

AN APPLICATION OF GEOTHERMAL ENERGY
FOR SALINE WATER CONVERSION

by

Robert H. Kirchhoff

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SIGNED: Robert H. Kirchhoff

APPROVAL BY THESIS DIRECTOR

This thesis has been approved on the date shown below:

Russell E. Petersen
RUSSELL E. PETERSEN
Associate Professor of Aerospace
and Mechanical Engineering

JAN. 28, 1963
Date

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CHAPTER 1

INTRODUCTION

The practice of obtaining potable water from the sea is quite old. For over two centuries man has been able to extract fresh water from saline water. As far back as 1375, seafarers used simple distilling devices. In 1624, Roger Bacon observed, "With a heat sufficient for distillation, salt will not rise in a vapor." In the late 18th Century, Thomas Jefferson described various experiments of his time. Benjamin Franklin obtained 5,000 pounds for a process using the theory of cooling by evaporation (1)* Since these early investigations, men have organized and classified the field of research needs for salt water conversion.

A logical way to investigate the problem is to study it from the point of view of sources of energy and processes of separation. Studies of this kind have been conducted and published. For a complete breakdown of the problem see references (1) and (2).

The source of energy to be considered here is

* Numbers in parentheses refer to REFERENCES.

geothermal and the separation process is evaporation. Evaporation is a common process in saline water conversion. Pilot plants are in operation in this country which use both multiple effect evaporation and vapor compression evaporation. However, these plants utilize conventional sources of energy such as combustion and electric power. As a source of energy, geothermal energy has received limited attention.

Two studies on the availability and utilization of geothermal energy for saline water conversion have been published (3), (4). The results of both of these studies have been negative, i.e., for one reason or another, the utilization of geothermal energy for saline water conversion was reported to be not economically feasible. Some of the prohibiting factors were high capital expenditures, high pumping costs, and significant temperature reductions in the heat collecting zones. Both studies proposed to utilize the energy by bringing it to the surface and then using a conventional means to apply it to distillation. There is a point of significant difference between the utilization reported in references (3) and (4) and the technique suggested in this study. While the two other studies proposed to carry the energy to the surface of the earth before using it, this paper will discuss the utilization of the

energy in the crust of the earth. Further, the thermal conductivity of the region under consideration here can be as much as four times greater than those of the regions studied in references (5) and (6). These two reasons are felt to be enough to warrant further study of geothermal energy as a source of energy for saline water conversion.

Aquifers (i.e., permeable geological formations having a structure that permits appreciable water movement) containing brine, exist at great depths in the earth's crust. Due to the natural geothermal gradient¹ the brine can be at a considerably higher temperature than that of the surface of the earth. To utilize this geothermal energy, it is proposed to evaporate the brine in the crust of the earth, bring the vapor to the surface, condense it, and thereby provide a supply of potable water. Since the heat transfer would take place in moist sedimentary strata, the process has added interest because of the high thermal conductivities of moist strata as compared with those of igneous rocks for which the previous studies have been made (3), (4).

Interest in this project was generated as a result of examining correspondence between Mr. Walter A. Hubbard of Ancho, New Mexico, and John W. Harshbarger, Geologist, at The University of Arizona (5).

¹ See Chapter 2 for a discussion of the natural geothermal gradient.

CHAPTER 2

PRELIMINARY GEOLOGICAL CONSIDERATIONS

2.1 Occurrence and Classification of Ground Waters

It is a known fact to those in the petroleum industry that water exists deep in the earth. As oil wells are depleted of their reserves, water often encroaches into the well, eventually making the removal of more oil economically unfeasible. One of the common types of oil reservoir drives is the "water drive mechanism." In the "water drive," oil is removed from the well due to pressure gradients in the expansion of compressed water.

The occurrence of ground waters may be divided into three classes: surface waters, confined aquifers, and connate waters. All ground waters occur in the strata of sedimentary geological formations. It is the nature of sedimentary formations that they were originally deposited by the action of water, wind or other agencies. However, in the final process of burial, the formations were exposed to ground waters. Thus, it is to be expected that all subsurface rocks having appreciable porosity must be saturated with fluid; this, in fact, is

found to be true (6). Igneous and metamorphic formations in general do not have any appreciable porosity for holding ground waters. Clastic sedimentary rocks such as gravel, sand, silt, and clay are most important because of their relatively high permeability and porosity. Organic sedimentary rocks such as limestone and chalk are of lesser importance because of their relatively lower permeability and porosity. Formations of stratified rocks may be a few feet to a hundred feet in thickness and may have an areal extent from a few miles to a few hundred miles (6).

A typical geological formation showing ground waters is shown in Figure 2.1. Surface waters are those which have one free surface, i.e., they are unconfined.

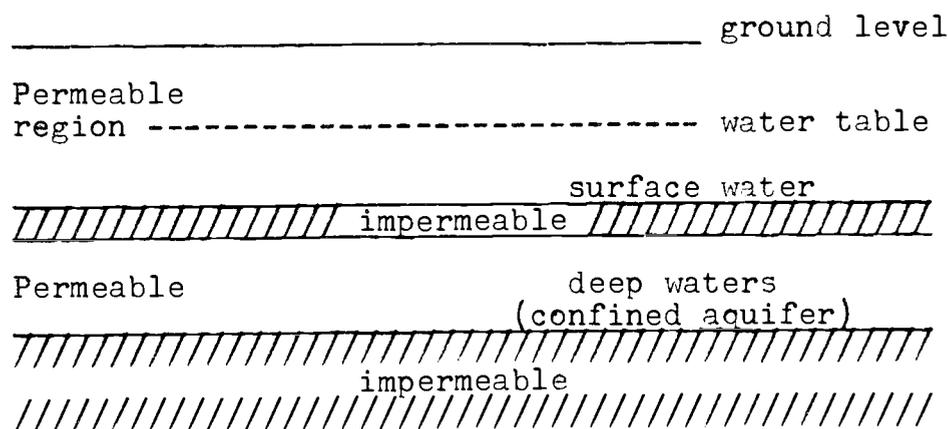


Figure 2.1

Waters which are confined by impermeable beds above and below the porous strata are usually referred to as confined aquifers.

The movement of waters in confined aquifers takes place under the influence of pressure gradients in the porous conduits defined by the upper and lower impermeable boundaries. Faults and other peculiarities in the geological structure may permit the entrance of surface waters into a confined aquifer. The geological formations of some confined aquifers have a surface outcrop as shown in Figure 2.2. Due to this surface outcrop, the pressure of the water in the aquifer is the hydrostatic pressure of the water in the sedimentary strata. In general, water in confined aquifers undergoes appreciable movement

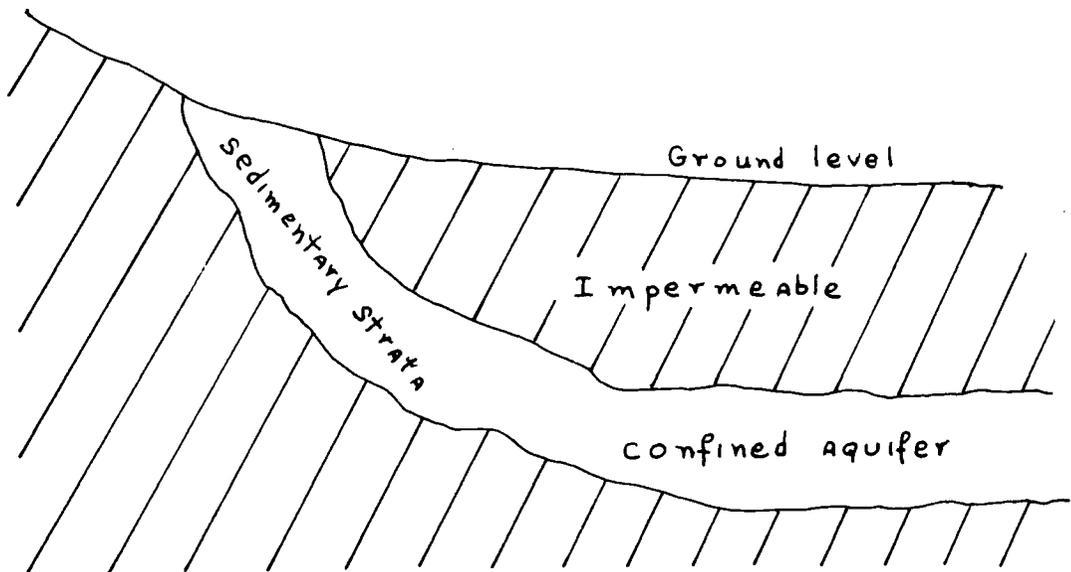


Figure 2.2

through porous conduits and sedimentary strata.

Connate water is water entrapped in the interstices of sedimentary rock at the time it was deposited. In contrast to confined aquifers, the geology of a connate water zone consists of a porous strata completely enclosed by an impermeable region. Water in connate zones undergoes almost no movement at all. Since connate waters are completely trapped, their pressure can be approximated by the weight of the overburden. This is approximately $1000\#/in^2$ per 1000 foot of descent (6). Confined aquifers and connate water zones are of primary interest in this study.

2.2 Physical Properties of Aquifers

There are two properties of porous media which are of fundamental importance: the permeability and the porosity. The porosity is a measure of the pore space or the fluid capacity of the medium. The porosity (7) is a dimensionless quantity defined as

$$\Theta = \frac{w}{V} \quad (2.1)$$

where w - volume of the void space
 V - total volume of the rock.

The permeability is a measure of the ease with which fluids may move in a porous medium under the influ-

ence of pressure. The permeability (7) is defined by Darcy's law

$$G = \frac{k\rho}{\mu} \frac{dp}{dn} \quad (2.2)$$

where k - permeability - ft.²

ρ - fluid density - lbm/ft.³

μ - fluid viscosity - (# - sec)/ft.²

G - mass flux - lbm/(sec - ft.²)

$\frac{dp}{dn}$ - pressure gradient #/ft.³

It is difficult to make any quantitative statements about the depth of confined aquifers and connate water zones. Depth figures can only be given in relation to specific geological formations. However, it can be safely stated that connate zones occur at greater depths than confined aquifers. It would be unusual to find a connate zone at a depth of less than 1000 ft.

Water in confined aquifers and connate zones is usually high in dissolved solids. Comparison is made of the composition of the brines in Tables 2.1 and 2.2 for a confined aquifer and common sea water respectively.

TABLE 2.1

CHEMICAL COMPOSITION OF THE BRINE IN THE CONFINED AQUI-
FER SURROUNDING THE EAST TEXAS OIL FIELD (8)

<u>Composition</u>	<u>P.P.M.²</u>
Sodium	22,050
Calcium	1,176
Magnesium	197
Sulfates	384
Chlorides	36,400
Carbonates	0
Bicarbonates	366

TABLE 2.2

CHEMICAL COMPOSITION OF SEA WATER (9)

<u>Composition</u>	<u>P.P.M.</u>
Chlorides	18,980
Bromides	65
SO ₄	2,649
Magnesium	1,772
Sodium	10,556

² P.P.M. - parts per million by weight.

Connate waters have considerably greater amounts of dissolved solids, which is to be expected since they are older and relatively immobile.

2.3 The Geothermal Gradient

There exists in the crust of the earth a natural geothermal gradient, i.e., as one descends into the earth, it becomes hotter. The magnitude of this gradient (10) is on the order of 1° F. in 70-100 ft. This means that ground waters at depths of 15,000 ft. would have temperatures in the range of 200° F. above surface temperatures. This fact, together with high pressures at these depths, gives the water a sizeable thermodynamic availability.³ This gives rise to the possibility of a saline water conversion process utilizing this hot water in the ground.

³ The availability of a system is defined as the maximum useful work that can be obtained in a process in which the system comes to equilibrium with its surroundings. John Francis Lee and Francis Weston Sears, Thermodynamics, (Reading: Addison-Wesley Publishing Co., Inc., 1955), p. 164.

CHAPTER 3

QUALITATIVE DISCUSSION OF THE SCHEME FOR DE-SALTING BRINE

3.1 The Gedanken-model

Consider a confined aquifer extending over a large area with a well bore sunk into it as shown in Figure 3.1

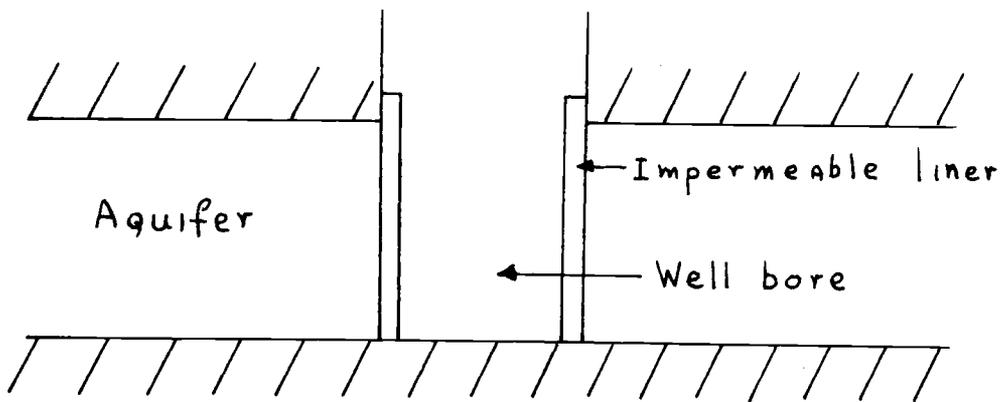


Figure 3.1

Further, assume that it is possible to remove all the water from the well bore and hold this condition by installing an impermeable liner in the well casing. Initially, there is no flow in the permeable strata and the pressure and temperature are constant at P_∞ and T_∞ respectively. The pressure in the well bore has been re-

duced to P_s which is a pressure less than the saturation pressure at T_w . If the impermeable liner is suddenly removed and a suction pump is turned on to keep the pressure at P_s , both heat and mass fluxes will occur at the well bore and extend into the aquifer. The mass flux will be caused by the pressure gradient in the aquifer, and the heat flux will be caused by the vaporization of the water at the well bore, together with the enthalpy transport of the mass flow. Due to these gradients, heat and mass will flow toward the well bore. Initially, at the well bore, both of these gradients will be high, but as time goes on, they will decrease. Although the pressure P_s is held constant by the pump, the temperature at the well bore will decrease in time. Eventually, when this temperature corresponds to the saturation temperature at a pressure slightly less than P_s the well bore will fill with water because vaporization has stopped.

From this thought model, some very important questions arise. How long will it take for vaporization to stop? How much water vapor will be entering the casing? Would it be possible to control the flow of water and heat to the well bore? What will happen to the dissolved solids that precipitate at the well bore? Could the pro-

cess be re-started once it has stopped? What would be the optimum ρ_s ? How could the vapor be removed?

3.2 Discussion of the Pertinent Questions

Some of these questions will be discussed qualitatively in this chapter and an attempt at some quantitative answers to some of the remaining questions will be made in succeeding chapters. In particular, a mathematical model will be constructed to predict the pressure and temperature in time and space.

Would it be possible to control the flow of heat and water to the well bore? From Darcy's law, Equation 2.2, it is seen that a change in permeability will change the flow rate. This fact offers a possible means to control the flow rate at the well bore. Consider Figure 3.1 with the impermeable liner replaced by a liner of permeability k_1 . The mass flux at the well bore would be

$$G = \frac{k_1}{\mu} \rho \frac{dp}{dr} \quad (3.1)$$

By making k_1 less than k it would be possible to decrease the flux at the well bore. This fact will be considered as one of the boundary conditions when studying the mathematical model.

What will happen to the dissolved solids that precipitate at the well bore? The economic feasibility of

many processes has been justified by utilizing secondary effects. It is proposed to design a well casing and permeable liner that not only will avoid clogging but also provide a means of reclaiming the dissolved solids. From a list of the dissolved solids in brines, it can be seen that many valuable materials could come out of the process secondary to the water obtained. The value of the by-products could possibly swing the economic balance of this process. The proposed design is outlined in Figure 3.2.

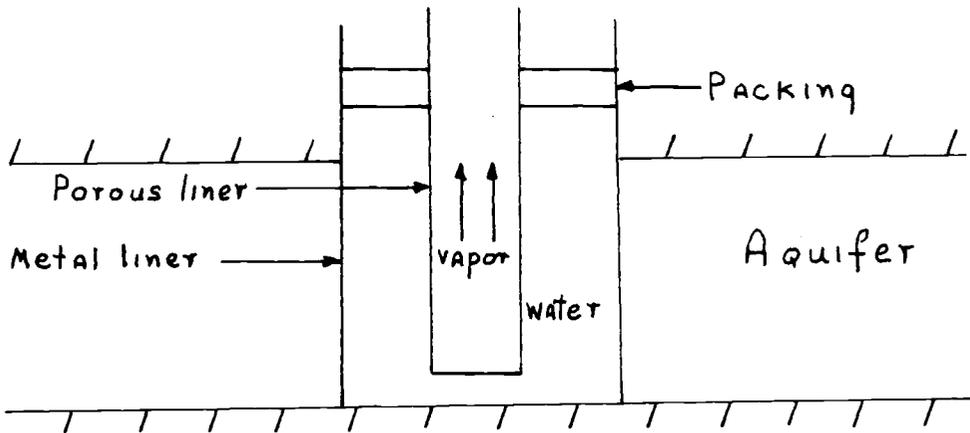


Figure 3.2

To remove the dissolved solids, the porous liner would periodically be removed and cleaned. At the same time it would be possible to flush the region between the porous liner and the metal liner to remove all the solids

which would precipitate in this region. The author has checked with experts in petroleum engineering and found that the proposed design is completely within the capabilities of current oil well practice.

Another important question is, what would be the optimum P_s ? One of the considerations in choosing P_s is the fact that vapor must be brought to the surface and condensed. If $P_s < P_a$ (where: $P_a = P$ atmosphere) the vapor must be pumped to the surface. If $P_s > P_a$, would it be possible to choose P_s large enough to provide all or some of the potential to lift the vapor to the surface?

To answer some of the other questions it is necessary to study the problem of the pressure and temperature distribution in the aquifer.

CHAPTER 4

THE THEORETICAL MODEL

4.1 Model Aquifers

Since every aquifer is a unique geological structure, it is difficult to generalize to a theoretical model. However, some basic geometries and boundaries will be chosen to set up a model for study.

The discussion is restricted to radial aquifers, i.e., those whose geometry may be defined by cylindrical polar coordinates. The aquifers are assumed to be horizontal in order to neglect gravitational components in Darcy's law. The geometry at the well bore is a cylinder as shown in Figure 4.1. The upper and lower boundaries are assumed to be impermeable strata; this restricts the flow to the radial direction only.

At the external boundary of an aquifer with a surface outcrop, it will be assumed that the pressure is constant as shown in Figure 4.1. This amounts to assuming that the aquifer is large enough to supply sufficient water to make up what has been removed without appreciable pressure drop at the external boundary.

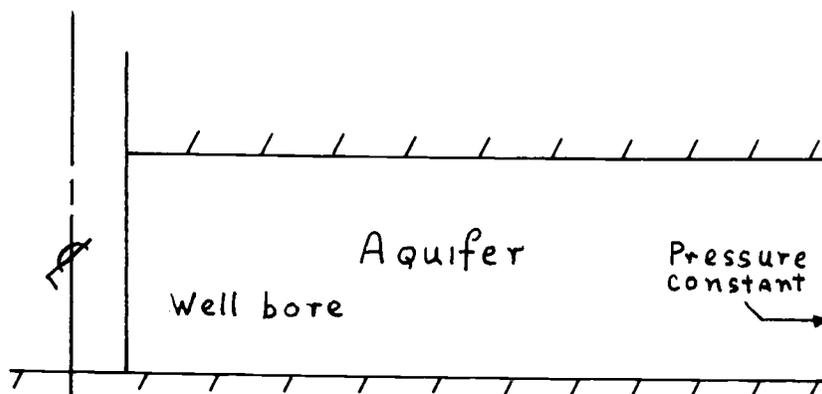


Figure 4.1

The physical condition at the external boundary of the connate zone is that the permeability has decreased to the point where a pressure gradient causes little or no flow. The mathematical counterpart is to assume that the pressure gradient is zero at the boundary.

It could happen that the areal extent and physical external boundary of the aquifer is unknown. Under this condition, it would be possible to require that the pressure remain finite as the boundary approaches infinity.

Of the three conditions mentioned, the first is the most likely to occur in practice.

The properties of permeability and porosity are

assumed to be homogenous and isotropic in the aquifer.

4.2 Darcy's Law

The flow of water in an aquifer has been found to be governed by Darcy's law, equation 2.2. The law is usually stated as an empirical law (11). However, there are some authors (6) who feel that it is a special case of the Navier-Stokes equations, and there are others (12) who feel that Darcy's law is not a strict consequence of the classical hydrodynamic equations. Be that as it may, it is necessary to be aware of its limitations.

The validity of any flow equation is usually given in terms of a range of Reynold's number. However, flows through porous media do not exhibit abrupt transitions from "laminar" to "turbulent" flow. Rather, the transition is gradual and, as would be expected, so is the deviation from Darcy's law (6). It is generally accepted that Darcy's law does not govern the flow when the inertial effects dominate the viscous effects. Experimenters have attempted Reynold's number correlation for Darcy's law but the results have not been satisfactory. One of the problems is the d in:

$$R_e = \frac{V_e d}{\mu} \quad (4.1)$$

The d is taken to be some average grain size which leads

to problems when trying to correlate data. There are many other reasons for the lack of correlation of experimental results which are adequately discussed in references (6) and (11).

This discussion is restricted to the region in which Darcy's law is valid. The same assumption is made in the fields of hydrology and petroleum engineering with satisfactory results.

4.3 The Equation of Fluid Motion in an Aquifer

The "pressure-diffusion" equation is generated by substituting the Darcy equation into the continuity equation. However, there are two fundamentally different approaches. One approach assumes that only the fluid is elastic, while the other assumes that both the fluid and the porous structure are elastic. In either case, the form of the resulting equation is the same. The derivation for the case of the elastic fluid alone will be given; the equation taking into account the elasticity of the fluid and the porous structure together follows along similar lines and will be stated without derivation.

The continuity equation for compressible unsteady flow in a rigid porous structure may be written (6):

$$\nabla \cdot (\rho \bar{v}) = -\theta \frac{d\rho}{dt} \quad (4.2)$$

where \bar{v} is the vectorial fluid velocity. Darcy's law may be written in the form:

$$\bar{v} = -\frac{k}{\mu} \nabla p \quad (4.3)$$

The usual thermodynamic pressure density relation for an isothermal compressible fluid is:

$$\rho = \rho_0 e^{\beta_1 (p - p_0)} \quad (4.4)$$

where β_1 is the isothermal compressibility and p_0 and ρ_0 are reference pressure and density. Substituting the Darcy equation into the continuity equation, eliminating the pressure by the ρ - p relation, and assuming the compressibility constant:

$$\nabla^2 \rho = \frac{\theta \beta_1 \mu}{k} \frac{d\rho}{dt} \quad (4.5)$$

Equation 4.5 is given by Muskat (6) for the flow of a compressible fluid in an aquifer.

Assuming that both the water and the porous structure are elastic, Jacob (13) (14) derived the follow-

4

Temperature differences of 50° F. have a negligible influence on the compressibility of water. Ovid W. Eshbach, ed., Handbook of Engineering Fundamentals, second edition, New York: John Wiley & Sons, Inc., 1952, p.6-05.

ing equation:

$$\nabla^2 p = \frac{\theta \mu}{k} \left[\frac{1}{E_w} + \frac{1}{\theta E_s} \right] \frac{dp}{dt} \quad (4.6)$$

where E_w and E_s are the moduli of elasticity of the water and the porous structure respectively. It can be seen that when the porous structure is incompressible, $E_s \rightarrow \infty$, equation 4.6 reduces to equation 4.5. Equation 4.6 may be written

$$\nabla^2 p = \frac{1}{\alpha^2} \frac{dp}{dt} \quad (4.7)$$

where:

$$\frac{1}{\alpha^2} = \left[\frac{1}{E_w} + \frac{1}{\theta E_s} \right] \frac{\theta \mu}{k} \quad (4.7a)$$

Further, Scheidegger (11) points out that if only the first two terms of the Taylor expansion of equation 4.4 are used, equation 4.7 may be written:

$$\nabla^2 p = \frac{1}{\alpha^2} \frac{dp}{dt} \quad (4.8)$$

Thus, only under this approximation does the same linear differential equation apply to the pressure and the density. Since β_1 for water is on the order of 10^{-4} , this is a good approximation.

If the fluid were incompressible, equation 4.2

would become:

$$\nabla \cdot \bar{v} = 0$$

and equation 4.8 would become:

$$\nabla^2 p = 0$$

Thus there would be no time history of the pressure distribution.

In addition to the pressure equation, it is necessary to derive an equation for the temperature distribution.

4.4 The Energy Equation

The energy equation is derived by applying the principle of conservation of energy for an element.

Consider a radial element of unit depth as shown in Figure 4.2. Assume that heat is transferred by conduction and convection in the radial direction only. Also, assume that the temperature of the fluid is the same as the temperature of the adjacent porous structure.

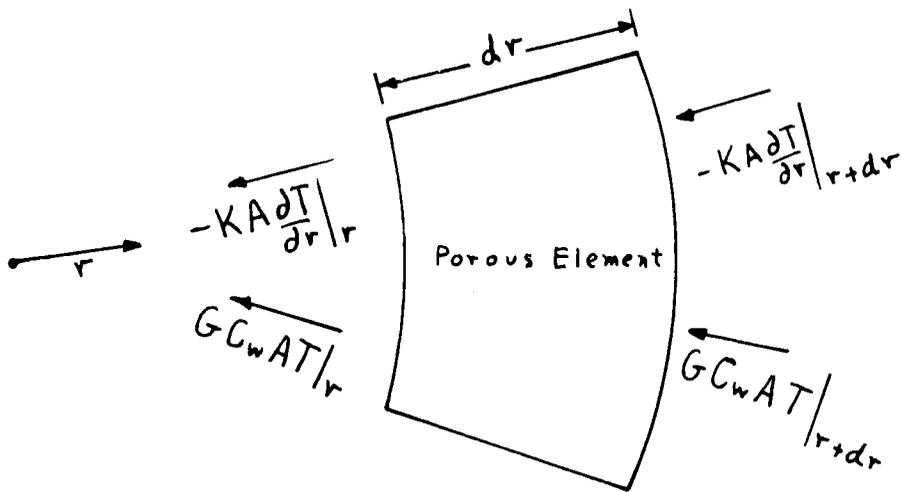


Figure 4.2

Flow In - Flow Out = Energy Storage Rate

$$-K 2\pi r \frac{dT}{dr} \Big|_r + G_w 2\pi r G T \Big|_r + K 2\pi (r+dr) \frac{dT}{dr} \Big|_{r+dr}$$

$$- G_w 2\pi (r+dr) G T \Big|_{r+dr} = V \left[\rho_w C_w \theta + \rho_s C_s (1-\theta) \right] \frac{dT}{dt}$$

Substituting the Darcy equation, neglecting higher order terms, combining and simplifying, the following results:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{C_w \rho_w k}{K \mu} \left[\frac{T}{r} \frac{\partial p}{\partial r} + \frac{\partial p}{\partial r} \frac{\partial T}{\partial r} + T \frac{\partial^2 p}{\partial r^2} \right] \quad (4.9)$$

$$= \left[\frac{(1-\theta) \rho_s C_s + \rho_w C_w \theta}{K} \right] \frac{\partial T}{\partial t}$$

Choosing the following:

$$\gamma^2 = \frac{C_w \rho_w k}{K \mu} \quad (4.10)$$

and

$$\frac{1}{\beta^2} = \left[\frac{(1-\theta) \rho_s C_s + \rho_w C_w \theta}{K} \right] \quad (4.11)$$

equation 4.9 becomes

$$\frac{1}{\beta^2} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \gamma^2 \left[\frac{T}{r} \frac{\partial p}{\partial r} + \frac{\partial p}{\partial r} \frac{\partial T}{\partial r} + T \frac{\partial^2 p}{\partial r^2} \right] \quad (4.12)$$

Equations 4.8 and 4.12 comprise the set of simultaneous partial differential equations for the flux of heat and mass in an aquifer.

4.5 Statement of the Problem

It is now possible to make a mathematical statement of the thought model discussed in 3.1. Consider Figure 4.3. Vaporization takes place at the well bore $r=a$; at $r=b$, the outer radius of the aquifer, the

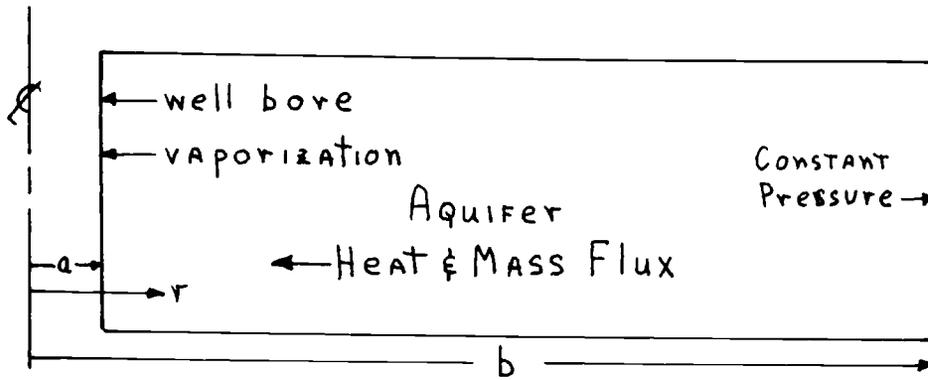


Figure 4.3

pressure and temperature are constant. In a real well, the dimension of a is on the order of one-half a foot while the size of b , as stated in 2.1, may be on the order of a few miles to a few hundred miles.

To describe the problem, equations 4.8 and 4.12 are stated with the following conditions:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} = \frac{1}{\alpha^2} \frac{\partial p}{\partial t} \quad (4.8)$$

$$\left. \begin{array}{l} t = 0 \quad p = p_{\infty}, \text{ All } r \\ t > 0 \quad \left. \begin{array}{l} r = a \quad p = p_s \\ r = b \quad p = p_{\infty} \end{array} \right\} \end{array} \right\} \quad (4.13)$$

and

$$\frac{1}{\beta^2} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \gamma^2 \left[\frac{T}{r} \frac{\partial p}{\partial r} + \frac{\partial p}{\partial r} \frac{\partial T}{\partial r} + T \frac{\partial^2 p}{\partial r^2} \right] \quad (4.12)$$

$$\left. \begin{array}{l} t = 0 \quad T = T_{\infty}, \text{ All } r \\ t > 0 \quad r = b \quad T = T_{\infty} \\ r = a \quad -K \frac{\partial T}{\partial r} \Big|_a + h_f G \Big|_a = h_v G \Big|_a \end{array} \right\} \quad (4.14)$$

h_f and h_v are the enthalpy of fluid and vapor respectively.

Equations 4.8, 4.12, 4.13, and 4.14 constitute the mathematical statement of the problem.

The vaporization boundary condition may be simplified by substituting the Darcy equation for the mass flux as follows:

$$+ K \frac{\partial T}{\partial r} \Big|_a + \frac{h_f k_i \rho}{\mu} \frac{\partial p}{\partial r} \Big|_a = \frac{h_v k_i \rho}{\mu} \frac{\partial p}{\partial r} \Big|_a \quad (4.17)$$

where k_i is the permeability of a porous liner installed at the well bore for control as discussed in 3.2. If the difference between the enthalpy of the fluid arriving at the well bore and the enthalpy of saturated fluid at vaporization conditions is neglected, equation 4.17 becomes:

$$\frac{dT}{dr}|_a = \frac{h_{fg} k_p}{K_\mu} \frac{dp}{dr}|_a \quad (4.18)$$

The enthalpy of vaporization h_{fg} is a function of both the temperature and the concentration of dissolved solids. Unterberg (15) gives several references for the enthalpy of vaporization of waters containing various concentrations of dissolved solids.

CHAPTER 5

ATTEMPTED SOLUTIONS TO THE EQUATIONS

5.1 Non-Dimensionalizing the Equations

Equations 4.8 and 4.12 can be non-dimensionalized by choosing the following parameters:

$$T' = \frac{T}{T_\infty} ; \rho' = \rho/\rho_\infty ; r' = \frac{r}{a} ; t' = t \left(\frac{\rho_\infty}{\mu} \right) \quad (5.1)$$

Substituting, and defining the dimensionless terms corresponding to α^2 , β^2 and γ^2 :

$$\frac{1}{\beta^2} \frac{\partial T'}{\partial t'} = \frac{\partial^2 T'}{\partial r'^2} + \frac{1}{r'} \frac{\partial T'}{\partial r'} + \gamma^2 \left[\frac{T'}{r'} \frac{\partial \rho'}{\partial r'} + \frac{\partial \rho'}{\partial r'} \frac{\partial T'}{\partial r'} + T' \frac{\partial^2 \rho'}{\partial r'^2} \right] \quad (5.2)$$

and

$$\frac{1}{\alpha'^2} \frac{\partial \rho'}{\partial t'} = \frac{\partial^2 \rho'}{\partial r'^2} + \frac{1}{r'} \frac{\partial \rho'}{\partial r'} \quad (5.3)$$

where

$$\left. \begin{aligned} \frac{1}{\beta'^2} &= \frac{a^2 \rho_\infty}{\beta^2 \mu} ; \gamma'^2 = \gamma^2 \rho_\infty \\ \frac{1}{\alpha'^2} &= \frac{a^2 \rho_\infty}{\alpha^2 \mu} \end{aligned} \right\} \quad (5.4)$$

The boundary conditions become:

$$T = T_{\infty} \quad T' = 1 \quad r' = b/a$$

$$P = P_{\infty} \quad P' = 1 \quad r' = b/a$$

$$P = P_s \quad P' = P_s/P_{\infty} \quad r' = 1$$

$$\left. \frac{\partial T'}{\partial r'} \right|_1 = \frac{h_{fg} k_1 P_{\infty}}{K \mu T_{\infty}} \left. \frac{\partial P'}{\partial r'} \right|_1$$

For simplicity, drop the primes keeping in mind that the equations are now in dimensionless form.

$$\frac{1}{\alpha^2} \frac{\partial P}{\partial t} = \frac{\partial^2 P}{\partial r^2} + \frac{1}{r} \frac{\partial P}{\partial r} \quad (5.5)$$

$$\left. \begin{array}{l} t = 0 \quad P = 1, \text{ All } r \\ t > 0 \quad P = P_s/P_{\infty}, r = 1 \\ \quad \quad P = 1, r = b/a \end{array} \right\} (5.6)$$

and

$$\frac{1}{\beta^2} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \gamma^2 \left[\frac{T}{r} \frac{\partial P}{\partial r} + \frac{\partial P}{\partial r} \frac{\partial T}{\partial r} + T \frac{\partial^2 P}{\partial r^2} \right] \quad (5.7)$$

$$t = 0 \quad T = 1 \quad \text{All } r$$

$$t > 0 \quad r = \frac{b}{a}, \quad T = 1 \quad (5.8)$$

$$r = 1, \quad \left. \frac{\partial T}{\partial r} \right|_1 = \frac{h_p q k_i \rho \rho_\infty}{k \mu T_\infty} \left. \frac{\partial p}{\partial r} \right|_1$$

Some problems can be simplified by neglecting terms of relatively small magnitude in the governing equations. However, the physics of this problem does not give rise to any simplification of either equations 5.5 or 5.7 by an order of magnitude analysis. Therefore, all the terms in both equations must be retained.

5.2 Separation of Variables

The pressure equation may be solved by the standard separation of variables technique. The solutions for several sets of boundary conditions are given in Carslaw and Jaeger (16). The solution to equation 5.5 with conditions 5.6 is:

$$\begin{aligned}
P = & \frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{\nu_n^2 J_0^2(\nu_n) e^{-\alpha^2 \nu_n^2 t}}{J_0^2(\nu_n) - J_0^2\left(\frac{b}{a} \nu_n\right)} \int_1^{b/a} r U_0(r \nu_n) dr \\
& - \pi \sum_{n=1}^{\infty} \frac{[J_0(\nu_n) - P^s/p_{\infty} J_0\left(\frac{b}{a} \nu_n\right)] U_0(r \nu_n) e^{-\alpha^2 \nu_n^2 t}}{J_0^2(\nu_n) - J_0^2\left(\frac{b}{a} \nu_n\right)} \quad (5.9) \\
& + \frac{P^s/p_{\infty} \ln\left(\frac{b}{a} \frac{1}{r}\right) + \ln r}{\ln\left(\frac{b}{a}\right)}
\end{aligned}$$

where:

$$U_0(r \nu_n) = J_0(\nu_n r) Y_0\left(\nu_n \frac{b}{a}\right) - J_0\left(\nu_n \frac{b}{a}\right) Y_0(\nu_n r) \quad (5.10)$$

Carslaw and Jaeger have calculated the first five eigenvalues ν_n for the solution. The solution to equation 5.5 when the pressure is specified only at the well bore is also given by Carslaw and Jaeger.

Since separable solutions of the form

$$p = T_i(t) R_i(r) \quad (5.11)$$

were valid for solving the pressure equation, it appeared logical to separate this form into the energy equation

and attempt to obtain separable solutions for $T(r,t)$ in the form $T = \tau_2(t) R_2(r)$. Following this approach:

$$\frac{\dot{\tau}_2 R_2}{\beta^2} = \tau_2 R_2'' + \frac{1}{r} \tau_2 R_2' + \gamma^2 \left[\tau_1 R_1' + \tau_1 R_1' \tau_2 R_2' + \tau_2 R_2 \tau_1 R_1'' \right] \quad (5.12)$$

At this point, it becomes apparent that separable solutions do not apply to the energy equation.

5.3 Similarity Solutions

A. Transformation

It is possible to transform equations 5.5 and 5.7 into two ordinary differential equations. A technique for accomplishing this transformation was developed by Dr. Russell E. Petersen.⁵ Using the dimensionless quantities of 5.1, assume the existence of a similarity parameter η , and new functions of the dependent variable of the following form:

$$\eta = t^{a'} r^{b'}, \quad T = r^c F(\eta) \quad (5.13)$$

$$\rho = r^e H(\eta)$$

⁵ Dr. Russell E. Petersen, Associate Professor of Mechanical Engineering at The University of Arizona. An undocumented technique presented in lectures on convective heat transfer, spring semester, 1962.

Substituting equations 5.13 into equation 5.5 yields:

$$r^2 b'^2 \eta^2 H'' + \left[(2b'e + b'^2) r^2 \eta - \frac{a't}{\alpha^2} a'^{-1} b' \right] H' + e^2 r^{-2} H = 0 \quad (5.14)$$

where the H denotes differentiation with respect to η . The only way to make equation 5.14 a function of η alone is to choose $a' = 1$ and $b' = -2$. With e still arbitrary, equation 5.5 transforms to:

$$4\eta^2 H'' + \left[(4 - 4e)\eta - \frac{1}{\alpha^2} \right] H' + e^2 H = 0 \quad (5.15)$$

Applying the same technique to equation 5.6 with F' denoting differentiation with respect to η :

$$\begin{aligned} \frac{r^{c-2}}{\beta^2} F' &= 4r^{c-2} \eta^2 F'' - 2[2c-3] r^{c-2} \eta F' + c(c-1) r^{c-2} F \\ &\quad - 2\eta r^{c-2} F' + c r^{c-2} F + \gamma^2 r^{c+e-2} b \eta H' F \\ &\quad + \gamma^2 e r^{c+e-2} H F - 2\gamma^2 b \eta^2 r^{c+e-2} F' H' + \gamma^2 c e r^{c+e-2} H F \\ &\quad - 2\gamma^2 \eta e r^{c+e-2} F' H + \gamma^2 c b \eta r^{c+e-2} F H' + \gamma^2 b^2 \eta^2 r^{c+e-2} H'' F \\ &\quad + \gamma^2 (2be + b^2 - b) \eta r^{c+e-2} H' F + e(e-1) r^{c+e-2} H F \end{aligned} \quad (5.16)$$

Since the term r^{c-2} appears in every term, it falls out

and C is arbitrary. In order for r^e to fall out of equation 5.16, e must be zero. Choosing c and e to be zero, equation, 5.7 finally transforms to:

$$4\eta^2 F'' + \left[4\eta + 4\gamma^2 \eta^2 H' - \frac{1}{\beta^2} \right] F' + \left[4\gamma^2 \eta H' + 4\gamma^2 \eta^2 H'' \right] F = 0 \quad (5.17)$$

C was not arbitrarily chosen zero. The second boundary condition of equation 5.20 is not independent of r unless C is chosen to be zero.

And equation 5.15 becomes;

$$4\eta^2 H'' + \left[(4 - 4e)\eta - \frac{1}{\alpha^2} \right] H' = 0 \quad (5.18)$$

The two simultaneous partial differential equations have now been transformed to ordinary differential equations.

The similarity variable is $\eta = t/r^2$ with $\rho = H(\eta)$ and $T = F(\eta)$ as the new dependent variables.

The boundary conditions must also be transformed. When time is zero, the similarity parameter η is zero. η is also zero when $r \rightarrow \infty$. The similarity parameter η is ∞ when $t = \infty$ or when $r \rightarrow 0$.

In order to use these simplifying conditions, it is necessary to assume that the physical boundaries of the problem are zero and infinity instead of a and b . From the discussion of the physical size of aquifers, this is a reasonable assumption. The boundary conditions

for equation 5.18 may be written:

$$\begin{aligned} \eta = 0 & \quad H(0) = 1 \\ \eta = \infty & \quad H(\infty) = \rho_s / \rho_\infty \end{aligned} \quad (5.19)$$

and the conditions on equation 5.17 may be written:

$$\begin{aligned} \eta = 0 & \quad F(0) = 1 \\ \eta = \infty & \quad F'(\infty) = \frac{h_{fg} k_f \rho_\infty}{K \mu T_\infty} H'(\infty) \end{aligned} \quad (5.20)$$

B. The Pressure Equation

The pressure equation 5.18 has an irregular singular point of class one according to the classification by Ince (17). Equations with irregular singular points usually require special techniques, but due to its simplicity, equation 5.18 is directly integratable. Rewriting equation 5.18 in the form:

$$H'' + \left[\frac{1}{\eta} - \frac{1}{4\alpha^2 \eta^2} \right] H' = 0 \quad (5.21)$$

Substitute $U = H'$

$$U' + \left[\frac{1}{\eta} - \frac{1}{4\alpha^2 \eta^2} \right] U = 0$$

and integrate once:

$$u_\eta = C e^{-\frac{1}{4\alpha^2}\eta}$$

where C is a constant of integration.

Integrate again

$$H = \int_0^\eta C \frac{e^{-\frac{1}{4\alpha^2}\eta}}{\eta} d\eta + C_1 \quad (5.22)$$

using the boundary conditions 5.19.

$C_1 = 1$ and

$$C = \frac{(P_s/P_\infty - 1)}{\int_0^\infty \frac{e^{-\frac{1}{4\alpha^2}\eta}}{\eta} d\eta} \quad (5.23)$$

However, the integral in the denominator of equation 5.23 is not bounded.

Under the transformation:

$$\beta = \frac{1}{4\alpha^2\eta}$$

the integral in equation 5.23 becomes:

$$\int_0^{\infty} \frac{e^{-\beta}}{\beta} d\beta \quad (5.24)$$

Equation 5.24 is one of the exponential integrals; it is unbounded at $\beta = 0$ as shown in the C.R.C. Mathematical Table (18). This fact eliminates the possibility of obtaining a solution to equation 5.18 with a boundary condition at $\eta = \infty$.

Another approach to obtaining a solution to 5.18 is to change the boundary condition at $\eta = \infty$. For the new boundary condition, assume:

$$\eta = \eta_1 \quad H(\eta_1) = \rho_s / \rho_{\infty} \quad (5.25)$$

where η_1 is very large but not infinity. From equation 5.22:

$$C = (\rho_s / \rho_{\infty} - 1) / \int_0^{\eta_1} \frac{e^{-\frac{1}{4\alpha^2}\eta}}{\eta} d\eta \quad (5.26)$$

And the solution to equation 5.18 becomes:

$$H = 1 + (\rho_s/\rho_\infty - 1) \frac{\int_0^\eta \frac{e^{-\frac{1}{4\alpha^2}\eta}}{\eta} d\eta}{\int_0^{\eta_1} \frac{e^{-\frac{1}{4\alpha^2}\eta}}{\eta} d\eta} \quad (5.27)$$

The integral in equation 5.26 is not tabulated over a large range of values; a technique for evaluating the integral is given in Appendix I.

Two things are apparent which indicate that equation 5.27 is not a valid similarity solution to the physical problem.

The boundary condition at $r = 1$ was $\rho = \rho_s/\rho_\infty$; it transformed to the similarity condition $\eta = \infty$, $H = \rho_s/\rho_\infty$. Instead of using $\eta = \infty$, η was chosen to be a finite, but very large, constant η_1 . Since $\eta = t/r^2$, the boundary condition at $r = 1$ is:

$$\eta_1 = \frac{t}{1} \quad (5.28)$$

This means that the solution to equation 5.27 is not unique, but is dependent on the value of time in 5.28.

A family of solutions for η_1 is obtained. This happens because the boundary condition is not a function

of the similarity parameter; rather, it is a function of one of the original parameters which make up the similarity parameter.

Another way of looking at it is to say that in order to keep η_1 constant, it is necessary that the physical boundary at $r = a$ moves with increasing time of the solution.

$$t = \eta_1 r^2$$

The boundary at r moves as

$$r = \sqrt{t/\eta_1}$$

which is not a physically acceptable model of the problem.

These two reasons are strong factors which indicate that a similarity solution to the pressure equation is not possible. The primary reason is that the physical problem does not generate similarity boundary conditions which, when applied to the transformed equations, will yield valid solutions.

C. The Energy Equation

Using equation 5.27, it is possible to obtain a solution to equation 5.18 by transforming the irregular singular point and using the method of Frobenius. How-

ever, it is not of any use since the solution to the pressure equation is not felt to be valid.

5.4 Laplace Transforms

The pressure equation 5.5 may be transformed by the Laplace technique and solved. However, no method was found to take the Laplace transform of the energy equation 5.7. In the energy equation, terms such as:

$$\int_0^{\infty} \frac{\partial p}{\partial r} \frac{\partial T}{\partial r} e^{-\Delta t} dt$$

appear. Thus, the Laplace transform technique was discarded.

5.5 Analogue Computer Solutions

Equations 5.17 and 5.18 were discussed with Granino A. Korn, Professor of Electrical Engineering, at The University of Arizona. It was his opinion that the equations could be solved on an analogue computer. However, since there are two independent variables, finite differences would have to be employed and the solution would be approximate. He also said that to do the problem, it is necessary to have a large industrial type computer. Neither his department nor the Mechanical Engineering Department has computer facilities large enough to handle the problem. Therefore, due to a lack

of facilities, an analogue computer solution was not attempted.

CHAPTER 6

CONCLUSIONS

6.1 Summary of the Work

The separation of variables technique was applicable to the pressure equation, but it would not work on the temperature equation. The Laplace transform approach was rejected for a similar reason. The analogue computer study was not attempted due to a lack of equipment.

Of all the approaches attempted, the similarity variable approach was pursued the farthest. However, it was concluded that the problem is fundamentally not a similarity problem.

6.2 Suggested Approach to the Solution

In a problem of this type, it is preferable to have an analytical solution in closed form. The primary reason is that a closed analytical solution can be studied for a wide range of the physical parameters involved. However, when no analytical solution is apparent, it is advisable to go to numerical techniques.

The solutions to this problem by finite differences are best generated on a digital computer. The statement of the problem in finite differences with sug-

gestions for particular techniques of solution will be given here.

It is possible to solve both equations 5.5 and 5.7 by finite differences but this is not the best way to arrive at an accurate solution of the problem. Since an analytical solution to equation 5.5 is known, it is better to use this than to calculate a finite difference solution. Consider equation 5.7:

$$\frac{1}{\beta^2} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \gamma^2 \left[\frac{T}{r} \frac{\partial p}{\partial r} + \frac{\partial p}{\partial r} \frac{\partial T}{\partial r} + T \frac{\partial^2 p}{\partial r^2} \right] \quad (5.7)$$

It is necessary to have both the

$$\frac{\partial p}{\partial r} \quad \text{and} \quad \frac{\partial^2 p}{\partial r^2}$$

to solve the energy equation. A great deal of accuracy can be lost in obtaining these derivatives from a numerical solution of $p = p(t, r)$. Considerable error can be eliminated if these partial derivatives are obtained from the analytical solution.

Following the technique of Salvadori and Baron (19), equation 5.7 may be written in finite differences. Let $T_{i,j}$ and $p_{i,j}$ be the temperature and pressure at any position $r = r_i$ and time $t = t_j$. Due to the symmetry of the problem, the temperature may be represented in a two-dimensional mesh as shown in Figure 6.1.

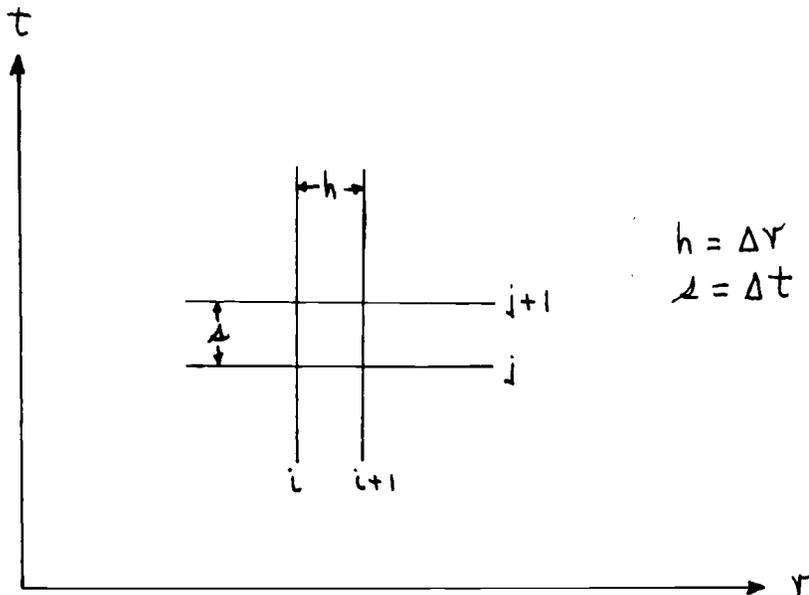


Figure 6.1

The $\frac{\partial T}{\partial t}$ is expressed in forward differences with an error on the order of Δ :

$$\frac{\partial T}{\partial t}{}_{i,j} = \frac{T_{i,j+1} - T_{i,j}}{\Delta} \quad (6.1)$$

The first two terms on the right-hand side of equation 5.7 may be written with central differences having an error term on the order of h^2 :

$$\frac{\partial^2 T_{i,j}}{\partial r_i^2} + \frac{1}{r_i} \frac{\partial T_{i,j}}{\partial r_i} = \frac{1}{h^2} \left[\left(1 + \frac{h}{2r_i}\right) T_{i+1,j} - 2T_{i,j} + \left(1 - \frac{h}{2r_i}\right) T_{i-1,j} \right] \quad (6.2)$$

and $\frac{\partial T_{i,j}}{\partial r_i}$ may also be written in central difference opera-

tors with an error on the order of h^2 as:

$$\frac{\partial T_{i,j}}{\partial r_i} = \frac{1}{2h} [T_{i+1,j} - T_{i-1,j}] \quad (6.3)$$

Combining terms, equation 5.7 may be expressed in finite differences as:

$$\begin{aligned} \frac{1}{\Delta \beta^2} [T_{i,j+1} - T_{i,j}] &= \frac{1}{h^2} \left[\left(1 + \frac{h}{2r_i}\right) T_{i+1,j} - 2T_{i,j} + \left(1 - \frac{h}{2r_i}\right) T_{i-1,j} \right] \\ &+ \frac{\gamma^2 T_{i,j}}{r_i} \rho'_{i,j} + \rho'_{i,j} \frac{\gamma^2}{2h} (T_{i+1,j} - T_{i-1,j}) + \gamma^2 T_{i,j} \rho''_{i,j} \end{aligned} \quad (6.4)$$

where $\rho'_{i,j}$ is the derivative of equation 5.8 with respect to r , the radial distance, at $r = r_i$ and $t = t_j$. A convenient recurrence equation which allows the determination of the temperature at the point $r = r_i$ and at time $t = t_{j+1}$, may be written from equation 6.4:

$$\begin{aligned} T_{i,j+1} &= \left[\frac{\Delta \beta^2}{h^2} \left(1 + \frac{h}{2r_i}\right) + \frac{\rho'_{i,j} \gamma^2 \Delta \beta^2}{2h} \right] T_{i+1,j} \\ &+ \left[1 - \frac{2\Delta \beta^2}{h^2} + \gamma^2 \frac{\rho'_{i,j} \Delta \beta^2}{r_i} + \gamma^2 \rho''_{i,j} \right] T_{i,j} \\ &+ \left[\frac{\Delta \beta^2}{h^2} \left(1 - \frac{h}{2r_i}\right) - \frac{\Delta \beta^2 \rho'_{i,j} \gamma^2}{2h} \right] T_{i-1,j} \end{aligned} \quad (6.5)$$

The boundary conditions may be written:

$$r_i = \frac{b}{a} \quad T_{i,j} = 1$$

$$r_i = 1 \quad \frac{1}{2h} [T_{i+1,j} - T_{i-1,j}] = \frac{h f_0 k_1 e \rho_\infty}{K \mu T_\infty} \beta'_{i,j}$$

$$t_j = 0 \quad T_{i,j} = 1$$

It is important to note that the $\frac{\partial T}{\partial t}$ was expressed with an error term on the order of Δ while the rest of equation 5.7 has error terms on the order of h^2 . Therefore, Δ should be chosen very small to insure accuracy. For every point in the region of Figure 6.1, the coefficients in equation 6.5 must be given numbers in order to compute the solution.

Equation 6.5 is the solution to the energy equation; however, computing it is a long, tedious process well suited to a digital computer.

APPENDIX I

$$I(\eta) = \int_0^{\eta_1} \frac{e^{-\frac{1}{4\alpha^2\eta}}}{\eta} d\eta$$

$$\beta = \frac{1}{4\alpha^2\eta}$$

$$I(\beta) = \int_{\frac{1}{4\alpha^2\eta_1}}^{\infty} \frac{e^{-\beta}}{\beta} d\beta$$

The exponential integral

$$\int_{0.1}^{\infty} \frac{e^{-\beta}}{\beta} d\beta = 1.823$$

The remaining part

$$\int_{\epsilon}^{0.1} \frac{e^{-\beta}}{\beta} d\beta \quad \text{where: } \epsilon = \frac{1}{4\alpha^2\eta_1}$$

may be calculated by expanding the exponential series and

integrating term by term

$$\ln \beta - \beta + \frac{\beta^2}{4} - \frac{\beta^3}{3 \cdot 3!} + \frac{\beta^4}{4 \cdot 4!} - \frac{\beta^5}{5 \cdot 5!} + \dots \Big]_{\epsilon}^{0.1}$$

The series is rapidly convergent and it affords a means of numerically evaluating $\Gamma(\eta_1)$.

GLOSSARY

a	well bore radius of the aquifer, ft.
A	area of an elemental surface, ft. ²
b	external radius of the aquifer, ft.
C _w	specific heat of water, (Btu/Lbm °F.).
C _s	specific heat of porous structure.
E _w	isothermal modulus of elasticity of water.
E _s	isothermal modulus of elasticity of a porous structure.
F	dimensionless similarity variable for temperature.
G	mass flux Lbm/(ft. ² -sec.)
h	mesh dimension for finite differences.
h _f	fluid enthalpy (Btu/Lbm).
h _n	vapor enthalpy.
h _{fg}	enthalpy of vaporization of brine.
H	dimensionless similarity variable for pressure.
J ₀	Bessel function of zero order and first kind.
k	permeability, defined by equation 2.1, ft. ²
K	thermal conductivity of an aquifer, (Btu/sec.ft.°F.)
k ₂	permeability of the liner in the well bore.
P	pressure #/ft. ²
P _a	atmospheric pressure.
P _s	pressure in the well bore.

P_{00}	initial pressure in the aquifer.
r	radial coordinate of the aquifer, ft.
R_e	Reynold's number.
s	mesh dimension for finite differences.
t	time, sec.
T	Temperature $^{\circ}\text{F}$.
T_{00}	initial temperature in the aquifer.
\bar{v}	vectorial fluid velocity, ft./sec.
V	total volume, ft. ³
w	volume of void space, ft. ³
Y_0	Bessel function of the second kind order zero.
α^2	constant defined by equation 4.7a, ft. ² /sec.
β^2	constant defined by equation 4.11, ft. ² /sec.
β_1	isothermal fluid compressibility
	$\beta_1 = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial p} \right)_T \quad \text{has dimensions } \frac{\text{Ft}^2}{\#}$
η	dimensionless similarity independent variable $\eta = t/r^2$
θ	dimensionless porosity defined by equation 2.1.
μ	viscosity, (#-sec)/ft. ²
ν_n	eigenvalue in equation 5.8.
ρ	mass density, Lbm/ft. ³
ρ_w	water density
ρ_s	density of the porous structure.

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