental observations at the macroscopic level. An alternative approach, which considers the properties of the material at a microstructural level, can be quite complex. Although experimentally observed phenomena are caused by microstructural processes, a macroscopic description of material behavior is often adequate and convenient for engineering application. Concepts of elastoplasticity can provide a firm theoretical foundation for the development of the phenomenological models. Elastoplastic models have also been shown to be quite versatile in terms of simulating various facets of observed behavior peculiar to geologic materials.

The development of constitutive relations involves the following steps:

1. Mathematical formulation
2. Identification of significant parameters and their determination
3. Verification.

The mathematical formulation of the model should be based on principles of continuum mechanics. The model should have the capacity to simulate, qualitatively, observed trends in material behavior. It is also necessary to strike a balance between theoretical rigor and simplicity. In other words, in addition to having the predictive capability, the model should be easy to use and should lend itself to easy calibration from a minimum number of commonly available tests. Therefore, an optimum
number of significant parameters and the tests most suitable for their
determination have to be identified. Before employing the model with
any degree of confidence, verification is necessary. This takes the
form of: (a) backprediction of laboratory test data used for parameter
determination and other test data not used for determination of material
constants and (b) comparisons between observed data or closed-form solu-
tions for practical boundary value problems and predications obtained
from a numerical scheme employing the proposed model.

A single model to describe all aspects of material behavior has
not been developed yet. Even if one is, it would be too unwieldy in
terms of the number of parameters involved. The trend, with the con-
stitutive modeling group at The University of Arizona, has been towards
developing a hierarchy of constitutive models (Desai, Somasundaram and
Frantziskonis, 1985) capable of handling a wide range of materials under
a variety of loading conditions. At one end of the spectrum it is
necessary to consider isotropic materials subjected to monotonically in-
creasing loads and, at the other, anisotropic materials subjected to
loading conditions that include stress reversals, complex stress paths
and rotations of the principal directions of stress. It is then
possible to select from the hierarchy as sophisticated a model as the
particular application demands.

**Scope of the Work**

This work deals with the formulation, parameter determination
and verification of a rate-independent plasticity-based constitutive
model for geologic materials. The objectives of this study may be summarized as follows:

(i) To develop a generalized constitutive model capable of simulating behavior of geologic materials under monotonic and cyclic loading, arbitrary three-dimensional stress paths and rotations of principal stresses. The first stage involves development of a basic isotropic hardening model to describe behavior of an initially isotropic material that hardens isotropically during plastic deformation. The second stage involves the development of the basic model to an anisotropic (kinematic) hardening model that accounts for unload-reload effects, memory of maximum prestress and induced anisotropy.

(ii) To determine the material parameters associated with the model using laboratory test data for a number of different geologic materials and to monitor the evolution of induced anisotropy during plastic deformation in a special series of laboratory tests. The testing device used is a stress-controlled cubical triaxial device (Sture and Desai, 1979; Desai, Phan and Sture, 1981; Desai, Siriwardane and Janardhanam, 1982). The tests are performed on dense specimens of Leighton Buzzard sand.

(iii) To verify the proposed constitutive model with respect to laboratory test data on a number of different soils subjected to various stress paths from different levels of initial confining pressure.

Chapter 2 of this dissertation reviews some fundamentals of the theory of plasticity and some of the more recent experimental and
analytical works carried out in the context of elastoplastic modeling of soil behavior.

Chapter 3 is devoted to the formulation and development of the general constitutive model. A measure to simulate induced anisotropy is defined and used as the basis for describing the deviation of actual material behavior from that predicted by a basic isotropic model. Some of the assumptions made in the formulation are verified from observed experimental data.

Chapter 4 describes the special series of tests carried out on Leighton Buzzard sand for purposes of parameter determination and investigation of induced anisotropy. A description of the cubical device and sample preparation techniques is provided. Chapter 5 provides the procedure for the determination of material constants for a number of soils.

Chapter 6 presents the verification of the model with respect to laboratory test data on various sands of different densities subjected to a variety of stress paths. The model is implemented in an integration routine in order to do this. A summary of this work and the conclusions reached in this study are briefly presented in Chapter 7.
CHAPTER 2

REVIEW OF SOME RELEVANT LITERATURE

The first part of this chapter contains a review of some fundamentals of the elastoplasticity theory. Some of the basic theoretical requirements of a material model are also discussed in the context of the theory of plasticity. The second part of the chapter contains a review of some of the recent work carried out in the field of constitutive modeling for geological media. Particular emphasis is placed on modeling for complex stress paths, cyclic loading and induced anisotropy.

This is by no means a comprehensive review. For detailed treatment of the various aspects reviewed here, the reader is referred to Hill (1950), Mroz and Norris (1982), Desai and Siriwardane (1984), Zienkiewicz (1981), Gudehus (1984) and the many other authors cited herein.

Some Fundamentals of the Theory of Elastoplasticity

This discussion is restricted to small deformations only. Details of the derivation of the incremental stress-strain relations, based on the theory of rate-independent hardening elastoplasticity, may be found in Hill (1950) and Desai and Siriwardane (1984). Only a summary is presented here.
Under the limitation of small deformations, total incremental strain at any stage of the deformation process may be linearly separated into elastic and plastic components as

\[ \text{d}e_{ij} = \text{d}e^e_{ij} + \text{d}e^p_{ij} \quad (2.1) \]

where \( \text{d}e_{ij} \) is the incremental strain and superscripts \( e \) and \( p \) refer to the elastic and plastic components, respectively. The elastic strain increment may be related to the stress increment \( \text{d}\sigma_{ij} \) through the generalized Hooke's law as

\[ \text{d}\sigma_{ij} = C_{ijkl}^{e} \text{d}e_{kl}^e \quad (2.2) \]

where \( C_{ijkl}^{e} \) is the elastic constitutive tensor.

The incremental plastic strains are evaluated based on the yield function \( F \) and the plastic potential function \( Q \), using the following equations (Hill, 1950):

\[ \text{d}e^p_{ij} = \kappa \lambda \frac{\partial Q}{\partial \sigma_{ij}} \quad (2.3) \]

\[ \kappa = \begin{cases} 1 & \text{if } F = 0 \text{ and } \frac{\partial F}{\partial \sigma_{ij}} \text{d}\sigma_{ij} > 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.4) \]

\[ \text{d}F = 0 \text{ during plastic flow} \quad (2.5) \]
where

\[ F = F \{ \sigma_{ij}, (a_{ijk \ldots})_r, (\xi)_s \} \]  \hspace{1cm} (2.6)

is the yield function and

\[ Q = Q \{ \sigma_{ij}, (a_{ijk \ldots})_r, (\xi)_s \} \]  \hspace{1cm} (2.7) 

is the plastic potential function. \((a_{ijk \ldots})_r (r = 1, 2, 3 \ldots)\) are tensor valued internal variables such as total plastic strain, \(\varepsilon_{ij}^P\), preferred orientations and induced anisotropy. \((\xi)_s (s = 1, 2 \ldots)\) are scalar valued internal variables such as plastic work, plastic strain trajectory and temperature. \(\lambda (> 0)\) is a scalar of proportionality.

The yield surface \(F\) represents a surface in the stress space enclosing a purely elastic region. It is a function of the state of stress and a number of internal state variables. When the state of stress is within \(F\) behavior is elastic. When the stress point reaches the boundary of \(F\), i.e., if \(F = 0\) is satisfied, plastic behavior initiates. With the occurrence of plastic flow, \(F\) may expand, translate or undergo a combination of the above.

The plastic potential \(Q\) represents a function to which the plastic strain increment \(d\varepsilon_{ij}^P\) is orthogonal. \(Q\) may or may not be a closed surface in the stress space. \(Q\) is also a function of \(\sigma_{ij}\) and the state variables and may undergo changes during plastic flow.
Equation (2.3) is known as the plastic flow rule and defines the magnitude and direction of the incremental plastic strain, $\text{d}e^p_{ij}$. The direction of $\text{d}e^p_{ij}$ coincides with the normal to the plastic potential at the particular state of stress and is given by $\partial Q/\partial \sigma_{ij}$. The magnitude of $\text{d}e^p_{ij}$ is governed by the proportionality constant $\lambda$.

Equation (2.4) is the load-unload criterion. It specifies plastic loading to occur when the stress point lies on the boundary of $F$ ($F = 0$), and the stress increment is directed outwards of surface $F$ ($\partial F/\partial \sigma_{ij} \text{d}\sigma_{ij} > 0$). When these conditions are satisfied, $\kappa = 1$ and $\text{d}e^p_{ij}$ is given by Eq. (2.3). If these conditions are not met, $\kappa = 0$, no plastic strains occur and the behavior of the material is purely elastic.

Equation (2.5) is commonly referred to as the consistency condition. It is the mathematical statement of the fact that during plastic flow the state of stress $\sigma_{ij}$ always lies on the boundaries of surface $F$. Whenever plastic flow occurs, $F$ expands, translates and/or distorts in such a manner that $\sigma_{ij}$ remains on $F$. This phenomenon is referred to as hardening.

Equations (2.1) to (2.7) may be combined, as shown subsequently in Chapter 3 and Appendix A, to give the incremental elastoplastic stress-strain relation as

\[
\text{d}\sigma_{ij} = C_{ijkl}^e \text{d}e_{kl}
\]  

(2.8)
From the foregoing discussion, it is seen that the plastic behavior is governed by the following:

i) yield and potential functions

ii) flow rule

iii) hardening rule.

Each one of these aspects is considered in detail.

The Yield and Potential Functions

In the interests of simplicity, \( F \) and \( Q \) are assumed to be functions of \( \sigma_{ij} \), one second order tensor valued internal variable \( a_{ij} \) and one scalar valued internal variable \( \xi \). It is further assumed that \( a_{ij} \) and \( \xi \) are functions of \( \varepsilon_{ij}^p \). Therefore, Eqs. (2.6) and (2.7) may be written as

\[
F = F (\sigma_{ij}, a_{ij}, \xi) \quad (2.9)
\]

\[
Q = Q (\sigma_{ij}, a_{ij}, \xi) \quad (2.10)
\]

\( F \) and \( Q \) are to be used to describe the constitutive behavior of the material. A constitutive law relates physical phenomenon; namely, the response of a material to applied loads. As such, the constitutive law must obey certain principles or axioms that govern physical phenomena. These axioms are referred to as the axioms of continuum
mechanics and are described in the works of Truesdell (1955), Truesdell and Noll (1965), and Eringen (1962, 1967, 1975). Details of these axioms are beyond the scope of this dissertation. A simple interpretation of these axioms may be found in Desai and Siriwardane (1984).

Of particular importance is the axiom dealing with objectivity or coordinate invariance. The axiom of objectivity states that the stress-strain relations must be invariant with respect to transformations of the space-time reference frame. This condition is satisfied if the stress-strain relations are represented by tensor valued functions and if the stress and strain tensors are measures that are unaffected by rigid body rotations (objective measures). For small deformation problems, \( \sigma_{ij} \) and \( \varepsilon_{ij} \) satisfy the latter condition. For large deformation formulations, however, objective measures, such as the Jaumann rate for stress, have to be defined.

In the absence of preferred orientations, all scalar or tensor valued state functions such as \( F \) and \( Q \) must be isotropic functions of the state variables, \( \sigma_{ij}, a_{ij} \), and \( \xi \) (Eringen, 1971, 1975; Baker and Desai, 1984). Note that the isotropy of the functional dependence does not imply isotropy in the mechanical sense (Dafalias, 1984). This may be expressed mathematically as

\[
F = F (\sigma_{ij}, a_{ij}, \xi) =
F (\beta_{ik}, \beta_{jl}, \beta_{ik}, \beta_{jl}, a_{kl}, \xi)
\]

(2.11)
\[ Q = Q (\sigma_{ij}, a_{ij}, \xi) = \]

\[ Q (\beta_{ik} \beta_{j\ell} \sigma_{k\ell}, \beta_{ik} \beta_{j\ell} a_{k\ell}, \xi) \]

where \( \beta_{ij} \) represents the full group of orthogonal transformations. Note that \( \xi \), being a scalar, is unaffected by the transformations.

On the basis of representation theorems, any function satisfying equations of the form Eq. (2.11) or Eq. (2.12) may be expressed in terms of the direct and joint invariants of its arguments (Rivlin and Ericksen, 1955; Eringen, 1962; Spencer and Rivlin, 1959; Shrivastava, Mroz and Dubey, 1973; Baker and Desai, 1984) as

\[ F = F (J_i, I_i^a, K_j, \xi) \]  \hspace{1cm} (2.13)

\[ Q = Q (J_i, I_i^a, K_j, \xi) \]  \hspace{1cm} (2.14)

where \( J_i \) and \( I_i^a \) \((i = 1, 2, 3)\) are the direct invariants of \( \sigma_{ij} \) and \( a_{ij} \), respectively. \( K_j \) \((j = 1, 2, 3, 4)\) are the joint invariants of \( \sigma_{ij} \) and \( a_{ij} \).

The direct invariants of a second order tensor may be defined as (with respect to \( \sigma_{ij} \))
The joint invariants of $G_{ij}$ and $\alpha_{ij}$ are defined as

\[ J_1 = \sigma_{ii} \]
\[ J_2 = \frac{1}{2} \sigma_{ij} \sigma_{ij} \]
\[ J_3 = \frac{1}{3} \sigma_{ik} \sigma_{kj} \sigma_{ji} \]

The joint invariants of $\sigma_{ij}$ and $\alpha_{ij}$ are defined as

\[ K_1 = \sigma_{ij} \alpha_{ij} \]
\[ K_2 = \sigma_{ik} \sigma_{kj} \alpha_{ji} \]
\[ K_3 = \sigma_{ik} \alpha_{kj} \alpha_{ji} \]
\[ K_4 = \sigma_{ik} \sigma_{kj} \alpha_{ji} \alpha_{kl} \alpha_{il} \]

It can be shown that of the ten direct and joint invariants of $\sigma_{ij}$ and $\alpha_{ij}$, only nine are functionally independent (Shrivastava, Mroz and Dubey, 1973). Therefore, the fourth joint invariant $K_4$ may be omitted in the expression for $F$ and $Q$.

Flow Rule

As mentioned earlier, when the loading criterion (Eq. (2.4) is satisfied, the incremental plastic strains are given by the flow rule, Eq. (2.3). The magnitude of $\lambda$ in Eq. (2.3) is determined from the consistency condition, Eq. (2.5), which is expressed as
\[ dF = \frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial F}{\partial \varepsilon^p_{ij}} d\varepsilon^p_{ij} = 0 \] (2.17)

since it was assumed, for the sake of simplicity, that the internal variables \( a_{ij} \) and \( \xi \) were functions of the plastic strain.

From Eqs. (2.17) and (2.3), \( \lambda \) can be expressed as

\[ \lambda = \frac{\frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij}}{M^p} \] (2.18)

where

\[ M^p = -\frac{\partial F}{\partial \sigma_{ij}} \frac{\partial Q}{\partial \sigma_{ij}} \] (2.19)

Since \( \lambda > 0 \) and \( \frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} > 0 \) during plastic loading, Eq. (2.4), \( M^p > 0 \).

The flow rule may be written as

\[ d\varepsilon^p_{ij} = \frac{\frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij}}{M^p} = \frac{\partial Q}{\partial \sigma_{ij}} \] (2.20)

**Associative Flow.** If the plastic potential function \( Q \) is adapted as identical to \( F \), the plastic flow is said to be associative; when \( Q \neq F \), the flow rule is nonassociative. Associative plastic flow has been observed in the case of metals, undrained behavior of saturated soils and drained behavior of certain geologic materials.
From the viewpoint of computational ease and stability requirements, the associative flow rule possesses a number of attractive features. Using an associative flow rule eliminates the need for establishing a separate potential function $Q$. The resulting elastoplastic constitutive tensor $C^{e-p}_{ijk\ell}$, Eq. (2.8), is symmetric. As a result, the stiffness matrix is symmetric when the constitutive model is used in a numerical (finite element) scheme. A symmetric stiffness matrix requires less storage and less computational effort to solve than a nonsymmetric matrix. The associative flow rule also conforms to the stability postulates (discussed later) of Drucker (1956) and Illiushin (1961).

Nonassociative Flow. For many frictional materials, experimental evidence indicates that plastic strain increments are not normal to the yield surface $F$ (Lade and Musante, 1978; Lade and Duncan, 1973, 1976; Poorooshasb, Holubec and Sherbourne, 1965, 1967). For such cases, it is necessary to define a plastic potential function $Q \neq F$ and use a nonassociative flow rule.

In general, the nonassociative flow rule may violate both Drucker's postulate and Illiushin's postulate. However, it can be shown that nonassociative flow may be stable under certain circumstances. Hill (1958) and Mroz (1963) have discussed the conditions under which nonassociative models exhibit uniqueness and stability.

The use of the nonassociative concept also leads to nonsymmetric constitutive and stiffness matrices. Therefore, nonsymmetric equation
solvers or special algorithms employing only symmetric solvers (Desai and Hashmi, 1985) are necessary for the solution of finite element equations.

Hardening Rules

The word hardening is used to describe the process of strength gain in a material due to permanent (plastic) strains. In contrast, a perfectly plastic material approaches failure once plastic behavior sets in.

In the context of the theory of plasticity, hardening behavior is allowed for by the expansion, translation and/or distortion of the initial yield surface in the stress space in such a manner that the stress point always lies on the boundary of surface $F$ during plastic flow. The hardening process continues until the state of stress reaches the ultimate (failure) envelope, at which time excessive strains occur with no corresponding increase in the state of stress, resulting in failure. When the material is subjected to a stress increment that forces the state of stress inside the surface $F$, the behavior becomes purely elastic. Plastic behavior and hardening resume once the state of stress reaches the boundaries of $F$ once again.

It is possible to identify various mechanisms through which hardening can be allowed for. Such mechanisms are referred to as hardening rules and may be categorized as

1. Isotropic hardening
2. Kinematic hardening
3. Anisotropic hardening.
Isotropic Hardening. The concept of isotropic hardening dates back to the development of work-hardening and strain-hardening plasticity theories for metals (Hill, 1950). This rule assumes that hardening is allowed for by the uniform (isotropic) expansion of the initial yield surface in the stress space. There are no rigid rotations or translations or changes in the shape of the yield surface. The expansion of the yield surface may be made a function of the plastic work or plastic strain measures such as volumetric plastic strain or total plastic strain trajectory.

As plastic deformation continues, the zone of purely elastic behavior grows. If the loading is now reversed, the yield surface remains fixed and elastic behavior resumes until the boundary of the expanded yield surface is reached once again. Thus, a substantial level of stress reversal has to take place before the isotropic hardening model predicts reverse plastic behavior.

For real materials, however, it has been observed that during reverse loading, plastic yielding commences at a relatively early stage, long before the stress level reaches the bounds of the isotropically expanded yield surface. This phenomenon is known as the Bauschinger effect (Hill, 1950; Fung, 1965; Malvern, 1969) and is caused by the anisotropy induced in the material by the permanent deformations.

The inability of an isotropic hardening model to simulate unloading phenomena such as the Bauschinger effect is a major drawback. Therefore, the applicability of plasticity models utilizing the isotropic hardening rule is generally limited to monotonic loading processes.
Kinematic Hardening. This rule assumes that the initial yield surface does not change its size or shape but undergoes rigid body translation in the stress space during plastic flow. Prager (1955) postulated the translation of the yield surface in the stress space to be in the direction of the plastic strain increment. Shield and Ziegler (1958) stipulated that the movement of F occurs in the direction of the radius vector joining the center of the yield surface to the stress point.

The major shortcoming associated with the kinematic hardening rule is that it assumes a constant value of plastic modulus during loading along any fixed stress path. This is contrary to observed behavior which indicates that the plastic modulus progressively decreases as the ultimate or failure state is approached.

Anisotropic Hardening. This is a general term which encompasses various combinations of isotropic hardening, kinematic hardening, rotations of the yield surface and changes in the shape of the yield surface. Such combinations of the various hardening mechanisms are necessary to overcome the shortcomings of the isotropic and kinematic hardening rules and to simulate various other facets of observed behavior. Several forms of anisotropic hardening have been proposed to deal with material behavior under arbitrary stress paths, load reversals and cyclic loading, principal stress rotations, initial anisotropy and induced anisotropy.

Anisotropic hardening is allowed for by introducing scalar valued and tensor valued internal variables that serve as the arguments
of $F$, Eq. (2.6). The scalar valued variables provide for the expansion of the yield surface. Tensor variables provide for translation, rotation and distortion of the surfaces.


Theoretical Requirements on Material Models

As mentioned earlier, a constitutive model should meet several requirements based on physics and continuum mechanics. Objectivity and invariance requirements were considered in the section on yield and potential functions. Some other basic theoretical requirements are considered here.

**Continuity Requirement.** This is one of the most basic requirements of a constitutive model. It may be stated as: differentially small changes in a finite stress (strain) path should lead to differentially small changes in strain (stress). The incremental elastoplastic stress-strain relation, Eq. (2.8), satisfies the continuity requirement if both the elastic and plastic stress-strain relations satisfy it. The elastic behavior, being linear, satisfies continuity. The plastic behavior satisfies this requirement if the plastic constitutive tensor is
an explicit continuous function of stress and the internal state variables used to define F and Q.

**Energy Requirements During Closed Stress Cycles.** This is a thermodynamical requirement that the material not generate energy during closed stress cycles. This may be expressed mathematically as

\[ \phi \sigma_{ij} \, d\varepsilon_{ij} > 0 \]  

(2.21)

where the symbol \( \phi \) represents integration over a closed stress cycle.

The work done due to elastic action during a closed stress path being zero, Eq. (2.21) may be expressed as

\[ \phi \sigma_{ij} \, d\varepsilon_{ij}^P > 0 \]  

(2.22)

substituting for \( d\varepsilon_{ij}^P \) from Eq. (2.20) in Eq. (2.22)

\[ \phi \left( \sigma_{ij} \frac{\partial Q}{\partial \sigma_{ij}} \frac{\partial F}{\partial \sigma_{km}} \, d\sigma_{km} \right) / M^P > 0 \]  

(2.23)

During plastic behavior, \( \partial F/\partial \sigma_{ij} \, d\sigma_{ij} > 0 \) and \( M^P > 0 \), Eqs. (2.4) and (2.19). Therefore, a sufficient condition for the above equation to be satisfied is

\[ \sigma_{ij} \frac{\partial Q}{\partial \sigma_{ij}} > 0 \]  

(2.24)
Equation (2.24) requires that the plastic strain increment direction make an acute angle with the total stress vector in the stress space, Fig. 2.1. This condition is satisfied if surface $Q$ is convex and it contains the origin of the stress space.

**Incremental Uniqueness and Stability.** An incremental stress-strain law is said to be unique if a one-to-one correspondence between stress increment and strain increment exists. Uniqueness is ensured if the incremental stress-strain law, Eq. (2.8), can be inverted; i.e., if the elastoplastic constitutive matrix $[C]^{e-p}$ can be inverted.

The stability requirement may be expressed in terms of the work done by the incremental stresses on the incremental strains defined as

\[ d^2W = d\sigma_{ij} \, dc_{ij} = d\varepsilon_{ij} \, C^{e-p}_{ijkl} \, dc_{kl} \]  \hspace{1cm} (2.25)

The famous stability postulate of Drucker (1956) states that for a stable material

\[ d^2W > 0 \]  \hspace{1cm} (2.27)

This may be expressed as

\[ d\sigma_{ij} \, \frac{\partial Q}{\partial \sigma_{ij}} > 0 \]  \hspace{1cm} (2.28)

In the case of associative plastic flow, $Q \equiv F$ and Eq. (2.28) is always satisfied. If the flow is nonassociative, however, there is a region in
Figure 2.1 Schematic of Incremental Stress and Plastic Strain Vectors in Stress Space
the stress space for which \( d\sigma_{ij} \frac{\partial Q}{\partial \sigma_{ij}} < 0 \). In Fig. 2.1, if \( d\sigma_{ij} \) falls in the region AOB, \( d\sigma_{ij} \) will make an obtuse angle with \( \partial Q/\partial \sigma_{ij} \) and as a result, \( d\sigma_{ij} \frac{\partial Q}{\partial \sigma_{ij}} < 0 \).

It is, therefore, evident that although associative plasticity is always stable, nonassociative models may or may not be stable. The conditional stability and uniqueness of nonassociative plasticity is discussed in detail by Hill (1958), Mroz (1963), Bleich (1972) and Nelson, Wagner and Ito (1985).

**Plasticity Models for Geologic Materials**

**Introduction**

The concept of plasticity has provided the theoretical basis for a number of constitutive models for geologic materials developed in recent years. Since the theory of plasticity was originally developed for metals, several modifications and extensions have to be incorporated in the classical theory before it can be used for geologic materials.

Geologic materials (soils and rock) have certain features that require special attention in constitutive modeling:

a. Soils and rock are two-phase materials. Applied total stresses are carried by both the solid phase (skeleton) (effective stress) and the pore fluid (pore pressure). Thus, when modeling the stress-strain behavior of the solid phase, formulation has to be in terms of effective stress.
b. Geological materials are frictional; their behavior depends on the mean normal effective stress.

c. Behavior is stress path dependent; for instance, material response in compression and extension are different.

d. Pure shear causes plastic volume changes. Loose sands and normally consolidated clays contract during shear while dense sands and overconsolidated clays dilate.

e. In their naturally occurring state, geologic materials are usually inherently anisotropic due to structure and depositional environment.

f. Considerable anisotropy is induced in the material during the deformation process. As a result, stress history has a profound influence on subsequent behavior.

A Review of Some Common Models for Geologic Materials

The literature abounds with different plasticity based constitutive models for soils, rock and concrete. Reviews of the various models and their relative merits may be found in Desai and Siriwardane (1984) and in the proceedings of several conferences and workshops dealing with the subject. Among them are The Workshop on Limit Equilibrium Plasticity and Generalized Stress-Strain in Geotechnical Engineering, McGill, 1980, The International Workshop on Constitutive Relations for Soils, Grenoble, 1982, the International Symposium on Soils Under Cyclic and Transient Loading, Swansea, 1980, the International Conference on

An attempt is made here to review in a very concise manner some of the common rate independent plasticity based approaches that have been used to model the behavior of soils and rock. The emphasis here is on the general concepts adapted rather than the details of the individual models.

Isotropic Hardening Models. Pioneering work on strain hardening theories of plasticity done by Drucker, Gibson and Henkel (1957) led to the development of critical state models by Roscoe, Schofield and Wroth (1958), Roscoe and Poorooshahb (1963), Roscoe, Schofield and Thurairajah (1963), Roscoe and Burland (1968), Schofield and Wroth (1968). The critical state theories were based on the observations that when soil samples were sheared, they passed through progressive states of yielding before approaching "failure" at a critical void ratio. Once the critical void ratio is reached, no further changes in void ratio occurred due to shear. Roscoe and his co-workers proposed an associative flow rule and an isotropically expanding yield surface that had as its hardening parameter the void ratio or volumetric plastic strains. The yield surfaces are expressed as

\[ F (\sigma_{ij}, \kappa) = 0 \]  

(2.29)

\[ \kappa = \varepsilon_p^V \]  

(2.30)
where $\kappa$ is the hardening parameter and $\varepsilon^p_v$ is the volumetric plastic strain. Equation (2.29) represents a family of yield surfaces. The locus of points on $F$ with $\frac{\partial F}{\partial \varepsilon^p_v} = 0$ $(d\varepsilon^p_v = 0)$ forms the critical state surface in the stress space.

DiMaggio and Sandler (1971) proposed the Cap model which is conceptually very similar to the critical state model. Here too the hardening parameter was assumed to be the volumetric plastic strain. Subsequent modifications on the Cap model may be found in Baron, Nelson and Sandler (1973), Sandler, DiMaggio and Baladi (1976) and Sandler and Baron (1979).

The disadvantage of these density hardening models (critical state and Cap) lies in their inability to simulate properly dilatancy in dense sands and the pore pressure variation in undrained tests on over-consolidated soils.

These shortcomings led to the development of combined deviatoric and density hardening models. Wilde (1977), Nova and Wood (1979), Prevost and Hoeg (1975) and Mroz and Norris (1982) proposed combined hardening parameters, $\kappa$, of the form

$$\kappa = \beta_1 \int d\varepsilon^p_{1j} d\varepsilon^p_{1j} + \beta_2 \int d\varepsilon^p_v$$

(2.31)

where $d\varepsilon^p_{1j}$ is the incremental deviatoric plastic strain and $\beta_1$ and $\beta_2$ are constants. With $\beta_1 = 0$, these models reduce to the critical state
type of models. Lade (1975) and Lade and Duncan (1975) used the plastic work $W^p$ instead of $\kappa$, Eq. (2.31), as the hardening parameter. Plastic work is defined as

$$W^p = \int \sigma_{ij} \, d\epsilon_{ij}^p \quad (2.32)$$

Early plasticity models were formulated in terms of two stress invariants ($J_1, J_2$). Such models were incapable of simulating the highly stress path dependent nature of soils. For instance, a two invariant model does not account for observed differences in behavior under compression, extension and simple shear. To accommodate these differences, the shape of the yield surface in the octahedral plane had to be modified from circular to more appropriate nonsymmetrical shapes. This was achieved by including the third invariant of stress ($J_3, J_{3D}$ or Lode angle $\theta$) in the formulation. Several researchers have proposed and used yield surfaces formulated in terms of all three stress invariants. They include Matsuoka and Nakai (1974), Lade (1975), Lade and Duncan (1975), Davis and Mullenger (1979), Wimam and Warnke (1974), Ghaboussi and Momen (1979, 1982), Van Eekelen (1980), Desai (1980), Desai and Faruque (1983), Faruque (1983), Desai, Frantziskonis and Somasundaram (1985).

From a series of experiments, Lade and Duncan (1973, 1976) and Poorooshashb et al. (1976) have shown that geologic media, in particular cohesionless soils, do not exhibit associative plastic flow. This led to the development of many nonassociative plasticity models. A concept
in which the yield function is corrected, constrained or controlled to obtain the plastic potential function was proposed by Desai and Siriwardane (1980) and Desai and Faruque (1983). A simple procedure by which the growth function of the yield surface $F$ is corrected to give the potential function $Q$ is presented in Frantziskonis, Desai and Somasundaram (1985).

Desai (1980) and Desai and Faruque (1983, 1984) have shown that by expressing the yield function $F$ as a complete polynomial of the direct invariants and choosing appropriate truncated forms of the polynomial, it is possible to derive a number of isotropic hardening yield functions used by various previous researchers. This procedure enables one to choose appropriate functions that are continuous, smooth and free of singularities caused by the intersection of yield and failure surfaces. This concept with continuing modifications is developed and used in this dissertation.

**Nested Surfaces Models.** Mroz (1967) and Iwan (1967) introduced the concept of nested surfaces by generalizing the kinematic hardening rules proposed by Ishilinsky (1954) and Prager (1956) for nonlinear strain hardening. This concept had considerable influence on subsequent developments in plasticity.

Prager's hardening rule calls for a yield surface of fixed size that translates in the stress space with the stress point. Since the size of the surface remains invariant, the model predicts a constant value of plastic modulus and, hence, a linear stress-strain relationship. The nested surfaces model consists of a family of such surfaces in the
stress space. The surfaces are of increasing size, each one enclosing all the yield surfaces smaller than itself. As the size of the surfaces increases, the plastic modulus associated with them decreases. Thus, the family of nested surfaces defines a field of strain hardening moduli. Since material behavior exhibits continuous hardening, theoretically, an infinite number of yield surfaces are required. For practical purposes, a finite number of surfaces are used. This gives a piecewise linear response which can be approximated to the observed nonlinear response.

Figure 2.2 gives a schematic of the nested surfaces in the stress space. The mechanics of the model are governed by a postulated set of rules. The surfaces \( f_0, f_1, \ldots, f_p \) may not intersect but may touch each other. The largest yield surface through the current stress point is referred to as the active yield surface, \( f_c \), and it defines the plastic modulus until a larger yield surface, \( f_{c+1} \), is encountered. All the yield surfaces that are in contact with each other at the stress point translate together during plastic flow. The translation of the nested surfaces is governed by a postulated translation rule.

Mroz (1967) originally developed the model for metals. Since then, many researchers have adopted this concept and developed modified versions of this model for metals and geologic materials. Prevost (1977) used the nested surfaces concept to develop a model for clay under undrained conditions. Subsequently, Prevost (1978, 1979) extended the model to account for the drained behavior of soils as well. The yield function is represented by
Figure 2.2 Schematic of Nested Surfaces Model
\[ f_m (\sigma_{ij} - a^{m}_{ij}) - k_m = 0 \]  

where \( m = 1, 2, \ldots, p \), \( p \) being the total number of yield surfaces, \( a_{ij} \) is the center of the yield surface and \( k \) its size. The initial positions and sizes of the yield surfaces reflect the past stress-strain history of the material. A plastic modulus, \( H_m \), is associated with each of the surfaces. The yield surfaces are allowed to change in size as well as translate; therefore, both \( k_m \) and \( H_m \) vary during the process of deformation.

As mentioned earlier, uniqueness requires that the surfaces do not overlap each other. Therefore, contact between adjacent surfaces can occur only at points with the same direction of outward normal; i.e., at conjugate points. This requirement governs the translation law for the surfaces.

The nested surfaces models fall within the general framework of classical plasticity. They can account for the anisotropic elasto-plastic stress path dependent stress-strain behavior of soils. They account for unload-reload effects and can simulate soil behavior under monotonic and cyclic loading conditions. The major disadvantage of this class of models is that the number of parameters is large (size and location of each one of the yield surfaces) and this requires substantial memory and 'bookkeeping.'

**Bounding Surface Models.** Dafalias and Popov (1975, 1976, 1977) and Krieg (1975) introduced the concept of bounding surface models.
Mroz, Norris and Zienkiewicz (1978, 1979, 1981) adopted the same approach and developed a model for soil mechanics problems. The bounding surface approach may be considered a modified and simplified adaptation of Mroz's multisurface concept. The bounding surface formulation involves only two surfaces; the yield and bounding surfaces. The field of hardening moduli and the numerous surfaces defining it are now replaced by an interpolation rule for the plastic moduli.

The formulation of this class of models involves establishing a bounding surface and a yield surface within the bounding surface. Both surfaces may be functions of plastic internal variables that reflect the stress history of the material. A mapping rule is established for associating any stress point, \( \sigma_c \), within the bounding surface with an 'image' point on the surface. An interpolation rule is postulated relating the plastic modulus for the current stress point to the plastic modulus of the image point. The translation of the yield surface is once again governed by the requirement that the two surfaces should not intersect. Figure 2.3 schematically illustrates the bounding surface concept.

Many variants of the bounding surface model have been proposed by researchers. They vary in terms of the form of the bounding and yield surfaces and the complexity of the mapping and interpolation rules. All of them allow plastic deformation to continue within the bounding surface and in some, no purely elastic domain exists. Some typical bounding surface models are described in the following. All the models considered here use the critical state ellipse as the bounding surface and assume associative plasticity.
Figure 2.3 Schematic of Bounding Surface Model
Mroz, Norris and Zienkiewicz (1978, 1979) proposed a two-surface anisotropic hardening model for soils. The yield surface, which has the same shape and orientation as the bounding surface, encloses a purely elastic region. The image point, \( \tilde{P} \), corresponding to any stress point, \( P \), within the bounding surface is taken as the conjugate point (defined earlier). The interpolation rule is expressed in terms of the distance, \( \delta \), in the stress space between the current state of stress and the image point, Fig. 2.3, as

\[
H = \tilde{H} + (\hat{H} - \tilde{H}) \left( \frac{\delta}{\delta_0} \right)^m
\]

(2.34)

where \( H \) is the plastic modulus at point \( P \), \( \tilde{H} \) is the plastic modulus at \( \tilde{P} \), \( \hat{H} \) is a large positive value, \( \delta_0 \) is the diameter of the bounding surface ellipse, Fig. 2.3, and \( m \) is a material constant. The incremental translation of the center of the yield surface is postulated to be in the direction of \( \tilde{P}P \), Fig. 2.3.

Dafalias and Popov (1977) and Dafalias and Herrmann (1980) proposed a further simplification of the bounding surface concept by eliminating the bounded yield surface in the formulation. The purely elastic region does not exist. The direction of plastic strains at any point \( P \) within the bounding surface is given by the normal to the bounding surface, \( \tilde{n} \), at the image point \( \tilde{P} \), Fig. 2.4. The plastic modulus, \( H \), at \( P \) is given by an interpolation rule such as the one used by Dafalias and Herrmann (1980):
Figure 2.4 Schematic of Bonding Surface Model with Vanishing Glandric Region
where \( H \) is now a function of the first stress invariant \( J_1 \) and the other quantities are as defined in Eq. (2.34). The image point, \( \tilde{P} \), is obtained by a simple 'radial' extension of line \( OP \), Fig. 2.4. Although this model is capable of simulating plastic behavior during unloading and reloading, it is incapable of accounting for induced anisotropy.

Mroz et al. (1980) introduced a modified version of the bounding surface concept as an infinite surface model. This concept was subsequently used for sands by Mroz and Pietruszczak (1983) and for \( K_0 \) consolidated clays by Pietruszczak and Mroz (1983). This concept allows for fading memory behavior in which cycles of bigger amplitude erase the memory of previous events while those of a smaller amplitude 'remember' previous larger amplitudes, Zienkiewicz (1981). This model involves the creation of a yield surface, known as a homologous surface, at the stress reversal point each time the load is reversed. These homologous surfaces expand with the stress point. Larger homologous surfaces 'erase' smaller ones. This model is fairly versatile but some loading histories may require a large number of the homologous reversal surfaces to be remembered.

Modeling for Induced Anisotropy

Motivation for modeling induced anisotropy is provided from experimental observations on both sands and clays reported by a number of researchers. Stipho (1978) reported that the stress-strain behavior of
clay samples that were anisotropically consolidated and subsequently sheared to failure under undrained conditions was significantly affected by the degree of anisotropy induced by consolidation. On the other hand, the slope of the critical state line at failure was relatively insensitive to the degree of anisotropy. Lewin (1978) found that if anisotropically consolidated samples were deviatorically unloaded and then subjected to hydrostatic loading, the incremental plastic strain vector deviated from the hydrostatic axis in the direction of negative deviatoric stress. He further observed that this deviation gradually disappeared as the hydrostatic loading continued. In the case of sands, a large degree of stress-induced stiffness anisotropy has been reported by Arthur, Chua and Dunstan (1977), Arthur, Bekenstein, Germain and Ladd (1980), Arthur, Chua, Dunstan and Rodriguez (1980), Mould (1982) and Symes, Hight and Gens (1982). The induced anisotropy in sands became apparent, especially when the principal stresses were rotated with respect to the material axes. Induced anisotropy was found to have a large influence on the strain required to achieve a given stress ratio and the secant modulus on reloading after a principal stress rotation. However, it was found to have negligible influence on the angle of shearing resistance.

Models for induced anisotropy have been proposed by a number of authors. Prevost's (1977, 1978) model, that was described in a previous section, accounts for initial anisotropy and, in a limited sense, induced anisotropy. Here, anisotropy is represented by the initial and subsequent positions of the nested surfaces in the stress space. This
model has the weakness of predicting non-fading anisotropy under continuous isotropic loading, contrary to observations made by Yamada and Ishihara (1982) and Lewin (1978). Pietruszczak and Mroz (1979) provide for the development and demise of anisotropy by deviatoric translation or rotation of the bounding surface. However, no definite procedure is given for the determination of anisotropic parameters. Kavvadas (1983) proposed a model, based on critical state concepts, that accounts for induced anisotropy by allowing continuous rotations of the yield surface. Anandarajah, Dafalias and Herrmann (1984) have proposed a bounding surface model capable of accounting for initial and evolving anisotropy by allowing the bounding surface to expand and rotate around the origin in the stress space. Dafalias (1981) has proposed a varying nonassociative flow rule to account for initial and induced anisotropy in cohesive soils during compressive loading. This is achieved by forcing the strain rate tensor to deviate from the normal to the yield surface during the course of the plastic deformation. The deviation from normality is assumed to increase with the deviatoric plastic strain and decrease with the volumetric plastic strain.

Desai, Somasundaram and Faruque (1984) and Desai, Frantziskonis and Somasundaram (1985) have proposed a general procedure for developing models that can simulate induced anisotropy. Desai, Somasundaram and Frantziskonis (1985) have developed a hierarchical approach whereby a basic isotropic hardening model may be modified to give models of progressively higher grades that account for initial and induced anisotropy as well as nonassociativeness and softening.
CHAPTER 3

DEVELOPMENT OF PROPOSED MODEL

Theoretical Considerations

In plasticity, the macroscopic description of the state of the material is achieved through the yield function, $F$, and plastic potential function, $Q$. In general, $F$ and $Q$ are functions of the state of effective stress and a number of other scalar valued or tensor valued internal variables. Scalar valued internal variables may include temperature, quantities such as plastic work and plastic strain trajectory to account for the deformation history of the material. Tensor valued internal variables may include plastic strains, preferred orientations, damage (microcracking) parameters and induced anisotropy parameters.

Considering a material with no preferred orientations or microcracking, under isothermal conditions $F$ and $Q$ may be expressed as

$$F = F (\sigma_{ij}, a_{ij}, \xi) \quad (3.1a)$$

$$Q = Q (\sigma_{ij}, a_{ij}, \xi) \quad (3.1b)$$

where $\sigma_{ij}$ is the effective stress tensor, $\xi$ is a scalar measure of the plastic strains and $a_{ij}$ is a continuously evolving tensor valued state variable that is dependent on the deformation history of the material. On the basis of representation theorems and invariance requirements
(Rivlin and Erickson, 1955; Truesdell, 1955; Truesdell and Noll, 1965; Eringen, 1962, 1967; Baker and Desai, 1984; Desai et al., 1985a, b) all scalar valued state functions such as $F$ and $Q$ must be isotropic functions of their arguments. Thus, $F$ and $Q$ are, in general, functions of the direct and joint or mixed invariants of $\sigma_{ij}$ and $a_{ij}$.

\[ F = F (J_r, I_s, K_t, \xi) \]  
\[ Q = Q (J_r, I_s, K_t, \xi) \]  

where $r, s, t = 1, 2, 3$, $J_r$ are the direct invariants of $\sigma_{ij}$, $I_s$ are the direct invariants of $a_{ij}$ and $K_t$ are the joint invariants of $\sigma_{ij}$ and $a_{ij}$.

If all the internal state variables had been scalars, i.e., if $a_{ij}$ was not a state variable, then $Q$ and $F$ would have been functions only of the direct invariants of stress and the scalar variables. Such a plasticity model would always predict strain increments that are coaxial with the total stress. Models that have a scalar measure of total plastic strains or volumetric plastic strains as their only internal variable fall into this category and are referred to as isotropic hardening models. In the stress space, such surfaces continuously expand isotropically with no change in shape, location or orientation. As a result, their application is limited to initially isotropic materials subject to monotonically increasing loading. During unloading and reloading, within the bounds of the isotropic yield surface corresponding to the maximum prestress level, the model does not predict any plastic strains.
In comparison, the presence of joint invariants in the expressions for F and Q allows for noncoaxiality between the strain rate tensor and the stress tensor (Shrivastava et al., 1973; Baker and Desai, 1984; Desai et al., 1985). The use of joint invariants results in anisotropically hardening plasticity models that are capable of simulating unloading-reloading phenomena and the anisotropic response of materials. In the stress space, this can be interpreted as translation, rotation or distortion along with isotropic expansion of the yield and potential surfaces.

It has been shown (Desai, 1980; Desai and Faruque, 1983, 1984; Desai, Somasundaram et al., 1985) that by expressing F and Q as complete polynomials of the direct and joint invariants and choosing appropriate truncated forms of the polynomials, one could derive a number of isotropic and anisotropic hardening functions used by various researchers: DiMaggio and Sandler, 1971; Lade, 1977; Matsuoka and Nakai, 1974; Mroz et al., 1978; Prevost, 1977, 1978; Dafalias and Popov, 1977; Ghaboussi and Momen, 1979.

The Hierarchical Approach to Modelling

The purpose of this study is to develop a general anisotropically hardening/kinematic model with nonassociative plasticity that incorporates initial and induced anisotropy. Such a model would be valid for a variety of geologic materials subjected to a variety of loading conditions including load reversals, cyclic loading and rotations of principal directions of stresses. This general model may then
be reduced to specialized models, each one of which would have a limited range of applications (Desai, Somasundaram and Frantziskonis, 1985). The simplest model in the hierarchy is the isotropic hardening model with associative plasticity, which assumes the material to be initially isotropic and to remain isotropic during deformation.

The first stage in the development of the general anisotropic representation is to define a basic isotropic hardening associative model that can be considered characteristic of the basic or fundamental state of the material. Actual observed behavior usually differs greatly from that implied by the basic model. This may be due to frictional characteristics of the material, inherent or induced anisotropy. Such departures from the fundamental isotropic characteristics may be handled by correcting or building up on the basic form through the concept of deviation from normality.

Associative plasticity implies normality of the strain rate tensor to the yield surface. However, observed behavioral trends which vary from that predicted by associative plasticity may be simulated by deviating from normality or, in other words, by employing nonassociative plasticity. One may then take the viewpoint that material peculiarities such as friction and anisotropy cause the deviation from normality and determine the extent of such deviation. For simple loading conditions, the deviation from normality, whether caused by frictional characteristics or anisotropy or both, may be taken care of by a material constant that corrects the plastic potential. For more complex loading situations, however, deviation from normality has to be related to some
quantity that can grow and evolve during the course of plastic deformation. An appropriate strain measure related to induced anisotropy is defined to serve this purpose. The correlations factor between this measure of induced anisotropy and deviation from normality is treated as a material property that may depend on the frictional nature of the material.

In addition to deviation from normality, kinematic capability is also introduced to the basic model. This permits plastic behavior during nonvirgin (unloading and reloading) loading and allows for the continuous evolution of induced anisotropy regardless of whether the material is under virgin conditions or not.

Both deviation from normality and kinematic capability are introduced by defining a modified form of $F$, which is adopted as the plastic potential surface $Q$. Surface $Q$ translates in the stress space. During nonvirgin loading, $Q$ acts as the loading surface as well.

The first part of this chapter deals with the definition of the basic model and the modification to allow for nonassociative behavior during simple monotonically increasing loading. The latter part of the chapter deals with the development of the general kinematic, anisotropic hardening nonassociative model.

The Basic Isotropic Hardening Model

A particular form of the yield function, Eq. (3.2a), containing as variables the direct invariants of stress and $\xi$, may be written as

(Desai, Frantziskonis and Somasundaram, 1985; Frantziskonis, Desai and Somasundaram, 1985)
\[ F = J_{2D} - \left( - \frac{\alpha}{(\alpha')^{n-2}} J_1^n + \gamma J_1^2 \right) (1 - \beta S_r)^m = 0 \] (3.3)

or

\[ F = J_{2D} - F_b F_s = 0 \] (3.4a)

where \( J_{2D} = \frac{1}{2} S_{ij} S_{ij} \) is the second invariant of the deviatoric stress tensor \( S_{ij} = \sigma_{ij} - \frac{1}{3} \delta_{ij} \). \( S_r \) is a stress ratio such as the Lode angle \( \theta \) and \( J_{3D}^{1/3}/J_{2D}^{1/2} \) where \( J_{3D} = \frac{1}{3} S_{ik} S_{kj} S_{ji} \) is the third invariant of the deviatoric stress tensor. \( \alpha, n, \gamma, \beta \) and \( m \) are response functionals. \( \alpha' \) has units of stress and is chosen equal to unity. \( \alpha' \) is introduced for the purpose of making \( \alpha \) dimensionless as are the parameters \( n, \gamma, \beta \) and \( m \). Since the magnitude of \( \alpha' \) is chosen to be unity, it is omitted from now onwards in the expression for \( F \). In general, all of the response functionals \( \alpha, \beta, \gamma, n \) and \( m \) may be functions of the history of deformation and other internal variables.

Equation (3.3) is more conveniently expressed as Eq. (3.4a) in terms of a basic \( (F_b) \) and a shape function \( (F_s) \) defined as

\[ F_b = (- \alpha J_1^n + \gamma J_1^2) \] (3.4b)

\[ F_s = (1 - \beta S_r)^m \] (3.4c)
The basic function can represent a number of different shapes of the yield surface in the $J_1 - \sqrt{J_{2D}}$ space. Examples include open ended yield surfaces (with $\gamma = 0$) as in models proposed by Drucker and Prager (1952) and Ghaboussi (1979), elliptic shapes (with $n = 1$) as in models proposed by Mroz et al. (1978) and Prevost (1978) and more complex shapes as those shown in this study. A yield function of the form $F = J_{2D} - F_b = 0$ has circular cross section on planes normal to the space diagonal (octahedral planes). The shape functions $F_s$ allow modification of the circular shape in the octahedral plane to shapes more appropriate for a given material. Thus, Eqs. (3.3) and (3.4) contain, as special cases, many of the yield functions proposed previously such as von Mises, Drucker-Prager, critical state, cap, Mroz et al. (1978), Wilde (1977), Ghaboussi and Momen (1979), Van Eekelen (1980), Lade (1977), Vermeer (1978) and Matsuoka and Nakai (1974).

In this study, a more specific form of Eq. (3.3) is adopted as the basic isotropic hardening yield function and may be written as

$$F = J_{2D} - (-\alpha J_1^n + \gamma J_1^2) (1 - \beta S_r)^{-1/2} \quad (3.5a)$$

with

$$S_r = J_{3D}/J_{2D}^{3/2} \quad (3.5b)$$

$m$ is chosen to be $-1/2$ based on the observed shape of yield surfaces for a number of geological materials. In the interest of simplicity,
only $\alpha$ is chosen to be dependent on the deformation history, whereas $n$, $\gamma$ and $\beta$ are treated as material constants. $\gamma$ and $\beta$ are determined from the ultimate envelope for the material. Ultimate envelope is defined as the locus, in stress space, of points corresponding to the asymptotic stress to the stress-strain curves for different tests. The classical states such as failure and critical fall below or coincide with the ultimate state (Desai and Faruque, 1984). Parameter $n$ is determined on the basis of the zero volume change state. Comprehensive treatment of parameter determination is presented in Chapter 5. $\alpha$ is adopted as the growth or hardening function and is, in general, dependent on quantities such as plastic work or plastic strain trajectory which define the deformation history of the material. Here $\alpha$ is assumed to be a function of the plastic strain trajectory $\xi$ which is defined as

$$\xi = \int (d\varepsilon_i^P \cdot d\varepsilon_j^P)^{1/2}$$

where $d\varepsilon_i^P$ is the incremental plastic strain tensor.

Typical plots of $F = 0$ (Eq. 3.5) corresponding to various values of the growth function and plotted in various stress spaces for a sand (described later) are shown in Fig. 3.1 (Frantziskonis, Desai and Somasundaram, 1985).

For materials that possess cohesive strength, such as clayey soils, rock and concrete, the yield function needs slight modification. This is achieved by shifting the surface $F$ along the negative $J_1$ axis
Figure 3.1 Plots of Yield Surfaces for Munich Sand in Various Stress Spaces
by an amount $J^T_1 = a P_a$ as shown schematically in Fig. 3.2. 'a' is a dimensionless parameter that depends on the cohesive strength of the material and $P_a$ is the atmospheric pressure. Applications of this model for rock and concrete may be found in Salami (1985).

Growth Function

The growth function may be expressed generally in terms of the plastic strain trajectories as

$$\alpha = \alpha (\xi, \xi_v, \xi_D, r_v, r_D)$$

(3.7)

where $\xi_v$ and $\xi_D$ are the volumetric and deviatoric components of $\xi$, respectively, $r_v = \xi_v/\xi$ and $r_D = \xi_D/\xi$. A simple form of Eq. (3.7) is given as

$$\alpha = \frac{a_1}{\eta_1 \xi}$$

(3.8)

where $a_1$ and $\eta_1$ are material constants for the hardening behavior. This form of hardening function has proven to be quite effective for the shear behavior of soils (Desai, Frantziskonis and Somasundaram, 1985; Desai, Somasundaram and Frantziskonis, 1985). However, in cases where influence of hydrostatic and proportional loadings predominates, it may be necessary to opt for improved forms for Eq. (3.7) (Desai and Faruque, 1984; Faruque and Desai, 1985; Hashmi, under preparation).
Figure 3.2 Plots of $F = 0$ for a Cohesive Material in $J_1 - J_{2D}$ Space
The yield function proposed above (Eq. 3.5) possesses a number of meaningful properties. The function describes a single closed continuous surface in the stress space that expands with increasing plastic strain trajectory. The shape of the surface may change from material to material but remains the same during deformation. Distortion of the shape during deformation may be achieved by varying \( n, \gamma \) or \( \beta \).

For \( J_{2D} = 0 \), the two roots of \( F \), defining the points at which \( F \) intersects the \( J_1 \) axis is given by \( J_1 = 0 \) and \( J_1 = \left( \frac{\gamma}{\alpha} \right)^{1/n-2} \).

As the material approaches the ultimate state the strains become large and \( \xi \to \infty \), causing \( \alpha \) to approach zero, as seen from Eq. (3.8). Thus, at the ultimate, the second intercept \( J_1 = \left( \frac{\gamma}{\alpha} \right)^{1/n-2} + \infty \). In other words, at the ultimate conditions, the yield surface degenerates into an open ended ultimate straight line envelope given by (Figs. 3.1a, 3.1b)

\[
J_{2D} - \gamma F_s J_1^2 = 0 \tag{3.9}
\]

Thus, a single function describes both the hardening and the ultimate responses. Hence, the need for multiple surfaces, singularity due to intersection of yield and ultimate (or failure) surfaces and the resulting computational difficulties are avoided.

The derivative of \( F \) with respect to \( \sqrt{J_{2D}} \) vanishes for \( \sqrt{J_{2D}} = 0 \) and \( J_1 \neq 0 \), giving orthogonal intersection of \( F \) with the \( J_1 \)-axis for \( J_1 \neq 0 \). This ensures purely volumetric plastic strains for an isotropic
material that is being loaded isotropically. The point of zero volume change or point of dilatancy on the yield surface is obtained by setting \( \partial F / \partial J_1 = 0 \), which gives

\[
J_1 = (2\gamma / n\alpha)^{1/n-2}
\]

Substitution of this value of \( J_1 \) in the expression for \( F \) yields

\[
J^{2D} - \gamma F_s \left( \frac{2}{n} + 1 \right) J_1^2 = 0
\]

Equation (3.10) represents the surface in stress space that demarcates the transition from compaction to dilation. A comparison of Eqs. (3.9) and (3.10) reveals the effect parameter \( n \) has on the relationship between ultimate state and zero volume change state.

It is, therefore, evident that Eq. (3.5) describes, in a simple and compact manner, a flexible yield surface. Also, the material constants associated with the yield surface have physical meanings. \( \gamma \) defines the ultimate envelope and \( \beta \) governs the stress-path dependent nature of the material; for instance, it defines the ratio of strengths in compression, extension and simple shear. \( n \) determines the dilation level, and \( \alpha \) controls the hardening behavior.

**Modification of Basic Model for Nonassociative Behavior**

As mentioned earlier in this chapter, the condition \( F = Q \) is not applicable for many geologic materials. This is especially true in the case of granular materials, as shown by experimental evidence (Frydman et al., 1973; Poorooshash et al., 1965, 1967). The intergranular friction and the continuously evolving state of isotropy (anisotropy) of the
material are among the factors that contribute to the deviation from normality to $F$. A concept in which the yield function $F$ is corrected, constrained or controlled to give the plastic potential $Q$ was proposed by Desai and Siriwardane (1980) and Desai and Faruque (1983). Accordingly, $Q$ can be expressed as

$$Q = F + h (J_i, \xi)$$  \hspace{1cm} (3.11)

where $h$ is a correction function of the stress invariants $J_i$ ($i = 1, 2, 3$) and $\xi$.

The yield function in Eq. (3.5) is modified to give $Q$ by correcting the growth function $\alpha$. $Q$ may then be written as

$$Q = J_{2d} - (\alpha Q J_1^n + \gamma J_1^2) F_s$$  \hspace{1cm} (3.12)

Here the growth function $\alpha$ in Eq. (3.5) is replaced by $\alpha_Q$. A simple expression for $\alpha_Q$ is given by

$$\alpha_Q = \alpha + \kappa (\alpha_I - \alpha) (1 - r_v)$$  \hspace{1cm} (3.13)

where $\kappa$ is a material constant and $\alpha_I$ is the value of $\alpha$ at the initiation of nonassociativeness. If there is initial hydrostatic compression on an isotropic material, initiation of nonassociativeness may be assumed to occur after hydrostatic compression. If not, $\alpha_I$ corresponds to the value of $\alpha$ at the initial yield surface demarcating the limit of the elastic zone.
It is seen that nonassociativeness is introduced through only one parameter $\kappa$. During hydrostatic loading, $r_v = 1$ and Eq. (3.13) gives $\alpha_Q = \alpha$. Therefore, nonassociativeness does not occur during isotropic compression of an initially isotropic material. With $\kappa = 0$, $\alpha_Q = \alpha$ and the nonassociative model specializes to the associative one.

The successful application of this nonassociative model to simulate behavior of dense sands is described in Frantziskonis, Desai and Somasundaram (1985).

**Development of General Anisotropic/ Kinematic Model**

The isotropic hardening associative and nonassociative plasticity models described in the previous section are incapable of representing the inelastic nonlinear phenomena observed during unloading and reloading. Such phenomena include the Bauschinger effect (Hill, 1950; Fung, 1965; Malvern, 1969) during load reversals and progressively changing state under cyclic loading. Thus, it is necessary to develop kinematic models that allow plastic deformations to continue within the isotropically expanded yield surfaces. Proposed herein is such a model based on nonassociative plasticity concepts. The deviation from normality or nonassociativeness is related to the evolution of induced anisotropy.

When geologic materials are subjected to a loading sequence, considerable changes may be induced in the fabric or structure of the material. Such changes can cause an initially isotropic material to
exhibit anisotropic characteristics. The degree of anisotropy so induced, continuously evolves as the loading progresses. In a phenomenological approach to constitutive modelling, induced anisotropy is accounted for by inclusion of scalar and tensor valued state variables that reflect the deformation history of the material in the model.

Mechanics of Proposed Model

Figure 3.3 illustrates schematically the details of the proposed model in two mutually orthogonal stress spaces. The model consists of a surface $Q$ that translates in a fixed field of infinite isotropic yield surfaces, $F$, in the stress space. $F$ range from an initial surface $F_0$ that demarcates the initial elastic region to the ultimate envelope. Equations (3.5) and (3.8) define the fixed field. Surface $Q$ moves with the stress point and intersects the fixed surfaces during loading, unloading and reloading. Throughout the plastic behavior the state of stress remains on the boundaries of $Q$.

Surfaces $F$ serve to (i) define virgin loading, unloading, and reloading, (ii) govern magnitude of plastic strain increments and (iii) retain memory of maximum prestress. Surface $Q$ acts as the plastic potential governing the directions of plastic strain increments. During nonvirgin loading (unloading or reloading) $Q$ also serves as a loading surface allowing for purely elastic behavior within its domain and plastic behavior at its boundaries.

As noted earlier, $F$ is a function of the state of stress and $\xi$ and is expressed in terms of the direct invariants of stress and $\xi$. $Q$ is
a function of the state of stress and its location, $a_{ij}$ (Fig. 3.3), in the stress space. $a_{ij}$ serves as the tensor valued state variable that reflects the development and evolution of induced anisotropy. Based on the reasoning that preceded Eq. (3.2), $Q$ may be expressed as

$$Q = Q (\sigma_{ij}, a_{ij}, \xi)$$

$$= Q (J_1, J_{2D}, J_{3D}, I_{1a}, I_{2Da}, I_{3Da}, K_{1D}, K_{2D}, K_{3D}, \xi)$$

(3.14)

where $I_{1a}$ is the first invariant of $a_{ij}$, $I_{2Da}$ and $I_{3Da}$ are the invariants of the deviatoric part of $a_{ij} = a'_{ij}$. $K_iD$ ($i = 1, 2, 3$) are the mixed invariants of the deviatoric stress tensor $S_{ij}$ and the deviatoric tensor $a'_{ij}$. The definition of these invariants is given below.

$$I_{1a} = a_{ii}$$

$$I_{2Da} = \frac{1}{2} a'_{ij} a'_{ij}$$

$$I_{3Da} = \frac{1}{3} a'_{ik} a'_{kj} a'_{ji}$$

$$K_{1D} = S_{ij} a'_{ij}$$

$$K_{2D} = S_{ik} a'_{kj} a'_{ji}$$

$$K_{3D} = S_{ik} S_{kj} a'_{ji}$$

(3.15)
Figure 3.3 Schematic of Proposed Model
In general, \( Q \) may be of any shape or size and may expand, translate or rotate in stress space. For convenience, \( Q \) is chosen to have the same shape as \( F \) and is assumed to remain unchanged in shape, size and orientation. The size of \( Q \) is assumed to be identical to the initial yield surface, \( F_0 \). Thus, the expression for \( Q \) takes the same form as \( F \), but in terms of a constant value for the hardening parameter and a modified 'stress' tensor \( \overline{\sigma}_{ij} \) given by

\[
\overline{\sigma}_{ij} = \sigma_{ij} - a_{ij}
\]  

(3.16)

Therefore, \( Q \) can be expressed as

\[
Q = Q (\overline{J}_i)
\]  

(3.17a)

where \( \overline{J}_i \) (\( i = 1, 2, 3 \)) are the invariants of \( \overline{\sigma}_{ij} \). More specifically,

\[
Q = \overline{J}_{2D} - (- \alpha_0 \overline{J}^n_1 + \gamma \overline{J}^2_1) (1 - \beta \overline{S}_r)^{-1/2}
\]  

(3.17b)

where \( \overline{S}_r = \overline{J}_{3D}/\overline{J}_{2D}^3/2 \), \( \overline{J}_1 \) is the first invariant of \( \overline{\sigma}_{ij} \) and \( \overline{J}_{2D} \) and \( \overline{J}_{3D} \) are the invariants of the deviatoric part of \( \overline{\sigma}_{ij} \). \( n, \gamma \) and \( \beta \) have the same values as in Eq. (3.5), while \( \alpha_0 \) is constant and corresponds to the hardening parameter of initial yield surface \( F_0 \). The expanded form for \( Q \), in terms of direct and mixed invariants, may be written as
Since $Q$ acts as the plastic potential, directions of plastic strain increments are governed by the translation tensor $a_{ij}$. The deviation between the normal to $Q$ at the stress point and the strain increment directions predicted by an associative model defines the deviation from normality. By appropriately postulating $a_{ij}$ in terms of $a_{ij}$ induced anisotropy, it is possible to define deviation from normality to any desired degree. Thus, the model has the capability to simulate qualitatively the effects of rotation or distortion of the yield surface.

The mechanism is relatively simple because only one translating surface is involved. The same potential surface remains active throughout the deformation history regardless of whether the loading is virgin or not. Continuous accumulation of induced anisotropy is provided for and its influence is reflected in the subsequent behavior.

Flow Rule

For every stress point in stress space, during virgin or non-virgin loading, there exists a corresponding yield surface $F_c$ (Fig. 3.4). In the context of this model, the material is said to be in a virgin
Figure 3.4 Magnitude of Plastic Strain Incement for Nonvirgin Loading
state if all the yield surfaces corresponding to its previous stress history are completely contained within the current yield surface, $F_c$. If the material is in a virgin state and a stress increment is directed outwards of $F_c$, the material is said to undergo virgin loading. If stress increment is not directed out of $F_c$, it ($F_c$) is retained in the material's memory as the maximum prestress surface, $F_{ps}$, until the stress point leaves its bounds once again, Fig. 3.4. During nonvirgin loading, one of the functions of surface $Q$ is to act as a loading surface: when stress state is within $Q$, behavior is elastic, and when it reaches the boundary of $Q$, plasticity resumes. During virgin and non-virgin loading, $Q$ serves as the plastic potential and translates in the stress space with the stress point whenever plastic behavior prevails. During elastic behavior, $Q$ remains fixed.

The translation of $Q$ is specified by means of an incremental translation rule described later. In general, $F_c$ and $Q$ may intersect.

The plastic flow rule with $Q$ as the plastic potential may be written as

$$d\varepsilon^p_{ij} = \lambda \frac{\partial Q/\partial \sigma_{ij}}{\| \partial Q/\partial \sigma \|}$$

(3.19)

where $d\varepsilon^p_{ij}$ is the incremental plastic strain and $\lambda$ is a scalar proportionality constant whose value is $\lambda_v$ or $\lambda_{nv}$, depending upon whether the loading is virgin or nonvirgin. $\| \|$ indicates norm.
The term $\partial Q/\partial \sigma_{ij}/\|\partial Q/\partial \sigma_{rs}\|$ in Eq. (3.19) describes the tensor-valued unit normal to the surface $Q$ at point $\sigma_{ij}$ in the six-dimensional stress space. If the origin of the stress space was translated to $a_{ij}$, the same stress point $\sigma_{ij}$ would now be expressed as $\bar{\sigma}_{ij} = \sigma_{ij} - a_{ij}$ and the same unit normal to $Q$ at $\sigma_{ij}$ may be expressed as $\partial Q/\partial \sigma_{ij}/\|\partial Q/\partial \bar{\sigma}_{rs}\|^{-1}$.

Therefore, it is possible to write

$$\frac{\partial Q/\partial \sigma_{ij}}{\|\partial Q/\partial \sigma_{rs}\|} = \frac{\partial Q/\partial \bar{\sigma}_{ij}}{\|\partial Q/\partial \bar{\sigma}_{rs}\|}$$

(3.20)

A proof of Eq. (3.20) is found in Appendix A. Since $Q$ is a function of the invariants of $\bar{\sigma}_{ij}$ (Eq. 3.17a), using the chain rule, it is possible to write

$$\frac{\partial Q}{\partial \bar{\sigma}_{ij}} = \frac{\partial Q}{\partial \bar{J}_1} \frac{\partial \bar{J}_1}{\partial \bar{\sigma}_{ij}} + \frac{\partial Q}{\partial \bar{J}_2} \frac{\partial \bar{J}_2}{\partial \bar{\sigma}_{ij}} + \frac{\partial Q}{\partial \bar{J}_3} \frac{\partial \bar{J}_3}{\partial \bar{\sigma}_{ij}}$$

(3.21a)

or

$$\frac{\partial Q}{\partial \bar{\sigma}_{ij}} = \frac{\partial Q}{\partial \bar{J}_1} \delta_{ij} + \frac{\partial Q}{\partial \bar{J}_2} \bar{\sigma}_{ij} + \frac{\partial Q}{\partial \bar{J}_3} \bar{\sigma}_{ik} \bar{\sigma}_{kj}$$

(3.21b)

which can be expanded as

$$\frac{\partial Q}{\partial \bar{\sigma}_{ij}} = \frac{\partial Q}{\partial \bar{J}_1} \delta_{ij} + \frac{\partial Q}{\partial \bar{J}} (\sigma_{ij} - a_{ij})$$

$$+ \frac{\partial Q}{\partial \bar{J}_3} (\sigma_{ik} - a_{ik}) (\sigma_{kj} - a_{kj})$$

(3.21c)
Now combining Eq. (3.19) and Eq. (3.21), the expression for incremental plastic strains may be written as

\[
d\varepsilon^p_{ij} = \bar{g}_1 \delta_{ij} + \bar{g}_2 (\sigma_{ij} - a_{ij}) + \bar{g}_3 (\sigma_{ik} - a_{ik})(\sigma_{kj} - a_{kj})
\]  
(3.22)

where \( \bar{g}_1 = \bar{g}_i (J_r, \lambda, \alpha, \beta, \gamma, n), (r = 1, 2, 3). \)

In contrast, the isotropic hardening model, which was mentioned earlier, is not a function of the joint invariants of \( \sigma_{ij} \) and \( a_{ij} \), and hence yields the flow rule of the form

\[
d\varepsilon^p_{ij} = g_1 \delta_{ij} + g_2 \sigma_{ij} + g_3 \sigma_{ik} \sigma_{kj}
\]  
(3.23)

where \( g_i = g_i (J_r, \lambda, \alpha, \beta, \gamma, n), (r = 1, 2, 3). \)

In Eq. (3.23), it is seen that \( d\varepsilon^p_{ij} \) is coaxial, i.e. they have the same principal directions with the total stress \( \sigma_{ij} \). In Eq. (3.22), however, \( d\varepsilon^p_{ij} \) is coaxial with \( (\sigma_{ij} - a_{ij}) \) and, therefore, not necessarily coaxial with the total \( \sigma_{ij} \). Comparing these two equations, it is apparent how a properly formulated value for \( a_{ij} \) could allow for deviation from normality of \( d\varepsilon^p_{ij} \), and non-coaxiality of \( \sigma_{ij} \) and \( d\varepsilon^p_{ij} \). Thus, a proper description of \( a_{ij} \) is a critical aspect of this model.
Magnitude of $dC_{ij}$ and the Interpolation Rule. During the virgin loading, the state of stress always remains on the isotropically hardening yield surface $F_c$. This is known as the consistency condition (Prager, 1951; Desai and Siriwardane, 1984) and mathematically stated as:

$$dF_c = 0$$  \hspace{1cm} (3.24)

using the chain rule of differentiation on $F_c = F_c (\sigma_{ij}, \xi)$.

$$dF_c = \left( \frac{\partial F_c}{\partial \sigma_{ij}} \right) d\sigma_{ij} + \left( \frac{\partial F_c}{\partial \xi} \right) d\xi = 0$$  \hspace{1cm} (3.25)

Since $d\xi = (d\varepsilon_{ij}^p d\varepsilon_{ij}^p)^{1/2}$ and $d\varepsilon_{ij}^p = \lambda_v \left[ \frac{\partial Q}{\partial \sigma_{ij}} \right]$, Eq. (3.25) reduces to

$$\lambda = \lambda_v \left( \frac{-\partial F_c}{\partial \sigma_{ij}} \right) d\sigma_{ij}$$  \hspace{1cm} (3.26)

Eq. (3.26) determines the magnitude of plastic strain increments during the virgin loading.

For plastic behavior during the nonvirgin loading (unloading and reloading), the concept used herein does not involve hardening yield surfaces. Hence, a consistency condition of the form in Eq. (3.24) is not applicable. Instead, an interpolation function is postulated for $\lambda = \lambda_{nv}$. The interpolation function satisfies a number of criteria.
In defining the nonvirgin loading, it has been ensured that (a) the degree of maximum prestress is incorporated in the formulation through surface $F_{ps}$ and (b) a smooth transition exists from the non-virgin to virgin loading and from elastic to plastic behavior. The effect of prestress is introduced to the formulation by identifying a stress level, $\sigma_{ij}^*$, on $F_{ps}$. $\sigma_{ij}^*$ is associated with the previous state of stress and the direction of the current stress increment. It is obtained by moving in the direction of $d\sigma_{ij}$ from $\sigma_{ij}$ until $F_{ps}$ is encountered, Fig. 3.4.

With these stipulations and on the basis of a study of observed unloading and reloading behavior under various stress paths and various levels of loading (see Appendix A), the following expression is postulated for $\lambda_{nv}$ in Eq. (3.19).

$$
\lambda_{nv} = \frac{\frac{\partial F_{ps}}{\partial \sigma_{ij}} \sigma_{ij}^* - d\sigma_{ij}}{\frac{\partial F_{ps}}{\partial \xi} \left[ 1 + \theta \frac{h^1}{R^1} \right]} = [\lambda_v]^* \left[ 1 + \theta \frac{h^1}{R^1} \right]^{h^1-1}
$$

(3.27)

where $h_1$ is a material constant and $\theta$ is given by

$$
\theta = h_2 \left( \frac{\Delta_{ps}}{J_1} \right)^{h^3}
$$

(3.28)
Where \( h_2, h_3 \) are material constants, \( \Delta_{ps} = \left(\frac{\gamma}{\alpha_{ps}}\right)^{n-2} \) denotes \( J_1 \) intercept of the prestress surface \( F_{ps} \) and \( J_1^* \) is the first invariant of \( \sigma_{ij}^* \), Fig. 3.4. \( \bar{R} \) is a ratio depending on the relative position of the current stress point within the prestress surface and is expressed in terms of the hardening parameters associated with the current surface, prestress surface and the surface from which the last stress reversal took place. \( \bar{R} \) is expressed as

\[
\bar{R} = \frac{(S_c^*)/\alpha_{ps} - (S_c^*)/\alpha_c}{(S_c^*)/\alpha_c - (S_R^*)/\alpha_R} \text{ } ^{1/\sqrt{F_s^*}}
\]

Where \( S_R \) is the stress ratio, Eq. (3.5b), subscripts *, c, and R refer to stress point; \( \sigma_{ij}^* \), current level of stress, \( \sigma_{ij} \) and the level of stress reversal, \( \sigma_{ij}^R \), respectively. \( F_s^* \) gives the value of the shape function (Eq. 3.4c) at \( \sigma_{ij}^* \). \( \alpha_c, \alpha_{ps} \) and \( \alpha_R \) are the values of the growth function corresponding to \( F_c, F_{ps} \) and the surface \( F \) from which the last stress reversal occurred, such as \( R_1 \) for \( C_2 \) and \( R_2 \) for \( C_3 \), Fig. 3.4. Conversely, \( \bar{R} \) could have also been specified in terms of the distances in stress space among the current stress point \( \sigma_{ij}, \sigma_{ij}^* \) and the stress point at which the last stress reversal took place.
Equation (3.27) is essentially an interpolation function which is capable of varying \( \lambda_{nv} \) from infinity to the value of \( \lambda_v \) at \( \sigma_{ij}^* \). The rate of interpolation is governed by the three nondimensional interpolation constants \( h_1, h_2, \) and \( h_3 \) which are determined from the observed unloading and reloading data.

At the initiation of unloading or stress reversals, such as point \( R \), in Fig. 3.4, \( \alpha_c = \alpha_R = \alpha_{R_1} \), \( (S_r)_c = (S_r)_R \) and from Eq. (3.29) \( \bar{R} \rightarrow \infty \) and only the elastic strains occur. As the current stress point gets further and further away from \( R_1 \), \( \alpha_c > \alpha_R \) and \( \bar{R} \) and \( \lambda_{nv} \) grow smaller, resulting in progressively increasing magnitudes of plastic strain increments. As the state of stress approaches \( F_{ps} \), \( \sigma_{ij}^* = \sigma_{ij}^* \) and according to Eq. (3.29), \( \bar{R} \rightarrow 0 \) since \( \alpha_c \rightarrow \alpha_{ps} \) and \( (S_r)_c = (S_r)_R \).

Then \( \lambda_{nv} \rightarrow \lambda_v \) and Eq. (3.27) tends to become identical to Eq. (3.26). Thus, a smooth transition from elastic to plastic behavior and nonvirgin to virgin loading is assured.

The motivation for the choice of the particular forms for \( \bar{R} \) and \( \Theta \), given by Eqs. (3.28) and (3.29), are discussed at length in Appendix A.

**Translation Rule for Associative Behavior**

From Eq. (3.22), it is evident how the location of \( Q, a_{ij} \), in the stress space, governs the direction of the incremental plastic
strains. Thus, it can be concluded that the most dominant feature of this model is a translation rule specifying the evolution of \( a_{ij} \). By choosing appropriate translation rules, it is possible to model (a) associative behavior, (b) nonassociative behavior based on the correction functions or deviatoric normality and (c) general deviation from normality based on induced anisotropy. The most simple case is that of associative behavior and the corresponding translation rule is treated as the basic rule from which deviations occur due to induced anisotropy.

**Virgin Loading.** Associative behavior during virgin loading and isotropic hardening can be obtained by ensuring that \( Q \) translates in such a manner that the unit normals to \( F \) and \( Q \) coincide, Fig. 3.5. Since the shape of \( Q \) and \( F \) are the same, it can be shown (see Appendix A) that if directions of \( \partial Q / \partial a_{ij} \) and \( \partial F / \partial a_{ij} \) coincide, then (a) directions of \( \sigma_{ij} \) and \( \bar{\sigma}_{ij} \) coincide and (b)

\[
\frac{||\sigma_{ij}||}{||\bar{\sigma}_{ij}||} = \left( \frac{\alpha_c}{\alpha_o} \right)^{1/2} \tag{3.30}
\]

From (a) and (b) above, it is possible to conclude that

\[
\frac{\sigma_{ij}}{\bar{\sigma}_{ij}} = \frac{\sigma_{ij}}{\sigma_{ij} - a_{ij}} = \left( \frac{\alpha_o}{\alpha_c} \right)^{1/2} \tag{3.31}
\]
Figure 3.5 Position of Q in Stress Space for Isotropic Associative Behavior
which yields the location of Q, \( (a_{ij})_{\text{iso}} \), for isotropic hardening associative behavior as

\[
(a_{ij})_{\text{iso}} = \sigma_{ij} [1 - (\alpha_c/\alpha_0)^{1/n-2}] \quad (3.32)
\]

The incremental form of Eq. (3.32) may be obtained as

\[
d(a_{ij})_{\text{iso}} = \frac{\partial (a_{ij})}{\partial \sigma_{rs}} d\sigma_{rs} + \frac{\partial (a_{ij})}{\partial \alpha} d\alpha \quad (3.33a)
\]

which reduces to

\[
d(a_{ij})_{\text{iso}} = p_1 d\sigma_{ij} + p_2 \sigma_{ij} \quad (3.33b)
\]

where

\[
p_1 = 1 - (\alpha_c/\alpha_0)^{1/n-2} \quad (3.33c)
\]

\[
p_2 = \frac{1}{2-n} \left( \frac{\alpha_c}{\alpha_0} \right)^{3-n} \quad (3.33c)
\]

When Eq. (3.33) is used as the translation rule, the flow rule would reduce to the isotropic hardening associative form shown in Eq. (3.23).

**Nonvirgin Loading.** During nonvirgin loading, the position of Q, consistent with the isotropic hardening associative behavior outlined above, is obtained by postulating a linear interpolation rule. If
(\(a_{ij}^0\)) is a known 'isotropic-associative' position of Q associated with a state of stress, \(\sigma_{ij}^0\), and if \(\sigma_{ij}^*\) is the point on \(F_{ps}\) corresponding to a particular direction of loading \(d\sigma_{ij}\), then the position of Q for any intermediate stress state \(\sigma_{ij}\) is given by Fig. 3.6,

\[
(a_{ij})_{iso} = (a_{ij})_{iso} + [(a_{ij})_{iso}^* - (a_{ij})_{iso}^0] \frac{||\sigma_{rs} - \sigma_{rs}^0||}{||\sigma_{rs}^* - \sigma_{rs}^0||}
\]

Where from Eq. (3.32)

\[
(a_{ij})_{iso}^* = \sigma_{ij}^* [1 - (\alpha_{ps}/\alpha_0)^{1/n-2}]
\]

Note that \(\sigma_{ij}^0\), \(a_{ij}^0\), \(\sigma_{ij}^*\), \(a_{ij}^*\) change each time there is a change in the direction of loading. The incremental form of Eq. (3.34) may be written as

\[
d(a_{ij})_{iso} = [(a_{ij})_{iso}^* - (a_{ij})_{iso}^0] \frac{d}{||\sigma_{rs} - \sigma_{rs}^0||} \frac{||\sigma_{rs}^* - \sigma_{rs}^0||}{||\sigma_{rs}^* - \sigma_{rs}||}
\]
Figure 3.6 Location of \((a_{ij})_{iso}\) during Nonvirgin Loading
Now, by substitution for \( (a_{ij}^0) \) from Eq. (3.34), the above equation simplifies to

\[
\frac{d (a_{ij}^*)}{iso} = \left[ (a_{ij}^*) - (a_{ij}) \right] \frac{|d\sigma_{rs}|}{|\sigma_{rs}^* - \sigma_{rs}|} \quad (3.36b)
\]

Equations (3.33) and (3.36) define translation rules for associative flow during virgin and nonvirgin loading, respectively. Therefore, when this basic form of the translation rule is adapted, no deviation from normality occurs. The next section described the correction of \( (a_{ij})_iso \) to yield desired levels of deviation from normality.

Translation Rule for Induced Anisotropy

In general, when an initially isotropic material is subject to non-hydrostatic loading, a certain degree of anisotropy is induced. Therefore, during deviatoric or shear loading, the material is, in general, not isotropic. The degree of induced anisotropy and its effect on the subsequent behavior may vary from negligible to significant amounts, depending on the type of material and the stress-history it is subjected to.

The effect of such induced anisotropy on the response of the material is introduced through a correction to the isotropic-associative position \( (a_{ij})_iso \) of \( Q \). This results in nonassociativeness or deviation from normality of the incremental plastic strains. The corrected form of \( a_{ij} \) can be written as
\[ a_{ij} = (a_{ij})_{iso} + d_{ij} \]  

(3.37)

where \( d_{ij} \) is the correction tensor to \( (a_{ij})_{iso} \). It serves as an indirect measure of deviation from normality or induced anisotropy, and may be written in terms of deviatoric and hydrostatic components as Fig. 3.7,

\[ d_{ij} = \Pi \delta_{ij} + \mu_{ij} \]  

(3.38)

\( \Pi \) specifies the magnitude of the correction along the \( J_1 \)-axis and \( \mu_{ij} \) represents the correction in the octahedral plane, as shown in Fig. 3.7.

The incremental form of Eq. (3.37) may be written as

\[ da_{ij} = d_{ij} + d(a_{ij})_{iso} \]  

(3.39)

where \( d_{ij} \) is written as

\[ d_{ij} = d\Pi \delta_{ij} + d\mu_{ij} \]  

(3.40)

d\( \Pi \delta_{ij} \) and d\( \mu_{ij} \) have to be defined in such a manner as to realistically describe the growth and demise of deviation from normality caused by induced anisotropy. Induced anisotropy is caused by strains due to applied stresses. Since plastic strains generally predominate in soils, it is assumed that anisotropy occurs mainly due to plastic straining. However, the state of plastic strains alone cannot define the degree of
Figure 3.7 Location of Q and $Q_{iso}$ in Stress Space
induced anisotropy. The same state of plastic strains could imply different degrees of induced anisotropy, depending on the corresponding state of stress. For example, a purely volumetric state of plastic strains under a hydrostatic state of stress implies isotropy. The same state of strains under a non-hydrostatic state of stress indicates that anisotropic behavior has occurred. Thus, the degree of induced anisotropy is also dependent on the state of stress, or through a transformation, $C_{ijkl}^e$, the state of elastic strains.

Although the degree of deviation from normality and induced anisotropy may vary during shear loading or unloading of an initially isotropic material, along any stress path, coaxiality between the total stress and plastic strain increments is maintained. Non-coaxiality is observed if principal directions of stress change or if the material is initially anisotropic. In the model, coaxiality is controlled by $a_{ij}$.

If $a_{ij}$ is coaxial with $\sigma_{ij}$, so is the plastic strain increment $d\varepsilon^p_{ij}$. Non-coaxiality between $d\varepsilon^p_{ij}$ and $\sigma_{ij}$ is possible only if $a_{ij}$ becomes non-coaxial with $\sigma_{ij}$. Thus, the translation rule for $a_{ij}$ should be capable of providing for the above phenomena, when necessary.

Based on the above premises and on experimental observations described in Chapter 2, rules for translation in the $J_1 - \sqrt{J_2/2D}$ plane and the octahedral plane are postulated.
Representation of Induced Anisotropy. In order to specify \( d\Pi \) in Eq. (3.40), a suitable measure, \( a_n \), related to induced anisotropy is defined and correlated to \( \Pi \). The measure to represent induced anisotropy chosen herein is based on the intuitive assumption that a sufficient condition for compliance with normality is obtained if the orientation of plastic strains (in strain space) coincide with the orientation of elastic strains. Thus, one measure that could be correlated to deviation from normality is the deviation of total plastic strain direction from the total elastic strain direction in the strain space. The following relation for \( \Pi \) is, therefore, postulated and its validity investigated.

\[
\Pi = \Pi \left\{ \left( \frac{I_1^{e}}{I_2^{e}} \right) - \left( \frac{I_1^{p}}{I_2^{p}} \right) \right\} \quad (3.41a)
\]

or

\[
\Pi = \Pi \left\{ R_e - R_p \right\} = \Pi \left\{ a_n \right\} \quad (3.41b)
\]

where

\[
R_e = \frac{I_1^{e}}{\sqrt{I_2^{e}}}, \quad R_p = \frac{I_1^{p}}{\sqrt{I_2^{p}}} \quad (3.42)
\]

is a measure used to represent induced anisotropy.

\( I_1 \) and \( I_2 \) are the first and second invariants of the strain tensor, and the superscripts \( e \) and \( p \) refer to elastic and plastic strains, respectively. \( I_1^{\sqrt{1/2}} \) is a measure of the orientation of the
state of strain in strain space. When $I_1^P$ and $I_2^P$ are both zero, $R_p$ becomes indeterminate and is then assumed equal to $R_e$. Thus, when the material is in a purely elastic state, $a_n = 0$ and no induced anisotropy occurs.

The relationship expressed by Eq. (3.41) was investigated for 'Munich Sand' (described later) on the basis of a number of tests in a truly triaxial device reported by Scheele and Desai, 1984. The value of $a_n$, Eq. (3.42), at various stages during compression (CTC and TC) and extension (TE and CTE) tests on Munich Sand were found. Contours of equal values of $a_n$ in the $J_1 - \sqrt{J_2D}$ stress space are shown for compression and extension in Figs. 3.8a and 3.8b, respectively.

Deviation from normality at various stages during the tests are given by the deviation of observed plastic strain increment directions from the directions predicted by the isotropic associative model, Eq. (3.23). The deviation from normality may also be expressed in terms of $a_{ii} - (a_{ii})_{iso}$, where $a_{ii}$ is the volumetric component of $a_{ij}$, which is the actual location of Q required to give the observed plastic strain increments. $(a_{ii})_{iso}$ is the volumetric component of $(a_{ij})_{iso}$ given by Eq. (3.32). The above measure quantifies deviation from normality in terms of deviation of the location of Q from the 'associative' position. Contours of equal deviation from normality during compression and extension tests are shown in Figs. 3.9a and 3.9b, respectively.
Figure 3.8a Contours of Equal $a_n$ for Munich Sand Compression Paths
Figure 3.8b  Contours of Equal $a_n$ for Munich Sand (Extension Paths)
Figure 3.9a Contours of Equal Deviation from Normality (Compression Paths)
Figure 3.9b Contours of Equal Deviation from Normality (Extension Paths)
It can be seen from the plots in Figs. 3.8 and 3.9 that deviation from normality and the proposed representation of induced anisotropy show very similar trends.

Figure 3.10 shows \( a_n \) plotted against a normalized value of \( \Pi, [-3\Pi/(\gamma/\alpha_0)(1/n-2)] \), for several stress paths under both virgin and non-virgin loading. A definite correlation between the two quantities can be observed. Similar correlation was found in the case of other sands as well, as described in Chapter 5. Although there appears to be some scatter, the relationship between \( a_n \) and \( \Pi \) can be idealized as linear, and may be expressed as

\[
\Pi = h_4 a_n = h_4 (R_e - R_p) \tag{3.43}
\]

where \( h_4 \) is the constant of proportionality with units of stress. \( h_4 \) is defined as the translational parameter for the material. The sensitivity of the model to \( h_4 \) is investigated later.

The expression for \( d\Pi \) in Eq. (3.40) can now be written as

\[
d\Pi = h_4 d a_n \tag{3.44}
\]

\( d\mu_{ij} \) in Eq. (3.40) is postulated as

\[
d\mu_{ij} = b dE_{ij}^p \tag{3.45}
\]
Figure 3.10. Plot of Normalized Value of $\pi$ vs. $a_n$ for Munich sand.

Numbers in parentheses refer to values of initial confining pressure in psi.

$\rho_0 = 2.03$ gm/cc

1.0 psi = 6.89 kPa
where \( \delta \bar{E}_{ij} = \delta \bar{E}_{ij} - 1/3 (\delta \bar{E}_{kk}) \delta_{ij} \) is the deviatoric part of the incremental plastic strain, and \( b \) is a scalar. \( \delta \bar{E}_{ij} \) and \( \delta \mu_{ij} \) cannot be totally independent of each other. They are subjected to the constraint that the translation is such that the stress point \( \sigma_{ij} \) has to remain on the surface of \( Q \) during plastic flow. This is true for both virgin and nonvirgin loading. This condition may be expressed as

\[
dQ = 0 \quad (3.46a)
\]

and is used to obtain magnitude of \( b \). Equation (3.46a) is expanded as

\[
dQ = \frac{\partial Q}{\partial \sigma_{ij}} \sigma_{ij} = \frac{\partial Q}{\partial \sigma_{ij}} [\sigma_{ij} - \delta_{ij}] = 0 \quad (3.46b)
\]

Substituting for \( \delta_{ij} \) from Eqs. (3.39), (3.40), (3.44) and (3.45), \( b \) is determined as

\[
b = \frac{\frac{\partial Q}{\partial \sigma_{ij}} [\sigma_{ij} - d (a_{ij}) - h_4 da_n \delta_{ij}]}{\frac{\partial Q}{\partial \sigma_{ij}} \delta \bar{E}_{ij}}
\]

Equations (3.38), (3.43), (3.44) and (3.45) satisfy a number of observed phenomena and intuitive expectations.

During hydrostatic loading of an initially isotropic material, no deviation from normality or induced anisotropy is observed. The state of total elastic and total plastic strains are both purely
volumetric and the deviatoric component of incremental plastic strains \( \text{d}E_{ij}^p \) is equal to zero. Therefore, in Eq. (3.43), \( R_e = R_p \) and \( a_n = \Pi = 0 \); in Eq. (3.45), \( \text{d}\mu_{ij} = 0 \) and hence \( \mu_{ij} = 0 \). As a result, \( \text{d}_{ij} \), Eq. (3.38), = 0 and no deviation from normality is predicted by the model. If, however, hydrostatic loading is preceded by a shear load-unload sequence, deviation from normality is observed; this is provided for in the model, as explained below. At the end of a shear load-unload cycle (from an original hydrostatic state of stress), the state of stress is hydrostatic and hence the total elastic strains are purely volumetric, whereas the total plastic strains are not. Thus, in Eq. (3.43), \( R_e \neq R_p \) and \( a_n \) has a non-zero value. From Eq. (3.45), \( \mu_{ij} \) has a non-zero value equal to \( \int b \text{d}E_{ij}^p \). As a result, \( \text{d}_{ij} \) in Eq. (3.38) has a non-zero value as well, and the model predicts deviation from normality. The degree of deviation in such cases is governed by the stress path and the maximum stress level of the preloading cycle.

Prior to shear loading of an initially isotropic material, \( R_e = R_p \), \( a_n = 0 \), \( \mu_{ij} = 0 \) and \( \text{d}_{ij} = 0 \), as explained above. During shear loading, the directions of incremental elastic and plastic strains do not coincide and, therefore, the values of \( R_e \) and \( R_p \) change at different rates, causing the magnitude of \( a_n \) to increase. Thus, induced anisotropy grows when an isotropic material is subjected to shear loading. If the loading is such that the orientation of total plastic strain,
represented by $R_p$, tends to approach the orientation of total elastic strains, $R_e$, the magnitude of $a_n$ decreases, Eq. (3.43), and a gradual demise of induced anisotropy and, hence, of deviation from normality occurs. This phenomenon may be observed in the case of hydrostatic loading following a shear load-unload cycle. Experimental evidence showing this fading process of induced anisotropy is reported by Lewin (1978) and Dafalias (1984) and discussed in Chapter 2.

When an initially isotropic material is loaded from a stress-free state, $a_{ij}$ is initially zero and from Eq. (3.22), $d\varepsilon_{ij}^P$ and hence $d\varepsilon_{ij}^P$ are coaxial with $\sigma_{ij}$. As a result, $d\varepsilon_{ij} = b d\varepsilon_{ij}^P$, $e_{ij} = f d\varepsilon_{ij}$, $d_{ij} (= \Pi \delta_{ij} + \mu_{ij})$ and $a_{ij}$, Eq. (3.37), are coaxial with $\sigma_{ij}$ after the first plastic strain increment. If the loading is continued with no change in the principal directions of stress, from Eqs. (3.22), (3.37), (3.38) and (3.45), the tensors $\sigma_{ij}$, $a_{ij}$ and $d\varepsilon_{ij}^P$ are all coaxial. Now, if the principal directions of stress were to change (rotate), $\sigma_{ij}$ and $a_{ij}$ are no longer coaxial. From Eq. (3.22), $d\varepsilon_{ij}^P$ is now coaxial with $(\sigma_{ij} - a_{ij})$, which is, in general, neither coaxial with $\sigma_{ij}$ nor $a_{ij}$.

Thus, the model is capable of maintaining coaxiality between $d\varepsilon_{ij}^P$ and $\sigma_{ij}$ during non-rotating (ex: straight line) stress paths and is capable of capturing non-coaxiality during rotations of principal directions of stress.
Translation Rule. Combining Eqs. (3.39), (3.40), (3.43), (3.44) and (3.45), the general anisotropic translation rule for \( Q \) can be expressed as

\[
d_{ij} = d(a_{ij})_\text{iso} + (h_4 \, d_a) \delta_{ij} + b \, dE_{ij}^p
\]  

(3.48)

where \( b \) is given by Eq. (3.47). The "isotropic" term \( d(a_{ij})_\text{iso} \) varies, depending on whether loading is virgin or nonvirgin, as given by Eqs. (3.33) and (3.36), respectively. Thus, the translation rule requires only one material constant, \( h_4 \), which is found as the average slope of the plots of \( a_n \) vs. \( \Pi \) for virgin and nonvirgin loading under different stress paths.

Equation (3.48) governs the translation of \( Q \) while plastic flow occurs. Whenever elastic behavior initiates (when load increment is directed inwards of \( Q \)), it is assumed that \( Q \) reverts to its isotropic position given by \( (a_{ij})_\text{iso} \). \( Q \) remains in this position until the state of stress reaches its boundary again. Once \( \sigma_{ij} \) reaches the boundary of \( Q \), plastic behavior is resumed, Eq. (3.37) applies once more and \( Q \) is reset accordingly. This is done to avoid any inconsistencies or ambiguities that might arise due to the fact that \( F \) and \( Q \) intersect each other. Such a possibility is illustrated in Fig. 3.11a where a nonvirgin loading sequence ABCD takes place following virgin loading to point A. At point C an ambiguity arises since the load increment CD satisfies the
Figure 3.11 Resetting of $Q$ during Elastic Behavior
condition for virgin plastic loading as well as nonvirgin elastic loading, the two of which are mutually exclusive. Resetting $Q$ at the initiation of elastic behavior to $(Q)_{iso}$ and at the conclusion of elastic behavior to $(Q)_{r}$, Fig. 3.11b, eliminates this ambiguity. The constant evolution and accumulation of induced anisotropy is not interrupted by this process. Any changes in induced anisotropy and $d_{ij}$ would be reflected when $Q$ is reset according to Eq. (3.37). The net effect of this procedure is to change the range of elastic behavior slightly.

Initial Anisotropy

Initial anisotropy in the field or due to sample preparation can be handled in an indirect manner within the framework of this model. The initial anisotropy can be considered to have occurred when an initially isotropic material was subjected to previous cycles of loading and unloading. Now, the state of 'initial' anisotropy for subsequent loading is completely defined if the initial position $a_{ij}^0$ of $Q$ and the corresponding stress surface $F_{ps0}$ are known.

Consider an initially isotropic state at $A$, Figs. 3.12a, 3.12d. The initial yield surface is $F_o$ and $Q$ is assumed to be of the same size as $F_o$. Now impose a stress history, $A$-$B$-$C$, schematically shown in Fig. 3.12d, that causes initial anisotropy. This history can be due to sample preparation where the load-unload sequence $A$-$B$-$C$ is equivalent to
Figure 3.12 Schematic to Illustrate Initial Anisotropy
compaction of the sample in the mold and subsequent dismantling and removal of sample from the mold. As a simplification, it can be assumed that only radial anisotropy is caused by this stress history.

Figure 3.12b shows the position of Q and the corresponding yield surface for point B at the end of loading. Upon unloading, this surface becomes the initial prestress surface $F_{pso}$. During unloading from B to C, Fig. 3.12d, Q moves to point C, Fig. 3.12c, where $a_{ij}^0$ denotes the translation tensor corresponding to the induced anisotropy during A-B-C. Now, if the sample is loaded hydrostatically, the measured strain increments in the three principal directions would be different, the difference being attributable to the state of initial induced anisotropy. The knowledge of incremental plastic strains during hydrostatic loading permits evaluation of $a_{ij}^0$ and $F_{pso}$.

The procedure outlined below can be used to determine the initial state of anisotropy of a material subject to hydrostatic loading from a stress state close to the unstressed state. Strictly speaking, this procedures yields $a_{ij}$ corresponding to a point close to the unstressed state. However, the $a_{ij}$ obtained by this procedure may be assumed to be approximately equal to $a_{ij}^0$. A similar procedure could also be adapted to find the initial anisotropy for materials under a particular state of in-situ stress and subjected to loading along a non-hydrostatic stress path.
\( a_{ij}^0 \) is found using the direction of the first plastic strain increment \( d\varepsilon_{ij}^P \) due to hydrostatic loading and the condition that stress point always lies on Q. The flow rule is expressed as, Eq. (3.19) and (3.20)

\[
d\varepsilon_{ij}^P = \lambda \frac{\delta Q/\delta \sigma_{ij}}{||\delta Q/\delta \sigma_{rs}||} \tag{3.49}
\]

The incremental volumetric plastic strains are, therefore, given by

\[
d\varepsilon_v^P = d\varepsilon_{ii}^P = \lambda \frac{\delta t/\delta \sigma_{ii}}{||\delta Q/\delta \sigma_{rs}||} \tag{3.50}
\]

The square of the ratio of the magnitudes of total and volumetric plastic strain increments is obtained from Eqs. (3.49) and (3.50) as

\[
\frac{d\varepsilon_{ij}^P}{(d\varepsilon_v^P)^2} = \frac{\delta Q/\delta \sigma_{ij}}{\delta Q/\delta \sigma_{ii}} \frac{\delta Q/\delta \sigma_{ij}}{(\delta Q/\delta \sigma_{ii})^2} \tag{3.51a}
\]

The ratio \( \frac{d\varepsilon_{ij}^P}{d\varepsilon_v^P} = k_0 \) can be evaluated from the first observed plastic strain increment \( d\varepsilon_{ij}^P \) due to hydrostatic loading. By specializing
the flow rule for hydrostatic loading and by evaluating $\frac{\partial Q}{\partial \overline{\sigma}_{ij}}$. Eq. (3.51a) may be expanded as

$$2\overline{J}_{2D} - (9K_o - 3) \overline{F}_S \left[ n \alpha_o \overline{J}_1^{n-1} - 2\gamma \overline{J}_1 \right]^2 = 0 \quad (3.51b)$$

The condition that stress point always lies on $Q$ may be expressed as

$$Q = \overline{J}_{2D} - (-\alpha \overline{J}_1^n + \gamma \overline{J}_1^2) \overline{F}_S = 0 \quad (3.52)$$

Equations (3.51) and (3.52) may now be solved to obtain $\overline{\sigma}_{ij}^o$ and hence the value of $\alpha_{ij}^o$.

To determine $F_{ps0}$, the corresponding hardening parameter $\alpha_{ps0}$ has to be found. From Eqs. (3.19) and (3.27) it is possible to write

$$m_1 = \left| \frac{dc_{ij}^p}{d\sigma_{ij}} \right| = \frac{\frac{\partial F_{ps}}{\partial \sigma_{ij}} \frac{d\sigma_{ij}}{|d\sigma_{rs}|}}{\frac{\partial F_{ps}}{\partial \sigma_{ij}} [1 + \theta \overline{R}_1]} \quad (3.53)$$

for the first hydrostatic stress increment. Considering two consecutive increments

$$m_1 = \frac{1 + \theta \overline{R}_2}{1 + \theta \overline{R}_1} h_1$$

$$m_2 = \frac{1 + \theta \overline{R}_2}{1 + \theta \overline{R}_1} h_2$$

(3.54)
Specializing Eq. (3.54) to hydrostatic loading and using Eqs. (3.28) and (3.29), it is possible to write

\[
\frac{m_1}{m_2} = \frac{1 + h_2 \left( \frac{\alpha_2 - \alpha_{pso}}{\alpha_0 - \alpha_2} \right)^{h_1}}{1 + h_2 \left( \frac{\alpha_1 - \alpha_{pso}}{\alpha_0 - \alpha_1} \right)^{h_1}}
\]

(3.55)

where \( \alpha_1 \) and \( \alpha_2 \) correspond to the current surface \( F_c \) at the beginning of the first and second hydrostatic load increments, respectively, and \( \alpha_0 \) corresponds to initial surface \( F_0 \). \( m_1 \) and \( m_2 \) are obtained from the first two hydrostatic increments, as given in Eq. (3.53). The only unknown in Eq. (3.55) is \( \alpha_{pso} \), and this may be determined by solving Eq. (3.55).

It has to be noted that the above approach is valid as long as the initial anisotropy observed evolves and either grows or diminishes as the deformation progresses. If, however, initial anisotropy is of the kind that persists, regardless of deformation, more rigorous approaches to modeling are required. Persisting anisotropy indicates the existence of inherent preferred directions. This involves including additional non-evolving tensor valued state variables (unlike \( a_{ij} \) which evolves) in the formulation or representing \( F \) and \( Q \) in terms of scalar values which are form invariant only under certain groups of orthogonal transformations that reflect material symmetries (Dafalias, 1981; Smith, 1983; Truesdell and Noll, 1965; Eringen, 1962, 1967).
Elasto Plastic Constitutive Relations

Before a constitutive law can be used in a numerical procedure like the finite element method, it has to be expressed in terms of a constitutive matrix or tensor relating incremental stress $d\sigma_{ij}$ to the incremental strain $d\epsilon_{ij}$, as

$$d\sigma_{ij} = C_{ijkl}^{e-p} d\epsilon_{kl}$$  \hspace{1cm} (3.56)

where $C_{ijkl}^{e-p}$ is the elasto plastic constitutive tensor expressed in terms of elastic and plastic parameters. Assuming small strains, the total incremental strain $d\epsilon_{kl}$ can be linearly decomposed into elastic and plastic components as

$$d\epsilon_{kl} = d\epsilon_{kl}^{e} + d\epsilon_{kl}^{p}$$  \hspace{1cm} (3.57)

where the superscripts $e$ and $p$ refer to elastic and plastic components, respectively. The stress-elastic strain relationship may be expressed as

$$d\sigma_{ij} = C_{ijkl}^{e} d\epsilon_{kl}^{e}$$  \hspace{1cm} (3.58)

where $C_{ijkl}^{e}$ is the elastic constitutive tensor. Combining Eqs. (3.58) and (3.57), it is possible to write

$$d\sigma_{ij} = C_{ijkl}^{e} (d\epsilon_{kl} - d\epsilon_{kl}^{p})$$  \hspace{1cm} (3.59)
The flow rule describing plastic strain increments is given by Eq. (3.19) as

\[ \text{Eq. (3.19)} \]

\[ \text{d}e_{ij}^p = \lambda \left( \frac{\partial Q}{\partial \sigma_{ij}} \right) \]

**Virgin Loading.** During virgin loading, \( \lambda \) in Eq. (3.19) is obtained from the condition \( dF = 0 \) and is given by \( \lambda_v \) in Eq. (3.26).

From Eqs. (3.26), (3.19) and (3.59), it is possible to write

\[ \lambda = \lambda_v = - \frac{\partial F_c}{\partial \sigma_{ij}} \frac{C_{ijkl} \text{d}e_{kl}}{\left| \frac{\partial Q}{\partial \sigma_{rs}} \right|} \]

(3.60)

and \( c_{ijkl}^e \) in Eq. (3.56) as

\[ c_{ijkl}^e = c_{ijkl}^{e-p} - c_{ijkl}^e \left( \frac{\partial F_c}{\partial \sigma_{uv}} \frac{\partial Q}{\partial \sigma_{mn}} c_{mnkl} \right) \]

(3.61)

**Nonvirgin Loading.** During nonvirgin loading, \( \lambda \) in Eq. (3.19) is postulated based on an interpolation rule, Eq. (3.27). From Eqs. (3.27), (3.19) and (3.59), it is possible to write
\[ \lambda = \lambda_{nv} = \frac{\frac{\partial F}{\partial \sigma^{*}} C_{ijkl} \frac{\partial \sigma}{\partial x}}{\frac{\partial F}{\partial \sigma^{*}} C_{ijkl} \frac{\partial \sigma}{\partial x} - \frac{\partial F}{\partial \sigma} [1 + \Theta R]} \]  

(3.62)

and \( C_{ijkl}^{e-p} \) in Eq. (3.56) as

\[ C_{ijkl}^{e-p} = C_{ijkl} \]

\[ C_{ijuv} \frac{\partial Q}{\partial \sigma_{uv}} \frac{\partial F}{\partial \sigma^{*}} C_{mnkl} - \frac{\partial F}{\partial \sigma} [1 + \Theta R] \left| \frac{\partial Q}{\partial \sigma_{rs}} \right| \]

(3.63)

By multiplying the numerator and denominator of the second term on the right hand side of Eqs. (3.61) and (3.63), by \( \left| \frac{\partial Q}{\partial \sigma_{rs}} \right| \) and utilizing Eq. (3.20), it is possible to replace \( \frac{\partial Q}{\partial \sigma_{ij}} \) by \( \frac{\partial Q}{\partial \sigma_{ij}} \) in the above equations. In matrix notation, Eq. (3.61) and Eq. (3.63) may be written as follows:

\[ [C]_{\text{virgin}}^{e-p} = [C]^{e} - \frac{\frac{\partial F}{\partial \sigma} \left\{ \frac{\partial Q}{\partial \sigma} \right\}^{T}}{\left\{ \frac{\partial F}{\partial \sigma} \right\} [C]^{e} \left\{ \frac{\partial Q}{\partial \sigma} \right\} - \frac{\partial F}{\partial \sigma} \left| \frac{\partial Q}{\partial \sigma} \right|} \]

(3.64)
\[ [C]_{\text{nonvirgin}}^{\varepsilon-P} = [C]^{\varepsilon} \]

\[
[C]^{\varepsilon} \left\{ \frac{\partial Q}{\partial \sigma} \right\}_{\text{ps}}^{T} \frac{\partial F}{\partial \sigma} [C]^{\varepsilon} \nonumber
\]

\[
- \left( \frac{\partial F}{\partial \sigma} \right)^{T} \left\{ \frac{\partial Q}{\partial \sigma} \right\}_{\text{ps}}^{T} [C]^{\varepsilon} - \frac{\partial F}{\partial \zeta} (1 + \theta \frac{R}{\mathcal{R}}) \left| \frac{\partial Q}{\partial \sigma} \right| \nonumber
\]

where square brackets indicate a 6 x 6 matrix and the vectors are of 6 x 1 dimension. Superscript T denotes transpose.

The gradient of \( F \) with respect to \( \sigma_{ij} \) may be expressed as

\[
\frac{\partial F}{\partial \sigma_{ij}} = \frac{\partial F}{\partial J_{1}} \delta_{ij} + \frac{\partial F}{\partial J_{2}} \sigma_{ij} + \frac{\partial F}{\partial J_{3}} \sigma_{ik} \sigma_{kj} \quad (3.66a)
\]

or in terms of the deviatoric stress tensor \( S_{ij} \) as

\[
\frac{\partial F}{\partial \sigma_{ij}} = \frac{\partial F}{\partial J_{1}} \delta_{ij} + \frac{\partial F}{\partial J_{2}} S_{ij} + \frac{\partial F}{\partial J_{3}} (S_{ik} S_{kj}) - \frac{2}{3} J_{2D} \delta_{ij} \quad (3.66b)
\]

Similarly, the gradient of \( Q \) with respect to \( \bar{\sigma}_{ij} = \sigma_{ij} - a_{ij} \) may be written as follows:

\[
\frac{\partial Q}{\partial \bar{\sigma}_{ij}} = \frac{\partial Q}{\partial J_{1}} \delta_{ij} + \frac{\partial Q}{\partial J_{2}} \bar{\sigma}_{ij} + \frac{\partial Q}{\partial J_{3}} \bar{\sigma}_{ik} \bar{\sigma}_{kj} \quad (3.67a)
\]

or
\[
\frac{\partial Q}{\partial \tilde{\sigma}_{ij}} = \frac{\partial Q}{\partial J_1} \delta_{ij} + \frac{\partial Q}{\partial J_{2D}} \tilde{S}_{ij} + \frac{\partial Q}{\partial J_{3D}} \left( \bar{S}_{1k} \bar{S}_{kj} - \frac{2}{3} J_{2D} \delta_{ij} \right)
\]

(3.67b)

where \(\bar{S}_{ij}\) is the deviatoric part of \(\tilde{\sigma}_{ij}\).

As a result of the nonassociative formulation, the \([C]^{e-p}\) matrix is nonsymmetric. Special algorithms are developed to implement nonsymmetric constitutive matrices in finite element schemes in such a manner that only symmetric equation solvers are used (Desai and Hashmi, 1985). This involves splitting up the constitutive and/or stiffness matrix into a symmetric and nonsymmetric part and using an iterative procedure. Decomposition of the nonsymmetric matrix should be such that the convergence at the iterative procedure is assured.
CHAPTER 4

LABORATORY TESTING PROGRAM

Introduction

This chapter describes the testing device, sample preparation and the results of the test series on dense specimens of Leighton-Buzzard Sand. Tests are carried out for purposes of parameter determination, model verification and investigation of induced anisotropy.

Multiaxial or Truly Triaxial Cubical Test Device

The test apparatus is a stress-controlled truly triaxial device capable of applying a three-dimensional state of stress to a 4 x 4 x 4 inch (10.16 x 10.16 x 10.16 cm) cubical sample (Desai et al., 1981; Sture and Desai, 1979; Desai et al., 1982). An "exploded view" of the device is shown in Fig. 4.1.

The sample is installed in the cubical cavity of the reaction frame with six flexible membranes on each side. A uniform (normal) stress on each face of the soil sample is applied by pneumatically pressurizing the membranes. The applied pressure in each of the three directions is controlled independently. The loading is quasi-static. Deformations are measured at three discrete points on each of the six faces of the sample by means of linear variable differential transformer (LVDT) probes.
Figure 4.1. 'Exploded View' of Multiaxial Cubical Device
The air pressure system consists of an air compressor, pressure regulators and Bourdon tube gages and is manually operated. Data acquisition, storage and processing is handled by a Hewlett-Packard data acquisition system.

Conventional laboratory testing devices such as the direct shear device and cylindrical triaxial device have the disadvantage of permitting only limited modes of stress application and deformation. The truly triaxial device permits application of three independent principal stresses, \( \sigma_1 \), \( \sigma_2 \), and \( \sigma_3 \). By varying the principal stresses appropriately, the sample can be subjected to any stress path in the three-dimensional stress space. Some of the commonly used (straight line) stress paths for geologic materials are depicted in Fig. 4.2.

**Description of Soil and Sample Preparation**

Leighton-Buzzard Sand is a subrounded, close graded (U.S. Sieve 20-30) material with a specific gravity of 2.66. The tests were conducted on dry, dense sand specimens with a void ratio of 0.53 and relative density of 95%.

Soil samples for testing in the cubical device were prepared as described by Desai et al. (1982), Sture and Desai (1979), and Mould (1983).

The sample is made inside a membrane bag constructed by using 0.007 in. (0.018 cm) thick pure latex dental dam glued together with rubber cement. The cubical sample preparation mold serves as an external membrane form. The membrane is held to the sides of the form by
Figure 4.2. Schematic of the Commonly Used Stress Paths
an applied vacuum. Dense, dry samples were prepared by aerial raining through a height of about six feet (two meters). The sand was held in a reservoir, throttled through a 0.15 in. (0.38 cm) opening and allowed to fall freely through a large diameter steel tube into the sample mold. Homogeneity of the sample was ensured by moving the steel tube back and forth in a regular pattern over the mold. This technique (Mould, 1983) yielded reasonably reproducible samples with a void ratio of 0.53. Several samples prepared in the same manner showed remarkably consistent stress-strain response curves during hydrostatic loading.

A 0.0625 in. (0.158 cm) outer diameter teflon tube with a filter paper protecting the open end was placed on the leveled top surface of the sample before sealing the top with a piece of latex membrane. A regulated internal vacuum 1.0 psi (6.9 kPa) was applied through this tube to confine the specimen during dismantlement of the mold and transfer of specimen to the cavity of the multiaxial device.

Presentation and Discussion of Results

Background

This section describes the special series of tests carried out on Leighton-Buzzard Sand and presents the test results along with some discussion of the results.

In general, laboratory testing is carried out for purposes of observing behavior patterns, providing calibration data for the model and establishing results against which the model can be verified. The standard tests used for determination of constants of the model are
conventional triaxial compression (CTC), conventional triaxial extension (CTE), triaxial compression (TC), triaxial extension (TE) and simple shear (SS) tests carried out to failure and including a few unloading and reloading cycles. The test series described herein does not concentrate on a comprehensive set of 'standard' tests. Rather, this special test series is described to provide varied data for verification of the model and to investigate effects of induced anisotropy and stress history. Table 4.1 summarizes the various tests carried out.

All the tests described were performed on dry samples of sand. The cubical sample was transferred to the apparatus in such a manner that the vertical direction during sample preparation coincided with the vertical z-direction of the apparatus, Fig. 4.1. The specimen preparation technique involving aerial raining causes a slight depositional anisotropy in the vertical direction. Therefore, initially the sample is isotropic in the x-y plane, whereas a slight depositional anisotropy exists in the z-direction. The principal directions of loading (and deformation), \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) \((\varepsilon_1, \varepsilon_2, \varepsilon_3)\), are defined to coincide with the z-, x- and y-directions, respectively, of the apparatus. In all of the tests performed in the series, \( \sigma_x \) was maintained equal to \( \sigma_y \).

In the plots of the test results, test points are connected by straight line segments. Although actual stress strain response is best described by a smooth curve, the piecewise linear representation adapted here is considered adequate for the purpose of plotting (by using the computer). The test results indicate a strong transverse isotropy
Table 4.1. Description of Test Series on Leighton Buzzard Sand (1.0 psi = 6.89 kPa)

<table>
<thead>
<tr>
<th>Test No.</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>HC loading to 30.0 psi, with unload-reload cycle</td>
</tr>
<tr>
<td>2</td>
<td>RTE with initial continuing pressure of 30.0 psi</td>
</tr>
<tr>
<td>3</td>
<td>RTC with initial continuing pressure of 30.0 psi</td>
</tr>
<tr>
<td>4</td>
<td>At various stages of a single CTC-RTE cycle (initial confining pressure = 13.0 psi) the sample is subjected to small 1.0 psi HC load increments to determine level of anisotropy</td>
</tr>
<tr>
<td>5</td>
<td>Cyclic triaxial test with four CTC-RTE cycles from an initial confining pressure of 13.0 psi</td>
</tr>
<tr>
<td>6</td>
<td>A series of CTC load-unload sequences with confining pressures of 5.0 psi, 10.0 psi and 15.0 psi</td>
</tr>
</tbody>
</table>
Table 4.1.--Continued

<table>
<thead>
<tr>
<th>Test No.</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>A series of TE load-unload sequences at confining pressures of 5.0, 10.0 and 15.0 psi</td>
</tr>
<tr>
<td>8</td>
<td>A series of CTC, HC and TE stress paths. CTC segments from confining pressures of 10.0 and 15.0 psi. TE segment from 10.0 psi</td>
</tr>
<tr>
<td>9</td>
<td>A series of CTC, HC and TE stress paths. TE segments from confining pressures of 15.0 and 20.0 psi. CTC segment from confining pressure of 15.0 psi</td>
</tr>
</tbody>
</table>
during all the tests with $\varepsilon_2 = \varepsilon_3$. Therefore, in some of the plots the average of the two strains is evaluated and plotted as $\varepsilon_2 = \varepsilon_3$. In representing stress-strain response during shear loading, the strain origin corresponds to the state of strain at the isotropic stress state immediately prior to commencement of shear loading. The various stress and strain measures appearing in the graphs in this chapter are defined below.

The sign convention adapted is that compressive stresses and strains are positive. All stresses are effective stress measures. During hydrostatic states of stress, the confining pressure is expressed as

$$\sigma_0 = \sigma_1 = \sigma_2 = \sigma_3$$

(4.1)

For general loading condition, stresses are frequently expressed in terms of an invariant quantity such as the octahedral shear stress defined as

$$\tau_{oct} = \frac{1}{3} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2}$$

(4.2)

Other stress parameters commonly used for triaxial test configurations ($\sigma_2 = \sigma_3$) are the mean principal stress $p$ and the deviator stress $q$ defined as
The volumetric strains are obtained from

\[ p = \frac{\sigma_1 + 2\sigma_3}{3} \]  \hspace{1cm} (4.3)

\[ q = \sigma_1 - \sigma_3 \]  \hspace{1cm} (4.4)

During triaxial test configurations \((\varepsilon_2 = \varepsilon_3)\), the shear strain is defined as

\[ \gamma = \varepsilon_1 - \varepsilon_3 \]  \hspace{1cm} (4.6)

The observed stress-strain response is usually plotted as \(\tau_{\text{oct}}\) versus \(\varepsilon_1, \varepsilon_2\) and \(\varepsilon_3\). The volumetric response is plotted in terms of the volumetric strain \(\varepsilon_v\) and \(\varepsilon_1\). The stress-strain results of cyclic triaxial tests are shown, as they are conventionally done, by plotting the stress ratio \(q/p\) against the shear strain \(\gamma\). The corresponding volumetric responses are depicted on \(q/p\) versus \(\varepsilon_v\) plots.

Discussion of Results

**Test 1.** Figure 4.3 shows the results of a hydrostatic compression (HC) test. It is seen that there is a small variation between the strains in the horizontal direction, \(\varepsilon_2\) and \(\varepsilon_3\), while the difference between the vertical and horizontal strains are somewhat larger. This
Figure 4.3. Stress-Strain Response for HC Test

\[ \rho \circ = 1.74 \text{ gm/cc} \]

\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
indicates a small depositional anisotropy during sample preparation. The degree of initial anisotropy may be quantified in terms of \( a_n \), Eq. (3.42), Chapter 3. From the measured stress-strain response, the value of \( a_n \) was found to be 0.05 at \( \sigma_0 = 2.0 \) psi (14 kPa). It gradually decreased to 0.04 at the pressure of \( \sigma_0 = 20.0 \) psi (138 kPa) and 0.03 at 30.0 psi (207 kPa). For an isotropic material, the value of \( a_n \) would have been zero. Thus, the samples used in this testing program have a small initial anisotropy. The level of anisotropy gradually decreases during the HC loading.

**Tests 2 and 3.** Figures 4.4a and 4.4b show the stress-strain response and volumetric response, respectively, during the reduced triaxial extension (RTE) test from the confining pressure of 30.0 psi (207 kPa). The results of a reduced triaxial compression (RTC) test from the confining pressure of 30.0 psi (207 kPa) are shown in Figs. 4.5a and 4.5b. Both tests involve hydrostatic compressive loading followed by unloading (nonvirgin loading) along different stress paths. Since nonvirgin loading is involved, the material response for these stress paths can be realistically backpredicted only by a model that allows for plastic behavior during unloading and reloading.

**Test 4.** Figures 4.6 and 4.7 illustrate the results of Test 4. It was carried out to investigate the correlation between \( a_n \) and measured anisotropic behavior. \( a_n \) was defined in Eq. (3.42), Chapter 3, to represent a relative measure of induced anisotropy and was correlated to the deviation from normality. It attempts to quantify the difference
\( \rho_0 = 1.74 \text{ gm/cc} \)
\( \sigma_0 = 30.0 \text{ psi} \)
\( 1.0 \text{ psi} = 6.89 \text{ kPa} \)

Figure 4.4a. Stress-Strain Response for RTE Test
\( \rho_o = 1.74 \text{ gm/cc} \)
\( \sigma_o = 30.0 \text{ psi} \)
\( 1.0 \text{ psi} = 6.89 \text{ kPa} \)

Figure 4.4b. Volumetric Response for RTE Test
Figure 4.5a. Stress-Strain Response for RTC Test

\[ \rho_o = 1.74 \text{ gm/cc} \]
\[ \sigma_o = 30.0 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 4.5b. Volumetric Response for RTC Test

\[ \rho_0 = 1.74 \text{ gm/cc} \]
\[ \sigma_0 = 30.0 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 4.6. Strain Increment Direction Due to Small Hydrostatic Stress Increments at Various Stages During a CTC-RTE Cycle

\[ \rho_0 = 1.74 \text{ gm/cc} \]
\[ \sigma_0 = 13.0 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 4.7. Comparison of $a_n$ and $\gamma$.
between the actual state of the material and the state required to produce the kind of response that an isotropic-associative model would predict.

An absolute measure of anisotropy may be found by subjecting the material to a (small) HC load increment and by measuring the strain increments in the three principal directions. If the response in all of the three directions were the same \( \varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} \), the material may be considered to be in an isotropic state. Therefore, any shear strains such as \( \varepsilon_{11} - \varepsilon_{22} = \gamma \) caused by a hydrostatic load increment may be considered indicative of the level of anisotropy in the material before the HC load was applied.

In this test, the sample was isotropically compressed to a pressure of 13.0 psi (90 kPa) and then subjected to a CTC-RTE cycle. The stress path followed and the schematic of the stress-strain curve are shown in Fig. 4.6. During various stages of the test, loading along the CTC-RTE stress path was halted and the material was subjected to an HC stress increment \( \sigma_0 = 1.0 \) psi (7.0 kPa). The incremental strains \( \varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33} \) caused by this stress increment were measured. The stress increment was then removed and the original stress path was resumed. This process was carried out at fourteen points during the cycle, as indicated in Fig. 4.6. It was found that at each one of these points \( \varepsilon_{11} \neq \varepsilon_{22} = \varepsilon_{33} \), indicating transverse isotropy. Since elasticity is isotropic, the elastic components of the stress increment are
equal; \( \Delta\varepsilon_1^e = \Delta\varepsilon_2^e = \Delta\varepsilon_3^e \). The direction of the plastic strain increments caused by HC loading at these points may be expressed as \( \tan^{-1} \left\{ \frac{(\Delta\varepsilon_1^p - \Delta\varepsilon_2^p)}{\Delta\varepsilon_1^p} \right\} \), where superscript \( p \) indicates plastic component of the strain increment, \( \Delta\varepsilon_1^p = \Delta\varepsilon_1^p + \Delta\varepsilon_2^p + \Delta\varepsilon_3^p \) and \( \Delta\varepsilon_2^p = \Delta\varepsilon_3^p \). These directions are indicated at the corresponding stress points on the stress path plot in Fig. 4.6. It is seen that at point 1, prior to the commencement of shear loading, the plastic strain increment caused by an HC stress increment is almost purely volumetric and its direction is approximately the same as the direction of the HC stress increment. This indicates the material to be very close to the isotropic state at point 1. Once shear loading begins, anisotropy is induced in the material and this is reflected in the deviation of the plastic strain increment from the direction of \( \Delta\gamma \) at each of the subsequent points considered. At point 8, even though the sample has returned to the hydrostatic state of stress, the direction of plastic strain increment (arrow 8 in Fig. 4.6) indicates that the material is not in the isotropic state. This is due to the load-unload cycle. Arrows 1, 8 and 12 indicate three significantly different directions of the plastic strain increments \( \Delta\gamma \) from the same state of hydrostatic stress. This illustrates the effect of stress history and induced anisotropy on material behavior.

As mentioned earlier, the shear strains (\( \Delta\gamma \)) caused by the hydrostatic stress increments reflect the level of anisotropy in the material. If \( \Delta\gamma = 0 \), material is in an isotropic state. \( a_n \) is a strain
measure that is adopted as a measure of induced anisotropy. It reflects the difference between the current state of the material and a hypothetical 'fundamental' state corresponding to the 'isotropic-associative' model. Thus, a direct correlation between $a_n$ and an absolute level of anisotropy is difficult to establish. However, if $a_n$ were a valid measure of anisotropy, it should correlate with another appropriate and direct measure of anisotropy. Here, the plastic strains due to a small hydrostatic stress at a given point is adapted as the direct measure of anisotropy at that point.

Consider the segment of the CTC stress path, Fig. 4.6, ranging from $\sigma_1 - \sigma_3 = 0$ to $\sigma_1 - \sigma_3 = 43$ psi (296 kPa). This portion of the stress path is traversed three times during this test, during virgin loading, unloading and reloading following an RTE cycle, Fig. 4.6. At each of the test points in this segment, the value of $a_n$ prior to the HC stress increment is evaluated from Eq. (3.42). The corresponding value of $d\gamma$ caused by the HC stress increment is evaluated as $d\gamma = d\epsilon_1 - d\epsilon_2$. During each of the three loading sequences (virgin loading, unloading and reloading) considered, the values of $a_n$ and $d\gamma$ are compared. To facilitate the comparison, the two quantities are normalized with respect to their respective maximum values $(a_n)_{\text{max}}$ and $(d\gamma)_{\text{max}}$ during each sequence and are plotted with respect to $\sigma_1 - \sigma_3$. Figures 4.7a, 4.7b and 4.7c show these plots during virgin loading, unloading and reloading, respectively. It is seen from all three plots that both
\( \frac{a_n}{(a_n)_{\text{max}}} \) and \( \frac{d\gamma}{(d\gamma)_{\text{max}}} \) show similar trends, indicating that the indirect measure of anisotropy \( a_n \) shows good correlation with the direct measure based on hydrostatic loads at various points.

**Test 5.** Figure 4.8a shows the stress-strain response of the material to cyclic triaxial loading. The response is expressed in terms of the stress ratio, \( q/p \), versus the shear strain \( \gamma \). Figure 4.8b shows the volumetric response, in terms of \( q/p \) versus volumetric strain, \( \varepsilon_v \), for the same test. In this test, the sample was first isotropically compressed to 13 psi (90 kPa) and then subjected to four CTC-RTE cycles. The stress application was quasi-static as in all the other tests run on this device. Thus, no significant loading rate effects are involved. The stress amplitudes in both the compression and extension directions were increased slightly for each cycle in order to observe the effect of amplitude on cyclic response. Since the sand used for the test is in a dense condition, very little compression takes place due to cyclic loading, Fig. 4.8b. As the stress amplitude increases, the material begins to dilate and large magnitudes of dilatant volumetric strains are observed. The cyclic test was performed to verify the ability of the model to predict response of materials to cyclic loading; the verification is presented in Chapter 6.

**Tests 6 to 9.** Tests 6, 7, 8 and 9 were performed to study the evolution of \( a_n \) with deformation and to observe the effect of induced anisotropy and stress history on subsequent material behavior under hydrostatic and shear loading.
Figure 4.8a. Stress-Strain Response for Cyclic Triaxial Test

-6 -4 -2 0 0.5 2
-2 -1 0 1 2
\( q/\rho \)

\( \rho_0 = 1.74 \text{ gm/cc} \)
\( \sigma_0 = 13.0 \text{ psi} \)
1.0 psi = 6.89 kPa
Figure 4.8b. Volumetric Response for Cyclic Triaxial Test

\[ \rho_0 = 1.74 \text{ gm/cc} \]
\[ \sigma_0 = 13.0 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 4.9 illustrates the stress path in the principal stress space, adapted for Test 6. The sample was hydrostatically compressed to $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_o = 5.0$ psi (34 kPa) and then loaded along the CTC stress path up to a stress ratio, $\sigma_1/\sigma_o = 3.0$. It was then unloaded along the CTC path to the hydrostatic state and then loaded hydrostatically to a confining pressure of $\sigma_o = 10$ psi (69 kPa). This load-unload-NC reload process was repeated from confining pressures of 10 psi (69 kPa) and 15 psi (103 kPa). During the second and third CTC loading sequences, loading was carried out to stress ratios of $\sigma_1/\sigma_o = 3.5$ and $\sigma_1/\sigma_o = 4.0$, respectively.

The values of $a_n$ (Eq. 3.42) evaluated at various stages during the test are displayed in Fig. 4.9. It is seen that the initial level of anisotropy represented by an $a_n$ value of 0.05 is small in comparison with the levels of anisotropy induced due to the shear loading. When the material is initially subjected to shear loading, the value of $a_n$ increases. Each load-unload cycle results in a net change in the value of $a_n$. During the hydrostatic loading, the value of $a_n$ tends to diminish, indicating a gradual demise of anisotropy induced previously by the shear loading. Figures 4.10a and 4.10b show the stress-strain and volumetric responses during the CTC loading-unloading segments of the tests. The level of maximum preload preceding each CTC sequence is indicated in the figures. All the responses are plotted from a common
All numbers shown are values of $a_n$ unless otherwise indicated.

Figure 4.9 Stress Path for Test 6
Max. prestress ($\sigma_1/\sigma_o$)  \quad \sigma_o
\begin{align*}
a. & \quad 0.0 \quad 5.0 \text{ psi} \\
b. & \quad 3.0 \quad 10.0 \text{ psi} \\
c. & \quad 3.5 \quad 15.0 \text{ psi}
\end{align*}

$\rho_o = 1.74 \text{ gm/cc}$

1.0 psi = 6.89 kPa

Figure 4.10a. Stress-Strain Response During CTC Segments of Test 6
Figure 4.10b. Volumetric Response During CTC Segments of Test 6
strain origin. The results shown in these figures and the behavior observed during the HC portions of the test are utilized later in this discussion for comparison with other test results.

Figure 4.11 illustrates the stress path adapted for Test 7. Here the sample is subjected to a series of TE load-unload cycles from confining pressures of 5, 10 and 15 psi (34, 69 and 103 kPa). Extension loading at these confining pressures was carried out to stress ratios of \( \sigma_1/\sigma_o = 0.4, 0.2 \) and 0.133, respectively. Once again the values of \( a_n \) at various stages of the test are evaluated and displayed in the figure. The remarks concerning \( a_n \) made in the previous section are applicable for this test as well. The stress-strain and volumetric responses of the sample during the TE loading-unloading cycles are shown in Figs. 4.12a and 4.12b, respectively. The response during hydrostatic loading is considered in the latter part of this section.

The stress paths adapted in Tests 8 and 9 are depicted in Figs. 4.13 and 4.14, respectively. It is seen that some of the stress path segments of Tests 6 and 7 are duplicated in these tests. However, the duplicated portions of the stress path are now preceded by different types and levels of preloading. In Test 8 (Fig. 4.13), the sample is loaded hydrostatically to 10 psi (69 kPa), then is subjected to a CTC stress path up to a stress ratio \( \sigma_1/\sigma_o = 3.5 \), unloaded to the hydrostatic stress level, reloaded along the TE path to \( \sigma_1/\sigma_o = 0.2 \) and unloaded again to the hydrostatic level at \( \sigma_o = 10 \) psi. The sample is
All numbers shown are values of $a_n$ unless otherwise indicated.

Figure 4.11 Stress Path for Test 7

$\sigma_0 = 20.0 \text{ psi}$

$\sigma_2 = 10.0 \text{ psi}$

$\sigma_3 = 5.0 \text{ psi}$

$\sigma_0/\sigma_0 = 0.133$

$\sigma_1/\sigma_0 = 0.2$

$\sigma_1/\sigma_0 = 0.4$

$1.0 \text{ psi} = 6.89 \text{ kPa}$
Figure 4.12a. Stress-Strain Response During TE Segments of Test 7
Figure 4.12b. Volumetric Response During TE Segments of Test 7

Max. prestress
\( \left( \frac{\sigma_1}{\sigma_o} \right) \)
\( \sigma_o \) (psi)
a 0.0 5.0
b 0.4 10.0
c 0.2 15.0
d 0.13 20.0

\( \rho_o = 1.74 \text{ gm/cc} \)
1.0 psi = 6.89 kPa
All numbers shown are values of $a_n$ unless otherwise indicated.

Figure 4.13 Stress Path for Test 8

1.0 psi = 6.89 kpa
All numbers shown are values of $a_0$ unless otherwise indicated.

Figure 4.14 Stress Path for Test 9
then loaded hydrostatically to \( \sigma_0 = 15 \) psi (104 kPa), loaded along the CTC path to \( \sigma_1/\sigma_3 = 4.0 \), unloaded to the hydrostatic level and reloaded hydrostatically to \( \sigma_0 = 20 \) psi (138 kPa).

The stress path used in Test 9, Fig. 4.14, consisted of the following series of paths: HC loading to \( \sigma_0 = 15 \) psi (104 kPa), CTC loading to \( \sigma_1/\sigma_0 = 4.0 \), unloading to the hydrostatic level, HC loading to \( \sigma_0 = 20 \) psi (140 kPa), HC unloading to \( \sigma_0 = 15 \) psi (104 kPa), TE loading to \( \sigma_1/\sigma_0 = 0.133 \), TE unloading, HC loading to \( \sigma_0 = 20 \) psi (140 kPa) and TE loading from 20 psi. The stress paths in Tests 8 and 9 were planned in this manner so as to provide sufficient data to compare with the results from Tests 7 and 8. Results from Tests 8 and 9 are displayed in conjunction with earlier test results in the figures that follow.

Effect of Stress History on HC Behavior. Figure 4.15 is a plot of confining pressure, \( \sigma_0' \), versus shear strain, \( \bar{\gamma} = \varepsilon_1 - \varepsilon_3 \), during hydrostatic loading from \( \sigma_0 = 15 \) to 20 psi (104 kPa to 140 kPa). This plot utilizes data from Tests 1, 6, 8 and 9. An isotropic material would exhibit no shear strains during hydrostatic loading. Thus, the response of an isotropic material would have plotted as a vertical line passing through \( \bar{\gamma} = 0 \) in Fig. 4.15. Thus, the magnitudes of shear strains produced due to the hydrostatic loading reflect the degree of anisotropy the material possessed initially. The larger the degree of
Figure 4.15. Effect of $a_n$ on HC Response Following Compression Preloading ($\sigma_o = 15$ to 20 psi)
anisotropy in the material, the larger the magnitudes of shear strain and the greater the deviation from the isotropic line in Fig. 4.15. The results from the hydrostatic compression test (Test 1) show a small deviation from the isotropic line. This is due to the small initial anisotropy represented by the initial $a_n$ value of 0.05. The samples in Tests 6, 8 and 9 were preloaded along the same compression loading (CTC) path to the same level of maximum preloading, $\sigma_1/\sigma_o = 4.0$. The level of maximum prestress governs the magnitudes of strains during subsequent loading. Since these three samples were prestressed to the same maximum state of stress, their responses may be compared from a common basis. However, due to differences in their exact stress history, each sample possessed a different value of $a_n$ immediately prior to hydrostatic loading from $\sigma_o = 15$ psi (104 kPa).

Samples from Tests 6, 8 and 9 had $a_n$ values of 2.09, 1.86 and 1.51, respectively, at $\sigma_o = 15$ psi (104 kPa). The effect the value of $a_n$ has on the subsequent behavior is apparent from Fig. 4.15. It is seen that for larger initial values of $a_n$ the behavior deviates further from the isotropic line. Thus, it can be concluded that $a_n$ is indeed a measure of induced anisotropy in this case.

Figure 4.16 once again is a plot of hydrostatic responses from a confining pressure of $\sigma_o = 15$ psi (104 kPa) to $\sigma_o = 20$ psi (140 kPa). Here the response of two samples (from Tests 7 and 9) that have been prestressed in extension loading (TE) to the same maximum prestress of
Figure 4.16. Effect of $a_n$ on HC Response Following Extension Preloading ($\sigma_o = 15$ to 20 psi)
$\sigma_1/\sigma_o = 0.133$ are compared. The response of the sample from Test 1, which reflects the 'virgin' behavior for this stress range, is also shown in the plot. The samples from Tests 7 and 9 are similar in every respect except that they have different values of $a_n = 2.21$ and 2.40, respectively, at $\sigma_o = 15$ psi (104 kPa). As a result, their behavior during hydrostatic loading is different. As in the previous case, their deviation from isotropic behavior is directly proportional to the initial values of $a_n$.

Figure 4.17 combines the plots from Figs. 4.15 and 4.16 to illustrate the contrasting effects of prestressing in compression and extension. The CTC preloading causes the material to yield negative values of $\gamma$ during hydrostatic loading; i.e., $d\varepsilon_3 > d\varepsilon_1$. Prestressing on the extension side has the opposite effect with $\gamma > 0.0$; i.e., $d\varepsilon_1 > d\varepsilon_3$ during subsequent hydrostatic loading.

The phenomena observed in Figs. 4.15 to 4.17 are apparent in Fig. 4.18 as well, which illustrates the hydrostatic response of samples from Tests 1, 7, 8 and 6, from $\sigma_o = 10$ to 15 psi (69 to 104 kPa). Samples from Tests 7 and 8 were subjected to a maximum TE prestress of $\sigma_1/\sigma_o = 0.2$. However, the sample from Test 7 has a smaller value of $a_n$ (= 2.31) at $\sigma_o = 10$ psi (69 kPa) than the sample from Test 8 which has $a_n = 2.40$ at $\sigma_o = 10$ psi. The sample with the larger initial (at 10 psi) value of $a_n$ exhibits greater anisotropic behavior than the
The numbers shown refer to values of $a_n$ at $\sigma_o = 15.0$ psi

$\rho_o = 1.74$ gm/cc

1.0 psi = 6.89 kPa

Figure 4.17. Effect of Preloading on HC Response ($\sigma_o = 15$ to 20 psi)
Figure 4.18. Effect of Preloading on HC Response ($\sigma_o = 10$ to 15 psi)
other. The sample from Test 8 was prestressed in compression to $\sigma_1/\sigma_o = 3.5$. It is seen that compressive prestressing forces the material to deviate from the isotropic line in the negative $\gamma$ direction, while extension prestressing causes deviation in the positive $\gamma$ direction.

**Effect of Stress History on Shear Behavior.** The next set of figures (Figs. 4.19 to 4.23) illustrate the effect of previous stress history on subsequent behavior in triaxial compression and extension. The results from Tests 6, 7, 8 and 9 are used in these figures.

Figures 4.19a and 4.19b illustrate the stress-strain and volumetric response, respectively, of two samples loaded from a confining pressure of $\sigma_o = 10$ psi (69 kPa) along the CTC stress path to the same stress level and then unloaded to the hydrostatic state. These two figures compare the behavior of a virgin sample to that of a sample prestressed in compression to $\sigma_1/\sigma_o = 3.0$. The prestressed sample shows a stiffer stress-strain response than the virgin sample. Both samples show very similar volumetric behavior.

Figure 4.20 a shows the stress-strain response of three different samples (from Tests 6, 7 and 9) during a CTC load-unload cycle from a confining pressure of $\sigma_o = 15$ psi (104 kPa). The corresponding volumetric responses are shown in Fig. 4.19b. The level of prestressing that each of the three samples were subjected to is indicated in the figure. The sample from Test 6 is in a virgin state, while samples from Tests 7 and 9 have been prestressed in compression ($\sigma_1/\sigma_o = 3.5$) and extension ($\sigma_1/\sigma_o = 0.2$), respectively. It is seen that compressive
0 - Virgin Sample (Test 8)
X - Prestressed to $\sigma_1/\sigma_0 = 3.0$ (Test 6)

$\rho_0 = 1.74 \text{ gm/cc}$
$\sigma_0 = 10.0 \text{ psi}$
$1.0 \text{ psi} = 6.89 \text{ kPa}$

Figure 4.19a. Effect on Preloading on CTC (10) Stress-Strain Response
Figure 4.19b. Effect of Preloading on CTC (10) Volumetric Response

- Virgin Sample (Test 8)
- Prestressed to $\sigma_1/\sigma_o = 3.0$ (Test 6)

- $\rho_o = 1.74$ gm/cc
- $\sigma_o = 10.0$ psi
- $1.0$ psi $= 6.89$ kPa
Figure 4.20a. Effect of Preloading on CTC (15) Stress-Strain Response

- Virgin Sample (Test 9)
- X - Prestressed to $\sigma_1/\sigma_o = 3.5$
  (Test 6)
- * - Prestressed to $\sigma_1/\sigma_o = 0.2$
  (Test 8)

$\rho_o = 1.74$ gm/cc
$\sigma_o = 15.0$ psi
1.0 psi = 6.89 kPa
0 - Virgin Sample (Test 9)
X - Prestressed to $\sigma_1/\sigma_o = 3.5$
  (Test 6)
* - Prestressed to $\sigma_1/\sigma_o = 0.2$
  (Test 8)

Figure 4.20b. Effect of Preloading on CTC (15) Volumetric Response

$\rho_o = 1.74 \text{ gm/cc}$
$\sigma_o = 15.0 \text{ psi}$
$1.0 \text{ psi} = 6.89 \text{ kPa}$
prestressing causes the material to exhibit a stiffer (than virgin) stress-strain response during subsequent compressive loading. On the other hand, prestressing in the extension direction causes the material to behave in a far less stiff manner during subsequent CTC loading. It is also apparent that the volumetric behavior of the virgin sample and the sample that was prestressed in compression are very similar. Extension prestressing, however, results in a big change in volumetric characteristics, as evidenced in Fig. 4.20b.

The results of TE tests on two samples, from a confining pressure of $\sigma_o = 10$ psi (69 kPa), are shown in Figs. 4.21a and 4.21b. One sample is prestressed in compression to $\sigma_1/\sigma_o = 3.5$ (Test 8), and the other prestressed in extension to $\sigma_1/\sigma_o = 0.4$ (Test 7), prior to this test. It is seen that the sample prestressed to $\sigma_1/\sigma_o = 0.4$ shows a stiffer stress-strain response than the sample prestressed to $\sigma_1/\sigma_o = 3.5$. Their volumetric responses are different too, with the former exhibiting more dilation than the latter.

Figures 4.22a and 4.22b show the results of two TE tests from a confining pressure of $\sigma_o = 15$ psi (104 kPa). One of the samples used in this test was prestressed in extension to $\sigma_1/\sigma_o = 0.2$ (Test 7), and the other in compression to $\sigma_1/\sigma_o = 4.0$ (Test 9). The remarks concerning the stiffness of the response and volumetric behavior made for Figs. 4.20a and 4.20b are equally applicable here.
Figure 4.21a. Effect of Preloading on TE (10) Stress-Strain Response
0 - Prestressed to $\sigma_1/\sigma_o = 0.4$
(Test 7)
X - Prestressed to $\sigma_1/\sigma_o = 3.5$
(Test 8)

$\rho_o = 1.74 \text{ gm/cc}$
$\sigma_o = 10.0 \text{ psi}$
$1.0 \text{ psi} = 6.89 \text{ kPa}$

Figure 4.21b. Effect of Preloading on TE (10) Volumetric Response
Figure 4.22a. Effect of Preloading on TE (15) Stress-Strain Response

0 - Prestressed to $\sigma_1/\sigma_o = 0.2$ (Test 7)
X - Prestressed to $\sigma_1/\sigma_o = 4.0$ (Test 9)

$\rho_o = 1.74 \text{ gm/cc}$
$\sigma_o = 15.0 \text{ psi}$
$1.0 \text{ psi} = 6.89 \text{ kPa}$
Figure 4.22b. Effect of Preloading on TE (15) Volumetric Response
From Figs. 4.19 to 4.22, the following conclusions can be made. The stress-strain response during a CTC test is stiffer when it is preceded by preloading in compression rather than preloading in extension. The volumetric behavior exhibits more dilation if preloading is in compression rather than in extension. In the case of triaxial extension tests, the response is stiffer and volumetric strains more dilatant when preloading is in extension rather than compression.
CHAPTER 5

DETERMINATION OF MATERIAL CONSTANTS

General

Once a constitutive law capable of simulating observed phenomena is formulated and its significant parameters are identified, it is necessary to determine them. These material parameters or constants are evaluated based on test results from appropriate laboratory tests.

The general three-dimensional constitutive law is first specialized to relevant test configurations and easily identifiable states of the material. Then, experimental data pertaining to these test configurations and states are substituted into the specialized forms of the constitutive law to determine the material constants. Test configurations commonly employed for soils include conventional triaxial compression and extension, triaxial compression and extension, simple shear, reduced triaxial extension and compression and hydrostatic compression, Fig. 4.2, Chapter 4. Major distinct states of the material that the model can be specialized to include ultimate, critical state and point of dilation.

It should be noted that the quantitative predictive capability of a model is governed by the quality, number, range and variety of the tests used for determination of the parameters. Thus, it is desirable
to have a comprehensive series of reliable tests carried out from various initial confining pressures along a variety of stress paths with unloading and reloading cycles.

When designing a test series for evaluation of parameters it is also important to keep in mind the type of boundary value problems that are to be solved because the test configurations should simulate all significant modes of loading and deformation expected in the boundary value problem. Although results from tests done on a truly triaxial device are preferred and used in this study, they are by no means a prerequisite for determination of model constants. Tests carried out on the conventional triaxial device can also be used.

The proposed constitutive model is, in general, applicable for any frictional material including sands, clays, rock and concrete. However, the parameter determination and verification carried out in this dissertation are limited to loose and dense sands. Three different sands are considered here. They are (i) "Munich" sand, (ii) Leighton Buzzard sand and (iii) Fuji River sand. The test data for "Munich" sand is obtained from Scheele and Desai (1984). Data for Leighton Buzzard sand was obtained from tests conducted by the author (Chapter 4) and those reported by Hashmi (under preparation). The source for Fuji River sand data is Tatsuoka and Ishihara (1974). A description of these sands, their physical properties and initial densities are provided later in this chapter.
Procedure for Determining Material Constants

The material constants associated with this model can be classified into the following categories:

1. Elastic Parameters, Eq. (3.58)
   \[ E \] and \[ \nu \] or \[ G \] and \[ K \]

2. Parameters for Ultimate Yielding, Eq. (3.3)
   \[ \gamma, \beta \] and \[ m \]

3. Shape Parameter, Eq. (3.3)
   \[ n \]

4. Parameters for the Growth Function, Eq. (3.8)
   \[ a_1, \eta_1 \]

5. Interpolation Parameters, Eqs. (3.27) and (3.28)
   \[ h_1, h_2, h_3 \]

6. Translation Parameter, Eq. (3.43)
   \[ h_4 \]

Detailed procedures for determining these parameters are described below.

1. Elastic Parameters

There are two independent constants for linear isotropic elasticity: Young's modulus \( E \) and Poisson's ratio \( \nu \). Instead of \( E \) and \( \nu \), the elastic parameters may also be expressed through the bulk modulus \( K \) and the shear modulus \( G \). The model assumes elastic behavior to occur whenever the state of stress is within the initial yield surface \( F_0 \) (enclosing the initial purely elastic region) or within the translating
surface, Q. This happens at the very early stages of loading from a stress-free state and at the early stages of unloading. Thus, the initial stages of hydrostatic loading and the initial stages of unloading from any stress path may be utilized for determination of elastic constants.

Usually, $E$ and $\nu$ are conveniently obtained from the initial unloading portion of a CTC test. The CTC test data is plotted as $(\sigma_1 - \sigma_3)$ vs. $e_1$ and $(\sigma_1 - \sigma_3)$ vs. $e_3$, as schematically shown in Fig. 5.1a. The slopes of the initial unloading portion of the curves give $E$ for the first plot and $E/\nu$ for the second plot, respectively, Fig. 5.1a. An alternative is to obtain $E$ as above and to obtain the bulk modulus $K$ from the slope of the initial unloading portion of a hydrostatic test plotted in the mean pressure versus volumetric strain space. $K$ is related to $E$ and $\nu$ through the following equation:

$$K' = \frac{E}{3(1-2\nu)} \quad (5.1)$$

If more than one test and if more than one instance of unloading is available, $E$ and $\nu$ are evaluated for each case and then averaged.

2. Parameters for Ultimate Yielding

The ultimate condition for a stress path test is defined as the asymptotic value of stress in the stress-strain curve, as schematically illustrated in Fig. 5.1. Failure and critical states could occur at or below the ultimate state. Parameter $\gamma$ governs the slope of the ultimate
Figure 5.1. Schematic Plot of Typical Stress-Strain and Volumetric Response
envelope and $\beta$ and $m$ govern the relative values of ultimate slope under different stress paths. As explained in Chapter 3, the value of $m$ is adapted as $-1/2$ for all materials.

As the ultimate state is approached, plastic strains and hence $\xi$ become large and the value of $\alpha$ approaches zero asymptotically. When $\alpha = 0$, the closed yield surfaces degenerate into an open ultimate envelope given by

$$J_{2D} - \gamma (1 - \beta S_\tau)^{-1/2} J_1^2 = 0 \quad (5.2)$$

For each available shear test carried out to ultimate conditions, the values of $J_1$, $J_{2D}$ and $S_\tau$ are known. Thus, there are only two unknowns in Eq. (5.2).

The results of two tests, along two different stress paths that have different values of $S_\tau$, are sufficient to determine $\gamma$ and $\beta$. For example, results of a compression test and an extension test are sufficient. However, if both stress paths have the same value of $S_\tau$ (as in the case of a CTC test and a TC test), the term $(1 - \beta S_\tau)^{-1/2}$ is common to both equations and only one independent equation results. Therefore, it is necessary to ensure that results of tests carried out along at least two different stress paths (in the octahedral plane) are available. Frequently the number of test results available outnumber the constants to be evaluated and the problem becomes overdetermine.
Several optimization techniques exist for the solution of such problems. A simple least-square procedure, described below, was used in this study.

Equation (5.2) may be written as

\[(1 - \beta S_r) J_{2D}^2 - \gamma^2 J_1^4 = 0\]  \hspace{1cm} (5.3a)

or

\[
\begin{bmatrix} S_r & J_{2D}^2 & J_1^4 \end{bmatrix} \begin{bmatrix} \beta \\ \gamma^2 \end{bmatrix} = J_{2D}^2
\]  \hspace{1cm} (5.3b)

Equation (5.3b) is evaluated for every available test. If N tests are used, the final set of equations may be assembled as

\[
[A] \begin{bmatrix} \beta \\ \gamma^2 \end{bmatrix} = \{\beta\}
\]  \hspace{1cm} (5.4)

where \([A]\) is a known N x 3 coefficient matrix and \(\{\beta\}\) is a known N x 1 vector. Equation (5.4) is solved using a least-square algorithm (Desai and Siriwardane, 1984) to obtain \(\beta\) and \(\gamma\).

3. Shape Parameter

The value of n governs the shape of the yield surface in \(J_1 - \sqrt{J_{2D}}\) space. For \(n = 2\), the surface is elliptical. As the value of n increases, the surface deviates more and more from the elliptical shape.
For the basic isotropic hardening model described in section 3.3, n is an important parameter. It determines the state of stress at which volume change is zero. This is the state at which dense materials begin to dilate and loose materials approach failure or ultimate states. For the general anisotropic model, n is no longer a dominant parameter since directions of plastic strain increments are governed by Q and its location rather than the shape of F.

For the basic isotropic hardening associative model, n is determined from the locus of zero volume change points in the stress space. At the zero volume change state

$$\frac{\partial F}{\partial J_1} = 0$$

which yields

$$(-\alpha n J_1^{n-1} + 2\gamma J_1) F_s = 0$$

and

$$J_1 = \frac{2\gamma}{\alpha n}$$

Substituting Eq. (5.5c) in the expression for F, Eq. (3.5), yields

$$J_{2D} = J_1^2 \gamma F_s \left(1 - \frac{2}{n}\right)$$
as the zero volume change state. By substituting the known zero volume change states for various stress paths in Eq. (5.6), the value of \( n \) for each test is obtained and then averaged.

The value of \( n \) can also be determined from the hydrostatic compression test. By specializing the flow rule, Eq. (3.23), Chapter 3, and the expression for \( F \), Eq. (3.5), for HC loading conditions, the following expression can be obtained for a hydrostatic load increment (Frantziskonis, Desai and Somasundaram, 1985).

\[
J_1 \, \varepsilon_{11}^p \, J_1^{n-2} = \sqrt{3} \, \gamma \, dJ_1 \, (n-2)
\]  

(5.7)

where \( dJ_1 \) is the increment of \( J_1 \) and \( \varepsilon_{11}^p \) is the increment of volumetric plastic strain. All other quantities being known in Eq. (5.7), \( n \) may be determined.

The value of \( n \) so obtained is adapted for the general anisotropic model as well. From experimental data on various sands, it is observed that the value of \( n \) lies between 2.5 and 4. Therefore, an assumed value of \( n \); e.g., \( n = 3 \), would be a significant simplification.

4. Hardening Parameters

The hardening parameters \( a_1 \) and \( \eta_1 \) are calculated based on the knowledge of stress-strain data during virgin loading along various stress paths. The growth function \( \alpha = a_1 / \varepsilon_1^{\eta_1} \), Eq. (3.8), may be written as
During virgin loading, the condition $F = 0$ is always satisfied. Since the state of stress at every point during virgin loading is known, using the above condition, the value of $\alpha$ at these points is evaluated. The corresponding value of $\xi$ at these points is calculated from the plastic strain history up to the present state of strain, using Eq. (3.6). For each available test, $-\ln(\xi)$ is plotted against $\ln(\alpha)$. The slope of the straight line that best fits this data gives the value of $\eta_1$ and its intercept along the ordinate is used to obtain $a_1$.

5. Interpolation Parameters

During nonvirgin loading, the interpolation rule postulated for $\lambda_{uv}$, Eq. (3.27), is controlled by the three interpolation parameters, $h_1$, $h_2$, and $h_3$. These parameters determine the magnitude of plastic strains within the prestress surface. They are determined from the unloading and reloading data for various shear stress paths.

From Eq. (3.27) and Eq. (3.19),

$$
\eta_1 \ln(\xi) + \ln(\alpha) = \ln(a_1) \quad (5.8)
$$

The magnitude of the plastic strain increment $||\varepsilon_{ij}^p||$ is obtained from the above equation as
During any unloading or reloading sequence, \( F_{ps} \) and \( \sigma_{ij}^{*} \) are known. For each stress increment, the corresponding plastic strain increment is known from the experimental stress-strain data, and, hence, \( H_1 \) can be evaluated. The value of \( \bar{R} \) for the particular stress increment is found from Eq. (3.29).

For every unloading or reloading path available, values of \( \ln(\bar{R}) \) are plotted against corresponding values of \( \ln(H_1) \). This results in a number of approximately parallel lines (shown later), each one corresponding to a particular nonvirgin path. The slopes of each one of the lines (which are found to be approximately equal) are obtained and averaged to give \( h_1 \). The ordinate intercept of each line is...
$\ln (\theta) = \ln (h_2) + h_3 \ln \left( \frac{\Delta_{ps}}{J_1} \right)$ \hspace{1cm} (5.13)

For each nonvirgin path considered, $\ln (\theta)$ is obtained from the previous plot as the ordinate intercept. The values of $\Delta_{ps}$ and $J_1^*$ for each path are known and, hence, $\ln (\Delta_{ps}/J_1^*)$ can be evaluated. The values of $\ln (\theta)$ are now plotted with respect to the corresponding values of $\ln (\Delta_{ps}/J_1^*)$. A straight line is fitted through the points so plotted. From Eq. (5.13), it is seen that the slope of this straight line gives $h_3$ while $h_2$ is determined from the ordinate intercept.

6. Translation Parameter

The translation parameter $h_4$ relates the measure chosen to represent induced anisotropy, $a_n$, to the deviation from normality, expressed as $\pi$, Eq. (3.43). $a_n$ is defined as $R_e - R_p$, Eq. (3.42). Equations (3.37) and (3.38) may be combined as

$$a_{ij} = (a_{ij})_{iso} + \pi \delta_{ij} + \nu_{ij}$$ \hspace{1cm} (5.14)
from which $\pi$ is obtained as

$$\pi = \frac{1}{3} [a_{ii} - (a_{ii})_{iso}]$$

(5.15)

From Eqs. (3.23) and (3.34), $(a_{ii})_{iso}$ may be expressed as

$$(a_{ii})_{iso} = J_1 [1 - (\alpha_0/\alpha_c)^{1/n-2}]$$

(5.16a)

and

$$(a_{ii})_{iso} = (J^0_{iso}) + ((J^*)_{iso} - (J^0_{iso})$$

(5.16b)

for virgin and nonvirgin loading, respectively. $a_{ij}$ represents the location of Q in stress space. At any point during a test, $a_{ij}$ can be found based on the current states of stress, strain, $\xi$ and the measured plastic strain increment for a given stress increment.

The procedure adapted here is valid for the particular case of an initially isotropic or transversely isotropic sample, subjected to hydrostatic loading followed by shear loading (and unloading or re-loading) along a fixed triaxial stress path. This implies that during
the shear deformation process, \( \sigma_{11} \neq \sigma_{22} = \sigma_{33}, \varepsilon_{11} \neq \varepsilon_{22} = \varepsilon_{33} \) and \( d\varepsilon_{11} \neq d\varepsilon_{22} = d\varepsilon_{33} \). Under these conditions, induced anisotropy would also exhibit transverse tendencies with \( a_{11} \neq a_{22} = a_{31} \). As a result, \( \bar{\sigma}_{11} \neq \bar{\sigma}_{22} = \bar{\sigma}_{33} \). Therefore, during a particular monotonic loading (virgin or nonvirgin) sequence, the stress ratio corresponding to \( \bar{\sigma}_{1j} \),

\[
\bar{S}_r = \frac{J_{3D}}{J_{2D}}^{1/3} \quad \text{is a constant.}
\]

The direction of incremental plastic strain given by \( \partial Q/\partial \bar{\sigma}_{1j} \) may be written as, Eq. (3.21),

\[
\frac{\partial Q}{\partial \bar{\sigma}_{1j}} = \frac{\partial Q}{\partial \bar{J}_1} \frac{\partial \bar{J}_1}{\partial \bar{\sigma}_{1j}} + \frac{\partial Q}{\partial \bar{J}_{2D}} \frac{\partial \bar{J}_{2D}}{\partial \bar{\sigma}_{1j}} + \frac{\partial Q}{\partial \bar{S}_r} \frac{\partial \bar{S}_r}{\partial \bar{\sigma}_{1j}} \tag{5.17}
\]

Since \( \bar{S}_r \) is a constant for the specialization considered here, \( \partial Q/\partial \bar{S}_r = 0 \) and Eq. (5.17) reduces to

\[
\frac{\partial Q}{\partial \bar{\sigma}_{1j}} = \frac{\partial Q}{\partial \bar{J}_1} \delta_{ij} + \frac{\partial Q}{\partial \bar{J}_{2D}} \bar{S}_{ij} \tag{5.18}
\]

From the expression for \( Q \), Eq. (3.17b)

\[
\frac{\partial Q}{\partial \bar{J}_1} = -F_s (-\alpha \bar{J}_1 n - 1 + 2\gamma \bar{J}_1) \tag{5.19}
\]
Now using the flow rule, Eq. (3.19) and Eq. (3.20), the ratio $\frac{d\epsilon_{ij}^{P}}{d\epsilon_{ij}^{P}/(d\epsilon_{ij}^{P})}$ can be expressed as

$$\frac{d\epsilon_{ij}^{P}}{(d\epsilon_{ij}^{P})} = k_{1} = \frac{\lambda^{2}}{\left|\frac{\partial Q}{\partial \sigma_{rs}}\right|^{2}} \left(\frac{\partial Q}{\partial \sigma_{rs}}\right)^{2}$$

(5.21)

Substituting Eq. (5.18) in Eq. (5.21)

$$\frac{d\epsilon_{ij}^{P}}{d\epsilon_{ij}^{P}/(d\epsilon_{ij}^{P})} = \frac{3}{9} \left(\frac{\partial Q}{\partial J_{1}}\right)^{2} + \frac{\partial Q}{\partial J_{2D}}^{2}$$

(5.22)

Using Eqs. (5.19) and (5.20) in Eq. (5.22), the following equation is obtained:

$$2\bar{J}_{2D} - (9k_{1} - 3) F_{s}^{2} \left[ -n \alpha_{o} \bar{J}_{1}^{n-1} + 2\gamma\bar{J}_{1} \right]^{2}$$

(5.23)

At this time, the condition that the stress point always lies on the boundaries of $Q$ during plastic flow is invoked. This is stated as

$$Q = 0 = \bar{J}_{2D} - (-\alpha_{o} \bar{J}_{1}^{n} + \gamma\bar{J}_{1}^{2}) F_{s}$$

(5.24)

Both Eqs. (5.23) and (5.24) may be rearranged as
Equations (5.25) and (5.26) are simultaneous equations in two variables, \( \bar{J}_{2D}/\bar{J}_1^2 \) and \( \bar{J}_1^{n-2} \). They are solved to give the values of \( \bar{J}_1 \), \( \bar{J}_{2D} \) and, hence, that of \( \sigma_{ij} \). The location of Q corresponding to the observed plastic strain increment is now found as \( a_{ij} = \sigma_{ij} - \bar{\sigma}_{ij} \).

Knowing the value of \( a_{ij} \), \( \bar{\pi} \) may now be determined from Eqs. (5.15) and (5.16). Since the state of elastic and plastic strains are known at any point during the loading sequence, \( a_n = R_e - R_p \) can be determined at the points for which \( \pi \) is known. From all available tests, values of \( a_n \) are now plotted with respect to the values of \( \pi \) normalized with respect to the \( \bar{J}_1 \) intercept of surface Q. The normalized values of \( \pi \) are given by \( \bar{\pi} = -3 \left( \frac{\alpha}{\gamma} \right)^{1/n-2} \).

As explained in Chapter 3, the relationship between \( \bar{\pi} \) and \( a_n \) is idealized as linear. When the material is in an isotropic state, it exhibits no deviation from normality. Therefore, when \( a_n = 0 \), \( \bar{\pi} = 0 \).

\[
2 \frac{\bar{J}_{2D}}{\bar{J}_1^2} - (9k_1 - 3) F_s \left[ -n \alpha_0 \bar{J}_1^{n-2} + 2\gamma \right]^2 = 0 \tag{5.25}
\]

and

\[
\frac{\bar{J}_{2D}}{\bar{J}_1^2} - F_s (-\alpha_0 \bar{J}_1^{n-2} + \gamma) = 0 \tag{5.26}
\]
and the straight line relating $a_n$ to $\overline{\tau}$ passes through the origin. Hence, a straight line passing through the origin is fitted to the $a_n - \overline{\tau}$ plot. $h_4$ is determined from the slope of this line, Eq. (3.43).

Comments on Determination of Material Constants

Parameters $\gamma$, $\beta$, $h$, $a_1$ and $\eta_1$ are sufficient to define the basic isotropic hardening model. A computer code for the evaluation of these parameters was developed by Frantziskonis (under preparation). Further details on the determination of these basic parameters, performance of the basic model and the extensions and modifications of the model may be found in Hashmi (under preparation).

This dissertation concentrates on the development and verification of the general anisotropic nonassociative model. Therefore, most of the information presented in the rest of this chapter is more relevant to the translation and interpolation parameters. The basic model parameters ($\gamma$, $\beta$, $h$, $a_1$ and $\eta_1$) for the various soils considered here were obtained using the code developed by Frantziskonis (1986).

**Computer Code.** A computer procedure has been developed to aid the determination of parameters $h_1$, $h_2$, $h_3$ and $h_4$. The inputs to the program are the discretized stress-strain data, the stress path followed, the elastic constants ($E$, $\nu$) and the material constants for the basic isotropic hardening model.

The program determines every plastic strain increment and the states of total plastic strain, total elastic strain and $a_n$ prior to
each increment. From the direction of incremental plastic strains, the actual position of $Q$, $a_{ij}$ and its deviation from the 'isotropic-associative' position, $(a_{ij})_{iso}$, is determined. The program then gives as output the plot of $a_n$ vs. a normalized value of $\pi$.

During each nonvirgin loading (unloading or reloading) sequence, the magnitudes of the plastic strain increment and the directions of the stress increment are utilized to obtain $H_1$, Eq. (5.11), and $\bar{R}$. The program plots these values in the form of $\ln (H_1)$ vs. $\ln (\bar{R})$. The ordinate intercept of each one of these plots, $\ln (\theta)$, is then plotted with respect to $\ln (\Delta_p s / J_1^*)$, which is determined from a knowledge of the prestress surface and the current stress path. The translation parameter $h_4$ is now determined from the first plot. Interpolation parameter $h_1$ is evaluated from the second plot. The last plot yields the values of $h_2$ and $h_3$.

Size of $Q \equiv F_0$. As mentioned in Chapter 3, the size of $Q$ is adapted as the size of $F_0$ which defines the initial elastic region. If a distinct elastic limit is available from laboratory tests, corresponding stresses are substituted in the expression for $F$, Eq. (3.5), to obtain $\alpha_0$. The hardening parameter, $\alpha_0$, defines the size of $F_0$ and is assumed to represent the size of $Q$ as well. If such a distinct state cannot be identified, it is desirable to choose a state of stress in the
early stages of the stress-strain response, to define the elastic limit, so that an appropriate size of \( Q \) is obtained; a very small size of \( Q \) may cause computational difficulties. However, the size of \( Q \) does not have a significant effect on magnitudes and directions of plastic strain increments. In this study, \( Q \) is evaluated by assuming that a hydrostatic loading with \( \sigma_0 = 0.2 \) psi (1.4 kPa) defines the elastic limit. This results in a \( Q \) surface with a \( J_1 \) intercept of 0.6 psi (4.1 kPa). For "Munich" sand (described later), for example, the corresponding value of the hardening parameter is evaluated as \( \alpha_0 = \alpha_Q = 0.32 \).

**Materials Considered**

"Munich" Sand

This is a coarse to medium, well graded sand with approximately round grains and with about 3% silty particles. It is named "Munich" sand after the area of its origin near Munich, West Germany. The soil has specific gravity of 2.758 and a coefficient of uniformity \( C_u \) of 9.0. The minimum and maximum dry densities of the material are 1.69 t/m\(^3\) and 2.07 t/m\(^3\). The tests were carried out on medium dense samples at an initial density of about 2.03 t/m\(^3\) (\( D_r = 70\% \)) and water content of about 4% (the natural moisture content of the sand).

The tests were performed on the truly triaxial cubical device described in Chapter 4 and are reported in Scheele and Desai (1983). The tests used for the determination of material constants consist of an
HC test carried out to 70 psi (482 kPa), CTC tests from initial confining pressures of 6.5 psi (415 kPa) and 13 psi (90 kPa), TE, TC and SS tests at an initial confining pressure of 13 psi (90 kPa), and a CTE test from an initial confining pressure of 30 psi (207 kPa). The tests included unloading and reloading cycles. These test results are reproduced from Scheele and Desai (1983) in Appendix B.

The elastic parameters E and ν for "Munich" sand are obtained (as schematically illustrated in Fig. 5.1a) from the initial unloading portions of the two CTC tests and averaged.

The ultimate states of stress for the CTC, TC, TE and SS tests are substituted in Eqs. (5.3) and (5.4) to obtain the values of the ultimate parameters β and γ. The states of stress at the zero volume change point (point at which dilation initiates) for each of the above tests are identified, based on the stress-strain and volumetric response curves. Then, Eq. (5.6) is employed to find the corresponding values of n. These values of n are averaged to give the shape parameter for "Munich" sand.

Figure 5.2 shows the plot of \(-\ln (\xi)\) vs. \(\ln (\alpha)\) obtained from tests under four different stress paths for the "Munich" sand. Only the virgin loading portions of the stress-strain curve are used for this purpose. The hardening parameters \(n_1\) and \(a_1\) are obtained from the slope and ordinate intercept, respectively, of the straight line that is fitted through the points on the plot, Eq. (5.8).

Figure 5.3 illustrates the plot of normalized values of \(\pi (\overline{\pi})\) against \(a_n\). This plot utilizes both virgin and nonvirgin stress-strain
Figure 5.2. Plot of $-\ln(\xi)$ vs. $\ln(\alpha)$ from tests under different stress paths for "Munich" sand. *Numbers indicate initial confining pressure in psi (1 psi = 6.89 kPa).
Figure 5.3. Plot of $\bar{\tau}$ vs. $a_n$ for "Munich" Sand
data from a number of tests on "Munich" sand. Although there appears to be some scatter, a straight line passing through the origin is fitted to this data. The slope of this line yields the value of the translation parameter $h_4$, Eq. (3.43).

Figure 5.4 shows plots of $\ln (R)$ vs. $\ln (H_1)$ for several non-virgin loading sequences under different stress paths. Each line on this plot refers to a particular unloading or reloading sequence. As mentioned earlier, all the lines are approximately parallel. The average slope of these lines gives $h_1$, Eq. (5.12). The ordinate intercept of each one of the lines corresponds to the value of $\ln (\theta)$ for that particular nonvirgin loading sequence, Eq. (5.12).

The values of $\ln (\theta)$ obtained from Fig. 5.4 are plotted with respect to corresponding values of $\ln (\Delta_{ps}/J_1^*)$ in Fig. 5.5. Each point in Fig. 5.5 corresponds to a particular line in Fig. 5.4. Parameters $h_2$ and $h_3$ are obtained from the ordinate intercept and slope, respectively, of the straight line that best fits these points, Eq. (5.13).

The material constants for "Munich" sand are listed in Table 5.1 along with the constants for the other soils considered.

Leighton Buzzard Sand

A description of the physical properties of this sand is given in Chapter 4. Data for Leighton Buzzard sand is obtained from tests performed in this study and by Hashmi (under preparation).
Figure 5.4. Plot of $\ln (H_1)$ vs. $\ln (\bar{R})$ for "Munich" Sand

Numbers in parentheses indicate initial confining pressure in psi

$\rho_0 = 2.03 \text{ gm/cc}$

1.0 psi = 6.89 kPa
Figure 5.5. Plot of $\theta$ vs. $\ln \left( \frac{\Delta_{ps}}{J_1^*} \right)$ for "Munich" Sand.

$\rho_0 = 2.03 \text{ gm/cc}$
Tests carried out in this study were on samples of initial relative density $D_r = 95\%$. Samples were prepared by means of an aerial raining technique described in Chapter 4. On the other hand, the tests reported by Hashmi (1986) were on samples of initial relative density $D_r = 87\%$, prepared using vibratory compaction. Since the two types of samples considered have different initial densities, their behavior is different and they possess different material constants. From here onwards, the sand of higher initial density, tested by the author, is referred to as Leighton Buzzard I, while the sand of lower density is identified as Leighton Buzzard II.

**Leighton Buzzard I.** These samples were prepared using the aerial raining technique (described in Chapter 4) to give an initial density of 1.74 gm/cc ($D_r = 95\%$). The test results used for determination of material constants are shown in Chapter 4.

The elastic parameters for Leighton Buzzard (I) sand are obtained as the average values evaluated from the stress-strain behavior during the initial unloading portions of the CTC segments of Test 6, Fig. 4.10a, Chapter 4.

The ultimate parameters $\beta$ and $\gamma$ are obtained by using the ultimate stress states of the RTE test (Fig. 4.4), RTC test (Fig. 4.5) and the final TE segment of Test 7 (Figs. 4.11, 4.12) in Eqs. (5.3) and (5.4). The zero volume change points during the CTC segments of Test 6 (Fig. 4.10) and the TE segments of Test 7 (Fig. 4.12) are used in Eq. (5.6) to determine an average value for the shape parameter $n$. The
virgin loading portions of the first CTC segments of Test 6 (Figs. 4.9, 4.10) and Test 5 (Fig. 4.8) are used to plot $-\ln (\xi)$ against $\ln (\alpha)$, Fig. 5.6. The slope and ordinate intercept of the straight line through these points yield the values of the hardening parameters $n_1$ and $a_1$, Eq. 5.8.

Figure 5.7 illustrates the plot of normalized $n$ vs. $a$ from three different stress paths. The stress paths used were the first CTC segment of Test 6 (Figs. 4.9, 4.10), CTC loading and unloading segments of Test 5 (Fig. 4.8) and the first TE segment of Test 7 (Figs. 4.11, 4.12). The translation parameter is obtained from the slope of the straight line that best fits these points and passes through the origin.

Figure (5.8) shows a plot of $\ln (H)$ against $\ln (R)$ for three CTC unloading segments of Test 5 (Fig. 4.8) and the first CTC unloading segment of Test 6 (Figs. 4.9, 4.10). The average slope of these lines gives the value of $h_1$, Eq. (5.12). Figure 5.9 shows the plot of $\ln (\theta)$ with respect to $\ln (\Delta ps / J^*)$ for the above tests. Parameters $h_2$ and $h_3$ are evaluated from the ordinate intercept and slope, respectively, of the straight line that best fits these points, Eq. (5.13).

**Leighton Buzzard II.** A description of the sample preparation techniques used and the results of tests on these samples may be found in Hashmi (1986). Dry Leighton Buzzard II samples were tested from an initial density of 1.71 gm/cc ($D_r = 87\%$).
Figure 5.6. Plot of $-\ln (\xi)$ vs. $\ln (\alpha)$ for Leighton Buzzard I

$\rho_o = 1.74$ gm/cc
1.0 psi = 6.89 kPa

$0 - \text{CTC (5)}$
$X - \text{CTC (13)}$

Numbers in parentheses indicate initial confining pressures in psi.
Figure 5.7. Plot of $\bar{\pi}$ vs. $a_n$ for Leighton Buzzard I

Numbers in parentheses refer to initial confining pressure in psi

$0$ CTC (5)
$X$ TE (5)
$\Delta$ CTC (10)
$\Box$ CTC (13)

$\rho = 1.74 \text{ gm/cc}$
$1.0 \text{ psi} = 6.89 \text{ kPa}$

$P$ Q 1.
Figure 5.8. Plot of $\ln (H_i)$ vs. $\ln (R)$ for Leighton Buzzard I

Numbers in parentheses refer to initial confining pressure in psi

$\rho_o = 1.74$ gm/cc

1.0 psi = 6.89 kPa
Figure 5.9. Plot of \( \ln (\Theta) \) vs. \( \ln \left( \frac{\Delta_{ps}}{J_1^*} \right) \) for Leighton Buzzard I

\[ \rho_o = 1.74 \text{ gm/cc} \]
The test results used for evaluation of parameters are the following: an HC test to a confining pressure of 40 psi (276 kPa), a CTC test from an initial confining pressure of 5 psi (34 kPa), CTC, TC and TE tests from an initial confining pressure of 13 psi (90 kPa) and a TE test from an initial confining pressure of 20 psi (138 kPa). All the tests included unloading and reloading cycles.

The elastic parameters for the model were determined from the results of the CTC test, as outlined in previous sections. The parameters $\beta$, $\gamma$, $n$, $a_1$ and $n_1$ of the basic isotropic model are obtained from Hashmi (1986). They were evaluated using the results of the above-mentioned tests.

Figure 5.10 shows the plot for determination of the translation parameter $h_4$. The virgin and nonvirgin loading sequences of all of the above tests are used for this plot. Figures 5.11 and 5.12 depict the plots for determination of the interpolation parameters. The stress-strain responses during the unloading and reloading portions of the above tests are utilized for these two plots. $h_1$ is obtained from Fig. 5.11, while $h_2$ and $h_3$ are obtained from Fig. 5.12, as outlined in the previous section.

Parameters for Leighton Buzzard sand I and II are displayed in Table 5.1.

Fuji River Sand

This sand, secured from the Fuji River bed near Tokyo, Japan, is described as subangular in grain shape with effective particle size $D_{10}$. 
Numbers in parentheses refer to initial confining pressure in psi

\[ \rho_0 = 1.71 \text{ gm/cc} \]

\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]

Figure 5.10. Plot of \( \pi \) vs. \( a_n \) for Leighton Buzzard I
Figure 5.11. Plot of $\ln (H_i)$ vs. $\ln (R)$ for Leighton Buzzard II.

Numbers in parentheses refer to initial confining pressure in psi.

- CTC (5)
- X CTC (13)
- $\Delta$ TC (13)
- $\Box$ TE (15)
- $\ast$ TE (20)

$\rho_o = 1.71$ gm/cc

1.0 psi = 6.89 kPa
Figure 5.12. Plot of $\ln (\theta)$ vs. $\ln (\Delta_{ps}/J^*_1)$ for Leighton Buzzard II

$\rho_o = 1.71 \text{ gm/cc}$
of 0.22 mm (.056 in) and specific gravity of 2.68. The maximum and minimum void ratios are 1.08 and 0.53, respectively. The uniformity coefficient is 2.21. The test results used here are obtained from drained conventional triaxial tests conducted on loose saturated samples of sand 10 cm in height and 5 cm in diameter. Loose specimens with a void ratio of 0.74 were obtained by spooning freshly boiled sand into a mold filled with drained water. Details of these tests are reported in Tatsuoka and Ishihara (1974).

The available test results for samples of this density were limited to a CTC test, an RTE test and a few CTC-RTE cycles, all from an initial confining pressure of 28.4 psi (196 kPa). These test results are reproduced in Appendix B.

In the absence of HC data, the volumetric strains at the beginning of shear loading were estimated on the basis of the value of the coefficient of compressibility \( C_c \). The approximate value of \( C_c \) for loose Fuji River sand is given as 0.042 by Ghaboussi and Momen (1982). Thus, a lot of confidence cannot be placed on the values of parameters obtained for the Fuji River sand since the data available for parameter determination was not sufficient.

The elastic parameters for Fuji River sand are obtained from the initial CTC unloading portions of the cyclic triaxial test. The ultimate parameters are evaluated from the ultimate conditions of the CTC and RTE tests. The value of the shape parameter \( n \) was assumed to be equal to 3.0.
The hardening parameters are evaluated from a plot of $-\ln (\xi)$ vs. $\ln (a)$ for the CTC and RTE tests, Fig. 5.13. The translation parameter $h_4$ is obtained as the slope of the line that best fits the points in the $\bar{\mu}$ vs. $a_n$ plot, Fig. 5.14. This plot utilizes stress-strain data from the CTC test and the CTC unloading segments of the cyclic triaxial test. It is seen that this plot exhibits a considerable degree of scatter.

Figure 5.15 shows the values of $\ln (H_1)$ plotted with respect to $\ln (\overline{\bar{\mu}})$ for four CTC unloading segments of the cyclic triaxial test. The average slope of these lines gives $h_1$. The ordinate intercept of these lines are plotted against $\ln (\Delta_{ps}/J_1^*)$ in Fig. 5.15. Parameters $h_2$ and $h_3$ are obtained by fitting a straight line through the points in this plot.

**Comments.** The material constants corresponding to Fuji River sand are shown in Table 5.1. It is interesting to note that the translation parameter $h_4$ for Fuji River sand is negative whereas it has positive values for the other three sands. This is due to the fact that $a_n$ steadily decreases from an initial value of zero as the sample is sheared, Fig. 5.14. For the other materials considered here, $a_n$ initially drops below zero but increases to positive values as the shearing progresses, Figs. 5.3, 5.7 and 5.10.
Figure 5.13. Plot of $-\ln (\xi)$ vs. $\ln (\alpha)$ for Fuji River Sand
Figure 5.14. Plot of $\bar{\omega}$ vs. $a_n$ for Fuji River Sand

- $0$ CTC loading
- $\times$ 1st unloading
- $\Delta$ 2nd unloading
- $\Box$ 3rd unloading

$\rho_o = 1.54 \text{ gm/cc}$
$\sigma_o = 28.4 \text{ psi}$
$1.0 \text{ psi} = 6.89 \text{ kPa}$
Figure 5.15. Plot of $H_i$ vs. $\ln (R)$ for Fuji River Sand

- $\rho_o = 1.54$ gm/cc
- $\sigma_o = 28.4$ psi
- 1.0 psi = 6.89 kPa
Figure 5.16. Plot of $\ln(\theta)$ vs. $\ln(\Delta_{ps/J_1^*})$ for Fuji River Sand

$\rho_o = 1.54 \text{ gm/cc}$
The Fuji River sand considered here is in a loose state and it shows compressive volumetric characteristics during shear. The other three sands, however, have high values of initial density. During shear they initially compress but begin dilating as shearing continues.

On the basis of this evidence, it is possible to speculate that negative values of $a_n$ during shear indicate compressive volumetric behavior, whereas positive values of $a_n$ indicate dilating behavior and that, as a consequence, the translation parameter $h_4$ is positive for dilating materials and negative for compressive materials. It should be noted, however, that more verification with respect to other loose and dense materials is required before such conclusions can be made.
Table 5.1 Constants for Different Materials
(All constants, except where indicated, are dimensionless)

<table>
<thead>
<tr>
<th>Constants</th>
<th>Munich Sand</th>
<th>Leighton Buzzard Sand I</th>
<th>Leighton Buzzard Sand II</th>
<th>Fuji River Sand</th>
</tr>
</thead>
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<tr>
<td><strong>Elastic:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E$ psi (kPa)</td>
<td>$8.49 \times 10^4$</td>
<td>$15.0 \times 10^4$</td>
<td>$15.0 \times 10^4$</td>
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<td></td>
<td>($5.85 \times 10^5$)</td>
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<td>($10.24 \times 10^5$)</td>
<td>($4.49 \times 10^5$)</td>
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<td>0.29</td>
<td>0.29</td>
<td>0.30</td>
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<td></td>
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<tr>
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<td>0.0965</td>
<td>0.0958</td>
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<td>0.747</td>
<td>0.612</td>
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<td>-0.5</td>
<td>-0.5</td>
<td>-0.5</td>
</tr>
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</tr>
<tr>
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<td>2.5</td>
<td>2.5</td>
<td>3.0</td>
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<tr>
<td>$a_1$</td>
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<td>$3.10 \times 10^{-1}$</td>
<td>$2.62 \times 10^{-7}$</td>
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<td>2.00</td>
<td>1.54</td>
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<tr>
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<td></td>
</tr>
<tr>
<td>$h_4$ psi (kPa)</td>
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<td>0.55</td>
<td>0.33</td>
<td>-0.60</td>
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<tr>
<td></td>
<td>($2.76$)</td>
<td>($3.80$)</td>
<td>($2.30$)</td>
<td>($-4.13$)</td>
</tr>
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</table>
CHAPTER 6

VERIFICATION OF MODEL

This chapter presents verification of the proposed model with respect to laboratory test data for three different sands. Sands with both high initial density ("Munich" sand, Leighton Buzzard sand) and low initial density (Fuji River sand) are considered. A description of these soils and the values of their material constants are given in Chapter 5.

The model is used to backpredict the stress-strain and volumetric responses of these materials under a variety of stress paths. These backpredictions are then compared with the behavior observed in the laboratory. Test configurations considered include straight line paths with unloading and reloading cycles, cyclic loading and complex paths consisting of a number of different straight line stress path segments.

Various other aspects of the model are also verified in this chapter. A comparison is made between the predictions of the basic isotropic-associative model and the general anisotropic nonassociative model. As mentioned in Chapters 3 and 5, a sensitivity study is undertaken to investigate the effect of parameter \( h_A \) on the model predictions. The procedure outlined in Chapter 3 is used to evaluate the level of initial anisotropy caused by sample preparation, and the
effect of incorporating initial anisotropy into the model is shown. The sensitivity study and comparisons with the basic isotropic associative model are carried out with respect to "Munich" sand. Effects of initial anisotropy are investigated for the Leighton Buzzard sand.

Predictions are compared with experimental observations by plotting them on the same graph. Both experimental and predicted points on the plot are connected by straight lines. This piecewise linear representation is assumed to be an adequate approximation of the actual nonlinear response. The various stress and strain measures adopted for the plots are defined in Chapter 4. The stress-strain responses are represented in terms of the octahedral shear stress \( \tau_{\text{oct}} \) vs. the major and minor principal strains \( \varepsilon_1 \) and \( \varepsilon_2 \), or in terms of the stress ratio \( q/p \) vs. the shear strain \( \gamma \). The volumetric responses are plotted either as \( \varepsilon_v \) vs. \( \varepsilon_1 \) or as \( q/p \) vs. \( \varepsilon_v \).

Some of the test results that are backpredicted were used in determining material constants; some were not. The test results that were utilized for evaluation of material constants are described in Chapter 5 and Appendix B. Model predictions are made by using a numerical integration procedure described in the next section.

Integration Routine

The proposed constitutive law may be expressed in terms of an explicit relationship between incremental stress and incremental strain as
In this form, it is possible to calculate the stress increment for a given strain increment at any given stage of the deformation process. In order to predict the stress-strain response during stress-controlled tests, it is more appropriate to compute strain increments from stress inputs. Therefore, Eq. (6.1) is written in its inverse form as

\[ d\epsilon_{ij} = D^{e-p}_{ijkl} d\sigma_{kl} \]  \hspace{1cm} (6.2a)

where \( D^{e-p}_{ijkl} \) is the elastoplastic compliance tensor, obtained by inverting \( C^{e-p}_{ijkl} \). In matrix notation, Eq. (6.2a) is written as

\[ \{d\epsilon\} = [D]^{e-p} \{d\sigma\} \]  \hspace{1cm} (6.2b)

The vectors \( \{d\epsilon\} \) and \( \{d\sigma\} \) have six independent components and \([D]^{e-p}\) is a 6 x 6 matrix. In general, the components of \([D]^{e-p}\) depend on the current state of stress and strain. During nonvirgin loading, the direction of the stress path also influences \([D]^{e-p}\). The detailed expressions for the derivation at the constitutive matrix, Eq. 6.2, are given in Chapter 3 and Appendix A.

The relationship between total stress and strain may be obtained by integrating Eq. (6.2b) along the loading path as

\[ \{\epsilon\} = \{\epsilon_o\} + \int \{d\epsilon\} = \int [D]^{e-p} \{d\sigma\} + \{\epsilon_o\} \]  \hspace{1cm} (6.3)
where \( \{\varepsilon_0\} = \text{initial state of strain.} \)

A computer code was developed to numerically integrate the above relationship. Numerical integration is carried out using the explicit forward difference procedure given by

\[
\{\varepsilon\} = \sum_{i=1}^{N} \{\Delta\varepsilon\} - \sum_{i=1}^{N} [C]^{e-p} \{\Delta\sigma\}_i + \{\varepsilon_0\} \tag{6.4}
\]

where \( N \) denotes the total number of linear segments used to approximate the actual nonlinear stress-strain response. The accuracy of this operation depends on the size of the increments; smaller the increments, more accurate the results. However, it is necessary to achieve a trade-off between higher accuracy and computational cost by choosing an optimum size of stress increment.

Another factor that influences the magnitude of stress increment \( d\sigma \) is the size of the plastic potential surface \( Q \). Since \( Q \) translates in the stress space along with the state of stress \( \mathcal{\bar{\sigma}} \) and governs the directions of plastic flow, it is reasonable to assume that \( d\mathcal{\bar{\sigma}} \) has to be small in comparison to the size of \( Q \). One measure of the size of \( Q \) is the length of its \( J_1 \) intercept \( \Delta_Q \). The value of \( \Delta_Q \) is obtained from the material constants as

\[
\Delta_Q = (\gamma/\alpha_0)^{1/n-2} \tag{6.5}
\]

\( \Delta_Q \) has units of stress. Based on a number of numerical experiments, the following criterion for the maximum step size was found to provide reasonable accuracy.
\[
\max ||(d\sigma)|| \leq 0.20 \Delta Q
\] (6.6)

This criterion is followed throughout the dissertation. For example, for the "Munich" sand, \(\Delta Q\) was found to be 0.5 psi (3.5 kPa); therefore, the maximum step size was restricted to 0.1 psi (0.7 kPa).

It should be noted that when the model is used in a finite element program, the increments of strain are known at a given incremental stage, and the constitutive matrix is used to obtain the corresponding stress increments. Then, it becomes necessary to numerically integrate Eq. (6.1) instead of Eq. (6.2). The integration scheme may then be expressed as

\[
\{\sigma\} = \{\sigma_0\} + \sum_{i=1}^{N} \{\Delta \sigma\}_i = \{\sigma_0\} + \sum_{i=1}^{N} [C]^{e-p}_{i-1} \{d\sigma\}_i
\] (6.7)

where \(\{\sigma_0\}\) is the initial state of stress,

**Verification for "Munich" Sand**

This section contains comparisons between predictions of the general anisotropic model and the isotropic associative model, a sensitivity study on parameter \(h_4\), and backpredictions of tests carried out on "Munich" sand in the cubical triaxial device. The initial density and material constants for "Munich" sand are given in Table 5.1.

**Comparison With Isotropic-Associative Model**

As mentioned in Chapter 3, the general anisotropic-nonassociative model is developed by modifying and building up on a
basic isotropic hardening associative model. The isotropic-associative model is the simplest in the hierarchy and is defined by the parameters \( \gamma, \beta, m, n, a_1 \) and \( \eta_1 \). The general anisotropic model is defined by the above parameters and an additional set of four constants: \( h_4 \), the translation parameter, and \( h_1, h_2, h_3 \), the interpolation parameters.

The basic model predicts associative plastic strains during virgin loading and elastic behavior during unloading and reloading. Therefore, the applicability of the basic model is limited to cases where loads increase monotonically.

The performance of the basic model and the general anisotropic model during virgin loading along several different stress paths are compared with the observed behavior. It should be noted that during virgin loading, the general model reduces to the isotropic-associative model if parameter \( h_4 \) is set equal to zero.

Figures 6.1 through 6.4 compare the observed stress-strain and volumetric responses with the predictions from the isotropic-associative model and the general anisotropic model. These comparisons are made during virgin loading for various stress paths. Figures 6.1a and 6.1b refer to a CTC test from a confining pressure of 6.5 psi (45 kPa). Figures 6.2 (a and b), 6.3 (a and b) and 6.4 (a and b) show the results for (another) CTC, TC and TE tests, respectively. These three tests had an initial confining pressure of 13.0 psi (90 kPa).
Figure 6.1a. Isotropic and Anisotropic Model Predictions of Stress-Strain Response for CTC (6.5) Test on "Munich" Sand

- observed
- isotropic model prediction
- anisotropic model prediction

\[ \rho_0 = 2.03 \text{ gm/cc} \]
\[ \sigma_0 = 6.5 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 6.1b. Isotropic and Anisotropic Model Predictions of Volumetric Response for CTC (6.5) Test on "Munich" Sand

- observed
- isotropic prediction
- anisotropic prediction

\( \rho_o = 2.03 \, \text{gm/cc} \)
\( \sigma_o = 6.5 \, \text{psi} \)
\( 1.0 \, \text{psi} = 6.89 \, \text{kPa} \)
Figure 6.2a. Isotropic and Anisotropic Model Predictions of Stress-Strain Response for CTC (13) Test on "Munich" Sand

\[ \rho_0 = 2.03 \text{ gm/cc} \]
\[ \sigma_0 = 13.0 \text{ psi} \]

1.0 psi = 6.89 kPa
Figure 6.2b. Isotropic and Anisotropic Model Predictions of Volumetric Response for CTC (13) Test on "Munich" Sand

- observed
--- isotropic prediction
- anisotropic prediction

\[ \rho_o = 2.03 \text{ gm/cc} \]
\[ \sigma_o = 13 \text{ psi} \]
1.0 psi = 6.89 kPa
Figure 6.3a. Isotropic and Anisotropic Model Predictions of Stress-Strain Response for TC Test on "Munich" Sand

\( \rho_0 = 2.03 \text{ gm/cc} \)

\( \sigma_0 = 13.0 \text{ psi} \)

1.0 psi = 6.89 kPa
Figure 6.3b. Isotropic and Anisotropic Model Predictions of Volumetric Response for TC Test on "Munich" Sand

- observed
- isotropic prediction
- anisotropic prediction

\( \rho_0 = 2.03 \text{ gm/cc} \)
\( \sigma_0 = 13.0 \text{ psi} \)
\( 1.0 \text{ psi} = 6.89 \text{ kPa} \)
Figure 6.4a. Isotropic and Anisotropic Model Predictions of Stress-Strain Response for TE Test on "Munich" Sand

\[ \rho_0 = 2.03 \text{ gm/cc} \]
\[ \sigma_0 = 13.0 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 6.4b. Isotropic and Anisotropic Model Predictions of Volumetric Response for TE Test on "Munich" Sand

- observed
- isotropic prediction
- anisotropic prediction

\[ \rho_o = 2.03 \text{ gm/cc} \]
\[ \sigma_o = 13.0 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
The points marked on the plots refer to experimental observations; the discontinuous curve represents the predictions from the basic isotropic-associative model, while the solid line depicts predictions from the general anisotropic model. It is seen from these figures that the stress-strain responses predicted by the two models are somewhat different. The predictions of the anisotropic model match the observations more closely than the other model. The volumetric responses simulated by the two models are vastly different. The isotropic-associative model predicts excessive dilation whereas the anisotropic model predictions agree very well with observations.

These comparisons suggest that the proposed general model has the capacity to successfully simulate the virgin behavior of materials that exhibit nonassociative characteristics.

Sensitivity of Model to Parameter $h_4$

As mentioned in Chapters 3 and 5, the parameter $h_4$ is determined as the slope of the average straight line passing through the points on the $\bar{\tau}$ vs. $a_n$ plot. However, the points on the $\bar{\tau}$ vs. $a_n$ plot show considerable scatter for all the materials considered, as seen in Figs. 5.3, 5.6, 5.9 and 5.12 in Chapter 5. The degree of scatter warrants a study on the sensitivity of the model predictions to the value of $h_4$ chosen.

Figure 5.3 in Chapter 5 depicts the plot of $\bar{\tau}$ vs. $a_n$ for the "Munich" sand. It was assumed that these points may be approximated by
a straight line with slope $\frac{\bar{\pi}}{a_n} = 0.40 = h_4$. It should be noted that
the straight line relating $\bar{\pi}$ to $a_n$ has to pass through the origin in
order to allow for zero deviation from normality ($\bar{\pi} = 0$) when the level
of induced anisotropy is zero ($a_n = 0$).

For the purpose of the sensitivity study, two extreme values of
$h_4$ equal to 0.2 and 0.60 were chosen. These two values reflect a 50%
decrease and increase, respectively, about the actual value of $h_4 = 0.4$.
The two lines with slopes 0.2 and 0.6 envelop almost all of the points
on the $\bar{\pi}$ vs. $a_n$ plot.

Predictions of the stress-strain and volumetric responses re-
sulting from assumed values of $h_4 = 0.2$ and $h_4 = 0.6$ are compared in
Figs. 6.5 and 6.6. Figure 6.5a shows the stress-strain behavior pre-
dicted with those two values of $h_4$ and compares the predictions with the
observations for a CTC test from an initial confining pressure of
6.5 psi (45 kPa). It is seen that both values of $h_4$ give very similar
stress-strain predictions and that the predictions agree reasonably well
with the observed behavior.

Figure 6.5b shows the comparison of volumetric observations and
predictions for the same test. The volumetric behavior predicted with
the isotropic-associative model ($h_4 = 0$) is also displayed in this
figure for the purpose of comparison. It is seen that the effects of
changing $h_4$ are more pronounced in the case of the volumetric behavior
than in the case of the stress-strain response. Larger values of $h_4$
Figure 6.5a. Sensitivity Study on $h_4$: Stress-Strain Response for CTC (6.5) Test

- observed
- $h_4 = 0.2$
- $h_4 = 0.6$

$\rho_0 = 2.03$ gm/cc
$\sigma_0 = 13.0$ psi
1.0 psi = 6.89 kPa
Figure 6.5b. Sensitivity Study on $h_4$: Volumetric Response for CTC (6.5) Test

- - observed

$--- - h_4 = 0.2$

$---- - h_4 = 0.6$

$\rho_0 = 2.03 \text{ gm/cc}$

$\sigma_0 = 13.0 \text{ psi}$

$1.0 \text{ psi} = 6.89 \text{ kPa}$
Figure 6.6a. Sensitivity Study on $h_4$: Stress-Strain Response for CTC (13) Test

- observed
- $h_4 = 0.2$
- $h_4 = 0.6$

$\rho_o = 2.03 \text{ gm/cc}$
$\sigma_o = 13.0 \text{ psi}$
$1.0 \text{ psi} = 6.89 \text{ kPa}$
Figure 6.6b. Sensitivity Study on $h_4$: Volumetric Response for CTC (13) Test

- observed
- $h_4 = 0.2$
- $h_4 = 0.6$

\[ \rho_o = 2.03 \text{ gm/cc} \]
\[ \sigma_o = 13.0 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
result in less dilatant and more contractive behavior. For a large enough value of $h_4$, the model may predict no dilation at all. However, when compared to the prediction of the isotropic-associative model, the sensitivity of the proposed model to small charges in $h_4$ is relatively small. This is especially so during the lower ranges of strain ($\varepsilon_1 < 2-3\%$).

Figures 6.6a and 6.6b depict the results of a similar comparison for a CTC test carried out from an initial confining pressure of 13.0 psi (70 kPa). The comments made in the previous two paragraphs regarding Figs. 6.5a and 6.5b are equally valid here.

From this sensitivity study, it can be concluded that, for a $\pm 50\%$ change in the value chosen for parameter $h_4$, the predicted stress-strain responses were found to be only slightly sensitive; the predicted volumetric response was found to be somewhat more sensitive. For $h_4 < 0.4$, the predicted volumetric response is slightly more dilatant than the observations. For $h_4 > 0.4$, the predicted volumetric response is more contractive than the observed behavior. However, an average value of $h_4$ gives satisfactory predictions.

Backpredictions of Test Results

Comparisons between observations and model predictions of cubical triaxial tests along a variety of stress paths are presented in this section. The predictions include unloading and reloading behavior.
Figure 6.7a shows the comparison of the predicted and observed stress-strain response of a CTC test carried out from an initial confining pressure of 6.5 psi (45 kPa). Figure 6.7b illustrates the corresponding volumetric behavior. The comparisons include an unloading-reloading cycle. Figures 6.8 through 6.12 present the comparisons of stress-strain and volumetric behavior for a CTC test, a TC test, an SS test, a TE test and an RTE test, respectively. All of the tests had an initial confining pressure of 13.0 psi (90 kPa). Each test includes an unloading-reloading cycle.

It is evident from these comparisons that the proposed model is capable of qualitatively predicting the stress-strain and volumetric responses of the material during virgin and nonvirgin loading along a variety of stress paths from different values of initial confining pressure. The predicted stress-strain responses compare well with the observed behavior. However, there are some discrepancies between observed and predicted volumetric responses. For the two CTC tests, Figs. 6.6b and 6.7b, when the material is unloaded, the model predictions show some initial dilation before contracting. This results in the 'kinks' seen in the predictions, Figs. 6.6b and 6.7b. These kinks are not very apparent in the other predictions, Figs. 6.9b, 6.10b, 6.11b, 6.12b. For the TC test, Figs. 6.9a and 6.9b, $\varepsilon_1$ is underpredicted during the initial virgin loading. This discrepancy is carried over and is reflected in the comparisons of the unloading-reloading responses. A similar phenomenon is seen in the predictions of the SS
Figure 6.7a. Comparison of Stress-Strain Responses for CTC (6.5) Test ("Munich" Sand)

---

\( \rho_0 = 2.03 \text{ g/cc} \)
\( \sigma_0 = 6.5 \text{ psi} \)
\( 1.0 \text{ psi} = 6.89 \text{ kPa} \)
Figure 6.7b. Comparison of Volumetric Response for CTC (6.5) Test ("Munich" Sand)

- ---- Observed
- ___ Predicted

\[ \rho_o = 2.03 \text{ g/cm}^3 \]
\[ \sigma_o = 6.5 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 6.8a. Comparison of Stress-Strain Response for CTC (13) Test ("Munich" Sand)

$\rho_0 = 2.03 \text{ gm/cc}$

$\sigma_0 = 13 \text{ psi}$

$1.0 \text{ psi} = 6.89 \text{ kPa}$
Observed
Predicted

--- Figure 6.8b. Comparison of Volumetric Response for CTC (13) Test ("Munich" Sand)

\[
\begin{align*}
\rho_o &= 2.03 \text{ gm/cc} \\
\sigma_o &= 13 \text{ psi} \\
1.0 \text{ psi} &= 6.89 \text{ kPa}
\end{align*}
\]
Figure 6.9a. Comparison of Stress-Strain Response for TC Test ("Munich" Sand)

\( \rho_o = 2.03 \text{ gm/cc} \)
\( \sigma_o = 13 \text{ psi} \)
\( 1.0 \text{ psi} = 6.89 \text{ kPa} \)
Figure 6.9b. Comparison of Volumetric Response for TC Test ("Munich" Sand)

\[ \rho_0 = 2.03 \text{ gm/cc} \]
\[ \sigma_0 = 13 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 6.10a. Comparison of Stress-Strain Response for SS Test ("Munich" Sand)

\[ \rho_0 = 2.03 \text{ gm/cc} \]
\[ \sigma_0 = 13 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 6.10b. Comparison of Volumetric Response for SS Test ("Munich" Sand)

\[ \rho_o = 2.03 \text{ gm/cc} \]
\[ \sigma_o = 13 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 6.11a. Comparison of Stress-Strain Response for TE Test ("Munich" Sand)

\[
\rho_o = 2.03 \text{ gm/cc} \\
\sigma_o = 13 \text{ psi} \\
1.0 \text{ psi} = 6.89 \text{ kPa}
\]
Figure 6.11b. Comparison of Volumetric Response for TE Test ("Munich" Sand)

\[ \rho_0 = 2.03 \text{ gm/cc} \]
\[ \sigma_0 = 13 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 6.12a. Comparison of Stress-Strain Response for RTE Test ("Munich" Sand)

- Observed
- Predicted

\[ \rho_0 = 2.03 \text{ gm/cc} \]
\[ \sigma_0 = 13 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 6.12b. Comparison of Volumetric Response for RTE Test ("Munich" Sand)

\[ \rho_0 = 2.03 \text{ gm/cc} \]
\[ \sigma_0 = 13 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
test, Figs. 6.10a and 6.10b, where $\varepsilon_1$ is slightly underpredicted and $\varepsilon_2$ slightly overpredicted.

**Verification For Leighton Buzzard Sand**

Leighton Buzzard sand at two different densities is considered here. Verification with respect to samples of density 1.71 gm/cc takes the form of backpredicting the results of cubical triaxial tests under various stress paths, reported by Hashmi (1986). Verification with respect to Leighton Buzzard sand of density 1.74 gm/cc involves backprediction results of the series of tests presented in Chapter 4. The material parameters for sands of both densities are given in Table 5.1 in Chapter 5. As mentioned in Chapter 5, the denser sample tested by the author is referred to as Leighton Buzzard I while the other is identified as Leighton Buzzard II.

**Backpredictions for Leighton Buzzard II**

This section presents the comparisons between predictions and observations of cubical triaxial tests carried out along a variety of stress paths from different values of confining pressure. All the tests include unloading and reloading cycles.

Figures 6.13 (a and b) and 6.14 (a and b) display the comparisons between observed and predicted stress-strain and volumetric behavior for two CTC tests. The two tests were conducted from confining pressures of 5.0 psi (34.5 kPa) and 13.0 psi (90 kPa), respectively. Figures 6.15a and 6.15b show the comparisons for a TC test from an
Figure 6.13a. Comparison of Stress-Strain Response for CTC (5) Test (Leighton Buzzard II)

\[ \rho_0 = 1.71 \text{ gm/cc} \]
\[ \sigma_0 = 5 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 6.13b. Comparison of Volumetric Response for CTC (5) Test (Leighton Buzzard II)

\( \rho_0 = 1.71 \text{ gm/cc} \)

\( \sigma_0 = 5 \text{ psi} \)

1.0 psi = 6.89 kPa
Figure 6.14. Comparison of Stress-Strain Response for CTC (13) Test (Leighton Buzzard II)

\[ \rho_0 = 2.03 \text{ gm/cc} \]
\[ \sigma_0 = 13 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 6.14.--Continued

$\rho_0 = 1.71 \text{ gm/cc}$

$\sigma_0 = 13 \text{ psi}$

$1.0 \text{ psi} = 6.89 \text{ kPa}$
Figure 6.15a. Comparison of Stress-Strain Response for TC Test (Leighton Buzzard II)

- Observed
- Predicted

\[ \rho_0 = 1.71 \text{ gm/cc} \]
\[ \sigma_0 = 13 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 6.15b. Comparison of Volumetric Response for TC Test (Leighton Buzzard II)

- Observed
- Predicted

$\rho_o = 1.71 \text{ gm/cc}$

$\sigma_o = 13 \text{ psi}$

1.0 psi = 6.89 kPa
Figure 6.16a. Comparison of Stress-Strain Response for SS Test (Leighton Buzzard II)
Figure 6.16b. Comparison of Volumetric Response for SS Test (Leighton Buzzard II)

\( \rho_0 = 1.71 \text{ gm/cc} \)

\( \sigma_0 = 20 \text{ psi} \)

1.0 psi = 6.89 kPa
initial confining pressure of 20.0 psi (138 kPa). Comparisons for two
TE tests with initial confining pressures of 13.0 psi (90 kPa) and
20.0 psi (138 kPa) are depicted in Figs. 6.17 (a and b) and 6.18 (a and
b), respectively.

It is seen that the model predicts the observed stress-strain
and volumetric responses of the material during virgin loading, un-
loading and reloading reasonably well. Overall, the predictions for
Leighton Buzzard II appear to be slightly better than the predictions
for "Munich" sand.

The stress-strain and volumetric response predictions for the
first CTC test ($\sigma_o = 5.0$ psi) compare well with the observations,
Figs. 6.13a, 6.13b. During unloading, a slight increase and then a de-
crease in dilation is observed; during reloading, there is a slight
drop and then a steady increase in dilation, Fig. 6.13b. This gives
rise to a small loop or kink in the volumetric response curve. The pre-
diction, Fig. 6.13b, follows a very similar trend. For the second CTC
test ($\sigma_o = 13.0$ psi), observed volumetric behavior, Fig. 6.14b, during
the unloading and reloading cycles is similar to the volumetric response
observed during the previous test. Once again the model predictions
show the same trends. During the second unloading-reloading cycle, how-
ever, the predicted volumetric response shows a larger 'loop' than the
observed response. From the stress-strain response curves for this
test, Fig. 6.14a, it is seen that $\varepsilon_2$ and $\varepsilon_3$ are slightly overpredicted.

As a result, more dilation than observed is predicted, Fig. 6.14b.
Figure 6.17a. Comparison of Stress-Strain Response for TE (13) Test (Leighton Buzzard II)

--- Observed
--- Predicted

\( \rho_o = 1.71 \text{ gm/cc} \)
\( \sigma_o = 13 \text{ psi} \)
1.0 psi = 6.89 kPa
Figure 6.17b. Comparison of Volumetric Response for TE (13) Test (Leighton Buzzard II)

--- Observed
--- Predicted

\[ \rho_0 = 1.71 \text{ gm/cc} \]

\[ 
\sigma_0 = 13 \text{ psi} \\
1.0 \text{ psi} = 6.89 \text{ kPa}
\]
Figure 6.18. Comparison of Stress-Strain Response for TE (20) Test (Leighton Buzzard II)

$\rho_o = 1.71 \text{ gm/cc} \quad \sigma_o = 13 \text{ psi} \quad 1.0 \text{ psi} = 6.89 \text{ kPa}
Observed
Predicted

\( \rho_0 = 1.71 \) gm/cc
\( \sigma_0 = 20 \) psi
1.0 psi = 6.89 kPa
The observations and predictions compare fairly well for the TC test, Figs. 6.15a and 6.15b. From the comparisons of the stress-strain responses for the SS test, Fig. 6.16a, it is seen that \( \varepsilon_1 \) is slightly underpredicted and \( \varepsilon_3 \) is overpredicted. As a result, the volumetric predictions are slightly more dilatant than observed, Fig. 6.16b. The predicted volumetric behavior during unloading and reloading does not compare favorably with the observations, Fig. 6.16b. From Figs. 6.17b and 6.18b, it is seen that the volumetric predictions during both TE tests show slightly less dilation than observed.

Backpredictions for Leighton Buzzard I

Tests on Leighton Buzzard I that are backpredicted include RTC and RTE tests, a CTC-RTE cyclic test, and special stress path tests consisting of a number of loading-unloading-reloading segments. Complete descriptions of these tests may be found in Chapter 4.

Figure 6.19a illustrates the observations and predictions of the stress-strain behavior during an RTE test from an initial confining pressure of 30.0 psi (207 kPa). Figure 6.19b shows the volumetric responses observed and predicted for the same test. It is seen that \( \varepsilon_1 \) is underpredicted and \( \varepsilon_2 \) and \( \varepsilon_3 \) are overpredicted during the unloading and reloading portions of the test, Fig. 6.19a. As a result, the model predicts more contractive volumetric behavior than observed, Fig. 6.19b.

Figures 6.20a and 6.20b display the comparisons between observations and predictions of the stress-strain response and the volumetric
Figure 6.19a. Comparison of Stress-Strain Response for RTE Test (Leighton Buzzard I)

\( \rho_0 = 1.74 \text{ gm/cc} \)
\( \sigma_0 = 30 \text{ psi} \)
\( 1.0 \text{ psi} = 6.89 \text{ kPa} \)
Figure 6.19b. Comparison of Volumetric Response for RTE Test (Leighton Buzzard 1)

--- Observed
--- Predicted

\[ \rho_0 = 1.74 \text{ gm/cc} \]
\[ \sigma_0 = 30 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 6.20a. Comparison of Stress-Strain Response for RTC Test (Leighton Buzzard I)

--- Observed
--- Predicted

\( \rho_0 = 1.75 \text{ gm/cc} \)
\( \sigma_0 = 30 \text{ psi} \)
\( 1.0 \text{ psi} = 6.89 \text{ kPa} \)
Figure 6.20b. Comparison of Volumetric Response for RTC Test (Leighton Buzzard I)

\[ \rho_o = 1.75 \text{ gm/cc} \]
\[ \sigma_o = 30 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
response, respectively, during an RTE test from an initial confining pressure of 30.0 psi (207 kPa). The predictions match the observed behavior satisfactorily.

**Backpredictions of Cyclic Test.** Results of a quasistatic cyclic triaxial test consisting of four CTC-RTE cycles of progressively increasing amplitude were presented in Figs. 4.8a and 4.8b in Chapter 4. The model predictions of this test are compared with the experimental observations in Figs. 6.21, 6.22, 6.23 and 6.24. Note that the stress-strain response is plotted in terms of the stress ratio q/p vs. the shear strain $\gamma$, while the volumetric response is plotted as q/p vs. the volumetric strain $\varepsilon_v$. These measures of stress and strain are conventionally used to represent cyclic triaxial test results.

The observed and predicted stress-strain responses are depicted in Figs. 6.21a and 6.21b, respectively. The corresponding volumetric observations and predictions are shown in Figs. 6.22a and 6.22b, respectively. For the purpose of comparison, strategic stages during the test are identified and marked on the observed and predicted response curves. Figures 6.23a and 6.23b show the comparisons between observed and predicted stress-strain responses for the first two and the last two cycles, respectively. The comparisons of the volumetric responses for cycles 1 and 2 are shown in Fig. 6.24a, while the comparisons for cycles 3 and 4 are illustrated in Fig. 6.24b.

From Figs. 6.21 (a and b) and 6.23 (a and b), it is evident that the model predictions of stress-strain response agree well with the observations. At higher amplitude cycles, (cycles 3 and 4), the predicted
Figure 6.21a. Observed Stress-Strain Response for Cyclic Triaxial Test (Leighton Buzzard I)

\[ \rho_o = 1.75 \text{ gm/cc} \]
\[ \sigma_o = 13.0 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 6.21b. Predicted Stress-Strain Response for Cyclic Triaxial Test (Leighton Buzzard I)

\[ p_0 = 1.74 \text{ gm/cc} \]
\[ \sigma_0 = 13.0 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 6.22a. Observed Volumetric Response for Cyclic Triaxial Test (Leighton Buzzard I)
Figure 6.22b. Predicted Volumetric Response for Cyclic Triaxial Test (Leighton Buzzard I)

\[ \rho_0 = 1.74 \text{ gm/cc} \]
\[ \sigma_0 = 13.0 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 6.23a. Comparison of Observed and Predicted Stress-Strain Response for Cycles 1 and 2.

- $\rho_o = 1.75 \text{ gm/cc}$
- $\sigma_o = 13.0 \text{ psi}$
- $1.0 \text{ psi} = 6.89 \text{ kPa}$
Figure 6.23b. Comparison of Observed and Predicted Stress-Strain Responses for Cycles 3 and 4

\[ \rho_0 = 1.74 \, \text{gm/cc} \]
\[ \sigma_0 = 13.0 \, \text{psi} \]
\[ 1.0 \, \text{psi} = 6.89 \, \text{kPa} \]
Figure 6.24a. Comparison of Observed and Predicted Volumetric Responses for Cycles 3 and 4

\[ \rho_0 = 1.74 \text{ gm/cc} \]
\[ \sigma_0 = 13.0 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 6.24b. Comparison of Observed and Predicted Volumetric Responses for Cycles 3 and 4

\( \rho_0 = 1.74 \text{ gm/cc} \)

\( \sigma_0 = 13.0 \text{ psi} \)

1.0 psi = 6.89 kPa
stress-strain response is less stiff than observed during compression unloading. On the other hand, the predicted stress-strain response during extension unloading is stiffer than the observed response.

Figures 6.22 (a and b) and 6.24 (a and b) indicate that the predicted volumetric response shows the same trends as the observed behavior. However, the predicted volumetric strains during the compression portion of the cycles are found to be less than the observed values, whereas the volumetric predictions on the extension segments of the cycles are greater than the corresponding observations, Figs. 6.24a and 6.24b. This discrepancy becomes more pronounced at higher amplitude cycles.

**Prediction of Special Stress Path Tests.** For further verification of the model, Tests 6 and 7 reported in Chapter 4 are backpredicted. These tests were performed to study the effects of induced anisotropy and stress history on subsequent material behavior. The stress path adapted in Test 6 consisted of a series of CTC loading-unloading cycles from increasing levels of initial confining pressure. Test 7 consisted of a series of TE loading-unloading cycles from different levels of initial confining pressure. The stress paths of Tests 6 and 7 are illustrated in Figs. 4.9 and 4.11 of Chapter 4, respectively.

Figure 6.25a shows the predicted stress-strain responses compared to the experimental observations for the three CTC load-unload cycles of Test 6, Fig. 4.9. The corresponding volumetric response comparisons are shown in Fig. 6.25b. The comparisons for the three cycles are identified as a, b and c in Figs. 6.25a and 6.25b. Comparison set
Figure 6.25a. Comparison of Stress-Strain Responses for the CTC Segments of Test 6 (Leighton Buzzard 1)

- - - - - - - Observed
- - - - - - - Predicted

a - $\sigma_o = 5$ psi ($\sigma_1/\sigma_o = 0$

b - $\sigma_o = 10$ psi ($\sigma_1/\sigma_o = 3.0$

c - $\sigma_o = 15$ psi ($\sigma_1/\sigma_o = 3.5$

$\rho_o = 1.75$ gm/cc

1.0 psi = 6.89 kPa
Figure 6.25b. Comparison of Volumetric Responses for the CTC Segments of Test 6 (Leighton Buzzard I)
'a' refers to the load-unload cycle of a virgin (no preloading) sample from an initial confining pressure $\sigma_o = 5.0$ psi (34 kPa). Set 'b' shows the comparisons for the same sample after it had been prestressed to $\sigma_1/\sigma_o = 3.0$ and then subjected to the CTC cycle from $\sigma_o = 10.0$ psi (69 kPa). Comparison 'c' pertains to the third CTC cycle from $\sigma_o = 15.0$ psi (103 kPa) after the sample had been prestressed to $\sigma_1/\sigma_o = 3.5$.

Strategic points during the test are identified and marked on both plots.

Figures 6.26a and 6.26b depict the comparisons between the observations and predictions during the three TE load-unload segments of Test 7, Fig. 4.11. Comparisons identified as a, b and c refer to the TE load-unload cycles from initial confining pressures, $\sigma_o$ of 5.0 psi (34 kPa), 10.0 psi (69 kPa) and 15.0 psi (103 kPa), respectively. Cycle 'a' was performed on the sample in a virgin state. Cycles b and c were preceded by extension prestressing to $\sigma_1/\sigma_o = 0.4$ and $\sigma_1/\sigma_o = 0.2$, respectively.

From Figs. 6.25a, 6.25b, 6.26a and 6.26b, it is seen that the model provides good predictions of material behavior during compression and extension following various levels of prestressing.

**Verification for Fuji River Sand**

The suitability of the model to simulate the behavior of dense sands has been verified in the previous sections with respect to "Munich" sand and Leighton Buzzard sand. In this section, model verification is carried out for loose samples of Fuji River sand. This sand
Figure 6.26a. Comparison of Stress-Strain Responses for the TE Segments of Test 7 (Leighton Buzzard I)

- Observed
- Predicted

- $\sigma_o = 5$ psi ($\sigma_1/\sigma_o)_{ps} = 0$
- $\sigma_o = 10$ psi ($\sigma_1/\sigma_o)_{ps} = 0.4$
- $\sigma_o = 15$ psi ($\sigma_1/\sigma_o)_{ps} = 0.2$

$\rho_o = 1.75$ gm/cc
1.0 psi = 6.89 kPa
Figure 6.26b. Comparison of Volumetric Responses for the TE Segments of Test 8 (Leighton Buzzard I)

- Observed
- Predicted

a. $\sigma_0 = 5$ psi ($\sigma_1/\sigma_3_{ps} = 0.0$

b. $\sigma_0 = 10$ psi ($\sigma_1/\sigma_3_{ps} = 0.4$

c. $\sigma_0 = 15$ psi ($\sigma_1/\sigma_3_{ps} = 0.2$

$\rho_0 = 1.75$ gm/cc

1.0 psi = 6.89 kPa
had an initial relative density, $D_r$, of 62%. In comparison, "Munich" sand had an initial $D_r$ value of 70% and Leighton Buzzard sand had initial $D_r$ values of 86% and 95%. Unlike the other sands considered in this study, Fuji River sand exhibited no dilation during shear loading. Test data on Fuji River sand used for verification was obtained from Tatsuoka and Ishihara (1974). A description of this soil and the values of its material constants can be found in Table 5.1, Chapter 5.

Figures 6.27a and 6.27b compare the predicted and observed stress-strain and volumetric responses, respectively, during a CTC test from an initial confining pressure of 28.4 psi (196 kPa). Predictions and observations are plotted in terms of $q/p$, $\gamma$ and $\varepsilon_v$ in order to be consistent with the stress and strain measures used by Tatsuoka and Ishihara (1974). Comparisons for an RTE test from the same initial confining pressure are shown in Figs. 6.28a and 6.28b. The model predictions compare well with the observed behavior, bearing in mind the fact that the material constants were evaluated on the basis of very limited experimental data available (Chapter 5).

Figures 6.29a and 6.29b depict the comparisons between observations and predictions for three cycles of a cyclic triaxial test on Fuji River sand. The test was conducted from an initial confining pressure of 28.4 psi (196 kPa) and consists of CTC-RTE cycles of increasing amplitude. Tests were carried out at an axial strain rate of 0.24% per minute. Further details on this test may be found in Tatsuoka and Ishihara (1974).
Figure 6.27a. Comparison of Stress-Strain Response for CTC Test (Fuji River Sand)

\[ \rho_o = 1.54 \text{ gm/cc} \]
\[ \sigma_o = 28.4 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 6.27b. Comparison of Volumetric Response for CTC Test
(Fuji River Sand)

\[ \rho_0 = 1.54 \text{ gm/cc} \]
\[ \sigma_0 = 28.4 \text{ psi} \]
1.0 psi = 6.89 kPa
Figure 6.28a. Comparison of Stress-Strain Response for RTE Test (Fuji River Sand)

\[ p_0 = 1.54 \text{ gm/cc} \]
\[ \sigma_o = 28.4 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 6.28b. Comparison of Volumetric Response for RTE Test (Fuji River Sand)

--- Observed

--- Predicted

\[ \rho_0 = 1.54 \text{ gm/cc} \]
\[ \sigma_0 = 28.4 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]

\[ \varepsilon_v \text{ (\%)} \]
Figure 6.29a. Comparison of Stress-Strain Response for Cyclic Triaxial Test (Fuji River Sand)

\[ \rho_0 = 1.54 \text{ gm/cc} \]
\[ \sigma_0 = 28.4 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 6.29b. Comparison of Volumetric Response for Cyclic Triaxial Test (Fuji River Sand)
Figure 6.29a illustrates the comparison of the stress-strain behavior plotted in terms of the stress ratio $q/p$ vs. the shear strain $\gamma$. The predictions match the observations fairly well. The model predicts smaller shear strains than observed on the compression side of the first cycle. The predictions for subsequent cycles are very close to the observed values.

Figure 6.26b compares the predicted and observed volumetric responses. The comparison appears to be satisfactory. The model overpredicts the volumetric strains during the initial CTC loading portion of the first cycle. This error in the volumetric strain is carried over and reflected in the predictions of the subsequent cycles. The model successfully simulates overall magnitudes and trends and the progressive densification observed in the material.

**Initial Anisotropy Due to Sample Preparation**

As mentioned in Chapter 3, initial anisotropy of laboratory test specimens caused due to sample preparation can be handled in an indirect manner within the framework of the proposed model. The process of sample preparation may be assumed to be equivalent to one or more load-unload cycles applied to an isotropic material. In the context of the model, this equivalent stress history results in initial anisotropy before application of load and (a) the creation of an initial prestress surface, $F_{ps0}$, and (b) a shift in the location of $Q$ in the stress space from the origin to $a_{ij}^0$ (Fig. 3.12, Chapter 3). Once $a_{ij}^0$ and $F_{ps0}$
determined, the initial anisotropy in the material is defined. The procedure for determining these two parameters is detailed in Chapter 3.

In this section, the initial anisotropy of Leighton Buzzard sand samples of 1.75 gm/cc initial density is evaluated. Initial anisotropy is then incorporated in the model and its effects on model prediction are studied.

The results of the hydrostatic test, Test 1 (Fig. 4.3, Chapter 4), on Leighton Buzzard sand are used for the evaluation of $a_{ij}^0$ and $F_{Pso}$. The test starts from a confining pressure of 1.0 psi (6.9 kPa) and is carried out by increasing the confining pressure in steps of 1.0 psi (6.9 kPa). The first plastic strain increment measured during the HC test is used along with Eqs. (3.51b) and (3.52) of Chapter 3 to obtain the value of $a_{ij}^0$. The plastic strains measured during the first two consecutive stress increments are used in Eq. (3.55), Chapter 3, to obtain the value of $a_{Pso}$ which defines the surface $F_{Pso}$.

The value of $a_{ij}^0$, obtained as outlined above, may be written in the vector form as $a_{ij}^0 = \{-0.181 -0.157 -0.157 0.0 0.0\}^T$. From Eq. (3.55), the value of $a_{Pso}$ was found to be 0.039. These two values define the level of initial anisotropy in samples of Leighton Buzzard sand prepared in the laboratory.
Effects of Initial Anisotropy on HC Test Predictions

Figure 6.30 presents the observed stress-strain responses during the HC test compared with the model predictions assuming initial isotropy. The stress-strain response is plotted in terms of $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$ vs. confining pressure, $\sigma_0$. Since the sample exhibits a fair amount of radial isotropy, observed values of $\varepsilon_2$ and $\varepsilon_3$ are averaged and plotted as $\varepsilon_2 = \varepsilon_3$. Since initial isotropy is assumed in this case, equal values of $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$ are predicted. It is seen that model predictions of the hydrostatic response are unsatisfactory. The observed responses show a yielding tendency at low levels of confining pressure but stiffen as $\sigma_0$ increases to higher values. The predictions, on the other hand, are almost linear in the stress range considered.

These unsatisfactory predictions may be due to the fact that the simple form of the growth function, Eq. (3.8), Chapter 3, adapted in this study is not suited for cases where the loading is predominantly hydrostatic. As pointed out in Chapter 3, improved forms of the growth function that are more suitable for both hydrostatic and shear behavior may be found in Desai and Faruque (1984), Faruque and Desai (1985), and Hashmi (1986). These versions of the growth function may be used instead of Eq. (3.8) to ensure better model predictions in cases where the influence of hydrostatic and proportional loadings is dominant.

Figure 6.31 illustrates the comparison between HC observations and predictions that take into account the initial anisotropy of the
Figure 6.30. Comparison of Stress-Strain Responses for HC Test Assuming Initial Isotropy
Figure 6.31. Comparison of Stress-Strain Responses for
HC Test Initial Anisotropy Incorporate $\rho_o = 1.74 \text{ gm/cc}$
$1.0 \text{ psi} = 6.89 \text{ kPa}$
sample. It is seen that, although the inherent shortcomings of the HC predictions still exist, the inclusion of initial anisotropy significantly improves the predictions. The model now predicts larger values of $\epsilon_1$ and smaller values of $\epsilon_2 = \epsilon_3$. These agree with the experimentally observed trends.

The effects of including initial anisotropy in the model are more vividly illustrated in Fig. 6.32. This is a plot of shear strains vs. the confining pressure during the hydrostatic test. The sample being initially anisotropic, exhibits shear strains during hydrostatic loading. The model predictions, assuming initial isotropy, curve 1, Fig. 6.32, show no shear strains during the test. When initial anisotropy is incorporated in the model, a significant improvement in the predictions can be seen, curve 2, Fig. 6.32.

Effect of Initial Anisotropy on Predictions of Shear Tests

The effect of including initial anisotropy on predictions under shear loading is considered now. This study is carried out with respect to a CTC test (Test 5, Chapter 4) from an initial confining pressure of 13.0 psi (90 kPa).

Figure 6.33a shows the comparison of the observed stress-strain response with the two predictions including and neglecting the initial anisotropy in the sample. It is seen that there is virtually no difference between the two predictions. Figure 6.33b shows the corresponding comparisons for the volumetric responses. There appears to be a very small difference between the predictions with and without initial anisotropy.
Figure 6.32. Plot of Confining Pressure vs. Shear Strain for HC Test

- Prediction Assuming Initial Isotropy
- Prediction with Initial Anisotropy
- Observed

$\rho_o = 1.74 \text{ gm/cc}$

$1.0 \text{ psi} = 6.89 \text{ kPa}$
Figure 6.33a. Comparison of Stress-Strain Response for CTC Test. Predicted With and Without Initial Anisotropy

\[ \rho_0 = 1.74 \text{ gm/cc} \]
\[ \sigma_0 = 13 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure 6.33b. Comparisons of Volumetric Response for CTC Test. Predictions With and Without Initial Anisotropy.

- Observed
- Predicted assuming initial isotropy
- Predicted with initial anisotropy

\( \rho_0 = 1.74 \text{ gm/cc} \)
\( \sigma_0 = 13 \text{ psi} \)
1.0 psi = 6.89 kPa
It is evident that the incorporation of initial anisotropy into the model has very little effect on the predictions under shear loading. This is due to the fact that the level of initial anisotropy is quite small.

Comments

Three different sands at different levels of initial relative density are considered for the verification of the model. The model is used to backpredict the results of laboratory tests carried out on these sands along a variety of stress paths from various levels of initial confining pressure. Test configurations used for verification include straight line stress paths with unloading-reloading cycles, cyclic tri-axial tests and special stress path tests. The model predictions are compared with observations. It is seen that the model can simulate the responses of these materials with reasonable accuracy.
The behavior of geologic materials is nonlinear, inelastic and affected by factors such as inherent and induced anisotropy, state of stress, deformation history and stress path. Suitable simulation of their behavior is very often the most critical aspect in the solution of soil-structure interaction problems.

The concept of elastoplasticity offers a sound theoretical basis for the development of constitutive relations for geologic materials. In this study, a generalized procedure for deriving constitutive models is outlined. The model is expressed in terms of the state of stress and tensor valued and scalar valued internal state variables that govern material behavior. The formulation is based on principles of continuum mechanics.

A hierarchical approach has been taken in developing the proposed constitutive model. It permits evolution of models of progressively higher grades from a basic isotropic hardening associative model. Factors such as nonassociativeness due to friction and induced anisotropy due to deformation history are introduced as corrections or modifications to the basic model.

The main objective of this study was to develop a general anisotropic nonassociative model that accounts for effects of initial
anisotropy, induced anisotropy, stress history and stress path, and simulates behavior of the material during load reversals (unloading and reloading), complex stress paths and cyclic loading.

The mechanism of the proposed model is fairly simple. The parameters have been kept to a minimum and they are easily determined based on commonly available laboratory test results. The model is verified with respect to laboratory multiaxial test data under various paths of virgin and nonvirgin loading for three different sands at dense and loose states.

The work carried out in this study and the conclusions drawn from it may be summarized as follows:

1. A general procedure for the development of a basic isotropic associative model is outlined. A specific form of the yield surface was adapted and this formed the basis for the development of the general anisotropic hardening nonassociative model.

2. The model consists of a translating surface $Q$ that moves with the stress point in a fixed field of infinite isotropic yield surfaces, $F_i$, in the stress space. Surfaces $F_i$ serve to define virgin and nonvirgin loading, govern magnitude of plastic strain increments and retain memory of maximum prestress. $Q$ serves as the plastic potential and doubles as a loading surface during non-virgin loading. This results in an anisotropic/
kinematic hardening nonassociative plasticity model that allows for plastic behavior during loading, unloading and reloading.

3. At any stage during the deformation process, the location of Q in the stress space (or the position of the stress point on surface Q) governs the direction of incremental plastic strains. By choosing appropriate translation rules for Q, it is possible to model (a) associative behavior and (b) nonassociative behavior with any desired degree of deviation from normality.

4. Here, the translation of Q is made a function of the deformation history and the level of induced anisotropy. A strain ratio $a_n$ that reflects the level of induced anisotropy in the material is defined. This measure is incorporated in the translation rule in such a manner that the deviation from normality is governed by the level of induced anisotropy.

5. During virgin loading, the magnitude of plastic strains is obtained by imposing the consistency condition. During nonvirgin loading, an interpolation rule, based on the relative position of the stress point within the prestress surface, is postulated for the magnitude of plastic strains.
6. An experimental study is carried out to investigate the evolution of induced anisotropy, effect of stress history on subsequent material behavior and the validity of $a_n$ as a measure representing induced anisotropy. Testing was carried out in the cubical device on dense specimens of Leighton Buzzard sand. It was found that $a_n$ was indeed a suitable measure to simulate induced anisotropy and that it described the development and demise of induced anisotropy adequately.

7. The significant parameters of the model were identified and the material constants were determined based on the results of commonly used straight line stress path tests. Materials considered were "Munich" sand, Leighton Buzzard sand and Fuji River sand.

8. Verification of the model was carried out for loose and dense sands by backpredicting the stress-strain and volumetric responses during laboratory tests. Test configurations used for verification included a variety of straight line stress paths with unloading and reloading, cyclic triaxial tests and special tests consisting of a number of different straight line stress path segments. The model was found to be capable of predicting all the significant observed trends in material behavior. Comparisons between observed and predicted responses were found to be satisfactory.
9. Initial anisotropy in the field or due to sample preparation can be handled in an indirect manner within the framework of this model. A procedure for determining the level of initial anisotropy in laboratory samples is outlined. The effects of incorporating initial anisotropy on model predictions is studied. It is seen that including initial anisotropy improves the predicted stress-strain response during HC loading significantly.
APPENDIX A

DERIVATION OF SOME SELECTED EQUATIONS

From the definitions used in Chapter 3,

\[ \bar{\sigma}_{ij} = \sigma_{ij} - a_{ij} \] (A.1)

and

\[ a_{ij} = (a_{ij})_{iso} + d_{ij} \] (A.2)

but

\[ (a_{ij})_{iso} = \sigma_{ij} [1 - (\alpha/\alpha_o)^{1/n-2}] \] (A.3)

Substituting Eqs. (A.2) and (A.3) in Eq. (A.1),

\[ \bar{\sigma}_{ij} = (\alpha/\alpha_o)^{1/n-2} \sigma_{ij} - d_{ij} \] (A.4)

Therefore,

\[ \frac{\partial \bar{\sigma}_{ij}}{\partial \sigma_{rs}} = (\alpha/\alpha_o)^{1/n-2} \frac{\partial \sigma_{ij}}{\partial \sigma_{rs}} - \frac{\partial d_{ij}}{\partial \sigma_{rs}}. \]

Since \( d_{ij} \) is not a function of \( \sigma_{ij} \),
Now using the chain rule

\[
\frac{\partial Q}{\partial \sigma_{ij}} = \frac{\partial Q}{\partial \sigma_{rs}} \cdot \frac{\partial \sigma_{rs}}{\partial \sigma_{ij}}
\]

Using Eq. (A.5)

\[
\frac{\partial Q}{\partial \sigma_{ij}} = \frac{\partial Q}{\partial \sigma_{rs}} \left( \frac{\alpha}{\alpha_0} \right)^{1/n-2} \delta_{iv} \delta_{js} = \left( \frac{\alpha}{\alpha_0} \right)^{1/n-2} \frac{\partial Q}{\partial \sigma_{ij}}
\]

\[
(A.6)
\]

and

\[
\left| \frac{\partial Q}{\partial \sigma_{ij}} \right| = \left( \frac{\alpha}{\alpha_0} \right)^{1/n-2} \left| \frac{\partial Q}{\partial \sigma_{ij}} \right|
\]

\[
(A.7)
\]

Therefore

\[
\frac{\partial}{\partial \sigma_{ij}} \left| \frac{\partial Q}{\partial \sigma_{kl}} \right| = \left( \frac{\alpha}{\alpha_0} \right)^{1/n-2} \left| \frac{\partial Q}{\partial \sigma_{ij}} \right| \left| \frac{\partial Q}{\partial \sigma_{kl}} \right| = \left| \frac{\partial Q}{\partial \sigma_{ij}} \right| \left| \frac{\partial Q}{\partial \sigma_{kl}} \right|
\]

\[
(A.8)
\]

Derivation of the Elastoplastic Constitutive Tensor \( C^{e-p}_{ijkl} \)

The elastic stress strain rule is

\[
d\sigma_{ij} = C^{e}_{ijkl} \left( d\varepsilon_{kl} - d\varepsilon_{kl}^P \right)
\]

\[
(A.9)
\]
the flow rule is given by

\[ \frac{d \varepsilon^P_{ij}}{d \sigma_{ij}} = \lambda \frac{3Q_{ij}}{\sqrt{3Q_{ij}}} \]  \hspace{1cm} (A.10)

**Virgin Loading**

During virgin loading, the consistency condition \( dF = 0 \) is satisfied; i.e.,

\[ dF = \frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial F}{\partial \varepsilon} d\varepsilon = 0 \]  \hspace{1cm} (A.11)

where

\[ d\varepsilon = (d\varepsilon^P_{ij} d\varepsilon^P_{ij})^{1/2} \]  \hspace{1cm} (A.12)

From Eqs. (A.10), (A.11) and (A.12)

\[ \frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial F}{\partial \varepsilon} \lambda = 0 \]  \hspace{1cm} (A.13)

multiplying Eq. (A.9) by \( \partial F / \partial \sigma_{ij} \) and substituting Eq. (A.10)

\[ \frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} = \frac{\partial F}{\partial \sigma_{ij}} C_{ijkl} (d\varepsilon_{kl} - \lambda \frac{\partial Q_{kl}}{\partial \sigma_{rs}}) \]  \hspace{1cm} (A.14)

substituting Eq. (A.13) in Eq. (A.14)
which results in

\[
\lambda = \lambda_v = \frac{\frac{\partial F}{\partial \sigma_{ij}} C_{ijkl} \frac{\partial \xi}{\partial k} \frac{\partial Q}{\partial \sigma_{kl}}}{\frac{\partial F}{\partial \sigma_{ij}} C_{ijkl} \frac{\partial \sigma_{kl}}{\partial \xi} - \frac{\partial F}{\partial \xi}} \tag{A.15}
\]

Now substituting Eqs. (A.10) and (A.15) in Eq. (A.9)

\[
d\sigma_{ij} = C_{ijkl} d\xi_{kl} - \lambda C_{ijkl} \frac{\partial Q/\partial \sigma_{kl}}{\left| \frac{\partial Q/\partial \sigma_{rs}}{\partial Q/\partial \sigma_{rs}} \right|}
\]

\[
= C_{ijkl} d\xi_{kl} - C_{ijkl} \frac{\partial Q/\partial \sigma_{kl}}{\left| \frac{\partial Q/\partial \sigma_{rs}}{\partial Q/\partial \sigma_{rs}} \right|} \frac{\partial F}{\partial \sigma_{mn}} C_{mnuv} d\xi_{uv}
\]

\[
C_{ijkl} \left| \frac{\partial Q/\partial \sigma_{rs}}{\partial Q/\partial \sigma_{rs}} \right| \frac{\partial F}{\partial \sigma_{pq}} C_{pqrs} \sigma_{rs} \frac{\partial \xi}{\partial \sigma_{kl}} - \frac{\partial F}{\partial \sigma_{kl}}
\]

This may be simplified as

\[
d\sigma_{ij} = [C_{ijkl} - \frac{\frac{\partial F}{\partial \sigma_{pq}} C_{pqrs} \frac{\partial \xi}{\partial \sigma_{rs}} - \frac{\partial F}{\partial \sigma_{rs}}}{\left| \frac{\partial Q/\partial \sigma_{kl}}{\partial Q/\partial \sigma_{rs}} \right|}] d\xi_{kl} \tag{A.16a}
\]

or

\[
d\sigma_{ij} = \left[ C_{ijkl} - \frac{\frac{\partial F}{\partial \sigma_{pq}} C_{pqrs} \frac{\partial \xi}{\partial \sigma_{rs}} - \frac{\partial F}{\partial \sigma_{rs}}}{\left| \frac{\partial Q/\partial \sigma_{kl}}{\partial Q/\partial \sigma_{rs}} \right|} \right] d\xi_{kl}
\]
\[ d \sigma_{ij} = c_{ijkl}^e \, dc_{kl} \]  

\( \text{(A.16b)} \)

where

\[ c_{ijkl}^e = [C_{ijkl}^e - \frac{\partial F}{\partial \sigma_{pq}} \, c_{pqrs} \frac{\partial Q}{\partial \sigma_{rs}} - \frac{\partial F}{\partial \sigma_{uv}} \, c_{uv} \frac{\partial Q}{\partial \sigma_{mn}} \, c_{mnkl}^e] \]  

\( \text{(A.17)} \)

**Nonvirgin Loading**

During nonvirgin loading \( \lambda \) is postulated as

\[ \lambda = \frac{\frac{\partial F}{\partial \sigma_{ij}} \, d \sigma_{ij}}{\frac{\partial F}{\partial \xi} \, \left[ 1 + \theta \, h_1 \right]} \]

Multiplying Eq. (A.9) by \( \frac{\partial F}{\partial \sigma_{ij}} \) and substituting Eq. (A.10)

\[ \frac{\partial F}{\partial \sigma_{ij}} \, d \sigma_{ij} = \frac{\partial F}{\partial \sigma_{ij}} \, c_{ijkl}^e \, (dc_{kl} - \lambda \, \frac{\partial Q/\partial \sigma_{kl}}{\partial Q/\partial \sigma_{rs}}) \]  

\( \text{(A.19)} \)

substituting Eq. (A.18) in (A.19)
which results in

\[
\lambda = \lambda_{nv} = \frac{\partial F_{ps} \cdot C_{ijkl} \cdot d\epsilon_{kl}}{\partial \sigma_{ij}} = \lambda_{nv} = \frac{\partial F_{ps} \cdot C_{ijkl} \cdot d\epsilon_{kl}}{\partial \sigma_{ij}} - \frac{\partial F_{ps} \cdot [1 + \theta \cdot h_1]}{\partial \epsilon_{ij}}
\]

Substitution of Eqs. (A.10 and (A.20) in Eq. (A.9) yields

\[
\frac{d\sigma_{ij}}{d\epsilon_{kl}} = \left[ C_{ijkl} - \frac{\partial F_{ps} \cdot C_{ijkl} \cdot d\epsilon_{kl}}{\partial \sigma_{ij}} \right]
\]

During nonvirgin loadings \( C_{ijkl}^{\text{ep}} \) is expressed as
The interpolation rule used for nonvirgin loading is expressed as

\[ \lambda_{nv} = \frac{\frac{\partial F_{ps}}{\partial \sigma_{ij}}}{F_{ps}} \frac{\partial \sigma_{ij}}{\partial \sigma_{ij}} \left[ 1 + \frac{h_1}{R} \right] \]  
(A.23)

Where

\[ \theta = h_2 \left( \frac{\Delta P_{ps}}{J_1} \right) \]  
(A.24)

and

\[ \bar{R} = \left( \frac{(S_r)_{cp}}{\alpha_{cp}} - \frac{(S_r)_{cr}}{\alpha_{cr}} \right)^{1/2} \]  
(A.25)

The reasoning and motivation for using these particular expressions for \( \theta \) and \( \bar{R} \) are given below.
Ratio $\bar{R}$

For reasons explained in Chapter 3, $\bar{R}$ should be such that $\bar{R} \rightarrow \infty$ at point of stress reversal $\sigma_{ij}^R$ and $\bar{R} \rightarrow 0$ when $\sigma_{ij} \rightarrow \sigma_{ij}^*$. For any other stress level, $\bar{R}$ should vary between 0 and $\infty$, depending on the relative position of $\sigma_{ij}$ with respect to $\sigma_{ij}^R$ and $\sigma_{ij}^*$. Equation (A.25) satisfies these requirements using the relative size of surfaces to provide the interpolation. The term $1/\alpha$ is directly proportional to the size of the yield surface. Thus, $1/\alpha_{ps} - 1/\alpha_c$ represents the difference in size between surfaces $F_{ps}$ and $F_c$, while $1/\alpha_c - 1/\alpha_R$ represents the size difference between surfaces $F_c$ and $F_R$. At any stress point, $\bar{R}$ should be larger when the next stress increment is directed inwards of $F_c$ than when it is directed out of $F_c$. It is for this reason that the stress ratio $S_r$ is included in the expression for $\bar{R}$. When stress increment is directed out of $F_c$, Eq. (A.25) reduces to

$$\bar{R} = \frac{1/\alpha_{ps} - 1/\alpha_c}{1/\alpha_c - 1/\alpha_R}$$

(A.26)

When next stress increment is directed inwards of $F_c$, Eq. (A.25) reduces to
Figure (5.4) in Chapter 5 shows plots of $\ln (H_I)$ vs. $\ln (R)$ for unloading and reloading segments of various stress paths on "Munich" sand. It is seen that all the plots are parallel. However, when $\ln (H_I)$ was plotted against $\ln \left[ \frac{\ln \left( \frac{S_r}{\alpha_c} \right)}{\alpha_c} - \frac{\ln \left( \frac{S_r}{\alpha_c} \right)}{\alpha_c} \right]$, it was found that, all plots corresponding to the same stress path were parallel, plots for dissimilar stress paths were nonparallel. The exponent $1/\sqrt{\phi}$ was necessary to incorporate the effect of stress path into the ratio $R$ and make all the plots of $\ln (H_I)$ vs. $\ln (R)$ parallel to each other.

The ratio $R$ can also be expressed in terms of distances in the stress space among $\sigma_{ij}^R$, $\sigma_{ij}^*$ and $\sigma_{ij}^*$. One simple expression for $R$ that was used successfully is expressed as

$$R = \left[ \frac{\left| \sigma_{ij}^* - \sigma_{ij}^R \right|}{\left| \sigma_{ij}^* - \sigma_{ij}^R \right|} \right]^{1/\sqrt{\phi}}$$

where $f_1$ is a factor to account for the noncircular shape of the yield surface in the octahedral plane. A suitable expression for $f_1$ that was found to give good predictions may be expressed as
\[ f_1 = \left( \frac{F_s}{F_{sR}} \right)^5 \]

where \( (F_{sR}) \) \text{comp} refers to the value of the shape function for compression stress paths and \( (F_s) \) is the value of the shape function at the point of stress reversal.

**Ratio \( \theta \)**

In plots of \( \ln (R) \) vs. \( \ln (H_i) \), the intercept along the ordinate gives \( \ln (\theta) \), Fig. 5.4. It was found that the value of \( \theta \) during unloading segments of CTC and CTE tests was larger than the values of \( \theta \) corresponding to reloading segments. For TC and TE tests, the values of \( \theta \) remained essentially the same during unloading and reloading. When the material was stressed to higher levels and unloaded, higher values of \( \theta \) were observed for all stress paths. Based on these observations, the expression for \( \theta \) given by Eq. (A.24) was formulated.

The values of \( J_1^* \) during unloading portions of CTC or CTE tests are smaller than the values of \( J_1^* \) during reloading. Hence, \( \theta \) during unloading is greater than \( \theta \) during loading. For TC and TE tests, \( J_1^* \) and, hence, \( \theta \) remain unchanged, regardless of whether its unloading or reloading. When material is stressed to larger levels, the value of \( \Delta_{ps} \) and, therefore, the value of \( \theta \) becomes higher.
APPENDIX B

TEST RESULTS USED FOR PARAMETER DETERMINATION

Results of tests on Munich Sand and Fuji River sand that were used for determination of parameters are reproduced here from Scheebe and Desai (1983) and Tatsuoka and Ishihara (1974) respectively.
$\rho_0 = 2.03 \text{ gm/cc}$

$\sigma_0 = 6.5 \text{ psi}$

$1.0 \text{ psi} = 6.89 \text{ kPa}$

Figure B.1. Results of CTC Test on "Munich" Sand
($\sigma_0 = 6.5 \text{ psi}$) (Scheele and Desai, 1983)
Figure B.2. Results of CTC Test on "Munich" Sand
($\sigma_0 = 13.0$ psi) (Scheele and Desai, 1983)

$\rho_o = 2.03$ gm/cc
$\sigma_0 = 13.0$ psi
1.0 psi = 6.89 kPa
Figure B.3. Results of TC Test on "Munich" Sand
(Scheele and Desai, 1983)

\[ \sigma_0 = 2.03 \text{ gm/cc} \]
\[ \sigma_\circ = 13.0 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure B.4. Results of TE Test on "Munich" Sand
(Scheele and Desai, 1983)

$\rho_o = 2.03 \text{ gm/cc}$
$\sigma_o = 13.0 \text{ psi}$
$1.0 \text{ psi} = 6.89 \text{ kPa}$
Figure B.5. Results of RTE Test on "Munich" Sand  
(Scheele and Desai, 1983)
Figure B.6. Results of SS Test on "Munich" Sand
(Scheele and Desai)

\[ \rho_o = 2.03 \text{ gm/cc} \]
\[ \sigma_o = 13.0 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure B.7a. Results of CTC Test on Fuji River Sand
(Tatsuoka and Ishihara, 1974)

$\rho_o = 1.54 \text{ gm/cc}$

$\sigma_o = 28.4 \text{ psi}$

$1.0 \text{ psi} = 6.89 \text{ kPa}$
Figure B.7b. Results of CTC Test on Fuji River Sand
(Tatsuoka and Ishihara, 1974)

\[ \rho_0 = 1.54 \text{ gm/cc} \]
\[ \sigma_0 = 28.4 \text{ psi} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kPa} \]
Figure B.8a. Results of RTE Test on Fuji River Sand (Tatsuoka and Ishihara, 1974)

\( \rho_o = 1.54 \text{ gm/cc} \)
\( \sigma_o = 28.4 \text{ psi} \)
\( 1.0 \text{ psi} = 6.89 \text{ kPa} \)
Figure B.8b. Results of RTE Test on Fuji River Sand
(Tatsuoka and Ishihara, 1974)

\( \rho_0 = 1.54 \text{ gm/cc} \)
\( \sigma_0 = 28.4 \text{ psi} \)
\( 1.0 \text{ psi} = 6.89 \text{ kPa} \)
Figure B.9a. Results of Cyclic Triaxial Test
(Tatsuoka and Ishihara, 1974)
Figure B.9b. Results of Cyclic Triaxial Test
(Tatsuoka and Ishihara, 1974)

$\rho_o = 1.54 \text{ gm/cc}$
$\sigma_o = 28.4 \text{ psi}$
$1.0 \text{ psi} = 6.89 \text{ kPa}$
LIST OF REFERENCES


