STRONG REALITY IN COXETER GROUPS

By

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STRONG REALITY IN COXETER GROUPS
HONORS THESIS

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Background to Coxeter Groups

To begin, it will be helpful to review and introduce some terms from group theory and linear algebra. Much of our notation will be adapted from Benson and Grove [1]. First, a **group**, G, is a set combined with an associative binary operation on G, such that G has an identity element, e, and each nonidentity element x of G has an inverse, x^{-1}. The **order** of G, $|G|$, is the number of elements in G. In this project, we will focus on groups with finite order. The **order** of an element $x \in G$ is the least positive integer n such that $x^n = e$. The **normalizer** of a subgroup, $H \in G$, is denoted $N_G(H)$ and is defined: $N_G(H) = \{x \in G | xHx^{-1} = H\}$

For a real vectorspace, V, we will denote the group of all orthogonal transformations of V by $O(V)$. For our purposes, we will consider a group to be a subgroup of $O(V)$ for some real vectorspace V. If $T \in O(V)$, then $\det(T) = \pm 1$. If $\det(T) = 1$, then T is called a **rotation**. If $\dim(V) = n$, then any $(n-1)$-dimensional subspace is called a **hyperplane**.

A **reflection** is an element $S \in O(V)$ which carries each vector to its mirror image with respect to a fixed hyperplane, P. In other words, $Sx = x$ for $x \in P$, and $Sx = -x$ for $x \in P^{\perp}$ (see [1]). Suppose that we have some $r \in P^{\perp}$ with $r \neq 0$. If we define a transformation S_r by: $S_r = x - \frac{2(x,r)r}{(r,r)}$ for all $x \in V$, then S_r is the reflection S, which we refer to as the reflection through P or along r. We define the **roots** of $G \subseteq O(V)$ corresponding to S to be the unit vectors $\pm r \in P^{\perp}$ such that $S = S_r = S_{-r}$.

Before we define a Coxeter Group, we must introduce one last term. We say a group G is **effective** if $V_0(G) = \bigcap\{V_T | T \in G\} = 0$, where $V_T = \{x \in V | Tx = x\}$. We can now define a **Coxeter Group** to be a finite effective subgroup $G \subseteq O(V)$ that is generated by a set of reflections (see [1]). The **root system** of G, which we will denote by Δ, is the set of roots corresponding to reflections in the set:

$$\{TST^{-1} | T \in G, S \text{ is a generating reflection}\}.$$

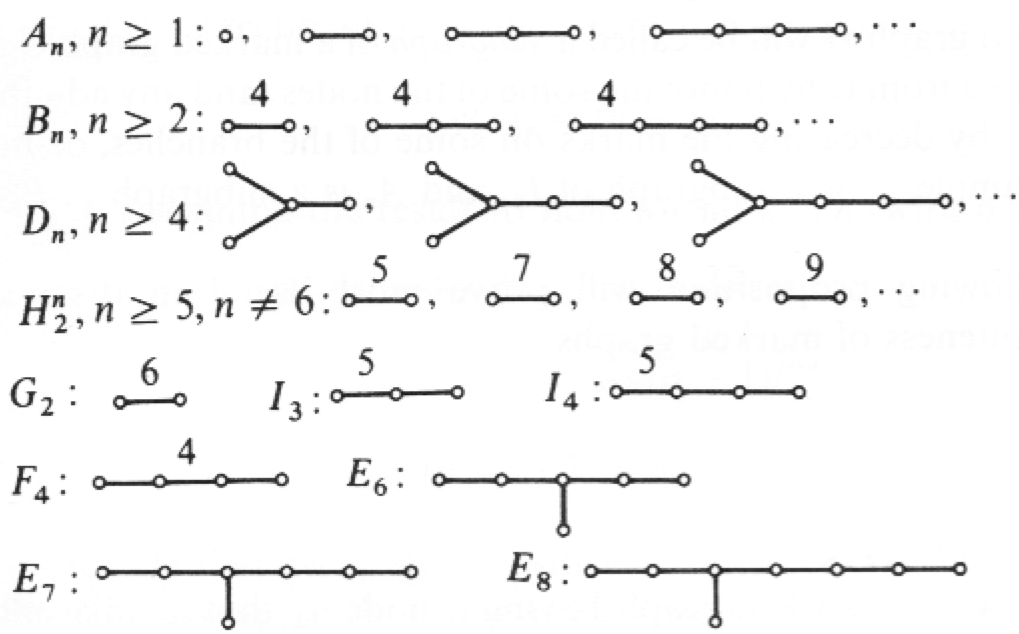
If we pick a vector $t \in V$ with the property that $(t,r) \neq 0$ for all $r \in \Delta$, we can partition Δ into $\Delta_t^+ = \{r \in \Delta | (t,r) > 0\}, \Delta_t^- = \{r \in \Delta | (t,r) < 0\}$. We can then define a **t-base** of Δ to be the basis of Δ_t^+. We say that the roots, $\{r_1, r_2, ..., r_n\}$ in the t-base of Δ are the **fundamental roots** of G. The reflections along these roots are called the **fundamental reflections** of G. It can be shown that G is generated by these fundamental reflections. (see [1], page 42 for proof).

Now, what happens when we take just a subset of these fundamental reflections? Clearly these would generate a subgroup of G. We call this a **parabolic subgroup**, and the normalizer of these subgroups is what we will focus on.

Coxeter Groups can be represented visually by diagrams called marked graphs, or specifically, **Coxeter Graphs**. These graphs are made up of a finite set of points, or nodes, that may be connected by lines. The nodes represent the fundamental roots of our Coxeter Group, and the lines represent the order of the product of the reflections corresponding to two nodes. Clearly, the reflection corresponding to each node has order 2. The product of who reflection whose corresponding nodes are not connected has order 2 also. Two reflections with corresponding notes connected by a line have a product of order 3, unless specified by a number above the line, which is then the order of the product.

Below is Figure 5.3 found in Benson and Grove ([1]) on page 57. This figure is the complete listing of irreducible Coxeter Groups (ie, those which are not made up of direct products of smaller Coxeter Groups) along with their graphs. For a more complete discussion of Coxeter Groups and their classifications, specifically the proof that these are in fact *all* of the irreducible Coxeter Groups, see page 62 of Grove [1].

Group Classifications.png



As you can see, there are infinite series of type A, D, B, and H_2. The groups $G_2, F_4, E_6, E_7, E_8, I_3$, and I_4 we will call the "exceptional cases". The classifications $A_n, B_n, D_n, E_6, E_7, E_8, F_4$,and G_2 constitute what we call the **Weyl Groups**. One may notice that the group A_n is the Symmetric Group of order $(n+1)!$ generated by n reflections, sometimes denoted $S_{(n+1)}$. Also, H_2^n is the

Dihedral Group of order $2n$, generated by 2 reflections. While the Dihedral Groups are often denoted D_n, we will refrain from using this notation so as not to confuse the reader. One may notice also that the Dihedral Group of order 12 is omitted from the list. This is because it is isomorphic to the exceptional group G_2. Thus, we will discuss G_2 as a member of the Dihedral Groups.

This paper will investigate the following conjectures involving these groups suggested by Dr. Ryan Vinroot.

(1) Let W be a finite Coxeter group, P a parabolic subgroup of W, and H the normalizer of P in W. We will check whether every element in H is strongly real, in the sense that for any $h \in H$, there are involutions s and t in H such that $h = st$.
(2) Any irreducible complex character χ of H has a Frobenius-Schur indicator of 1.
(3) If W is a Weyl group, any irreducible complex character χ of H has Schur index 1 over $\mathbb{Q}$.

Thus far in the semester, we have focused primarily on the first of these conjectures.

The First Conjecture

Let W be a finite Coxeter group, P a parabolic subgroup of W, and $H = N_W(P)$ be the normalizer of P in W. Then for any element, $h \in H$, there are involutions $s, t \in H$ such that $h = st$.

We will approach this problem by considering each irreducible Coxeter Group, beginning with the Dihedral Groups. In investigation of the subsequent classifications, we will utilize the computer algebra package GAP.

1. The Dihedral Groups, H_n

Recall the Coxeter Graph of these groups:

$$\overset{n}{\circ\!\!-\!\!-\!\!\circ}$$

Let W be the Dihedral Group of order $2n$. We know that each Dihedral Group is generated by the reflections along only two fundamental roots. Call these roots r_1 and r_2. If we denote by R_1, R_2 the reflections along r_1, r_2, respectively, then we have that $W = \langle R_1, R_2 \rangle$. Clearly, to form a parabolic subgroup of W, we have only the options of $P = \langle R_1 \rangle$ and $P = \langle R_2 \rangle$, and thus without loss of generality let us call $P = \langle R \rangle$.

Claim. *For any group G, $R \in G$ with order 2, $N(\langle R \rangle) = C(R)$ where $C(R)$ is the centralizer of R in G.*

Proof. Suppose $g \in N(\langle R \rangle)$. This means that

$$g\langle R \rangle g^{-1} \in \langle R \rangle.$$

But

$$g\langle R \rangle g^{-1} = gRg^{-1}, geg^{-1}$$

where e is the identity. We know that $geg^{-1} = e$, and $gRg^{-1} = e$ or R since $\langle R \rangle = e, R$. Suppose then that $gRg^{-1} = e$, and thus $gR = g$. Then this would imply that $R = e$, and since $o(R) = 2$, this cannot be the case. Therefore $gRg^{-1} = R$, which implies that $gR = Rg$. Then

$$g \in C(R)$$

and

$$N(\langle R \rangle) \subseteq C(R).$$

Now suppose $g \in C(R)$. We know from this that

$$gR = Rg,$$

$$ge = eg,$$

so

$$gRg^{-1} = R, geg^{-1} = e,$$

so

$$g \in N(\langle R \rangle)$$

and therefore

$$N(\langle R \rangle) \supseteq C(R).$$

Therefore $N(\langle R \rangle) = C(R)$, as stated. □

Now consider the normalizer of P in W, $N_W(P)$, and let $h \in N_W(P)$. We just showed that $N_W(P) = C(R)$, so we know that $h \in C(R)$, and hence we just need to understand this centralizer. Consider the conjugacy class of R, ie the set $\{gRg^{-1} | g \in W$. We know that in the Dihedral Group, g must be either a reflection or a rotation, and thus have determinant 1 or -1. Since $\det(R) = -1, \det(g) = \det(g^{-1})$, we know that any element in this conjugacy class must be a reflection, since its determinant is -1 and it must be in W. Looking at these conjugacy classes, consider the stabilizer of R, Stab(R). We know that this must be the conjugation for which $gRg^{-1} = R$, ie, $gR = Rg$. Then Stab$(R) = C(R)$.

If we consider the conjugacy class of R as an orbit, then by the Orbit-Stabilizer Theorem, we have that $|W| = |\text{Conj}(R)| * |\text{Stab}(R)|$, which implies that

$$\frac{|W|}{|\text{Conj}(R)|} = |\text{Stab}(R)| = |C(R)| = |N_W(P)|.$$

Consider first the case where W has order $2n$ for odd n. Then since there are n reflections in W, we know that the conjugacy class must be of order n, and thus $|N_W(P)| = \frac{2n}{n} = 2$. Then the normalizer of our parabolic subgroup is isomorphic to the cyclic group of order 2.

Now consider the case where n is even. In this case, there are two types of reflections in W, ones where the roots are on corners and ones where the roots are on the edges of the regular n-gon whose symmetries are described by W. Two reflections of different types cannot possibly be conjugates of one-another, since this would contradict the symmetry of the n-gon. This means the conjugacy class described above now has order $\frac{n}{2}$, and thus $|N_W(P)| = 4$. Now, a group of order 4 is either isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \bigoplus \mathbb{Z}_2$. Observing that the center of a dihedral group with even n includes a rotation of order 2, we know that both R and this rotation are in the normalizer, and thus it cannot be isomorphic to $\mathbb{Z}_4$, since there are at least 2 elements of order 2.

Thus since the normalizer is a Coxeter group itself, we have proven the conjecture for the dihedral groups: for any $h \in N_W(P)$, there exist involutions $s, t \in H$ such that $h = st$.

2. The Symmetric Groups, A_n

We now change our focus to another infinite series of Coxeter Groups, those of the form A_n. Below are the diagrams of A_n for $n = 1, 2, 3, 4$. We know from the classification of these groups that $W = A_n$ is generated by n reflections and is the Symmetric Group S_{n+1}. Therefore for this section, we will consider the Symmetric Groups S_n.

$$\circ,\ \circ\!-\!\circ,\ \circ\!-\!\circ\!-\!\circ,\ \circ\!-\!\circ\!-\!\circ\!-\!\circ$$

Taking a parabolic subgroup, P of S_n, we find that it is of the form

$$S_{k_1} \times S_{k_2} \times ... \times S_{k_m},$$

where

$$\sum_{i=1}^{m} k_i = n.$$

The following discussion will allow us to better understand the role of the normalizer of this parabolic subgroup.

Lemma 2.1. *Given an arbitrary subgroup $H \subseteq S_m$ and H-orbit,*

$$H\omega = \{h\omega | h \in H\}$$

for some element ω of $\{1, 2, ..., m\}$, if $n \in N_{S_m}(H)$ then

$$|nH\omega| = |H\omega|$$

where $nH\omega = \{nh\omega | h \in H\}$.

Proof. Since n is an element of the normalizer of H, we have that for any element $nh\omega \in nH\omega$, $nh\omega = h_2 n\omega$ for some other $h_2 \in H$. Clearly, then, $nH\omega$ is another H-orbit for the element $n\omega$. In addition, we can see that this new H-orbit is the same size as our original $H\omega$, since if some $h_1\omega \neq h_2\omega$, then $nh_1\omega$ cannot be equal to $nh_2\omega$ either. Thus for any distinct element $h\omega$ in $H\omega$, there is a corresponding distinct element $nh\omega$ in $nH\omega$. Similar logic shows that the same is true the other direction, and thus $|H\omega| = |nH\omega|$. □

Theorem. *Suppose P is a parabolic subgroup of S_m where*

$$P = (S_{k_1} \times S_{k_1} \times ... \times S_{k_1}) \times (S_{k_2} \times ... \times S_{k_2}) \times ... \times (S_{k_r} \times ... \times S_{k_r}),$$

for n_1 copies of S_{k_1}, n_2 copies of S_{k_2}, etc, and $k_1 > k_2 > ... > k_r$. Then

$$N_{S_m}(P) = S_{k_1} \wr S_{n_1} \times ... \times S_{k_r} \wr S_{n_r},$$

where $\wr$ denotes the wreath product.

Proof. For proof, use Lemma 2.1 with $H = P$ and see Section 2.1 for a discussion of Wreath Products. □

Now that we know how the normalizer is structured, we need to show that any element is in fact the product of two involutions. In order to complete this task, we familiarize ourselves with wreath products and their conjugacy classes.

2.1. **Wreath Products.** We adapt the following notation and definitions from Kerber's paper on Wreath Products [3].

Let G be a group. Then we define the **wreath product**, $G \wr S_n$, of G and the symmetric group S_n by the set:

$$\{[f;\pi] : f \text{ maps } \{1,...n\} \text{ into } G, \pi \in S_n\}$$

joined with the composition law:

$$[f;\pi] \cdot [f';\pi'] := [ff'_\pi; \pi \cdot \pi']$$

where

$$f_\pi(i) := f(\pi^{-1}(i)), \quad \forall i \in \{1,...,n\}$$

and ff' is defined by

$$ff'(i) := f(i)f'(i), \quad \forall i \in \{1,...,n\}.$$

We will often denote an element by $[f_1, f_2, ..., f_n; \pi]$, where $f_i = f(i)$ is an element of G. One can verify that $G \wr S_n$ is indeed a group (see Kerber [3], pages 24-25).

Consider an element, π of S_n. For $i = 1, ..., n$, let a_i denote the number of i-cycles among the cyclic factors of π. Then we define the **type of** $\pi \in S_n$ by

$$T\pi := (a_1, ..., a_n)$$

and note that $\sum_{i=1}^{n} i \cdot a_i = n$. Obviously, these types describe the conjugacy classes of S_n (see Kerber [3]).

Example:

To demonstrate the concept of types, we describe the conjugacy classes of the Symmetric Group S_3.

$$S_3 = \{(1), (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$$

We can see that

$$T(1) = (3,0,0)$$
$$T(1\ 2) = (1,1,0)$$
$$T(1\ 3) = (1,1,0)$$
$$T(2\ 3) = (1,1,0)$$
$$T(1\ 2\ 3) = (0,0,1)$$
$$T(1\ 3\ 2) = (0,0,1)$$

So we define the conjugacy classes to be

$$C^1 := \{(1)\}$$
$$C^2 := \{(1\ 2), (1\ 3), (2\ 3)\}$$
$$C^3 := \{(1\ 2\ 3), (1\ 3\ 2)\}$$

$\Diamond\Diamond\Diamond$

Similarly, the conjugacy classes of $G \wr S_n$ will be defined by the type of an element in $G \wr S_n$. Let $[f;\pi] \in G \wr S_n$. Define the **cycleproduct** associated with the cyclic factor $(j...\pi^r(j))$ of π with respect to f to be the element

$$ff_\pi...f_{\pi^r}(j) = f(j) \cdot f(\pi^{-1}(j)) \cdot ... \cdot f(\pi^{-r}(j)) \in G.$$

We know that π has a_k cycles of length k, which correspond to a_k cycleproducts with respect to f. If $C^1, ..., C^s$ is an ordering of the conjugacy classes of G, and exactly a_{ik} of these cycleproducts belong to C^i, then we define the **type of** $[f;\pi] \in G \wr S_n$ by the $s \times n$-matrix

$$T[f;\pi] := (a_{ik})$$

where i is the row index, and k is the column index.

Example

To illustrate the types in wreath products, we will find the types for representatives in $S_3 \wr S_2$.

In the element $[f;\pi] = [(1),(1);(1)]$, we see that

$$f(1) = (1)$$

$$f(2) = (1)$$

so

$$T[(1),(1);(1)] = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Using a similar method, we see that

$$T[(1\ 2),(1);(1)] = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T[(1\ 2\ 3),(1);(1)] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$T[(1\ 2),(1\ 2);(1)] = \begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T[(1\ 2\ 3),(1\ 2\ 3);(1)] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 2 & 0 \end{pmatrix}$$

$$T[(1),(1);(1\ 2)] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T[(1\ 2),(1\ 2\ 3);(1)] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$T[(1\ 2\ 3),(1\ 2\ 3);(1\ 2)] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T[(1\ 2),(1\ 2\ 3);(1\ 2)] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$\Diamond\Diamond\Diamond$

Now that we have introduced the basic concept of wreath products, we can begin with a few lemmas.

Lemma 2.2. *Let G be a group, $a \in G$ an element of order greater than 2. Then there exist involutions τ_1, τ_2 such that $a = \tau_1 \cdot \tau_2$ if and only if there exists an involution τ such that $\tau \cdot a \cdot \tau^{-1} = a^{-1}$, ie, which conjugates a to its inverse.*

Proof. Consider an element a such that $\tau_1 \cdot \tau_2 = a$ for involutions τ_1 and τ_2. Then $\tau_1 = a \cdot \tau_2 \Rightarrow \tau_2 \cdot \tau_1 = a^{-1} \Rightarrow \tau_1 = \tau_2 \cdot a^{-1} \Rightarrow a \cdot \tau_2 = \tau_2 \cdot a^{-1} \Rightarrow \tau_2 \cdot a \cdot \tau_2 = a^{-1}$. Now suppose that $\tau \cdot a \cdot \tau = a^{-1}$ for some involution τ. Then $(\tau \cdot a)(\tau \cdot a) = (\tau \cdot a \cdot \tau)(a) = (a^{-1})(a) = 1$. Then $\tau \cdot a$ is itself an involution, and $(\tau)(\tau \cdot a) = a$. Thus, our conjecture holds if and only if there is some involution in the normalizer that conjugates x to its inverse. □

Lemma 2.3. *Let G be the internal direct product $N_1 \times N_2$ of two normal subgroups N_1 and N_2, and suppose $\tau \in N_1$ conjugates $n_1 \in N_1$ to its inverse and $\sigma \in N_2$ conjugates $n_2 \in N_2$ to its inverse. Then $\tau \cdot \sigma$ conjugates $n_1 \cdot n_2 \in G$ to its inverse.*

Proof. First notice that if $x \in N_1, y \in N_2$, then $x \cdot y = y \cdot x$.
Now, we have that

$$\tau \cdot n_1 \cdot \tau^{-1} = n_1^{-1}$$

and

$$\sigma \cdot n_2 \cdot \sigma^{-1} = n_2^{-1}.$$

Thus,

$$\begin{aligned} (\tau \cdot \sigma) \cdot (n_1 \cdot n_2) \cdot (\tau \cdot \sigma)^{-1} & \\ &= (\tau \cdot n_1 \cdot \tau^{-1}) \cdot (\sigma \cdot n_2 \cdot \sigma^{-1}) \\ &= n_1^{-1} \cdot n_2^{-1}. \end{aligned}$$

□

Note that an induction argument shows that Lemma 2.3 holds for arbitrary direct products.

We now focus on $S_m \wr S_n$. By Kerber [3], page 44, any element $a \in S_m \wr S_n$ is conjugate to an element of the form

$$[(f_1, (1), ..., (1)), (f_2, (1), ..., (1)), ..., (f_k, (1), ..., (1)); \pi_1 \cdot \pi_2 \cdot ... \cdot \pi_k]$$

where the π_i's are disjoint cycles of the form

$$\pi_1 = \begin{pmatrix} 1 & ... & j_1 \end{pmatrix},$$

$$\pi_2 = \begin{pmatrix} j_1+1 & ... & j_1+j_2 \end{pmatrix},$$

up to

$$\pi_k = \begin{pmatrix} j_{k-1} & ... & n \end{pmatrix}. \tag{1}$$

This conjugate element can then be separated into a product of k factors of commuting elements:

$$a_1 = [(f_1, (1), ..., (1)), (1), ..., (1); \pi_1],$$

$$a_2 = [((1), ..., (1)), (f_2, (1), ..., (1)), (1), ..., (1); \pi_2],$$

(2) $$..., a_k = [(1), ..., (1), (f_k, (1), ..., (1)); \pi_k]$$

It suffices to prove our conjecture for the conjugacy class representatives.

Lemma 2.4. *Let $x \in S_m$. For any element of the form*

$$[x, (1), (1), ..., (1); \begin{pmatrix} 1 & ... & n \end{pmatrix}] \in S_m \wr S_n,$$

there is an involution t which conjugates the element to its inverse.

Proof. Consider the wreath product $S_m \wr S_n$. Let $x \in S_m$ and consider the element $a = [x, (1), (1), ..., (1); \begin{pmatrix} 1 & ... & n \end{pmatrix}$ $S_m \wr S_n$. (i.e., the function in this element is $f(i) = (1), i \neq 1$ and $f(1) = x$.) First, we claim that the inverse of this element is $[(1), (1), ..., x^{-1}; \begin{pmatrix} n & ... & 1 \end{pmatrix}]$.

Proof of Claim.

$$a \cdot [(1), (1), ..., x^{-1}; \begin{pmatrix} n & ... & 1 \end{pmatrix}]$$
$$= [x, (1), (1), ..., (1); \begin{pmatrix} 1 & ... & n \end{pmatrix}] \cdot [(1), (1), ..., x^{-1}; \begin{pmatrix} n & ... & 1 \end{pmatrix}]$$
$$= [(1), (1), ..., (1); (1)]$$

So, $a^{-1} = [(1), (1), ..., x^{-1}; \begin{pmatrix} n & ... & 1 \end{pmatrix}]$. □

In S_m, we know that there exists some involution $\pi \in S_m$ such that $\pi \cdot x \cdot \pi = x^{-1}$. Define the permutation, rev, as follows:

$$rev(i) := n - i, \forall i = 1, ..., n.$$

Then $|\text{rev}| = 2$, and

$$\text{rev} \begin{pmatrix} 1 & ... & n \end{pmatrix} \text{rev} = \begin{pmatrix} n & ... & 1 \end{pmatrix} = \begin{pmatrix} 1 & ... & n \end{pmatrix}^{-1}.$$

So consider the nontrivial element of $S_m \wr S_n$:

$$t = [\pi, \pi, ..., \pi; \text{rev}].$$

Note that

$$\begin{aligned} t^2 &= [\pi, \pi, ..., \pi; \text{rev}] \cdot [\pi, \pi, ..., \pi; \text{rev}] \\ &= [\pi\pi, \pi\pi, ..., \pi\pi; \text{rev} \cdot \text{rev}] \\ &= [(1), (1), ..., (1); (1)]. \end{aligned}$$

ie, t^2 is the identity in $S_m \wr S_n$. Thus, t is an involution in $S_m \wr S_n$. Also notice that:

$$\begin{aligned} t \cdot a \cdot t &= [\pi, \pi, ..., \pi; \text{rev}] \cdot [\text{x}, (1), (1), ..., (1); \begin{pmatrix} 1 & ... & n \end{pmatrix}] \cdot [\pi, \pi, ..., \pi; \text{rev}] \\ &= [\pi, \pi, ..., \pi x; \text{rev} \begin{pmatrix} 1 & ... & n \end{pmatrix}] \cdot [\pi, \pi, ..., \pi; \text{rev}] \\ &= [\pi\pi, \pi\pi, ..., \pi x \pi; \text{rev} \begin{pmatrix} 1 & ... & n \end{pmatrix} \text{rev}] \\ &= [(1), (1), ..., x^{-1}; \begin{pmatrix} n & ... & 1 \end{pmatrix}]. \end{aligned}$$

and thus $t \cdot a \cdot t = a^{-1}$. □

We are now have the necessary tools to prove a generalized result:

Theorem 2.5. *Any element a of $S_m \wr S_n$ is conjugate to its inverse. Moreover, there is an involution which conjugates a to a^{-1}.*

Proof. Notice that we can write the set of indices $\{1, ..., n\}$ as the union of disjoint subsets of indices $B_1 \cup B_2 \cup ... \cup B_k$, where

$$B_1 = \{1, ..., j_1\}, B_2 = \{j_1 + 1, ..., j_1 + j_2\}, ..., B_k = \{j_{k-1}, ..., n\}$$

using the j_i's from (1). Now define:

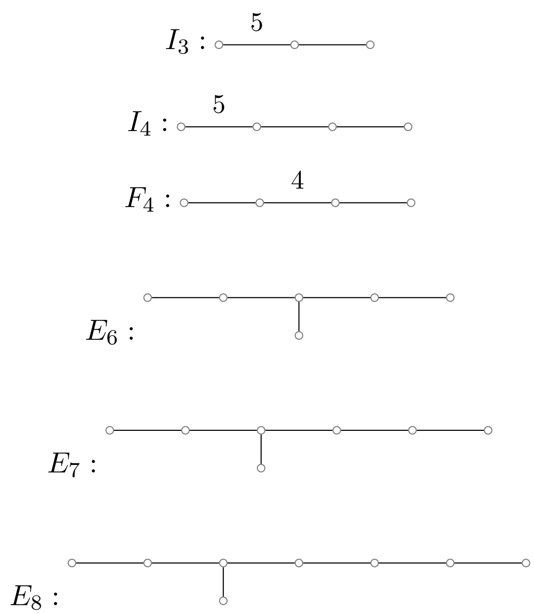
$$H_i := \{[f_1, ..., f_n; \pi] \in S_m \wr S_n : \pi \text{ fixes } \{1, ..., n\} \setminus B_i,$$
$$f_j = (1) \text{ if } j \text{ is not in } B_i\}.$$

Then $H_i \cong S_m \wr S_{|B_i|}$. For each factor, $a_i, i = 1, ..., k$ of our conjugate element in (2) above, $a_i \in H_i$ by construction. Thus from Lemma 2.4, we can find an involution $t_i \in H_i \cong S_m \wr S_{|B_i|}$ which conjugates the factor to its inverse. The H_i's are normal in $S_m \wr S_n$ and they form an internal direct product so that $S_m \wr S_n = H_1 \times H_2 \times ... \times H_k$. Then by Lemma 2.3, $t_1 \cdot t_2 \cdot ... \cdot t_k$ conjugates a to its inverse. Note that $t_1 \cdot t_2 \cdot ... \cdot t_k$ is an involution, since elements of distinct normal subgroups in an internal direct product commute. □

Then by Lemma 2.2, we have proven that arbitrary elements of $S_m \wr S_n$ are products of two involutions, and thus our conjecture is satisfied for the Symmetric groups.

3. The Exceptional Groups

Recall the Coxeter diagrams for the exceptional cases:



In order to test our first conjecture for the exceptional groups, that is, the groups F_4, I_3, I_4, E_6, E_7, and E_8, we use the computer algebra system, GAP. The code for our program can be found in Appendix A. Our main idea is that given one of these groups, we want to run through all possible parabolic subgroups, testing our conjecture. So, for each parabolic, we want to find the normalizer, which we can do in GAP using the Normalizer function, which takes in two arguments - the group and the subgroup whose normalizer we wish to find.

Once we have the normalizer of a particular parabolic, we want to find the conjugacy classes and test the conjecture for each conjugacy class. Since conjugacy classes partition the normalizer into sets of elements of the same order, we can find all elements of order two by finding the conjugacy classes whose representatives are of order two. We then test whether given a conjugacy class whose elements are of order greater than 2 (since elements of order 2 are trivially strongly real), there are two elements of order two which multiply into that conjugacy class.

In the next few sections, we describe the methods in GAP which we invoked to prove the conjecture.

3.1. **Finding All Parabolic Subgroups of an Exceptional Group.**

In order to define a parabolic subgroup in GAP, we use the Subgroup function. This function takes in two arguments-the original group and the list of generators which we want to generate our parabolic subgroup. We can find the generators of our original group (that is, the reflections which correspond to the roots denoted by nodes in the Coxeter diagram) using the function GeneratorsOfGroup. This function takes only the argument of the group. For example, if we look at the group $W = F_4$, the following is an excerpt from GAP which creates a parabolic subgroup on the first 2 nodes:

```
gap> W;
<matrix group with 4 generators>
gap> gen:=GeneratorsOfGroup(W);
[ [ [ -1, 0, 1, 0 ], [ 0, 1, 0, 0 ], [ 0, 0, 1, 0 ], [ 0, 0, 0, 1 ] ],
  [ [ 1, 0, 0, 0 ], [ 0, -1, 0, 1 ], [ 0, 0, 1, 0 ], [ 0, 0, 0, 1 ] ],
  [ [ 1, 0, 0, 0 ], [ 0, 1, 0, 0 ], [ 1, 0, -1, 1 ], [ 0, 0, 0, 1 ] ],
  [ [ 1, 0, 0, 0 ], [ 0, 1, 0, 0 ], [ 0, 0, 1, 0 ], [ 0, 1, 2, -1 ] ] ]
gap> P:=Subgroup(W,[gen[1],gen[2]]);
<matrix group with 2 generators>
gap> quit;
```

In order to find all parabolic subgroups we have written a method called ”addone” (see Appendix A). This method takes in a number in binary with the number of digits determined by the number of generators and adds one. In our ”MakeParSubgps” method (see Appendix A) combined with ”addone”, we then loop through all binary numbers which can be written using that number of digits. With each of these binary numbers, we use the Subgroup function to make the subgroup generated by the generators with the indices which have a one in the binary number. We store all of these parabolic subgroups in a list in GAP, and can then loop through them to test the conjecture for each parabolic.

3.2. **Testing the Conjugacy Classes.** Once we have the list of parabolic subgroups for our Coxeter Group, we loop through them in ”MakeParSubgps” and for each one read in our method ”ConjRepOrd2” (see Appendix A), which finds the normalizer using the Normalizer function in GAP, the conjugacy classes of the normalizer using the ConjugacyClasses function (which takes in the group as an argument-in our case the normalizer), and the conjugacy class representatives using a list and the Representative function. ”ConjRepOrd2” then creates a list of all representatives of order 2 using a filtered list. We create the character table for the normalizer using the CharacterTable function in GAP, and then read in our ”NumInvMult” method (see Appendix A). In ”NumInvMult”, we loop through each conjugacy class and within this loop run through each possible pair of conjugacy classes with elements of order two. Inside these loops we count the number of elements from, for example, conjugacy class j which multiply with an element from conjugacy class i to get an element in conjugacy class k. To do this, we use the ClassMultiplicationCoefficient function in GAP. For example, given a character table ”ct”, in the following code the number c is the number of elements from the ith conjugacy class which multiply with an element from the jth conjugacy class to form an element in the kth conjugacy class.

```
c:=ClassMultiplicationCoefficient(ct, i, j, k);
```

Given a conjugacy class k, we run through the i and j conjugacy classes which represent those whose elements are of order 2, and we sum up the quantities c for each i and j. If this number is nonzero, then this particular conjugacy class k satisfies the conjecture. If the sum is 0, then the conjecture fails for the entire parabolic subgroup, since there is an entire conjugacy class of elements which are not strongly real. We return a boolean value ”existinv” which is true if the sum is nonzero for all conjugacy classes and false if the sum is 0 for any conjugacy class. In "MakeParSubgps" we then run through all parabolic subgroups and create another boolean value "theanswer", which returns true if "existinv" is true for all parabolic subgroups and false if "existinv" is false for any parabolic subgroup.

3.3. **Results for the Exceptional Groups.** We tested each of the groups F_4, I_3, I_4, E_6, E_7, and E_8 using the programs in the Appendix A. The result was that each of these groups satisfied our conjecture. That is, for every parabolic subgroup, the normalizer holds the property that every element of order larger than 2 can be written as the product of two involutions. See Appendix B for the GAP logs of the tests.

4. Type B_n

We now investigate the conjecture for the Coxeter group of type B_n. This section will require the discussion on wreath products from Section 2. It will also require the following definition of a **semidirect product**.

Definition 4.1. Suppose a group G has a subgroup K and normal subgroup N with the following properties:

(i) $N \cap K = \{1\}$,
(ii) $G = NK$.

Then G is the **internal semidirect product** of N and K, denoted $G = N \rtimes K$.

We call K a **complement** of H in G.

Given two groups, $\widetilde{N}$ and $\widetilde{K}$, and a homomorphism of groups $\varphi : \widetilde{K} \to \mathrm{Aut}(\widetilde{N})$, we can construct a new group called the **external semidirect product** $N \rtimes_\varphi K$ with the group operation:

$$(n, k)(n', k') = \big(n\varphi(k)(n'), kk'\big)\,.$$

We may also use $k \circ n'$ to denote $\varphi(k)(n')$. In this case our operation is denoted:

$$(n, k)(n', k') = \big(n(k \circ n'), kk'\big)\,.$$

Note that the internal semidirect product is isomorphic to the external direct product where the mapping φ takes k to the automorphism on N defined by conjugation by k.

Given a group G, notice that the wreath product, $G \wr S_n$ can also be described as the external semidirect product $(G \times ... \times G) \rtimes S_n = G^n \rtimes S_n$. Recall from Section 2 that for $[f;\pi], [f';\pi'] \in G \wr S_n$, the group operation is

$$[f;\pi] \cdot [f';\pi'] = [ff'_\pi; \pi\pi']$$

where $f_\pi(\pi(i)) = f(i)f'(i)$.

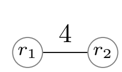
In pages 67-68 of Benson and Grove [1], the authors describe B_n as a semidirect product isomorphic to

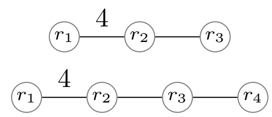
$$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times ... \times \mathbb{Z}_2) \rtimes S_n$$

where there are n copies of $\mathbb{Z}_2$. Furthermore, the action of S_n on $\mathbb{Z}_2^n$ which the authors describe shows that in fact this is the wreath product. That is,

$$B_n \cong \mathbb{Z}_2 \wr S_n.$$

Recall that the Coxeter Diagram for B_n looks like that of A_{n-1} (or S_n) with an extra node attached by a double line, denoting a product of order 4. The diagrams of B_2, B_3, B_4 are shown below.





Now, we wish to find the normalizer of a parabolic subgroup of $B_n \cong \mathbb{Z}_2 \wr S_n$. First, note that for any semidirect product $G = N \rtimes K$ for N an abelian group whose elements are all of order ≤ 2, the inverse of an element (n, k) is of the form

$$(n,k)^{-1} = (k^{-1} \circ n, k^{-1}).$$

We first look at a parabolic subgroup P of $B_n = \mathbb{Z}_2 \wr S_n$ whose generating set does not include the first node of the diagram, meaning that P is isomorphic to a parabolic subgroup of $S_n = \langle r_2, ..., r_n \rangle$.

Definition 4.2. Let P be a parabolic subgroup of $B_n = \mathbb{Z}_2 \wr S_n$. Then we define the subgroup N^P to be the set of elements in $\mathbb{Z}_2 \times ... \times \mathbb{Z}_2$ which are fixed under permutations in P.

Lemma 4.3. *An element of $N_{B_n}(P)$, where P does not include the reflection corresponding to the first node, r_1 of B_n, is of the form $[n, k]$ where $n \in N^P$ and $k \in N_{S_n}(P)$. Moreover, N^P is normal in $N_{B_n}(P)$, so that $N_{B_n}(P) \cong N^P \rtimes N_{S_n}(P)$.*

Proof. Let $\pi \in P$ as a subgroup of S_n. Then $[0; \pi]$ is an element of P as a subgroup of $\mathbb{Z}_2 \wr S_n$. Now suppose that $[n, k] \in \mathbb{Z}_2 \wr S_n$ is in the normalizer of P in $\mathbb{Z}_2 \wr S_n$, i.e.,

$$[n,k] \cdot [0,\pi] \cdot [k^{-1} \circ n, k^{-1}] = [0, \pi']$$

for some $\pi' \in P$. Then we must have that $k\pi k^{-1} = \pi'$, i.e. $k \in N_{S_n}(P)$. But also,

$$\begin{aligned} 0 &= n + k\pi k^{-1} \circ n \\ &= n + \pi' \circ n \end{aligned}$$

But this means that $\pi' \circ n = -n = n$ since $n \in \mathbb{Z}_2 \times ... \times \mathbb{Z}_2$. So n is fixed under permutations in P of the form $k\pi k^{-1}$ for $k \in N_{S_n}(P)$ and $\pi \in P$. But since $kPk^{-1} = P$, this is all permutations in P. Then n must be fixed under all permutations in P. Thus $n \in N^P$, $k \in N_{S_n}(P)$.

Now, let $n \in N^P$, $x \in N_{B_n}(P)$. Then we know that for any $p \in P$, $pnp^{-1} = p \circ n = n$. To show that N^P is normal in $N_{B_n}(P)$, we want to show that $p(xnx^{-1})p^{-1} = xnx^{-1}$. Well, we have that $p(xnx^{-1})p^{-1} = xp_1np_1^{-1}x^{-1}$ for some $p_1 \in P$, since P is normal in $N_{B_n}(P)$. Then this is the same as xnx^{-1} since n is fixed under P. Thus we have that N^P is normal in $N_{B_n}(P)$.

□

We next consider the simplified case where P is of the form

$$P \cong S_k \times S_k \times ... \times S_k$$

where there are n/k copies of S_k, ie

$$P \cong S_k^{n/k}.$$

Recall from Section 2 on the symmetric groups that $N_{S_n}(P) = S_k \wr S_{n/k}$. Also, elements of N^P are the elements $(x_1, ..., x_n)$ of $\mathbb{Z}_2^n$ where

$$x_{ik+j} = x_{ik+k}$$

for all $0 \leq i < n/k$ and $1 \leq j < k$, because they must be unchanged by any element $(\pi_1, ...\pi_{n/k})$ where pi_1 permutes only the first k indices, π_2 permutes the second set of k indices, and so on. We can define a mapping $N^P \to \mathbb{Z}_2^{n/k}$ by

$$(x_1, ..., x_n) \mapsto (x_k, x_{2k}, ..., x_n).$$

This mapping defines an isomorphism, so N^P is isomorphic to $\mathbb{Z}_2 \times ... \times \mathbb{Z}_2$, where there are n/k copies of $\mathbb{Z}_2$. We claim in the next theorem that the normalizer of P in B_n is a wreath product.

Theorem 4.4. *Let G be B_n and P be the parabolic isomorphic to $S_k \times S_k \times ... \times S_k$, ($n/k$ copies), then $N_G(P)$ is isomorphic to $(\mathbb{Z}_2 \times S_k) \wr S_{n/k}$.*

Proof. From Lemma 4.3, we have that $(n,k) \in N_G(P)$ is of the form $n \in N^P \cong \mathbb{Z}_2^{n/k}$, $k \in N_{S_n}(P) \cong S_k \wr S_{n/k}$, so that $N_G(P) = N^P \rtimes N_{S_n}(P)$. From the above discussion, we know that $|N_G(P)| = |(\mathbb{Z}_2 \times S_k) \wr S_{n/k}|$, so we just need to find an epimorphism between the two groups. Define a mapping φ from $(\mathbb{Z}_2 \times S_k) \wr S_{n/k} \to N_G(P)$ which sends elements π in the complement, $S_{n/k}$, to elements of the form $[0,...,0;[1,...,1;\pi]]$ in $N_G(P)$ and elements $[(x_1,\pi_1),...,(x_{n/k},\pi_{n/k});1]$ in $(\mathbb{Z}_2 \times S_k)^{n/k}$ to $[x_1,...,x_1,x_2,...,x_2,...,x_{n/k},...x_{n/k};[\pi_1,...,\pi_{n/k};(1)]]$ in $N_G(P)$. Since any element in $(\mathbb{Z}_2 \times S_k) \wr S_{n/k}$ can be written as

$$[(x_1,\pi_1),...,(x_{n/k},\pi_{n/k});\pi] = [(x_1,\pi_1),...,(x_{n/k},\pi_{n/k});1] \cdot [(0,1),...,(0,1);\pi],$$

and any element in $N_G(P)$ can be written as

$$\begin{aligned}&[x_1,...,x_1,x_2,...,x_2,...,x_{n/k},...x_{n/k};[\pi_1,...,\pi_{n/k};\pi]]\\ &= [x_1,...,x_1,x_2,...,x_2,...,x_{n/k},...x_{n/k};[\pi_1,...,\pi_{n/k};(1)]] \cdot [0,...0;[1,...,1;\pi]],\end{aligned}$$

we can define

$$\begin{aligned}&\varphi\left([(x_1,\pi_1),...,(x_{n/k},\pi_{n/k});\pi]\right)\\ &= [x_1,...,x_1,x_2,...,x_2,...,x_{n/k},...x_{n/k};[\pi_1,...,\pi_{n/k};\pi]]\end{aligned}$$

to see that this indeed defines a map from $(\mathbb{Z}_2 \times S_k) \wr S_{n/k}$ onto $N_G(P)$. We next check that this is a homomorphism. Let $[(x_1,\pi_1),...,(x_{n/k},\pi_{n/k});\pi]$ and $[(y_1,\sigma_1),...,(y_{n/k},\sigma_{n/k});\sigma]$ be elements in $(\mathbb{Z}_2 \times S_k) \wr S_{n/k}$. Then

$$\begin{aligned}&\varphi\left([(x_1,\pi_1),...,(x_{n/k},\pi_{n/k});\pi]\right)\varphi\left([(y_1,\sigma_1),...,(y_{n/k},\sigma_{n/k});\sigma]\right)\\ =\ &\left[x_1,...,x_1,x_2,...,x_2,...,x_{n/k},...x_{n/k};[\pi_1,...,\pi_{n/k};\pi]\right] \cdot \left[y_1,...,y_1,...,y_{n/k},...,y_{n/k};[\sigma_1,...,\sigma_{n/k};\sigma]\right]\\ =\ &\left[x_1\pi \cdot y_1,...x_1\pi \cdot y_1,...,x_{n/k}\pi \cdot y_{n/k},...,x_{n/k}\pi \cdot y_{n/k};[\pi_1\pi \cdot \sigma_1,...,\pi_{n/k}\pi \cdot \sigma_{n/k};\pi\sigma]\right]\end{aligned}$$

since $\pi_i \in S_k \times ... \times S_k$ and $(y_1,...,y_1,...,y_{n/k},...,y_{n/k}) \in N^P$ and therefore is unchanged by π_i for all i. Also,

$$\begin{aligned}&\varphi\left([(x_1,\pi_1),...,(x_{n/k},\pi_{n/k});\pi] \cdot [(y_1,\sigma_1),...,(y_{n/k},\sigma_{n/k});\sigma]\right)\\ =\ &\varphi\left([(x_1,\pi_1)\pi \cdot (y_1,\sigma_1),...,(x_{n/k},\pi_{n/k})\pi \cdot (y_{n/k},\sigma_{n/k});\pi\sigma]\right)\\ =\ &\varphi\left(\left[(x_1\pi \cdot y_1,\pi_1\pi \cdot \sigma_1),...,(x_{n/k}\pi \cdot y_{n/k},\pi_{n/k}\pi \cdot \sigma_{n/k});\pi\sigma\right]\right)\\ =\ &\left[x_1\pi \cdot y_1,...x_1\pi \cdot y_1,...,x_{n/k}\pi \cdot y_{n/k},...,x_{n/k}\pi \cdot y_{n/k};[\pi_1\pi \cdot \sigma_1,...,\pi_{n/k}\pi \cdot \sigma_{n/k};\pi\sigma]\right]\end{aligned}$$

and thus φ is an epimorphism. This proves that $N_G(P) \cong (\mathbb{Z}_2 \times S_k) \wr S_{n/k}$.

□

Before the statement of the next theorem, recall from Section 2 on type A that any parabolic subgroup of a symmetric group is of the form

$$P = S_1^{k_1} \times S_2^{k_2} \times ... \times S_n^{k_n}$$

where $0 \leq k_i \leq n/i$ for each $i = 1,...,n$ and $\sum_{i=1}^{n} i \cdot k_i = n$. Therefore, if P is a parabolic of B_n whose generating set does not include the first node, then P is also of this form. Here the summand $S_1^{k_1}$ represents k_1 nodes (aside from the first node) which are not included in the generating set of P and do not separate nodes which are found in the generating set. Since $S_1 \cong \{1\}$, we could instead say that $P = S_2^{k_2} \times ... \times S_n^{k_n}$, but we leave the copies of S_1 for convenience.

Theorem 4.5. *Let P be a parabolic of B_n of the form*

$$P = S_1^{k_1} \times S_2^{k_2} \times ... \times S_n^{k_n}.$$

Then

$$N_{B_n}(P) \cong B_{k_1} \times (\mathbb{Z}_2 \times S_2) \wr S_{k_2} \times ... \times (\mathbb{Z}_2 \times S_n) \wr S_{k_n}.$$

Proof. We know from Lemma 4.3 that $N_{B_n}(P) = N^P \rtimes N_{S_n}(P)$ and from Section 2 that $N_{S_n}(P) = S_1 \wr S_{k_1} \times ... \times S_n \wr S_{k_n}$. From similar reasoning to that of the proof in the more simple case of Theorem 4.4, we see that elements of N^P are the elements $(x_1, ..., x_n) \in \mathbb{Z}_2^n$ such that after the first k_1 coordinates, the x_i's are blocked so that there are k_2 sets such that $x_i = x_{i+1}$ followed by k_3 sets where $x_j = ... = x_{j+2}$, and so on. Then N^P is isomorphic to $\mathbb{Z}_2^{k_1} \times \mathbb{Z}_2^{k_2} \times ... \times \mathbb{Z}_2^{k_n}$. This means that

$$N_{B_n}(P) \cong (\mathbb{Z}_2^{k_1} \times \mathbb{Z}_2^{k_2} \times ... \times \mathbb{Z}_2^{k_n}) \rtimes (S_1 \wr S_{k_1} \times ... \times S_n \wr S_{k_n}).$$

So the order of this normalizer is the same as the order of $(\mathbb{Z}_2 \times S_1) \wr S_{k_1} \times (\mathbb{Z}_2 \times S_2) \wr S_{k_2} \times ... \times (\mathbb{Z}_2 \times S_n) \wr S_{k_n}$.

As in the proof of Theorem 4.4, it suffices to find an epimorphism

$$\varphi : (\mathbb{Z}_2 \times S_1) \wr S_{k_1} \times (\mathbb{Z}_2 \times S_2) \wr S_{k_2} \times ... \times (\mathbb{Z}_2 \times S_n) \wr S_{k_n} \to N_{B_n}(P).$$

Let φ send the element $[(n_1, \pi_1), (n_2, \pi_2), ..., (n_{k_1}, \pi_{k_1}); \pi]$ in $(\mathbb{Z}_2 \times S_1) \wr S_{k_1}$ to the element $[(n_1, n_2, ..., n_{k_1}, 0, ..., 0); ([\pi_1 \pi_2 \cdots \pi_{k_1}; \pi], 1, ..., 1)]$ in $N_{B_n}(P) \cong (\mathbb{Z}_2^{k_1} \times \mathbb{Z}_2^{k_2} \times ... \times \mathbb{Z}_2^{k_n}) \rtimes (S_1 \wr S_{k_1} \times ... \times S_n \wr S_{k_n})$. Similarly, let φ map $[(x_1, \sigma_1), ..., (x_{k_2}, \sigma_{k_2}); \sigma] \in (\mathbb{Z}_2 \times S_2) \wr S_{k_2}$ to the element $[(0, ..., 0, x_1, x_1, x_2, x_2, ..., x_{k_2}, x_{k_2}, 0, ..., 0); (1, [\sigma_1 ... \sigma_{k_2}; \sigma], 1, ..., 1)]$ in $N_{B_n}(P)$ where there are 0's in the first k_1 coordinates. Continuing in this manner, we get a canonical generalization of the epimorphism φ used in the proof of Theorem 4.4. Due to the same reasoning as in the proof of Theorem 4.4, this is again an epimorphism, so

$$N_{B_n}(P) \cong (\mathbb{Z}_2 \times S_1) \wr S_{k_1} \times (\mathbb{Z}_2 \times S_2) \wr S_{k_2} \times ... \times (\mathbb{Z}_2 \times S_n) \wr S_{k_n}.$$

But $(\mathbb{Z}_2 \times S_1) \wr S_{k_1} \cong (\mathbb{Z}_2) \wr S_{k_1} \cong B_{k_1}$, so we have

$$N_{B_n}(P) \cong B_{k_1} \times (\mathbb{Z}_2 \times S_2) \wr S_{k_2} \times ... \times (\mathbb{Z}_2 \times S_n) \wr S_{k_n},$$

as stated. □

Theorem 4.6. *Let $P \subseteq \langle r_2, ..., r_n \rangle$ be a parabolic subgroup of B_n. Then any element in $N_{B_n}(P)$ is conjugate to its inverse by an involution.*

Proof. By Theorem 4.5, we have that $N_{B_n}(P)$ is a wreath product, which by Theorem 2.5 satisfies the statement. □

Theorem 4.7. *Let P be the parabolic subgroup generated by the reflections corresponding to the first m nodes of B_n, that is, P is isomorphic to B_m. Then $N_{B_n}(P) \cong B_m \times B_{n-m}$.*

Proof. Let $[n; k]$ be an element in P. Then $n = (n_1, n_2, ..., n_m, 0, ..., 0) \in \mathbb{Z}_2^n$ and $k \in S_m$. Suppose $[a; \pi] \in B_n = \mathbb{Z}_2 \wr S_n$ normalizes $[n, k]$. Then we have that

$$[a; \pi][n; k][\pi^{-1} \circ a; \pi^{-1}] = [n'; k'] \in P$$

where $n' = (n'_1, n'_2, ..., n'_m, 0, ..., 0) \in \mathbb{Z}_2^n$ and $k' \in S_m$. Then as in the case with the parabolic not including the first node, it must be the case that $\pi k \pi^{-1} = k'$, so that $\pi \in N_{S_n}(S_m) \cong S_m \times S_{n-m}$. We also find that

$$a + \pi \circ \big(n + k \circ (\pi^{-1} \circ a)\big) = n'.$$

But notice that if $a = (a_1, ..., a_n)$, then since $\pi, \pi^{-1} \in S_m \times S_{n-m}$, we know π^{-1} permutes separately the first m coordinates and the other $n - m$. Then since $k \in S_m$, we have that it only permutes the first m coordinates. Now, when added to n, we still have the last $n-m$ coordinates of a unchanged aside from the permutation from π^{-1}. When π is applied to $\left(n+k\circ(\pi^{-1}\circ a)\right)$, we then have that the last $n-m$ coordinates are permuted back to their original position, so that we end up with

$$\begin{aligned} a+\pi\circ\left(n+k\circ(\pi^{-1}\circ a)\right) &= (a_1+n_{i_1}+a_{k_1},...,a_m+n_{i_m}+a_{k_m},2a_{m+1},...,2a_n) \quad \text{where } 1\leq i_j,k_j\leq m \text{ for all} \\ &= (a_1+n_{i_1}+a_{k_1},...,a_m+n_{i_m}+a_{k_m},0,...,0) \\ &= (n'_1,...,n'_m,0...,0)=n' \end{aligned}$$

and therefore a can be any element of $\mathbb{Z}_2^n$.

Therefore, we have that an element in $N_{B_n}(P)$ is of the form $[a,\pi]$ for $\pi\in N_{S_n}(S_m)$ and $a\in\mathbb{Z}_2^n$. Thus

$$N_{B_n}(P)\cong\mathbb{Z}_2^n\rtimes(S_m\times S_{n-m})=(\mathbb{Z}_2^m\times\mathbb{Z}_2^{n-m})\rtimes(S_m\times S_{n-m}).$$

Because of this, we know $N_{B_n}(P)$ has a subgroup isomorphic to $\mathbb{Z}_2^m\rtimes S_m$ and another subgroup isomorphic to $\mathbb{Z}_2^{n-m}\rtimes S_{n-m}$, whose intersection is the trivial subgroup. Furthermore, each of these are normal, since elements in $\mathbb{Z}_2^m\rtimes S_m$ commute with the elements in $\mathbb{Z}_2^{n-m}\rtimes S_{n-m}$ and $N_{B_n}(P)=(\mathbb{Z}_2^m\rtimes S_m)\cdot(\mathbb{Z}_2^{n-m}\rtimes S_{n-m})$. Then

$$\begin{aligned} N_{B_n}(P) &\cong (\mathbb{Z}_2^m\rtimes S_m)\times(\mathbb{Z}_2^{n-m}\rtimes S_{n-m}) \\ &\cong \mathbb{Z}_2\wr S_m\times\mathbb{Z}_2\wr S_{n-m} \\ &\cong B_m\times B_{n-m}. \end{aligned}$$

□

Remark 4.8. Notice that since B_n satisfies our conjecture, Theorem 4.7 tells us that $N_{B_n}(P)$ does as well for $P\cong B_m$.

Before the next theorem, note that any parabolic P of B_n can be written

$$P=B_m\times S_1^{k_1}\times S_2^{k_2}\times...\times S_n^{k_n}$$

where $m+\sum_{i=1}^n i\cdot k_i=n$.

Theorem 4.9. *Let P be a parabolic of B_n of the form*

$$P\cong B_m\times S_1^{k_1}\times S_2^{k_2}\times...\times S_{n-m}^{k_{n-m}}.$$

Then

$$N_{B_n}(P)\cong B_m\times B_{k_1}\times(\mathbb{Z}_2\times S_2)\wr S_{k_2}\times...\times(\mathbb{Z}_2\times S_{n-m})\wr S_{k_{n-m}}.$$

Proof. Let P be a parabolic of the form described. Then we can view P as $B_m\times P^*$ where

$$P^*=S_1^{k_1}\times S_2^{k_2}\times...\times S_{n-m}^{k_{n-m}}$$

is a parabolic of B_{n-m} of the type described in Theorem 4.5. So notice that the conclusion of the theorem is equivalent to

$$N_{B_n}(P)=N_{B_n}(B_m\times P^*)\cong B_m\times N_{B_{n-m}}(P^*)$$

by Theorem 4.5.

Suppose that $\tau=[\tau_1,...,\tau_n;\sigma]$ is an element of $N_{B_n}(P)$. Then for any element $x=[x_1,...,x_n;\pi]\in P$ we have $\tau x\tau^{-1}\in P$. Notice that for any $x\in P$ for $i>m$, we have $x_i=0$ and $\pi\in S_m\times S_1^{k_1}\times S_2^{k_2}\times...\times S_{n-m}^{k_{n-m}}$. In particular, consider the element $p=[0,...,0;\pi]$ where $\pi\in S_m\times S_1^{k_1}\times S_2^{k_2}\times...\times S_{n-m}^{k_{n-m}}$. For any such π, this is an element of P. Then for

$$\tau p\tau^{-1}=[\tau_1,...,\tau_n;\sigma][0,...,0;\pi][\sigma^{-1}(\tau_1),...,\sigma^{-1}(\tau_n);\sigma^{-1}=[\tau_1+\sigma\pi\sigma^{-1}(\tau_1),...,\tau_n+\sigma\pi\sigma^{-1}(\tau_n);\sigma\pi\sigma^{-1}]$$

to be an element of P, we need that $\sigma\pi\sigma^{-1}\in S_m\times S_1^{k_1}\times S_2^{k_2}\times...\times S_{n-m}^{k_{n-m}}$, and therefore $\sigma\in N_{S_n}(S_m\times P^*)$. Also, for every $i>m$, we require that $\tau_i+\sigma\pi\sigma^{-1}(\tau_i)=0$. Then $\tau_i=\sigma\pi\sigma^{-1}(\tau_i)$, i.e. for any $i>m$ τ_i is fixed under $\sigma\pi\sigma^{-1}$ for any $\pi\in S_m\times P^*$.

Now consider the element $x = [x_1, ..., x_n; \pi] = [1, ..., 1, 0, ..., 0; \pi] \in P$. I.e., this element has $x_i = 1$ for $i \leq m$ and $x_i = 0$ for $i > m$, and π is any permutation in $S_m \times P^*$. So we know

$$\tau x \tau^{-1} = [\tau_1 + \sigma(x_1) + \sigma\pi\sigma^{-1}(\tau_1), ..., \tau_n + \sigma(x_n) + \sigma\pi\sigma^{-1}(\tau_n); \sigma\pi\sigma^{-1}$$

must be an element of P. Then for $i > m$ we have

$$0 = \tau_i + \sigma(x_i) + \sigma\pi\sigma^{-1}(\tau_i) = \sigma(x_i)$$

from above. Then $\sigma(x_i) = 0$ so σ must take x_i to x_j for some other $j > m$ since these are the only coordinates which are 0. Similarly, for $i \leq m$ we have

$$1 = \tau_i + \sigma(x_i) + \sigma\pi\sigma^{-1}(\tau_i) = \sigma(x_i),$$

so here σ takes x_i to another x_j for $j \leq m$. Then σ must permute indices 1 through m separately from indices $m+1$ through n. This means that $\sigma \in S_m \times S_{n-m}$, so σ can be written as the product of $\sigma_{<m}$ which permutes 1 through m and $\sigma_{>m}$ which permutes $m+1$ through n. Since any $\pi \in S_m \times P^*$ can similarly be written $\pi = (\pi_{<m})(\pi_{>m})$ we have that

$$\begin{aligned} & \sigma\pi\sigma^{-1} = \pi' \in S_m \times P^* \\ \Rightarrow\quad & (\sigma_{<m})(\sigma_{>m})(\pi_{<m})(\pi_{>m})(\sigma_{>m}^{-1})(\sigma_{<m}^{-1}) = \pi'_{<m}\pi'_{>m} \\ \Rightarrow\quad & (\sigma_{<m})(\pi_{<m})(\sigma_{>m}^{-1}) = \pi'_{<m} \quad \text{and} \quad (\sigma_{>m})(\pi_{>m})(\sigma_{>m}^{-1}) = \pi'_{>m} \end{aligned}$$

Thus we have that $\sigma_{<m} \in S_m$ and $\sigma_{>m} \in N_{S_{n-m}}(P*)$ so

$$\sigma \in S_m \times N_{S_{n-m}}(P*).$$

Recall from above that for $i > m$ we have $\tau_i = \sigma\pi\sigma^{-1}(\tau_i)$. Then

$$\tau_i = \left[(\sigma_{<m})(\sigma_{>m})(\pi_{<m})(\pi_{>m})(\sigma_{>m}^{-1})(\sigma_{<m}^{-1})\right](\tau_i) = \left[(\sigma_{>m})(\pi_{>m})(\sigma_{>m}^{-1})\right](\tau_i)$$

and thus $\tau_i \in N^{P*}$ in the notation of Lemma 4.3. So we can write τ as

$$\tau = [\tau_1, ..., \tau_m, 0, ..., 0; \sigma_{<m}] \cdot [0, ..., 0, \tau_{m+1}, ..., \tau_n; \sigma_{>m}]$$

with $\tau_{m+1}, ..., \tau_n \in N^{P^*}$ and $\sigma_{>m} \in N_{S_{n-m}}(P^*)$. So as there is a canonical isomorphism between elements of this form and $B_m \times N_{B_{n-m}}(P^*)$ we have shown that

$$N_{B_n}(P) = N_{B_n}(B_m \times P^*) \cong B_m \times N_{B_{n-m}}(P^*).$$

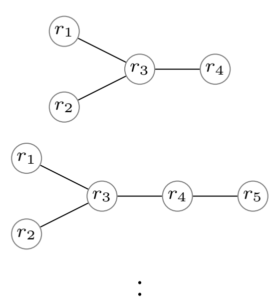
□

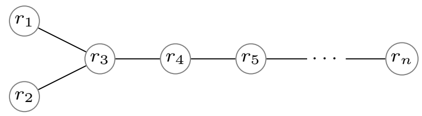
Theorem 4.10. *Any element in the normalizer of a parabolic subgroup of B_n is conjugate to its inverse by an involution.*

Proof. This follows directly from Theorem 4.9 and the fact that any element in a group of type B and any element in a wreath product is conjugate to its inverse by an involution. □

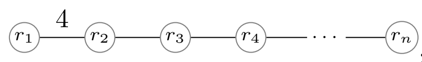
5. Type D_n

Recall the Coxeter Diagram for type D_n:

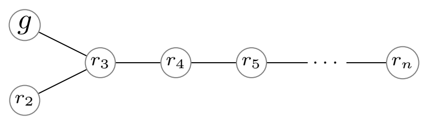




Claim. *If the fundamental reflections of B_n are labeled as follows:*



then we claim that we can write the fundamental reflections of D_n in terms of the $r_1, ..., r_n$ in the following way:



where $g = r_1 r_2 r_1$.

Proof. To prove this claim, we need to show that g has order 2, gr_2 has order 2 and is equal to $r_2 g$, gr_3 has order 3, and that gr_i has order 2 and is equal to $r_i g$ for $i \geq 4$. Consider g^2:

$$\begin{aligned} g^2 &= (r_1 r_2 r_1)(r_1 r_2 r_1) \\ &= r_1 r_2 r_2 r_1 \\ &= r_1 r_1 \\ &= 1. \end{aligned}$$

So $|g| = 2$. Now consider $(gr_2)^2$.

$$\begin{aligned} (gr_2)^2 &= (r_1 r_2 r_1 r_2)(r_1 r_2 r_1 r_2) \\ &= (r_1 r_2)^4 \\ &= 1 \qquad \text{since } r_1 r_2 \text{ has order 4 in } B_n. \end{aligned}$$

Then $|gr_2| = 2$. We want to show that g commutes with r_2. To do this, we use the wreath product notation for these elements. I.e.

$$r_1 = [1, 0, ..., 0; (1)],$$

$$r_2 = [0, ..., 0; (12)], r_3 = [0, ..., 0; (23)], \ldots r_n = [0, ..., 0; (n-1 \quad n)]$$

Using this notation, we have

$$\begin{aligned} gr_2 &= r_1 r_2 r_1 r_2 \\ &= [1, 0, ..., 0; (1)] \cdot [0, ..., 0; (12)] \cdot [1, 0, ..., 0; (1)] \cdot [0, ..., 0; (12)] \\ &= [1, 0, ..., 0; (12)] \cdot [1, ..., 0; (12)] \\ &= [1, 1, 0, ..., 0; (1)] \\ &= [0, 1, 0, ..., 0; (12)] \cdot [0, 1, 0, ..., 0; (12)] \\ &= [0, ..., 0; (12)] \cdot [1, 0, ..., 0; (1)] \cdot [0, ..., 0; (12)] \cdot [1, 0, ..., 0; (1)] \\ &= r_2 r_1 r_2 r_1 \\ &= r_2 g \end{aligned}$$

Now consider $(gr_3)^3$.

$$\begin{aligned}(gr_3)^3 &= (r_1r_2r_1r_3)(r_1r_2r_1r_3)(r_1r_2r_1r_3) \\ &= r_1r_2r_3r_2r_3r_2r_3r_1 \quad \text{since } r_1r_3 = r_3r_1 \text{ in } B_n \\ &= r_1(r_2r_3)^3r_1 \\ &= r_1r_1 \quad \text{since } |r_2r_3| = 3 \\ &= 1.\end{aligned}$$

Now let $i \geq 4$ and consider $(gr_i)^2$.

$$\begin{aligned}(gr_i)^2 &= (r_1r_2r_1r_i)(r_1r_2r_1r_i) \\ &= r_1r_2r_ir_2r_ir_1 \\ &= r_1r_ir_ir_2r_2r_1 \\ &= 1.\end{aligned}$$

Finally, all we have left is to show that $gr_i = r_ig$ for i larger than 3.

$$\begin{aligned}gr_i &= r_1r_2r_1r_i \\ &= r_1r_2r_ir_1 \\ &= r_1r_ir_2r_1 \\ &= r_ir_1r_2r_1\end{aligned}$$

since r_i commutes with r_1 and r_2. Then our claim is proved. □

Lemma 5.1. *If τ is an element of order 2 in B_n, then $\tau' = \tau \cdot [1, ..., 1; (1)]$ also has order 2.*

Proof. Since $[1, ..., 1; (1)]$ commutes with every element in B_n, we have

$$\tau'^2 = \tau \cdot [1, ..., 1; (1)]\tau \cdot [1, ..., 1; (1)] = \tau^2 = 1.$$

□

Lemma 5.2. *If $\tau \in N_{B_n}(P)$ conjugates $x \in N_{B_n}(P)$ to its inverse, then $\tau' = \tau \cdot [1, ..., 1; (1)]$ also conjugates x to its inverse.*

Proof.

$$\begin{aligned}\tau'x\tau' &= (\tau \cdot [1, ..., 1; (1)])x(\tau \cdot [1, ..., 1; (1)])^{-1} \\ &= (\tau \cdot [1, ..., 1; (1)])x([1, ..., 1; (1)] \cdot \tau^{-1}) \\ &= \tau x \tau^{-1} \\ &= x^{-1}.\end{aligned}$$

□

5.1. **Case 1: n odd.**

Remark 5.3. We can view D_n as the subgroup of $B_n \cong \mathbb{Z}_2 \wr S_n$ whose elements always contain an even number of 1s in their $\mathbb{Z}_2^n$ coordinates.

Proposition 5.4. *When n is odd,*

$$B_n \cong \mathbb{Z}_2 \times D_n.$$

Proof. From Remark 5.3 we see that D_n is of index 2 in B_n, and therefore is normal. Notice also that since n is odd, the element $[11, ..., 1; (1)]$ is not in D_n and is of order 2. Then $[11, ..., 1; (1)]$ generates a subgroup of B_n of order two whose intersection with D_n is trivial. Now let $[x_1, ..., x_n; \pi]$ be any element of B_n. Then $[x_1, ..., x_n; \pi] \cdot [1, ..., 1; (1)] = [x_1 + 1, ..., x_n + 1; \pi] = [1, ..., 1; (1)] \cdot [x_1, ..., x_n; \pi]$. Then $[1, ..., 1; (1)]$ commutes with every element in B_n, so the subgroup it generates is normal. Then

$$B_n = \langle [1, ..., 1; (1)] \rangle \times D_n \cong \mathbb{Z}_2 \times D_n.$$

□

Notice that if P is a parabolic subgroup of D_n whose generating set does not include the node g, then it is also a parabolic of B_n. Also, $N_{D_n}(P) \subseteq N_{B_n}(P)$, and from the section on type B, we know that given $x \in N_{D_n}(P)$ there is an involution τ in $N_{B_n}(P)$ such that $\tau x \tau = x^{-1}$. If this τ is already in D_n, then we are done, but if it is in B_n but not D_n, i.e. it has an odd number of 1's in its $\mathbb{Z}_2^n$ coordinate, then we can multiply $\tau = [\tau_1, ..., \tau_n; \pi]$ by $[1, 1, ..., 1; (1)]$, resulting in an odd number of 0's and thus an even number of 1's (since n is odd). Then this new element, call it τ', is in D_n. In the following series of lemmas, we claim that in fact τ' is an element of $N_{D_n}(P)$, that its order is unchanged, and that $\tau' x \tau' = x^{-1}$.

Lemma 5.5. *If $\tau \in N_{B_n}(P)$ but is not an element of D_n, then the element $\tau' = \tau \cdot [1, 1, ..., 1; (1)] = [\tau_1, ..., \tau_n; \pi] \cdot [1, 1, ..., 1; (1)]$ is in the normalizer of P in D_n.*

Proof. Since $[1, ..., 1; (1)]$ commutes with every element of B_n, and therefore every element of D_n, if $\tau P \tau^{-1} = P$, then

$$\begin{aligned} \tau' P \tau'^{-1} &= (\tau \cdot [1, ..., 1; (1)]) P (\tau \cdot [1, ..., 1; (1)])^{-1} \\ &= (\tau \cdot [1, ..., 1; (1)]) P ([1, ..., 1; (1)] \cdot \tau^{-1}) \\ &= \tau P \tau^{-1} \\ &= P \end{aligned}$$

so $\tau' \in N_{D_n}(P)$. □

Theorem 5.6. *If P is a parabolic subgroup of D_n such that $P \subseteq \langle r_2, ..., r_n \rangle$, then for every element $x \in N_{D_n}(P)$, there is an involution $\tau \in N_{D_n}(P)$ such that $\tau x \tau = x^{-1}$.*

Proof. This follows directly from Lemma 5.5, Lemma 5.1, and Lemma 5.2. □

Remark 5.7. Note that Theorem 5.6 also applies to the case where $P \subset \langle g, r_3, ..., r_n \rangle$ since conjugating each node by r_1 gives us $\langle r_2, ..., r_n \rangle$ again.

Theorem 5.8. *Let P be a parabolic subgroup of D_n with n odd which contains both of the first two nodes. That is, if $\{r_i\}_{i=1}^n$ denote the generators of B_n , we have*

$$P = \langle r_1 r_2 r_1, r_2, X \rangle$$

where X is some subset of $\{r_3, ..., r_n\}$. Then

$$N_{B_n}(\widetilde{P}) \cong N_{D_n}(P) \times \mathbb{Z}_2$$

where $\widetilde{P}$ is the parabolic subgroup of B_n with

$$\widetilde{P} = \langle r_1, r_2, X \rangle.$$

Proof. We know that $N_{D_n}(P) \subset N_{B_n}(P)$, and in fact, we claim that $N_{D_n}(P) \times \mathbb{Z}_2 \cong N_{B_n}(P)$. Suppose that $g \in N_{B_n}(P)$ but not in $N_{D_n}(P)$. Then $g[1, ..., 1; (1)] \in D_n$ and in fact is an element of $N_{D_n}(P)$ since $[1, ..., 1; (1)]$ commutes with every element of D_n. So $N_{D_n}(P)$ is of index 2 in $N_{B_n}(P)$, with

$$N_{B_n}(P) = N_{D_n}(P) \times \langle [1, ..., 1; (1)] \rangle \cong N_{D_n}(P) \times \mathbb{Z}_2.$$

It is clear that $P \subset \widetilde{P}$ in B_n. Now, we claim that $N_{B_n}(P) = N_{B_n}(\widetilde{P})$.

Suppose that $x \in N_{B_n}(P)$. Say $x = [x_1, ..., x_n; \pi]$. Then since $r_1r_2r_1r_2 = [1, 1, 0, ..., 0; (1)] \in P$, we have that

$$x[1, 1, 0, ..., 0; (1)]x^{-1} \in P.$$

We will now denote the element $r_1r_2r_1r_2$ using subscripts to show indices in the $\mathbb{Z}_2^n$ coordinate:

$$r_1r_2r_1r_2 = [1_1, 1_2, 0_3, ..., 0_n; (1)].$$

We have that

$$[x_1, ..., x_n; \pi][1_1, 1_2, 0_3, ..., 0_n; (1)][\pi^{-1}(x_1), ..., \pi^{-1}(x_n); \pi^{-1}]$$
$$= [x_1 + \pi\pi^{-1}(x_1) + \pi(1_1), x_2 + \pi\pi^{-1}(x_2) + \pi(1_2), x_3 + \pi\pi^{-1}(x_3) + \pi(0_3), ..., x_n + \pi\pi^{-1}(x_n) + \pi(0_n); (1)]$$
$$= [\pi(1_1), \pi(1_2), \pi(0_3), ..., \pi(0_n); (1)].$$

But then we also see that

$$xr_1x^{-1} = x[1, 0, 0, ..., 0; (1)]x^{-1} = [\pi(1_1), \pi(0_2), \pi(0_3), ..., \pi(0_n); (1)]$$

and notice that this element xr_1x^{-1} differs from $x(r_1r_2r_1r_2)x^{-1}$ in exactly one spot, since π will permute the indices, so that $x(r_1r_2r_1r_2)x^{-1}$ still has two 1s but xr_1x^{-1} has only a single 1. Say π makes the j'th index a 1 in $x(r_1r_2r_1r_2)x^{-1}$, but a 0 in xr_1x^{-1}. Then we see that

$$xr_1x^{-1} = \left[x(r_1r_2r_1r_2)x^{-1}\right][0_1, ...0_{j-1}, 1_j, 0_{j+1}...0_n; (1)].$$

Recall that for $x = [x_1, ..., x_n; \pi]$ to be in $N_{B_n}(P)$, we require that π be in the normalizer of $S_k \times P^*$, where $P \cong D_k \times P^*$. Then since j is determined by the permutation π, we have that the element $[0_1, ...0_{j-1}, 1_j, 0_{j+1}...0_n; (1)]$ is an element of $\widetilde{P}$. We already know that $x(r_1r_2r_1r_2)x^{-1} \in P \subset \widetilde{P}$, so we have that their product is contained in $\widetilde{P}$. Then we see that

$$xr_1x^{-1} \in \widetilde{P}.$$

Now also, we have that $xr_ix^{-1} \in P \subset \widetilde{P}$ for $i > 1$ with $r_i \in \widetilde{P}$, since for $i > 1$, $r_i \in \widetilde{P} \iff r_i \in P$. Then x conjugates every generator in $\widetilde{P}$ to another element of $\widetilde{P}$, and therefore $x \in N_{B_n}(P)$. So we see that

$$N_{B_n}(P) \subseteq N_{B_n}(\widetilde{P}).$$

Conversely, suppose $x \in N_{B_n}(\widetilde{P})$. Then the only possibility which would make x not be an element in $N_{B_n}(P)$ is if $xyx^{-1} = r_1$ for some $y \in P$. This is because the generating sets of P and $\widetilde{P}$ differ only in that $\widetilde{P}$ contains r_1 and P does not. So suppose $xyx^{-1} = r_1$ for some $y = [y_1, ..., y_n; \sigma]$. Then

$$[1, 0, ..., 0; (1)] = [x_1, ..., x_n; \pi][y_1, ..., y_n; \sigma][\pi^{-1}(x_1), ..., \pi^{-1}(x_n); \pi^{-1}]$$
$$= [x_1 + \pi(y_1) + \pi\sigma\pi^{-1}(x_1), ..., x_n + \pi(y_n) + \pi\sigma\pi^{-1}(x_n); \pi\sigma\pi^{-1}]$$

Then since $\pi\sigma\pi^{-1} = (1)$, we have that this is

$$[x_1 + \pi(y_1) + x_1, ..., x_n + \pi(y_n) + x_n; (1)] = [\pi(y_1), ..., \pi(y_n); (1)].$$

Then we see that $\pi(y_1) = 1$ and $\pi(y_i) = 0$ for $1 < i \leq n$. This means that y has only a single 1, which means that it cannot be an element of D_n, as elements of D_n have an even number of indices which are 1. Thus $x \in N_{B_n}(\widetilde{P})$ sends every element of P to another element of P, as if it sends an element to an element $\widetilde{P}$ which is not in P, then the original element was also in $\widetilde{P}$ but not P. So we see that

$$N_{B_n}(\widetilde{P}) \subseteq N_{B_n}(P).$$

Therefore, we have shown that

$$N_{B_n}(\widetilde{P}) = N_{B_n}(P) = N_{D_n}(P) \times \langle [1, ..., 1; (1)] \rangle \cong N_{D_n}(P) \times \mathbb{Z}_2.$$

□

Theorem 5.9. *If P is a parabolic of D_n containing the first two nodes, then every element of $N_{D_n}(P)$ is conjugate to its inverse by an involution.*

Proof. From Theorem 5.8 we know that

$$N_{B_n}(\widetilde{P}) = N_{D_n}(P) \times \langle [1, ..., 1; (1)] \rangle.$$

From the previous section, we know that every element in $N_{B_n}(\widetilde{P})$ is conjugate to its inverse by an involution, which means that since $N_{D_n}(P) \subset N_{B_n}(\widetilde{P})$, for every element $x \in N_{D_n}(P)$, there is some involution $\tau \in N_{B_n}(\widetilde{P})$ which conjugates x to its inverse. If this τ is also an element in $N_{D_n}(P)$, we are done, but if $\tau \notin D_n$, then $\tau[1, 1, ..., 1; (1)] \in D_n$, and this element conjugates x to its inverse as shown in Lemma 5.2 and is an involution as seen in Lemma 5.1. By Lemma 5.5, we also have that $\tau[1, ..., 1; (1)] \in N_{D_n}(P)$. Then we see that $\tau[1, 1, ..., 1 : (1)] \in N_{D_n}(P)$ is an involution which conjugates x to its inverse. Then every element of $N_{D_n}(P)$ is conjugate to its inverse by an involution. □

Theorem 5.10. *If P is a parabolic of D_n for n odd, then for every element in $N_{D_n}(P)$, there is an involution in $N_{D_n}(P)$ which conjugates it to its inverse.*

Proof. This follows directly from Theorem 5.9 and Theorem 5.6. □

5.2. **Case 2: n even.**

Theorem 5.11. *Any element in D_n is strongly real. That is, for any $x \in D_n$, there exists $\tau \in D_n$ with $|\tau| = 2$ such that $\tau x \tau = x^{-1}$.*

Proof. Since $x \in D_n \subseteq B_n \cong \mathbb{Z}_2 \wr S_n$, we know from Section 2.1 that x is conjugate in B_n to an element of the form

$$\widetilde{x} = \prod_i [(0, ..., 0), ..., (j_i, 0, ..., 0), ..., (0, ..., 0); \pi_i],$$

where the π_is are disjoint cycles, each (...) is a tuple corresponding to the positions permuted by some π_k, the j_i is in the first position permuted by π_i, and where j_i is some element of $\mathbb{Z}_2$. (Note that for $i = 1$, the factor is $[j_1, 0, ..., 0; \pi_1]$). This means that there is some $y \in B_n$ such that

$$yxy^{-1} = \prod_i [0, ..., 0, j_i, 0, ..., 0; \pi_i] = \widetilde{x}.$$

Now, for each factor $x_i = [0, ..., 0, j_i, 0, ..., 0; \pi_i]$ of $\widetilde{x}$, recall again from Section 2.1 that the element $\tau_i := [1, ..., 1; rev_i]$ is an involution which conjugates x_i to its inverse. (Note here that rev_i is the reversing permutation for π_i as described in Lemma 2.4 in Section 2.1, and there is a 1 in each position in the $\mathbb{Z}_2^n$ coordinate of τ_i.) That is

$$\tau_i x_i \tau_i = [1, ..., 1; rev_i][0, ..., 0, (j_i, 0, ..., 0), 0, .., 0; \pi_i][1, ..., 1; rev_i] = [0, ..., 0, (0, ..., 0, j_i), 0, ..., 0; \pi_i^{-1}] = x_i^{-1}.$$

Since n is even, we know that $\tau_i \in D_n$ for all i. Also, since rev_i and rev_j permute different sets for $i \neq j$, we have that the τ_i commute with one another. Thus the product

$$\widetilde{\tau} = \prod_i \tau_i$$

conjugates $\widetilde{x}$ to its inverse and is an involution. So we have found an involution $\widetilde{\tau}$ in D_n such that $\widetilde{\tau}\widetilde{x}\widetilde{\tau} = \widetilde{x}^{-1}$. Now, recall that $x = y^{-1}\widetilde{x}y$ so $x^{-1} = y^{-1}\widetilde{x}^{-1}y$. Define $\tau := y^{-1}\widetilde{\tau}y$. Then τ is still an involution, and moreover $\tau \in D_n$ since D_n is a normal subgroup of B_n. Also,

$$\tau x \tau = y^{-1}\widetilde{\tau}yy^{-1}\widetilde{x}yy^{-1}\widetilde{\tau}y = y^{-1}\widetilde{\tau}\widetilde{x}\widetilde{\tau}y = y^{-1}\widetilde{x}^{-1}y = x^{-1}.$$

Then $\tau \in D_n$ is an involution which conjugates x to its inverse, and therefore x is strongly real.

□

Lemma 5.12. *If P is a parabolic subgroup of D_n, then*

$$N_{D_n}(P) = N_{B_n}(P) \cap D_n.$$

Proof. If $x \in N_{D_n}(P)$, then $x \in D_n \subseteq B_n$ and $xPx^{-1} = P$, so $x \in N_{B_n}(P)$ and $x \in D_n$. Conversely, if $x \in N_{B_n}(P) \cap D_n$, then $xPx^{-1} = P$, but also $x \in D_n$, so $x \in N_{D_n}(P)$. □

Remark 5.13. In particular, recall that for $P \subseteq \langle r_2, r_3, ..., r_n \rangle$, P is also a parabolic of B_n.

Theorem 5.14. *If P is a parabolic subgroup of D_n such that $P \subseteq \langle r_2, r_3, ..., r_n \rangle$ which is isomorphic to a subgroup of S_n of the following form:*

$$P \cong S_{k_1} \times S_{k_2} \times ... \times S_{k_m}$$

where k_i is even for all $i = 1, ..., m$, then every element in $N_{D_n}(P)$ is strongly real.

Proof. Recall from Section 4 that the normalizer in B_n of P is of the form $N^P \rtimes N_{S_n}(P)$ where N^P is the set of elements invariant under P, which looks like m blocks of k_i coordinates of $\mathbb{Z}_2^n$ which are all the same. That is, such an element has all 1s or all 0s in the first k_1 positions, and likewise for the subsequent k_i positions. Thus since here k_i is even for all i, we know that any element has an even number of 1s, and therefore by Lemma 5.12 we have

$$N_{D_n}(P) = N_{B_n}(P) \cap D_n = N_{B_n}(P).$$

Since we proved the conjecture for all parabolic subgroups of B_n, this completes the proof. □

Remark 5.15. Note that from Section 4, if $P \subseteq \langle r_2, r_3, ..., r_n \rangle$, with $P \cong S_{k_1}^{n_1} \times S_{k_2}^{n_2} \times ... \times S_{k_m}^{n_m}$, then

$$N_{B_n}(P) \cong (\mathbb{Z}_2 \times S_{k_1}) \wr S_{n_1} \times ... \times (\mathbb{Z}_2 \times S_{k_m}) \wr S_{n_m}.$$

This means that if we reorder the k_i so that k_i is even for $i \leq j$ and k_i is odd for $i > j$, then since this is a direct product, we can consider elements of $N_{B_n}(P)$ to be of the form $x_1 x_2$ where $x_1 \in (\mathbb{Z}_2 \times S_{k_1}) \wr S_{n_1} \times ... \times (\mathbb{Z}_2 \times S_{k_j}) \wr S_{n_j}$ and $x_2 \in (\mathbb{Z}_2 \times S_{k_{j+1}}) \wr S_{n_{j+1}} \times ... \times (\mathbb{Z}_2 \times S_{k_m}) \wr S_{n_m}$. Say

$$N_{B_n}(P) = N_1 \times N_2$$

where

$$N_1 \cong (\mathbb{Z}_2 \times S_{k_1}) \wr S_{n_1} \times ... \times (\mathbb{Z}_2 \times S_{k_j}) \wr S_{n_j}$$

and

$$N_2 \cong (\mathbb{Z}_2 \times S_{k_{j+1}}) \wr S_{n_{j+1}} \times ... \times (\mathbb{Z}_2 \times S_{k_m}) \wr S_{n_m}.$$

From Theorem 5.14 we know there is an involution $\tau_1 \in N_1 \cap D_n$ which conjugates x_1 to its inverse. Then if we can find $\tau_2 \in N_2 \cap D_n$ which conjugates x_2 to its inverse, then $\tau_1 \tau_2$ will be an involution conjugating x to its inverse, since τ_1 and τ_2 will commute. Thus is suffices to show that for $P \subseteq \langle r_2, r_3, ..., r_n \rangle$ which is isomorphic to a subgroup of S_n of the form:

$$P \cong S_{k_1} \times S_{k_2} \times ... \times S_{k_m}$$

where k_i is odd for all $i = 1, ..., m$, that every element in $N_{D_n}(P)$ is strongly real.

Proposition 5.16. *If P is a parabolic subgroup of D_n of the form $P \subseteq \langle r_2, r_3, ..., r_n \rangle$, with $P \cong S_{k_1}^{n_1} \times S_{k_2}^{n_2} \times ... \times S_{k_{m-1}}^{n_{m-1}} \times S_{k_m}$ with k_i even for $i < m$ and k_m odd, then every element in $N_{D_n}(P)$ is strongly real.*

Proof. Recall that $N_{B_n}(P) = N^P \rtimes N_{S_n}(P)$. Notice that the factor $S_{k_1}^{n_1} \times S_{k_2}^{n_2} \times ... \times S_{k_{m-1}}^{n_{m-1}}$ always contributes an even number of 1s in the N^P coordinate of $N_{B_n}(P)$, since k_i is even for $i < m$. Then an element in $N_{B_n}(P) = N^P \rtimes N_{S_n}(P)$ is in D_n if and only if the S_m factor contributes no 1s to the N^P coordinate, as m is odd. That is, the N^P coordinate of such an element would be $[X, 0, ..., 0]$

where X is a vector containing an even number of 1s and the last m positions contain a 0. Thus we can see that since

$$N_{B_n}(P) \cong (\mathbb{Z}_2 \times S_{k_1}) \wr S_{n_1} \times ... \times (\mathbb{Z}_2 \times S_{k_{m-1}}) \wr S_{n_{m-1}} \times (\mathbb{Z}_2 \times S_{k_m}),$$

we have

$$N_{D_n}(P) = N_{B_n}(P) \cap D_n \cong (\mathbb{Z}_2 \times S_{k_1}) \wr S_{n_1} \times ... \times (\mathbb{Z}_2 \times S_{k_{m-1}}) \wr S_{n_{m-1}} \times (S_{k_m}).$$

Then since elements in the symmetric group and elements in the wreath product $(\mathbb{Z}_2 \times S_k) \wr S_n$ are strongly real, we have that elements in $N_{D_n}(P)$ are strongly real. □

Proposition 5.17. *If P is a parabolic subgroup of D_n of the form $P \subseteq \langle r_2, r_3, ..., r_n \rangle$, with $P \cong S_{k_1}^{n_1} \times S_{k_2}^{n_2} \times ... \times S_{k_{m-2}}^{n_{m-2}} \times S_{k_{m-1}} \times S_{k_m}$ with k_i even for $i < m-1$ and k_m, k_{m-1} odd, then every element in $N_{D_n}(P)$ is strongly real.*

Proof. Note that by Remark 5.15, it suffices to show that if $P \cong S_{k_1} \times S_{k_2}$ with k_1 and k_2 odd, then the statement is true. In this case, N^P contains the following 4 elements: the all-zero vector $\bar{0} := [0, ..., 0]$, the all-1 vector $\bar{1} := [1, ..., 1]$, the vector which contains k_1 1s and k_2 zeros $[1, ..., 1, 0, ..., 0]$, and the vector which contains k_2 1s and k_1 zeros $[0, ..., 0, 1, ..., 1]$. Since k_1 and k_2 are both odd, we know that the latter two elements of N^P cannot be the N^P coordinate of an element of $N_{D_n}(P) = N_{B_n}(P) \cap D_n$ since elements in D_n contain an even number of 1s. Since $k_1 + k_2$ is even, we have that the all-zero and all-1 vectors can be the N^P coordinate. Thus elements of $N_{D_n}(P)$ are of the form

$$[\bar{1}; \pi] \quad \text{or} \quad [\bar{0}; \pi]$$

where $\pi \in N_{S_n}(P)$. Then we see that

$$N_{D_n}(P) \cong \mathbb{Z}_2 \times N_{S_n}(P).$$

Then by Section 2, this completes the proof.

□

6. Summary of Results

We have shown that if W is a Coxeter group of type A, B, E, F, G, H, or I, then if P is a parabolic subgroup of W, then any element of $N_W(P)$ is strongly real. That is, for $x \in N_W(P)$, there is some $\tau \in N_W(P)$ of order 2 such that $\tau x \tau = x^{-1}$. We have also shown that if $W = D_n$ for n odd, then the statement still holds. In addition, if $W = D_n$ for n even, and P is a parabolic which does not contain the reflections corresponding to both of the first two nodes, then if P is isomorphic to the direct product of symmetric groups on an even number of symbols, then every element of $N_W(P)$ is strongly real. If instead P is the direct product of symmetric groups, all but at most two of which are on an even number of symbols, then every element of $N_W(P)$ is strongly real.

In order to complete the proof of our conjecture, it would suffice to show the following:

(1) If P is a parabolic subgroup of $W = D_{2n}$ which contains both of the first two nodes, then every element of $N_W(P)$ is strongly real;
(2) If P is a parabolic subgroup of D_{2n} which does not contain both of the first two nodes and can be written as the direct product of $m \geq 3$ symmetric groups on an odd number of symbols, then every element of $N_W(P)$ is strongly real.

Although they have not been proven, we see no reason that these two statements would not be true, and believe them to be true.

7. Appendix A

7.1. "MakePermGroup.txt".

```
Read("C:/gap4r4/MakeCoxGroup.txt");
p:=PositiveRoots(R);
orb:=Orbit(W,p[1]);
gen:=GeneratorsOfGroup(W);
pgens:=List(gen,y->PermList(List(orb,x->Position(orb,x^y))));
W:=Group(pgens);
gen:=GeneratorsOfGroup(W);
```

7.2. "MakeCoxGroup.txt".

```
#Creates our coxeter group ie, x:="F"; y:=4;

L:=SimpleLieAlgebra(x, y, Rationals);
R:=RootSystem(L);
W:=WeylGroup(R);
gen:=GeneratorsOfGroup(W);
```

7.3. "MakeParSubgps.txt".

```
l:=List([1..Length(gen)], c->0);
P:=[1..2^Length(gen)-1];

#here we create a list of all possible parabolic
#subgroups of W

for m in [1..2^Length(gen)-1]
  do
    Read("C:/gap4r4/addone.txt");
    lfilt:=Filtered([1..Length(gen)], x->l[x]=1);
    genforP:=List(lfilt, x->gen[x]);
    P[m]:=Subgroup(W, genforP);
  od;


theanswer:=true;

#for each parabolic subgroup, we read the file ConjRepOrd2,
#which creates the normalizer and reads NumberInvMult,
#returning whether our condition is satisfied for that
#subgroup.  If that answer is false for any subgroup, we
#define a new value "theanswer" to be false, meaning the
#conjecture is incorrect for this group.

for m in [1..2^Length(gen)-1]
  do
    Read("C:/gap4r4/ConjRepOrd2.txt");
    if existinv = false then
       theanswer:=false; break;
    else
```

```
      theanswer:=true;
    fi;
  od;
```

7.4. **"addone.txt".**

```
#adds one in binary to a list of size Length(gen) = number of generating reflections

for i in [1..Length(gen)]
do
  if l[i] = 0 then
    l[i]:=1; break;
  else
    l[i]:=0;
  fi;
od;
```

7.5. **"ConjRepOrd2.txt".**

```
N:=Normalizer(W, P[m]);
conj:=ConjugacyClasses(N);
rep:=List(conj, x->Representative(x));
ord2:=Filtered([1..Length(rep)], x->Order(rep[x])=2);

ct:=CharacterTable(N);

Read("C:/gap4r4/NumberInvMult.txt");
```

7.6. **"NumberInvMult.txt".**

```
#This file creates a list of the sums
#of Class Multiplication Coefficients

existinv:=true;
sum:=List([1..Length(conj)], x->0);
for k in [1..Length(rep)]
do
  for i in ord2
  do
     for j in ord2
     do
     c:=ClassMultiplicationCoefficient(ct, i, j, k);
sum[k] := sum[k]+c;
     od;
  od;
od;

#for each k (ie, each conjugacy class), we find whether
#there are two involutions multiplying into that class
#we return a value of "false" if a conj. rep. of order >2
#does not have 2 such involutions.
```

```
for k in [1..Length(rep)]
do
  if sum[k] = 0 then
    if order(rep[k]) > 2 then
existinv := false; break;
    fi;
  fi;
od;
```

8. Appendix B

8.1. Testing F_4.

```
gap> x:="F";y:=4;
"F"
4
gap> Read("C:/gap4r4/MakeCoxGroup.txt");
gap> Read("C:/gap4r4/MakeParSubgps.txt");
gap> theanswer;
true
```

8.2. Testing I_3.

```
gap> fg:=FreeGroup(3);
<free group on the generators [ f1, f2, f3 ]>
gap> x1:=fg.1; x2:=fg.2; x3:=fg.3;
f1
f2
f3
gap> f:=[x1^2, x3^2, x2^2, (x1*x2)^5, (x1*x3)^2, (x2*x3)^3];
[ f1^2, f3^2, f2^2, f1*f2*f1*f2*f1*f2*f1*f2*f1*f2, f1*f3*f1*f3,
  f2*f3*f2*f3*f2*f3 ]
gap> W:=fg/f;
<fp group on the generators [ f1, f2, f3 ]>
gap> Size(W);
120
gap> gen:=GeneratorsOfGroup(W);
[ f1, f2, f3 ]
gap> Read("C:/gap4r4/MakeParSubgps.txt");
gap> theanswer;
true
```

8.3. Testing I_4.

```
gap> fg:=FreeGroup(4);
<free group on the generators [ f1, f2, f3, f4 ]>
gap> x1:=fg.1; x2:=fg.2; x3:=fg.3; x4:=fg.4;
f1
f2
f3
f4
gap> $x4)^2,(x2*x4)^2]
```

```
> ;
[ f1^2, f2^2, f3^2, f4^2, f1*f2*f1*f2*f1*f2*f1*f2*f1*f2, f2*f3*f2*f3*f2*f3,
  f3*f4*f3*f4*f3*f4, f1*f3*f1*f3, f1*f4*f1*f4, f2*f4*f2*f4 ]
gap> W:=fg/f;
<fp group on the generators [ f1, f2, f3, f4 ]>
gap> gen:=GeneratorsOfGroup(W);
[ f1, f2, f3, f4 ]
gap> Read("C:/gap4r4/MakeParSubgps.txt");
gap> theanswer;
true
```

8.4. **Testing** E_6.

```
gap> x:="E";y:=6;
"E"
6
gap> Read("C:/gap4r4/MakeCoxGroup.txt");
gap> Size(W);
51840
gap> Read("C:/gap4r4/MakeParSubgps.txt");
gap> theanswer;
true
```

8.5. **Testing** E_7.

```
gap> x:="E"; y:=7;
"E"
7
gap> Read("C:/gap4r4/MakePermGroup.txt");
gap> Size(W);
2903040
gap> Read("C:/gap4r4/MakeParSubgps.txt");
gap> theanswer;
true
```

8.6. **Testing** E_8.

```
gap> x:="E"; y:=8;
"E"
8
gap> Read("C:/gap4r4/MakePermGroup.txt");
gap> Size(W);
696729600
gap> Read("C:/gap4r4/MakeParSubgps.txt");
gap> theanswer;
true
```

References

[1] L. C. Grove and C. T. Benson. *Finite reflection groups*, volume 99 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1985.

[2] Robert B. Howlett. Normalizers of parabolic subgroups of reflection groups. *J. London Math. Soc. (2)*, 21(1):62–80, 1980.

[3] Adalbert Kerber. *Representations of permutation groups. I*. Lecture Notes in Mathematics, Vol. 240. Springer-Verlag, Berlin, 1971.