1 Introduction

The Standard Model explains all the observed forces (except gravity) to some degree of success based on gauge fields arising from the requirement of local gauge invariance. For consistency with this picture, the masses of the fundamental particles are explained by interactions with a scalar field which spontaneously breaks the local gauge invariance through the Higgs mechanism.

Interactions with the Higgs field are not limited to the generation of mass. It is well known, for instance, that the theory predicts a new particle, the Higgs boson. Experimental and theoretical work suggest strongly that the Higgs mass lay roughly between 100 and 200 GeV, and many expect the Higgs to be detected at the LHC [1].

Also of interest are possible non-perturbative interactions with the Higgs field. The top quark is predicted to have a Yukawa coupling constant to the Higgs field of \( g_t = 0.99 \pm 0.01 \). This remarkable value strongly suggests that non-perturbative effects may be observable. Froggatt and coworkers, in a series of several papers [6,4,5], make a non-relativistic calculation assuming
the dynamical restoration of the electroweak symmetry, and come to the conclusion that large numbers of top quarks can bind very tightly through Higgs exchange and even form massless bound states.

The physical intuition that guided Froggat’s work is as follows. First, scalar forces are always attractive. Therefore, the Higgs-exchange binding energy of a many-body system grows with the square of the number of participants. In contrast, binding due to vector interactions is screened by like charges as more participants are added. Second, due to the color and spin degrees of freedom, many quarks can bind without any need to decrease binding by symmetrizing the spatial wave function. Specifically, up to six top quarks (3 colors × 2 spin projections) can bind with up to six anti-top quarks without any symmetrization. The resulting state would be a spin 0 color singlet.

While the other quarks form QCD bound states such as charmonium and bottomium, it is believed that the top lifetime of $\tau = 0.5 \times 10^{-24}$ seconds [1] is too short for such a state to form. In a Higgs top quark bound state, QCD would play a small role due to the discussed suppression by a factor of $1/N$ and the asymptotic freedom of the QCD coupling, which gives a relevant values of $\alpha_{QCD} \approx 0.1$. The electroweak gauge bosons will contribute even less to the binding, with couplings of order $\alpha = 1/137$.

In this work, we have made a fully relativistic study of a system of many top quark in interaction with the Higgs field and in doing so given much more careful consideration of the electroweak symmetry restoration than have others. Our method is based on techniques used in earlier works to find soliton solutions for single fermions in interaction with the Higgs field [8], and in other works involving relativistic quark bag models [11,10].

In the following, I first set the context for our work with a description of the standard model and electroweak symmetry breaking via the Higgs mechanism. Next, based on [13], I describe in detail the derivation of our approach from a variational principal, focusing on the nuances arising from considering antiparticles. Then after shortly discussing the numerical methods used, I present the results. In general I find that given the experimental value of the top mass and the constraints on the Higgs mass, the strongly bound solutions predicted by Froggatt and coworkers do not exist. This is in agreement with a similar nonrelativistic mean-field treatment of the problem [9]. I conclude with future directions to improve the accuracy of the calculation and modifications to the theory that may realize the tightly bound states sought for by Froggatt et al. In the appendices I describe some of the notation used and
attempt to resolve a conflict between two similar works [8,3].

2 Background

2.1 Standard Model

The Standard Model Lagrangian for the interacting Higgs and third generation quark sector is, ignoring gauge couplings,

\[ \mathcal{L} = \mathcal{L}_f + \mathcal{L}_i - U(\Phi) \]  

(1)

The free quark and scalar contribution is

\[ \mathcal{L}_f = \overline{\Psi}_i \not{\partial} \Psi + \partial^\mu \phi^\dagger \partial_\mu \phi \]  

(2)

where \( \Psi = (t, b)^T \) is the doublet of top and bottom quark Dirac spinors \( t \) and \( b \) respectively. The Feynman dagger is interpreted \( \not{\partial} = \gamma^\mu \partial_\mu \), where the Dirac matrices \( \gamma^\mu \) are defined according to common convention. The overline is understood as \( \overline{\Psi} = (\overline{t}, \overline{b}) \) where \( \overline{t} = t^\dagger \gamma^0 \).

The interaction term is [7]

\[ \mathcal{L}_i = g_t (t_R^C \Phi^\dagger \Psi_L + \overline{\Psi}_L \Phi_C t_R) + g_b (b_R^C \Phi^\dagger L + \overline{L} \Phi b_R) \]  

(3)

The left-handed projection \( \Psi_L = (t_L, b_L)^T = \frac{1}{2} (1 - \gamma^5) \Psi \) is a SU(2)_L doublet, and the right-handed projection \( t_R = \frac{1}{2} (1 + \gamma^5) \Psi \) is a SU(2)_L singlet. The Higgs scalar SU(2) doublet is \( \Phi = (\phi^+, \phi^0)^T \), and \( \Phi_C = -i \sigma_2 \Phi^* \) also transforms as a SU(2) doublet. The top and bottom Yukawa couplings are \( g_t \) and \( g_b \) respectively.

The Higgs self-interaction potential is

\[ U(\Phi) = -\mu^2 \Phi^\dagger \Phi + \frac{\lambda_2}{2} (\Phi^\dagger \Phi)^2 \]  

(4)

where \( \mu \) and \( \lambda \) are parameters of the model. The relation of these parameters to physically observable quantities is described in the following section.

2.2 Electroweak Symmetry Breaking

When gauge terms are included, the Lagrangian Eq. 3 exhibits a SU(2)_L \times U(1)_Y local gauge symmetry. This symmetry is however not reflected in the observed fields. Specifically, observed fermions and gauge bosons have non-gauge invariant masses. This can be understood because the true vacuum of the model breaks the gauge symmetry.
Consider Eq. 4 in the context of classical fields. The equilibrium point at $|\Phi| = 0$ is unstable. Through differentiation with respect to $\Phi^\dagger \Phi$ we find the stable minimum occurs at

$$(\Phi^\dagger \Phi)_0 = \frac{\mu^2}{\lambda} \equiv \frac{v^2}{2},$$

and is degenerate under SU(2) transformations of the Higgs doublet. We therefore expect degenerate vacua in the quantum field theory with vacuum expectation values for the Higgs field. We choose a gauge such that the Higgs field of the observed true vacuum is

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \phi \end{pmatrix}$$

(6)

with $\phi$ a real scalar. The VEV of $\phi$ is then $\langle \phi \rangle = v$.

Expanding $\phi$ about its VEV,

$$\phi(x^\mu) = v + h(x^\mu),$$

(7)

we make the connection to the observed standard model particles. The gauge bosons acquire masses through their couplings to the higgs field, which lead to terms of the form $v^2 A^\mu A_\mu$. Experimental measurements of weak interactions give $v = 246$ GeV [1].

Fermions gain masses through the Yukawa coupling in Eq. 3. With Eq. 7, using $\bar{t}_L t_R + \bar{t}_R t_L = \bar{t} t$ we obtain for the mass term

$$\mathcal{L}_m = \frac{g v}{\sqrt{2}} \bar{t} t.$$

(8)

We therefore identify

$$m_t = \frac{g v}{\sqrt{2}}$$

(9)

The Higgs boson $h$ will also have a mass. Substituting Eq. 7 into the self-interaction Eq. 4 we find

$$U(h) = -\frac{\mu^4}{2\lambda} + \mu^2 h^2 + ....$$

(10)

Since $h$ is a real scalar, we therefore identify

$$m_h^2 = 2\mu^2 = \lambda v^2.$$

(11)

To summarize, our parameters are related to the observable values $v$, $m_t$, and $m_h$ via

$$\lambda = \frac{m_h^2}{v^2}; \quad g = \sqrt{\frac{\lambda}{v}}; \quad \mu^2 = \frac{m_h^2}{2}$$

(12)
3 Many-Top Bound States

3.1 Variational Approach

Disregarding the bottom quark contribution, the energy component of the stress energy tensor corresponding to Eq. 1 is

$$T_0^0 = \psi^\dagger \gamma^0 (-i \gamma^i \partial_i - g \phi) \psi + |\partial_0 \phi|^2 + |\nabla \phi|^2 + U(\phi),$$

(13)

where $| \bullet |^2$ indicates $\dot{\bullet} \bullet$ and from now on we write $\psi \equiv t$ for the top quark Dirac spinor. We have chosen a gauge where the SU(2) Higgs doublet is $\Phi = (0, \phi)^T$ with a real field $\phi$. The Dirac part of the q-number Hamiltonian corresponding to Eq. 13 is

$$\hat{H} = \int d^3x \hat{\psi}^\dagger(\vec{x}) \gamma^0 (-i \gamma^i \partial_i - g \hat{\phi}) \hat{\psi}(\vec{x}),$$

(14)

from which we identify the classical single particle Hamiltonian

$$\mathcal{H} = -i \gamma^0 \gamma^i \partial_i - \gamma^0 g \phi.$$  

(15)

Let us consider an arbitrary complete c-number basis of dirac Spinors $a_n(\vec{x})$ for the Hilbert space of Eq. 15. We define corresponding states

$$|a_n\rangle = \hat{a}_n^\dagger |0_a\rangle \equiv \int d^3x a_n(\vec{x}) \hat{\psi}^\dagger(\vec{x}) |0_a\rangle,$$

(16)

where $\hat{\psi}(\vec{x}) |0_a\rangle = 0$ defines $|0_a\rangle$.

In this basis, the Hamiltonian Eq. 14 becomes

$$\hat{H} = \sum_{i,j} \mathcal{H}_{ij} \hat{a}_i^\dagger \hat{a}_j$$  

(17)

where $\mathcal{H}_{ij} = \int d^3x a_i^\dagger(\vec{x})a_j(\vec{x})$ is the matrix element of the c-number fields. Splitting the positive and negative spectra we have

$$\hat{H} = \sum_{i,j | \mathcal{H}_{ij} > 0} \mathcal{H}_{ij} \hat{a}_i^\dagger \hat{a}_j + \sum_{i,j | \mathcal{H}_{ij} < 0} \mathcal{H}_{ij} \hat{a}_i^\dagger \hat{a}_j.$$  

(18)

Since the positive and negative portion of the Dirac spectra can be diagonalized independently, we find for the true vacuum state

$$\hat{H} |0\rangle = \left( \sum_{j | \mathcal{H}_{jj} < 0} \mathcal{H}_{jj} \right) |0\rangle.$$  

(19)
Subtracting as usual the vacuum energy Eq. 19 from the Hamiltonian Eq. 18 we obtain

$$\hat{H} \Rightarrow \sum_{i,j|\mathcal{H}_{ij}>0} \mathcal{H}_{ij} \hat{a}_i^\dagger \hat{a}_j + \sum_{i\neq j|\mathcal{H}_{ij}<0} \mathcal{H}_{ij} \hat{a}_i^\dagger \hat{a}_j + \sum_{j|\mathcal{H}_{jj}<0} \mathcal{H}_{jj} (\hat{a}_j^\dagger \hat{a}_j - 1).$$

(20)

With the anti-commutation relations,

$$\{ \hat{\psi}(\vec{x}), \hat{\psi}(\vec{x}') \} = \delta(\vec{x} - \vec{x}') \Rightarrow \{ \hat{a}_i^\dagger, \hat{a}_j \} = \delta_{ij},$$

(21)

we obtain

$$\hat{H} = \sum_{i,j|\mathcal{H}_{ij}>0} \mathcal{H}_{ij} \hat{a}_i^\dagger \hat{a}_j + \sum_{i\neq j|\mathcal{H}_{ij}<0} \mathcal{H}_{ij} \hat{a}_i^\dagger \hat{a}_j + \sum_{j|\mathcal{H}_{jj}<0} \mathcal{H}_{jj} (\hat{a}_j^\dagger \hat{a}_j).$$

(22)

Considering the diagonal part of Eq. 22, we are motivated to define respectively quasi and anti-quasi particle creation operators

$$\hat{\alpha}_j^{(s,c)}^\dagger = \hat{a}_j^\dagger = \int d^3x \hat{a}_j^{(s,c)}(\vec{x}) \hat{\psi}^{(s,c)}(\vec{x}) \quad (H_{jj} > 0),$$

(23)

$$= 0 \quad (H_{jj} < 0),$$

(24)

$$\hat{\beta}_j^{(s,c)}^\dagger = 0 \quad (H_{jj} > 0),$$

(25)

$$\hat{\beta}_j^{(s,c)}^\dagger = \hat{a}_j = \int d^3x \hat{a}_j^{(s,c)}(\vec{x}) \hat{\psi}^{(s,c)}(\vec{x}) \quad (H_{jj} < 0)$$

(26)

where we now write explicitly the spin and color degrees of freedom in the superscript. The corresponding annihilation operators have the property

$$\hat{\alpha}_j |0\rangle = \hat{\beta}_j |0\rangle = 0.$$
where \( N = N_t + N_{\tilde{t}} \), \( m_t \) is the top quark mass, and \( \hat{H} \) includes both Eq. 22 and the scalar field contributions from Eq. 13.

With these restrictions in mind, we follow [13] in constructing a trial state. The conditions Eq. 29 and Eq. 30 are satisfied by choosing a product of quasi and anti-quasi particles for the quark content of the state:

\[
|X_q\rangle = \hat{X}_q^\dagger |0\rangle = \sum \{s_i\}, \{c_i\} C_i \prod_j |\hat{\alpha}_{n_j}^{(s_i,c_i)}\rangle \prod_j |\hat{\beta}_{n_j}^{(s_i,c_i)}\rangle |0\rangle.
\] (32)

The sums over spins and color configurations build up the desired quantum numbers for the state. The constants \( C_i \) include the normalization of the state.

Since the Lagrangian Eq. 1 is independent of the spin and color degrees of freedom, for \( N_t, N_{\tilde{t}} \leq 6 \), we expect all particles to occupy the same state \( \psi_n \equiv \psi_p \) and likewise for antiparticles \( \psi_{\bar{n}} \equiv \psi_{\bar{p}} \). For larger top or anti-top numbers, Fermi statistics prevents the particles from occupying the same state. We are otherwise not concerned with the spin and color degrees of freedom, and we henceforth drop these explicit indices. Note that \( \psi_p \) and \( \psi_n \) are not yet specified but will rather be treated as variational parameters of our trial state.

The simplest choice for the Higgs field is a static coherent state [13]

\[
|X_H\rangle = \hat{X}_H^\dagger |0\rangle = \exp \left( -i \int d^3x \hat{\pi}(\vec{x},0) \psi(\vec{x}) |0\rangle \right); \quad \pi = \frac{\delta L}{\delta \phi}.
\] (33)

Such a state is conveniently an eigenstate of the annihilation operator,

\[
\hat{\pi}(\vec{x})|X_H\rangle = \phi_C(\vec{x})|X_H\rangle \Rightarrow \langle X_H|\hat{\pi}|X_H\rangle = \phi_C
\] (34)

This ansatz is essentially the classical limit of the Higgs field. Like the Dirac spinors, the field \( \phi_C \) is not specified and remains a variational parameter.

We have now our complete trial state from Eq. 33 and Eq. 32,

\[
|X\rangle = \hat{X}_H^\dagger \hat{X}_q^\dagger |0\rangle
\] (35)

and the complete top-Higgs Hamiltonian pieced together from Eq. 22 and Eq. 13:

\[
\hat{H} = \sum_{i,j:|\mathcal{H}_{ij}|>0} \mathcal{H}_{ij} \hat{a}_i^\dagger \hat{a}_j + \sum_{i \neq j:|\mathcal{H}_{ij}|<0} \mathcal{H}_{ij} \hat{a}_i^\dagger \hat{a}_j + \sum_{j:|\mathcal{H}_{jj}|<0} \left| \mathcal{H}_{jj} \right| \hat{a}_j^\dagger \hat{a}_j + |\partial_0 \hat{\phi}|^2 + |\vec{\nabla} \hat{\phi}|^2 + U(\hat{\phi}).
\] (36)
Only the diagonal elements of Eq. 36 contribute to the mass Eq. 31, yielding

\[
M = \int d^3x \left[ N_t \psi_p^\dagger(x) \gamma^0 \left( -i \gamma^i \partial_i - g \phi_C(x) \right) \psi_p(x) \right.
\]

\[
- N_t \psi_n^\dagger(x) \gamma^0 \left( -i \gamma^i \partial_i - g \phi_C(x) \right) \psi_n(x) + |\bar{\nabla} \phi_C|^2 + U(\phi_C) \right].
\]

(37)  

(38)  

Notice that the particle \( \psi_p \) and antiparticle \( \psi_n \) spinors enter with opposite signs so that both contribute positively to the energy.

The variation \( \delta M = 0 \) of Eq. 37 with respect to the c-number fields \( \psi_p, \psi_n \) and \( \phi_C \) leads to the classical equations of motions for the fields,

\[
\gamma^0 \left( -i \gamma^i \partial_i - g \phi_C \right) \psi_p = 0 \quad \text{(39)}
\]

\[
\gamma^0 \left( -i \gamma^i \partial_i - g \phi_C \right) \psi_n = 0 \quad \text{(40)}
\]

\[
\bar{\nabla}^2 \phi_C - \frac{1}{2} \frac{\partial U}{\partial \phi_C} = \frac{g(N_t \bar{\psi}_p \psi_p - N_t \bar{\psi}_n \psi_n)}{2}. \quad \text{(41)}
\]

The fields are further constrained by the normalization condition Eq. 29, which leads to

\[
\int d^3x \psi_p^\dagger \psi_p = \int d^3x \psi_n^\dagger \psi_n = 1. \quad \text{(42)}
\]

At first glance it looks like there might be a cancellation in Eq. 41 between the contributions from particles and antiparticles in the RHS. However, in the absence of a vector potential, the Dirac wave functions can be simply transformed between particles and antiparticles:

\[
\psi_n = \gamma_5 \psi_p^\ast. \quad \text{(43)}
\]

Since we are interested in the lowest energy state, we assume \( E_n = E_p = E_0 \), and therefore we can use the transformation Eq. 43 which leads to \( \psi_n \psi_n = -\bar{\psi}_p \psi_p \). With this Eq. 41 becomes,

\[
\bar{\nabla}^2 \phi_C - \frac{1}{2} \frac{\partial U}{\partial \phi_C} = \frac{g(N_t \bar{\psi}_p \psi_p)}{2}. \quad \text{(44)}
\]

In the future we drop the subscript \( \psi_p \rightarrow \psi \) and it is understood that we consider only the positive energy spectrum.

Lastly we assume that the groundstate wavefunctions are spherically symmetric so that we can write, following the notation of [8],

\[
\psi = \frac{1}{r} \begin{pmatrix} u_1(r) \chi_+ (\hat{r}) \\ u_1(r) \chi_- (\hat{r}) \end{pmatrix} e^{-iEt}, \quad \text{(45)}
\]

\[
\phi = v(r)/r, \quad \text{(46)}
\]
where $\chi$ are the standard spherical harmonics normalized to $|\chi|^2 = 1/4\pi$. With the equations of motion Eq. 39 and with $\kappa = -1$ for the ground state, we obtain

$$\frac{d}{dx} \left( -\frac{\kappa}{x} \right) u_1 = \left( \epsilon + \frac{gv}{x} \right) u_2$$

(47)

$$\frac{d}{dx} \left( \frac{\kappa}{x} \right) u_2 = \left( -\epsilon + \frac{gv}{x} \right) u_1$$

(48)

$$\left[ \frac{d^2}{dx^2} - \lambda \left( 1 - \frac{v^2}{x^2} \right) \right] v = \frac{Ng}{8\pi x} (u_1^2 - u_2^2).$$

(49)

The energy and length scales are now $x = \phi_0 r$ and $\epsilon = E/\phi_0$, with $\phi_0 = v/\sqrt{2} = 174.1$ GeV the amplitude of the higgs field $\phi$ at the minimum of $U(\phi)$ in Eq. 4. The solutions are subject to the boundary condition that the Higgs field reach its constant VEV as $x \to \infty$, or equivalently $v \to x$ as $x \to \infty$.

### 3.2 Physical Interpretation

We interpret the equations of motion Eq. 39 for the classical fields as a set of self-consistent equations. Each top/anti-top exists in the single particle ground state of the average field of the (N-1) other particles, and the field is generated by the (N-1) other particles in the same ground state. We expect such a picture to become more accurate for large N.

Furthermore, from such a picture, we would not expect there to exist $N=1$ boundstates. However, the structure of the equations of motion Eq. 39 indicates that for sufficient coupling these states do exist. Such states have been studied in the context of the relation between single particle soliton solutions and quantized fields [8,3], but in this many body context they do not have physical meaning. We therefore adopt a small modification to the equation motion for the Higgs field Eq. 49,

$$\left[ \frac{d^2}{dx^2} - \lambda \left( 1 - \frac{v^2}{x^2} \right) \right] v = \frac{(N-1)g}{8\pi x} (u_1^2 - u_2^2).$$

(50)

For $N = 1$ there now remain only free particle solutions, just as we desired.

However, we introduce a nuance now in calculating the mass of the solution. When determining the field kinetic energy and self-potential, we are interested in the field generated by all N quarks. In this case, the original
Eq. 49 describes the correct field, and must be used instead. To summarize, Eq. 50 is used in the interaction term of the Dirac equations, and Eq. 49 is used to calculate the field energy aside from the interaction term.

Although we veer away from the original variational principle with these prescriptions, we have done so with physical intuition and in hopes of generating more physically relevant solutions.

4 Results

The coupled nonlinear boundary value problem for the field equations of motion was solved using the COLSYS package. All results presented were checked to 0.1% by comparing numerically integrated Dirac energies to the eigenenergy and by comparison of results with a virial theorem [12].

We found that there was a minimum coupling $g_c$ for which bound states emerged, in agreement with the similar results of [8]. As can be seen in Fig 1, the necessary coupling naturally becomes lower with larger $N$, but the slope levels off rapidly. We have extended our plots past $N = 12$ to show that not much gain in coupling is made, even if there are other low-lying states that can be occupied along with the ground state. Series for reasonable values of the Higgs mass have been chosen. Because in all cases $g_t < g_c$, we do not expect the top quark bound state predicted by Froggatt and coworkers to exist.

The likely problem with Froggatt and coworkers’ calculations is the assumption that the Higgs field vanishes in the region of the bound state. While we see in Fig. 2 that this is true for very large couplings, for coupling close to the critical coupling the Higgs field is barely perturbed from its VEV. As we see, even at $m_t = 450$ GeV, the Higgs field does not drop far below its VEV in the region of the solution.

Fig. 3 shows the general behavior of the solutions. Notice that for large top quark masses, the solutions become localized at a finite radius and very relativistic.

Finally, Fig 4 shows the bound state mass per top quark at $N = 12$ as a function of the top quark mass. Note that the mass of the bound state grows near the same speed as the mass of the free solution until the peak, at which point it levels off while the binding energy compared to the free solution continues to grow.
5 Outlook

We have considered the bound state of many tops and anti-tops due to the interaction with the Higgs field, and shown that for realistic top and Higgs masses, Froggatt et. al. greatly overestimate the perturbation of the Higgs field from its VEV due to the interaction with the fermions. As a result, the bound states they predict do not exist except at much higher couplings (i.e. higher than observed top masses).

Our calculation could be refined to include gauge interactions. Although one expects the perturbation to be small due to the relative magnitudes of the coupling constants, this may not be for certain the case. Specifically, Dirac fields interact in fundamentally different ways with vector potentials than with scalar potentials. For instance, because the spectrum is symmetric for particles and antiparticles in a scalar potential, bound states in external scalar fields can never dive below $E = 0$. Furthermore, since the equations are nonlinear, any small effect could be further magnified. It is therefore possible that adding gauge interactions will affect the solutions enough to reach the predicted bound states within realistic parameters.

Another possibility to expand our calculation is to consider a more compi-
Figure 2: Value of $\phi$ vs. distance from origin. Notice that $\phi$ returns to its VEV as $x \to \infty$. The sudden jump in the solution occurs at $r \approx 1/(174 \text{ GeV} \approx 0.001 \text{ fm}$. Both series are for $N = 12$ and $m_H = 246 \text{ GeV}$.)

Figure 3: Scalar and vector densities for $N = 12$ and $m_H = 246 \text{ GeV}$.)
In conclusion, our result rules out the simplest possible realization of Froggatt et. al’s strongly bound top-Higgs states. Further research will be directed toward seeing how elusive these bound states are within reasonable modification to the standard model.

6 Appendix

6.1 Conflicts over Couplings and Parameters

Different Lagrangians have been used in the literature of Higgs-Fermion interactions. The comparison of results necessitates an identification of the correspondences of parameters between models and physical observables. The
initial Lagrangian from which the equations of motion are derived in [3] is
\[ \mathcal{L} = \overline{\Psi} (i\partial - g_1 \sigma) \Psi + \frac{1}{2} (\partial \sigma)^2 - \lambda_1 (\sigma^4 - 2\sigma^2 v^2 + v^4). \] (51)
\( \sigma \) is the real scalar Higgs field obtained via a gauge transformation from the complex SU(2) doublet:
\[ \Phi = \begin{pmatrix} \frac{\phi_1 + i\phi_2}{\sqrt{2}} \\ \frac{\phi_3 + i\phi_4}{\sqrt{2}} \end{pmatrix} \quad \phi_1 = \phi_2 = 0; \quad \phi_3 \equiv \sigma \]
(52)
and \( \langle \sigma \rangle = v = 246 \text{ GeV} \) is the Higgs vacuum expectation value (VEV).

The Yukawa coupling constant \( g \) and Higgs self coupling constant \( \lambda \) carry subscripts to indicate that the definition of these parameters differ between notations.

To identify the correspondence between \( \lambda \)'s and \( g \)'s in the Eq. 53 and Eq. 51, we make the substitution Eq. 52 in Eq. 53, leading to
\[ \mathcal{L} = \overline{\Psi} (i\partial + \frac{g_2}{\sqrt{2}} \sigma) \Psi + \frac{1}{2} (\partial \sigma)^2 - \frac{\lambda_2 v^2}{2} \Phi^4 - \frac{\lambda_2}{2} (\Phi^4)^2. \] (53)
where we have used the identity \( \overline{\Psi} R \Psi + \overline{\Psi} L \Psi = \overline{\Psi} \Psi \). We therefore identify the following correspondences,
\[ 8\lambda_1 = \lambda_2 = \frac{m_H^2}{v^2}, \]
\[ \sqrt{2}g_1 = g_2 = \sqrt{2}m_F/v, \]
where \( m_H \) is the Higgs Boson mass after spontaneous symmetry breaking.

6.2 Notation
The Pauli matrices are:
\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \] (55)
The Dirac gamma matrices are:
\[ \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & i \sigma_i \\ -i \sigma_i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \] (56)
References


