Semiclassical Asymptotics of the Focusing Nonlinear Schrödinger Equation for Square Barrier Initial Data

by
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Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copies of the dissertation to the Graduate College.

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Signed: Robert Jenkins

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Dedication

For Amanda, thanks for waiting.

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ABSTRACT

The small dispersion limit of the focusing nonlinear Schroödinger equation (fNLS) exhibits a rich structure with rapid oscillations at microscopic scales. Due to the non self-adjoint scattering problem associated to fNLS, very few rigorous results exist in the semiclassical limit. The asymptotics for reflectionless WKB-like initial data was worked out in [KMM03] and for the family $q(x,0) = \text{sech}^{1+\frac{i}{\epsilon}\mu}$ in [TVZ04]. In both studies the authors observed sharp breaking curves in the space-time separating regions with disparate asymptotic behaviors.

In this paper we consider another exactly solvable family of initial data, specifically the family of centered square pulses, $q(x,0) = q\chi_{[-L,L]}$ for real amplitudes q. Using Riemann-Hilbert techniques we obtain rigorous pointwise asymptotics for the semiclassical limit of fNLS globally in space and up to an $\mathcal{O}(1)$ maximal time. In particular, we find breaking curves emerging in accord with the previous studies. Finally, we show that the discontinuities in our initial data regularize by the immediate generation of genus one oscillations emitted into the support of the initial data. This is the first case in which the genus structure of the semiclassical asymptotics for fNLS have been calculated for non-analytic initial data.

Chapter 1

Introduction

1.1 Problem Statement and Background Material

We consider the semi-classical limit of the cubic focusing nonlinear Schrödinger (NLS) equation

$$i\epsilon\frac{\partial q}{\partial t}+\frac{\epsilon^2}{2}\frac{\partial^2 q}{\partial x^2}+|q|^2 q=0, \tag{1.1}$$

subject to the square-barrier initial condition,

$$q(x,0)=q(x)=\begin{cases} q & : \quad -L<x<L \\ 0 & : \quad |x|\geq L \end{cases}. \tag{1.2}$$

That is, we consider the family of solutions $q(x,t;\epsilon)$ parameterized by ϵ and we study their limiting behavior as the parameter $\epsilon\to 0^+$. For general initial data, and in particular for our square barrier, when we study the semiclassical limit we consider data which is independent of the dispersion parameter ϵ so as to avoid solutions which follow from trivial rescalings.

The focusing NLS equation is one of the canonical examples of nonlinear hamiltonian PDEs. It models many physical wave-like phenomena in fields as diverse as nonlinear optics and the study of wave waves. The small ϵ limit of focusing NLS arises naturally in the study of nonlinear fiber optics where it describes the propagation of waves in Kerr media subject to anomalous dispersion.

Solutions of the initial value problem naturally exhibit both the macrostructure inherited from the spacial scale of the initial condition and microstructure inherited from the intrinsic, ϵ-dependent, time and spacial scale of the PDE. The interplay between theses scales results in solutions exhibiting a complex interplay between these vastly disparate length scales. In fact one can show that the limiting dynamics are modulationally unstable. If we write the solution $q(x,t)$ in fluid like density $\rho(x,t)$ and momentum $\mu(x,t)$ variables defined by

$$\rho=|q|^2, \qquad \mu=\epsilon\,\mathrm{Im}(\bar{q}\partial_x q),$$

then in terms of these variables the NLS equation becomes the Euler like system of equations,

$$\partial_t\rho+\partial_x\mu=0, \qquad \partial_t\mu+\partial_x\left(\frac{\mu^2}{\rho}-\frac{\rho^2}{2}\right)=\frac{\epsilon^2}{4}\partial_x\left(\rho\partial_x^2\log\rho\right), \tag{1.3}$$

with initial condition $\rho(x,0)=|q(x,0)|^2$ and $\mu(x,0)=\epsilon\,\mathrm{Im}(\bar{q}(x,0)\partial_x q(x,0))$. If we now set ϵ to zero, the limiting system of equations is elliptic. It follows that the initial value problem we consider is ill-posed in the zero dispersion limit. However, for nonzero ϵ the focusing NLS equation (1.1) on the real line is globally well posed for initial data $q_0(x)\in L^2(\mathbb{R})$, of which our square barrier initial condition (1.2) is certainly a member, in the following

sense: there exist a function $q(x,t) \in C(\mathbb{R}_+, L^2(\mathrm{d}x)) \cap (L^\infty(\mathrm{d}x) \otimes L^4_{\mathrm{loc}}(\mathrm{d}t))$ solving the weak (integral) form of (1.1), such that $q(x,0) = q_0(x)$ and $\|q(x,t)\|_{L^2(\mathrm{d}x)}$ is conserved [**?**]. In the semiclassical limit we are studying a family of solutions $q(x,t;\epsilon)$ whose gradients are bounded but grow as ϵ decreases, leading to an ill-posed limit. The limiting version of (1.3) can only be studied for initial data which yield analytic initial densities $\rho_0(x)$ and momentums $\mu_0(x)$. This is an overly restrictive subset of the initial data for which the NLS equation is well-posed before we take the formal limit. Moreover, the discontinuous square-barrier initial condition we study here certainly falls well outside the class of analytic initial data and the limiting Euler system is definitely ill-posed.

It has been known since 1972 that the focusing NLS equation is completely integrable. In their work [ZS71], Zakharov and Shabat demonstrate that the focusing NLS equation is equivalent to the solution of the Lax Pair,

$$\epsilon \Psi_x = \mathcal{L}\Psi, \tag{1.4}$$

$$i\epsilon \Psi_t = \mathcal{B}\Psi, \tag{1.5}$$

where the operators $\mathcal{L}$ and B are given by

$$\mathcal{L} = -iz\sigma_3 + Q, \tag{1.6}$$

$$\mathcal{B} = iz\mathcal{L} + \frac{1}{2}\,\sigma_3\left(Q^2 - \epsilon\, Q_x\right), \tag{1.7}$$

where z is a spectral parameter and the 2×2 matrices σ_3 and Q are

$$\begin{aligned} \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ Q \;=\; Q(x,t) &= \begin{pmatrix} 0 & q(x,t) \\ -\overline{q(x,t)} & 0 \end{pmatrix}. \end{aligned}$$

The equivalence of NLS follows from the Lax equation; if we can find a simultaneous solution Ψ of the Lax Pair (1.4) and (1.5) then they must satisfy the compatibility condition

$$i\epsilon\mathcal{L}_t - \epsilon\mathcal{B}_x + [\mathcal{L}, \mathcal{B}] = 0, \tag{1.8}$$

which reads for the Zakharov-Shabat Lax Pair,

$$i\epsilon Q_t + \frac{\epsilon^2}{2}\sigma_3 Q_{xx} - \sigma_3 Q^3 = 0, \tag{1.9}$$

which is simply a matrix form of the focusing NLS equation.

The integrable structure of the NLS equation allows one to use the methods of scattering theory to linearize the PDE and formally solve the initial value problem. In chapters 2 and 3 we introduce the scattering and inverse scattering formalism and describe them in the context of focusing NLS. In practice both the forward and inverse scattering problems are prohibitively difficult to analyze for general initial data. This is particularly true in the focusing NLS case due to the fact that the Zakharov-Shabat (ZS) spectral problem, (1.4)

one must analyze in the scattering step is not self-adjoint. Understanding the spectral data generated by the ZS problem from arbitrary initial data is still an open area of research. Moreover, the semiclassical problem only aggravates the issue. The number of eigenvalues generated in the scattering step typically scales like ϵ^{-1}. This is not only an issue in the scattering step, where we have to search for an ever growing family of point spectra, but also in the inverse scattering step. There the eigenvalues become the poles of a Riemann-Hilbert problem which we must analyze as the growing number of poles collect on some subset of the complex spectral plane.

The first rigorous analytic results describing the semiclassical limit of focusing NLS were given by Kamvissis, Miller, and McLaughlin in [KMM03]. Here the authors considered the family of initial data

$$q(x,0) = A(x), \tag{1.10}$$

for real analytic "bell-shaped" function $A(x)$. In this study the authors used WKB approximations of the spectral data to generate a reflectionless Riemann-Hilbert problem which they were able to rigorously analyze using the Deift-Zhou method of steepest descent. In the analysis the original spectral data is replaced by its WKB approximation, this step was later made rigorous in [Mil02]. In the special case where the $q(x,0)$ was the Satsuma-Yajima initial data

$$A(x) = A\,\mathrm{sech}(x), \quad A \text{ constant}, \tag{1.11}$$

the forward scattering can be calculated explicitly [SY74] and exactly matches the WKB approximation. In a later study Tobvis, and Venakides considered another family of sech-like initial data generalizing the Satsuma-Yajima initial data; in [TV00] they show how one can exactly calculate the scattering data for the analytic initial data

$$q(x,0) = A\,\mathrm{sech}^{1+i\frac{\mu}{\epsilon}}(x). \tag{1.12}$$

Together with Zhou in [TVZ04] and [TVZ06], the authors analyzed the Riemann-Hilbert problem recovering rigorously the semiclassical asymptotic behavior.

In both the [KMM03] study and the [TVZ04] study the authors observed that the asymptotic behavior of the solution $q(x,t)$ exhibits sharp transitions in space time. That is, both studies found the existence of *caustics* or *breaking curves* in space-time across which the qualitative description of the asymptotic limit was sharply different. In each region the limiting behavior was described in terms of slowly modulated multi-phase wave trains of a particular genus G; as the solution evolves, and crosses each caustic the genus of the limiting solution increases, and the behavior becomes more complex.

1.2 Statement of Main Results

The square barrier initial data problem is an attractive case study for two reasons. First, the square barrier is manifestly non-analytic. In this case the ill-posedness of the limiting Euler system ((1.3) after dropping the ϵ-dependence terms) becomes an issue. This distinguishes the square barrier from the sech like initial data studied in [KMM03] and [TVZ04] where the initial conditions, (1.10) and (1.12) respectively, are analytic. To our knowledge our

results are the first example where rigorous semiclassical asymptotics have been computed for focusing NLS given non-analytic initial data.

The other reason to consider square barrier data is that we can actually solve the Zakharov-Shabat spectral problem and exactly characterize the entire spectral data. This is a more than just a practical nicety; if we were to use approximate spectral data, as was done in the [KMM03] study, we have no reason to expect that the modified initial data, $\tilde{q}(x)$, would retain the discontinuities of the exact initial data. Additionally, the compact support of the square barrier endows the spectral data with several nice properties which we use to simplify the Riemann-Hilbert analysis.

The main result of this study is the following theorem.

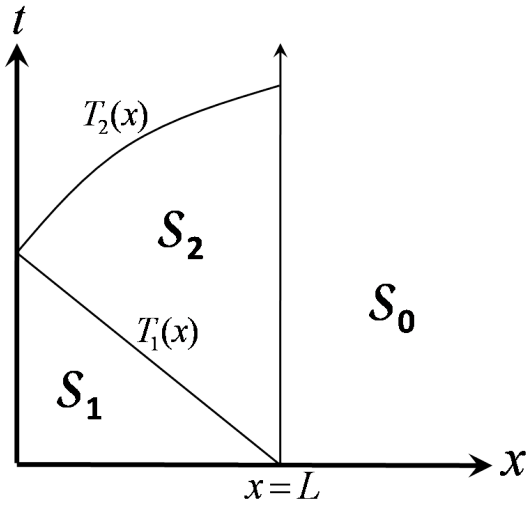

FIGURE 1.1. The sets $\mathcal{S}_j$ for which Theorem 1 provides strong semiclassical asymptotics. The set $\mathcal{S}_0$ is unbounded in time. The curves $T_1(x)$ and $T_2(x)$ sharply divide regions where the asymptotic limiting behavior $q_0(x,t)$ is completely different.

Theorem 1. *Consider the curves $t = T_1(x)$ and $t = T_2(x)$ defined by* (5.29) *and* (6.32) *respectively. Define the three following subsets of space-time, $\mathbb{R} \times [0, \infty)$:*

$$\mathcal{S}_0 = \{(x,t) \,:\, x > L,\ t > 0\},$$
$$\mathcal{S}_1 = \{(x,t) \,:\, 0 \le x < L,\ 0 < t < T_1(x)\},$$
$$\mathcal{S}_2 = \{(x,t) \,:\, 0 \le x < L,\ T_1(x) < t < T_2(x)\},$$

which are depicted in Figure 1.1. Then for each $(x,t) \in \mathcal{S}_j,\ j = 0,1,2$, the solution $q(x,t;\epsilon)$ of the focusing NLS equation (1.1) *given the square barrier initial data* (1.2) *satisfies the following semiclassical ($\epsilon \to 0^+$) limit:*

$$|q(x,t) - q_0(x,t)| = \mathcal{O}(\epsilon^{1/2-\rho})$$

for any $\rho > 0$. The limiting function $q_0(x,t)$ is given in each set $\mathcal{S}_k$ by

1. *For $(x,t) \in \mathcal{S}_0$,*
$$q_0(x,t) := 0$$
2. *For $(x,t) \in \mathcal{S}_1$,*
$$q_0(x,t) := qe^{\frac{i}{\epsilon}q^2 t}$$
3. *For $(x,t) \in \mathcal{S}_2$,*
$$q_0(x,t) = i\,\mathrm{Re}(\alpha(x,t))\frac{\Theta(\mathbf{Z})}{\Theta(\mathbf{Y})}\frac{\Theta(\mathbf{Y}+iU-i\Omega/\epsilon)}{\Theta(\mathbf{Z}-iU+i\Omega/\epsilon)}e^{-\frac{2i}{\epsilon}(g(\infty)-\epsilon\chi)} + \mathcal{O}(\epsilon^{1/2-\rho}).$$
Here, $\Theta(\cdot)$ is the genus one Riemann Theta function with period matrix H given by (6.91)*, $\mathbf{Z} = u(\infty) - V_1$ and $\mathbf{Y} = u(\infty) + V_1$; the function $u(z) = \int_{-iq}^{z} \nu$ is defined by* (6.94)*; the numbers V_1, U, and χ are functions of (x,t) defined by* (6.95)*,* (6.92)*, and* (6.97)*, respectively; the function $\alpha(x,t)$ is defined in Prop. 20, and the numbers Ω and $g(\infty)$ also depend on (x,t) and are defined by* (6.28) *and* (6.25)*.*

Furthermore, the convergence of $q(x,t;\epsilon)$ to $q_0(x,t)$ is uniform on any compact subset $K_j \subset \mathcal{S}_j$, $j = 0,1,2$.

We make the following observations concerning our results:

1. We have provided asymptotics only for positive values of x. However, because the square barrier initial data is even, the solution $q(x,t)$ must also be even at all later times. Therefore, by symmetry we have provided global asymptotics in space up to an order one time which depends on x.

2. The barrier does not spread out in time. At any positive time $t > 0$ the solution is no longer compactly supported but the solution outside the support consists only of radiative terms whose amplitude decays with ϵ.

3. In regions $\mathcal{S}_0$ and $\mathcal{S}_1$ the asymptotic limiting solution $q_0(x,t)$ is the genus-zero plane wave solution consistent with solving the NLS equation for constant amplitude initial data whose amplitude is equal to the amplitude of the square barrier in each region.

4. The initial shocks in the square barrier initial data at $x = \pm L$ regularize for all positive times by shedding genus one oscillations into region $\mathcal{S}_2$. This is consistent with the previous studies [KMM03] and [TVZ04] where the mechanism for formation of the first caustic is the formation of a shock in the genus zero solution valid for times less than the breaking time.

5. We do not yet understand the genus structure beyond the second caustic $T_2(x)$. Preliminary work in that direction indicates that a single new band emerges in the Riemann Hilbert analysis causing the genus of the solution beyond the secondary caustic to jump to genus two for t just beyond the secondary caustic. This is only a preliminary remark, we do not pursue a description of the solution beyond the secondary caustic in this study.

Organization of the Paper

The remainder of this paper is divided into five chapters. In chapter two we present the forward scattering theory for the focusing NLS equation in general and use this theory to explicitly calculate the spectral data in the special case of square barrier data. In chapter three we introduce the Riemann-Hilbert formalism and describe how one can cast the inverse scattering problem in terms of a Riemann-Hilbert Problem. The analysis of the inverse problem is the subject of the final three chapters. In each chapter we present the analysis necessary to prove one part of our main result Chapter four presents the analysis for $(x,t) \in \mathcal{S}_0$, that is, outside the support of the square barrier. In chapter five we consider $(x,t) \in \mathcal{S}_1$, and identify the mechanism by which the first breaking curve occurs and explicitly calculate its value. In the final chapter we present the Riemann-Hilbert analysis describing the genus one oscillations and characterize the second caustic as the simultaneous solution of a pair of algebraic equations. The analysis in each of the final three chapters is largely independent of the others but whenever possible we use calculations from the previous chapters to keep things as succinct as possible.

Before we begin we include a brief note on notation. In all that follows we will make extensive use of the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Also we will make frequent use of the somewhat standard notation $\bar{f}(z)$, which for any function f is a shorthand for $\overline{f(\bar{z})}$. In particular, this notation will appear in the factorization of the jump matrices which arise in the analysis of the Riemann-Hilbert problems associated with NLS. These are standard notations in the literature of Riemann Hilbert problems which we use throughout, largely without comment.

Chapter 2

Forward Scattering Theory for Focusing NLS

The forward scattering problem begins by seeking vector solutions, ψ, to the Zakharov Shabat (ZS) eigenvalue problem for the given initial potential:

$$i\epsilon\sigma_3\psi_x - i\begin{pmatrix} 0 & q_0(x) \\ \overline{q}_0(x) & 0 \end{pmatrix}\psi = z\psi. \tag{2.1}$$

A particular value of the spectral parameter z will be considered an eigenvalue of (2.1) if it admits a solution, $\psi(x, t = 0; z) \in L^2(\mathbb{R})$. In practice it is difficult to directly solve for the spectral data. Instead, we first seek a pair of matrix solutions of (2.1) for z restricted to the real line.

$$\begin{cases} \Psi(x;z)e^{\frac{i}{\epsilon}zx\sigma_3} \to I & x \to \infty \\ \Psi(x;z)e^{\frac{i}{\epsilon}zx\sigma_3} \text{ bounded} & x \to -\infty, \end{cases}$$
$$\begin{cases} \Phi(x;z)e^{\frac{i}{\epsilon}zx\sigma_3} \text{ bounded} & x \to \infty, \\ \Phi(x;z)e^{\frac{i}{\epsilon}zx\sigma_3} \to I & x \to -\infty. \end{cases} \tag{2.2}$$

Though these normalization may seem strange at first, they are the natural normalization for potentials which decay at infinity. We call such solutions the *Jost solutions* of the system.

2.1 An integral representation of the Jost solutions

Let $N = \Psi e^{\frac{i}{\epsilon}xz\sigma_3}$ (this scaling approaches identity as $x \to \infty$), then N satisfies,

$$N_x = -\frac{iz}{\epsilon}[\sigma_3, N] + \frac{1}{\epsilon}\begin{pmatrix} 0 & q(x) \\ -q(x) & 0 \end{pmatrix}N. \tag{2.3}$$

We will call the differential equation for $N(x;z)$ the *associated ZS problem.*

Formally, we can integrate,

$$N(x,0;z) = I + \int_{\infty}^{x} e^{\frac{i}{\epsilon}z(s-x)\sigma_3}\begin{pmatrix} 0 & q(s) \\ -q(s) & 0 \end{pmatrix}N(s,0;z)e^{-\frac{i}{\epsilon}z(s-x)\sigma_3}\,\mathrm{d}s. \tag{2.4}$$

Similarly, we can introduce $O = \Phi e^{\frac{i}{\epsilon}xz\sigma_3}$ yielding,

$$O(x,0;z) = I + \int_{-\infty}^{x} e^{\frac{i}{\epsilon}z(s-x)\sigma_3}\begin{pmatrix} 0 & q(s) \\ -q(s) & 0 \end{pmatrix}e^{-\frac{i}{\epsilon}z(s-x)\sigma_3}O(s,0;z)\,\mathrm{d}s. \tag{2.5}$$

Whether solutions to (2.4) and (2.5) exist for all x depends on how quickly the potential $q(x)$ decays. In Lemma 2, whose proof is the content of Appendix A, we show that for

$q(x) \in L^1(\mathbb{R})$, or more naturally $q(x) \in L^2(\mathbb{R}, (1+|x|^2)\mathrm{d}x)$, that solutions to these integral equations exist and are bounded. In particular, this will imply that the entire real axis is part of the continuous spectrum for initial data in $L^2(\mathbb{R}, (1+|x|^2)\mathrm{d}x)$.

How do we solve (2.1) when $\mathrm{Im}(z) \neq 0$? If we write our matrix solutions as pairs of column vectors $N = (N_1, N_2)$ and $O = (O_1, O_2)$, the columns may be extended analytically into different regions of the complex plane. The existence of solutions and the extension of each column into the complex plane is the subject of the following lemma.

Lemma 2. *Solutions $N = (N_1, N_2)$ and $O = (O_1, O_2)$ to (2.4) and (2.5) exist for each $z \in \mathbb{R}$. Under the following conditions on $q(x)$ the column vectors have the following analytic extensions:*

1. *If $q(x) \in L^1(\mathbb{R})$, then N_1 and O_2 extend to $\mathbb{C}^-$ while N_2 and O_1 extend to $\mathbb{C}^+$.*
2. *If $\exists\ A, b$ such that $|q(x)| < Ae^{-bx}$ then N_1 and O_2 extend to $\left\{z|\ Im(z) < \frac{\epsilon}{2}b\right\}$ while N_2 and O_1 extend to $\left\{z|\ Im(z) > -\frac{\epsilon}{2}b\right\}$.*
3. *If $q(x)$ is compactly supported than both N and O are entire functions of z.*

Proof: The proof follows from showing that the kernel of the integrals in formulae (2.4) and (2.5) are bounded. The details of the proof are provided in appendix A.

2.2 The Scattering Matrix

Both $\Phi(x;z)$ and $\Psi(x;z)$ are fundamental solutions of the ZS system and so must be linearly related for each z for which they simultaneously exist. That is, there exists a matrix $S(z)$, which we call the *scattering matrix*, such that

$$\Phi(x;z) = \Psi(x;z)S(z). \tag{2.6}$$

Remark: In the general case the scattering matrix exists only for $z \in \mathbb{R}$. However, as implied by Lemma 2, if the initial data is compactly supported equation (2.6) is valid for every $z \in \mathbb{C}$.

Properties of the Scattering Matrix and Jost solutions

Observe the following easily verifiable facts which we will use as we derive the scattering theory of NLS.

1. *Symmetry:* If $\Psi(x;z)$ is a matrix solution of the ZS system then so is $\widetilde{\Psi} = \sigma_2\overline{\Psi(x;\bar{z})}\sigma_2$.
2. *Unimodularity:* Both Jost solutions, $\Psi(x;z)$ and $\Phi(x;z)$, and the scattering matrix $S(z)$ satisfy $\det(\Psi) = \det(\Phi) = \det(S) = 1$ wherever they exist.
3. *Unitarity:* For $z \in \mathbb{R}$ we have $\Psi\Psi^* = I,\ \Phi\Phi^* = I$, and $SS^* = I$.

The symmetry property is particularly useful because it implies that eigenvalues must either be real or come in complex conjugate pairs. This allows us to restrict our search for eigenvalues to $z \in \overline{\mathbb{C}^+}$.

2.3 Extending the Jost solution to the upper half plane

In the following we assume that $q(x) \in L^2(\mathbb{R}, (1+|x|^2)\mathrm{d}x)$ unless explicitly placed elsewhere. Define the matrix $\widetilde{M}(x;z) = (O_1(x;z)|N_2(x;z))$. Since both column vectors are analytic for $z \in \mathbb{C}^+$, this matrix defines an analytic solution of (2.3) for each z in the upper half-plane. However, it is not a Jost solution; as $x \to \infty$ we know that $N_2(x;z) \to \binom{0}{1}$ but as yet we know only that O_1 stays bounded as $x \to \infty$. A careful examination of O_1 reveals a bit of luck; we can show that $o_{21}(x;z) \to 0$ as $x \to \infty$ for each $z \in \mathbb{C}^+$.

Claim 1. For $O(x;z)$ a Jost solution of (2.3) as given by (2.5), $\lim_{x\to+\infty}\ o_{21}(x;z) = 0$ for every $z \in \mathbb{C}^+$ provided $q(x) \in L^2(\mathbb{R}, (1+|x|^2)\mathrm{d}x)$.

Proof. From (2.5) it follows that the integral equation for o_{21} is

$$o_{21}(x;z) = \int_{-\infty}^{x} -\overline{q(s)}e^{2iz\epsilon^{-1}(x-s)}o_{11}(s)\ ds. \tag{2.7}$$

For $z \in \mathbb{C}^+$ and $s < x$, $e^{2iz\epsilon^{-1}(x-s)} < 1$. Also, from the proof of Theorem 2 it follows that $\|o_{11}\|_\infty < \infty$. Using the inclusion $L^2(\mathbb{R}, 1+|x|^2\mathrm{d}x) \subset L^1(\mathbb{R})$ we have the inequality,

$$\begin{aligned} |o_{21}(x;z)| &\leq \|o_{11}\|_\infty \left(e^{-\epsilon^{-1}\operatorname{Im}(z)x} \int_{-\infty}^{x/2} |q(s)|\,\mathrm{d}s + \int_{x/2}^{x} |q(s)|\,\mathrm{d}s \right) \\ &\leq \|o_{11}\|_\infty \left(e^{-\epsilon^{-1}\operatorname{Im}(z)x} \|q\|_{L^1(\mathbb{R})} + \int_{x/2}^{\infty} |q(s)|\,\mathrm{d}s \right) \end{aligned}$$

Then taking the limit as $x \to \infty$ we get the desired result. □

At this point we know that as $x \to \infty$, $\widetilde{M}(x;z) \to \begin{pmatrix} a(z) & 0 \\ 0 & 1 \end{pmatrix}$, for some complex $a(z)$, bounded for z real. It then follows that

$$M^{(up)}(x;z) = \left(\frac{O_1(x;z)}{a(z)} \mid N_2(x;z) \right) \tag{2.8}$$

is also a solution of the associated ZS problem (2.3) which, by direct construction, satisfies the Jost normalization for each value of the parameter $z \in \mathbb{C}^+$.

What about the analyticity of $M^{(up)}$? This depends on the properties of $a(z)$. To understand $a(z)$, we first consider $\Delta(x,z) = \det(\widetilde{M}(x;z))$. Using (2.3) we have

$$\begin{aligned}\frac{d\Delta}{dx} &= \det\left(\frac{dO_1}{dx} \mid N_2\right) + \det\left(O_1 \mid \frac{dN_2}{dx}\right) \\ &= \frac{-iz}{\epsilon}\left[\det\left((\sigma_3 - I)O_1 \mid N_2\right) + \det\left(O_1 \mid (\sigma_3 + I)N_2\right)\right] \\ &\qquad + \frac{1}{\epsilon}\left[\det\left(QO_1 \mid N_2\right) + \det\left(O_1 \mid QN_2\right)\right] \\ &= 0.\end{aligned}$$

The last equality above follows from direct inspection of the quantities in brackets. This calculation shows that $\Delta(x;z) = \Delta(z)$ is independent of x. We evaluate $\Delta(z)$ by letting $x \to \infty$, $\Delta(z) = \det\begin{pmatrix} a(z) & 0 \\ 0 & 1 \end{pmatrix} = a(z)$. In particular, since $\widetilde{M}$ is analytic for $z \in \mathbb{C}^+$ and the determinate preserves this analyticity, it follows that $a(z)$ is an analytic function in $\mathbb{C}^+$ and may be expressed as $a(z) = \det\big(O_1(x;z) \mid N_2(x;z)\big)$. Thus, $M^{(up)}(x;z)$ is analytic for all z in the upper half plane except those z_k where $a(z_k) = 0$, at these points $M^{(up)}$ develops poles.

2.3.1 Using symmetry to get a solution for $\mathrm{Im}(z) < 0$

As stated previously, solutions of the ZS problem have a Schwartz-like symmetry, this is also true of the associated problem: If $M(x;z)$ is one solution, then $M^\dagger(x) = \sigma_2\overline{M(x;\overline{z})}\sigma_2$ is another solution. We will quickly show here that this holds for $N(x;z)$ and $O(x;z)$ as well. Taking the complex conjugate of (2.3) and multiplying from the left and right by σ_2 we get

$$N_x^\dagger = \sigma_2\overline{N}_x\sigma_2 = \frac{i\overline{z}}{\epsilon}\sigma_2[\sigma_3, \overline{N}]\sigma_2 + \frac{1}{\epsilon}\sigma_2\begin{pmatrix} 0 & -\overline{q(x)} \\ q(x) & 0 \end{pmatrix}\overline{N}\sigma_2$$

and since $\sigma_2\sigma_3\sigma_2 = -\sigma_3$ and $\sigma_2\overline{Q(x)}\sigma_2 = Q(x)$ it follows that

$$N_x^\dagger = -\frac{i\overline{z}}{\epsilon}[\sigma_3, N^\dagger] + \frac{1}{\epsilon}\begin{pmatrix} 0 & q(x) \\ -\overline{q(x)} & 0 \end{pmatrix}N^\dagger.$$

We can use this fact to generate a Jost solution in the lower half-plane: Given $M^{(up)}(x;z)$, a meromorphic solution of the associated ZS problem in the upper half-plane with identity asymptotics for large x, the above argument shows that

$$M^{(down)}(x;z) = M^\dagger(x;z) = \sigma_2\overline{M^{(up)}(x,\overline{z})}\sigma_2$$

is also a solution of the associated ZS problem with the same asymptotic behavior for large x but, by the Schwartz reflection principle, meromorphic for each $z \in \mathbb{C}^-$.

2.4 The reflection coefficient

For $z \in \mathbb{R}$ we can use the scattering matrix to simplify $M^{(up)}$. The scattering matrix implies that

$$O_1(x;z) = N_1(x;z)s_{11}(z) + N_2(x;z)s_{21}(z)e^{2i\epsilon^{-1}xz}.$$

Inserting this into the relation $a(z) = \det(\widetilde{M}(x;z))$ we get

$$\begin{aligned} a(z) &= \det\left(N_1(x;z)s_{11}(z) + N_2(x;z)_{21}(z)e^{2i\epsilon^{-1}xz} \;\middle|\; N_2(x;z)\right) \\ &= s_{11}(z)\cdot\det(N(x;z)) = s_{11}(z). \end{aligned} \tag{2.9}$$

We can then reexpress $M^{(up)}$ in terms of only $N(x;z)$:

$$\begin{aligned} M^{(up)}(x;z) &= \left(\frac{O_1(x;z)}{a(z)} \;\middle|\; N_2(x;z)\right) \\ &= \left(N_1(x;z) + N_2(x;z)\frac{s_{21}(z)}{s_{11}(z)}e^{2i\epsilon^{-1}xz} \;\middle|\; N_2(x;z)\right). \end{aligned} \tag{2.10}$$

The expression,

$$\mathfrak{r}(z) = \frac{s_{21}(z)}{s_{11}(z)}, \tag{2.11}$$

defined *a priori* only for $z \in \mathbb{R}$, is called the *reflection coefficient*. This is the first component of what together with the eigenvalues and normalization constants (to be defined later) are called the scattering data.

Before moving onto a discussion of the eigenvalues we make the following observation. Each half-plane solution takes continuous boundary values on the real axis and through the scattering relation we can write each boundary value in terms of $N(x;z)$ only. For $z \in \mathbb{R}$, coupling the symmetry and unitarity properties of $N(x;z)$ we have,

$$\begin{aligned} \lim_{\delta\to 0^+} M^{(up)}(x;z+i\delta) &= N(x;z)\begin{pmatrix} 1 & 0 \\ \mathfrak{r}(z)e^{2i\epsilon^{-1}xz} & 1 \end{pmatrix}, \\ \lim_{\delta\to 0^-} M^{(down)}(x;z+i\delta) &= N(x;z)\begin{pmatrix} 1 & -\overline{\mathfrak{r}(z)}e^{-2i\epsilon^{-1}xz} \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

These calculations will be useful in Chapter 3 where we will build a Riemann Hilbert problem from the Jost solutions.

2.5 Eigenvalues of the ZS system

The Jost solution of (2.3) we have constructed, $M^{(up)}(x;z) = \left(\frac{O_1(x;z)}{a(z)} \mid N_2(x;z)\right)$, is analytic and well defined for each $z \in \mathbb{C}^+$ except for z_k such that $a(z_k) = 0$. At these point $M^{(up)}(x;z_k)$ has a pole in its first column. What can we say about such z_k? At these points $a(z_k) = \det\left(O_1(x;z_k) \;\middle|\; N_2(x;z_k)\right) = \det\left(\Phi_1(x;z_k) \;\middle|\; \Psi_2(x;z_k)\right) = 0$, implying that $\Psi_2(x;z_k)$ and $\Phi_1(x;z_k)$ are linearly dependent, *i.e.* there exists a *normalization constant*,

c_k, such that $\Phi_1(x;z_k) = c_k \cdot \Psi_2(x;z_k)$. For $z \in \mathbb{C}^+$ this implies that $\psi_k = \Psi_2(x;z_k)$ decays exponentially as $|x| \to \infty$, or that each z_k is an eigenvalue of the ZS equation (2.1) with corresponding eigenvector ψ_k. This gives us an alternative characterization of the eigenvalues of the ZS system:

The eigenvalues of the ZS system in (2.1) are precisely those $z \in \mathbb{C}^+$ which solve the equation $a(z) = 0$ together with their complex conjugates[1].

However, we cannot rule out the existence of $z \in \mathbb{R}$ solving $a(z) = 0$. In this case the corresponding solution $\Psi_2(x;z)$ does not decay as $|x| \to \infty$. If a real eigenvalue exists it is called an emerging eigenvalue; we exclude this situation from our analysis.

2.5.1 Normalization constants for compactly supported initial data

In general, calculating the normalization constants is difficult and must be dealt with on a case-by-case basis. However, for compactly supported initial data there is a nice relationship between the normalization constants and the scattering matrix.

For compact potentials $q(x)$ Lemma 2 tells us that both Jost solutions,$\Psi(x,z)$ and $\Phi(x;z)$, are entire functions of z and therefore the scattering relation (2.6) is valid in the entire complex plane. In particular applying (2.9) and (2.6) for general $z \in \mathbb{C}^+$ we write,

$$\Phi_1(x;z) = a(z)\Psi_1(x;z) + s_{21}(z)\Psi_2(x;z).$$

Coupling this with the eigenvalue condition $a(z_k) = 0$ we have

$$\Phi_1(x;z_k) = s_{21}(z_k)\Psi_2(x;z_k),$$

from which it is immediately apparent that for compact initial data,

$$c_k = s_{21}(z_k). \tag{2.12}$$

2.6 Time Evolution of the Scattering Data

Thus far we have described how the initial data $q(x,t=0) = q(x)$ generates the scattering data: the scattering matrix, $S(z)$; the eigenvalues, z_k; and the normalization constants, c_k. The goal of this section is to describe how these quantities evolve in time. The eigenvalues of the system do not evolve; this is a consequence of the isospectrality of the Lax-Pair. The evolution of the other quantities is governed by the t-flow operator (1.7), which together with the ZS equation make up the Lax-Pair for NLS. An amazing property of the integrable structure of NLS, is that the time evolution of the scattering data is linear. In this sense the scattering map, taking the potential $q(x)$ to the collection of scattering data, acts as a nonlinear Fourier transform. Specifically, in this section we will show that the spectral data

[1] Eigenvalues come in complex conjugate pairs because of the symmetry solution $M(x;z) = \sigma_2\overline{M(x,\overline{z})}\sigma_2$

evolves as follows:

$$\begin{aligned} S(z,t) &= e^{-\frac{i}{\epsilon}z^2 t\sigma_3} S(z,0) e^{\frac{i}{\epsilon}z^2 t\sigma_3}, \\ \mathfrak{r}(z,t) &= \frac{s_{21}(z,0)}{s_{11}(z)} e^{\frac{2i}{\epsilon}z^2 t}, \\ c_k(t) &= c_k(0) e^{\frac{2i}{\epsilon}z^2 t}. \end{aligned}$$

2.6.1 Time evolution of the reflection coefficient

Suppose that we can find $\Psi(x,t;z)$ and $\Phi(x,t;z)$, Jost solutions of the ZS equation for $t > 0$, where $\Psi(x,t=0;z) = \Psi(x;z)$ and $\Phi(x,t=0;z) = \Phi(x;z)$ are the solutions we found above for the initial data $q(x)$. Then, for real z, these solutions will be related by a time-dependent scattering matrix $S(z,t)$,

$$\Phi(x,t;z) = \Psi(x,t;z) S(z,t). \tag{2.13}$$

From the time evolving Jost solutions we would like to generate simultaneous solutions of the Lax-pair which we call $\widehat{\Psi}$ and $\widehat{\Phi}$. As solutions of the ZS equation our simultaneous solutions must have the representation

$$\widehat{\Psi}(x,t;z) = \Psi(x,t;z) C(t), \tag{2.14}$$

$$\widehat{\Phi}(x,t;z) = \Phi(x,t;z) K(t). \tag{2.15}$$

Observe that these functions are still solutions of the ZS problem because the matrices $C(t)$ and $K(t)$ are independent of x, but they are generically not Jost solutions; they violate the Jost normalization except for the special case $C(t) = K(t) = I$.

Inserting $\widehat{\Psi}$ into the t-flow equation (1.7) we get

$$i\epsilon \frac{\partial}{\partial t} \widehat{\Psi}(x,t;z) = \mathcal{B}\Big(\widehat{\Psi}(x,t;z) \Big),$$

which becomes after inserting (2.14),

$$i\epsilon \, \frac{\partial}{\partial t} \Big(\Psi(x,t;z) C(t) \Big) = \mathcal{B}\Big(\Psi(x,t;z) C(t) \Big).$$

We now wish to take the limit as $x \to \infty$. If the potential $q(x,t)$ decays fast enough we can pass the limits through the operators to get the simplified equation,

$$i\epsilon \frac{\partial}{\partial t} \Big(e^{-\frac{i}{\epsilon}xz\sigma_3} C(t) \Big) = z^2 \sigma_3 e^{-\frac{i}{\epsilon}xz\sigma_3} C(t), \tag{2.16}$$

which after canceling out the exponential factors from both sides has the simple solution

$$C(t) = e^{-\frac{i}{\epsilon}z^2 t\sigma_3}. \tag{2.17}$$

An identical argument applied to $\widehat{\Phi}(x,t;z) = \Phi(x,t;z)K(t)$, shows that

$$K(t) = e^{-\frac{i}{\epsilon}z^2 t\sigma_3}. \tag{2.18}$$

Since $\widehat{\Psi}$ and $\widehat{\Phi}$ are also solutions of the ZS equation, they must be related by a scattering matrix

$$\widehat{\Phi}(x,t;z) = \widehat{\Psi}(x,t;z)\widehat{S}(z,t) \tag{2.19}$$

Applying the operator $i\epsilon\partial_t$ to both sides of (2.19) and using (1.5) we have

$$\begin{aligned} i\epsilon\widehat{\Phi}_t &= i\epsilon\widehat{\Psi}_t\widehat{S} + i\epsilon\widehat{\Psi}\widehat{S}_t, \\ \mathcal{B}\widehat{\Phi} &= \mathcal{B}\widehat{\Psi}\widehat{S} + i\epsilon\widehat{\Psi}\widehat{S}_t, \\ \mathcal{B}(\widehat{\Phi} - \widehat{\Psi}\widehat{S}) &= i\epsilon\widehat{\Psi}\widehat{S}_t, \\ 0 &= i\epsilon\widehat{\Psi}\widehat{S}_t. \end{aligned}$$

Since $\Psi(x,t;z)$ is nondegenerate, $\widehat{S}_t = 0$ so,

$$\widehat{S}(z,t) = \widehat{S}(z,0) = S(z,0). \tag{2.20}$$

Finally, inserting (2.14), (2.15), and (2.20) into (2.19) we get

$$\Phi(x,t;z)K(t) = \Psi(x,t;z)C(t)S(z,0).$$

Solving for $\Phi(x,t;z)$ we have

$$\Phi(x,t;z) = \Psi(x,t;z)\Big[C(t)\widehat{S}(z,0)K^{-1}(t)\Big],$$

which implies that

$$S(z,t) = e^{-\frac{i}{\epsilon}z^2 t\sigma_3}S(z,0)e^{\frac{i}{\epsilon}z^2 t\sigma_3}. \tag{2.21}$$

Using equation (2.21) we can recover the time evolution of the reflection coefficient $r(z,t) = \frac{s_{21}(z,t)}{s_{11}(z,t)}$:

$$\begin{aligned} s_{11}(z,t) &= s_{11}(z,0) = s_{11}(z) \\ s_{21}(z,t) &= s_{21}(z,0)e^{\frac{2i}{\epsilon}z^2 t} \\ \mathfrak{r}(z,t) &= \frac{s_{21}(z,0)}{s_{11}(z)}e^{\frac{2i}{\epsilon}z^2 t} \end{aligned} \tag{2.22}$$

2.6.2 Time evolution of the normalization constants

Recall that for any eigenvalue $z = z_k \in \mathbb{C}^+$ the two vector solutions Φ_1 and Ψ_2 are linearly related by a scalar normalizing constant $c_k(t)$:

$$\Phi_1(x,t;z_k) = c_k(t)\cdot\Psi(x,t;z_k). \tag{2.23}$$

As was shown during the discussion of the evolution of the reflection coefficient, if $\Psi(x,t;z)$ and $\Phi(x,t;z)$ solve the ZS problem for all positive times, then $\widehat{\Psi}(x,t;z) = \Psi(x,t;z)e^{-i\epsilon^{-1}z^2t\sigma_3}$ and $\widehat{\Phi}(x,t;z) = \Phi(x,t;z)e^{-i\epsilon^{-1}z^2t\sigma_3}$ solves the t-flow equation. In particular this implies that the column vectors solve the t-flow equation:

$$\begin{aligned} i\epsilon\partial_t(\Phi_1(x,t,z)e^{-\frac{i}{\epsilon}z^2t}) &= \mathcal{B}\Big(\Phi_1(x,t;z)e^{-\frac{i}{\epsilon}z^2t}\Big), \\ i\epsilon\partial_t(\Psi_2(x,t,z)e^{\frac{i}{\epsilon}z^2t}) &= \mathcal{B}\Big(\Psi_2(x,t;z)e^{\frac{i}{\epsilon}z^2t}\Big). \end{aligned} \tag{2.24}$$

Multiplying expression (2.23) by $e^{-i\epsilon^{-1}z_k^2t}$ and applying the operator $i\epsilon\partial_t$ we get

$$i\epsilon\partial_t\left(\Phi_1(x,t;z_k)e^{-\frac{i}{\epsilon}z_k^2t}\right) = i\epsilon\partial_t\left(c_k(t)e^{-\frac{2i}{\epsilon}z_k^2t}\cdot\Psi(x,t;z_k)e^{\frac{i}{\epsilon}z_k^2t}\right). \tag{2.25}$$

Using (2.24) and a little rearranging of terms we are led to

$$\begin{aligned} \mathcal{B}\Big[\Phi_1(x,t;z_k) - c_k(t)\cdot\Psi_2(x,t;z_k)\Big]e^{-\frac{i}{\epsilon}z_k^2t} &= i\epsilon\frac{\partial}{\partial t}\left[c_k(t)e^{-\frac{2i}{\epsilon}z_k^2t}\right] \\ 0 &= i\epsilon\frac{\partial}{\partial t}\left[c_k(t)e^{-\frac{2i}{\epsilon}z_k^2t}\right], \end{aligned} \tag{2.26}$$

where we used (2.23) to eliminate the left-hand side of the equation in the second step. From (2.26) the evolution of $c_k(t)$ is particularly simple; Given the initial normalizing constant $c_k(0)$,

$$c_k(t) = c_k(0)e^{\frac{2i}{\epsilon}z_k^2t}. \tag{2.27}$$

This completes our review of the generation and evolution of the scattering data. Our description is general, in practice what one needs are not implicit integral formulae for the scattering data but explicit closed formulae. In practice finding the Jost solutions for generic initial data $q(x)$ is problematic. In the special case we consider, the square barrier potential, all of these issues are resolved. As we will see in the next section, the scattering problem for the square barrier potential is solved in a few lines of computation.

2.7 Scattering of a Square Barrier

In this section we derive the scattering data for the particular initial data

$$q(x,t=0) = q(x) = \begin{cases} q & : \quad -L \leq x \leq L \\ 0 & : \quad |x| > L \end{cases} \tag{2.28}$$

for given constants $q,\ L > 0$. As we will see in this special case, the ZS-eigenvalue problem is exactly solvable so many of the difficulties that present themselves for general data are immediately resolved.

We begin by calculating the Jost solutions $\Psi(x;z)$ and $\Phi(x;z)$ to the ZS problem (2.1)

for the square-pulse potential. Recall that these solutions are normalized so that

$$\Psi(x;z)e^{\frac{i}{\epsilon}xz\sigma_3} \to I \qquad \text{for } x \to \infty, \tag{2.29}$$

$$\Phi(x;z)e^{\frac{i}{\epsilon}xz\sigma_3} \to I \qquad \text{for } x \to -\infty. \tag{2.30}$$

The problem naturally splits into three regions:

$$\Psi(x;z) = \begin{cases} \Psi_{III}(x;z) & : \quad x > L \\ \Psi_{II}(x;z) & : \quad -L \le x \le L \\ \Psi_{I}(x;z) & : \quad x < -L \end{cases}. \tag{2.31}$$

In regions I and III the potential matrix Q is identically zero and in region II is the constant $Q_0 = \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix}$, so the solutions of (2.1) in each region are

$$\begin{aligned} \Psi_I(x;z) &= e^{-\frac{i}{\epsilon}xz\sigma_3}C_I(z), \\ \Psi_{II}(x;z) &= e^{-\frac{1}{\epsilon}(iz\sigma_3 - Q)x}C_{II}(z), \\ \Psi_{III}(x;z) &= e^{-\frac{i}{\epsilon}xz\sigma_3}C_{III}(z). \end{aligned} \tag{2.32}$$

The Jost normalization (2.29) sets $C_{III} = I$. We determine the other constants by demanding that Ψ be continuous at each junction. The continuity conditions imply that

$$\begin{aligned} C_I(z) &= e^{-\frac{i}{\epsilon}Lz\sigma_3}e^{\frac{2L}{\epsilon}(iz\sigma_3 - Q_0)}e^{-\frac{i}{\epsilon}Lz\sigma_3}, \\ C_{II}(z) &= e^{\frac{L}{\epsilon}(iz\sigma_3 - Q_0)}e^{-\frac{i}{\epsilon}Lz\sigma_3}. \end{aligned}$$

If we break $\Phi(x;z)$ into regions as in (2.31) and use the Jost normalization (2.30) and continuity conditions we get

$$\Phi(x;z) = \begin{cases} e^{-\frac{i}{\epsilon}xz\sigma_3}D_{III}(z) & : \quad x > L \\ e^{-\frac{1}{\epsilon}(iz\sigma_3 - Q_0)x}D_{II}(z) & : \quad -L \le x \le L \\ e^{-\frac{i}{\epsilon}xz\sigma_3} & : \quad x < L \end{cases}, \tag{2.33}$$

where,

$$\begin{aligned} D_{II}(z) &= e^{-\frac{L}{\epsilon}(iz\sigma_3 - Q_0)}e^{\frac{i}{\epsilon}Lz\sigma_3}, \\ D_{III}(z) &= e^{\frac{i}{\epsilon}Lz\sigma_3}e^{-\frac{2L}{\epsilon}(iz\sigma_3 - Q_0)}e^{\frac{i}{\epsilon}Lz\sigma_3}. \end{aligned}$$

Using $\Psi(x;z)$ and $\Phi(x;z)$ we can now calculate the scattering matrix $S(z)$. Exploiting the unitary property of the Jost solutions for real z, we write:

$$S(z) = \Psi^*(x;z)\Phi(x;z). \tag{2.34}$$

Inserting our formulas for Φ and Ψ we have,

$$S(z) = e^{\frac{i}{\epsilon}Lz\sigma_3} e^{-\frac{2L}{\epsilon}(iz\sigma_3 - Q_0)} e^{\frac{i}{\epsilon}Lz\sigma_3}, \tag{2.35}$$

which becomes after expanding the exponentials,

$$S(z) = \begin{pmatrix} \frac{\nu(z)\cos\left(\frac{2L}{\epsilon}\nu(z)\right) - iz\sin\left(\frac{2L}{\epsilon}\nu(z)\right)}{\nu(z)} e^{\frac{2izL}{\epsilon}} & \frac{q_0\sin\left(\frac{2L}{\epsilon}\nu(z)\right)}{\nu(z)} \\ -\frac{q_0\sin\left(\frac{2L}{\epsilon}\nu(z)\right)}{\nu(z)} & \frac{\nu(z)\cos\left(\frac{2L}{\epsilon}\nu(z)\right) + iz\sin\left(\frac{2L}{\epsilon}\nu(z)\right)}{\nu(z)} e^{-\frac{2izL}{\epsilon}} \end{pmatrix}. \tag{2.36}$$

Here $\nu(z) = \sqrt{q^2 + z^2}$ is finitely branched and $\sim z$ for large z. We need not specify the particular branch cut of ν yet because we observe that $S(z)$ is an entire function; This is a consequence of Lemma 2 and the compact support of the square barrier; it may also be checked directly.

Now that we have the scattering matrix, we can quickly write down all the scattering data. From $S(z)$ and (3.18) the initial reflection coefficient is

$$\mathfrak{r}(z) = -\frac{q\sin\left(\frac{2L}{\epsilon}\nu(z)\right)}{\nu(z)\cos\left(\frac{2L}{\epsilon}\nu(z)\right) - iz\sin\left(\frac{2L}{\epsilon}\nu(z)\right)} e^{-\frac{2i}{\epsilon}Lz}. \tag{2.37}$$

Using the time evolution derived in (2.13) the reflection coefficient at any later time is given by

$$\mathfrak{r}(z,t) = -\frac{q\sin\left(\frac{2L}{\epsilon}\nu(z)\right)}{\nu(z)\cos\left(\frac{2L}{\epsilon}\nu(z)\right) - iz\sin\left(\nu(z)\right)} e^{\frac{i}{\epsilon}(2z^2t - 2zL)}. \tag{2.38}$$

The eigenvalue condition becomes $a(z) = s_{11}(z) = 0$ or

$$\frac{\nu(z)\cos\left(\frac{2L}{\epsilon}\nu(z)\right) - iz\sin\left(\frac{2L}{\epsilon}\nu(z)\right)}{\nu(z)} = 0. \tag{2.39}$$

Finally if we denote the eigenvalues by $z = z_k$, then because the initial data is compactly supported the normalizing constants are given by $c_k(t) = s_{21}(z_k, t)$. So using (2.13) and (2.36) the normalizing constants are

$$c_k(t) = -q\frac{\sin\left(\frac{2L}{\epsilon}\nu(z_k)\right)}{\nu(z_k)} e^{\frac{2i}{\epsilon}z_k^2 t} \tag{2.40}$$

2.7.1 Location of the eigenvalues in the complex plane

We have already characterized the poles as the set of zeros of the first entry in the scattering matrix. In this section we will describe in detail where these zeros lie in the complex plane and describe there distribution. Our work is greatly reduced by the following fact: for real positive potentials $q(x)$ with a single global maximum, of which the square barrier is a degenerate case, the eigenvalues of the Zakharov-Shabat operator are confined to the imaginary axis [KS02], [KS03].

We will show that in particular the eigenvalues lie in segment $i(-q,\ q)$ on the imaginary axis. Recall, that the eigenvalue condition is

$$\frac{\nu(z)\cos\left(\frac{2L}{\epsilon}\nu(z)\right) - iz\sin\left(\frac{2L}{\epsilon}\nu(z)\right)}{\nu(z)} = 0.$$

The function $\nu(z) = \sqrt{z^2+q^2}$ has a zero at $z = iq$ but the eigenvalue condition is continuous there; as $z \to iq_0$,

$$a(z) \to 1 + \frac{2Lq}{\epsilon} \neq 0.$$

Thus, $z = iq$ cannot be an eigenvalue.

Now suppose z is purely imaginary and $|z| > |q|$, write $z = iq\xi$ for $\xi > 1$, then the eigenvalue condition becomes

$$\sqrt{\xi^2-1}\cosh\left(\frac{2Lq}{\epsilon}\sqrt{\xi^2-1}\right) + \xi\sinh\left(\frac{2Lq}{\epsilon}\sqrt{\xi^2-1}\right) = 0,$$

where we have eliminated the nonzero denominator from the previous expression. The above expression clearly has no solutions for real ξ; this implies that any eigenvalue of the ZS problem for the square-pulse potential lies in the interval $i(-q,\ q)$. But are there any eigenvalues in this interval and how do they behave as we let $\epsilon \to 0^+$?

For $z \in i(-q,\ q)$ again write $z = iq\xi$ for $0 < \xi < 1$. The eigenvalue condition in our new variables is

$$\sqrt{1-\xi^2}\cos\left(\frac{2Lq}{\epsilon}\sqrt{1-\xi^2}\right) + \xi\sin\left(\frac{2Lq}{\epsilon}\sqrt{1-\xi^2}\right) = 0. \tag{2.41}$$

We now exclude a special case. For ϵ in the sequence

$$\epsilon_n = \frac{2}{\pi}\frac{2Lq}{2n-1}, \quad n \in \mathbb{N} \tag{2.42}$$

the eigenvalue condition has a zero at the origin. This amounts to an emerging eigenvalue of the spectral problem which will become a pair of eigenvalues in the upper and lower half-planes for any small decrease in the dispersion parameter. These emerging eigenvalues complicate the analysis without providing any significant contribution to the semiclassical limit. In everything that follows we will exclude the possibility of an emerging eigenvalue from our analysis, or put another way, we will consider ϵ going to zero but avoiding the special sequence (2.42).

The eigenvalue condition (2.41) can be rewritten in the following form more convenient for counting zeros:

$$n\pi - \arctan\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) = \frac{2Lq}{\epsilon}\sqrt{1-\xi^2}. \tag{2.43}$$

This leads naturally to the number of eigenvalues: there are n eigenvalues of the ZS problem

for the square pulse potential provided that

$$\frac{2n-1}{2} \;<\; \frac{2Lq}{\pi\epsilon} \;<\; \frac{2n+1}{2},$$

or equivalently,

$$\frac{2Lq}{\pi\epsilon} - \frac{1}{2} \;<\; n \;<\; \frac{2Lq}{\pi\epsilon} + \frac{1}{2}. \tag{2.44}$$

Note, in particular, that if $\frac{2Lq_0}{\epsilon} \;<\; \frac{\pi}{2}$ then there are no eigenvalues of this problem. This threshold for the creation of eigenvalues' is general and another result of Klaus and Shaw, see [KS03]. However, in the small dispersion limit this threshold is quickly overcome, if fact from (2.44) we see that the number of eigenvalues of the scattering problem, and thus poles of the matrix $M(z;x,t)$ build from the Jost functions, scales like $1/\epsilon$. Though we will make little use of this fact, it is interesting to write down the limiting distribution of eigenvalues, if we treat n as a continuous variable then we can define a limiting eigenvalue density ρ by the equation

$$\rho(\xi) = \lim_{\epsilon\to 0^+} -\epsilon\frac{dn}{d\xi} = \frac{2Lq}{\pi}\frac{\xi}{\sqrt{1-\xi^2}}. \tag{2.45}$$

Examining the limiting density function we see that the poles are accumulating at the endpoints $z = \pm iq$ and are sparse near the origin. This seems to validate our choice to exclude the special case when a new pole emerges from the origin.

We now have a complete description of the scattering data for the special case of the square barrier initial data. In what follows we will begin to consider the much more difficult inverse question. Given the initial scattering data and its time evolution, can we recover the solution of NLS at a later time $t > 0$? The very short answer is yes, though it will take us the bulk of the remainder of this work to arrive at a satisfactory answer to this question.

Chapter 3

Inverse Scattering: The Riemann Hilbert Problem for focusing NLS

In this chapter we describe the rudiments of the inverse scattering transform. The concept of a Riemann-Hilbert problem will be introduced, and we will speak briefly about their solvability, and more importantly, how the inverse scattering transform can be phrased in the language of Riemann-Hilbert problems (RHPs).

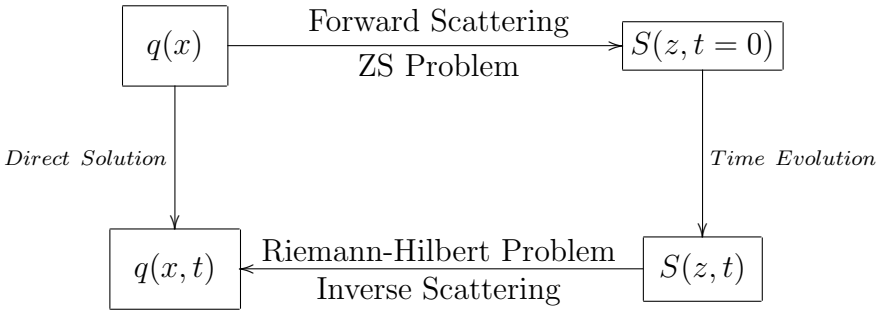

Figure 3.1. Schematic representation of the inverse scattering method for NLS

Recall the scattering/inverse scattering process, depicted in Figure 3.1, for solving our nonlinear Schrödinger equation. Thus far, we have described the forward scattering problem–the generation of scattering data from the initial data $q(x)$ of the NLS problem via the ZS eigenvalue problem–and the time evolution of the scattering data. In the inverse scattering step we want to recover the matrix $m(z)$, depending parametrically on (x, t), built from the time evolved Jost solutions of the Zakharov-Shabat eigenvalue problem.

In Section 2.6 we showed that if we can find the time evolving Jost solutions then building simultaneous solutions of the Lax-Pair, and thus recovering the solution $q(x, t)$ of the focusing NLS equation, is a triviality. Finding the Jost solutions for positive times directly is out of the question; the forward scattering step requires that we know the time evolved potential $q(x, t)$ to produce $S(z, t)$–we would already need to know the solution of NLS to re-derive it from the Jost functions. Instead, we can use the properties of the time evolving scattering data, which we described in the previous chapter, to characterize the analytical structure of the matrix $m(z; x, t)$ built from time evolved Jost functions. The characterization is exactly in the form of a meromorphic Riemann-Hilbert problem, which we will shortly define. By solving the RHP we find the Jost solutions which easily generate the solution $q(x, t)$ of NLS.

3.1 Matrix Riemann-Hilbert Problems

Though in principle their are many different Riemann-Hilbert problems one can consider, for our purposes we will mainly be concerned with the following type of (2×2) matrix RHP:

Definition: (**Holomorphic Matrix Riemann-Hilbert Problem**). Find a matrix valued function $M(z)$ of the complex variable z subject to the following constraints:

- **Analyticity:** $M(z)$ is analytic for $z \in \mathbb{C} \backslash \Sigma$, where Σ is a given oriented contour, or set of contours, in the complex plane.
- **Jump Matrix:** On the contour Σ, $M(z)$ has well defined boundary values, denoted $M_+(z)$ and $M_-(z)$, defined for $z \in \Sigma$ as the limit as z approaches the contour from the left or right respectively as defined by the contour's orientation. The boundary values are related by an equation of the form

$$M_+(z) = M_-(z)V(z), \tag{3.1}$$

  where $V(z)$ is a given 'jump matrix' defined along the contour.
- **Normalization:** As $|z| \to \infty$, the unknown matrix $M(z)$ has prescribed asymptotic behavior given by $M(z) = M_\infty + \mathcal{O}(1/z)$ for some given M_∞. Occasionally we will have a different decay rate than $\mathcal{O}(1/z)$.

The existence of solutions to problems of this type depends on the jump matrices and the collection of contours Σ. For the Riemann Hilbert problem for NLS, which we will define shortly, existence of solutions exists under weak assumptions on the scattering data, we refer the reader to [DZ02], [Zho98] for more details. Uniqueness, on the other hand, is typically easy to establish. Given two solutions M_1 and M_2, one usually has that $E = M_1 M_2^{-1}$ is an entire bounded function approaching identity at infinity. Uniqueness follows from Liouville's theorem.

In our study of the inverse scattering problem we will need to extend the class of holomorphic RHPs by allowing for the possibility of poles. A meromorphic RHP includes each of the conditions for the holomorphic type and one additional condition:

- **Poles** At each pole z_k of $M(z)$ in $\mathbb{C} \backslash \Sigma$ the largest term in the Laurent expansion of $M(z)$ must be specified. In particular for simple poles this implies that the problem statement includes a value for

$$\lim_{z \to z_k} (z - z_k) M(z). \tag{3.2}$$

3.2 Constructing a RHP from the scattering data

We will show now how the following meromorphic RHP arises naturally from the time evolved scattering data of the matrix $m(x, t; z)$ built from the Jost solution of the ZS eigenvalue problem. The problem we present assumes that all the eigenvalues of the ZS problem are simple; this is a generic assumptions which in particular holds for square

barrier initial data. In addition we will exclude the possibility that the ZS problem has real eigenvalues. For the square barrier this means that we will exclude the particular sequence of values ϵ_n defined by (2.42) for which new eigenvalues are born at the origin.

The NLS Riemann-Hilbert Problem.

For each value of (x,t) the matrix $m(z;x,t)$ built from the Jost solutions of the Zakharov-Shabat eigenvalue problem has the following analytic properties:

- $m(z;x,t)$ is meromorphic in $\mathbb{C}\backslash\mathbb{R}$.
- On $\mathbb{R}$, $m(z)$ takes continuous boundary values $m_+(z)$ and $m_-(z)$. The boundary values satisfy the jump relation $m_+(z) = m_-(z)V(z)$, where

$$V(z) = \begin{pmatrix} 1+|\mathfrak{r}(z)|^2 & \bar{\mathfrak{r}}(z)e^{-\frac{i}{\epsilon}\vartheta(x,t)} \\ \mathfrak{r}(z)e^{\frac{i}{\epsilon}\vartheta(x,t)} & 1 \end{pmatrix}, \tag{3.3}$$

  where $\mathfrak{r}(z) = \mathfrak{r}(z,t=0)$ is the initial reflection coefficient, and

$$\vartheta(x,t) = 2tz^2 + 2xz \tag{3.4}$$

  contains all the space-time dependence in the problem.

- As $z \to \infty$,

$$m(z;x,t) = I + O(z^{-1}). \tag{3.5}$$

- At each pole $z = z_k \in \mathbb{C}^+$ or $z = \bar{z}_k \in \mathbb{C}^-$ the matrix $m(z;x,t)$ has simple poles such that

$$\begin{aligned} \lim_{z\to z_k}(z-z_k)m(z;x,t) &= \lim_{z\to z_k} m(z;x,t)\begin{pmatrix} 0 & 0 \\ \operatorname*{Res}_{z=z_k}\mathfrak{r}(z)e^{\frac{i}{\epsilon}\vartheta(x,t)} & 1 \end{pmatrix}, \\ \lim_{z\to \bar{z}_k}(z-\bar{z}_k)m(z;x,t) &= \lim_{z\to \bar{z}_k} m(z;x,t)\begin{pmatrix} 1 & -\operatorname*{Res}_{z=\bar{z}_k}\bar{\mathfrak{r}}(z)e^{-\frac{i}{\epsilon}\vartheta(x,t)} \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{3.6}$$

3.2.1 The contour Σ and the jump matrix

In Chapter 2 we showed that the matrix,

$$M^{(up)}(x,t=0;z) = \left(\frac{O_1(x;z)}{a(z)} \mid N_2(x;z)\right),$$

was a meromorphic solution of the associated ZS-problem (2.3) at $t=0$ for $z \in \mathbb{C}^+$ with poles $z = z_k$. We then used the ZS problem's reflection symmetry to show that $M^{(down)}(x,t;z) = \sigma_2\overline{M^{(up)}}(x,t;\bar{z})\sigma_2$ was a meromorphic solution in $\mathbb{C}^-$ with poles at each $\bar{z}_k$. If we now define the matrix

$$m(z;x,t=0) = \begin{cases} M^{(up)}(x,t=0;z) & : \quad \mathrm{Im}(z) > 0, \quad z \neq z_k \\ M^{(down)}(x,t=0;z) & : \quad \mathrm{Im}(z) < 0, \quad z \neq \bar{z}_k \end{cases} \tag{3.7}$$

then the new matrix $m(z, x, t = 0)$ is meromorphic in $\mathbb{C}\backslash\mathbb{R}$. This is the origination of the contour Σ of our Riemann Hilbert problem. What is the jump matrix on $\mathbb{R}$?

During the discussion of the reflection coefficient in Section 2.4 it was shown that for $z \in \mathbb{R}$

$$\lim_{\delta\to 0^+} M^{(up)}(x; z + i\delta) = N(x; z) \begin{pmatrix} 1 & 0 \\ \mathfrak{r}(z)e^{2i\epsilon^{-1}xz} & 1 \end{pmatrix}, \tag{3.8}$$

$$\lim_{\delta\to 0^-} M^{(down)}(x; z + i\delta) = N(x; z) \begin{pmatrix} 1 & -\overline{\mathfrak{r}(z)}e^{-2i\epsilon^{-1}xz} \\ 0 & 1 \end{pmatrix}. \tag{3.9}$$

From these formula we can immediately calculate the jump matrix V. For $z \in \mathbb{R}$,

$$\begin{aligned} V(z; x, t) &= \begin{pmatrix} 1 & \overline{\mathfrak{r}}(z)e^{-\frac{i}{\epsilon}\vartheta(x,t)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mathfrak{r}(z)e^{\frac{i}{\epsilon}\vartheta(x,t)} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 + |\mathfrak{r}(z)|^2 & \overline{\mathfrak{r}}(z)e^{-\frac{i}{\epsilon}\vartheta(x,t)} \\ \mathfrak{r}(z)e^{\frac{i}{\epsilon}\vartheta(x,t)} & 1 \end{pmatrix}. \end{aligned} \tag{3.10}$$

Note the factorization in (3.10); the right matrix extends analytically into $\mathbb{C}^+$ while the matrix on the left extends analytically into $\mathbb{C}^-$. This fact will be useful later when we want to remove the poles from the RHP.

3.2.2 The residues

At each eigenvalue $z = z_k$ the function $a(z) = \det((O_1(x; z), N_2(x; z))$ vanishes; recall that the eigenvalues are constants of the evolution. The time evolved matrix $m(z; x, t) = \left(a^{-1}(z)O_1(x, t; z) \mid N_2(x, t; z)\right)$ develops poles in its first column at each eigenvalue. In principle the order of these poles is arbitrary, however, generically, and for the square barrier, each pole is simple. To specify a RHP we must characterize the residue at each (simple) pole.

At each eigenvalue the column vectors O_1 and N_2 are linearly dependent. In the forward scattering step we found constants $c_k(t)$, such that

$$O_1(x, t; z_k) = c_k(t)N_2(x, t; z_k)e^{2i\epsilon^{-1}xz}.$$

If we then write the first column of $m(z; x, t)$ as,

$$= \frac{c_k(t)}{a(z)}N_2(x, t; z_k)e^{2i\epsilon^{-1}xz} + \frac{O_1(x, t; z) - c_k(t)N_2(x, t; z_k)e^{2i\epsilon^{-1}xz}}{a(z)},$$

the first term has a pole while the second term's singularity is removable under our assumption that the zeros of $a(z)$ are all simple. From this representation the residue at each

$z_k \in \mathbb{C}^+$ can be represented as follows:

$$\begin{aligned}\lim_{z\to z_k}\ (z-z_k)m(z;x,t) &= \left(\frac{c_k(t)}{a'(z_k)}N_2(x,t;z_k)e^{2i\epsilon^{-1}xz_k}\,\middle|\, 0\right)\\ &= \lim_{z\to z_k}\ m(z;x,t)\begin{pmatrix}0 & 0\\ \frac{c_k(t)}{a'(z_k)}e^{2i\epsilon^{-1}xz_k} & 0\end{pmatrix}.\end{aligned}$$

A similar calculation at each pole $z=\bar{z}_k \in \mathbb{C}^-$ gives the residues at $z=\overline{z_k}$ as

$$\lim_{z\to \bar{z}_k}\ (z-\bar{z}_k)m(z;x,t) = \lim_{z\to \bar{z}_k}\ m(z;x,t)\begin{pmatrix}0 & -\frac{\bar{c}_k(t)}{\bar{a}'(z_k)}e^{-2i\epsilon^{-1}x\bar{z}_k}\\ 0 & 0\end{pmatrix}.$$

While these expression are true for simple poles in general, they simplify nicely for initial data with compact support. In Section 2.5.1 we showed that for compact initial data $c_k(0) = s_{21}(z, t=0)$. Combining this identification with the formula for the time evolving reflection coeffient (2.22) the residue condition becomes

$$\lim_{z\to z_k}\ (z-z_k)m(z,x,t) = \lim_{z\to z_k}\ m(z,x,t)\begin{pmatrix}0 & 0\\ \operatorname*{Res}_{z=z_k}\mathfrak{r}(z)e^{\frac{i}{\epsilon}\vartheta(x,t)} & 0\end{pmatrix}. \tag{3.11}$$

$$\lim_{z\to \bar{z}_k}\ (z-\bar{z}_k)m(z;x,t) = \lim_{z\to \bar{z}_k}\ m(z;x,t)\begin{pmatrix}0 & -\operatorname*{Res}_{z=\bar{z}_k}\bar{\mathfrak{r}}(z)e^{-\frac{i}{\epsilon}\vartheta(x,t)}\\ 0 & 0\end{pmatrix}. \tag{3.12}$$

3.2.3 The normalization at infinity

The columns of the matrix $m(z;x,t)$ are built from $N_2(x,t;z)$ and $O_1(x,t;z)$, the columns of the renormalized Jost functions. These column vectors have representations as the solutions of the following integral equations:

$$\begin{aligned}N_2(x,t;z) &= \begin{pmatrix}0\\1\end{pmatrix} + \int_\infty^x \begin{pmatrix}0 & q(s,t)e^{2i\epsilon^{-1}z(s-x)}\\ -\overline{q(s,t)} & 0\end{pmatrix} N_2(s,t;z)\ ds,\\ O_1(x,t;z) &= \begin{pmatrix}1\\0\end{pmatrix} + \int_{-\infty}^x \begin{pmatrix}0 & q(s,t)\\ -\overline{q(s,t)}e^{-2i\epsilon^{-1}z(s-x)} & 0\end{pmatrix} O_1(s,t;z)\ ds.\end{aligned}$$

In Appendix A we show that both column vectors $O_1(x,t;z)$ and $N_2(x,t;z)$ are bounded functions of x for each $z \in \mathbb{C}^+$. Knowing this, a simple dominated convergence argument shows that each of the integrals in the above expressions vanish as $z \to \infty$ away from the

real axis in $\mathbb{C}^+$. Since both O_1 and N_2 are analytic in $\mathbb{C}^+$ this leads to the conclusion that

$$\begin{aligned} O_1(x,t=0;z) &= \begin{pmatrix}1\\0\end{pmatrix} + O(z^{-1}) \qquad \text{as} \quad z\to\infty, \\ N_2(x,t=0;z) &= \begin{pmatrix}0\\1\end{pmatrix} + O(z^{-1}) \qquad \text{as} \quad z\to\infty, \\ a(z) = &= 1 + O(z^{-1}) \qquad \text{as} \quad z\to\infty. \end{aligned}$$

The last expression follows from the formula $a(z) = \det(O_1|N_2)$. Putting these formulae together tell us that the matrix $m(z;x,t)$ approaches identity for large z. Specifically we have the normalization condition,

$$m(z;x,t) \sim I + O(z^{-1}) \qquad \text{as} \quad z\to\infty \qquad \text{for any pair (x,t).} \tag{3.13}$$

3.3 Recovering the solution of NLS from the solution of the Riemann Hilbert problem

Suppose that we can find the solution $m(z;x,t)$ of the RHP at some later time. Then it must also solve the associated ZS problem at that time. Using equations (2.14) and (2.17) the matrix

$$m(z;x,t)e^{-\frac{i}{\epsilon}(z^2t+xz)\sigma_3} \tag{3.14}$$

is a simultaneous solution of the Lax-pair. The compatibility condition (1.8) then guarantees that the function $q(x,t)$ in the lax operators $\mathcal{L}$ and $\mathcal{B}$ solves the focusing NLS equation for the given initial data. We can use (3.14) to find $q(x,t)$.

The solution of the Riemann-Hilbert problem $m(z;x,t)$ has an asymptotic expansion for large z,

$$m(z;x,t) = I + \frac{m_1(x,t)}{z} + O(z^{-2}). \tag{3.15}$$

If we insert this into our solution of the Lax-pair (3.14), the first lax equation $\epsilon\Psi_x = \mathcal{L}\Psi$ becomes, after a few simplifications,

$$\begin{pmatrix} 0 & q(x,t) \\ -\overline{q(x,t)} & 0 \end{pmatrix} - i[\sigma_3, m_1(x,t)] = O(z^{-1}). \tag{3.16}$$

from which it follows that

$$q(x,t) = 2i(m_1)_{12}(x,t). \tag{3.17}$$

This completes the inverse scattering diagram. If we can find the solution of the Riemann-Hilbert problem for NLS defined by (3.3), (3.5), and (3.6), then by calculating the first moment at infinity we recover the solution of NLS at any later time.

3.4 Notation and Preliminaries for RHP analysis

In the next chapter we will begin the analysis of NLS RHP for the spectral data associated with square barrier initial data. We take time now to observe several simple calculations which we will use often in our analysis. In addition, we define several notational conventions that will be used throughout the text, in particular we define a short hand for naming particular matrices which appears throughout. From time-to-time we will remind the reader of these definitions but the reader comfortable with this notational baggage will find the text much easier to parse.

3.4.1 The Reflection Coefficient

Recall that the reflection coefficient for square barrier initial data (which we first derived in (2.37) of Section 2.7), is given at $t = 0$ by the equation

$$\mathfrak{r}(z) = \frac{-q_0 \sin\left(\frac{2L}{\epsilon}\nu(z)\right)}{\nu(z)\cos\left(\frac{2L}{\epsilon}\nu(z)\right) - iz\sin\left(\frac{2L}{\epsilon}\nu(z)\right)} e^{-\frac{2i}{\epsilon}Lz}. \tag{3.18}$$

This is a meromorphic function of z with simple poles at each eigenvalue of the associated spectral problem. Everywhere else in the complex plane, $\mathfrak{r}(z)$ is analytic, a consequence of the initial data being compactly supported.

Let us momentarily ignore the final exponential factor in $\mathfrak{r}(z)$ and consider the function

$$r(z) = \frac{-q_0 \sin\left(\frac{2L}{\epsilon}\nu(z)\right)}{\nu(z)\cos\left(\frac{2L}{\epsilon}\nu(z)\right) - iz\sin\left(\frac{2L}{\epsilon}\nu(z)\right)}. \tag{3.19}$$

We will prefer to work with the function $r(z)$ instead of the full reflection coefficient $\mathfrak{r}(z)$. To compensate for this change, we must also make a change to the function $\vartheta(x,t)$ (defined by (3.4)), which contains the space-time dependence of the RHP and appears in both the jump relation (3.3) and the residue conditions (3.6) of the RHP. We define

$$\theta(x,t) = 2tz^2 + 2(x-L)z. \tag{3.20}$$

Usually we will repress the dependence x and t and write simply θ. In terms of the new functions $r(z)$ and θ the NLS RHP originally defined by (3.3), (3.5), and (3.6) is given as follows.

The Riemann-Hilbert problem for NLS

Find a 2×2 matrix-valued function $m(z; x, t)$ that satisfies the following collection of properties:

- $m(z; x, t)$ is meromorphic in $\mathbb{C}\backslash\mathbb{R}$.
- $m(z)$ takes continuous boundary values $m_+(z)$ and $m_-(z)$ for $z \in \mathbb{R}$. The boundary

values satisfy the jump relation $m_+(z) = m_-(z)V(z)$, where

$$V(z) = \begin{pmatrix} 1+|r(z)|^2 & \bar{r}(z)e^{-\frac{i}{\epsilon}\theta(x,t)} \\ r(z)e^{\frac{i}{\epsilon}\theta(x,t)} & 1 \end{pmatrix}, \tag{3.21}$$

- As $z \to \infty$,
$$m(z;x,t) = I + O(z^{-1}). \tag{3.22}$$

- At each pole $z = z_k \in \mathbb{C}^+$ or $z = \bar{z}_k \in \mathbb{C}^-$ the matrix $m(z;x,t)$ has simple poles such that

$$\begin{aligned} \lim_{z\to z_k}(z-z_k)m(z;x,t) &= \lim_{z\to z_k} m(z;x,t)\begin{pmatrix} 1 & 0 \\ \operatorname*{Res}_{z=z_k} r(z)e^{\frac{i}{\epsilon}\theta(x,t)} & 1 \end{pmatrix}, \\ \lim_{z\to \bar{z}_k}(z-\bar{z}_k)m(z;x,t) &= \lim_{z\to \bar{z}_k} m(z;x,t)\begin{pmatrix} 1 & -\operatorname*{Res}_{z=\bar{z}_k} \overline{r(\bar{z})}e^{-\frac{i}{\epsilon}\theta(x,t)} \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{3.23}$$

To study the semiclassical limit of this Riemann-Hilbert problem we need a detailed understanding of how the reflection coefficient $r(z)$ behaves in the complex plane. Unlike short or long-time asymptotic problems where the reflection coefficient plays little role in the analysis, in the semiclassical limit its role is essential. This is due to the fact that the reflection coefficient depends nonlinearly on the dispersion parameter ϵ, while the time evolution appears only in the phase $\theta(x,t)$.

Consider the function $r(z)$ defined above by (3.19). For any $z \in \mathbb{C}^+\backslash\{iy : 0 \le y \le q\}$, $\mathrm{Im}(\nu(z)) > 0$. Thus as $\epsilon \to 0^+$, $r(z)$ is exponentially near the limiting function

$$r_0(z) = \frac{-iq}{z+\nu(z)}, \tag{3.24}$$

while for $z \in \mathbb{R} \cup [-iq, iq]$, $r(z)$ is rapidly oscillatory. More can be said; for $z \in \mathbb{C}^+\backslash(0, iq)$, $r(z)$ can be expanded in a convergent series

$$r(z) = r_0(z) + \sum_{k=1}^{\infty} r_k(z)e^{\frac{4ikL\nu(z)}{\epsilon}}, \qquad r_k(z) = \frac{2i\nu(z)}{q}\left(\frac{-q}{\nu(z)+z}\right)^{2k}, \quad k \ge 1. \tag{3.25}$$

This series expansion captures one of the essential difficulties in studying the semiclassical problem. Unlike the long or short-time problems where there is exactly one phase, in the semiclassical problem their are infinitely many phases, from which the dominant behavior has to be extracted.

We would like to separate the components of the reflection coefficient which depend on ϵ from those which are independent. Using the expansion of the reflection coefficient above, let us introduce the family of phase functions

$$\theta_k = 2tz^2 + 2(x-L)z + 4kL\nu, \tag{3.26}$$

to reexpress the product $r(z)e^{i\theta/\epsilon}$ appearing in the jump matrix (3.21) and the residue condition (3.23) as

$$r(z)e^{\frac{i}{\epsilon}\theta} = \sum_{k=0}^{\infty} r_k(z)e^{\frac{i}{\epsilon}\theta_k} \tag{3.27}$$

wherever the expansion is valid. The phase functions, θ_k, are real-valued along the real axis and each has a unique critical point along the real axis. We denote these critical points by z_k; they are the unique real solutions of the equations

$$\left.\frac{d\theta_k}{dz}\right|_{z=z_k} = 4tz_k + 2(x-L) + \frac{4kLz_k}{\nu(z_k)} = 0. \tag{3.28}$$

One easily checks that these points are ordered along the real axis: $z_0 > z_1 > z_2 > \ldots$. Though in principle we might need to consider the entire infinite family of phase functions and their critical points, this is not the case. Throughout the text we will only need to use the first two phase functions and critical points, $\{\theta_0,\ z_0\}$ and $\{\theta_1,\ z_1\}$. Finally, comparing (3.26) and (3.20) we have the obvious equality $\theta_0 = \theta$, thus we will usually drop the subscript notation on θ_0 and simply write θ instead.

3.4.2 Some useful and much used matrix factorizations

The jump matrix V_m (defined by (3.21)) associated with the RHP for NLS has two typical factorizations: a two term factorization,

$$V_m = \begin{pmatrix} 1+|r|^2 & \bar{r}e^{-\frac{i}{\epsilon}\theta} \\ re^{\frac{i}{\epsilon}\theta} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \bar{r}e^{-\frac{i}{\epsilon}\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ re^{\frac{i}{\epsilon}\theta} & 1 \end{pmatrix} = R^*R, \tag{3.29}$$

and a three term factorization,

$$V_m = \begin{pmatrix} 1 & 0 \\ \frac{re^{\frac{i}{\epsilon}\theta}}{1+|r|^2} & 1 \end{pmatrix} \begin{pmatrix} 1+|r|^2 & 0 \\ 0 & \frac{1}{1+|r|^2} \end{pmatrix} \begin{pmatrix} 1 & \frac{\bar{r}e^{-\frac{i}{\epsilon}\theta}}{1+|r|^2} \\ 0 & 1 \end{pmatrix} = \widehat{R}D\widehat{R}^*. \tag{3.30}$$

These factorizations, which are used to open lenses in the RHP, are a mainstay of the Deift-Zhou method and we will make frequent use of them. To open lenses, we need to know that the functions $\frac{r(z)}{1+|r(z)|^2}$ and $\frac{\bar{r}(z)}{1+|r(z)|^2}$ which appear in the three term factorization for $z \in R$ can be analytically extended off the real line; the extension is given by replacing $\overline{r(z)}$ by $\bar{r}(z)$ and $|r(z)|^2$ by $r(z)\bar{r}(z)$.

We make the following notation observations. The matrix factors R and $\widehat{R}$ defined above in (3.29) and (3.30) will be used throughout the text to keep the formulas succinct. Also note that, the "$*$" operator acting on the matrices above, and matrices throughout the later text, is not quite the usual hermitian conjugate operator. Instead, given a matrix function $M(z)$, the function $M^*(z)$ is the hermitian conjugate evaluated at $\bar{z}$,

$$M^*(z) = \overline{M(\bar{z})}^T,$$

which becomes the standard hermitian conjugate for z real. By the reflection principle, if $M(z)$ in analytic in a set $\mathcal{U}$, $M^*(z)$ is analytic in the complex conjugate set $\mathcal{U}^*$.

The series expansion of the reflection coefficient leads immediately to the product expansion

$$R(z) = \begin{pmatrix} 1 & 0 \\ r(z)e^{\frac{i}{\epsilon}\theta} & 1 \end{pmatrix} = \prod_{k=0}^{\infty} \begin{pmatrix} 1 & 0 \\ r_k(z)e^{\frac{i}{\epsilon}\theta_k} & 1 \end{pmatrix} = \prod_{k=0}^{\infty} R_k(z), \tag{3.31}$$

which converges on the same domain as the series for $r(z)$. We will use the notation R_k and $\widehat{R}_k$, along with their starred versions to identify matrices identical to those in (3.29) and (3.30) with the reflection coefficient and phase replaced with the k^{th} term in the series expansion of $r(z)$ and the associated phase θ_k. For example we will write

$$R_1(z) = \begin{pmatrix} 1 & 0 \\ r_1(z)e^{\frac{i}{\epsilon}\theta_1} & 1 \end{pmatrix}, \qquad \widehat{R}_0^* = \begin{pmatrix} 1 & \frac{\bar{r}_0(z)e^{-\frac{i}{\epsilon}\theta_0}}{1+|r_0(z)|^2} \\ 0 & 1 \end{pmatrix}.$$

3.4.3 Removing the poles

To arrive at a small-norm RHP we must first remove the poles from the original RHP for $m(z)$. For each finite ϵ there are finitely many poles appearing in complex conjugate pairs, and for the square barrier initial data considered here, they all reside on the imaginary axis between $-iq$ and iq. For a special decreasing sequence of values, ϵ_n defined by (2.42), a new pole is born at the origin which then splits into a conjugate pairs as ϵ decreases. In our analysis we exclude these special values of ϵ and consider the generic case in which no pole lies at the origin. Recall that the poles of $m(z)$ are simple and in the upper half-plane satisfy the residue condition (cf. Section 2.7.1),

$$\begin{aligned} \lim_{z\to z_k}(z-z_k)m(z;x,t) &= \lim_{z\to z_k} m(z;x,t) \begin{pmatrix} 0 & 0 \\ \frac{s_{21}(z,t=0)}{s_{11}'(z)}e^{\frac{i}{\epsilon}(2tz^2+xz)} & 0 \end{pmatrix} \\ &= \lim_{z\to z_k} m(z,x,t) \begin{pmatrix} 0 & 0 \\ \operatorname*{Res}_{z_k}\left\{r(z)e^{i\epsilon^{-1}\theta(x,t)}\right\} & 0 \end{pmatrix}. \end{aligned} \tag{3.32}$$

The final equality is a special property of the scattering problem for compactly supported initial data, for general data $r(z)$ need not be analytic, and this equality need not hold. In any case, for the square barrier, the relation between the poles of the matrix $m(z)$ and the reflection coefficient $r(z)$ yield an effective method for removing the poles from the Riemann Hilbert problem.

Let Ω be an open subset of the upper half-plane that contains the poles and $\Gamma = \partial\Omega\backslash\mathbb{R}$ be the boundary of Ω away from the real axis. Define a new unknown

$$m(z) = \begin{cases} M(z)R(z) & : \quad z \in \Omega \\ M(z)R^{-*}(z) & : \quad z \in \Omega^* \\ M(z) & : \quad \text{elsewhere} \end{cases} \tag{3.33}$$

It is easy to check that the new unknown $M(z)$ has no poles. The factorization is not

without cost, as the new unknown $M(z)$ incurs a new jump $V_M(z) = R(z)$, for $z \in \Gamma$. For this factorization to be useful it should move us closer to, or at least no further from, the "small-norm" class of Riemann Hilbert problems. That is, we need $R(z)$ ($R^*(z)$) to approach identity in Ω (Ω^*) as $z \to \infty$ and ideally we would like $R(z)$ to be near identity in some sense along Γ. The validity of this pole removing scheme depends on (x, t) through the matrix $R(z)$. In different regions of space-time different pole removal schemes will be necessary. As we will see, finding an appropriate pole removing factorization akin to (3.33) is the crucial step toward solving our Riemann Hilbert problem for $m(z; x, t)$.

Chapter 4

Riemann Hilbert Analysis Outside the Initial Support

4.1 Outside the Support: Zero Cut Solution

In this chapter we consider points (x,t) in space time that lay outside the support of the initial data, that is $x > L$ and $t \geq 0$. In this region the analysis of the RHP is considerably simpler than for x chosen inside the region of support. The simplifying feature of this region is the method by which the poles are removed. Here, unlike points inside the support, the critical points of each phase function lie to the left of the poles; this allows one to remove the poles without introducing a g-function to stabilize the RHP. The resulting asymptotic expression for the solution of NLS in this region is particularly simple.

Theorem 3. *For each value* $(x,t) \in \mathcal{S}_0 = \{(x,t) \ : \ |x| > L, \ \ t > 0\}$ *the solution* $q(x,t)$ *to the semiclassically focusing NLS equation*

$$i\epsilon q_t + \frac{\epsilon^2}{2} q_{xx} + |q|^2 q = 0, \qquad q(x,0) = q(x),$$

where $q(x)$ *is the square barrier initial data* (1.2)*, has the following asymptotically valid description as* $\epsilon \to 0^+$*:*

$$q(x,t) = \mathcal{O}(\epsilon^{1/2-\rho}). \tag{4.1}$$

for any number $\rho > 0$*. Additionally, the convergence is uniform on any compact subset of* $\mathcal{S}_0$*.*

Essentially this theorem indicates that the solution $q(x,t)$ does not spread out in time, the only contribution to the solution in this region is asymptotically small radiation emitted by the bulk. The proof of this theorem is contained in the following sections of this chapter.

4.1.1 The first transformation $\mathbf{m(z)} \mapsto M(z)$

For the purpose of asymptotic analysis, the original RHP for $m(z)$ is non-ideal for two reasons. The function $m(z)$ has poles and the jump matrix V_m is a rapidly oscillatory function along the real axis. Our first transformation seeks to accomplish two things: 1. to remove the poles from the RHP for $m(z)$, and 2. to use several factorization of the jump matrix V_m to open lenses that move the oscillatory terms in V_m into regions of exponential decay.

Such a factorization requires that we understand the behavior of the reflection coefficient $r(z)$ and its associated phase functions θ_k defined by (3.18) and (3.26) respectively. The phase factors $\theta_0 = 2tz^2 + 2(x-L)z$ and $\theta_1 = 2tz^2 + 2(x-L)z + 4L\nu$ and their critical points z_0 and z_1 defined by (3.28) divide the plane into six distinct regions in which each

exponential either grows or decays. In each region we want to open an appropriate lens to reduce the jump matrices to near-identity factors. The regions and the growth or decay of each exponential are shown in Figure 4.1.

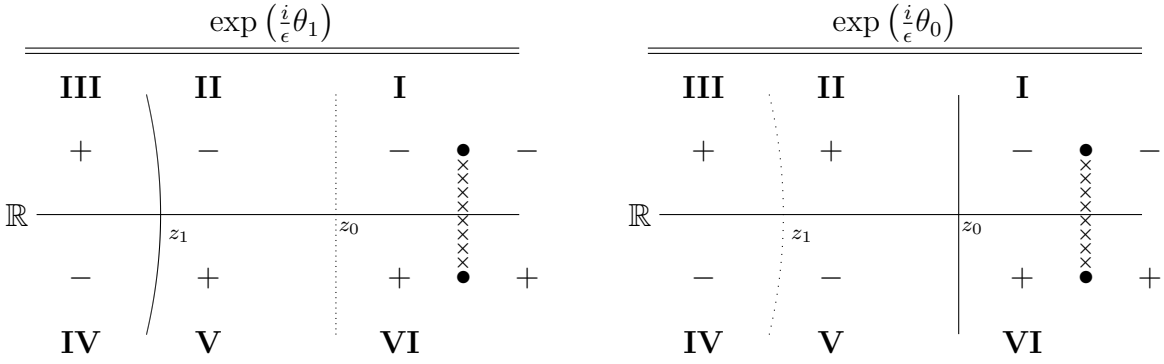

FIGURE 4.1. Regions of decay (-) and growth (+) of the exponential functions $e^{\frac{i}{\epsilon}\theta_0}$ and $e^{\frac{i}{\epsilon}\theta_1}$, valid for x and t in the parameter regime $\{(x,t) : x > L, t > 0\}$. In this regime, the poles of $m(z)$ (denoted by x's) lie to the right of the critical point z_0.

For $x > L$ the critical point z_0 lies to the left of the poles. As we will see, this allows us to remove the poles as was described in Section 3.4.3 without introducing any large jump matrices. We begin by opening lenses to simplify the jumps along the real axis and to remove the poles from the problem. Consider contours Γ_0, Γ_1, and Γ_2 as in Figure 4.2 such that Ω_0 contains the interval $(0, iq_0)$ and stay in regions of decay of the respective phase functions. The exact form of the contours is flexible, what is important is that each contour stay in a region where the jump matrices along them are exponentially close to the identity away from the real axis.

Define a new matrix $\widetilde{M}(z)$ via the following formulae

$$m(z) = \begin{cases} \widetilde{M}(z)R & : \quad z \in \Omega_0 \\ \widetilde{M}(z)RR_0^{-1} & : \quad z \in \Omega_1 \\ \widetilde{M}(z)\widehat{R}^* & : \quad z \in \Omega_2 \\ \widetilde{M}(z)R^{-*} & : \quad z \in \Omega_0^* \\ \widetilde{M}(z)R^{-*}R_0^* & : \quad z \in \Omega_1^* \\ \widetilde{M}(z)\widehat{R}^{-1} & : \quad z \in \Omega_2^* \\ M(z) & : \quad \text{elsewhere} \end{cases} \tag{4.2}$$

where the matrices R, $\widehat{R}$, and R_0 are those defined previously in Section 3.4.

Proposition 4. *Let* $\Sigma = (-\infty, z_0) \bigcup_{i=0}^{2} (\Gamma_i \cup \Gamma_i^*)$. *Then* $\widetilde{M}(z)$ *is analytic in* $\mathbb{C}\backslash\Sigma$ *with jump*

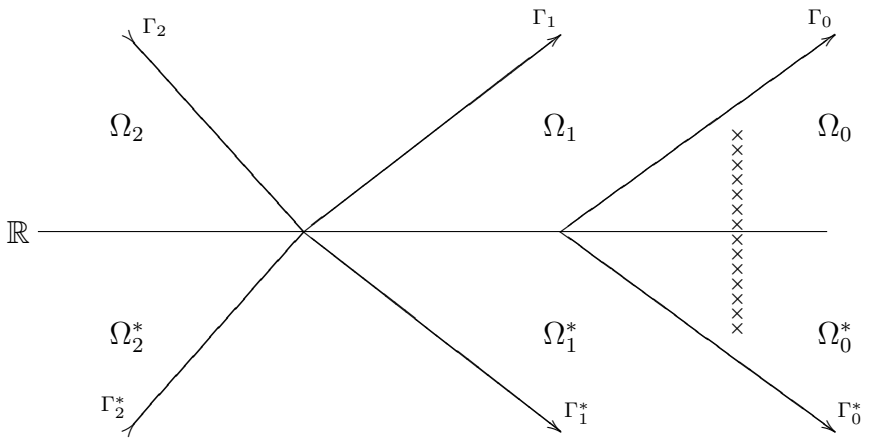

FIGURE 4.2. Contours and regions for opening lenses in the first transformation, outside the support

matrices given by

$$V_{\widetilde{M}} = \begin{cases} R_0 & : \quad z \in \Gamma_0 \\ RR_0^{-1} & : \quad z \in \Gamma_1 \\ \widehat{R}^* & : \quad z \in \Gamma_2 \\ \begin{pmatrix} 1+|r_0|^2 & \bar{r}_0 e^{-\frac{i}{\epsilon}\theta_0} \\ re^{\frac{i}{\epsilon}\theta_0} & 1 \end{pmatrix}^{Tspace} & : \quad z \in (z_1,\ z_0) \\ (1+|r(z)|^2)^{\sigma_3} & : \quad z \in (-\infty, z_1) \\ R_0^* & : \quad z \in \Gamma_0^* \\ R^* R_0^{-*} & : \quad z \in \Gamma_1^* \\ \widehat{R} & : \quad z \in \Gamma_2^* \end{cases} . \tag{4.3}$$

Proof. The jump matrices follow from direct calculation. To check analyticity one observes that the away from the poles each of the functions in (4.3) is analytic and that from (4.3) and (3.32) that the residue of $\widetilde{M}$ at each pole vanishes. Since the poles were at most of first order, $\widetilde{M}$ is a holomorphic function in each sector. □

Stabilizing the jump along $(-\infty, z_0)$ The jumps of the matrix $\widetilde{M}$ away from the real axis are exponentially close to identity, but near and on the real axis the jump matrices are still rapidly oscillatory. To complete the transformation to $M(z)$ we need to further factor the jumps along the real axis so that what remains on the axis has a well defined semiclassical limit. The jump along the interval (z_1, z_0) is built out of the function $r_0(z)$ and the exponential $\exp\left(\frac{i}{\epsilon}\theta_0\right)$ which grows exponentially in region **II**, thus it is appropriate in this region to use the triple factorization of the jump matrix akin to (3.30) with $r(z)$ replaced

by $r_0(z)$:

$$\begin{aligned}\begin{pmatrix} 1+|r_0|^2 & \bar{r_0}e^{-\frac{i}{\epsilon}\theta_0} \\ r_0 e^{\frac{i}{\epsilon}\theta_0} & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ \frac{r_0 e^{\frac{i}{\epsilon}\theta_0}}{1+|r_0|^2} & 1 \end{pmatrix} (1+|r_0|^2)^{\sigma_3} \begin{pmatrix} 1 & \frac{\bar{r_0}e^{-\frac{i}{\epsilon}\theta_0}}{1+|r_0|^2} \\ 0 & 1 \end{pmatrix} \\ &= \widehat{R}_0 (1+|r_0|^2)^{\sigma_3} \widehat{R}_0^*. \end{aligned} \tag{4.4}$$

The final factorization we will need is of the jump left along $(-\infty, z_1)$. For fixed ϵ this jump is typically removed from the RHP by explicit solution of the diagonal problem. However, due to the rapid oscillations in $r(z)$ this would be a poor choice. Instead, we choose to factor the diagonal matrix $(1+|r|^2)^{\sigma_3}$ in order to push the oscillations into regions of the plane in which they will decay. To arrive at a good factorization we recall some of the algebraic properties of the reflection coefficient.

The reflection coefficient is built out of the scattering functions

$$a(z) = \frac{\nu(z)\cos\left(\frac{2L}{\epsilon}\nu(z)\right) - iz\sin\left(\nu(z)\right)}{\nu(z)}, \tag{4.5}$$

$$b(z) = \frac{-iq\sin\left(\frac{2L}{\epsilon}\nu(z)\right)}{\nu(z)}, \tag{4.6}$$

by taking the quotient, $r(z) = b(z)/a(z)$. One can also check that the scattering functions satisfy the equation $a(z)\overline{a}(z) + b(z)\overline{b}(z) = 1$ for all z. The quantities appearing in the triple factorization (3.30) can be reduced to simple expressions built out of the scattering functions:

$$\begin{aligned} 1+|r(z)|^2 &= 1 + \frac{b(z)\overline{b}(z)}{a(z)\overline{a}(z)} = \frac{a(z)\overline{a}(z) + b(z)\overline{b}(z)}{a(z)\overline{a(\bar{z})}} = \frac{1}{a(z)\overline{a}(z)}, \\ \frac{r(z)}{1+|r(z)|^2} &= \frac{b(z)}{a(z)} \cdot a(z)\overline{a}(z) = b(z)\overline{a}(z). \end{aligned}$$

In addition to these formulae we need to understand the asymptotics of the function $a(z)$. Expanding in exponentials and collecting terms, we can factor $a(z)$ as follows

$$a(z) = \frac{z+\nu(z)}{2\nu(z)} e^{-\frac{2Li\nu(z)}{\epsilon}} \left[1 - r_0^2(z) e^{\frac{4Li\nu(z)}{\epsilon}}\right], \tag{4.7}$$

where we identify the term

$$a_0(z) = \frac{z+\nu(z)}{2\nu(z)} e^{-\frac{2Li\nu(z)}{\epsilon}} = \frac{e^{-\frac{2Li\nu(z)}{\epsilon}}}{1+|r_0(z)|^2} \tag{4.8}$$

as the leading behavior of $a(z)$ in $\mathbb{C}^+$. The other factor, which we denote by

$$a_1(z) = 1 - r_0^2(z) e^{\frac{4Li\nu(z)}{\epsilon}}, \tag{4.9}$$

captures the oscillations and zeros of $a(z)$. Note that away from the set $\mathbb{R} \cup [-iq, iq]$ $a_1(z)$

is asymptotically near 1.

With the factorizations above in mind, we factor the diagonal jump on $(-\infty, z_1)$ as

$$\begin{aligned}(1+|r(z)|^2)^{\sigma_3} &= (a(z)\overline{a}(z))^{-\sigma_3} = \overline{a}_1(z)^{-\sigma_3}\left\{a_0(z)\overline{a}_0(z)\right\}^{-\sigma_3} a_1(z)^{-\sigma_3} \\ &= \overline{a}_1(z)^{-\sigma_3}(1+|r_0(z)|^2)^{2\sigma_3} a_1(z)^{-\sigma_3}\end{aligned} . \tag{4.10}$$

Definition of M(z)

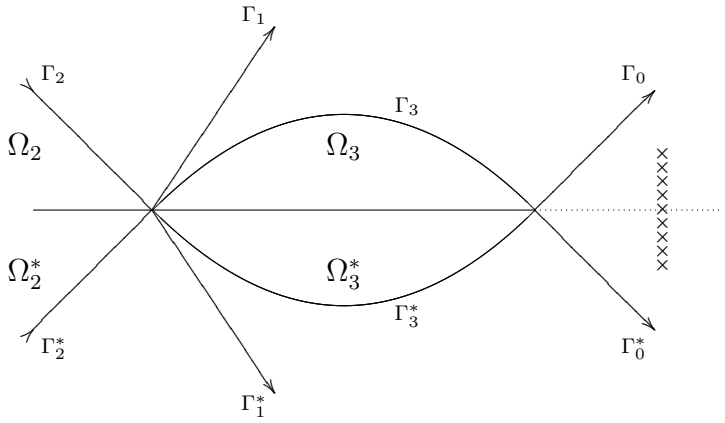

FIGURE 4.3. Diagram of the contours Γ_k and regions Ω_k used to define the transformation from $\widetilde{M}(z)$ to $M(z)$. The new contour Γ_3 is a smooth arc in $\mathbb{C}^+$ beginning at z_1 and terminating at z_0. The contour does not intersect Γ_0 or Γ_1 and approaches each endpoint from a nonzero acute angle non-tangential to the the other contours and the real axis. The contours and regions in $\mathbb{C}^-$ are obtained by symmetry.

Let Γ_3 be the simple arc in $\mathbb{C}^+$ connecting z_1 and z_0 which is depicted in Figure 4.3, (see the caption to the figure for a more a more precise definition of Γ_3), and let Ω_3 be the region enclosed by Γ_3 and the real axis. Using the factorizations in (4.10) and (4.4) together with the function $\widetilde{M}$ defined in (4.2) we arrive at our first transformation of the RHP. We define $M(z)$ by the formula

$$M(z) = \begin{cases} \widetilde{M}(z)a_1^{-\sigma_3}(z) & : \quad z \in \Omega_2 \\ \widetilde{M}(z)\widehat{R}_0^* & : \quad z \in \Omega_3 \\ \widetilde{M}(z)\bar{a}_1^{\sigma_3}(z) & : \quad z \in \Omega_2^* \\ \widetilde{M}(z)\widehat{R}_0^{-1} & : \quad z \in \Omega_3^* \\ \widetilde{M}(z) & : \quad \text{elsewhere} \end{cases} \tag{4.11}$$

The new unknown matrix $M(z)$ satisfies the following holomorphic RHP:

RHP for M: Seek a function $M(z)$ satisfying the following properties

- $M(z)$ is analytic in $\mathbb{C}\backslash(-\infty, z_0)\bigcup_{i=0}^{3}(\Gamma_i \cup \Gamma_i^*)$ on which it assumes continuous boundary values $M_+(z)$ and $M_-(z)$.

- $M(z) = I + O(z^{-1})$ as $|z| \to \infty$
- On each contours $M(z)$ satisfies the jump relationship $M_+(z) = M_-(z)V_M(z)$, where

$$V_M(z) = \begin{cases} R_0 & : \quad z \in \Gamma_0 \\ RR_0^{-1} & : \quad z \in \Gamma_1 \\ a_1^{-\sigma_3}\widehat{R}^* & : \quad z \in \Gamma_2 \\ \widehat{R}_0^* & : \quad z \in \Gamma_3 \\ (1+|r_0|^2)^{\sigma_3} & : \quad z \in (z_1, z_2) \\ (1+|r_0|^2)^{2\sigma_3} & : \quad z \in (-\infty, z_2) \\ R_0^* & : \quad z \in \Gamma_0^* \\ R^*R_0^{-*} & : \quad z \in \Gamma_1^* \\ \widehat{R}\bar{a}_1^{-\sigma_3} & : \quad z \in \Gamma_2^* \\ \widehat{R}_0^1 & : \quad z \in \Gamma_3^* \end{cases} \tag{4.12}$$

The new RHP for $M(z)$ is well suited to asymptotic analysis. In addition to removing the poles, the lenses we have opened have removed all of the oscillatory features of the jump matrices. Along the axes we have jumps independent of ϵ while on the contours Γ_i we have jumps that are asymptotically near identity for each fixed z as $\epsilon \downarrow 0$. However, the jumps are not uniformly near identity on Γ_i as z approached the two critical points z_0 and z_1. Near these points more delicate analysis is required.

4.2 The inner and outer model problems: a global approximate solution

Away from the real axis all of the jump matrices are exponentially close to the identity matrix. It seems intuitive that away from the critical points z_1 and z_0, where the contours Γ_i return to the real axis, that we should be able to ignore these small corrections. Near z_0 and z_1 the phases θ_0 and θ_1 vanish quadratically and so near these points we should expect that a different approximation is needed. Motivated by this idea we want to introduce a global approximation $A(z)$ of the form

$$A(z) = \begin{cases} A_0(z) & : \quad z \in \mathcal{U}_0 \\ A_1(z) & : \quad z \in \mathcal{U}_1 \\ O(z) & : \quad \text{elsewhere} \end{cases} \tag{4.13}$$

where $\mathcal{U}_0$ and $\mathcal{U}_1$ are small fixed size neighborhoods of the respective critical point and the functions A_0, A_1 and O solve RHPs that closely approximate the true solution uniformly in their region of validity in such a way that the error is uniformly small in ϵ everywhere in the complex plane.

Before we introduce these approximations though, it is advantageous to first explicitly remove the remaining jumps along the real axis. Consider the scaler RHP,

The δ-RHP,

Find a scalar function $\delta(z)$ satisfying the following properties.

- δ is analytic in $\mathbb{C}\backslash(-\infty, z_0)$.
- As $|z| \to \infty$,
$$\delta(z) = 1 + \mathcal{O}(1/z). \tag{4.14}$$
- For $z \in (-\infty, z_0)$, the boundary values of δ satisfy the relationship
$$\delta_+(z)/\delta_-(z) = \begin{cases} (1 + |r_0(z)|^2)^2 & : \quad z \in (-\infty, z_1) \\ (1 + |r_0(z)|^2) & : \quad z \in (z_1, z_0) \end{cases} \tag{4.15}$$

If we can find a such a function, then we define a new unknown $N(z)$ by the formula

$$M(z) = N(z)\delta^{\sigma_3}(z). \tag{4.16}$$

The effect of this change of variables is to remove the jumps from the real axis at the cost of modifying the jumps on the off axis contours. Introducing some new notation borrowed from Lie algebras, let

$$\lambda^{\mathrm{ad}\,\sigma_3} v = \lambda^{\sigma_3} v \lambda^{-\sigma_3}, \tag{4.17}$$

for any nonzero scalar λ and 2×2 matrix v. Using this notation, we can compactly write down the new RHP for the unknown $N(z)$:

RHP for $N(z)$:

Find a matrix valued function $N(z)$ such that

- $N(z)$ us analytic in $\mathbb{C}\backslash \bigcup_{i=0}^{3}(\Gamma_i \cup \Gamma_i^*)$ on which it assumes continuous boundary values $N_+(z)$ and $N_-(z)$.
- As $|z| \to \infty$,
$$N(z) = I + O(z^{-1}) \tag{4.18}$$
- For $z \in \bigcup_{i=0}^{3}(\Gamma_i \cup \Gamma_i^*)$, the boundary values $N_+(z)$ and $N_-(z)$ satisfy the jump relationship $N_+(z) = N_-(z)V_N(z)$, where
$$V_N(z) = \delta(z)^{\mathrm{ad}\,\sigma_3} V_M(z). \tag{4.19}$$

Looking at the new RHP, the unknown $N(z)$ has no jumps on the real axis. By explicit transformations we have removed all of the oscillatory jumps on the real axis, trading them for jumps on complex contours where the perturbations from identity are exponential small. If these perturbations were uniformly small we could appeal to the small-norm theory of RHPs to conclude that the solution itself was near identity. However, the exponential perturbation is not uniform, near the points z_0 and z_1 the contours come back to the real axis where the jump matrices are oscillatory and the phase functions are stationary. To arrive

at a problem whose jump matrices are uniformly near identity we need to introduce local approximations that address the local behavior of the jump matrices near the stationary phase points in such a way that the resulting difference possesses jumps uniformly near identity. This is the driving motivation for the model problems we will soon introduce locally near $z = z_0$ and $z = z_1$.

Strictly speaking, the above comments rely on the fact that by introducing $\delta(z)$ and the factorization (4.16) removing the jumps on the real axis we have not significantly altered the behavior of the jump matrices away from the real axis. To verify this fact we now provide an explicit formula for the solution $\delta(z)$ to the δ-RHP defined by (4.15) and (4.14) and explore its analytic properties.

The function $\delta(z)$, and its endpoint behavior. The solution of the δ-RHP is standard; one can easily check using the Plamelj formula that

$$\delta(z) = \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{z_1} \frac{\log(1+|r_0(s)|^2)}{s-z}\,\mathrm{d}s + \frac{1}{2\pi i}\int_{-\infty}^{z_0} \frac{\log(1+|r_0(s)|^2)}{s-z}\,\mathrm{d}s\right], \tag{4.20}$$

is analytic in $\mathbb{C}\backslash(-\infty, z_0]$ and satisfies the jump condition for $z \in (-\infty, z_0)$. Observing that $r_0(z) \in L^2(\mathbb{R})$ one may immediately conclude that $\delta(z)$ is bounded and satisfies the asymptotic condition (4.14).

Later when we build local approximation near the stationary phase points, we will need to understand the local behavior of $\delta(z)$ in neighborhoods of z_0 and z_1. From (4.20) we observe that these are endpoints of Cauchy-type integrals and so the local behavior is necessarily singular. We record here the local behavior to be referred to later as needed.

Let us introduce the following functions. Define

$$\chi(z,a) = \frac{1}{2\pi i}\int_{-\infty}^{a} \frac{\log(1+|r_0(s)|^2)}{s-z}\,\mathrm{d}s, \tag{4.21}$$

and

$$\kappa(z) = -\frac{1}{2\pi}\log(1+|r_0(z)|^2). \tag{4.22}$$

Using this notation we can write the function $\delta(z)$ as $\delta(z) = \exp\{\chi(z,z_0)+\chi(z,z_1)\}$. We now introduce the following proposition which describes the local behavior of the function $\chi(z,a)$ for z near the contour $(-\infty, a)$.

Proposition 5. *Consider $\chi(z,a)$ as defined by (6.97) and fix a values $a,\ b \in \mathbb{R}$, with $b < a$. Suppose that the points $z = b$ and $z = a$ are bounded away from the branch cut of $r_0(z)$ by some positive distance 2ρ. Then in suitably small neighborhoods of the points $z = a$ and $z = b$ we have the following local expansions of the function $\chi(z,a)$.*

1. *Near $z = a$, as z approaches a we have the uniform expansion:*

$$\chi(z,a) = i\kappa(a)\log(z-a) + \hat{\chi}(a,a) + \mathcal{O}\Big((z-a)\log(z-a)\Big), \tag{4.23}$$

where,

$$\hat{\chi}(z,a) = \chi(z,a) - \frac{1}{2\pi i}\log(1+|r_0(z)|^2)\log(z-a), \tag{4.24}$$

is a bounded analytic function for z near a.

2. *Near $z=b$, suppose that b is bounded away from a and the branch cut of the function $r_0(z)$. As z approaches b from above (+) or below (-) we have*

$$\chi(z,a) = \chi_{\pm}(b,a) + \mathcal{O}(|z-b|). \tag{4.25}$$

The numbers $\chi_{\pm}(b,a)$ are bounded constants satisfying the relation

$$\chi_+(b,a) - \chi_-(b,a) = \log(1+|r_0(b)|^2). \tag{4.26}$$

Proof. Our proof takes advantage of the fact that $r_0(z)$ is analytic in a neighborhood of every real z bounded away from the branch cut of $r_0(z)$ which connects the points $\pm iq$. For such z we extend $\log(1+|r_0(z)|^2)$ using the formula $\log(1+r_0(z)\bar{r}_0(z))$ for $z \notin \mathbb{R}$. For brevity, in the proof let us write $f(z)$ for $\log(1+|r_0|^2)$.

Consider first z near a. We have,

$$\chi(z,a) = \int_{a-\rho}^{a} \frac{f(z)}{s-z}\,\mathrm{d}s + \int_{a-\rho}^{a} \frac{f(s)-f(z)}{s-z}\,\mathrm{d}s + \int_{-\infty}^{a-\rho} \frac{f(s)}{s-z}\,\mathrm{d}s.$$

We now claim that only the first integral behaves singularly near $z=a$. The last integral is clearly analytic near $z=a$; to see that the same is true of the middle integral, we use the analyticity of $f(z)$ to rewrite the integral in the form,

$$\int_{a-\rho}^{a} \frac{f(s)-f(z)}{s-z}\,\mathrm{d}s = \int_{a-\rho}^{a}\int_{0}^{1} \frac{df(\lambda(s-z)+z)}{d\lambda}\,\mathrm{d}\lambda\mathrm{d}s,$$

which is clearly analytic for $|z-a|<\rho$. Finally, integrating the first term yields the only singular part of the expansion, $f(z)\log(z-a)$. The result follows from expanding $f(z)$ at $z=a$.

Now suppose b is bounded away from a and the branch cut of $r_0(z)$, define $\eta = \min\{\rho,(a-b)/2\}$. Let $C_b^{\pm}$ be the contour obtained by taking the contour $(-\infty,a)$ oriented left-to-right excising the interval $(b-\eta,b+\eta)$ and adding the circular arc $b+\eta e^{\pi\pm i\phi}$, $0<\phi<\pi$, see Figure 4.4. Then as z approaches b off the real axis from above (+) or below (-) we have by residue calculus,

$$\chi(z,b) = \pm\log(1+|r_0(z)|^2) + \int_{C_b^{\pm}} \frac{\log(1+|r_0(s)|^2)}{s-z}\,\mathrm{d}s. \tag{4.27}$$

Each term above is analytic inside the circle $|z-b|<\eta$ which establishes the estimate in (4.25). Taking the difference of the expressions for z above and z below the real axis gives relation (4.26).

□

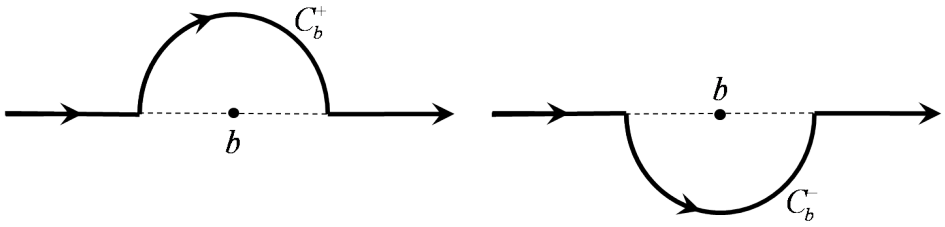

FIGURE 4.4. The contours $C_b^{\pm}$. As z approaches b from above (or below) by deforming the real contour of integration $(-\infty, z_0)$ in (6.97) to C_b^+ (or C_b^-) one picks up a residue term which accounts for the extra term in formula (4.27).

Using the results of Prop. 5 we can completely isolate the singular behavior of $\delta(z)$ near each endpoint of integration. Locally near the endpoints we have

$$\begin{aligned} \delta(z) &= (z-z_0)^{i\kappa(z_0)}\delta_0\left[1+\mathcal{O}\Big((z-z_0)\log(z-z_0)\Big)\right], \\ \delta_\pm(z) &= (z-z_1)^{i\kappa(z_1)}\delta_{1\pm}\left[1+\mathcal{O}\Big((z-z_1)\log(z-z_1)\Big)\right], \end{aligned} \tag{4.28}$$

where δ_0 and $\delta_{1\pm}$ are constants give by the formulae

$$\begin{aligned} \delta_0 &= \exp\left(\hat{\chi}(z_0,z_0)+\chi(z_0,z_1)\right), \\ \delta_{1\pm} &= \exp\left(\hat{\chi}(z_1,z_1)+\chi_\pm(z_1,z_0)\right). \end{aligned} \tag{4.29}$$

Understanding the local behavior of the function δ, we are now in a position to build local and outer model problems in such a way that we will be able to uniformly control the error everywhere in the complex plane.

4.2.1 The outer model problem

Away from the real axis the jump matrices for $N(z)$ are exponential perturbations of the identity. Thus, excising neighborhoods of the stationary phase points z_0 and z_1 the correction to approximating the jump matrix V_N by identity vanishes to all algebraic orders. Therefore, outside fixed neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ we have the outer approximation $O(z) = I$.

4.2.2 Local problem near $z = z_0$

Near the critical point $z = z_0$ the jumps of the function $N(z)$ are given in Figure 4.5. As the contours Γ_0 and Γ_3 approach the critical point $z = z_0$ on the real axis the outer estimates break down. The phase function $\theta_0 = 2t(z-z_0)^2 - 2tz_0^2$ grows quadratically as z moves away from the stationary phase point. Because of this behavior we expect that the algebraic terms in the jump matrix only meaningfully contribute inside a neighborhood of size $\epsilon^{1/2}$, outside this region the exponential decay should dominate the algebraic terms.

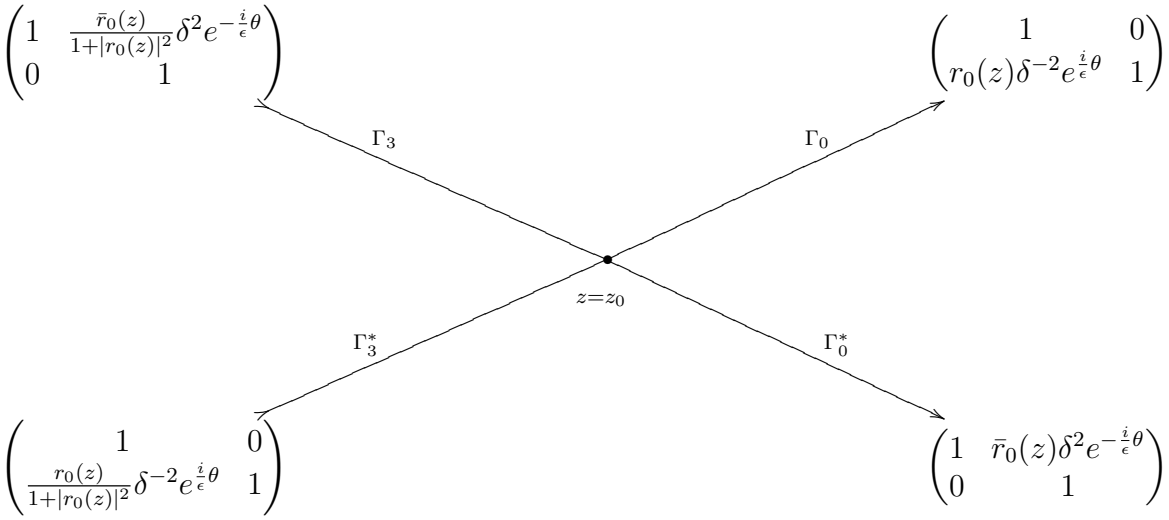

FIGURE 4.5. Local (exact) structure of the jump matrices of $N(z)$ near the stationary point $z = z_0$.

Our first step toward building such a function is to introduce a change of variables that rescales to local coordinates near $z = z_0$. Inside the local neighborhood $\mathcal{U}_0$ define a new variable ζ_0 by the formula

$$\frac{1}{2}\zeta_0^2 = \frac{1}{\epsilon}\left(\theta_0(z) - \theta_0(z_0)\right) = \frac{2t}{\epsilon}(z - z_0)^2. \tag{4.30}$$

This is an analytic and invertible change of variables mapping the interior of $\mathcal{U}_0$ to the disc $\zeta_0(\mathcal{U}_0)$ centered at the origin with a radius scaling like $\epsilon^{-1/2}$. We use the freedom in choosing the contours Γ_i so that the image contours $\zeta(\Gamma_i \cap \mathcal{U})$ lie on the straight line rays of steepest ascent of the function $\exp(i\zeta^2/2)$.

We build our local approximation near z_0 by replacing the reflection coefficient $r(z)$ by its evaluation at the stationary phase point z_0, $r_0(z) \mapsto r_0(z_0)$ and, using Prop. 5 and (4.30), we replace $\delta(z)$ by the local approximation,

$$\delta(z) \mapsto \left(\sqrt{\frac{\theta_0''(z_0)}{\epsilon}}\zeta_0(z)\right)^{i\kappa_0}\delta_0.$$

That is, we seek a function $A_0(z)$ which satisfies the jump relationship $A_{0+} = A_{0-}V_{z_0}$,

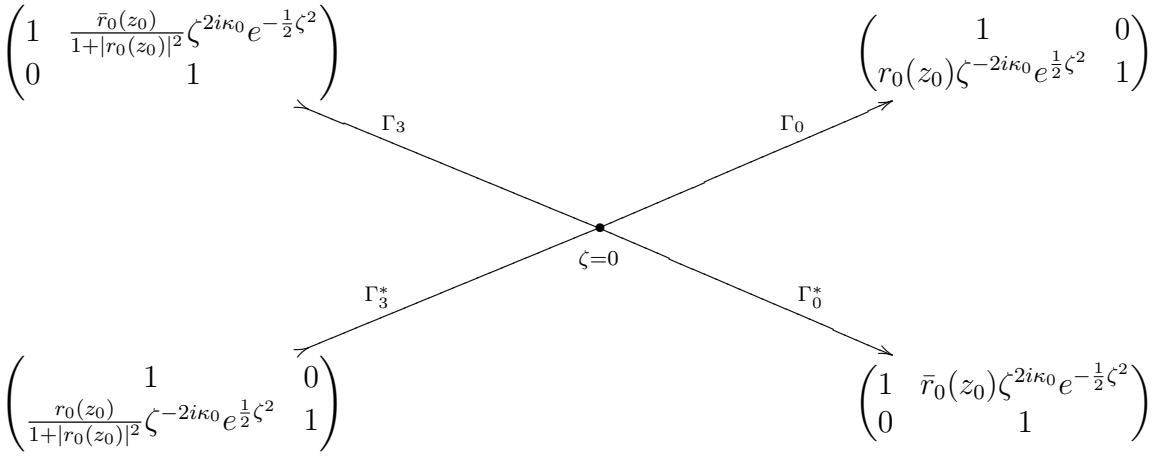

FIGURE 4.6. Local problem for z near z_0 after mapping to the ζ-plane. Observe, that the contours have been extended to ∞. The problem is approximate because we have replaced the reflection coefficient by its value at $z = z_0$ and the function $\delta(z)$ by its singular behavior at z_0, see Prop. 5.

where

$$V_{z_0} = \begin{cases} \begin{pmatrix} 1 & 0 \\ r_0(z_0)\left(\frac{\epsilon}{\theta_0''(z_0)}\right)^{-i\kappa_0}\zeta_0(z)^{-2i\kappa_0}\delta_0^{-2}e^{\frac{i}{\epsilon}\theta_0(z)} & 1 \end{pmatrix} & : \quad z\in\Gamma_0\cap\mathcal{U}_0 \\ \begin{pmatrix} 1 & \frac{\bar r_0(z_0)}{1+|r_0(z_0)|^2}\left(\frac{\epsilon}{\theta_0''(z_0)}\right)^{i\kappa_0}\zeta_0(z)^{2i\kappa_0}\delta_0^{2}e^{-\frac{i}{\epsilon}\theta_0(z)} \\ 0 & 1 \end{pmatrix} & : \quad z\in\Gamma_3\cap\mathcal{U}_0 \\ \begin{pmatrix} 1 & \bar r_0(z_0)\left(\frac{\epsilon}{\theta_0''(z_0)}\right)^{i\kappa_0}\zeta_0(z)^{2i\kappa_0}\delta_0^{2}e^{-\frac{i}{\epsilon}\theta_0(z)} \\ 0 & 1 \end{pmatrix} & : \quad z\in\Gamma_0^*\cap\mathcal{U}_0 \\ \begin{pmatrix} 1 & 0 \\ \frac{r_0(z_0)}{1+|r_0(z_0)|^2}\left(\frac{\epsilon}{\theta_0''(z_0)}\right)^{-i\kappa_0}\zeta_0(z)^{-2i\kappa_0}\delta_0^{-2}e^{\frac{i}{\epsilon}\theta_0(z)} & 1 \end{pmatrix} & : \quad z\in\Gamma_3^*\cap\mathcal{U}_0 \end{cases} . \tag{4.31}$$

To find a formula for $A_0(z)$ we introduce the following rescaling which removes the constant factors in (4.31). For $K_0(z)$ an analytic function to be chosen later, we write the approximation $A_0(z)$ in the form

$$A_0(z) = K_0(z)\Psi_0(\zeta_0(z))\left(\frac{\epsilon}{\theta_0''(z_0)}\right)^{-i\kappa_0\sigma_3/2}\delta_0^{-\sigma_3}e^{\frac{i}{2\epsilon}\theta_0(z_0)\sigma_3}, \tag{4.32}$$

where we recall that δ_0 is the scalar constant defined by (4.29). The new unknown Ψ_0, as function of ζ, satisfies the following well known RHP .

Local model problem $z = z_0$

Find a 2×2 matrix $\Psi_0(\zeta)$ that satisfies the following properties:

- Ψ_0 is an analytic function for $\zeta \in \mathbb{C}\backslash\{z : \arg(z) = \pm\pi/4, \pm 3\pi/4\}$ with continuous boundary values on each ray.
- As $|\zeta| \to \infty$,
$$\Psi_0 = \left[I + \frac{\psi_0^{(1)}}{\zeta} + O(1/\zeta^2) \right]. \tag{4.33}$$
- On the jump contours, the boundary values of Ψ_0 satisfy $\Psi_{0+} = \Psi_{0-} V_{\Psi_0}$ where
$$V_{\Psi_0} = \begin{cases} \begin{pmatrix} 1 & 0 \\ r_0(z_0)\zeta^{-2i\kappa_0} e^{i\zeta^2/2} & 1 \end{pmatrix} & : \quad \arg(\zeta) = \pi/4 \\ \begin{pmatrix} 1 & \bar{r}_0(z_0)\zeta^{2i\kappa_0} e^{-i\zeta^2/2} \\ 0 & 1 \end{pmatrix} & : \quad \arg(\zeta) = -\pi/4 \\ \begin{pmatrix} 1 & \frac{\bar{r}_0(z_0)}{1+|r_0(z_0)|^2}\zeta^{2i\kappa_0} e^{-i\zeta^2/2} \\ 0 & 1 \end{pmatrix} & : \quad \arg(\zeta) = 3\pi/4 \\ \begin{pmatrix} 1 & 0 \\ \frac{r_0(z_0)}{1+|r_0(z_0)|^2}\zeta^{-2i\kappa_0} e^{i\zeta^2/2} & 1 \end{pmatrix} & : \quad \arg(\zeta) = -3\pi/4 \end{cases} \tag{4.34}$$
and $\kappa_0 = -\frac{1}{2\pi}\log(1 + |r_0(z_0)|^2)$.

A diagram of the contours and jump functions is given for the model problem $\Psi(\zeta)$ is given in Figure 4.6.

The RHP defined by (4.33) and (4.34) is one of the standard parametrices that arise in the study of integrable systems and random matrix theory. Its solution, built out of parabolic cylinder functions, is given below.

Proposition 6. *The RHP problem for $\Psi_0(\zeta)$ has a unique solution given by the equation*

$\Psi_0(\zeta) = \Psi_{P.C.}(\zeta, r_0(z_0))$. *The function* $\Psi_{P.C.}(\zeta, a)$ *is given by the formula*

$$\Psi_{P.C.}(\zeta;a) = \begin{cases} P(\zeta;a)\begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} e^{\frac{i}{4}\zeta^2\sigma_3}\zeta^{-i\kappa\sigma_3} & : \quad 0 < \arg(\zeta) < \pi/4 \\ P(\zeta;a)\begin{pmatrix} 1 & \frac{-\bar{a}}{1+|a|^2} \\ 0 & 1 \end{pmatrix} e^{\frac{i}{4}\zeta^2\sigma_3}\zeta^{-i\kappa\sigma_3} & : \quad 3\pi/4 < \arg(\zeta) < \pi \\ P(\zeta;a)e^{\frac{i}{4}\zeta^2\sigma_3}\zeta^{-i\kappa\sigma_3} & : \quad \pi/4 < |\arg(\zeta)| < 3\pi/4 \\ P(\zeta;a)\begin{pmatrix} 1 & 0 \\ \frac{a}{1+|a|^2} & 1 \end{pmatrix} e^{\frac{i}{4}\zeta^2\sigma_3}\zeta^{-i\kappa\sigma_3} & : \quad -\pi < \arg(\zeta) < -3\pi/4 \\ P(\zeta;a)\begin{pmatrix} 1 & \bar{a} \\ 0 & 1 \end{pmatrix} e^{\frac{i}{4}\zeta^2\sigma_3}\zeta^{-i\kappa\sigma_3} & : \quad -\pi/4 < \arg(\zeta) < 0 \end{cases} \tag{4.35}$$

where the function $P(\zeta, a)$*, built out of the parabolic cylinder functions* $D_{\pm i\kappa}(\cdot)$*, is*

$$P(\zeta;a) = \begin{cases} \begin{pmatrix} e^{-\frac{3\pi\kappa}{4}} D_{i\kappa}\left(e^{\frac{-3i\pi}{4}}\zeta\right) & \frac{\partial_\zeta - \frac{i}{2}\zeta}{\beta_{21}} e^{\frac{\pi\kappa}{4}} D_{-i\kappa}\left(e^{-\frac{i\pi}{4}}\zeta\right) \\ \frac{\partial_\zeta + \frac{i}{2}\zeta}{\beta_{12}} e^{\frac{-3\pi\kappa}{4}} D_{i\kappa}\left(e^{\frac{-3i\pi}{4}}\zeta\right) & e^{\frac{\pi\kappa}{4}} D_{-i\kappa}\left(e^{\frac{-i\pi}{4}}\zeta\right) \end{pmatrix} & : \zeta \in \mathbb{C}^+ \\ \begin{pmatrix} e^{\frac{\pi\kappa}{4}} D_{i\kappa}\left(e^{\frac{i\pi}{4}}\zeta\right) & \frac{\partial_\zeta - \frac{i}{2}\zeta}{\beta_{21}} e^{\frac{-3\pi\kappa}{4}} D_{-i\kappa}\left(e^{\frac{3i\pi}{4}}\zeta\right) \\ \frac{\partial_\zeta + \frac{i}{2}\zeta}{\beta_{12}} e^{\frac{\pi\kappa}{4}} D_{i\kappa}\left(e^{\frac{i\pi}{4}}\zeta\right) & e^{\frac{-3\pi\kappa}{4}} D_{-i\kappa}\left(e^{\frac{3i\pi}{4}}\zeta\right) \end{pmatrix} & : \zeta \in \mathbb{C}^- \end{cases} \tag{4.36}$$

and the constants κ*,* β_{12}*, and* β_{21} *are given by the formulae:*

$$\begin{aligned} \kappa &= -\frac{1}{2\pi}\log(1+|a|^2), \\ \beta_{12} &= \frac{\sqrt{2\pi}e^{i\pi/4}}{\Gamma(-i\kappa)}\frac{1+|a|^2}{a}, \\ \beta_{21} &= \kappa\frac{\Gamma(-i\kappa)}{\sqrt{2\pi}e^{i\pi/4}}\frac{a}{1+|a|^2}. \end{aligned}$$

Using the classical properties of the parabolic cylinder functions, $D_{i\kappa}(\zeta)$, one can show by direct calculation that (4.35) solves the parabolic cylinder RHP. We avoid doing this here and instead refer the reader to [DZ94] for a complete, pedagogical, derivation of the solution.

The Matching condition on $\partial\mathcal{U}_0$*.*

To complete our local model near $z = z_0$ we need to turn to a consideration of the matching function $K_0(z)$. The criterion that selects $K_0(z)$ is a matching condition on $\partial\mathcal{U}_0$; in order to control the error where the inner and outer approximation meet we ask that,

$$O^{-1}(z)A_0(z) = I + \text{ vanishing corrections.} \tag{4.37}$$

In addition the function $K_0(z)$ should be analytic in $\mathcal{U}_0$. As was shown above, for x outside the support the outer approximation is trivial, $O(z) = I$. Using (4.32) and (4.33) we have for $z \in \partial\mathcal{U}_0$,

$$O^{-1}(z)A_0(z) = K_0(z)\left[I + \mathcal{O}(1/\zeta(z))\right]\left(\frac{\epsilon}{\theta_0''(z_0)}\right)^{-i\kappa_0\sigma_3/2}\delta_0^{-\sigma_3}e^{\frac{i}{2\epsilon}\theta_0(z_0)\sigma_3}.$$

Setting

$$K_0(z) = e^{-\frac{i}{2\epsilon}\theta_0(z_0)\sigma_3}\delta_0^{\sigma_3}\left(\frac{\epsilon}{\theta_0''(z_0)}\right)^{i\kappa_0\sigma_3/2}, \tag{4.38}$$

we have uniformly for $z \in \partial\mathcal{U}_0$,

$$O^{-1}(z)A_0(z) = I + \mathcal{O}(\epsilon^{1/2}). \tag{4.39}$$

Here we have used (4.30) to exchange the uniform correction in ζ for a uniform correction in the dispersion parameter ϵ. Observe that $K_0(z)$ is simply a bounded constant and thus trivially analytic in $\mathcal{U}_0$.

4.2.3 Local problem near $z = z_1$

Recall the collection of phase functions θ_k defined by (3.26) and their stationary phase points z_k given by (3.28). Near the stationary phase point $z = z_1$ the outer approximation loses uniformity as the jump contours come back to the real axis. As the point $z = z_1$ is a stationary point of the phase function θ_1, one might expect that the appropriate parametrix near z_1 would be a parabolic cylinder problem like we introduced in a neighborhood of z_0. However, unlike z_0, near z_1 there are two competing phase functions. Both θ_1, which is locally quadratic, and θ_0, which is locally linear, appear in the contours that emerge from z_1, see Figure 4.7. The competition between the linear and quadratic phases requires a more involved model problem near $z = z_1$.

The organizing principle that guides us to the correct model in this region is that there are disparate regions in which the linear and quadratic phases dominate the analysis. In each region a different model becomes appropriate. To capture the linearity of θ_0, we rewrite θ_0 using (3.26) by the equivalent quantity $\theta_1 - 4L\nu$. The following easily verified proposition describes the behavior of each of the exponential phases as one moves away from z_1 and delineates the regions in which each phase dominates the analysis.

Proposition 7. *Fix real numbers β_1, β_2, and ρ such that $0 < \beta_1 < 1/2 < \beta_2 < 1$ and $\rho > 0$. Also, suppose that $\exists\ \alpha \in (0, \pi/2)$ such that the the contours Γ_i, $i = 1, 2, 3$ lie entirely in the set $S_\alpha = \{z \in \mathbb{C}^+ :\ \alpha < |\arg(z - z_1)| < \pi/2 - \alpha\}$. Then the following estimates hold.*

- *For $|z - z_1| < \rho\epsilon^{\beta_2}$,*

$$\exp\left(\frac{i}{\epsilon}\theta_1(z)\right) = \exp\left(\frac{i}{\epsilon}\theta_1(z_1)\right)\left[1 + \mathcal{O}(\epsilon^{2\beta_2 - 1})\right].$$

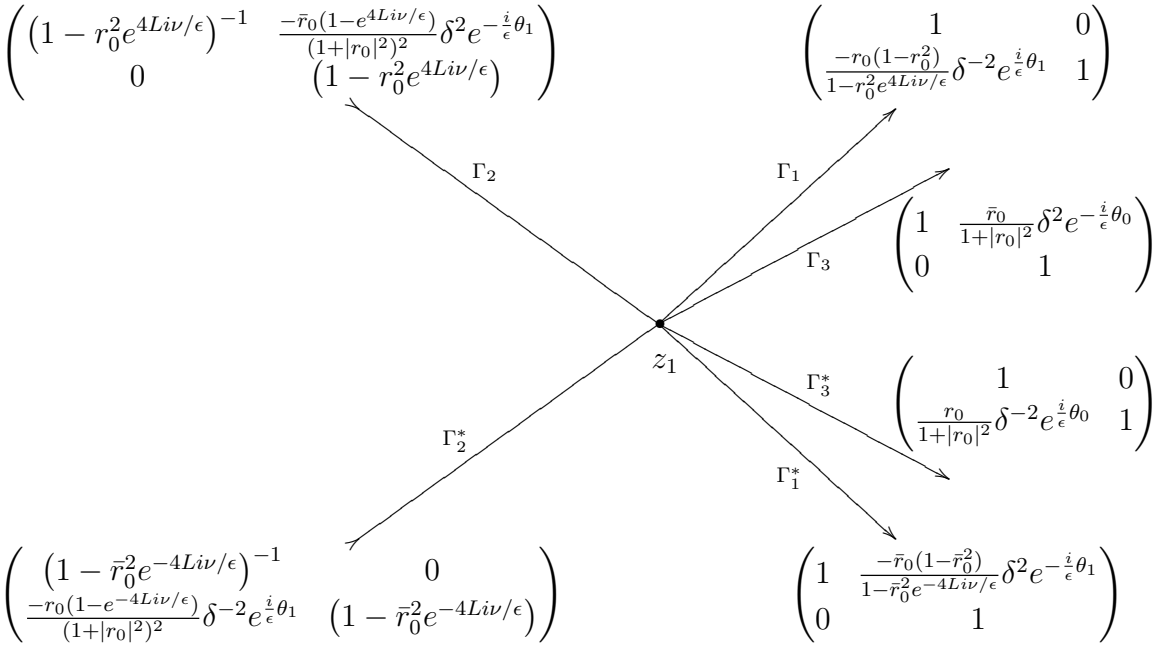

FIGURE 4.7. Local (exact) structure of the jump matrices of $N(z)$ near the stationary phase point $z = z_0$.

- *For $z \in \Gamma_i$ and $|z - z_1| > \rho\epsilon^{\beta_2}$,*

$$\exp\left(\frac{4Li\nu(z)}{\epsilon}\right) = \mathcal{O}\left\{\exp\left(-\frac{c\sin\alpha}{\epsilon^{1-\beta_2}}\right)\right\}.$$

- *For $z \in \Gamma_i$ and $|z - z_1| > \rho\epsilon^{\beta_1}$,*

$$\exp\left(\frac{i}{\epsilon}\theta_1(z)\right) = \mathcal{O}\left\{\exp\left(-\frac{c\sin 2\alpha}{\epsilon^{1-2\beta_1}}\right)\right\}.$$

Let us fix $\rho > 0$ suitably small and β, $1/2 < \beta < 1$. Define the nested neighborhoods $\mathcal{U}_1(z_1) = \{z : |z-z_1| < \rho\}$ and $\mathcal{D} = \{z : |z-z_1| < \rho\epsilon^{\beta}\}$. Let $\mathcal{A}$ be the annular region $\mathcal{U}_1 \backslash \mathcal{D}$, see Figure 4.8. Then essentially, the meaning of Proposition 7 is that outside $\mathcal{U}$ the outer approximation is uniformly valid, in $\mathcal{A}$ the linear exponentials with phase $4Li\nu(z)/\epsilon$ vanishes to all orders, while in the shrinking disk $\mathcal{D}$, the quadratic phase θ_1 is well approximated by a constant and the linear terms dominate. Thus we are led to build a two-level local parametrix $A_1(z)$ for $M(z)$ defined for $z \in \mathcal{U}_1$ by

$$A_1(z) = \begin{cases} B_{\mathcal{A}}(z) & : \quad z \in \mathcal{A} \\ B_{\mathcal{D}}(z) & : \quad z \in \mathcal{D} \end{cases} \tag{4.40}$$

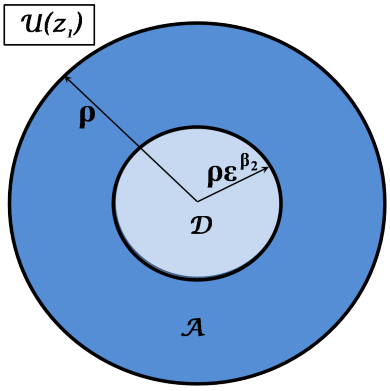

FIGURE 4.8. The set $\mathcal{U}_1$ is divided into the disk $D(z_1)$ (lightly shaded region) and the annulus $\mathcal{A}$ (darkly shaded region). In $D(z_1)$ the linear phase dominates while in $\mathcal{A}$ only the quadratic phase θ_1 meaningfully contributes.

Model in the annular region, $\mathcal{A}$

In the annular region $\mathcal{A}$ any terms that have a linear phase are exponentially small and can be eliminated from the jump matrices without incurring any significant errors. With these terms removed, the local model RHP is given by Figure 4.9. The only exponential phase that remains is θ_1 which is stationary at $z = z_1$. Comparing Figure 4.9 to Figure 4.5 one sees that the structure of the annular model problem is nearly identical to the model near $z = z_0$. It should come as no surprise that the solution in the annulus is again built from parabolic cylinder functions.

We arrive at the parabolic cylinder problem by introducing approximations analogous to those introduced for the model problem near $z = z_0$. We introduce in $\mathcal{U}_1$ the following analytic and invertible change of variables for $z \in \mathcal{U}_1$:

$$\frac{1}{2}\zeta_1^2(z) = \frac{1}{\epsilon}\left[\theta_1(z) - \theta_1(z_1)\right]. \tag{4.41}$$

Under $\zeta_1(z)$, $\mathcal{U}_1$ maps to an expanding neighborhood of $\zeta = 0$. We use the freedom we have in choosing the contours Γ_1, Γ_2, Γ_1^*, and Γ_2^* so that the image contours $\zeta_1(\Gamma_i \cap \mathcal{U}_1)$ and $\zeta_1(\Gamma_i^* \cap \mathcal{U}_1)$, $i = 1, 2$, lie on the rays $\arg(\zeta) = \pm\pi/4$, $\pm 3\pi/4$ respectively, see Figure 4.10. As before, we replace $r_0(z)$ with its value at the stationary phase point, $r_0(z) \mapsto r_0(z_1)$, and we use the local expansion formula (4.28) to approximate $\delta(z)$ near $z = z_1$. The annular model $B_{\mathcal{A}}(z)$ should then satisfy the jump relationship $B_{\mathcal{A}+}(z) = B_{\mathcal{A}-}(z)V_{z_1}$, where V_{z_1} is

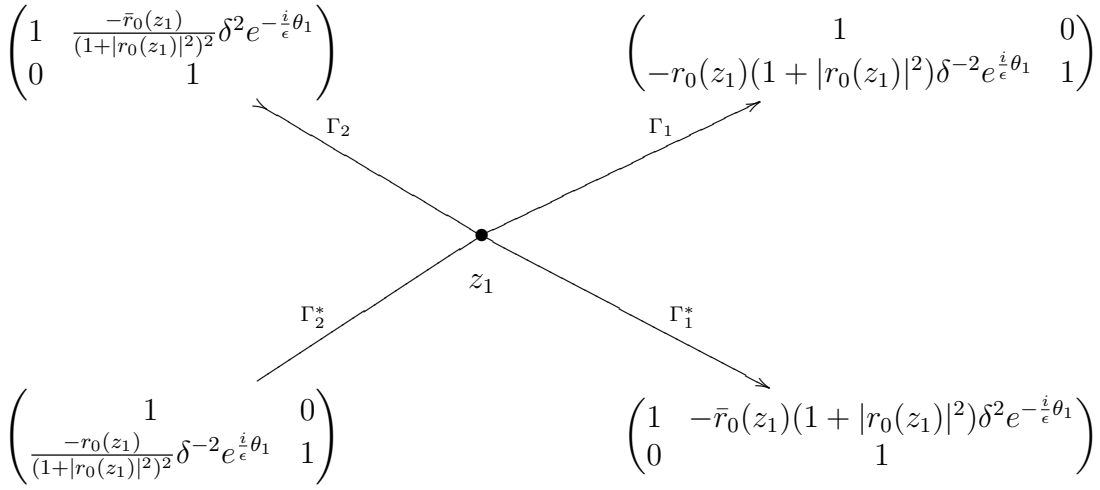

FIGURE 4.9. Annular approximation of contours and jump matrices near $z = z_1$

the function

$$V_{z_1} = \begin{cases} \begin{pmatrix} 1 & 0 \\ -r_0(z_1)(z-z_1)^{-2i\kappa_1}(\delta_{1+}\delta_{1-})^{-1}e^{\frac{i}{\epsilon}\theta_1(z)} & 1 \end{pmatrix} & : \quad z \in \Gamma_1 \cap \mathcal{U}_1 \\ \begin{pmatrix} 1 & \frac{-\bar{r}_0(z_1)}{1+|r_0(z_1)|^2}(z-z_1)^{2i\kappa_1}\delta_{1+}\delta_{1-}e^{-\frac{i}{\epsilon}\theta_1(z)} \\ 0 & 1 \end{pmatrix} & : \quad z \in \Gamma_2 \cap \mathcal{U}_1 \\ \begin{pmatrix} 1 & -\bar{r}_0(z_1)(z-z_1)^{2i\kappa_1}\delta_{1+}\delta_{1-}e^{-\frac{i}{\epsilon}\theta_1(z)} \\ 0 & 1 \end{pmatrix} & : \quad z \in \Gamma_1^* \cap \mathcal{U}_1 \\ \begin{pmatrix} 1 & 0 \\ \frac{-r_0(z_1)}{1+|r_0(z_1)|^2}(z-z_1)^{-2i\kappa_0}(\delta_{1+}\delta_{1-})^{-1}e^{\frac{i}{\epsilon}\theta_1(z)} & 1 \end{pmatrix} & : \quad z \in \Gamma_2^* \cap \mathcal{U}_1 \end{cases} . \tag{4.42}$$

We then write the inner model $B_{\mathcal{A}}(z)$ as

$$B_{\mathcal{A}}(z) = K_{\mathcal{A}}(z)\Psi_1(\zeta_1(z))c_\delta^{-\sigma_3}\left(\frac{\epsilon}{\theta_1''(z_1)}\right)^{-i\kappa_1\sigma_3/2} e^{\frac{i}{2\epsilon}\theta_1(z_1)\sigma_3}, \tag{4.43}$$

where $K_{\mathcal{A}}(z)$ is an analytic matrix that will be chosen later to match the outer parametrix $O(z)$ and c_δ is the constant

$$c_\delta = \delta_{1+}^{1/2}\delta_{1-}^{1/2}, \tag{4.44}$$

built from δ_{1+} and δ_{1-} which we recall are associated with the local behavior of $\delta(z)$ near $z = z_1$ (cf. (4.28)).

The new unknown Ψ_1 solves the following Riemann Hilbert problem,

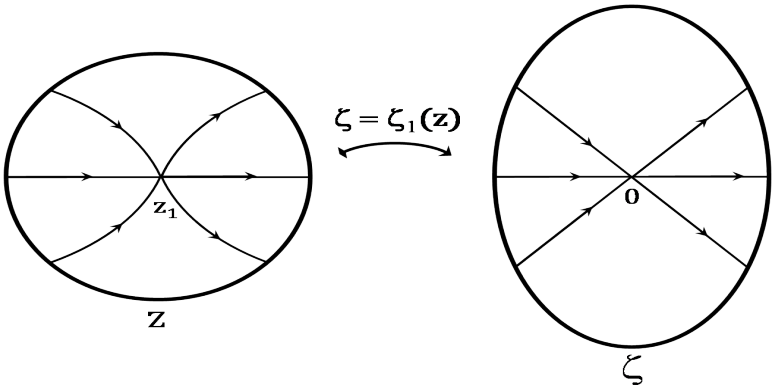

FIGURE 4.10. The mapping $\zeta = \zeta_1(z)$. The contours inside $\mathcal{U}_1$ are chosen so that in the image contours $\zeta(\Gamma_i)$ lie on the rays $\arg\zeta = \pm\pi/4 + k\pi/2,\ k = 0, \ldots, 3$.

The RHP for $\Psi_1(\zeta)$

Find a 2×2 matrix $\Psi(\zeta)$ satisfying the following properties:

- Ψ_1 is analytic for $\zeta \in \mathbb{C}\backslash\{z : \arg(z) = \pm\pi/4,\ \pm 3\pi/4\}$ with continuous boundary values as z approaches each contour.
- As $|\zeta| \to \infty$,
$$\Psi_1(\zeta) = [I + \mathcal{O}(1/\zeta)]\,\zeta^{i\kappa_1\sigma_3}, \tag{4.45}$$
where $\kappa_1 = -\frac{1}{2\pi}\log(1 + |r_0(z_1)|^2)$.
- On the jump contours, the boundary values of Ψ_1 satisfy $\Psi_{1+} = \Psi_{1-}V_{\Psi_1}$, where
$$V_{\Psi_1} = \begin{cases} \begin{pmatrix} 1 & 0 \\ -r_0(z_1)\zeta^{-2i\kappa_1}e^{i\zeta^2/2} & 1 \end{pmatrix} & : \quad \arg(\zeta) = \pi/4 \\ \begin{pmatrix} 1 & -\bar{r}_0(z_1)\zeta^{2i\kappa_1}e^{-i\zeta^2/2} \\ 0 & 1 \end{pmatrix} & : \quad \arg(\zeta) = -\pi/4 \\ \begin{pmatrix} 1 & \frac{-\bar{r}_0(z_1)}{(1+|r_0(z_1)|^2)}\zeta^{2i\kappa_1}e^{-i\zeta^2/2} \\ 0 & 1 \end{pmatrix} & : \quad \arg(\zeta) = 3\pi/4 \\ \begin{pmatrix} 1 & 0 \\ \frac{-r_0(z_1)}{(1+|r_0(z_1)|^2)}\zeta^{-2i\kappa_1}e^{i\zeta^2/2} & 1 \end{pmatrix} & : \quad \arg(\zeta) = -3\pi/4 \end{cases} \tag{4.46}$$

Comparing the asymptotic and jump conditions (4.45) and (4.46) for $\Psi_1(\zeta)$ to those of the model $\Psi_0(\zeta)$ for z near z_0 (cf. (4.33) and (4.34)) we see that the problems differ only by a single constant parameter; the constant $r_0(z_0)$ that appears in the problem for $\Psi_0(\zeta)$ is replaced in the problem for $\Psi_1(\zeta)$ by $-r_0(z_1)$. Thus, the solution $\Psi_1(\zeta)$ is also built out of

parabolic cylinder functions and the matching condition selecting $K_{\mathcal{A}}(z)$ is analogous to the matching condition selection $K_0(z)$. We state this explicitly in the following proposition:

Proposition 8. *The RHP problem for $\Psi_1(\zeta)$ has a unique solution given by the equation*

$$\Psi_1(\zeta) = \Psi_{P.C.}(\zeta, -r_0(z_1)) \tag{4.47}$$

where $\Psi_{P.C.}(\zeta, a)$ is the parabolic cylinder parametrix defined by (4.35)*. The analytic matching function $K_{\mathcal{A}}(z)$ is given for z outside the support by*

$$K_{\mathcal{A}}(z) = e^{-\frac{i}{2\epsilon}\theta_1(z_1)\sigma_3} c_\delta^{\sigma_3} \left(\frac{\epsilon}{\theta_1''(z_1)}\right)^{i\kappa_1\sigma_3/2} \tag{4.48}$$

and uniformly for $z \in \partial\mathcal{U}_1$,

$$O^{-1}(z)B_{\mathcal{A}}(z) = I + \mathcal{O}(\epsilon^{1/2}). \tag{4.49}$$

To verify the provided formula for the function $K_{\mathcal{A}}(z)$ and the resulting matching (4.49) for $z \in \partial\mathcal{U}_1$ one simply repeats the previous calculations producing $K_0(z)$ (cf. the discussion immediately following (4.37)), replacing $A_0(z)$ with $B_{\mathcal{A}}(z)$. The results are immediate.

The model problem on $\mathcal{D}$, the interior disk

Inside the disk $\mathcal{D}$, Proposition 7 implies that the quadratic phase θ_1 is well approximated by $\theta_1(z_1)$, its value at the stationary phase point. Thus, it should play no significant role in the behavior of the local model inside $\mathcal{D}$. Inside the shrinking disk $\mathcal{D}$ we expect that the phase factor, $e^{4Li\nu/\epsilon}$, and the local behavior of $\delta(z)$ are the major contributors to the local behavior of the solution. For these reasons we seek an inner approximation $B_{\mathcal{D}}$ of the form

$$B_{\mathcal{D}} = K_{\mathcal{D}}(z)L(z)e^{\frac{i}{2\epsilon}\theta_1(z)\sigma_3} \tag{4.50}$$

where $K_{\mathcal{D}}(z)$ is an analytic matrix-valued function to be chosen later.

Let us now consider the Riemann-Hilbert problem for $L(z)$. The contours that emerge from z_1 and the real left half-axis divide the disk into seven regions denoted by $\mathcal{D}_0$, $\mathcal{D}_i$ and $\mathcal{D}_i^*$, $i = 1, 2, 3$. On each contour we arrive at model jumps by replacing the reflection coefficient with its value at z_1, $r_0(z) \mapsto r_0(z_1)$, and we use (4.28) to approximate $\delta(z)$ near z_1. The jump matrices (conjugated by the local behavior of δ, cf (4.51)) inside $\mathcal{D}$ and the regions $\mathcal{D}_i$ are given in Figure 4.11.

RHP for $L(z)$. Find a 2×2 matrix $L(z)$ that satisfies the following properties.

- $L(z)$ is analytic in $\mathcal{D} \setminus \bigcup_{i=1}^{3} (\Sigma_i \cup \Sigma_i^*)$
- On the jump contours, the boundary values of L satisfy

$$L_+(z) = L_-(z) \left[(z - z_1)^{i\kappa_1} C_\delta(z)\right]^{\operatorname{ad}\sigma_3} V_L(z), \tag{4.51}$$

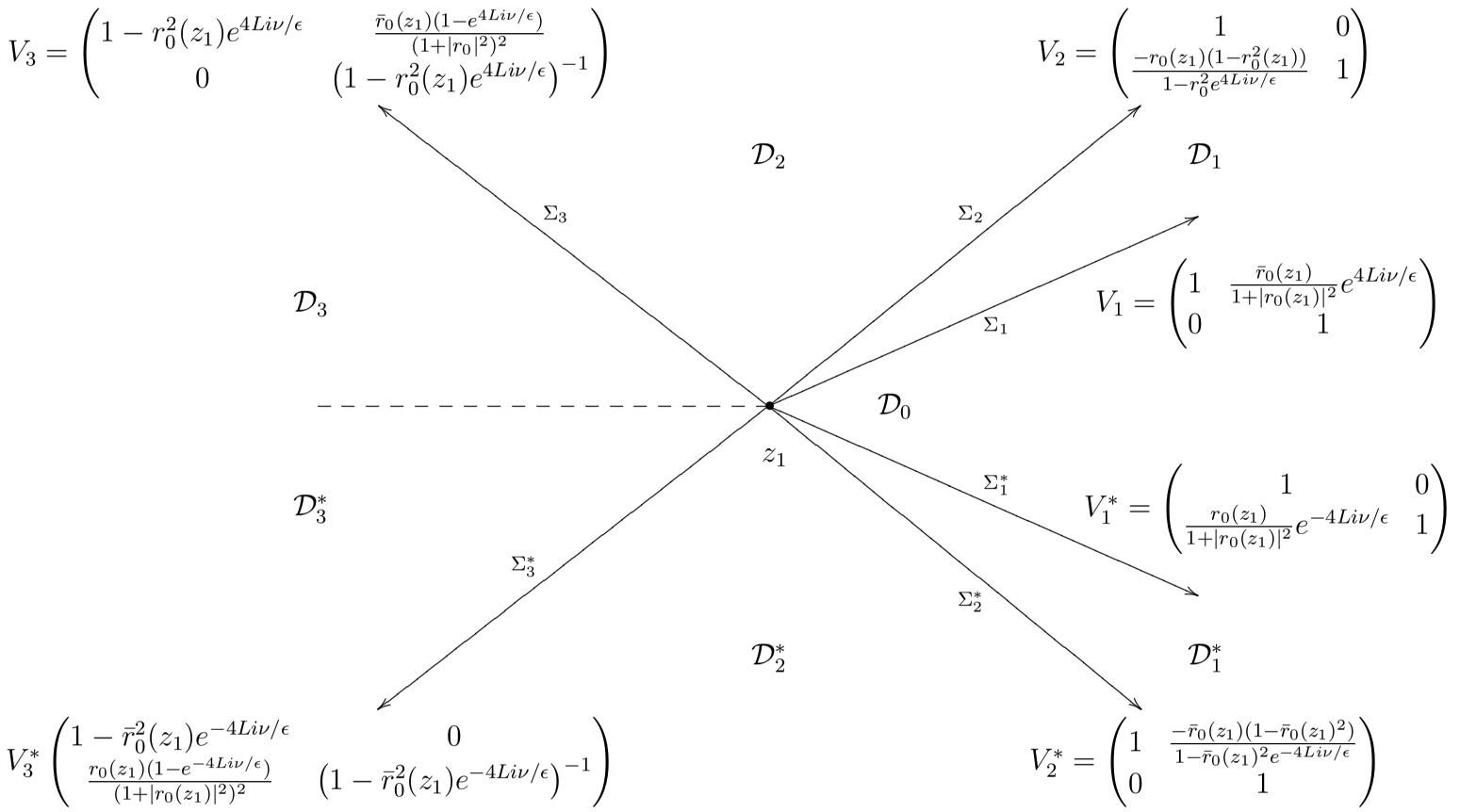

FIGURE 4.11. The regions $\mathcal{D}_i$; contours Σ_i and Σ_i^*; and the jump matrices, V_L, for the model problem inside the disk $\mathcal{D}$. Observe that relative to the outer and annular models, the contours Γ have been relabeled Σ. Also, the contours Σ_2 and Σ_2^* have been reoriented.

where κ_1 is defined by (4.45), and $C_\delta(z)$ is the piecewise constant function

$$C_\delta(z) = \begin{cases} \delta_{1+} & : \quad z \in \mathbb{C}^+ \\ \delta_{1-} & : \quad z \in \mathbb{C}^- \end{cases}, \tag{4.52}$$

where we recall that $\delta_{1\pm}$ are the constants defined in (4.29). Finally, V_L is given on each contour by:

$$V_L(z) = \begin{cases} \begin{pmatrix} 1 & \frac{\bar{r}_0(z_1)}{1+|r_0(z_1)|^2} e^{\frac{4Li\nu}{\epsilon}} \\ 0 & 1 \end{pmatrix} & : \quad z \in \Sigma_1 \\ \begin{pmatrix} 1 & 0 \\ -r_0(z_1)(1+|r_0(z_1)|^2) & 1 \end{pmatrix} & : \quad z \in \Sigma_2 \\ \begin{pmatrix} 1 - r_0^2(z_1) e^{\frac{4Li\nu}{\epsilon}} & \frac{\bar{r}_0(z_1)(1-e^{\frac{4Li\nu}{\epsilon}})}{(1+|r_0(z_1)|^2)^2} \\ 0 & (1 - r_0^2(z_1) e^{\frac{4Li\nu}{\epsilon}})^{-1} \end{pmatrix} & : \quad z \in \Sigma_3 \\ \begin{pmatrix} 1 & 0 \\ \frac{r_0(z_1)}{1+|r_0(z_1)|^2} e^{-\frac{4Li\nu}{\epsilon}} & 1 \end{pmatrix} & : \quad z \in \Sigma_1^* \\ \begin{pmatrix} 1 & -\bar{r}_0(z_1)(1+|r_0(z_1)|^2) \\ 0 & 1 \end{pmatrix} & : \quad z \in \Sigma_2^* \\ \begin{pmatrix} 1 - \bar{r}_0^2(z_1) e^{-\frac{4Li\nu}{\epsilon}} & 0 \\ \frac{r_0(z_1)(1-e^{-\frac{4Li\nu}{\epsilon}})}{(1+|r_0(z_1)|^2)^2} & (1 - \bar{r}_0^2(z_1) e^{-\frac{4Li\nu}{\epsilon}})^{-1} \end{pmatrix} & : \quad z \in \Sigma_3^* \end{cases}$$

The statement of this RHP is irregular for two reasons. First, we are interested in solving the problem not in the entire complex plane but just on the shrinking disk $\mathcal{D}$. Second, there is not a well-defined normalization condition. Instead, we only ask that $L(z)$ match the annular solution on $\partial\mathcal{D}$; we require that the two approximations $\mathcal{B}_\mathcal{A}(z)$ and $\mathcal{B}_\mathcal{D}(z)$ satisfy the relation

$$\mathcal{B}_\mathcal{A}^{-1}(z)\mathcal{B}_\mathcal{D}(z) = I + \mathcal{O}(\epsilon^l),$$

for some $l > 0$. As before this matching condition will select the analytic function $K_\mathcal{D}(z)$. For now, let us build a particular solution of the jump conditions well suited to matching.

Let us denote the matrix $V_L(z)$ along the contour Σ_k by $V_k(z)$ and along Σ_k^* by $V_k^*(z)$. Similarly we will write $L_k(z)$ ($L_k^*(z)$) for the solution $L(z)$ in the sector $\mathcal{D}_k$ ($\mathcal{D}_k^*$). Each $V_k(z)$ is clearly analytic in $\mathcal{D}$; either $V_k(z)$ is constant or the product of constants with the phase factor $e^{4Li\nu/\epsilon}$. We build our solution $L(z)$ simply by continuing our solution in $\mathcal{D}_0$ to adjacent sectors using the jump relations. We choose our initializing function in $\mathcal{D}_0$ to match the singular behavior of $\delta(z)$ near $z = z_1$. We make this procedure explicit as follows:

Proposition 9. *Let $L_k(z)$ and $L_k^*(z)$ denote the function elements of $L(z)$ in the sectors $\mathcal{D}_k$ and $\mathcal{D}_k^*$ analytically extended to functions defined on $\mathcal{D} \cap (-\infty, z_1)$. Similarly, let $V_k(z)$*

and V_k^* denote the elements of the jump function $V_L(z)$ along the contours Σ_k and Σ_k^* analytically extended to a function on $\mathcal{D}$. Then one solution to the RHP for $L(z)$ defined by (4.51) *is given by the formula*

$$L(z) = \begin{cases} (z-z_1)^{-i\kappa_1\sigma_3} & : \quad z \in \mathcal{D}_0 \\ L_{k-1}(z)\left[(z-z_1)^{i\kappa_1}C_\delta(z)\right]^{\mathrm{ad}\,\sigma_3} V_k(z) & : \quad z \in \mathcal{D}_k, \quad k=1,2,3 \\ L_{k-1}^*(z)\left[(z-z_1)^{i\kappa_1}C_\delta(z)\right]^{\mathrm{ad}\,\sigma_3} V_k^{-*}(z) & : \quad z \in \mathcal{D}_k^*, \quad k=1,2,3. \end{cases} \tag{4.53}$$

In addition, setting

$$K_{\mathcal{D}}(z) = K_{\mathcal{A}}(z) P_+(0; -r_0(z_1)) \begin{pmatrix} 1 & 0 \\ r_0(z_1) & 1 \end{pmatrix} c_\delta^{-\sigma_3}, \tag{4.54}$$

where we recall the definition (4.44) *of the constant c_δ, the annular and disc model solutions $B_{\mathcal{A}}(z)$ and $\mathcal{B}_{\mathcal{D}}(z)$ satisfy the matching estimate*

$$B_{\mathcal{A}}^{-1}(z) B_{\mathcal{D}}(z) = I + \mathcal{O}(\epsilon^{\beta_2 - 1/2}) \tag{4.55}$$

uniformly for $z \in \partial\mathcal{D}$.

Proof. The jump matrix elements $V_k(z)$ and $V_k^*(z)$ are all analytic functions in $\mathcal{D}$ so $L(z)$ is well defined and analytic in each open sector. Further, using (4.54) as a definition, $K_{\mathcal{D}}$ inherits analyticity in $\mathcal{D}$ from $K_{\mathcal{A}}$, which we showed was analytic in the larger disc $\mathcal{U}_1$ in Prop. 8. Thus $B_{\mathcal{D}}(z)$ will solve the inner model provided that $L(z)$ as defined by (4.53) satisfies the jump relation (4.51) and that $K_{\mathcal{D}}$ gives us the matching error (4.55). Clearly, $L(z)$ satisfies the jump relations on $\Sigma_k \cup \Sigma_k^*$, $k = 1, 2, 3$. We must verify that $L(z)$ is analytic across the half-line $(-\infty, z_1)$. This is equivalent to showing that,

$$V_3^* V_2^* V_1^* (1 + |r_0(z_1)|^2)^{\sigma_3} V_1 V_2 V_3 = (1 + |r_0(z_1)|^2)^{2\sigma_3}$$

This is a straightforward calculation which essentially amounts to showing the jump relations for $L(z)$ are consistent. We omit the details of the calculation, commenting only that the work is simplifies by observing that $r_0(z_1)$ is pure imaginary so $\bar{r}_0(z_1) = -r_0(z_1)$ and for $z \in \mathbb{R}$, $V_k^*(z)$ is the hermitian conjugate of $V_k(z)$.

Finally, we must show that $B_{\mathcal{D}}$ matches the annular solution $B_{\mathcal{A}}$ on the boundary $\partial\mathcal{D}$. Consider first $z \in \mathbb{C}^+ \cap \partial\mathcal{D}_0$. For $z \in \mathbb{C}^+ \cap \mathcal{D}_0$ using formulas (4.50) and (4.53) the function $B_{\mathcal{D}}(z)$ is given by

$$B_{\mathcal{D}} = K_{\mathcal{D}}(z)(z-z_1)^{-i\kappa_1\sigma_3} e^{\frac{i}{2\epsilon}\theta_1(z)\sigma_3}.$$

Similarly, for z in the annular region, $\mathcal{A}$, adjoining $\mathbb{C}^+ \cap \partial\mathcal{D}_0$, the annular solution $B_{\mathcal{A}}(z)$ defined by equations (4.43), (4.35), and Prop. 8, is given by

$$B_{\mathcal{A}} = K_{\mathcal{A}}(z) P(\zeta_1(z); -r_0(z_1)) \begin{pmatrix} 1 & 0 \\ r_0(z_1) & 1 \end{pmatrix} c_\delta^{-\sigma_3} (z-z_1)^{-i\kappa_1\sigma_3} e^{\frac{i}{2\epsilon}\theta_1(z)\sigma_3}$$

where we have used the change of variables (4.41) to move between $\zeta_1(z)$ and $(z-z_1)$. Again using the formula for $\zeta_1(z)$, for $z \in \partial\mathcal{D}$ we have for $z \in \mathbb{C}^+$ or $\mathbb{C}^-$ the uniform expansion

$$P(\zeta_1(z); -r_1(z_1)) = P_\pm(0; -r_1(z_1)) + \mathcal{O}(\epsilon^\beta - 1/2).$$

Comparing the formulae above and setting $K_\mathcal{D}(z)$ equal to the proposed function (4.54), we get the claimed matching error (4.55) in this sector. One then checks that as we move into the various arcs $\partial\mathcal{D}_k$, the matching condition is preserved. This is a straightforward calculation that we omit. Finally, we observe that $K_\mathcal{D}(z)$ is analytic as required since it is simply a constant multiple of the analytic function $K_\mathcal{A}(z)$. □

4.3 Estimating the global error

We now have a complete description of the global parametrix,

$$A(z) = \begin{cases} A_0(z) & : \quad z \in \mathcal{U}_0 \\ B_\mathcal{A}(z) & : \quad z \in \mathcal{A} \\ B_\mathcal{D}(z) & : \quad z \in \mathcal{D} \\ I & : \quad \text{elsewhere} \end{cases}, \tag{4.56}$$

where the function $A_0(z)$ and the neighborhood $\mathcal{U}_0$ of z_0 were described in Section 4.2.2 and the annular and disc sets $\mathcal{A}$ and $\mathcal{D}$ near $z = z_1$ and the functions $B_\mathcal{A}(z)$ and $B_\mathcal{D}(z)$ were described in Section 4.2.3. We now want to prove that our global parametrix (4.56) provides a uniform asymptotic approximation to the function $N(z)$ for each (x,t) in any compact subset of

$$\mathcal{S}_0 = \{(x,t) \,:\, x > L,\ t > 0\}. \tag{4.57}$$

Recall that $N(z)$ is the solution of the RHP defined by (4.18) and (4.19). By inverting the explicit series of transformation which mapped $m(z)$ to $N(z)$ we will end up with a uniform asymptotic approximation to the function $m(z)$ on every compact subset of $\mathcal{S}_0$ which through (3.17) implies a uniform asymptotic approximation to the solution of NLS, $q(x,t)$, for each (x,t) in a compact subset of $\mathcal{S}_0$.

Specifically we will prove that the error matrix $E(z)$ defined by

$$E(z) = N(z)A^{-1}(z) \tag{4.58}$$

falls into the class of "small norm" Riemann-Hilbert problems for which their exists a concrete theory for deriving and estimating solutions.

4.3.1 Results from the small-norm theory of Riemann-Hilbert problems

We now present a collection of results from the small-norm, or more appropriately near-identity, theory of Riemann-Hilbert problems. The material presented here comes from the appendices of [DKM$^+$99] and [KMM03] where the authors collect the necessary classical facts needed to analyze the small-norm problem.

Let Σ be a finite collection of oriented smooth arcs on the Riemann sphere which intersect at most a finite number of points such that all intersections are transversal. Fix a matrix norm $\|\cdot\|$. For each p, $1 \leq p \leq \infty$ define $L^p(\Sigma)$ to be the usual space of 2×2 matrix-valued functions f such that the corresponding norm is finite:

$$\begin{aligned} \|f\|_{L^p(\Sigma)} &:= \left(\int_\Sigma \|f(s)\|^p \, |ds| \right)^{1/p} && 1 \leq p < \infty, \\ \|f\|_{L^\infty(\Sigma)} &:= \operatorname*{ess\,sup}_{s \in \Sigma} \|f(s)\| && p = \infty. \end{aligned}$$

The near-identity theory of Riemann-Hilbert problems is intimately connected with the properties of the Cauchy transform $\mathcal{C}$,

$$\mathcal{C}f(z) = \frac{1}{2\pi i} \int_\Sigma \frac{f(s)}{s - z} ds. \tag{4.59}$$

Under the above assumptions on the contour Σ, the Cauchy transform $\mathcal{C}$ satisfies the following properties as an operator acting on functions $f \in L^2(\Sigma)$.

1. $\mathcal{C}f$ is analytic in $\mathbb{C} \backslash \Sigma$.
2. As z approaches Σ transversely from the $+$ or $-$ side, $\mathcal{C}f$ converges to a unique element $\mathcal{C}f_\pm \in L^2(\Sigma)$.
3. The operators $\mathcal{C}_\pm f \equiv (\mathcal{C}f)_\pm$ are bounded operators from $L^2(\Sigma)$ to $L^2(\Sigma)$ and $\mathcal{C}_+ - \mathcal{C}_- = \mathbf{1}$. The operator bounds depend on the contour Σ.

Now consider the following general RHP: Find a 2×2 matrix $Y(z)$ which satisfies the following properties:

$$\begin{cases} Y(z) & \text{is analytic in } \mathbb{C} \backslash \Sigma, \\ Y_+(s) = Y_-(s) v(s) & \text{for } s \in \Sigma, \\ Y(z) = I + \mathcal{O}(1/z) & \text{as } z \to \infty. \end{cases} \tag{4.60}$$

Now suppose that the jump matrix $v(z)$ has a factorization

$$v = b_-^{-1} b_+ = (I - w_-)^{-1} (I + w_+), \tag{4.61}$$

and define the operator $\mathcal{C}_w$ by

$$\mathcal{C}_w f = \mathcal{C}_-(f w_+) + \mathcal{C}_+(f w_-). \tag{4.62}$$

Then the following lemma establishes the existence of a solution to (4.60) provided that the functions $w_\pm$ are suitably small in the appropriate sense.

Lemma 10. *Assume that $w_\pm \in L^2(\Sigma) \cup L^\infty(\Sigma)$, and that*

$$\sup_{s \in \Sigma} \|w_+(s)\| < \frac{1}{2 \|\mathcal{C}_-\|_{L^2(\Sigma)}} \qquad \textit{and} \qquad \sup_{s \in \Sigma} \|w_-(s)\| < \frac{1}{2 \|\mathcal{C}_+\|_{L^2(\Sigma)}}.$$

Then, the RHP defined by (4.60) *has a unique solution defined for* $z \notin \Sigma$ *by*

$$Y(z) = I + \mathcal{C}\Big((\eta + I)(w_+ + w_-)\Big) = I + \frac{1}{2\pi i}\int_\Sigma \frac{(I+\eta(s))(w_+(s)+w_-(s))}{s-z}\,ds,$$

where the function $\eta \in L^2(\Sigma)$ *is the unique solution of*

$$(\mathbf{1} - \mathcal{C}_w)\eta = \mathcal{C}_w I.$$

η *satisfies the upper bound*

$$\|\eta\|_{L^2(\Sigma)} \leq \frac{\|w_+\|_{L^2(\Sigma)} + \|w_-\|_{L^2(\Sigma)}}{1 - \|\mathcal{C}_+\|_{L^2(\Sigma)}\|w_-\|_{L^\infty(\Sigma)} - \|\mathcal{C}_-\|_{L^2(\Sigma)}\|w_+\|_{L^\infty(\Sigma)}}.$$

If in addition we would like to control the size of the moments of the solution $Y(z)$ for z near ∞, we need extra control on the rate at which the function $w_\pm$ decay.

Lemma 11. *Suppose, in addition to the properties listed in Lemma 10, that for each* k, $0 \leq k \leq n$, *the functions* $s^k \cdot w_\pm(s) \in L^1(\Sigma) \cap L^2(\Sigma)$. *Then we have the following expansion valid for each* $z \in \mathbb{C}\backslash\Sigma$

$$Y(z) = I + \sum_{k=1}^{n} \frac{1}{z^k} Y_k + \frac{1}{2\pi i z^n}\int_\Sigma \frac{(\eta(s)+I)(s^n w_+(s) + s^n w_-(s))}{s-z} ds,$$

where the constants Y_k *are each given by the formula*

$$Y_k = \frac{1}{2\pi i}\int_\Sigma (\eta(s)+I)(s^{k-1}w_+(s) + s^{k-1}w_-(s))ds.$$

Each constant Y_k *is bounded by,*

$$\|Y_k\| \leq \frac{1}{2\pi}\Bigg[\|\eta\|_{L^2(\Sigma)}\left(\left\|s^{k-1}w_+\right\|_{L^2(\Sigma)} + \left\|s^{k-1}w_-\right\|_{L^2(\Sigma)}\right) + \left\|s^{k-1}w_+\right\|_{L^1(\Sigma)} + \left\|s^{k-1}w_-\right\|_{L^1(\Sigma)}\Bigg].$$

4.3.2 The Global Error RHP

The definition of $E(z)$, (4.58), generates the following RHP for $E(z)$. Find a 2×2 matrix-valued function $E(z)$ that satisfies the following properties

- $E(z)$ is analytic away from the contours $\bigcup_{k=0}^{3}\Gamma_k \cup \Gamma_k^*$, and the boundaries $\partial\mathcal{U}_0$, $\partial\mathcal{U}_1$, and $\partial\mathcal{D}$, see Figure 4.12.
- As $|z| \to \infty$,
$$E(z) = I + \mathcal{O}(1/z). \tag{4.63}$$

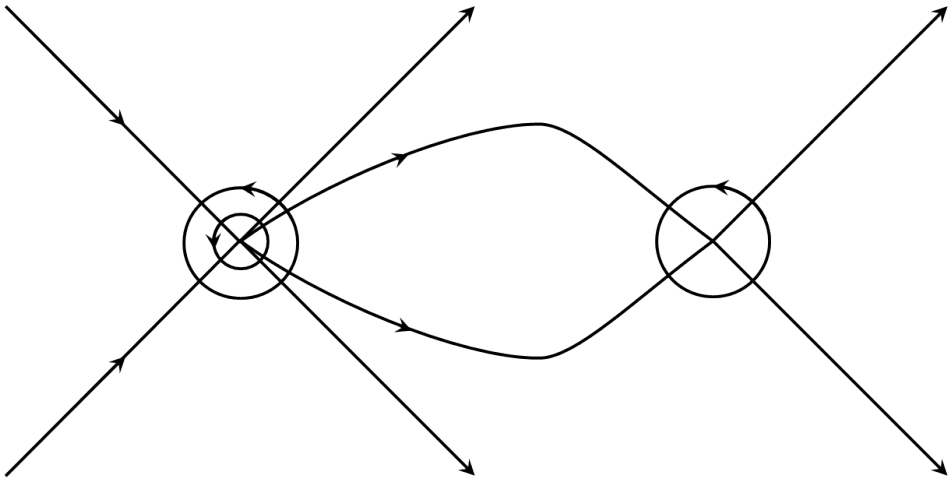

FIGURE 4.12. The collection of contours Γ_E for the Error RHP. Note that the three circular contours are given a counterclockwise orientation.

- On each of these contours $E(z)$ assumes continuous boundary values $E_+(z)$ and $E_-(z)$ which satisfy the jump relation $E_+(z) = E_-(z)V_E(z)$ where the jump matrix $V_E(z)$ is given by

$$V_E(z) = A_-(z)\left(V_N(z)V_A^{-1}(z)\right)A_-^{-1}(z). \tag{4.64}$$

The RHP for $E(z)$ follows directly from its definition (4.58), the properties of our global parametrix $A(z)$, and the RHP for $N(z)$ defined by (4.18) and (4.19). In (4.64) the matrices $V_N(z)$ and $V_A(z)$ are understood to be extended to contours where the respective functions $N(z)$ and $A(z)$ are analytic by setting the jump matrix to identity on these contours.

Let us denote the entire collection of contours on which $E(z)$ has jumps by Γ_E. These contours fall into three natural categories: the portions of the contours Γ_k and Γ_k^* lying outside the discs $\mathcal{U}_0$ and $\mathcal{U}_1$, which we will denote by Γ_E^{out}; the portions of these contours inside each disc, which we will denote by Γ_E^{in}; and the circluar boundaries $\partial\mathcal{U}_0$, $\partial\mathcal{U}_1$, and $\partial\mathcal{D}$ which we endow with a clockwise orientation and denote by Γ_E^{circ}.

Lemma 12. *Fix $K \subset \mathcal{S}_0$ compact and $\ell \in \mathbb{N}_0$, there exists $M_K^\ell > 0$ such that for any $\rho \ll 1$, the jump matrix $V_E(z)$ defined by* (4.64) *satisfies the following inequalities. For every sufficiently small value of ϵ,*

$$\|s^\ell(V_E - I)\|_{L^p(\Gamma_E)} \leq M_K^\ell \epsilon^{1/2-\rho}, \tag{4.65}$$

for $p = 1, 2,$ or ∞.

Proof. Fix $(x,t) \in K$. We will verify (4.65) independently on each sub-collection of contours Γ_E^{out}, Γ_E^{in}, and Γ_E^{circ}. The calculations needed to verify the lemma within each

sub-collection are similar. We will summarize these calculations by presenting a representative from each collection. We begin by considering the exterior collection of contours Γ_E^{out}. On these contours we can write the jump V_E in the form $V_E = A_- V_N A_-^{-1}$ since $A(z)$ is analytic outside the sets $\mathcal{U}_0$ and $\mathcal{U}_1$. Observing that $A(z)$ is everywhere bounded, we need only consider the quantity $\|V_N - I\|$. Using (4.19), for $z \in \Gamma_0^{out}$ we have $\|V_N - I\| \leq M \left|\delta^{-2}(z) r_0(z)\right| e^{-\frac{1}{\epsilon}\operatorname{Im}(\theta(z))}$. The functions δ and r are each bounded, and there exists $c > 0$ such that $\operatorname{Im}(\theta(z)) > c$ for all $z \in \Gamma_0^{out}$. Using these estimates we have for $p = 1, 2, \infty$,

$$\|V_N - I\|_{L^p(\Gamma_0^{out})} < M' \exp\left(-\frac{c}{\epsilon}\right) \left\| s^\ell e^{\frac{1}{\epsilon}(i\theta(s)+c)} \right\|_{L^p(\Gamma_0^{out})} \leq M^\ell \exp\left(-\frac{c}{\epsilon}\right).$$

The last equality follows from the fact that $\exp(\frac{1}{\epsilon}(i\theta + c))$ is bounded independent of ϵ and decays exponentially in z for each fixed ϵ as $z \to \infty$ along Γ_0. The constant c may depend on (x, t) but is independent of ϵ. Similar results hold on each of the other external contours in Γ_E^{out}.

On the remaining compact contours the desired estimates are much easier to verify. First, we need only verify the L^∞ bound, once we have it, the compactness of the contours give the L^1 and L^2 bounds for free. Also, because the compact contours have a finite radius, once we establish the bound for $\|V_E - I\|$ the same estimate holds for $\|s^\ell(V_E - I)\|$ albeit with a different constant prefactor M^ℓ.

Let us now consider the set of circular contours, Γ_E^{circ}. On these contours V_N is analytic so the error jump takes the form $V_E = A_- A_+^{-1}$. We have already done the essential analysis on these contours. For $z \in \partial\mathcal{U}_0$ and using (4.39) and for $z \in \partial\mathcal{U}_1$ using (4.49) we have $\|V_E - I\| \leq M\epsilon^{1/2}$ independent of z. On the shrinking contour $\partial\mathcal{D}$, (4.55) gives $\|V_E - I\| \leq M\epsilon^{\beta - 1/2}$ where we recall $1/2 < \beta < 1$ is related to the rate at which the disk $\mathcal{D}$ shrinks, $\mathcal{D} = \{z : |z - z_1| < \epsilon^\beta\}$. Putting these together we have $\|V_E - I\|_{L^\infty(\Gamma_E^{circ})} < M\epsilon^{\beta - 1/2}$.

It remains to consider the collection of finite contours Γ_E^{in}. Here we write the error jump in the form $V_E = A_- V_N V_A^{-1} A_-^{-1}$. To establish the error bound we need only consider the quantity $\left\|V_N V_A^{-1} - I\right\|_{L^\infty(\Gamma_E^{in})}$ since $A(z)$ is bounded independent of ϵ. On these contours the perturbation from identity comes from the approximations introduced in the local model problems on $\mathcal{U}_1$, $\mathcal{A}$, and $\mathcal{D}$. Consider first the jumps in the $\mathcal{U}_0$. Comparing (4.19) to (4.31) we see that $\|V_N V_A^{-1} - I\|$ is proportional in $\mathcal{U}_0$ to the quantities

$$\left|(r_0(z) - r_0(z_0))\, e^{\frac{i}{2}\zeta_0(z)}\right| \quad \text{and} \quad \left|\left(\delta(z) - \left(\frac{\epsilon}{\theta_0''(z_0)}\right)^{i\kappa_0/2} \zeta_0(z)^{i\kappa_0} \delta_0\right) e^{\frac{i}{2}\zeta_0(z)}\right|.$$

Recalling the scaling by $\sqrt{\epsilon}$ in the definition (4.30) of $\zeta_0(z)$ and (4.28) which describes the local behavior of the function $\delta(z)$ near the points z_1 and z_0, we have by elementary estimates,

$$\left\|V_N V_A^{-1} - I\right\|_{L^\infty(\Gamma_{\mathcal{U}_0^j n})} \leq M\epsilon^{1/2} \log \epsilon.$$

Similar calculations give the same estimate for the contours in the annular set $\mathcal{A}$.

For the contours in shrinking disk $\mathcal{D}$ the estimates are instead generated by virtue of

the fact that radius of the set is ϵ^β. On these contours we observe along each contour in $\mathcal{D}$ that using (4.19) and (4.51) we have $\left\|V_N V_A^{-1}\right\| \leq M(f(z) - f(z_0))$, where the function $f(z)$ depends on the choice of contour and is smooth and independent of ϵ. Simply by virtue of the shrinking radius this allows us to conclude that for $z \in \Gamma_E^{in} \cap \mathcal{D}$ we have $\|V_N V_A^{-1} - I\| \leq M\epsilon^\beta$

Collecting the estimates along each sub-collection of Γ_E, we minimize the perturbation of the jumps V_E from identity by taking β as near 1 as the possible. Writing $\beta = 1 - \rho$ for $\rho \ll 1/2$ and collecting our estimates we see that the largest error comes from the matching on the shrinking disc $\partial\mathcal{D}$ which now takes the form

$$\left\| s^\ell (V_E - I) \right\|_{L^\infty(\Gamma_E)} \leq M_K^\ell \epsilon^{1/2-\rho}.$$

In this final step we have used the compactness of K to choose the constant M_K^ℓ such that the bound above is uniformly valid for each $(x,t) \in K$.

□

For a fixed collection of contours Σ Lemma 12 and Lemma 10 would allow us to immediately conclude a solution $E(z)$ exists to the error RHP and allow us to bound its size for ϵ sufficiently small. However, our contours are not fixed. Among the contours in Γ_E is the shrinking circle $\partial\mathcal{D}$ whose radius we chose to be $\epsilon^{1-\rho}$ to minimize the perturbation of the error jumps from identity. The operator norm $\|\mathcal{C}\|_{L^2(\Gamma_E)}$ inherits ϵ dependence through this contour. However, in [KMM03] the authors show that the L^2 norm of the Cauchy boundary operators $\mathcal{C}_\pm$ acting on a circle of radius are bounded above as the radius $r \to 0$. This allows us to conclude that there exists a $\mathbf{C} > 0$ such that for any $\epsilon > 0$,

$$\|\mathcal{C}_\pm\|_{L^2(\Gamma_E)} \leq \mathbf{C}.$$

Given this bound on the on the Cauchy operators we can prove that the error RHP has a solution which is everywhere near identity.

Theorem 13. *For each (x,t) outside the support of the initial data, that is for $(x,t) \in \mathcal{S}_0$, the global error Riemann-Hilbert problem defined by* (4.63) *and* (4.64) *has a unique solution given for $z \notin \Gamma_E$ by the formula*

$$E(z) = I + \frac{1}{2\pi i} \int_{\Gamma_E} \frac{(\eta(s) + I)(V_E(s) - I)}{s - z} ds,$$

where $\eta(s)$ is the unique solution of the equation

$$\eta - \mathcal{C}_-(\eta(V_E - I)) = \mathcal{C}_-(V_E - I).$$

In addition as $z \to \infty$ away from Γ_E, $E(z)$ has a valid asymptotic expansion,

$$E(z) = I + \frac{E^{(1)}}{z} + \ldots,$$

where each moment $E^{(k)}$ is bounded, and in particular,

$$\left\| E^{(1)} \right\| \leq M\epsilon^{1/2-\rho}$$

for any $\rho > 0$. The bound is uniform on any compact $K \subset \mathcal{S}_0$.

Proof. The proof follows directly from the fact that the Cauchy operators $\mathcal{C}_\pm$ are L^2-bounded operators over Γ_E, Lemma 12 controlling the jump matrices, and Lemmas 10 and 11 where we have used the trivial factorization in (4.61): $w_+ = V_E - I$, $w_- = 0$. Uniqueness follows from the usual Liouville argument. □

Having found an explicit formula for the error matrix $E(z)$ we can now unravel the series of explicit transformations

$$m \mapsto M \mapsto N \mapsto E$$

to get a asymptotic expansion for the solution $m(z)$ to the original NLS RHP. Using the large z expansion formula we can then recover a asymptotic formula for the time evolved solution of NLS as described by (3.17). This is the content of Theorem 3 which we stated at the beginning of this chapter which we are now in a position to prove. We reproduce the Theorem here and provide its proof.

Theorem. *For each value $(x,t) \in \mathcal{S}_0 = \{(x,t) \ : \ x > L, \quad t > 0\}$ the solution $q(x,t)$ to the semiclassically focusing NLS equation*

$$i\epsilon q_t + \frac{\epsilon^2}{2} q_{xx} + |q|^2 q = 0, \qquad q(x,0) = q(x),$$

where $q(x)$ is the square barrier initial data (1.2)*, has the following asymptotically valid description as $\epsilon \to 0^+$:*

$$q(x,t) = \mathcal{O}(\epsilon^{1/2-\rho}).$$

for any number $\rho > 0$. Additionally, the convergence is uniform on any compact subset of $\mathcal{S}_0$.

Proof. Let $(x,t) \in K \subset \mathcal{S}_0$, for K compact. Using the explicit transformation formulae (4.2), (4.11), and (4.16), and the global parametrix $A(z)$ built in Section 4.2 the solution $m(z)$ to the original NLS RHP is given for suitably large z along the positive imaginary axis by the formula

$$m(z) = E(z).$$

The theorem then follows immediately from Theorem 13. □

Chapter 5

The Single Band Solution Inside the Support

For $0 \leq x < L$, that is x inside the support of the initial square barrier data, the analysis of the problem changes. The mechanism of this change is the location of the stationary phase point z_0 defined in the previous chapter by $z_0 = -(x-L)/2t$. For $x > L$ this point lays on the negative real line and we were able to introduce the standard pole removing factorization (cf. (4.2) and Section (3.4.3)) in such a way that the jumps of the new unknown $M(z)$ were all exponentially decaying off the real axis. Once we allow $0 \leq x < L$, z_0 moves onto the positive real line. The effect of this change is to make the jumps resulting from the standard pole factorization exponentially large.

As we will see, we will need to introduce new machinery to control the jump matrices and arrive at a small-norm RHP for $0 \leq x < L$. The new machinery we will need is the idea of the so called g-function which we will introduce as it becomes necessary in the analysis. The result of these changes is a new and completely different asymptotic description of the semiclassical limit of NLS for x inside the support. In this chapter we will prove the second of our main results:

Theorem 14. *For each value $(x,t) \in \mathcal{S}_1$, where*

$$\mathcal{S}_1 = \left\{ (x,t) \ : \ 0 \leq |x| < L, \quad t < \frac{1}{2\sqrt{2}q}\,(L-x) \right\},$$

the solution $q(x,t)$ to the semiclassically focusing NLS equation

$$i\epsilon q_t + \frac{\epsilon^2}{2} q_{xx} + |q|^2 q = 0, \qquad q(x,0) = q(x),$$

where $q(x)$ is the square barrier initial data (1.2)*, has the following asymptotically valid description as $\epsilon \to 0^+$:*

$$q(x,t) = q e^{\frac{i}{\epsilon} q^2 t} + \mathcal{O}(\epsilon^{1/2-\rho}). \tag{5.1}$$

for any number $\rho > 0$. Additionally, the convergence is uniform on any compact subset of $\mathcal{S}_1$.

We begin the steps toward proving Theorem 14 by returning to the original RHP for NLS defined by (3.21), (3.22) and (3.23). We will reuse much of the notation from the previous chapter, such as using Γ_k to denote contours and z_k to denote stationary phase points, but their meanings may be different from their previous incarnations.

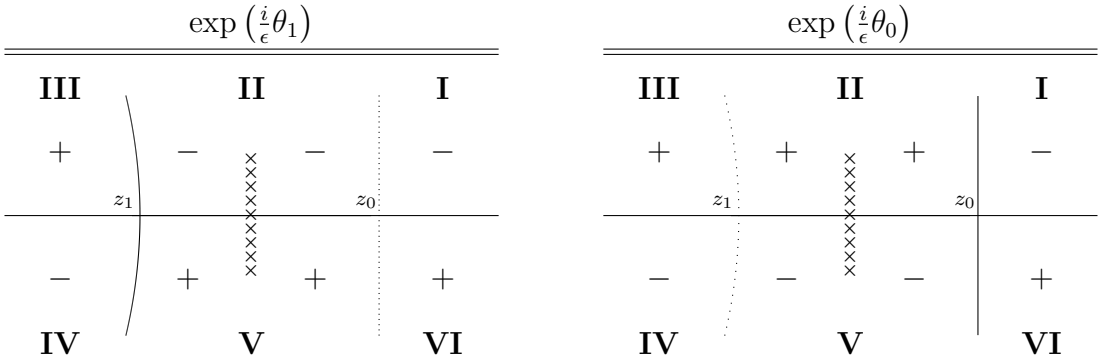

FIGURE 5.1. Regions of decay (-) and growth (+) of the exponential functions $e^{\frac{i}{\epsilon}\theta_0}$ and $e^{\frac{i}{\epsilon}\theta_1}$. The poles (x) lie to the left of the critical point z_0 defined in the previous chapter by (cf. (3.28))

5.1 Inside the support: the one-cut regime

For $0 < x < L$ the regions of growth and decay of the exponentials $e^{\frac{i}{\epsilon}\theta_0(z)}$ and $e^{\frac{i}{\epsilon}\theta_0(z)}$ are given in Fig 5.1.

Observe that the poles lie entirely in regions **II** and **V**, where the pole removing factorization (3.33) that we used for x outside the support grows exponentially. Thus, we cannot remove the poles using the factorization (3.33) described in 3.4.3 as we did for $x > L$ without introducing exponentially large jump matrices. For $0 \leq x < L$ we need an alternative scheme for removing the poles that does not introduce regions where the unknown grows exponentially.

Recall the product expansion (cf. (3.31)) of the matrix $R(z)$

$$R(z) = \prod_{i=0}^{\infty} R_k(z) = \prod_{i=0}^{\infty} \begin{pmatrix} 1 & 0 \\ r_k(z)e^{\frac{i}{\epsilon}\theta_k} & 1 \end{pmatrix}, \qquad r_k(z) = r_0^{2k+1}(z) - r_0^{2k-1}(z).$$

In region **II**, only the leading term, $R_0(z)$, contains an exponentially growing term; the remaining terms each decay exponentially. Thus, the matrix RR_0^{-1} converges exponentially to identity in region **II** (Similarly, the matrix $R^*R_0^{-*}$ converges exponentially to identity in region **V**). This matrix, which was used for $x > L$ to open lenses off the real axis, will take the place of $R(z)$ in our pole removal scheme for $0 \leq x < L$.

5.1.1 The first transformation $m(z) \mapsto M(z)$

Just as outside the support, the first transformation seeks to deform the contours in such a way that they rapid oscillations on the real axis are replaced by contours in the complex plane along which the jumps rapidly decay. In addition, this transformation should removed the poles from the problem. Consider the set of contours and open sets in Figure 5.2. The contours Γ_k, $k = 0, 1, 2$, and their complex conjugates are straight line rays except near the real axis where we will later need to deform their shape. We call the point from which Γ_0

leaves the real axis z_0 and the point where Γ_1 and Γ_2 meet the real axis z_1. These points are no longer the critical points of the phase functions θ_k as we defined them in the the previous chapter (cf. (3.28)); for now they are simply two points on the real axis such that $z_0 > 0$ and $z_1 < 0$. Their values will be determined later.

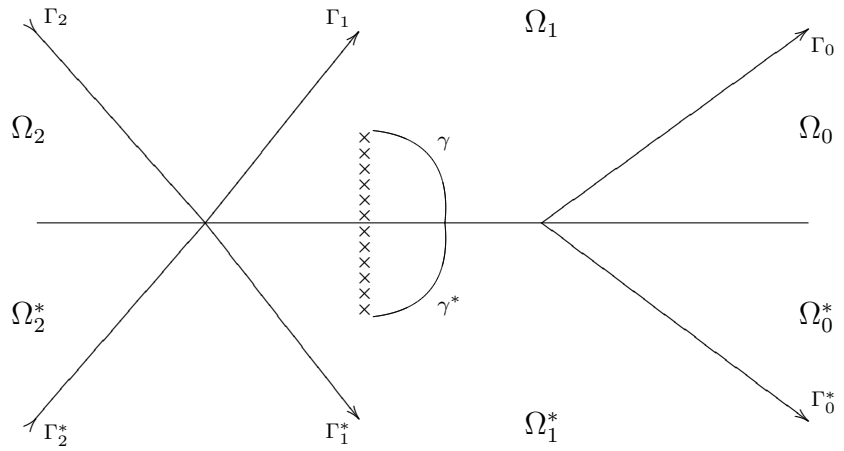

FIGURE 5.2. Contours and regions used to define $M^{(1)}(z)$ for x inside the support. The contour $\gamma^* \cup \gamma$ connects $-iq$ to iq. A jump exists here because the matrix $R_0(z)$ used to define $M^{(1)}(z)$ is branched along $\gamma \cup \gamma^*$.

As before, we make the transformation to $M(z)$ in two steps. First, define a new unknown matrix $M^{(1)}(z)$ by the following relation:

$$m(z) = \begin{cases} M^{(1)}(z)R & : \quad z \in \Omega_0 \\ M^{(1)}(z)RR_0^{-1} & : \quad z \in \Omega_1 \\ M^{(1)}(z)\widehat{R}^* & : \quad z \in \Omega_2 \\ M^{(1)}(z)R^{-*} & : \quad z \in \Omega_0^* \\ M^{(1)}(z)R^{-*}R_0^* & : \quad z \in \Omega_1^* \\ M^{(1)}(z)\widehat{R}^{-1} & : \quad z \in \Omega_2^* \\ M^{(1)}(z) & : \quad \text{elsewhere} \end{cases} . \tag{5.2}$$

Claim 2. The new unknown matrix $M^{(1)}(z)$ satisfies the following RHP:

Find a 2×2 matrix $M^{(1)}(z)$ that satisfies the following properties.

- $M^{(1)}(z)$ is analytic in the set $\mathbb{C} \setminus \left\{ \bigcup_{i=0}^{2} (\Gamma_i \cup \Gamma_i^*) \cup \gamma \cup \gamma^* \cup (-\infty, z_0) \right\}$ and has continuous boundary values as z approaches the contours.
- $M^{(1)}(z) = I + \mathcal{O}(1/z)$ as $|z| \to \infty$.

- On the contours, the boundary values of $M^{(1)}$ satisfy the relation $M^{(1)}_+ = M^{(1)}_- V_{M^{(1)}}$ where

$$V_{M^{(1)}} = \begin{cases} R_0 & : \quad z \in \Gamma_0 \\ R R_0^{-1} & : \quad z \in \Gamma_1 \\ \widehat{R}^* & : \quad z \in \Gamma_2 \\ R_{0-}^{-1} R_{0+} & : \quad z \in \gamma \\ R_0^* R_0 & : \quad z \in (z_1,\ z_0) \\ (1+|r(z)|^2)^{\sigma_3} & : \quad z \in (-\infty, z_1) \\ R_{0-}^* R_{0+}^{-*} & : \quad z \in \gamma^* \\ R_0^* & : \quad z \in \Gamma_0^* \\ R^* R_0^{-*} & : \quad z \in \Gamma_1^* \\ \widehat{R}^1 & : \quad z \in \Gamma_2^* \end{cases} . \tag{5.3}$$

Proof. The definition of $M^{(1)}$ in (5.2) is identical to the definition of $\widetilde{M}$ in (4.2) except that the locations of the regions Ω_i have moved in relation to the poles. The proof of the asymptotic behavior and jump relationships away from the contours γ and γ^* are identical to the previous proof for $\widetilde{M}$. Here we will verify that $M^{(1)}$ has no poles in $\mathbb{C}^+$ and the prescribed jumps on γ. The proof in the lower half-plane is analogous.

In Ω_1, $M^{(1)}(z) = m(z)R^{-1}R_0$. In Prop. 4 we showed that the product $m(z)R^{-1}$ is analytic in the upper half-plane. The function $R_0(z)$ has no poles, but does have a branch cut along $\gamma \cup \gamma^*$ due to its dependence on the function $\nu(z) = \sqrt{z^2+q^2}$. It follows that $M^{(1)}$ as defined in (5.2) has no poles. However, the interpolate that removes the poles is cut along γ and therefore $M^{(1)}$ inherits a jump along this contour. We calculate the jump along γ using the formula $V_{M^{(1)}} = (M^{(1)}_-)^{-1} M^{(1)}_+$:

$$\begin{aligned} V_{M^{(1)}} &= R_{0-}^{-1} R_- m_- m_+ R_+^{-1} R_{0+} = \\ R_{0-}^{-1} R_{0+} &= \begin{pmatrix} 1 & 0 \\ -r_{0-}(z) e^{\frac{i}{\epsilon}\theta_0} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r_{0+}(z) e^{\frac{i}{\epsilon}\theta_0} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ w(z) e^{\frac{i}{\epsilon}\theta_0} & 1 \end{pmatrix}, \end{aligned} \tag{5.4}$$

where we have defined

$$w(z) = r_{0+}(z) - r_{0-}(z) = \frac{2\nu_+}{iq}. \tag{5.5}$$

Similar arguments show that (5.3) holds on γ^*. The remaining jumps follow from direct computation. □.

Remark 1. The careful reader will observe that the jumps along γ and γ^* are exponentially large due to the presence of the phase θ_0 in the exponentials. This is an important detail and must be carefully addressed. However, we defer a discussion of how to handle these unwanted jumps for a moment to finish our discussion of stabilizing the jumps along the real axis.

Remark 2. On the contour γ and γ^* the function $w(z)$, which appears in the off-diagonal entries of the jump matrix, can be related to boundary values of the function $\frac{\bar{r}_0}{1+|r_0|^2}$. Recalling formula (3.24) for $r_0(z)$ and (5.5) defining $w(z)$ we have

$$\frac{r_0(z)}{1+|r_0(z)|^2} = -\frac{iq}{2\nu}, \quad \frac{\bar{r}_0(z)}{1+|r_0(z)|^2} = \frac{iq}{2\nu}, \quad \text{and} \quad w(z) = \frac{2\nu_+}{iq}. \tag{5.6}$$

where the first two formulae are understood in $\mathbb{C}\backslash(\gamma \cup \gamma^*)$ and the last restricted to the cut $\gamma \cup \gamma^*$. From these formulae we have the obvious equivalences for $z \in \gamma \cup \gamma^*$:

$$\begin{aligned} [w(z)]^{-1} &= -\left(\tfrac{r_0(z)}{1+|r_0(z)|^2}\right)_+ = \left(\tfrac{r_0(z)}{1+|r_0(z)|^2}\right)_- \\ &= \left(\tfrac{\bar{r}_0(z)}{1+|r_0(z)|^2}\right)_+ = -\left(\tfrac{\bar{r}_0(z)}{1+|r_0(z)|^2}\right)_-. \end{aligned} \tag{5.7}$$

We will use these identities later to open lenses which surround the contours $\gamma \cup \gamma^*$.

We turn now to a consideration of the jump matrices left of the real axis, which are still rapidly oscillatory. We need to further factor these jumps to reach a RHP whose jumps have stable pointwise limits as $\epsilon \downarrow 0$. As in chapter 4 we will use the factorization (cf (4.7)-(4.9)),

$$(1+|r(z)|^2)^{\sigma_3} = \overline{a_1(\bar{z})}(1+|r_0(z)|^2)^{2\sigma_3} a_1(z)^{-\sigma_3} \tag{5.8}$$

for $z \in (\infty, z_1)$ and

$$R_0^* R_0 = \widehat{R}_0 (1+|r_0|^2)^{\sigma_3} \widehat{R}_0^* \tag{5.9}$$

for $z \in (z_1, z_0)$, to push the remaining oscillations for z real into regions of decay. However, the contour $\gamma \cup \gamma^*$ crossing the real axis forces us to work a little harder to open these lenses. The jump matrix on γ has the following factorization:

$$\begin{pmatrix} 1 & 0 \\ we^{\frac{i}{\epsilon}\theta_0} & 1 \end{pmatrix} = \begin{pmatrix} 1 & w^{-1}e^{-\frac{i}{\epsilon}\theta_0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -w^{-1}e^{-\frac{i}{\epsilon}\theta_0} \\ we^{\frac{i}{\epsilon}\theta_0} & 0 \end{pmatrix} \begin{pmatrix} 1 & w^{-1}e^{-\frac{i}{\epsilon}\theta_0} \\ 0 & 1 \end{pmatrix}. \tag{5.10}$$

We can rewrite this using the equivalences in Remark 2 as

$$\begin{pmatrix} 1 & 0 \\ we^{\frac{i}{\epsilon}\theta_0} & 1 \end{pmatrix} = \left(\widehat{R}_0^{-*}\right)_- \begin{pmatrix} 0 & -w^{-1}e^{-\frac{i}{\epsilon}\theta_0} \\ we^{\frac{i}{\epsilon}\theta_0} & 0 \end{pmatrix} \left(\widehat{R}_0^*\right)_+. \tag{5.11}$$

Similarly, on γ^* we have the factorization,

$$\begin{pmatrix} 1 & we^{-\frac{i}{\epsilon}\theta_0} \\ 0 & 1 \end{pmatrix} = \left(\widehat{R}_0\right)_- \begin{pmatrix} 0 & we^{-\frac{i}{\epsilon}\theta_0} \\ -w^{-1}e^{\frac{i}{\epsilon}\theta_0} & 0 \end{pmatrix} \left(\widehat{R}_0^{-1}\right)_+ \tag{5.12}$$

To use these factorizations we need a new contour. Let Γ_3 be a simple arc in $\mathbb{C}^+$ beginning at z_1 and terminating at z_0 such that the region Ω_3 enclosed between Γ_3 and the real axis contains the arc γ, see Figure 5.3. As usual, let Γ_3^* denote the complex conjugate contour and Ω_3^* the conjugate set. Motivated by the above factorizations we define a new

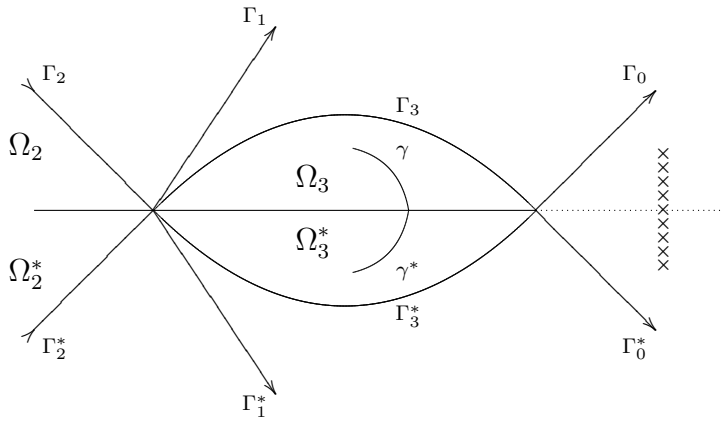

FIGURE 5.3. The set of contours on which $M(z)$ has jumps and the regions Ω_2 and Ω_3 used to define $M(z)$. The contour Γ_3 is oriented from z_1 to z_0, lies entirely in $\mathbb{C}^+$, and encloses the contour γ.

unknown $M(z)$ by

$$M^{(1)}(z) = \begin{cases} M(z)a_1^{-\sigma_3}(z) & : \quad z \in \Omega_2 \\ M(z)\widehat{R}_0^* & : \quad z \in \Omega_3 \\ M(z)\bar{a}_1^{\sigma_3}(z) & : \quad z \in \Omega_2^* \\ M(z)\widehat{R}_0^{-1} & : \quad z \in \Omega_3^* \\ M(z) & : \quad \text{elsewhere} \end{cases} . \tag{5.13}$$

Remark 3. Since $\widehat{R}_0$ is given by

$$\widehat{R}_0 = \begin{pmatrix} 1 & 0 \\ -\frac{iq}{2\nu}e^{\frac{i}{\epsilon}\theta} & 1 \end{pmatrix},$$

(cf. (5.6) & (4.4)), we see that the change of variables above is well defined away from the contours, and specifically away from the points $z = \pm iq$. Indeed, the off diagonal entry of $\widehat{R}_0$ behaves like $|z \pm iq|^{-1/2}$. Because $M^{(1)}(z)$ is bounded near $z = \pm iq$, and the definition (5.13) of the new unknown $M(z)$ near $z = iq$ and $z = -iq$ are given by

$$M(z) = M^{(1)}(z)\widehat{R}_0^{-*}, \quad \text{and} \quad M(z) = M^{(1)}(z)\widehat{R}_0,$$

respectively, the new unknown $M(z)$ must be allowed to blow up at a rate prescribed by the singular change of variables. We include the behavior of M near the singular endpoints in the resulting RHP for $M(z)$; this additional information is recorded in formula (5.15).

Denote by Γ_M the collection of contours on which M has a nontrivial jump

$$\Gamma_M = (-\infty, z_0) \cup \gamma \cup \gamma^* \bigcup_{i=0}^{3} (\Gamma_i \cup \Gamma_i^*)$$

RHP for $M(z)$ Find a 2×2 matrix $M(z)$ that satisfies the following properties.

- $M(z)$ is analytic in the set $\mathbb{C}\backslash\Gamma_M$ and has continuous boundary values as z approaches the contours away from the endpoints $z = \pm iq$.
- $M(z) = I + \mathcal{O}(1/z)$ as $|z| \to \infty$.
- On the contours, the boundary values of M satisfy the relation $M_+ = M_- V_M$ where

$$V_M = \begin{cases} R_0 & : \quad z \in \Gamma_0 \\ RR_0^{-1} & : \quad z \in \Gamma_1 \\ a_1^{-\sigma_3}(z)\widehat{R}^* & : \quad z \in \Gamma_2 \\ \widehat{R}_0^* & : \quad z \in \Gamma_3 \\ \begin{pmatrix} 0 & -w^{-1}(z)e^{-\frac{i}{\epsilon}\theta_0} \\ w(z)e^{\frac{i}{\epsilon}\theta_0} & 0 \end{pmatrix} & : \quad z \in \gamma \\ (1+|r_0(z)|^2)^{\sigma_3} & : \quad z \in (z_1,\ z_0) \\ (1+|r_0(z)|^2)^{2\sigma_3} & : \quad z \in (-\infty, z_1) \\ \begin{pmatrix} 0 & w(z)e^{-\frac{i}{\epsilon}\theta_0} \\ -w^{-1}(z)e^{\frac{i}{\epsilon}\theta_0} & 0 \end{pmatrix} & : \quad z \in \gamma^* \\ R_0^* & : \quad z \in \Gamma_0^* \\ R^* R_0^{-*} & : \quad z \in \Gamma_1^* \\ \widehat{R}\bar{a}_1^{-\sigma_3}(z) & : \quad z \in \Gamma_2^* \\ \widehat{R}_0 & : \quad z \in \Gamma_3^* \end{cases} . \tag{5.14}$$

- Near $z = iq$ and $z = -iq$, the entries of $M(z)$ grow at the following prescribed rates:

$$\begin{aligned} M(z) &= \mathcal{O}\begin{pmatrix} 1 & |z-iq|^{-1/2} \\ 1 & |z-iq|^{-1/2} \end{pmatrix} \\ M(z) &= \mathcal{O}\begin{pmatrix} |z+iq|^{-1/2} & 1 \\ |z+iq|^{-1/2} & 1 \end{pmatrix}. \end{aligned} \tag{5.15}$$

Verifying that the new unknown $M(z)$ satisfies the above RHP follows from straightforward, but tedious, calculation using the RHP for $M^{(1)}(z)$ defined by (5.3) and the definition (5.13) of $M(z)$. In particular to verify the jump conditions for $M(z)$ one needs the factorizations formulae (5.8)-(5.12). The other conditions to check are the new growth conditions near $z = \pm iq$. These follow directly from the definition (5.13) of $M(z)$ and the singularities of the off diagonal entries of the function $\widehat{R}_0$ defined by (4.4) as we commented on in Remark 3.

5.2 The transformation $M(z) \mapsto N(z)$: introducing the g-function

The RHP for $M(z)$ is nearly in a form amenable to approximation by parametrices. The jumps along the real line are independent of ϵ and along the contours Γ_i the jumps are ex-

ponentially near the identity. However, the jumps on γ and γ^* grow exponentially. We need to introduce a transformation that can remove this exponential growth without destroying the good features of the RHP for $M(z)$.

Consider the global change of variables

$$M(z) = e^{-\frac{i}{\epsilon}g(\infty)\sigma_3} N(z) e^{\frac{i}{\epsilon}g(z)\sigma_3}. \tag{5.16}$$

This change of variables is meant to remove the exponential growth of the jump V_M along $\gamma \cup \gamma^*$ without altering the positive properties of the RHP for $M(z)$. To that end, we demand that g satisfy each of the following properties.

1. **Dependence on ϵ:** $g(z)$ is independent of ϵ.
2. **Large z Behavior:** $g(z) = \mathcal{O}(1)$ as $|z| \to \infty$.
3. **Symmetry:** $\overline{g(z)} = g(\bar{z}),\ \forall\ z \in \mathbb{C}$

Beyond these basic properties, we want to choose g so that the resulting problem no longer contains jumps which grow exponentially; the choice of g should control the jump along the contour $\gamma \cup \gamma^*$. We will assume that initially g is cut on a single symmetric interval $(\ell \cup \ell^*) \subseteq (\gamma \cup \gamma^*)$.

The new jumps of $N(z)$ take the form:
for $z \in \Sigma \backslash (\ell \cup \ell^*)$,

$$V_N = \begin{pmatrix} \sharp & \sharp e^{-\frac{i}{\epsilon}(\theta_i - 2g)} \\ \sharp e^{\frac{i}{\epsilon}(\theta_i - 2g)} & \sharp \end{pmatrix} \tag{5.17}$$

For $z \in \ell \cup \ell^*$ the jump is,

$$V_N = \begin{pmatrix} 0 & \sharp e^{-\frac{i}{\epsilon}(\theta - g_+ - g_-)} \\ \sharp e^{\frac{i}{\epsilon}(\theta - g_+ - g_-)} & 0 \end{pmatrix} \tag{5.18}$$

here the $\sharp$'s indicate the values of the corresponding entries in the jump matrix $V_M(z)$.

Examining these jumps, we demand that g satisfy the following set of conditions:

$$\begin{array}{ll} 1. & g \text{ is analytic in } \mathbb{C}\backslash\ell. \\ 2. & \theta(z) - g_+(z) - g_-(z) = 0, \text{ for } z \in \ell. \\ 3. & g(z) = \mathcal{O}(1), \text{ as } |z| \to \infty. \\ 4. & \mathrm{Im}(g_+ - g_-) = \mathrm{Im}(\theta - 2g_-) = 0, \text{for } z \in \ell. \\ 5. & \mathrm{Im}(\theta - 2g) > 0, \text{ for } z \in \gamma\backslash\ell. \end{array} \tag{5.19}$$

5.2.1 A partial formula for $g(z)$ and existence of the contour ℓ

Denote the endpoints of the symmetric contour ℓ by α and $\bar{\alpha}$, $\alpha = \alpha_r + \alpha_i$, with $\alpha_i > 0$. For any fixed contour ℓ, the Sokhotskyi-Plemelj formula give the unique solution of the RHP specified in conditions (1) - (3) above as the integral

$$g(z) = \frac{S(z)}{2\pi i} \int_{\ell \cup \ell^*} \frac{\theta(s)}{S_+(s)} \frac{ds}{s - z}, \tag{5.20}$$

which can be evaluated via residue calculus

$$g(z) = \frac{1}{2}\theta(z) - S(z)\left[t(z-\alpha_r) - (x-L)\right], \tag{5.21}$$

where we have defined

$$S(z) = \sqrt{z-\alpha}\sqrt{z-\bar{\alpha}} \tag{5.22}$$

to be branched along ℓ and normalized so that $S(z) \sim z$ for $z \to \infty$.

By introducing the g-function, we have modified the exponential phases θ_i. In their place we now have new functions ϕ_i defined by the equations

$$\phi_k(z) = \theta_k(z) - 2g(z). \tag{5.23}$$

In particular, using the form of the single-cut g-function in (5.21) the main phase function ϕ_0 is given by

$$\phi_0(z) = 2S(z)\left[t(z-\alpha_r) + (x-L)\right]. \tag{5.24}$$

These new phase functions ϕ_i determine the stationary phase points z_0 and z_1 where the contours Γ_i meet the real axis in the same way that the θ_i phases chose these points for x outside the support. For now we must leave these points ambiguously defined. Once the g-function is completely specified we must come back and describe the points z_0 and z_1 more carefully.

The last two conditions that must be satisfied by g are truly conditions on the function $\phi_0(z)$. Conditions (4) and (5) above, described in terms of the new phase function ϕ_0, become

4. $\mathrm{Im}(\phi_{0-}) = 0$ for $z \in \ell \cup \ell^*$,
5. $\mathrm{Im}(\phi_0) > 0$ for $z \in \gamma \backslash \ell$.

The problem of choosing a g-function is now reduced to choosing the contours ℓ and γ; the function g as defined in (5.21) is the unique function that satisfies properties (1) - (3), but its definition is incomplete in that we have not yet specified the contour ℓ. We must now ask if there exists contours ℓ and γ such that properties (4) and (5) can be satisfied. The answer to this question is a consequence of the following proposition.

Proposition 15. *Fix $b \in \mathbb{R}$ and $\alpha = \alpha_r + i\alpha_i \in \mathbb{C}^+$, and consider the function*

$$\phi(z) = S(z)[t(z-\alpha_r) + b]$$

where $S(z)$ is defined in (5.22)*. The following facts about the level set $L(t) = \{z \,:\, \mathrm{Im}(\phi) = 0\}$ and the function $\mathrm{Im}(\phi(z))$ hold true.*

1. *$L(0) = \mathbb{R} \cup \{z = \alpha_r + i\tau \;:\; -\alpha_i \le \tau \le \alpha_i\}$*
2. *For $0 < t < T_b < \infty$ the level set $L(t)$ consist of the union of the real axis, a simple contour connecting α and $\bar{\alpha}$ and a third contour emerging from infinity and crossing*

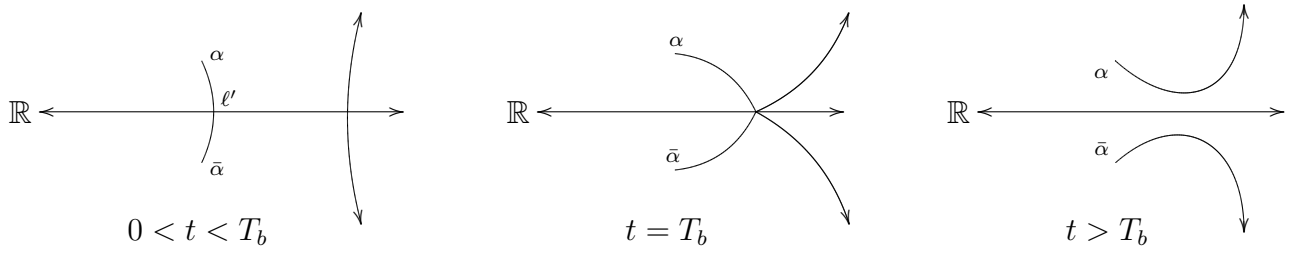

FIGURE 5.4. The level set $\mathrm{Im}(\phi) = 0$ before, at, and after the breaking time T_b for $b < 0$.

the real axis at a point $p_1 = -\frac{b}{2t} - \alpha_r + \mathcal{O}(t/b)$. *The upper time limit* T_b, *which we call the breaking time, is given by*

$$T_b = \frac{|b|}{\sqrt{8}\alpha_i}. \tag{5.25}$$

T_b *is a true upper limit in the sense that for* $t > T_b$, $L(t)$ *does not contain a contour connecting* α *and* $\bar{\alpha}$. *The topological structure of the set* $\mathrm{Im}(\phi)$ *before, at, and after the breaking time* T_b *is depicted in Figure 5.4*

3. *For* $0 < t < T_b$, *fix the branch cut of* ϕ *to lie on the trajectory of* $L(t)$ *connecting* α *and* $\bar{\alpha}$. *The sign structure of* $\mathrm{Im}(\phi(z;t))$ *is then given in Figure 5.5*

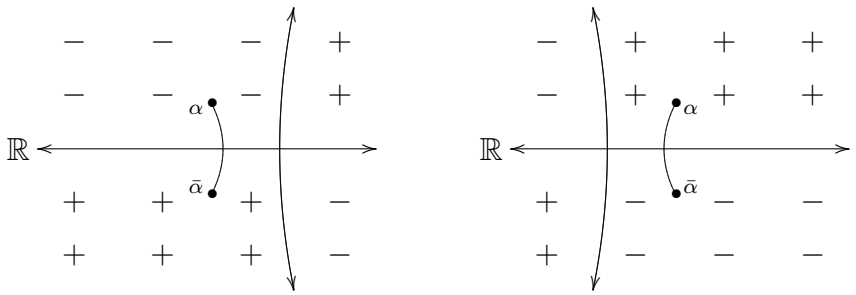

FIGURE 5.5. The sign structure of $\mathrm{Im}(\phi)$ with $b < 0$ (left) and $b > 0$ (right) for $0 < t < T_b$

We will defer the proof of this proposition to the end of the section so as to focus on its consequences for choosing the endpoints α and $\bar{\alpha}$ and the corresponding contours ℓ and γ. The phase function $\phi_0 = 2S(z)[t(z - \alpha_r) + (x - L)]$ is exactly of the type considered in Prop. 15 with $b = (x - L) < 0$. So what does the proposition buy us? The first two parts of Prop. 15 tell us that, up to a positive time T condition (4) in (5.19) is satisfied regardless of our choice of α. This is good in the sense that it tells us that our putative g-function satisfies the reality condition, but bad in that it gets us no closer to a unique choice of the endpoint α (and thus the contours γ and ℓ). The situation gets much clearer

when we consider the final condition on g, that $\mathrm{Im}(\phi_0(z;t)) > 0,\ z \in \gamma\backslash\ell$. The last fact in Prop. 15 implies that in any suitably small neighborhood B_α of the point α, the opposite inequality holds: $z \in \mathcal{B}_\alpha\backslash\ell, \Rightarrow \mathrm{Im}(\phi_0(z;t)) < 0$. The only possible way to satisfy condition (5) is to set $\alpha = iq$, in which case $\gamma = \ell$ and the condition is vacuously satisfied.

5.2.2 The one-cut g-function, final formulation

The identification of the endpoint $\alpha = iq$ completes the definition of the function $g(z)$. Though all of the details lie in what came before, we collect those facts here: for each (x,t) in the space-time wedge $\{\ (x,t) \mid 0 \le x < L,\ 0 \le t < T_1(x)\}$ we define the function

$$g(z) = \frac{1}{2}\theta - \nu(z)\left[tz + (x-L)\right], \tag{5.26}$$

which has the large z expansion,

$$g(z) = -\frac{1}{2}q^2 t + \mathcal{O}(1/z); \tag{5.27}$$

the associated phase functions ϕ_k, defined by

$$\phi_k(z) = \theta_k - 2g(z) = 2\nu(z)\left[tz + (x + (2k-1)L)\right]; \tag{5.28}$$

and the curve $T_1(x)$, called the first breaking curve, is given by

$$T_1(x) = -\frac{1}{2\sqrt{2}q}(x-L). \tag{5.29}$$

The function $\nu(z) = \sqrt{z^2+q^2}$ used in the definition of $g(z)$ is branched on the unique contour $\gamma \cup \gamma^*$ connecting $\pm iq$ along which $\mathrm{Im}\,\phi_0 = 0$, the existence of this contour is guaranteed by Prop. 15, and normalized such that $g(z) \sim z$ for z large. Finally, we observe that the function $g(z)$ above is unique in the sense that it is the only function that satisfies each of the conditions listed in (5.19) and that T_1 is optimal in the sense that for $t > T_1$, there does not exist a function with a single branch cut that can satisfy the conditions in (5.19).

The RHP for $N(z)$

Now that the g-function has been completely determined we can write down the resulting RHP for the unknown $N(z)$, defined by (5.16), explicitly. The set of jump contours is not changed by the introduction of $g(z)$:

$$\Gamma_N = \Gamma_M = (-\infty, z_0) \cup \gamma \cup \gamma^* \bigcup_{i=0}^{3} (\Gamma_i \cup \Gamma_i^*). \tag{5.30}$$

RHP for $N(z)$: For $(x,t) \in \mathcal{G}_1 = \{(x,t) \mid 0 \le x < L,\ 0 < t < T_1(x)\}$ with $T_1(x)$ defined in (5.25) find a 2×2 matrix values function $N(z)$ such that

- $N(z)$ is analytic in $\mathbb{C}\backslash\Gamma_N$ and takes continuous boundary values $N_\pm(z)$ as z approaches the contour away from the endpoints $z = \pm iq$.
- $N(z) = I + \mathcal{O}(1/z)$ as $|z| \to \infty$.
- For $z \in \Gamma_N$ the boundary values satisfy the relationship $N_+ = N_- V_N$ where

$$V_N = \begin{cases} e^{g\,\mathrm{ad}\,\sigma_3} V_M & : \quad z \in \Gamma_N \backslash (\gamma \cup \gamma^*) \\ \begin{pmatrix} 0 & -w^{-1}(z) \\ w(z) & 0 \end{pmatrix} & : \quad z \in \gamma \\ \begin{pmatrix} 0 & w(z) \\ -w^{-1}(z) & 0 \end{pmatrix} & : \quad z \in \gamma^* \end{cases} \tag{5.31}$$

  where V_M is defined in (5.14).
- Near $z = iq$, the unknown $N(z)$ grows at the prescribed rate:

$$N(z) = \mathcal{O}\begin{pmatrix} 1 & |z - iq|^{-1/2} \\ 1 & |z - iq|^{-1/2} \end{pmatrix} \tag{5.32}$$

- Near $z = -iq$, the growth of $N(z)$ is prescribed by:

$$N(z) = \mathcal{O}\begin{pmatrix} |z + iq|^{-1/2} & 1 \\ |z + iq|^{-1/2} & 1 \end{pmatrix} \tag{5.33}$$

In the above we have made use of the Lie algebra notation

$$\lambda^{\mathrm{ad}\,\sigma_3} v = \lambda^{\sigma_3} v \lambda^{-\sigma_3}$$

defined for any nonzero scalar λ.

5.2.3 The endpoints $z_0,\ \xi_0,$ and z_1

The RHP problem for N is incomplete in that the collection of contours Γ_M have not yet been explicitly defined. As is usual in these problems, we allow some flexibility so that we can deform contours as needed later. However, the points $\{z_1, z_0, \xi_0\}$ where the contours come back to the real axis must be rigidly defined. Our factorization of the initial jump matrix defining $M^{(1)}(z)$ was predicated on the existence of stationary points where the exponential phases transition from regions of exponential growth to regions of exponential decay. Up to this point, we have tacitly assumed the existence of these stationary points without explicitly defining them. We will do so now.

The new phase functions ϕ_k which appear in the off diagonal entries of the jump matrix V_N are

$$\phi_0 = 2\nu(z)\left[tz + (x - L)\right] \tag{5.34}$$

$$\phi_1 = 2\nu(z)\left[tz + (x + L)\right]. \tag{5.35}$$

using the results of Prop. 15 we can exactly characterize the regions of exponential growth and decay of these functions for $(x,t) \in \mathcal{G}_1$, which we summarize in Figure 5.6.

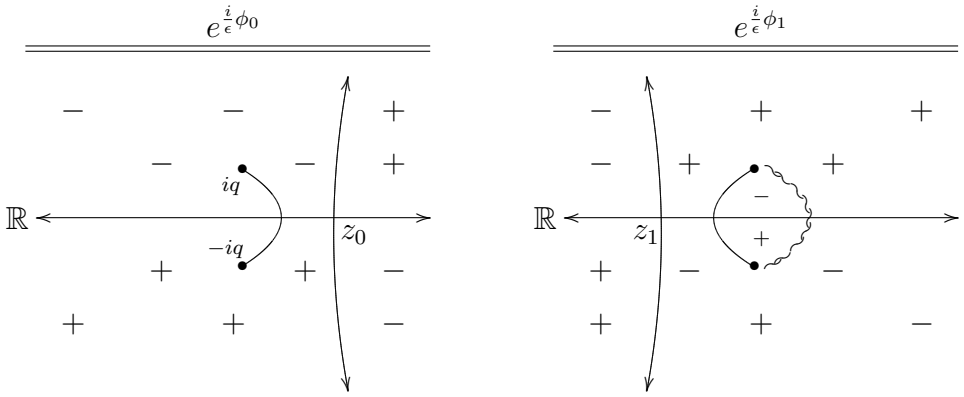

FIGURE 5.6. Regions of growth (+) and decay (-) for the exponential functions $e^{\frac{i}{\epsilon}\phi_0}$ and $e^{\frac{i}{\epsilon}\phi_1}$. The positions of the stationary phase points z_0 and z_1 are labeled on each figure.

Comparing the regions of growth and decay in Figure 5.6 to the contours Γ_N we see that points z_k must be the points along the real axis at which the level sets $\mathcal{L}_k = \{z \mid \mathrm{Im}(\phi_k(z)) = 0\}$ have trajectories emerging that connect to $i\infty$ and $-i\infty$. In the proof of Prop. 15 which appears at the end of this section the values of these stationary points are explicitly calculated

$$z_{0,1} = -\frac{x \mp L}{4t}\left[1 + \sqrt{1 - \frac{8t^2q^2}{(x \mp L)^2}}\right] \tag{5.36}$$

where the minus in the above formula corresponds to z_0 and the plus to z_1.

Remark 4. Note that for t near 0 these endpoints agree to leading order with the critical points z_0 and z_1 computed in Section 3.4 for x outside the support. Indeed, these points have convergent asymptotic expansions on every compact subset of $\mathcal{G}_1$ given by

$$z_{0,1} = -\frac{x \mp L}{2t} + \mathcal{O}\left(\frac{t}{|x \mp L|}\right). \tag{5.37}$$

The leading order term in (5.37) are the exact values of the of z_0 and z_1 calculated for x outside the support by formula (3.28).

The point ξ_0 is defined in the transformation $m(z) \mapsto M(z)$ to be the point where the contour γ meets the real axis. This is of course the other critical value of the phase ϕ_0 which is given by

$$\xi_0 = -\frac{x - L}{4t}\left[1 - \sqrt{1 - \frac{8t^2q^2}{(x - L)^2}}\right] \tag{5.38}$$

and has the asymptotic expansion for $t \ll 1$ and $x < L$

$$\xi_0 = -\frac{tq^2}{x-L} + \mathcal{O}\left(\frac{t^2}{|x-L|^2}\right). \tag{5.39}$$

We now consider the RHP defining $N(z)$ for contours Γ_i which connect to the real axis at the points z_0, z_1 and ξ_0 and stay in regions where the associated exponentials are decaying.

The new RHP for $N(z)$ has a well defined limit as $\epsilon \to 0$. Along the contours Γ_i, the jump matrices are exponentially close to identity at any finite distance from their endpoints and so these contours should not provide any meaningful contribution to the asymptotic description of the solution except near the endpoints. Along the contours $(-\infty, z_0)$ and $\gamma \cup \gamma^*$ the jump matrices are not small, but are independent of the dispersion parameter ϵ. Thus it is reasonable to expect that the solution of the RHP whose jumps includes only those which do not approach identity as $\epsilon \to 0$ should provide the leading order asymptotics for $N(z)$, and thus for the solution $q(x,t)$ of NLS by following the explicit transformations that lead from the original RHP for $m(z)$ to $N(z)$.

5.2.4 Proof of Proposition 15

There are several routes to proving the statements in Proposition 15. Because we have an explicit, relatively simple phase function $\phi(z;t)$ we choose to give a geometric hands-on proof of Prop. 15. The proof relies heavily on local analysis of ϕ near its critical points and on the restricted structure of the level sets of harmonic functions.

To verify the first statement, observe that $\phi(z;t=0) = bS(z)$ and thus $\mathrm{Im}(\phi) = \mathrm{Im}(S(z))$. By symmetry the result is immediate. For $t > 0$ we begin by observing that $\phi(z;t)$ is analytic away from a finite branch cut connecting the branch points α and $\bar{\alpha}$. However, the location of the branch cut has no effect on the shape of the level set $L(t)$. If λ and λ' are two different choices of the branch cut connecting α to $\bar{\alpha}$ and $\lambda \circ \lambda'$ the closed oriented loop formed by following λ and then $(\lambda')^{-1}$ the function $\phi(z)$ simply changes sign in either $\mathrm{Ext}(\lambda \circ \lambda')$ or $\mathrm{Int}(\lambda \circ \lambda')$; in either case the zero level set $L(t)$ is unaffected. For the purpose of determining the structure of $L(t)$ we will fix the branch cut to be the vertical contour $\lambda = \{z = \bar{\alpha}_i s + (\alpha - \bar{\alpha}(1-s) \,:\, 0 \le s \le 1\}$. Once the structure is determined we can move the branch cut as we please.

In what follows we will need to understand how the critical points p such that $\phi_z(p) = 0$ behave as function of t. Taking the derivative

$$\frac{d\phi}{dz} = \frac{1}{S(z)}\left[2tz^2 + (b - 4t\alpha_r)z + t|\alpha|^2 + t\alpha_r^2 - b\alpha_r\right],$$

we see that $\phi(z,t)$ has at most two critical points in the complex plane given by

$$\frac{-b + \alpha_r \pm \sqrt{b^2 - 8t^2\alpha_i^2}}{4t}.$$

The location of these critical points depends on the value of t relative to the breaking time,

$T_b = \frac{|b|}{\sqrt{8}\alpha_i}$:

(1) For $0 < t < T_b$ there are two real-valued critical points p_0 and p_1 which have the following asymptotic behavior for $t/b \ll 1$

$$p_0 = \alpha_r - \alpha_i^2 \frac{t}{b} + \mathcal{O}\left(\frac{t^2}{b^2}\right) \tag{5.40}$$

$$p_1 = -\frac{b}{2t} + \alpha_r + \mathcal{O}\left(\frac{t}{b}\right). \tag{5.41}$$

Near each of these points $\phi(z) = \phi(p_i) + c(z-p_i)^2 + \mathcal{O}((z-p_i)^2)$ for a real constant $c \neq 0$.

(2) For $t > T_b$ the zeros become a complex-conjugate pair p and $\bar{p}$. Though obvious, we stress that near any point on the real axis the local behavior of ϕ is linear for $t > T_b$.

(3) For $t = T_b$ there is exactly one real critical point

$$p_c = -\operatorname{sgn}(b)\sqrt{2}\alpha_i + \alpha_r \tag{5.42}$$

at which $\phi(z) = \phi(p_c) + c(z-p_c)^3 + \mathcal{O}((z-p_c)^4)$ for a real constant $c \neq 0$

The remainder of the proof uses the following basic fact about the structure of the level sets of analytic functions which we record in the following lemma.

Lemma 16. *Let $\mathcal{D}$ be a connected open subset of $\mathbb{C}$. Suppose $u(z)$ is a non-constant real-valued harmonic function on $\mathcal{D}$. Then for any $c \in \mathbb{R}$, the level set $L_c = \{z \in \mathcal{D} : u(z) = c\}$ consists of a collection of smooth arcs which do not form closed loops and may terminate only at $z \in \partial\mathcal{D}$. At any point $p \in \mathcal{D}$ where two or more arcs of the level set intersect the derivative must vanish, i.e. $u_z(p) = 0$.*

In our case, the harmonic function is $u(z) = \operatorname{Im}(\phi(z))$ which is harmonic in the set $\mathbb{C}\backslash\lambda$. To understand the structure of $L(t)$ we need to understand the local behavior near the branch points α and $\bar{\alpha}$, ∞, and the critical points p_0 and p_1:

Local structure of $L(t)$ for $0 < t < T_b$

- Near $z = \alpha$, $\phi(z) = \rho e^{i\tau}(z-\alpha)^{1/2}\left[1 + \mathcal{O}((z-\alpha))\right]$ for real constants ρ and τ. A single trajectory of $L(t)$ emerges from α at an angle of -2τ. By symmetry, near $z = \bar{\alpha}$ a single trajectory emanates from $\bar{\alpha}$ at an angle of 2τ.

- Near $z = \infty$, $\phi(z) = tz^2 + \mathcal{O}(z)$. It follows that four trajectories of $L(t)$ approach infinity and that the trajectories are asymptotically parallel to the coordinate axes.

- Near $z = p_i$, $\phi(z) = \phi(p_i) + c(z-p_i)^2 + \mathcal{O}((z-p_i)^2)$ for a nonzero real constant c. Four trajectories of $L(t)$ pass through p_i at angles $k\pi/2$ for $k = 0, 1, 2, 3$.

Global structure of $L(t)$ for $0 < t < T_b$
Using the local structure analysis and the symmetry $\overline{\phi(z;t)} = \phi(\bar{z};t)$ the structure of the level set $L(t)$ for $0 < t < T_b$ is given in Figure 5.7. There are exactly two possibilities for trajectory 1 leaving α: either it leaves $\mathbb{C}^+$ through the real axis at one of the p_i or it approaches ∞. Suppose that this trajectory approaches ∞. Then trajectories 2 and 3 cannot leave $\mathbb{C}^+$ at ∞ nor can they terminate at α, their only option is to form a closed loop. By Lemma 16, this loop must enclose α to be allowable. However, if the loop encloses α, it intersects trajectory 1 connecting α to infinity at some point p^*. Because trajectories can only cross at critical points we rule this possibility out. Thus, trajectory 1 leaving α must leave $\mathbb{C}^+$ at p_1 or p_0. By symmetry the trajectory leaving $\bar{\alpha}$ leaves $\mathbb{C}^-$ at the same point. These paths form a symmetric contour $\ell \cup \ell^* \subset L(t)$ connecting α and α^*.

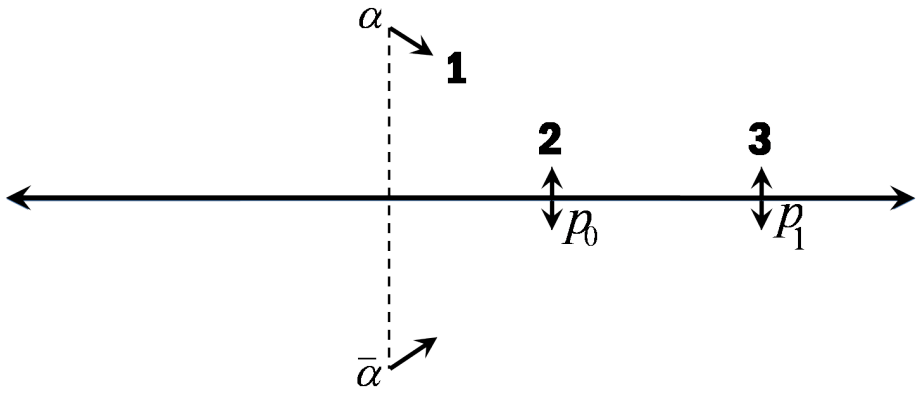

FIGURE 5.7. The local structure of the level set $L(t) = \{z : \mathrm{Im}(\phi(z;t)) = 0\}$ for $0 < t < T_b$ and $b < 0$. The trajectories of $L(t)$ leaving α, p_0 and p_1 into $\mathbb{C}^+$ are labeled 1, 2, and 3 respectively. They may terminate only by connecting or approaching ∞ in $\mathbb{C}^+$.

We now want to prove that trajectory 1 leaves $\mathbb{C}^+$ at p_0. Suppose instead that trajectory 1 leaves $\mathbb{C}^+$ through p_1. Then necessarily, trajectory 3 leaves $\mathbb{C}^+$ at ∞. However, observe that for $t > 0$ and $\eta \in (0, \alpha_i)$, $|\mathrm{Im}(\phi(\alpha_r + i\eta; t)| = t\eta|\alpha_i^2 - \eta^2|^{1/2} > 0$, thus trajectories of $L(t)$ cannot cross the vertical line segment connecting α_r and α. The trajectory connecting α to p_1, the real axis, and the segment connecting α_r to α enclose all possible paths leading from trajectory 2 to ∞. Because these paths cannot cross we are lead to a contradiction. It follows that for each $0 < t < T_b$ the connection $\ell \cup \ell^*$ passes through p_0 while a second contour passing through p_1 connects $i\infty$ to $-i\infty$. See Figure 5.4.

Structure of $L(t)$ for $t = T_b$
For $t = T_b$, the local structure at α, $\bar{\alpha}$, and ∞ are unchanged. On the real axis there is exactly one critical point p_c whose value is given by (5.42). Near $z = p_c$, $\phi(z) = \phi(p_c) + c(z - p_c)^3 + \mathcal{O}((z - p_c)^4)$ for a nonzero real constant c. Thus six trajectories of

$L(t)$ leave p_c: two along the real line and four others at angles of $\pi/3$ from the real axis. Repeating the arguments above one sees that $L(T_b)$ consists of the real axis and four contours emerging from α, $\bar{\alpha}$, $i\infty$, and $-i\infty$ which all meet at the cubic critical point $z = p_c$.

Structure of $L(t)$ for $t > T_b$
For $t > T_b$, the local structure at α, $\bar{\alpha}$, and ∞ are unchanged. However, for each $p \in \mathbb{R}$, the local behavior of $\phi(z;t)$ is linear. Thus, a trajectory beginning in $\mathbb{C}^+$ may only exit at $i\infty$. It follows that the structure of $L(t)$ consist of the real axis plus two symmetric contours connecting α and $\bar{\alpha}$ to $i\infty$ and $-i\infty$ respectively.

Finally, we must determine the sign structure of $\mathrm{Im}(\phi(z;t))$ for $0 < t < T_b$. To this point, we have fixed the branch cut of $\phi(z;t)$ to lie on the vertical line connecting α to $\bar{\alpha}$. We now move the branch cut to lie on the contour $\ell \cup \ell^* \subset L(t)$. With this choice of branch cut, $\mathrm{sgn}(\mathrm{Im}(\phi))$ changes only across the set $L(t)\backslash(\ell \cup \ell^*)$. We can then completely determine the sign in the complex plane by examining the large z behavior of $S(z)$,

$$\phi(z;t) = tz^2[1 + \mathcal{O}(1/z)],$$

and simply continue the sign inwards changing the sign only when we cross $L(t)\backslash(\ell \cup \ell^*)$. The sign structure for $b > 0$ and $b < 0$ are given in Figure 5.5. □

5.3 $N(z) \mapsto Q(z)$: Constant Jump Reduction

Before we begin the approximation procedure we would like to be in a position where the limiting jump matrices are constant function of z. Motivated by this desire, consider the following pair of scalar RHPs:

1. **The δ RHP** Find a scalar function $\delta(z)$ such that
   - δ is analytic in $\mathbb{C}\backslash(-\infty, z_0)$.
   - As $|z| \to \infty$,
   $$\delta(z) = 1 + \mathcal{O}(1/z). \tag{5.43}$$
   - For $z \in (-\infty, z_0)$ the boundary values of δ satisfy the relationship
   $$\delta_+(z)/\delta_-(z) = \begin{cases} (1 + |r_0(z)|^2)^2 & : \quad z \in (-\infty, z_1) \\ (1 + |r_0(z)|^2) & : \quad z \in (z_1, z_0) \end{cases} \tag{5.44}$$

2. **The Szegő RHP** Find a scalar function $s(z)$ such that
   - $s(z)$ is analytic in $\mathbb{C}\backslash(\gamma \cup \gamma^*)$.
   - $s(z) = s_\infty + \mathcal{O}(1/z)$ as $|z| \to \infty$ where s_∞ is a bounded constant.
   - For $z \in \gamma \cup \gamma^*$ the boundary values of $s(z)$ satisfy the relationship
   $$s_+(z)s_-(z) = \begin{cases} w(z)\delta^{-2}(z) & : \quad z \in \gamma \\ -(w(z))^{-1}\delta^{-2}(z) & : \quad z \in \gamma^* \end{cases} \tag{5.45}$$

If we can find solutions δ and s to these scalar RHPs (which we can easily do using the Plamelj formula) then we reduce then making the following change of variables

$$N(z) = Q(z)s^{\sigma_3}(z)\delta^{\sigma_3}(z) \tag{5.46}$$

it is easy to check that on the contours $(-\infty, z_0)$ and $\gamma \cup \gamma^*$, the contours whose jumps remain large in the limit as $\epsilon \to 0$, the new jumps of the unknown $Q(z)$ are constant in z. Usually at this point we would write down the resulting RHP for $Q(z)$. However, we first must take the time to explicitly construct the functions $s(z)$ and $\delta(z)$ and explore their properties.

The RHP for $\delta(z)$ that appears in (5.44) is the same type that appeared for z outside the support of the initial data. The solution and its properties are described in detail in Section 4.2. For our convenience we recopy the formula for the solution $\delta(z)$ from the previous chapter,

$$\delta(z) = \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{z_1} \frac{\log(1+|r_0(\lambda)|^2)}{\lambda - z}\,\mathrm{d}\lambda + \frac{1}{2\pi i}\int_{-\infty}^{z_0} \frac{\log(1+|r_0(\lambda)|^2)}{\lambda - z}\,\mathrm{d}\lambda\right]. \tag{5.47}$$

We observe that the formula here for $\delta(z)$ is identical to the formula used in the previous chapter for $x > L$ outside the support, with one subtle difference. The points z_1 and z_0 are now the critical points of the phase functions $\phi_1(z)$ and $\phi_0(z)$ respectively, defined by (5.36). This has no effect on the properties derived in Section 4.2 other than to move the location of the points z_0 and z_1.

The RHP for $s(z)$ was not present in the previous chapter. Its appearance is a consequence of the introduction of the g-function. The solution to the RHP for $s(z)$ is built in a manner analogous to the problem for δ, but it is useful to first make a transformation that removes the square root behavior of the jump (5.45) near the endpoints $\pm iq$. To do this, we introduce the function $a(z)$ defined by,

$$a(z) = \left(\frac{z - iq}{z + iq}\right)^{1/4} \tag{5.48}$$

where the root is understood to be branched along the contour $\gamma \cup \gamma^*$ and approaches unity at infinity. We use $a(z)$ to define a new unknown $\tau(z)$,

$$s(z) = a(z)\tau(z).$$

Because $a(z)$ approaches unity at infinity the asymptotic condition on τ is identical to the condition on s. The jump relation for $\tau(z)$ becomes

$$\tau_+\tau_- = \frac{s_+ s_-}{-ia_+^2} = v(z) = \begin{cases} \frac{2(z+iq)}{q}\delta(z)^{-2} & : \quad z \in \gamma \\ \frac{q}{2(z-iq)}\delta(z)^{-2} & : \quad z \in \gamma^* \end{cases}.$$

This problem is solved by applying the Plamelj formula; the resulting Szegő function $s(z)$ is given by

$$s(z) = a(z) \exp\left(\frac{\nu(z)}{2\pi i} \int_{\gamma \cup \gamma^*} \frac{\log(v(\lambda))}{\nu_+(\lambda)} \frac{d\lambda}{\lambda - z} \right). \tag{5.49}$$

it is not difficult to show from (5.49) that $s(z)$ is a bounded nonzero function everywhere in the plane except the endpoints $z = \pm iq$ where its local behavior is, up to a constant, identical to the behavior of $a(z)$.

The limiting value, s_∞, of the function $s(z)$ can be found by geometrically expanding the integrand in (5.49) for z large. The constant term in this expansion is

$$s_\infty = \exp\left(-\frac{\nu(z)}{2\pi i} \int_{\gamma \cup \gamma^*} \frac{\log(v(\lambda))}{\nu_+(\lambda)} d\lambda \right) = e^{-\frac{i\pi}{4}}, \tag{5.50}$$

where the last equality follows from a contour deformation calculation.

We now write down the resulting RHP for $Q(z)$. Let Γ_Q represent the totality of contours on which $Q(z)$ fails to be analytic. The transformation (5.46) mapping $N(z)$ to $Q(z)$ does not introduce any new contours so $\Gamma_Q = \Gamma_N$. Recall that Γ_N is defined by (5.30). Using the properties of the functions $s(z)$ and $\delta(z)$ we have just derived, the resulting RHP for the unknown $Q(z)$ takes the following form.

RHP for $Q(z)$:

Find a 2×2 matrix valued function $Q(z)$ that satisfies the following properties.

- $Q(z)$ is analytic in $\mathbb{C} \backslash \Gamma_Q$ and takes continuous boundary values as z approaches the contours away from $z = \pm iq$.
- As $|z| \to \infty$,
$$Q(z) = I + \mathcal{O}(1/z). \tag{5.51}$$
- The boundary values of $Q(z)$ satisfy the jump relationship $Q_+(z) = Q_-(z) V_Q$ where
$$V_Q = \begin{cases} \{\delta(z) s(z)\}^{\operatorname{ad} \sigma_3} V_N & : \quad z \in \Gamma_Q \backslash (\gamma \cup \gamma^*) \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & : \quad z \in \gamma \cup \gamma^* \end{cases} \tag{5.52}$$
- Near $z = \pm iq$, $Q(z)$ admits 1/4-root singularities in each entry.

As we now see, the change of variable to $Q(z)$ results in a new RHP whose jumps are either constant or are exponentially near identity. In addition, the local behavior of $Q(z)$ near the endpoints $z = \pm iq$ has changed; the square root singularities allowed for $N(z)$ at $\pm iq$ have become 1/4-root singularities for the new unknown $Q(z)$. This is a technical nicety that allows us to easily conclude that the solution $Q(z)$ is unique.

5.4 Building a Global Approximation and Error Estimates

As in the previous chapter we now want to build a global approximation to the RHP for $Q(z)$ composed of an outer parametrix $O(z)$ which considers only the limiting jump contours and local parametrices near the endpoints z_0 and z_1 where the limiting jump matrices are no longer uniformly close to the full problem. Explicitly, we build a piecewise defined matrix

$$A(z) = \begin{cases} A_{z_0}(z) & : \quad z \in \mathcal{U}_0 \\ A_{z_1}(z) & : \quad z \in \mathcal{U}_1 \\ O(z) & : \quad \text{elsewhere} \end{cases} \tag{5.53}$$

where the sets $\mathcal{U}(z_i)$ are sufficiently small fixed size open discs centered at the points z_i.

5.4.1 The outer parametrix

Examining the jump relations for $Q(z)$ given by (5.52), away from the points z_1 and z_0 and the contour $\gamma \cup \gamma^*$ the jump matrices are exponentially near identity as $\epsilon \to 0$. Our outer parametrix is built by simply ignoring these jump matrices, or if one likes, by taking the pointwise limits of the jump matrix V_Q as $\epsilon \to 0$. The resulting problem for $O(z)$ takes the following form.

RHP for outer parametrix, $O(z)$: Find a 2×2 matrix valued function $O(z)$ that satisfies the following properties.

- $O(z)$ is analytic in $\mathbb{C} \backslash (\gamma \cup \gamma^*)$ and takes continuous boundary values as z approaches the contour away from the points $z = \pm iq$ where it admits 1/4-root singularities.
- As $z \to \infty$,
$$O(z) = I + \mathcal{O}(1/z) \tag{5.54}$$
- For $z \in \gamma \cup \gamma^*$, the boundary values of $O(z)$ satisfy the jump relationship $O_+(z) = O_-(z)V_O$ where
$$V_O = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{5.55}$$

This RHP shows up in a number of different problems in integrable PDE and random matrices. One builds the solution by diagonalizing the jump matrix and solving the resulting (simple) scalar problem. One finds that the solution is given by

$$O(z) = \begin{pmatrix} \frac{a(z)+a(z)^{-1}}{2} & -\frac{a(z)-a(z)^{-1}}{2i} \\ \frac{a(z)-a(z)^{-1}}{2i} & \frac{a(z)+a(z)^{-1}}{2} \end{pmatrix} \tag{5.56}$$

where the function $a(z)$ is the ration of 1/4-roots given before by formula (5.48).

Later, when we seek to recover the solution to NLS, we will need the large z behavior of the outer model $O(z)$. Expanding (5.56) for z large we have

$$O(z) = I + \begin{pmatrix} 0 & q/2 \\ -q/2 & 0 \end{pmatrix} \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right). \tag{5.57}$$

Note that the there is no (x,t) dependence in the first moment; q is the constant amplitude of the initial data.

5.4.2 Local problems near $z = z_0$ and $z = z_1$

Near z_0 and z_1 the jump matrices are no longer uniformly exponentially near identity. For this reason we need to introduce local model problems near each of these points which account for the local behavior of the jump matrices. Examining the local behavior of the jumps of $Q(z)$ defined in (5.52) we observe that the behavior is nearly identical to the local jumps we previously built models for in Chapter 4 for x outside the support (cf. (4.19) and Figures 4.5 and 4.7). The only differences are that in $Q(z)$ the phase functions appearing in the exponentials have been modulated by the introduction of the g-function and we have additional factors of $s(z)$ appearing in the off-diagonal entries of the jumps of $Q(z)$. These difference, however, are only cosmetic. Locally, near the points $z = z_0$ and $z = z_1$, the modulated phase functions are stationary, i.e.

$$\phi_k(z) = \phi_k(z_k) + \frac{1}{2}\phi_k''(z_k)(z - z_k)^2 + \dots, \quad \phi_k''(z_k) \neq 0, \quad k = 0, 1. \tag{5.58}$$

This behavior is identical to that of the phase functions θ_0 and θ_1 appearing in the local jumps for $x > L$. Additionally, the Szegő function, $s(z)$, is analytic in neighborhoods of z_1 and z_0 and is thus well approximated by its value at these points.

The close similarity of the jump matrices leads us to believe that the correct local parametrices near z_0 and z_1 are identical to those previous introduced for $x > L$. The models will again be built from the solution $\Psi_{P.C.}$, the solution of the parabolic cylinder RHP. The minor differences we will see stem from the modulation of the phase functions, the corresponding change in the locations of the stationary phase points, and the presence of the factors of $s(z_k)$ which easily factor out. Without repeating the details of the calculations, or their motivation we will simply record the local model problems and their solutions. The reader can find all the necessary details in sections 4.2.2 and 4.2.3.

Model near $z = z_0$

We seek a function $A_0(z)$ such that for $z \in \mathcal{U}_0$, $A_0(z)$ satisfies the jump relationship $A_{0+}(z) = A_{0-}(z)V_{z_0}(z)$, where

$$V_{z_0} = \begin{cases} \begin{pmatrix} 1 & 0 \\ r_0(z_0)s^{-2}(z_0)(z-z_0)^{-2i\kappa_0}\delta_0^{-2}e^{\frac{i}{\epsilon}\phi_0(z)} & 1 \end{pmatrix} & : \quad z \in \Gamma_0 \cap \mathcal{U}_0 \\ \begin{pmatrix} 1 & \frac{\bar{r}_0(z_0)}{1+|r_0(z_0)|^2}s^2(z_0)(z-z_0)^{2i\kappa_0}\delta_0^2 e^{-\frac{i}{\epsilon}\phi_0(z)} \\ 0 & 1 \end{pmatrix} & : \quad z \in \Gamma_3 \cap \mathcal{U}_0 \\ \begin{pmatrix} 1 & \bar{r}_0(z_0)s^2(z_0)(z-z_0)^{2i\kappa_0}\delta_0^2 e^{-\frac{i}{\epsilon}\phi_0(z)} \\ 0 & 1 \end{pmatrix} & : \quad z \in \Gamma_0^* \cap \mathcal{U}_0 \\ \begin{pmatrix} 1 & 0 \\ \frac{r_0(z_0)}{1+|r_0(z_0)|^2}(z-z_0)^{-2i\kappa_0}s^{-2}(z_0)\delta_0^{-2}e^{\frac{i}{\epsilon}\phi_0(z)} & 1 \end{pmatrix} & : \quad z \in \Gamma_3^* \cap \mathcal{U}_0 \end{cases}. \tag{5.59}$$

Here we recall that the constant κ_0 is given by the formula $\kappa_0 = -\frac{1}{2\pi}\log(1+|r_0(z_0)|^2)$.

We build such a function by first defining a local variable. Define $\zeta_0(z)$ by the relation

$$\frac{1}{2}\zeta_0^2 = \frac{1}{\epsilon}\left(\phi_0(z) - \phi_0(z_0)\right). \tag{5.60}$$

Using (5.58) it follows that locally this defines a one-to-one analytic map from a neighborhood of $z = z_0$ to an expanding neighborhood of the origin in the ζ-plane. Note that the neighborhood of z_0 where this map is analytic is independent of ϵ. We use the freedom to deform our contours to choose Γ_0, Γ_0^*, Γ_3, and Γ_3^* such that inside $\mathcal{U}_0$ their images under ζ_0 lie on the straight lines $\arg(\zeta) = i\pi/4, -i\pi/4, 3i\pi/4$, and $-3i\pi/4$ respectively. Comparing (5.59) to (4.31) our local model $A_0(z)$ is given by the formula

$$A_0(z) = K_0(z)\Psi_{P.C.}(\zeta_0(z); r_0(z_0))\left(\frac{\epsilon}{\phi_0''(z_0)}\right)^{-i\kappa_0\sigma_3/2}\delta_0^{-\sigma_3}s(z_0)^{-\sigma_3}e^{\frac{i}{2\epsilon}\phi_0(z_0)\sigma_3}, \tag{5.61}$$

where we recall that $\Psi_{P.C.}$ is the solution of the parabolic cylinder RHP whose definition is given by (4.35). The pre-factor $K_0(z)$ is an analytic function we now choose to match the outer model on the boundary $\partial\mathcal{U}_0$. The condition selecting $K_0(z)$ is that the outer model $O(z)$ and the local model $A_0(z)$ should be equal to leading order on the boundary $\partial\mathcal{U}_0$. Explicitly, for $z \in \partial\mathcal{U}_0$, we ask that

$$O^{-1}(z)A_0(z) = I + \text{terms vanishing in } \epsilon.$$

Comparing (5.56) and (5.61), setting

$$K_0(z) = O(z)\left(\frac{\epsilon}{\phi_0''(z_0)}\right)^{i\kappa_0\sigma_3/2}\delta_0^{\sigma_3}s(z_0)^{\sigma_3}e^{-\frac{i}{2\epsilon}\phi_0(z_0)\sigma_3} \tag{5.62}$$

we have for $z \in \partial\mathcal{U}_0$,

$$O(z)A_0^{-1}(z) = I + \mathcal{O}\left(\epsilon^{1/2}\right). \tag{5.63}$$

The error term follows from the definition (5.60) of $\zeta_0(z)$ and the boundedness of $O(z)$ and $A_0(z)$ near $z = z_0$. Finally, we make the obvious observation that as defined $K_0(z)$ is analytic in a neighborhood of z_0. This follows from the fact that $O(z)$ is analytic away from $\gamma \cup \gamma^*$ and that for $t < T_1(x)$, $z_0 > \xi_0$ (cf. (5.36) and (5.38)).

Annular model near $z = z_1$

Near $z = z_1$ the presence of phase factor $e^{4Li\nu/\epsilon}$ in addition to the stationary function $e^{i\phi_1/\epsilon}$ forces us to build a two level model problem. Let us introduce a sufficiently small, fixed size, neighborhood $\mathcal{U}_1$ and a smaller shrinking disk $\mathcal{D} = \{z : |z - z_1| < \epsilon^\beta\}$ where $\beta = 1 - \eta$, $\eta \ll 1/2$. Let $\mathcal{A}$ denote the annular region $\mathcal{U}_0 \backslash \mathcal{D}$. We then seek a local model $A_1(z)$ in the form

$$A_1(z) = \begin{cases} B_{\mathcal{A}}(z) & : \quad z \in \mathcal{A} \\ B_{\mathcal{D}}(z) & : \quad z \in \mathcal{D} \end{cases}.$$

In the annular region we ask that $B_{\mathcal{A}}(z)$ satisfy the jump relationship $B_{\mathcal{A}+}(z) =$

$B_{\mathcal{A}-}(z)V_{\mathcal{A}}(z)$, where

$$V_{\mathcal{A}} = \begin{cases} \begin{pmatrix} 1 & 0 \\ -r_0(z_1)s^{-2}(z_1)(z-z_1)^{-2i\kappa_1}(c_\delta)^{-2}e^{\frac{i}{\epsilon}\phi_1(z)} & 1 \end{pmatrix} & : \quad z \in \Gamma_1 \cap \mathcal{U}_1 \\ \begin{pmatrix} 1 & \frac{-\bar{r}_0(z_1)}{1+|r_0(z_1)|^2}s^2(z_1)(z-z_1)^{2i\kappa_1}c_\delta^2 e^{-\frac{i}{\epsilon}\phi_1(z)} \\ 0 & 1 \end{pmatrix} & : \quad z \in \Gamma_2 \cap \mathcal{U}_1 \\ \begin{pmatrix} 1 & -\bar{r}_0(z_1)s^2(z_1)(z-z_1)^{2i\kappa_1}c_\delta^2 e^{-\frac{i}{\epsilon}\phi_1(z)} \\ 0 & 1 \end{pmatrix} & : \quad z \in \Gamma_1^* \cap \mathcal{U}_1 \\ \begin{pmatrix} 1 & 0 \\ \frac{-r_0(z_1)}{1+|r_0(z_1)|^2}s^{-2}(z_1)(z-z_1)^{-2i\kappa_0}c_\delta^{-2}e^{\frac{i}{\epsilon}\phi_1(z)} & 1 \end{pmatrix} & : \quad z \in \Gamma_2^* \cap \mathcal{U}_1 \end{cases} . \tag{5.64}$$

Here we have made use of the constant c_δ previously defined by (4.44).

This problem is directly mapped onto the parabolic cylinder RHP defined by (4.33) and (4.34) by the introduction of the appropriate local variable and factoring off constants. To wit, define the map $\zeta = \zeta_1(z)$ by the relation

$$\frac{1}{2}\zeta_1^2 = \frac{1}{\epsilon}\left(\phi_1(z) - \phi_1(z_1)\right). \tag{5.65}$$

Using (5.58) it is easy to see that this is a analytic one-to-one map from any suitably small, but ϵ independent, neighborhood of z_1 to an expanding neighborhood of the origin in the ζ-plane. Using our ability to deform our lensing contours, we choose the contours Γ_1, Γ_1^*, Γ_2, and Γ_2^* such that, restricted to $\mathcal{U}_1$, their images under $\zeta_1(z)$ line on the straight line rays $\arg(\zeta) = i\pi/4$, $-i\pi/4$, $3i\pi/4$, and $-3i\pi/4$ respectively. The annular model $B_{\mathcal{A}}(z)$ is then given by

$$B_{\mathcal{A}}(z) = K_{\mathcal{A}}(z)\Psi_{P.C.}(\zeta_1(z); -r_1(z_1)) \left(\frac{\epsilon}{\phi_1''(z_1)}\right)^{-i\kappa_1\sigma_3/2} c_\delta^{-\sigma_3} s(z_1)^{-\sigma_3} e^{\frac{i}{2\epsilon}\phi_1(z_1)\sigma_3}, \tag{5.66}$$

where we recall the constants $\kappa_1 = -\frac{1}{2\pi}\log(1+|r_0(z_1)|^2)$ and c_δ, defined by (4.44), related to the approximation of $\delta(z)$ near $z = z_1$. The pre-factor $K_{\mathcal{A}}(z)$ is an analytic matching function. Its calculation is identical to the previous calculation determining $K_0(z)$. Repeating that calculation with $B_{\mathcal{A}}$ replacing A_0 we find setting

$$K_{\mathcal{A}}(z) = O(z) \left(\frac{\epsilon}{\phi_1''(z_1)}\right)^{i\kappa_1\sigma_3/2} c_\delta^{\sigma_3} s(z_1)^{\sigma_3} e^{-\frac{i}{2\epsilon}\phi_1(z_1)\sigma_3}, \tag{5.67}$$

that uniformly on the boundary $\partial\mathcal{U}_1$,

$$O^{-1}(z)B_{\mathcal{A}}(z) = I + \mathcal{O}(\epsilon^{1/2}). \tag{5.68}$$

Model on the disk $\mathcal{D}$ near $z = z_1$

Inside the shrinking disc $\mathcal{D}$ the annular model $B_{\mathcal{A}}(z)$ is no longer a good approximation

of the true jumps. Inside $\mathcal{D}$, the linear oscillations generated by $\exp(4Li\nu/\epsilon)$ are no longer negligible while the quadratic phase $\phi_1(z)$ is nearly constant; we need a model that accounts for these oscillations. We seek our model $B_{\mathcal{D}}$ in the form

$$B_{\mathcal{D}}(z) = K_{\mathcal{D}}(z)L(z)e^{\frac{i}{2\epsilon}\phi_1(z)}, \tag{5.69}$$

where, as always, the prefactor $K_{\mathcal{D}}(z)$ is an analytic function, here chosen to match the inner disc model to the annular model on the boundary $\partial\mathcal{D}$. The function $L(z)$ we choose to satisfies the jump relation $L_+(z) = L_-(z)V_L$, where

$$V_L = [s(z_1)(z-z_1)^{i\kappa_1}C_\delta(z)]^{\operatorname{ad}\sigma_3}\widehat{V}_L, \tag{5.70}$$

where we recall $\kappa_1 = -\frac{1}{2\pi}\log(1+|r_0(z_1)|^2)$, C_δ is the piecewise constant given by (4.52), and the matrix $\widehat{V}_L$ is given on each contour by

$$\widehat{V}_L(z) = \begin{cases} \begin{pmatrix} 1 & \frac{\bar{r}_0(z_1)}{1+|r_0(z_1)|^2}e^{\frac{4Li\nu}{\epsilon}} \\ 0 & 1 \end{pmatrix} & : \quad z \in \Sigma_1 \\ \begin{pmatrix} 1 & 0 \\ -r_0(z_1)(1+|r_0(z_1)|^2) & 1 \end{pmatrix} & : \quad z \in \Sigma_2 \\ \begin{pmatrix} 1-r_0^2(z_1)e^{\frac{4Li\nu}{\epsilon}} & \frac{\bar{r}_0(z_1)(1-e^{\frac{4Li\nu}{\epsilon}})}{(1+|r_0(z_1)|^2)^2} \\ 0 & (1-r_0^2(z_1)e^{\frac{4Li\nu}{\epsilon}})^{-1} \end{pmatrix} & : \quad z \in \Sigma_3 \\ \begin{pmatrix} 1 & 0 \\ \frac{r_0(z_1)}{1+|r_0(z_1)|^2}e^{-\frac{4Li\nu}{\epsilon}} & 1 \end{pmatrix} & : \quad z \in \Sigma_1^* \\ \begin{pmatrix} 1 & -\bar{r}_0(z_1)(1+|r_0(z_1)|^2) \\ 0 & 1 \end{pmatrix} & : \quad z \in \Sigma_2^* \\ \begin{pmatrix} 1-\bar{r}_0^2(z_1)e^{-\frac{4Li\nu}{\epsilon}} & 0 \\ \frac{r_0(z_1)(1-e^{-\frac{4Li\nu}{\epsilon}})}{(1+|r_0(z_1)|^2)^2} & (1-\bar{r}_0^2(z_1)e^{-\frac{4Li\nu}{\epsilon}})^{-1} \end{pmatrix} & : \quad z \in \Sigma_3^* \end{cases}$$

To simplify the notation we have relabeled and reoriented the contours inside $\mathcal{D}$, see Figure 4.11 in section 4.2.3 for the labeling and orientation of the contours Σ_i and Σ_i^*.

To complete our model in $\mathcal{D}$ we need to specify the function $L(z)$ satisfying the above jump conditions and the matching function $K_{\mathcal{D}}(z)$. These functions are given by

$$L(z) = \begin{cases} (z-z_1)^{-i\kappa_1\sigma_3} & : \quad z \in \mathcal{D}_0 \\ L_{k-1}(z)\left[(z-z_1)^{i\kappa_1}C_\delta(z)\right]^{\operatorname{ad}\sigma_3}V_k(z) & : \quad z \in \mathcal{D}_k, \quad k=1,2,3 \\ L_{k-1}^*(z)\left[(z-z_1)^{i\kappa_1}C_\delta(z)\right]^{\operatorname{ad}\sigma_3}V_k^{-*}(z) & : \quad z \in \mathcal{D}_k^*, \quad k=1,2,3. \end{cases} \tag{5.71}$$

and

$$K_{\mathcal{D}}(z) = K_{\mathcal{A}}(z) P_+(0; -r_0(z_1)) \begin{pmatrix} 1 & 0 \\ r_0(z_1) & 1 \end{pmatrix} c_\delta^{-\sigma_3} s(z_1)^{-\sigma_3}. \tag{5.72}$$

Here we recall that the function $P(\zeta; a)$, defined by (4.36), is part of construction of the solution $\Psi_{P.C.}$ to the parabolic cylinder RHP. The proof that $L(z)$ and $K_{\mathcal{D}}(z)$ satisfy the jump conditions and the matching function $K_{\mathcal{D}}$ are virtually identical to case for $x > L$ which is given in Prop. 9. In the proof we show that the matching condition on the boundary $\partial\mathcal{D}$ satisfies:

$$B_{\mathcal{A}}^{-1}(z) B_{\mathcal{D}}(z) = I + \mathcal{O}\left(\epsilon^{1/2-\beta}\right). \tag{5.73}$$

5.4.3 The Global Error Riemann-Hilbert Problem

We now have a complete description of the global parametrix $A(z)$,

$$A(z) = \begin{cases} A_0(z) & : \quad z \in \mathcal{U}_0 \\ B_{\mathcal{A}}(z) & : \quad z \in \mathcal{A} \\ B_{\mathcal{D}}(z) & : \quad z \in \mathcal{D} \\ O(z) & : \quad \text{elsewhere} \end{cases}. \tag{5.74}$$

The outer solution $O(z)$ was defined in Section 5.4.1 by (5.56); the parabolic cylinder models $A_0(z)$ and $B_{\mathcal{A}}(z)$ defined by (5.61) and (5.66) respectively, and the disc model $B_{\mathcal{D}}$ defined by (5.71) were described in Section 5.4.2.

We now want to prove that our parametrix provides a uniform asymptotic approximation of the true solution $Q(z)$ to the RHP defined by (5.51) and (5.52). In doing so we will recover a valid asymptotic expansion of the solution $m(z)$ to the original RHP for NLS and thus an asymptotic description of the solution $q(x,t)$ to NLS for square barrier initial data in the space-time set $\mathcal{S}_1$, see (5.78) below, where the one-cut g-function controls the inverse spectral analysis.

As in the previous chapter we consider the error matrix, $E(z)$, defined by

$$E(z) = Q(z) A^{-1}(z). \tag{5.75}$$

Using the RHP for $Q(z)$ and the properties of our global parametrix, this definition generates a RHP for the error matrix, $E(z)$. Let Γ_E consist of the contours $\Gamma_Q \backslash (\gamma \cup \gamma^*)$, that is those contours on which $Q(z)$ had jumps which were ignored in our construction of the outer model $O(z)$, together with the three counter-clockwise oriented circles $\partial\mathcal{U}_0$, ∂U_1, and $\partial\mathcal{D}$ where the global parametrix is patched together, see Figure 5.8.

RHP for E(z):

Find a 2×2 matrix $E(z)$ which satisfies the following properties.

- $E(z)$ is analytic in $\mathbb{C} \backslash \Gamma_E$.
- As $z \to \infty$,

$$E(z) = I + \mathcal{O}(1/z). \tag{5.76}$$

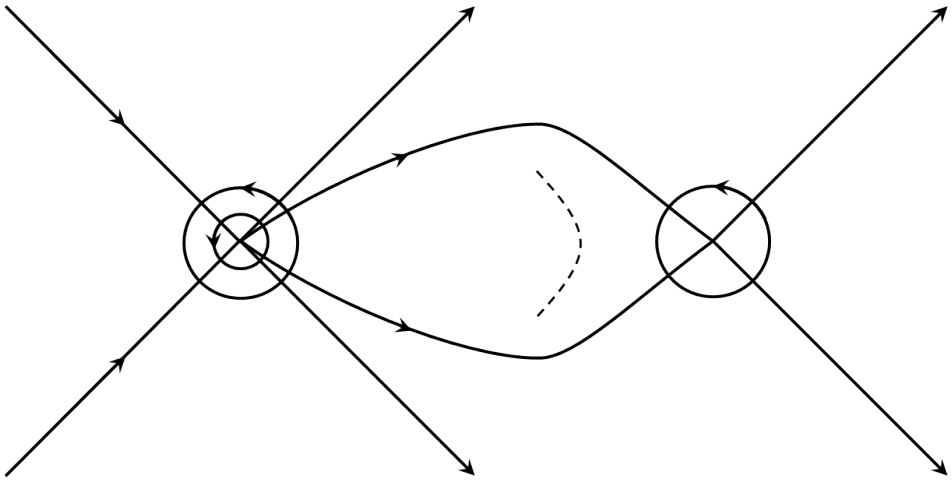

FIGURE 5.8. The collection of contours Γ_E for the Error RHP. The dashed contour $\gamma \cup \gamma^*$ is not a part of Γ_E, our global parametrix solves the exact jumps of the full problem for $Q(z)$ along this contour so the ratio $E = QA^{-1}$ is analytic across $\gamma \cup \gamma^*$.

- For $z \in \Gamma_E$, $E(z)$ assumes continuous boundary values $E_\pm(z)$ which satisfy the jump relation $E_+(z) = E_-(z)V_E(z)$, where

$$V_E = \begin{cases} A_-(z)V_Q(z)V_A^{-1}A_-^{-1} & : \quad z \in \Gamma_Q \backslash (\gamma \cup \gamma^*) \\ O(z)A_0^{-1}(z) & : \quad z \in \partial\mathcal{U}_0 \\ O(z)B_{\mathcal{A}}^{-1}(z) & : \quad z \in \partial\mathcal{U}_1 \\ B_{\mathcal{A}}(z)B_{\mathcal{D}}^{-1}(z) & : \quad z \in \partial\mathcal{D} \end{cases} \tag{5.77}$$

Remark 5. Observe that along the contour $\gamma \cup \gamma^*$ the error $E(z) = Q(z)A^{-1}(z)$ is analytic. Comparing (5.52) with (5.55), we see that the outer model problem $O(z)$ has exactly the same jump relation as $Q(z)$ along $\gamma \cup \gamma^*$. The ratio $E(z)$ has an identity jump (i.e. E is analytic) along this contour. Additionally, because the outer model $O(z)$ has the 1/4-root singularities prescribed in the RHP for $Q(z)$ (cf. (5.56)), the resulting ratio $E(z) = Q(z)A^{-1}(z)$ is bounded in any neighborhood of $z = \pm iq$.

The RHP for $E(z)$ is solvable using the theory of small norm RHPs which we described in Section 4.3.1 of the previous chapter. The key estimate we need to find the solution is contained in the following lemma. Recall, the space-time set

$$\mathcal{S}_1 = \left\{ (x,t) \, : \, 0 \le x < L, \; 0 < t < \frac{1}{2\sqrt{2}q}(L-x) \right\} \tag{5.78}$$

in which we were able to find a g-function cut on a single band which stabilized the RHP (cf. Section 5.2.2), the lemma estimating the perturbation of the error jumps from identity is as follows:

Lemma 17. *Let $K \subset \mathcal{S}_1$ be compact and $\ell \in \mathbb{N}_0$. Then there exists a constant $M_K^\ell < \infty$ such that for any $(x,t) \in K$ and $\rho \ll 1/2$, the jump matrix $V_E(z)$ defined by (5.77) satisfies the inequality,*

$$\left\| s^\ell (V_E - I) \right\|_{L^p(\Gamma_E)} \leq M_K^\ell \epsilon^{1/2-\rho}, \tag{5.79}$$

for $p = 1, 2,$ or ∞ for every sufficiently small value of ϵ.

Proof. We could of course rehash previous arguments to prove this lemma directly. Instead, we observe that the calculations are essentially identical to those used in the previous chapter to prove the analogous statements in Lemma 12. One only needs to observe that the factors of $s^2(z)$ or $s^{-2}(z)$ appearing in the off-diagonal entries of V_E in the one-cut case are bounded smooth functions along every contour in Γ_E and that the phase functions ϕ_k appearing in the new estimates have all the same properties as the θ_k which appear in the proof of the previous lemma. We leave the details to the reader. □

Using Lemma 17 and the small-norm theory presented in Section 4.3.1 we can write down the solution $E(z)$ to the error RHP defined by (5.76) and (5.77) as the unique solution of a singular integral equation.

Theorem 18. *For each $(x,t) \in \mathcal{S}_1$, defined by (5.78), the global error Riemann-Hilbert problem defined by (5.76) and (5.77) has a unique solution given for $z \notin \Gamma_E$ by the formula*

$$E(z) = I + \frac{1}{2\pi i} \int_{\Gamma_E} \frac{(\eta(s) + I)(V_E(s) - I)}{s - z} ds,$$

where $\eta(s)$ is the unique solution of the equation

$$\eta - \mathcal{C}_-(\eta(V_E - I)) = \mathcal{C}_-(V_E - I).$$

In addition as $z \to \infty$ away from Γ_E, $E(z)$ has a valid asymptotic expansion,

$$E(z) = I + \frac{E^{(1)}}{z} + \dots,$$

where each moment $E^{(k)}$ is bounded, and in particular,

$$\left\| E^{(1)} \right\| \leq M \epsilon^{1/2-\rho}$$

for any $\rho > 0$. The bound is uniform on any compact $K \subset \mathcal{S}_1$.

Proof. Let $w_-(s) = V_E - I$ and $w_+(s) = 0$ and choose $K \subset \mathcal{S}_1$, compact. Then for each $(x,t) \in K$, the proof is a direct consequence of Lemma 17 controlling the size of $V_E - I$ and Lemmas 10 and 11 which relate the Cauchy transform to the solution of small-norm RHPs. □

Now that we have our hands on the solution $E(z)$ to the error RHP, we can use the series of explicit transformation used to transform the original RHP for $m(z)$ into $E(z)$:

$$m \mapsto M \mapsto N \mapsto Q \mapsto E,$$

to recover a formula for the solution $m(z)$ of the NLS RHP we originally sought. What's more, because the error is asymptotically near identity, we get an asymptotic description of the solution $m(z)$ from which an asymptotic expression for the solution $q(x,t)$ of NLS follows by (3.17). These results are the content of Theorem 14 which we began this chapter by stating. We reproduce the theorem here and now give the culmination of its proof.

Theorem. *For each value $(x,t) \in \mathcal{S}_1$, where*

$$\mathcal{S}_1 = \left\{ (x,t) \ : \ 0 \leq |x| < L, \quad t < \frac{1}{2\sqrt{2}q} \left(L - x \right) \right\},$$

the solution $q(x,t)$ to the semiclassically focusing NLS equation

$$i\epsilon q_t + \frac{\epsilon^2}{2} q_{xx} + |q|^2 q = 0, \qquad q(x,0) = q(x),$$

where $q(x)$ is the square barrier initial data (1.2)*, has the following asymptotically valid description as $\epsilon \to 0^+$:*

$$q(x,t) = q e^{\frac{i}{\epsilon} q^2 t} + \mathcal{O}(\epsilon^{1/2-\rho}). \tag{5.80}$$

for any number $\rho > 0$. Additionally, the convergence is uniform on any compact subset of $\mathcal{S}_1$.

Proof. Let $(x,t) \in K \subset \mathcal{S}_1$, for K compact. Using the explicit transformation formulae (5.2), (5.13), (5.16), and (5.46), together with the global parametrix $A(z)$ which is summarized in (5.74), the solution $m(z)$ to the original NLS RHP is given for suitably large z along the positive imaginary axis by the formula

$$m(z) = e^{-\frac{i}{\epsilon} g(\infty) \sigma_3} s_\infty^{-\sigma_3} E(z) O(z) \delta^{\sigma_3}(z) s^{\sigma_3}(z) e^{\frac{i}{\epsilon} g(z) \sigma_3}.$$

The theorem then follows from the large z expansion formulae of $g(z)$, $\delta(z)$, $s(z)$, and $O(z)$, given by (5.27), (5.43), (5.50), and (5.57) respectively and the bound on the moments of $E(z)$ given by Theorem 18. $\square$

Chapter 6

The Two Banded Solution Inside the Support

Thus far we have successfully described the semiclassical asymptotics of the NLS Riemann Hilbert problem in the space-time regions $\mathcal{S}_0$ and $\mathcal{S}_1$, both inside and outside the support of the initial square barrier initial data, see Figure 6.1. In this chapter we complete the description of our main results by describing the Riemann-Hilbert analysis in region $\mathcal{S}_2$. In doing so we will complete our initial objective: to provide a description of the semiclassical asymptotic behavior of the focusing NLS evolution of the square barrier initial data globally in space up to order-one times. Beyond the boundary curve $T_2(x)$, which we define later in this chapter, a new asymptotic description is needed, but we do not explore this problem in this study.

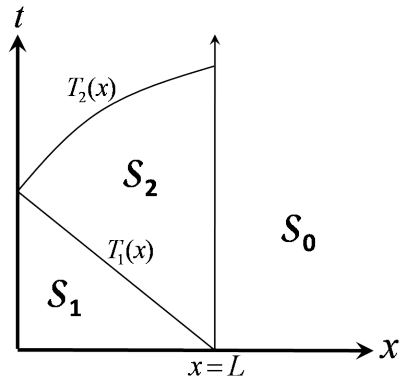

Figure 6.1. The regions of space-time in which we can successfully described the analysis of the RHP. Previously we have described the analysis in the regions $\mathcal{S}_0$ and $\mathcal{S}_1$. In this chapter we describe the Riemann-Hilbert analysis in region $\mathcal{S}_2$ thereby providing a complete description of the evolution of the initial data everywhere in space up to order-one times.

We begin our analysis in this chapter by recalling the machinery we used in the previous chapter to described the RH analysis in $\mathcal{S}_1$. In $\mathcal{S}_1$ we needed to introduce a g-function cut on a single band connecting the endpoints, $z = \pm iq$, of the imaginary interval on which the poles of the RHP accumulate. The g-function was the essential element of our analysis which allowed us to stabilize the jump matrices and arrive at a problem which was amenable to approximation by a parametrix. A necessary requirement of the single band ansatz was that the modified phase $\phi_0 = \theta_0 - 2g$ take purely real values along the single cut. Using

the results of Prop. 15 we found that beyond the breaking time

$$T_1(x) = -\frac{x-L}{2\sqrt{2}q}. \tag{6.1}$$

this cannot happen. For $t > T_1(x)$ the level set $\mathrm{Im}(\phi_0) = 0$ no longer connects the endpoints $\pm iq$, a gap has emerged through which the one cut g-function cannot pass (Fig. 6.2).

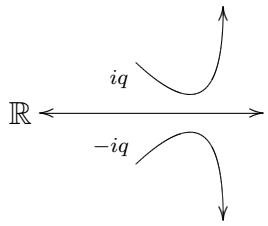

FIGURE 6.2. The level set $\mathrm{Im}(\phi) = 0$ beyond the breaking time. Observe that there is no longer a connection of the endpoints $\pm iq$.

To successfully analyze the Riemann Hilbert problem for (x,t) in $\mathcal{S}_2$ we have to answer the following question: How do we modify our g-function to study the problem for $t > T_1(x)$? As we will see, the resolution is to introduce a new g-function supported on two bands γ_b and γ_b^* which no longer connect at the real axis. Instead the interval γ_b will have a fixed endpoint at iq, as before, and an endpoint $\alpha \in \mathbb{C}^+$ whose value will vary with space-time, $\alpha = \alpha(x,t)$. As we will see, the resulting g-function resolves the issue of the emerged gap, and allows us to build a new global parametrix for $(x,t) \in \mathcal{S}_2$. The new outer model, which provides the leading order asymptotics to the solution of NLS, will be built out of genus one Θ-functions. The analysis in this chapter will ultimately result in the proof of the following theorem.

Theorem 19. *Define the set $\mathcal{S}_2$ by,*

$$\mathcal{S}_1 = \{(x,t) \ : \ 0 \le |x| < L, \ \ T_1(x) < t < T_2(x)\},$$

for the functions $T_1(x)$ and $T_2(x)$ defined by (6.1) *and* (6.32) *respectively. Then for each value $(x,t) \in \mathcal{S}_2$ the solution $q(x,t)$ to the semiclassically focusing NLS equation*

$$i\epsilon q_t + \frac{\epsilon^2}{2} q_{xx} + |q|^2 q = 0, \qquad q(x,0) = q(x),$$

where $q(x)$ is the square barrier initial data (1.2)*, has the following asymptotically valid description as $\epsilon \to 0^+$:*

$$q(x,t) = q_0(x,t) + \mathcal{O}(\epsilon^{1/2-\rho}) \tag{6.2}$$

for any number $\rho > 0$. Additionally, the convergence is uniform on any compact subset of $\mathcal{S}_2$. The function $q_0(x,t)$ is described in terms of genus-one theta functions:

$$q_0(x,t) = i\,\mathrm{Re}(\alpha(x,t))\frac{\Theta(\mathbf{Z})}{\Theta(\mathbf{Y})}\frac{\Theta(\mathbf{Y}+iU-i\Omega/\epsilon)}{\Theta(\mathbf{Z}-iU+i\Omega/\epsilon)}e^{-\frac{2i}{\epsilon}(g(\infty)-\epsilon\chi)} + \mathcal{O}(\epsilon^{1/2-\rho}), \tag{6.3}$$

where, $\mathbf{Z} = u(\infty) - V_1$ *and* $\mathbf{Y} = u(\infty) + V_1$*; the function* $u(z) = \int_{-iq}^{z} \nu$ *is defined by* (6.94)*; the constants* V_1*,* U*, and* χ *are defined by* (6.95)*,* (6.92)*, and* (6.97)*, respectively; the function* $\alpha(x,t)$ *is defined in Prop. 20, and the constants* Ω *and* $g(\infty)$ *are defined by* (6.28) *and* (6.25)*.*

6.1 The Transformation $m \mapsto M$

We begin our analysis from the original NLS RHP defined by (3.21), (3.22), and (3.23). We will recycle many of the symbols from the previous chapters though their meaning may be different than in their previous incarnations.

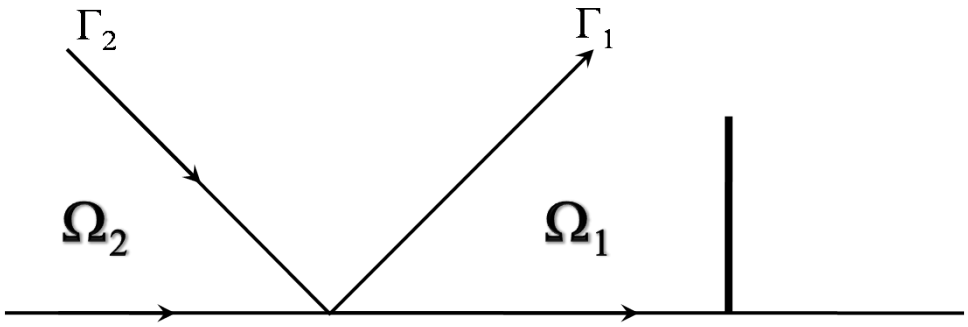

FIGURE 6.3. Diagram of the contours Γ_1 and Γ_2 emerging from z_0 and the corresponding regions Ω_1 and Ω_2. These regions are used in the transformation $m \mapsto M$, defined by (6.4), which removes the poles from the RHP. In the figure the poles are represented by the bold vertical line.

As always, our first step is to introduce a transformation $m \mapsto M$ which results in a new RHP for $M(z)$ which has no poles. Recall the pole removing scheme used inside the support before the first breaking time (cf. 5.1.1); the point z_1 from which the lenses in the pole removing factorization (5.2) still lies in the negative half line at the breaking time. At breaking time this pole removing factorization is still robust, so we expect that it should still work beyond the first breaking time. Specifically we define by the relationship

$$m(z) = \begin{cases} M(z)RR_0^{-1} & : \quad z \in \Omega_1 \\ M(z)a_1(z)^{\sigma_3}\widehat{R}^* & : \quad z \in \Omega_2 \\ M(z)R^*R_0^* & : \quad z \in \Omega_1^* \\ M(z)\widehat{R}^{-1}\overline{a}_1(z)^{\sigma_3} & : \quad z \in \Omega_2^* \\ M(z) & : \quad \text{elsewhere} \end{cases} \tag{6.4}$$

where the regions Ω_j and Ω_j^*, $j = 1, 2$ are defined in figure 6.3. We haven't yet specified the endpoint z_1; we only require that $z_1 < 0$ so that the regions Ω_1 and Ω_1^* completely contain the interval $[-iq, iq]$ where the eigenvalues accumulate. As was described in the one cut case this factorization removes the poles from the problem at the cost of introducing a cut

along the interval $\gamma \cup \gamma^*$ along which the function $r_0(z)$ is branched. The resulting RHP for $M(z)$ is given below.
Find a 2×2 matrix $M(z)$ that satisfies the following properties.

- $M(z)$ is analytic in the set $\mathbb{C} \backslash \{(\Gamma_1 \cup \Gamma_1^*) \bigcup (\gamma \cup \gamma^*) \bigcup \mathbb{R}\}$ and has continuous boundary values as z approaches the contours.
- $M(z) = I + \mathcal{O}(1/z)$ as $|z| \to \infty$.
- On the contours, the boundary values of M satisfy the relation $M_+ = M_- V_M$ where

$$V_M = \begin{cases} R R_0^{-1} & : \quad z \in \Gamma_1 \\ a_1(z)^{\sigma_3} \widehat{R}^* & : \quad z \in \Gamma_2 \\ R_{0-}^{-1} R_{0+} & : \quad z \in \gamma \\ R_0^* R_0 & : \quad z \in (z_1, \infty) \\ (1 + |r_0(z)|^2)^{2\sigma_3} & : \quad z \in (-\infty, z_1) \\ R_{0-} R_{0+}^{-1} & : \quad z \in \gamma^* \\ \overline{a}_1(z)^{\sigma_3} \widehat{R} & : \quad z \in \Gamma_2^* \\ R^* R_0^{-*} & : \quad z \in \Gamma_1^* \end{cases} . \tag{6.5}$$

Comparing the definition of $M(z)$ here to the definition before the breaking time (see (5.2) and (5.13)) one sees that we have not introduced a second critical point z_0 on the real line from which factorizations emerge. This is because at the breaking time $T_1(x)$ z_0 collides with the contour γ creating a bifurcation. For $t > T_1(x)$ it isn't clear that there is a critical point z_0 on the positive real line. After introducing a g-function we will see what must be done to remedy the remaining oscillations on the interval (z_1, ∞).

6.2 Introducing a g-function with two cuts

As before we introduce a g-function to stabilize the growth along the interval $\gamma \cup \gamma^*$. However, we now assume that the g-function is supported on two symmetric bands γ_b and γ_b^*. The band γ_b has an endpoint at the edge iq and an endpoint $\alpha \in \mathbb{C}^+$. By symmetry, the endpoints of γ_b^* are $-iq$ and $\bar{\alpha}$. We will denote the gap between these bands by $\gamma_g \cup \gamma_g^*$ where $\gamma_g = \gamma \backslash \gamma_b$ has an endpoint at α and another on the real axis at a point we will denote z_0. *A posteriori* knowledge of these types of problems leads us to guess that the function g is also not analytic in the gap $\gamma_g \cup \gamma_g^*$, but that the jump relation is constant, $g_+ - g_- = \Omega$, for some constant Ω. Using this ansatz, we make the change of variables

$$M(z) = e^{-\frac{i}{\epsilon} g_2(\infty) \sigma_3} N(z) e^{\frac{i}{\epsilon} g_2(z) \sigma_3}. \tag{6.6}$$

The new jumps for $N(z)$ are given by the formula

$$V_N(z) = \begin{cases} e^{\frac{i}{\epsilon} g_2 \operatorname{ad} \sigma_3} V_M & : \quad z \in \Gamma_M \backslash (\gamma \cup \gamma^*) \\ e^{\frac{i}{\epsilon} g_{2-} \sigma_3} V_M e^{-\frac{i}{\epsilon} g_{2+} \sigma_3} & : \quad z \in \gamma \cup \gamma^* \end{cases}$$

away from the bands of support the effect of this transformation is simply to replace the phases θ_k by

$$\varphi_k = \theta_k - 2g_2. \tag{6.7}$$

Along the bands and gaps the function $g_2(z)$ is cut and the jump matrices are more significantly altered. The jumps on γ and γ^* are

$$V_N = \begin{cases} \begin{pmatrix} e^{-\frac{i}{\epsilon}(g_+ - g_-)} & 0 \\ we^{\frac{i}{\epsilon}(\theta - g_+ - g_-)} & e^{\frac{i}{\epsilon}(g_+ - g_-)} \end{pmatrix} & : \quad z \in \gamma \\ \begin{pmatrix} e^{-\frac{i}{\epsilon}(g_+ - g_-)} & we^{-\frac{i}{\epsilon}(\theta - g_+ - g_-)} \\ 0 & e^{\frac{i}{\epsilon}(g_+ - g_-)} \end{pmatrix} & : \quad z \in \gamma^* \end{cases} \tag{6.8}$$

As before to arrive at a problem whose jumps are stable we require that the function $g_2(z)$ satisfy the following properties:

1. g is analytic in $\mathbb{C}\backslash(\gamma \cup \gamma^*)$.
2. $g = g(\infty) + \mathcal{O}(1/z)$ as $|z| \to \infty$, where $g(\infty)$ is a bounded constant.
3. For $z \in \gamma_b \cup \gamma_b^*$, $\theta - g_{2+} - g_{2-} = 0$.
4. For $z \in \gamma_g \cup \gamma_g^*$, $g_{2+} - g_{2-} = \Omega, \quad \Omega$ constant.
5. For $z \in \gamma_b \cup \gamma_b^*$, $\mathrm{Im}(g_{2+} - g_{2-}) = \mathrm{Im}(\varphi_-) = 0$.

(6.9)

We could try to find a function satisfying these properties directly, but for the two cut g-function it is simpler to first find a formula for the derivative $h = g_2'$. The function h satisfies the following properties

1. h is analytic in $\mathbb{C}\backslash(\gamma_b \cup \gamma_b^*)$.
2. $h = \mathcal{O}(1/z^2)$ as $|z| \to \infty$.
3. For $z \in \gamma_b \cup \gamma_b^*$, $\theta' - h_+ - h_- = 0$.
4. $\mathrm{Im}\left(\int_{\bar{\alpha}}^{\alpha} \theta' - 2h\mathrm{d}z\right) = 0$

(6.10)

Each of the conditions on h is clearly a direct consequence of the corresponding condition on g_2, except perhaps the last condition. If $\mathrm{Im}(g_{2+} - g_{2-}) = 0$ on both bands then across the gap the change of this function must be zero. Condition 4 on h is just the expression of this no change condition written in integral form.

6.2.1 A formula for $h = g'(z)$

Conditions $1 - 3$ in (6.10) define a scalar RHP for h which we solve using the Sokhotskyi-Plemelj formula, yielding

$$h(z) = \frac{S_2(z)}{2\pi i} \int_{\gamma_b \cup \gamma_b^*} \frac{\theta'(s)}{S_{2+}(s)} \frac{\mathrm{d}s}{s - z}, \tag{6.11}$$

where we have defined

$$S_2(z) = \left(\frac{z - \alpha}{z - iq}\right)^{1/2} \left(\frac{z - \bar{\alpha}}{z + iq}\right)^{1/2} \tag{6.12}$$

to be branched along $\gamma_b \cup \gamma_b^*$ and normalized such that $S_2(z) \to 1$ for $z \to \infty$.

The integral representation (6.11) can be simplified using residue calculus to yield the nicer formula

$$h(z) = \frac{1}{2}\theta'(z) - S_2(z)\left[2t(z+\alpha_r) + (x-L)\right]. \tag{6.13}$$

The function h, thus defined, solves the necessary jump relations, conditions (1) and (3) in (6.10) but does not yet satisfy the moment condition (2) or the gap condition (4). These conditions, we will show directly, uniquely select the value of α for each point (x,t) beyond the breaking curve.

6.2.2 Moment-Gap conditions: Determining the endpoint α

Our description of $h(z)$ is complete up to the determination of the points α and $\bar{\alpha}$ and the curves γ_b and γ_b^* connecting the endpoints. We leave the existence of the curves for later and here focus on how one determines the values of the endpoint α. The condition that $h = \mathcal{O}(1/z^2)$ is nontrivial; for an arbitrary choice of α the large z behavior of h is $\mathcal{O}(1/z)$. Expanding (6.11) for large z we require that the coefficient of the $1/z$ term vanish:

$$\mathbf{M}(x,t,\alpha) := \frac{1}{2\pi i}\int_{\gamma_b \cup \gamma_b^*} \frac{\theta'(s)}{S_{2+}(s)}\mathrm{d}s = 0. \tag{6.14}$$

Using residue calculus we can evaluate the integral to reduce the condition to an algebraic condition on the endpoints:

$$\mathbf{M}(\mu,\alpha) := \frac{\alpha_i^2}{2} - (\alpha_r - \mu)^2 - \left(\frac{q^2}{2} - \mu^2\right) = 0. \tag{6.15}$$

In this formula we have introduced a new variable

$$\mu = -\frac{x-L}{4t} \tag{6.16}$$

which encodes the dependence of the moment condition on space-time. In the (x,t)-plane μ should be thought of as an angle-like variable, with vertex at $x = L$ and measured counterclockwise from vertical. Using μ we can reexpress the breaking curve $T_1(x) = -\frac{x-L}{2\sqrt{2}q}$ as the simpler expression $\mu = \frac{q}{\sqrt{2}}$. In this language we would like to determine α for each value of μ in the interval $[0, \frac{q}{\sqrt{2}}]$.

Condition 4, which we call the gap condition, constitutes a second condition on the endpoints. Written using (6.13), condition 4 in (6.10) becomes:

$$\mathrm{Im}\left(\int_{\bar{\alpha}}^{\alpha} S_2(z)\left[z + \alpha_r - 2\mu\right]\mathrm{d}z\right) = 0.$$

The reflectional symmetry of the integrand implies that the real part of this integral must

be zero, thus we can simply write the gap condition as

$$\mathbf{G}(\mu,\alpha) := \int_{\bar{\alpha}}^{\alpha} S_2(z)\,[z+\alpha_r-2\mu]\,\mathrm{d}z = 0. \tag{6.17}$$

Unlike the moment condition, we cannot use residue calculus to evaluate this integral. However, the integration contour can be deformed away from α and $\bar{\alpha}$ to show that $\mathbf{G}$ is an differentiable function of α and $\bar{\alpha}$. Explicitly, define $\widetilde{S}$ by the formula

$$\widetilde{S}(z) = \begin{cases} -S_2(z) & : \quad z \in U \\ S_2(z) & : \quad \text{elsewhere} \end{cases}, \tag{6.18}$$

where U is the region bounded by the vertical segments connecting the pairs $\pm iq$ and $\{\alpha,\bar{\alpha}\}$ and the bands, $\gamma_b \cup \gamma_b^*$. It follows that $\widetilde{S}$ is cut along the vertical segments and the gap condition can be rewritten as

$$\mathbf{G}(\mu,\alpha,\bar{\alpha}) = \int_{\bar{\alpha}}^{\alpha} \widetilde{S}_-(z)\,[z-\alpha_r-2\mu]\,\mathrm{d}z = \frac{1}{2}\int_C \widetilde{S}(z)\,[z-\alpha_r-2\mu]\,\mathrm{d}z = 0 \tag{6.19}$$

where C is any closed loop enclosing α and $\bar{\alpha}$ and has $\pm iq$ on its exterior.

The two equations $\{\mathbf{M}=0,\ \mathbf{G}=0\}$ constitute a system of two real nonlinear equations relating the two real unknowns of α to the known angle $\mu = -\frac{x-L}{4t}$. The following proposition states that this system uniquely determines α for each pair (x,t) beyond the breaking curve $T_1(x)$.

Proposition 20 (Existence of the endpoint α)**.** *Let $\mathbb{C}^{++} = \{z:\ \mathrm{Re}(z)>0,\ \mathrm{Im}(z)>0\}$. There exists a unique map $\alpha : [0,\frac{q}{\sqrt{2}}] \to \overline{\mathbb{C}^{++}}$ such that the moment-gap conditions*

$$\mathbf{M}(\mu,\alpha(\mu)) = \frac{\alpha_i^2}{2} - (\alpha_r-\mu)^2 - \left(\frac{q^2}{2}-\mu^2\right) = 0$$
$$\mathbf{G}(\mu,\alpha(\mu)) = \frac{1}{2}\int_C \widetilde{S}(z)\,[z-\alpha_r-2\mu]\,\mathrm{d}z = 0$$

are satisfied for each point $(\mu,\alpha(\mu))$ on the graph of α. Further, the function $\alpha(\mu)$ is smooth and its endpoints are known explicitly:

$$\alpha(0) = iq \qquad\qquad \alpha(q/\sqrt{2}) = q/\sqrt{2}.$$

Proof. We expect that at the breaking time the two-cut and one-cut g-functions should be equivalent. In the language of the moment and gap conditions, we believe that at $\mu = q/\sqrt{2}$, the breaking time, $\alpha = z_0 = q/\sqrt{2}$ solves the moment gap conditions. One easily can check that these values satisfy the moment and gap conditions. It is tempting to then invoke the implicit function theorem to prove that $\alpha(\mu)$ exists in a neighborhood of $\mu = q/\sqrt{2}$. However, the Jacobian determinant

$$\left|\frac{\partial(M,G)}{\partial(\alpha,\bar{\alpha})}\right| = (\bar{\alpha}-\alpha)\frac{\partial M}{\partial\alpha}\frac{\partial M}{\partial\bar{\alpha}}\int_{\bar{\alpha}}^{\alpha}\frac{\widetilde{S}(z)}{(z-\alpha)(z-\bar{\alpha})}\mathrm{d}z$$

which is nonzero for $\alpha \in \mathbb{C}^+$, vanishes along the real line. More careful analysis is needed near this endpoint.

Writing small perturbations of the endpoint in the form $\mu = q/\sqrt{2} + \eta$ and $\alpha_r = q/\sqrt{2} + a(\eta)$ we use the moment condition to solve for α_i:

$$\alpha_i^2 = -4\mu^*\eta - 4\eta a + 2a^2.$$

We can insert these expressions into the gap condition in form (6.17) and shift and re-scale variables $z = \alpha_r + i\alpha_i\xi$ to zoom in on the opening gap. This yields a new equation implicitly defining a as a function of η.

$$\widehat{G}(\eta, a) = \frac{1}{i\alpha_i^2} G(\mu, \alpha) = \int_{-1}^{1} \frac{2(a-\eta) + i\alpha_i\xi}{(z(\xi) - iq)^{1/2}(z(\xi) + iq)^{1/2}} \sqrt{1-\xi^2}\mathrm{d}\xi = 0. \tag{6.20}$$

This new equation in now invertible at the origin

$$\left.\frac{\partial \widehat{G}}{\partial a}\right|_{(\eta=0,\ a=0)} = \int_{-1}^{1} \frac{2 + i\sqrt{2}\xi}{\sqrt{\mu^{*2} + q^2}} \sqrt{1-\xi^2}\mathrm{d}\xi = \frac{\pi}{\sqrt{\mu^{*2} + q^2}} \neq 0.$$

Expanding (6.20) in powers of η and keeping only linear terms, we get an expansion for α of the form:

$$\alpha_r(\eta) = \mu^* + \frac{4}{3}\eta + \mathcal{O}(\eta^2) \tag{6.21}$$

$$\alpha_i(\eta) = 2\sqrt{-\mu^*\eta}\,[1 + \mathcal{O}(\eta)]\,. \tag{6.22}$$

As we move μ away from the first breaking time α moves into $\mathbb{C}^{++}$ as expected. For $\alpha \in \mathbb{C}^{++}$ the Jacobian determinant is finite and the implicit function theorem implies the existence of a $\alpha(\mu)$ and bounds its derivative. At the very least, the solution exists until α leaves $\mathbb{C}^{++}$. This could happen in exactly three ways: α could come back to the real axis, α could go to ∞, or α could hit the imaginary axis. We can rule out the first two possibilities. The moment condition places a lower bound on α_i:

$$\alpha_i \geq 2\sqrt{\frac{q^2}{2} - \mu^2}$$

which is strictly positive for $0 \leq \mu < \mu^*$. Similarly, for any fixed μ if we allow $\alpha \to \infty$ along the manifold defined by the moment condition then the gap integral (6.19) approaches a nonzero constant. The only possible way for α to leave $\mathbb{C}^{++}$ is to hit the imaginary axis. In fact, this must happen. By rewriting the integrand of $\mathbf{G}$ in real and imaginary parts one can show that to satisfy the gap condition a necessary condition is $\alpha_r \leq \mu$. So as we decrease μ to zero there exists a first time that α hits the imaginary axis. However, there is exactly one solution of the moment-gap conditions for α purely imaginary: $\mu = 0,\ \alpha = iq$. This completes the proof of the proposition. □

6.2.3 The band contour γ_b, and gap contour γ_g

With α chosen to satisfy the moment and gap conditions we define the g-function with two cuts by integrating $h(z)$:

$$g_2(z) = \frac{1}{2}\theta(z) - 2t\int_{-iq}^{z} S_2(\lambda)\,[\lambda + \alpha_r - 2\mu]\,\mathrm{d}\lambda, \tag{6.23}$$

and let $\varphi = \theta - 2g_2(z)$, or

$$\varphi = 4t\int_{-iq}^{z} S_2(\lambda)\,[\lambda + \alpha_r - 2\mu]\,\mathrm{d}\lambda. \tag{6.24}$$

In both integrals the contour of integration is understood to avoid the contour $\gamma \cup \gamma^*$. Thus defined, g_2 is a single-valued analytic function in $\mathbb{C}\backslash(\gamma \cup \gamma^*)$ with continuous boundary values on the cut $\gamma \cup \gamma^*$. To see this let C_1 and C_2 be two allowable contours connecting $-iq$ to a point $z \notin \gamma \cup \gamma^*$. Then the difference $C_1 - C_2$ is an integer number of loops enclosing $\gamma \cup \gamma^*$. The moment condition (6.14) implies that integrating the integrand of (6.23) over such a contour gives zero. We also need the limiting value of $g_2(z)$ at ∞ which the moment condition guarantees exists and is well defined,

$$g(\infty) = \frac{1}{2}\theta(-iq) + 2t\int_{-iq}^{\infty} (\lambda - 2\mu)(1 - S_2(\lambda)) - \alpha_r S_2(\lambda)\,d\lambda. \tag{6.25}$$

For $z \in \gamma \cup \gamma^*$ the boundary values of $g_2(z)$ satisfy the following jump relations:

- For $z \in \gamma_b \cup \gamma_b^*$,

$$\begin{aligned} g_{2+}(z) + g_{2-}(z) &= \theta(z) \\ g_{2+}(z) - g_{2-}(z) &= \varphi_{0-}(z) = -\varphi_{0+}(z) \end{aligned} \tag{6.26}$$

- For $z \in \gamma_g \cup \gamma_g^*$,

$$\begin{aligned} g_{2+}(z) - g_{2-}(z) &= \Omega \\ g_{2+}(z) + g_{2-}(z) &= \theta(z) - \varphi_{0+}(z) - \Omega \end{aligned} \tag{6.27}$$

  where Ω is a real constant defined by the loop integral

$$\Omega = 2t\int_{C_{\gamma_b^*}} S_2(\lambda)[\lambda + \alpha_r - 2\mu]\mathrm{d}\lambda \tag{6.28}$$

  and the contour $C_{\gamma_b^*}$ is any positively oriented loop enclosing γ_b^* and avoiding γ_b.

Verifying the jump relations is simple algebra with the exception of (6.26) which follows from the gap condition,

$$\mathbf{G}(\mu, \alpha) = \int_{\bar{\alpha}}^{\alpha} S_2(\lambda)\,[\lambda + \alpha_r - 2\mu]\,\mathrm{d}\lambda = 0,$$

for $z \in \gamma_b$.

Local behavior of φ_0 at branch and critical points It remains to show that we can chose the contour γ_b connecting α to iq and its conjugate in the lower half-plane to satisfy the condition 4 in (6.9). That is, we need to establish the existence of contours connecting the endpoints of support along which the phase

$$\varphi = \theta - 2g_2 = 4t \int_{-iq}^{z} S_2(\lambda) \left[\lambda + \alpha_r - 2\mu\right] \mathrm{d}\lambda$$

is purely real. Let

$$L(\mu) = \{z \ : \ \mathrm{Im}(\varphi(z)) = 0\}.$$

As in the one-cut case, determining the structure of $L(\mu)$ follows from analysis of the local behavior near the critical and branching points of φ and the simplicity of the level sets of harmonic functions.

The function φ has four branching points at $\pm iq$, α, and $\bar{\alpha}$ and one critical point, z_0, given by

$$z_0 = 2\mu - \alpha_r = -\frac{x - L}{2t} - \alpha_r(x, t). \tag{6.29}$$

The local behavior at the points is $\overline{\mathbb{C}^-}$ is given below. By symmetry the behavior at points in $\mathbb{C}^+$ are the same. Let $k \neq 0$, then

- Near $z = -iq$, $\varphi(z) = k(z - iq)^{1/2}[1 + \mathcal{O}((z - iq))]$. A single trajectory leaves $z = iq$ along which $\mathrm{Im}(\varphi_0) = 0$.
- Near $z = \bar{\alpha}$, $\varphi_\pm(z) = \mp\Omega + k(z - \alpha)^{3/2}[1 + \mathcal{O}((z - \alpha))]$, where Ω is the real constant defined by (6.28). There are three trajectories of $L(\mu)$ emerging from $z = \alpha$ separated by angles of $\pi/3$.
- Near $z = \infty$, $\varphi(z) = tz^2 + \mathcal{O}(z)$. It follows that that there are four trajectories of $L(\mu)$ approaching infinity that are asymptotic to the coordinate axes.
- Near $z = z_0$, $\varphi(z) = \varphi(z_0) + k(z - z_0)^2 + \mathcal{O}((z - z_0)^3)$, $c \neq 0$. So there are four trajectories of $L(\mu)$ through $z = z_0$ at angles of $k\pi/2$ for $k = 0, 1, 2, 3$.

Global Structure of $L(\mu)$. Selecting the band and gap contours.
The symmetry $\varphi_0(z) = \overline{\varphi_0(\bar{z})}$ implies that $\mathbb{R} \subset L(\mu)$. The question remains how the trajectories in the upper and lower half-planes connect. There are three trajectories emerging from α and exactly three termini for these trajectories: one at $i\infty$, one at iq, and one at z_0 (See Fig. 6.4). Because the level sets of harmonic functions do not admit homoclinic orbits, each trajectories emerging from α connect to exactly one of $i\infty$, iq, z_0. This exhaust the possible endpoints for trajectories of $L(\mu)$ and thus completes the structure of the level set. In particular, for each value of $\mu \in [0, \mu^*]$ the level set $L(\mu)$ contains a arc connecting α to iq. Since $L(\mu)$ is independent of the branch cut of g_2, we now select the branch cut to lie along this contour and denote it by γ_b and its counterpart in $\mathbb{C}^-$ by γ_b^*.

To complete our choice definitions we need to selected the gap contour γ_g connecting α to z_0. The exact nature of this contour is not important as the jump matrix V_N is

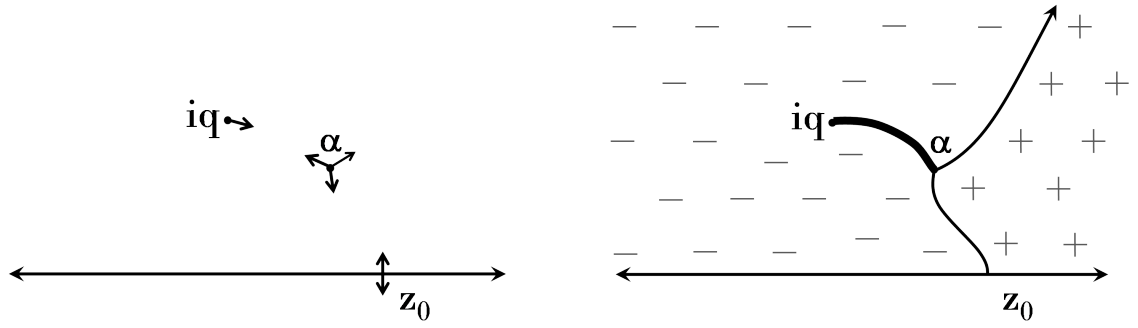

FIGURE 6.4. *Left*: Local structure of $L(\mu) = \{z : \mathrm{Im}(\varphi(z)) = 0\}$ near the branch and stationary phase points in $\mathbb{C}^+$. *Right*: The global structure of $L(\mu)$ in $\mathbb{C}^+$ and the resulting sign structure of $\mathrm{Im}(\varphi(z))$. The symmetry $\varphi(z) = \overline{\varphi(\bar{z})}$ completes the picture in $\mathbb{C}^-$.

exponentially near the identity away from the endpoints. We only require that γ_g be choose so that the off-diagonal entries of V_N decay as fast as possible near the endpoints α and z_0. By examining the local behavior of φ_0 near these points we see that γ_g must leave z_0 at an angle of $\pi/4$ from the real axis and approaches α at an angle π with respect to γ_b to maximize the local decay.

The other phase function, $\varphi_1(z)$

Up to this point we have supposed that the pole removal scheme in (6.4) is still a valid scheme after conjugating the phases by $g_2(z)$. The validity of this scheme is equivalent to requiring that the phase $\varphi_1(z) = \varphi + 4L\nu$ has a stationary phase point $z = z_1$ in the left half plane from which to open the lenses in equation (6.4) that defined the transformation $m(z) \mapsto M(z)$ such that $\mathrm{Im}(\varphi_1(z))$ satisfy the correct inequalities to guarantee that the jump matrices defined on Γ_1, Γ_2, and their complex conjugates, are near identity at any fixed distance from z_1. Up to the first breaking time $T_1(x)$, the one-cut phase function $\phi_1 = \theta - 2g_1(z) + 4L\nu(z)$ satisfies the correct inequalities. This leads one to believe that for some positive amount of time the inequality should persist in the two-cut case, where $\phi_1(z)$ is replaced by $\varphi_1(z)$. Indeed, as we now show, for each μ in the two cut regime, that is $0 < \mu < q/\sqrt{2}$, there exists a maximal positive time $T_2(\mu)$ such that for $0 < t < T_2(\mu)$ the pole removal scheme we have introduced in (6.4) remains valid.

Consider the function φ_1, which using formula (6.24) and (6.29) takes the form,

$$\varphi_1(z) = \varphi(z) + 4L\nu = 4t \int_{-iq}^{z} S_2(\lambda)(\lambda - z_0)\mathrm{d}\lambda + 4L\nu(z). \tag{6.30}$$

The symmetry $\varphi_1(z) = \overline{\varphi_1(\bar{z})}$ tells us both that the level set $\mathrm{Im}(\varphi_1) = 0$ is symmetric and contains the real line. To determine the remaining structure of the set $\mathrm{Im}(\varphi_1) = 0$ we need to understand the location of the zeros of the derivative φ_1'.

Proposition 21. *For each $\mu \in (0, q/\sqrt{2})$ there exists a $T_2(x) > T_1(x)$ such that for each t, $T_1(x) < t < T_2(x)$, the derivative*

$$\frac{d\varphi_1}{dz} = \frac{4t}{\nu}\Big[(z-\alpha)^{1/2}(z-\bar{\alpha})^{1/2}(z-z_0) + \frac{L}{t}z\Big]. \tag{6.31}$$

has exactly two zeros labeled z_1 and ξ_1 which lie on the negative half line such that $z_1 < \xi_1 < 0$. The upper time limit which we call the second breaking time *is characterized as the unique time $T_2(x)$ for each fixed μ such that*

$$\begin{cases} \varphi_1'(z) = 0 \\ \varphi_1''(z) = 0 \end{cases} \tag{6.32}$$

has a simultaneous real solution $z = z_c$.

Proof. Clearly the derivative $\varphi'(z)$ given by (6.31) has a zero at z if and only if

$$f(z) = (z-\alpha)^{1/2}(z-\bar{\alpha})^{1/2}(z-z_0) + \frac{L}{t}z \tag{6.33}$$

vanishes at z. We note that the square-root $(z-\alpha)^{1/2}(z-\bar{\alpha})^{1/2}$ is cut on $\gamma_g \cup \gamma_g^*$, which passes through the real axis at z_0, and $\sim z^2$ for z large. Now, f lifts naturally to a meromorphic function on the genus zero Riemann surface S associated with the function $(z-\alpha)^{1/2}(z-\bar{\alpha})^{1/2}$. As a function over S, f has exactly four poles counting multiplicity: f has a double pole at both pre-images of ∞ and is elsewhere holomorphic. Thus over S, because f is meromorphic it must also have exactly four zeros.

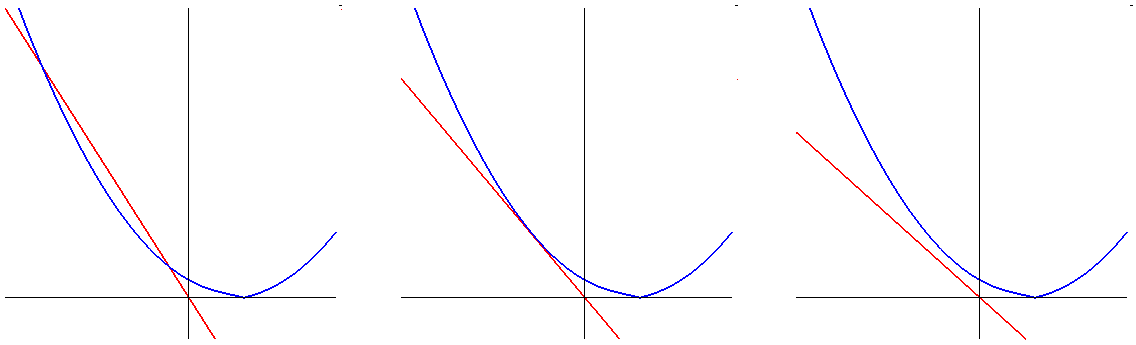

FIGURE 6.5. The motion of the critical points z_1 and ξ_1 as t increases for a fixed value of μ. Each graph depicts the functions $(z-\alpha)^{1/2}(z-\bar{\alpha})^{1/2}(z-z_0)$ and $-Lz/t$ for increasing values of t as we move left to right. The values of z at which intersections occur are the critical points z_1 and ξ_1. As t increases the two distinct real zeros (left picture) converge to a single higher order critical point (middle picture) after which there are no real critical points (right picture).

Now fix $\mu \in (0, q/\sqrt{2})$. This fixes the values of α and z_0 so t is the only free parameter in

(6.31). By elementary estimates, graphing suffices, one can easily show that on the second sheet, that is the sheet on which $(z-\alpha)^{1/2}(z-\bar{\alpha})^{1/2} \sim -z^2$ for z large, there are always two positive real zeros. Using the same logic, one can show that for t suitably small, there are exactly two real zeros $z_1 < \xi_1 < 0$ on the first sheet, see Figure 6.5. This exhaust the zeros. As we let t increase the values z_1 and ξ_1 converge on a unique point z^*, $T_2(x)$ is then the time at which $z_1 = z^* = \xi_1$, but at this point the second derivative φ_1'' also vanishes. Finally, note that it makes sense to write $T_2(x)$ instead of $T_2(\mu)$ because when we fix μ to find the critical value of t, we can then use (6.16) to change variables from (μ, t) to (x, t). □

For $t < T_2(\mu)$, we determine the structure of the level set $L^1 = \{z : \mathrm{Im}(\varphi_1(z)) = 0\}$, by patching together the local structure near the critical and branch points. Near z_1 and ξ_1, φ_1 is quadratic; from each point a single trajectory of L^1 enters $\mathbb{C}^+$. The local behavior of $\mathrm{Im}(\varphi_1)$ near iq and near ∞ is identical to the local structure of φ_0 given above. However, at the branching points α and $\bar{\alpha}$, $\mathrm{Im}(\varphi_1) = 4L\,\mathrm{Im}(\nu) \neq 0$ and so the level $\mathrm{Im}(\varphi_1) = 0$ is bounded away from these branch points. In fact more is true, recalling the sign structure of $\mathrm{Im}(\varphi(z))$, see Figure 6.4, and since $\mathrm{Im}(4L\nu(z)) > 0$ for all z in $\mathbb{C}^+ \backslash \{iy : 0 < y < q\}$, the zero-level set of $\mathrm{Im}(\varphi_1(z))$ must lie completely to the left of the set $L(\mu) = \{z : \mathrm{Im}(\varphi(z)) = 0\}$, and may only approach it at z_0 and iq, see Figure 6.6.

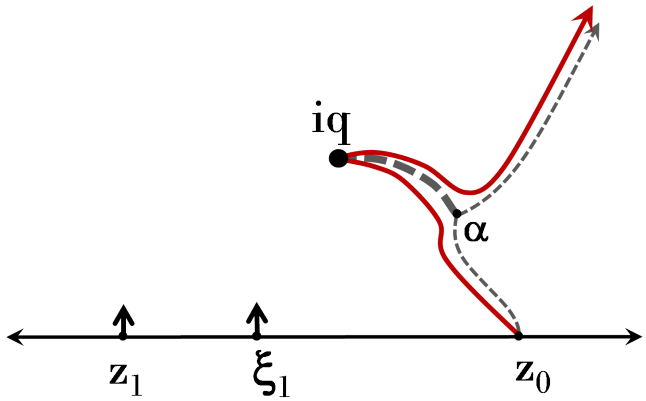

FIGURE 6.6. The local structure of the level set $L^1 = \{z : \mathrm{Im}(\varphi_1(z)) = 0\}$ in $\mathbb{C}^+$ near the critical points z_1 and ξ_1 and the branch point iq. The zero-level of $\mathrm{Im}(\varphi(z))$ is shown as a dashed line. The zero-level set of $\varphi_1(z)$, L^1, must lie everywhere to the left of the solid red line.

Let $\mathcal{U}$ be the subset of $\mathbb{C}^+$ such that $\mathrm{Im}(\varphi(z)) < 0$, see Figure (6.4). This is a connected open subset of $\mathbb{C}^+$ which contains the level set $L^1_+ = \{z \in \mathbb{C}^+ : \mathrm{Im}(\varphi_1(z)) = 0\}$. From our local analysis, two trajectories of L^1 enter $\mathcal{U}$ from the real axis, one at z_1 and the other at ξ_1. Because φ_1 is analytic in $\mathcal{U}$ and L^1 is bounded away from $L(\mu)$ except near the branching point iq, these two trajectories may leave $\mathcal{U}$ only at ∞ or iq and due to the local structure of φ_1 only one trajectory may leave at either point. Suppose that the trajectory entering $\mathcal{U}$ at z_1 leaves at iq. Then, necessarily the trajectory entering $\mathcal{U}$ at ξ_1

leaves at ∞. The trajectory from z_1 to iq, the real axis, and the boundary of $\mathcal{U}$ enclose ξ_1. Thus, the two trajectories must intersect at some point $\zeta \in \mathcal{U}$, forming a closed loop on which $\mathrm{Im}(\varphi_1(z)) = 0$. However, by the maximum principle this would imply that $\varphi_1(z) = 0$ everywhere in the interior of the loop which is false. Therefore, the trajectory entering $\mathcal{U}$ at z_1 must leave at ∞. The resulting structure of the level set $L^1 = \{z : \mathrm{Im}(\varphi_1(z)) = 0\}$ and the sign of $\mathrm{Im}(\varphi_1)$ off L^1 are given in Figure 6.7.

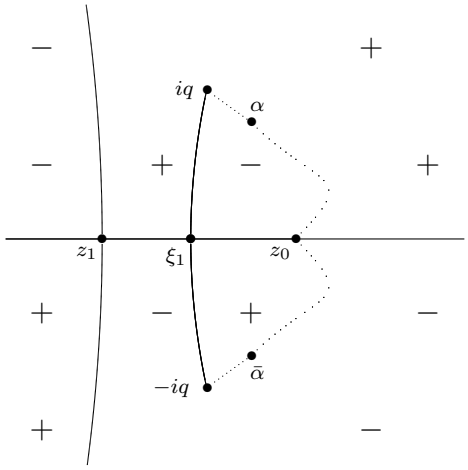

FIGURE 6.7. The sign structure of $\mathrm{Im}(\varphi_1)$. The solid lines indicate the level set $\mathrm{Im}(\varphi_1) = 0$ while the dotted lines indicate its branch cut.

Recalling the transformation defined by (6.4) which removed the poles from the RHP, we see that we may choose the contours Γ_1 and Γ_2 to lie completely regions where the resulting jump matrices are exponentially near identity. Thus for each (x, t) such that $0 < \mu < q/\sqrt{2}$ and $0 < t < T_2(\mu)$, the pole removal scheme introduced in (6.4) results in jumps exponentially near identity away from the real axis and the contour $\gamma \cup \gamma^*$ where the g-function is supported. This defines the region of space-time where our twice-cut g-function will allow us to solve the RHP. For $t > T_2(\mu)$, our pole removal scheme no longer results in jumps near identity near $z = z_1$. New analysis is needed to move into this new region of space time. We leave this to future work.

6.2.4 Opening lenses in the two-cut case

The g-function with two-cuts stabilizes the jumps along $\gamma \cup \gamma^*$. However, there are still rapid oscillations left on the real interval (z_1, ∞). In this section we open lenses to move these oscillations into regions of decay. The factorizations (5.11) and (5.12) we used in Section 5.1.1 to remove the oscillations on the interval (z_1, z_0) remain valid here, however the new endpoints α and $\bar{\alpha}$ deform the contours that define the lenses. Consider the contours Γ_3 and Γ_0 as defined in Figure 6.8. As with the contour γ_g the exact definition of these contours are flexible, we will take advantage of this flexibility when constructing parametrices near each endpoint.

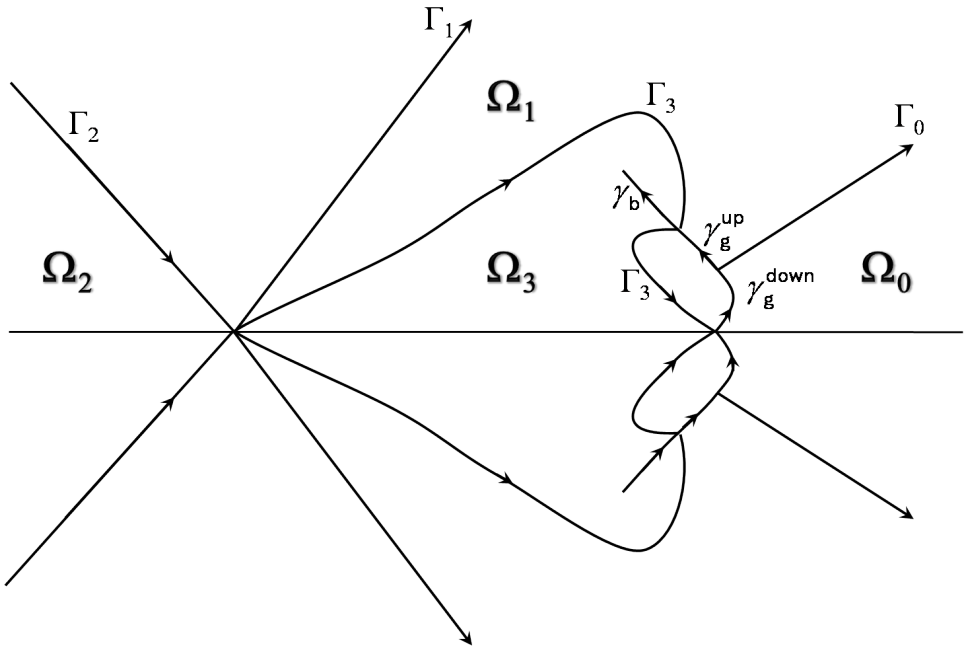

FIGURE 6.8. Diagram of the contours Γ_3 and Γ_0 and the regions Ω_3 and Ω_0. Note that Γ_3 is oriented from z_1 to z_0 through α. By introducing the contour Γ_0 we divide the gap contour γ_g into two pieces which we denote γ_g^{up} and γ_g^{down}. Symmetry defines the contours and regions in the lower half-plane.

With these contours in place, let Ω_3 be the region enclosed by Γ_3 and the real axis and Ω_0 the region enclosed by Γ_0, γ_g, and the real axis; let Ω_3^* and Ω_0^* be their complex conjugates. We now open lenses to define a new unknown $P(z)$ which has no oscillatory jump matrices.

$$N(z) = \begin{cases} P(z)e^{\frac{i}{\epsilon}g_2 \operatorname{ad}\sigma_3} R_0 & : \quad z \in \Omega_0 \\ P(z)e^{\frac{i}{\epsilon}g_2 \operatorname{ad}\sigma_3} \widehat{R}_0^* & : \quad z \in \Omega_3 \\ P(z)e^{\frac{i}{\epsilon}g_2 \operatorname{ad}\sigma_3} R_0^{-*} & : \quad z \in \Omega_0^* \\ P(z)e^{\frac{i}{\epsilon}g_2 \operatorname{ad}\sigma_3} \widehat{R}_0^{-1} & : \quad z \in \Omega_3^* \\ P(z) & : \quad \text{elsewhere} \end{cases} \tag{6.34}$$

The resulting unknown $P(z)$ has jumps on the set of contours $\Gamma_P = (-\infty, z_0) \cup (\gamma \cup \gamma^*) \bigcup_{i=0}^{3} \Gamma_i$. As in the one-cut case, the definition of $P(z)$ in Ω_3 has a square root singularity at $z = iq$, which imposes a growth condition on the entries of $P(z)$ (cf. (6.35) and (6.36)).

RHP for $P(z)$: Find a 2×2 matrix P that satisfies the following properties:

- $P(z)$ is analytic in the set $\mathbb{C} \backslash \Gamma_P$ and has continuous boundary values as z approaches the contours.

- $P(z) = I + \mathcal{O}(1/z)$ as $|z| \to \infty$.
- Near $z = iq$, the growth of $P(z)$ is prescribed by

$$P(z) = \mathcal{O}\begin{pmatrix} 1 & |z-iq|^{-1/2} \\ 1 & |z-iq|^{-1/2} \end{pmatrix}. \tag{6.35}$$

  Near $z = -iq$, the growth of $P(z)$ is prescribed by

$$P(z) = \mathcal{O}\begin{pmatrix} |z+iq|^{-1/2} & 1 \\ |z+iq|^{-1/2} & 1 \end{pmatrix}. \tag{6.36}$$

- For $z \in \Gamma_P$, the boundary values of P satisfy the relation $P_+ = P_- V_P$ where

$$V_P = \begin{cases} e^{\frac{i}{\epsilon} g_2 \operatorname{ad} \sigma_3} & : \quad z \in \Gamma_0 \\ e^{\frac{i}{\epsilon} g_2 \operatorname{ad} \sigma_3} \widehat{R}_0^* & : \quad z \in \Gamma_3 \\ \begin{pmatrix} 0 & -w^{-1} \\ w & 0 \end{pmatrix} & : \quad z \in \gamma_b \\ \begin{pmatrix} e^{-\frac{i}{\epsilon}\Omega} & 0 \\ we^{\frac{i}{\epsilon}(\varphi_+ + \Omega)} & e^{\frac{i}{\epsilon}\Omega} \end{pmatrix} & : \quad z \in \gamma_g^{up} \\ \begin{pmatrix} e^{-\frac{i}{\epsilon}\Omega} & 0 \\ r_{0+}(z)e^{\frac{i}{\epsilon}(\varphi_+ + \Omega)} & e^{\frac{i}{\epsilon}\Omega} \end{pmatrix} & : \quad z \in \gamma_g^{down} \\ (1+|r_0|^2)^{\sigma_3} & : \quad z \in (z_1, z_0) \\ \begin{pmatrix} e^{-\frac{i}{\epsilon}\Omega} & -\bar{r}_{0+}(z)e^{-\frac{i}{\epsilon}(\varphi_+ + \Omega)} \\ 0 & e^{\frac{i}{\epsilon}\Omega} \end{pmatrix} & : \quad z \in \gamma_g^{*\,down} \\ \begin{pmatrix} e^{-\frac{i}{\epsilon}\Omega} & we^{-\frac{i}{\epsilon}(\varphi_+ + \Omega)} \\ 0 & e^{\frac{i}{\epsilon}\Omega} \end{pmatrix} & : \quad z \in \gamma_g^{*\,up} \\ \begin{pmatrix} 0 & w \\ -w^{-1} & 0 \end{pmatrix} & : \quad z \in \gamma_b^* \\ e^{\frac{i}{\epsilon} g_2 \operatorname{ad} \sigma_3} \widehat{R}_0 & : \quad z \in \Gamma_3^* \\ e^{\frac{i}{\epsilon} g_2 \operatorname{ad} \sigma_3} R_0^* & : \quad z \in \Gamma_0^* \\ V_N & : \quad \text{elsewhere} \end{cases} \tag{6.37}$$

The new RHP for $P(z)$ has a complicated cut structure (see Figure 6.8 for a diagram of the contours), but the jumps all have stable semi-classical limits; away from the real axis and the contour $\gamma \cup \gamma^*$ the jumps are exponentially close to the identity. As in the one-cut case the jumps on the real axis and the bands are ϵ independent. The new wrinkle in the two cut case is the jump in the gap $\gamma_g \cup \gamma_g^*$. As we will see, the sensitive, but simple, dependence of these jumps on ϵ complicates the analysis of the outer problem and leads to significantly more interesting behavior in the asymptotics for NLS.

Before we pursue the construction of a global approximation we introduce a final transformation that reduces the jumps on the limiting contours to constants. Consider the following two scalar RHPs:

1. **The δ RHP,** find a scalar function $\delta(z)$ such that
   - δ is analytic in $\mathbb{C}\backslash(-\infty, z_0)$.
   - As $|z| \to \infty$,
   $$\delta(z) = 1 + \mathcal{O}(1/z). \tag{6.38}$$
   - For $z \in (-\infty, z_0)$, the boundary values of δ satisfy the relationship
   $$\delta_+(z)/\delta_-(z) = \begin{cases} (1+|r_0(z)|^2)^2 & : \quad z \in (-\infty, z_1) \\ (1+|r_0(z)|^2) & : \quad z \in (z_1, z_0) \end{cases} \tag{6.39}$$

2. **The s RHP,** find a scalar function $s(z)$ such that
   - $s(z)$ is analytic in $\mathbb{C}\backslash(\gamma_b \cup \gamma_b^*)$
   - s takes continuous boundary values away from the endpoints where 1/4-root singularities are admitted. The boundary values satisfy the following relations:
   $$s_+(z)s_-(z) = \begin{cases} -iw(z)\delta^{-2}(z) & : \quad z \in \gamma_b \\ -iw^{-1}(z)\delta^{-2}(z) & : \quad z \in \gamma_b^* \end{cases} \tag{6.40}$$

If we can find the solutions of these problems – the first has already been described in Section 5.4.1 – we reduce the outer model parametrix by introducing the new variable $Q(z)$ using the relation

$$P(z) = \beta(z)Q(z)s^{\sigma_3}(z)\delta^{\sigma_3}(z). \tag{6.41}$$

where $\beta(z)$ is the scalar function,

$$\beta(z) = \left(\frac{z-iq}{z-\alpha}\right)^{1/4}\left(\frac{z+iq}{z-\bar{\alpha}}\right)^{1/4}, \tag{6.42}$$

cut on $\gamma_b \cup \gamma_b^*$ normalized to approach unity at ∞.

It is easy to check using (6.37)-(6.42) that this transformation removes the jump from the real axis and replaces the jumps in the bands γ_b and γ_b^* with the constant twist jump

$$V_Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

To properly write down the rest of the new RHP for $Q(z)$ we need to first understand the functions $\delta(z)$ and $s(z)$.

The δ problem given above was described in full detail in the Section 4.2. We remind the reader that using Plemelj formula we can write its solution in the form

$$\delta(z) = \exp\left(\frac{1}{2\pi i}\int_{-\infty}^{z_1}\log(1+|r_0(\lambda)|^2)\frac{d\lambda}{\lambda - z} + \frac{1}{2\pi i}\int_{-\infty}^{z_0}\log(1+|r_0(\lambda)|^2)\frac{d\lambda}{\lambda - z}\right). \tag{6.43}$$

The function is everywhere bounded, has no zeros, and its behavior near the endpoints of integration is summarized in Prop. 5 and formula (4.28).

The problem for s, is similar, but not identical, to the problem for $s(z)$ we considered in the previous chapter. As in the previous chapter we write its solution by first introducing the function

$$a(z) = \left(\frac{z - iq}{z + iq}\right)^{1/4}, \tag{6.44}$$

which is cut along the contour $\gamma \cup \gamma^*$ and approachs unity for$z$ large. Define

$$\mathcal{R}(z) = \sqrt{(z - iq)(z - \alpha)}\sqrt{(z + iq)(z - \bar{\alpha})} \tag{6.45}$$

branched on $\gamma_b \cup \gamma_b^*$ and normalized so that $\mathcal{R} \sim z^2$ for z large. If we now define a new scalar unknown $h(z)$ using the relation $s(z) = a(z) \exp\left(\mathcal{R}(z)h(z)\right)$ the resulting RHP for $h(z)$ has jumps that are both bounded and nonzero away from the endpoints of the cuts. The jump relations for h are given by

$$h_+(z) - h_-(z) = \frac{v(z)}{\mathcal{R}_+(z)}, \tag{6.46}$$

where,

$$v = \begin{cases} \log\left(\frac{2(z+iq)}{iq}\delta^{-2}(z)\right) & : \quad z \in \gamma_b \\ -\frac{i\pi}{2} & : \quad z \in \gamma_g \cup \gamma_g^* \\ \log\left(\frac{iq}{2(z-iq)}\delta^{-2}(z)\right) & : \quad z \in \gamma_b^* \end{cases}$$

Using the Plemelj formula we can solve for the $h(z)$, from which we recover a formula for the desired function $s(z)$:

$$s(z) = a(z) \exp\left(\frac{\mathcal{R}(z)}{2\pi i} \int\limits_{\gamma \cup \gamma^*} \frac{v(\lambda)}{\mathcal{R}_+(\lambda)} \frac{\mathrm{d}\lambda}{\lambda - z}\right). \tag{6.47}$$

In our construction of the function $s(z)$ we did not specify a normalization condition on $s(z)$ for large z. In fact, (6.47) has an essential singularity at infinity. The function $\mathcal{R}$ has the following asymptotic expansion for large z,

$$\mathcal{R}(z) = z^2 - \mathrm{Re}(\alpha)z + \frac{q^2 - \mathrm{Im}(\alpha)^2}{2} + \mathcal{O}(1/z). \tag{6.48}$$

Expanding the integral in (6.47) for large z we obtain the following asymptotic expansion valid for z suitably large,

$$s(z) = e^{ip(z)}\left[1 + \mathcal{O}(z^{-1})\right], \tag{6.49}$$

where p is the linear function

$$\begin{aligned} p(z) &= p_1 z + p_0 \\ p_1 &= -\frac{1}{2\pi}\int_{\gamma\cup\gamma^*} \frac{v(\lambda)}{\mathcal{R}_+(\lambda)}\mathrm{d}\lambda \\ p_0 &= -\frac{1}{2\pi}\int_{\gamma\cup\gamma^*} (\lambda - \mathrm{Re}(\alpha))\frac{v(\lambda)}{\mathcal{R}_+(\lambda)}\mathrm{d}\lambda. \end{aligned} \tag{6.50}$$

Using the symmetry of the jump matrices, $\overline{v(\lambda)} = -v(\bar{\lambda})$, and the conjugation anti-symmetry of γ, $\bar{\gamma} = -\gamma^*$ imply that the coefficients p_1 and p_0 are purely real.

Later, when we consider the necessary local model problems, we will need to understand the behavior of $s(z)$ near the endpoints α and $\bar{\alpha}$.

To this end, let us introduce the following notation. Let $v_b(z)$ denote the restriction of $v(z)$ to the bands $\gamma_b \cup \gamma_b^*$ analytically extended to each half-plane and $v_g(z)$ the restriction to the gaps $\gamma_g \cup \gamma_g^*$ extended to $\mathbb{C}$, that is

$$v_b(z) = \begin{cases} \log\left(\frac{2(z+iq)}{iq}\delta^{-2}(z)\right) & : \quad z \in \mathbb{C}^+ \\ \log\left(\frac{iq}{2(z-iq)}\delta^{-2}(z)\right) & : \quad z \in \mathbb{C}^- \end{cases},$$

and,

$$v_g(z) = -i\pi/2.$$

Finally, let us extend $\gamma \cup \gamma^*$ by introducing the contour $\gamma_{ext} = \{z = iy : |y| \geq q\}$ oriented upward. We use this contour to define another branch of the square root $\mathcal{R}(z)$. Define

$$\widehat{\mathcal{R}}(z) = \begin{cases} -\mathcal{R}(z) & : \quad z \text{ left of } \gamma \cup \gamma^* \cup \gamma_{ext} \\ \mathcal{R}(z) & : \quad z \text{ right of } \gamma \cup \gamma^* \cup \gamma_{ext} \end{cases}, \tag{6.51}$$

this function is branched along the gap $\gamma_g \cup \gamma_g^*$ and the extension γ_{ext}.

Using the notation we have just developed we write the function $s(z)$ as

$$s(z) = a(z)\exp\frac{\mathcal{R}(z)}{2\pi i}\left(\int_{\gamma_b\cup\gamma_b^*}\frac{v_b(\lambda)}{\mathcal{R}_+(\lambda)}\frac{\mathrm{d}\,\lambda}{\lambda - z} - \int_{\gamma_g\cup\gamma_g^*}\frac{v_g(\lambda)}{\widehat{\mathcal{R}}_+(\lambda)}\frac{\mathrm{d}\,\lambda}{\lambda - z}\right).$$

Now for z near the endpoint α or $\bar{\alpha}$ we use residue calculus to write $s(z)$ as

$$s(z) = s_{\mathrm{hol}}(z)e^{\frac{1}{2}\mathcal{R}(z)f(z)}, \tag{6.52}$$

where s_{hol}, given by

$$s_{\mathrm{hol}}(z) = a(z)\exp\left(\frac{i\pi}{4}\frac{\mathcal{R}(z)}{\widehat{\mathcal{R}}(z)} + \frac{1}{2}v_b(z)\right),$$

is holomorphic in neighborhoods of $z = \alpha$ and $z = \bar{\alpha}$. While, $f(z)$ is the function analytic

near $z = \alpha$ and $z = \bar{\alpha}$ given by

$$f(z) = \frac{\mathcal{R}(z)}{4\pi i}\left(\int_{C_g} \frac{v_g(\lambda)}{\widehat{\mathcal{R}}(\lambda)}\frac{\mathrm{d}\,\lambda}{\lambda - z} - \int_{C_b \cup C_b^*} \frac{v_b(\lambda)}{\mathcal{R}(\lambda)}\frac{\mathrm{d}\,\lambda}{\lambda - z}\right). \tag{6.53}$$

Here C_b is a simple positively oriented loop in $\mathbb{C}^+$ enclosing γ_b, C_b^* its complex conjugate, and C_g a simple positively oriented loop enclosing $\gamma_g \cup \gamma_g^*$ without intersecting γ_{ext}.

Let us now consider the resulting RHP for $Q(z)$ which, we recall, was built from $P(z)$ through the transformation defined by (6.41).

RHP for $Q(z)$:

Let $\Gamma_Q = \gamma \cup \gamma^* \bigcup_{k=0}^{3}(\Gamma_k \cup \Gamma_k^*)$. Find a 2×2 matrix $Q(z)$ that satisfies the following properties:

- $Q(z)$ is analytic in the set $\mathbb{C}\backslash\Gamma_Q$ and assumes continuous boundary values as z approaches the contour away from endpoints. At the endpoints $z = iq$ and $z = -iq$, $Q(z)$ admits 1/2-root singularities.

- As $z \to \infty$,
$$Q(z)e^{ip(z)\sigma_3} = I + \mathcal{O}(1/z) \tag{6.54}$$
where $p(z)$ is the polynomial defined by (6.50).

- For $z \in \Gamma_Q$, the boundary values satisfy the relation $Q_+(z) = Q_-(z)V_Q(z)$ where,
$$V_Q(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & : \quad z \in \gamma_b \cup \gamma_b^* \\ [s(z)\delta(z)]^{\mathrm{ad}\,\sigma_3} V_P(z) & : \quad z \in \Gamma_Q\backslash(\gamma_b \cup \gamma_b^*) \end{cases}, \tag{6.55}$$
and $V_P(z)$ is the jump matrix for $P(z)$ defined by (6.37).

Notice the new normalization condition on $Q(z)$. The essential singularity that appears at infinity is a consequence of the asymptotic behavior of $s(z)$ at infinity which we derived in formula (6.49). The asymptotics of $Q(z)$ are such that the product $Q(z)s^{\sigma_3}(z)$ has identity asymptotics at ∞.

We now are in an ideal position to introduce outer and local approximations to arrive at a global parametrix. Unlike the problems outside the support and inside before the breaking time, building the outer model is no longer a triviality. Building this solution will require the introduction of functions over Riemann surfaces and the so called Baker-Akhiezer functions. We pursue these ideas in the following section.

6.3 Building the Outer Parametrix

As before, we build a global parametrix consisting of an outer approximation that solves the limiting problem and inner approximates near those points where the limiting problem

is not uniform. Specifically, we build a matrix

$$A(z) = \begin{cases} A_{z_0} & : \quad z \in \mathcal{U}_0 \\ A_{z_1} & : \quad z \in \mathcal{U}_1 \\ A_\alpha & : \quad z \in \mathcal{U}(\alpha) \\ A_{\bar{\alpha}} & : \quad z \in \mathcal{U}(\bar{\alpha}) \\ O(z) & : \quad \text{elsewhere} \end{cases} \tag{6.56}$$

where the set $\mathcal{U}(p)$ is a suitably small open neighborhood of the point p.

Away from the real axis and the endpoints α and $\bar{\alpha}$ the jump matrices on the contours $\bigcup_{k=0}^{3}(\Gamma_k \cup \Gamma_k^*)$ are exponentially near identity. By simply taking the limit as $\epsilon \to 0^+$ along these contours we arrive at our outer model problem.

RHP for the outer parametrix, $O(z)$: Find a 2×2 matrix valued function $O(z)$ that satisfies the following properties.

- $O(z)$ is analytic in $\mathbb{C}\backslash(\gamma \cup \gamma^*)$ and assumes continuous boundary values as z approaches the contour away from the endpoints $\pm iq$. At $z = \pm iq$, $O(z)$ admits square root singularities.
- As $z \to \infty$,
$$O(z)e^{ip(z)\sigma_3} = I + \mathcal{O}(1/z), \tag{6.57}$$
the function $p(z)$ is the polynomial defined by (6.50).
- The boundary values of $O(z)$ satisfy the jump relationship $O_+(z) = O_-(z)V_O$, where
$$V_O = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & : \quad z \in \gamma_b \cup \gamma_b^* \\ \begin{pmatrix} e^{-\frac{i}{\epsilon}\Omega} & 0 \\ 0 & e^{\frac{i}{\epsilon}\Omega} \end{pmatrix} & : \quad z \in \gamma_g \cup \gamma_g^* \end{cases} . \tag{6.58}$$

6.3.1 Reduction to Hyperelliptic Riemann Hilbert problem

The introduction of the gap, and the corresponding new jump over $\gamma_g \cup \gamma_g^*$, makes the outer problem much more difficult to solve than the corresponding problem in the genus zero case, as described in Section 5.4.1. In the following we build the solution of the outer problem in terms of Θ-functions built on a genus one Riemann surface. The procedure we follow is taken from [DKM$^+$99] and [Dub81].

Before we introduce the Riemann surface, we would like to move the twist jumps off of the bands $\gamma_b \cup \gamma_b^*$. We do this by first re-introducing the contour Γ_{ext} defined by

$$\Gamma_{ext} = \{ z = iy \, : \, |y| \geq q \}$$

oriented up the imaginary axis; together with the contour $\gamma \cup \gamma^*$, Γ_{ext} separates the plane

into left and right halves. Let us also introduce the notation

$$\Gamma_g = \gamma_g \cup \gamma_g^*, \qquad \Gamma_b = \gamma_b \cup \gamma_b^*,$$

to simplify our notation moving forward. We push the twist jumps of $O(z)$ off the bands by making the change of variables,

$$O(z) = \begin{cases} O^{(1)}(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & : \quad z \text{ to the left of } \Gamma_{ext} \cup \Gamma_b \cup \Gamma_g \\ O^{(1)}(z) & : \quad z \text{ to the right of } \Gamma_{ext} \cup \Gamma_b \cup \Gamma_g \end{cases} . \tag{6.59}$$

Using the RHP for $O(z)$ defined by (6.57) and (6.58) we have the following equivalent RHP for $O^{(1)}(z)$:

Riemann Hilbert Problem for $O^{(1)}(z)$:

Find a 2×2 matrix valued function $O^{(1)}(z)$ for $z \in \mathbb{C}$, such that

- $O^{(1)}(z)$ is analytic in $\mathbb{C} \backslash (\Gamma_{ext} \cup \Gamma_g)$ and assumes continuous boundary values as z approaches the contour away from the endpoints $\pm iq$. At $z = \pm iq$, $O^{(1)}(z)$ admits square root singularities.
- As $|z| \to \infty$, $Q(z)$ has the following behavior

$$\begin{aligned} O^{(1)}(z) e^{ip(z)\sigma_3} &= I + \mathcal{O}(1/z) && \text{Re}(z) > 0 \\ O^{(1)}(z) e^{-ip(z)\sigma_3} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \mathcal{O}(1/z) && \text{Re}(z) < 0 \end{aligned} \tag{6.60}$$

- For $z \in \Gamma_g \cup \Gamma_{ext}$ the boundary values satisfy the jump relationship $O_+^{(1)}(z)(z) = O_-^{(1)}(z) V_{O^{(1)}}(z)$

$$V_{O^{(1)}}(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & : \quad z \in \Gamma_{ext} \\ \begin{pmatrix} 0 & e^{-\frac{i}{\epsilon}\Omega} \\ e^{\frac{i}{\epsilon}\Omega} & 0 \end{pmatrix} & : \quad z \in \Gamma_g \end{cases} . \tag{6.61}$$

Next, we introduce the following Riemann surface. Let Σ be the two sheeted cover of the function

$$\widehat{\mathcal{R}}(z) = \sqrt{(z - iq)(z + iq)(z - \alpha)(z - \bar{\alpha})} \tag{6.62}$$

cut along Γ_{ext} and Γ_g which we previously introduced in (6.51). The surface Σ is formed by first opening up the cuts Γ_{ext} and Γ_g and then identifying opposing edges in each sheet, gluing the sheets together along the identified edges to generate the genus one surface Σ. In order to produce smooth contours on Σ from the pre-images of Γ_{ext} and Γ_g we orient the contours as follows: On Σ_1 we orient the left side of each cut upward as we have previously,

on the right side of each cut we reverse the orientation to move downward. On the second sheet Σ_2 we reverse all orientations from those on Σ_1. The opened contours and their orientations are depicted in Figure 6.9.

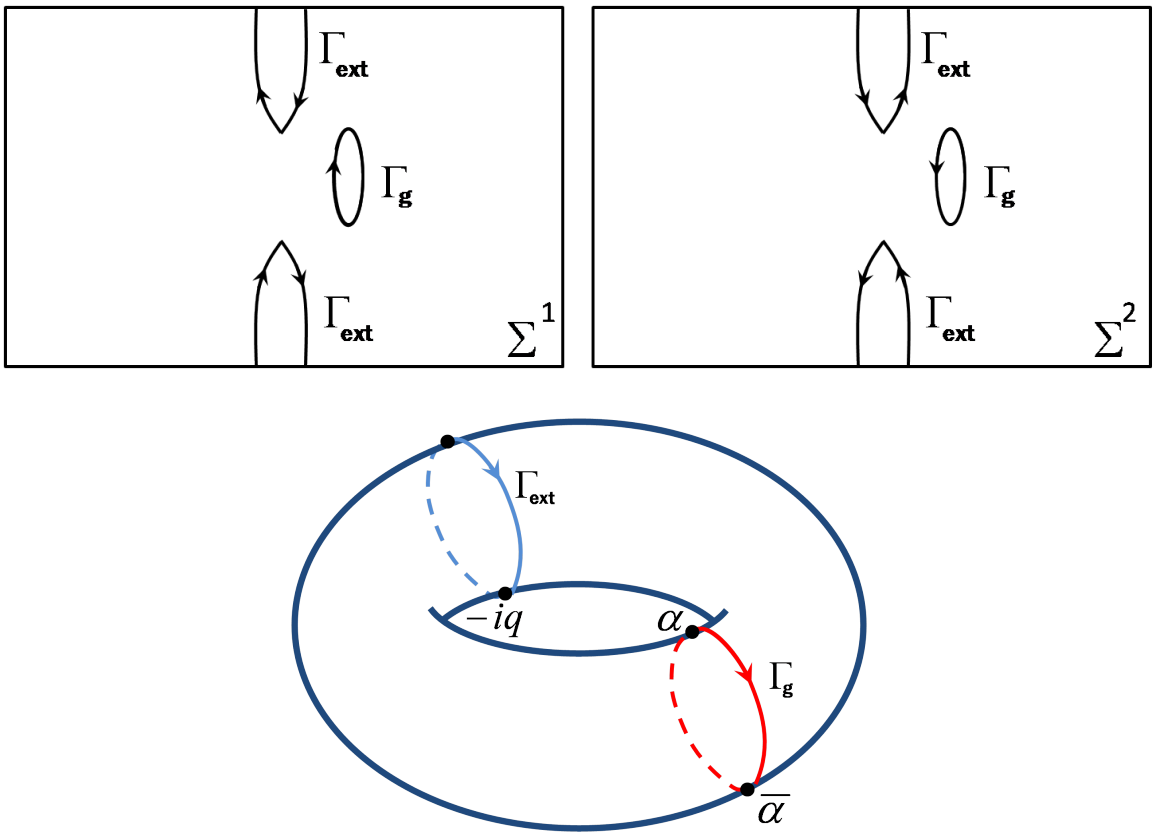

FIGURE 6.9. Above: The Riemann surface Σ represented as two cut copies of the complex plane. Note the orientations, the surfaces are glued together so that each edge attaches to the edge on the opposing sheet with the same orientation. Below: the Riemann surface Σ represented as a torus with the pre-images of the contours Γ_g and Γ_{ext} included.

On the Riemann surface Σ we will denote points by P and the pre-images of z in the first and second sheet by $P^1(z)$ and $P^2(z)$. The sheets Σ^1 and Σ^2 are labeled so that $\widehat{\mathcal{R}}(z(P)) \sim z^2$ for $P \in \Sigma^1$ as $z \to \infty$, with $\mathrm{Re}(z) > 0$. We will also need to denote the two pre-images of ∞. Let ∞_1 be the pre-image associated with $z \to +\infty$ (the real infinity) on Σ_1, and ∞_2 the pre-image associated with $z \to -\infty$ on Σ_1.

Using our chosen ordering of the sheets we define a vector-valued function $v(P)$ for each $P \in \Sigma$ by

$$v(P) = \begin{cases} O_1^{(1)}(z(P)) & : \quad P \text{ in sheet 1 of } \Sigma \\ O_2^{(1)}(z(P)) & : \quad P \text{ in sheet 2 of } \Sigma \end{cases} . \tag{6.63}$$

where $O_j^{(1)}(z)$ is the j^{th} column of the matrix $O^{(1)}(z)$. The new unknown $v(P)$ acquires jumps along the pre-images of the cuts Γ_g and Γ_{ext} and asymptotic behavior near each ∞ by transporting the normalization (6.54) for $Q(z)$ through (6.63). We will abuse notation slightly by referring to the lifting of the contours Γ_g and Γ_{ext} to Σ by their names back on the complex plane. Note that the orientation choices made in the gluing create from each cut a smooth closed loop on Σ, see Figure 6.9.

The resulting jump relations for $v(P)$ are quite simple. Across Γ_{ext} $v(P)$ is analytic, this follows from the twist jump relations on Γ_{ext}: $O_{1+}^{(1)} = O_{2-}^{(1)}$ and $O_{2+}^{(1)} = O_{1-}^{(1)}$. For $z \in \Gamma_g$, we get $v_+(P) = e^{\frac{i}{\epsilon}\Omega} v_-(P)$ from the relations $O_{1+}^{(1)} = e^{\frac{i}{\epsilon}\Omega} O_{2-}^{(1)}$ and $O_{1-}^{(1)} = e^{\frac{i}{\epsilon}\Omega} O_{2+}^{(1)}$.

In addition to the jump across Γ_g, $v(P)$ also admits poles at $z = \pm iq$. Recall, that the RHP for $O^{(1)}$ allowed each entry to admit square-root singularities at $z = \pm iq$. Due to the double-ramification of these points, on the Riemann surface $v(P)$ must admit a pole at each of these points. The resulting RHP for $v(P)$ on the Riemann surface Σ is as follows.

Problem for $v(P)$:

Find a vector-valued function $v : \Sigma \to \overline{\mathbb{C}}$ such that

- $v(P)$ is meromorphic on $\Sigma \backslash \Gamma_g \cup \{\infty_1, \infty_2\}$
- $v(P)$ takes continuous boundary values $v_\pm(P)$ on Γ_g. The boundary values satisfy the jump relations
$$v_+(P) = e^{\frac{i}{\epsilon}\Omega} v_-(P), \quad P \in \Gamma_g \tag{6.64}$$
- $v(P)$ has the following essentially singular behavior at ∞_1 and ∞_2:
$$\begin{aligned} v(P)e^{ip(z)} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathcal{O}(1/z(P)), \qquad P \to \infty_1 \\ v(P)e^{-ip(z)} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathcal{O}(1/z(P)), \qquad P \to \infty_2 \end{aligned} \tag{6.65}$$
where $p(z)$ is the polynomial defined in (6.50).

6.3.2 Explicit construction of the hyperelliptic functions

In this section we derive the solution to the Riemann-Hilbert problem defined by (6.64) and (6.65) for the unknown $v(P)$ over the Riemann surface Σ. To do so we will need to use several results from the general theory of meromorphic functions over compact Riemann Surfaces. We collect the necessary results and standardize notation in Appendix B. Using the notation developed in Appendix B we now restate the problem for our unknown $v(P)$:

Problem for $v(P)$

Find a function $v : \Sigma \to \mathbb{C}$ meromorphic on $\Sigma \backslash \Gamma_g \cup \{\infty_1, \infty_2\}$ such that:

- For $z \in \Gamma_g$, the boundary values of $v(P)$ satisfy

$$v_+(P) = e^{\frac{i}{\epsilon}\Omega} v_-(P) \tag{6.66}$$

- The finite pole divisor $(v)_-$, that is the divisor of v's poles in $\Sigma \backslash \{\infty_1, \infty_2\}$, satisfies the inequality

$$\mathcal{D}^0 - (v)_- \geq 0, \tag{6.67}$$

where,

$$\mathcal{D}^0 = P_{iq} + P_{-iq}. \tag{6.68}$$

- $v(P)$ is essentially singular at $P = \infty_1$ and $P = \infty_2$ and the behavior at these points is given by (6.65).

Our first step towards solving this problem is to introduce functions $f_k(P)$ which removes the poles and normalizes the behavior at each infinity. To wit, consider the divisors $\mathcal{D}^0 - \infty_2$ and $\mathcal{D}^0 - \infty_1$ and the associated spaces of meromorphic functions $\mathcal{L}(\infty_2 - \mathcal{D}^0)$ and $\mathcal{L}(\infty_1 - \mathcal{D}^0)$. Let $k, k' \in \{1, 2\},\ k \neq k'$. A function f_k is in $\mathcal{L}(\infty_{k'} - \mathcal{D}^0)$ precisely when f_k admits at worst simple poles at the points $P = iq$ and $P = -iq$ and $f_k(\infty_{k'}) = 0$. We would like to find particular functions in each space normalized so that $f_k(\infty_k) = 1$.

Claim 3. Let $\mathcal{D}^0$ be defined as in (6.68), and $k, k' \in \{1, 2\},\ k \neq k'$. There exists a unique function $f_k \in \mathcal{L}(\infty_{k'} - \mathcal{D}^0)$ such that $f_k(\infty_k) = 1$. In particular, these functions are given by the formulae

$$f_1(P) = \frac{1}{2}\left[1 + \frac{(z-\alpha)(z-\bar{\alpha})}{\widehat{\mathcal{R}}_\Sigma(P)}\right] \tag{6.69}$$

$$f_2(P) = \frac{1}{2}\left[1 - \frac{(z-\alpha)(z-\bar{\alpha})}{\widehat{\mathcal{R}}_\Sigma(P)}\right] \tag{6.70}$$

where $\widehat{\mathcal{R}}_\Sigma(P)$ is the lifting of $\widehat{\mathcal{R}}(z)$ defined by (6.62) to the Riemann surface Σ.

Proof: By inspection the functions $f_1(P)$ and $f_2(P)$ satisfy the required properties, it remains to show that these are the only such functions. By the Riemann-Roch theorem $r(\infty_{k'} - \mathcal{D}^0) = i(\mathcal{D}^0 - \infty_{k'}) + 1$, if we can show that $i(\mathcal{D}^0 - \infty_{k'}) = 0$ then we are done. Consider first $\mathcal{L}^1(\mathcal{D}^0 - \infty_2)$. This space consists of meromorphic differentials with zeros at $P = iq$ and $P = -iq$ and at worst a single pole at $P = \infty_2$. Since the residues of any meromorphic differential must sum to zero, it follows that $\nu \in \mathcal{L}^1(\mathcal{D}^0 - \infty_2)$ implies that ν is holomorphic. The holomorphic differentials on Σ form a one dimensional vector space that can be represented on each sheet by the formula

$$\nu(P) = \frac{c}{\widehat{R}_\Sigma(P)} \mathrm{d}z(P),$$

where c is a constant to be chosen. For ν to have zeros at both P_{iq} and P_{-iq} we much have $c \equiv 0$, thus $i(\mathcal{D}^0 - \infty_2) = 0$ and $\mathcal{D}^0 - \infty_2$ is nonspecial. An analogous argument shows that

$\mathcal{D}^0 - \infty_1$ is nonspecial. □

The functions f_1 and f_2 are meromorphic with exactly two simple poles at $P = \pm iq$. In addition, each has a simple zero fixed at the opposing infinity. That leaves $f_1(P)$ and $f_2(P)$ each with a single addition zero which we denote P_1 and P_2 respectively. These points are easily calculated to be the distinct pre-images of the same real value,

$$z_f = \frac{|\alpha|^2 - q^2}{\alpha + \bar{\alpha}}, \qquad P_1 = P^1(z_f), \qquad P_2 = P^2(z_f). \tag{6.71}$$

Let $\mathcal{D}_k$ be the divisors of these extra zeros, i.e. $\mathcal{D}_k = P_k$. Because $\mathcal{D}_k$ and $\infty_{k'} - \mathcal{D}^0$ necessarily belong to the same divisor class it follows that the divisors $\mathcal{D}_k$ are nonspecial as well.

Using the function $f(P)$ we define a new function $\xi(P)$ as follows:

$$\xi_1(P) = \frac{v_1(P)}{f_1(P)}, \qquad \xi_2(P) = \frac{v_2(P)}{f_2(P)}. \tag{6.72}$$

The new function $\xi(P)$ must have the following properties.

Properties of the function $\xi(P)$:

- $\xi(P)$ is a meromorphic function on $\Sigma \backslash (\Gamma_g \cup \{\infty_1, \infty_2\})$.
- $\xi(P)$ takes continuous boundary values $\xi_\pm(P)$ for $P \in \Gamma_g$ which satisfy the jump relationship
$$\xi_+(P) = \xi_-(P) e^{\frac{i}{\epsilon}\Omega} \tag{6.73}$$
- The components $\xi_k(P)$ each admit at worst a simple pole at the single point P_k defined by (6.71).
- $\xi(P)$ has the following asymptotic behavior:
$$\begin{aligned} \xi(P)e^{ip(z)} &= \begin{pmatrix} 1 \\ c_2 \end{pmatrix} + \mathcal{O}\left(1/z(P)\right) \qquad P \to \infty_1 \\ \xi(P)e^{-ip(z)} &= \begin{pmatrix} c_1 \\ 1 \end{pmatrix} + \mathcal{O}\left(1/z(P)\right) \qquad P \to \infty_2, \end{aligned} \tag{6.74}$$
where $p(z)$ is the polynomial (6.50) and c_1 and c_2 are undetermined constants.

The scalar components ξ_k are generalizations of *Baker-Akhiezer* function. One can represent Baker-Akhiezer functions as ratios of Θ-functions on Σ. We begin this program here using the notations developed in Appendix B.

To build the Baker-Akhiezer functions we need to select a homology basis on Σ. Since Σ is genus one, this is just a pair of closed curves $\{a_1, b_1\}$ on Σ satisfying $a \circ b = 1$ where we recall the intersection operator $\circ$ defined in section B. Our particular choice of homology basis is given in Figure 6.10.

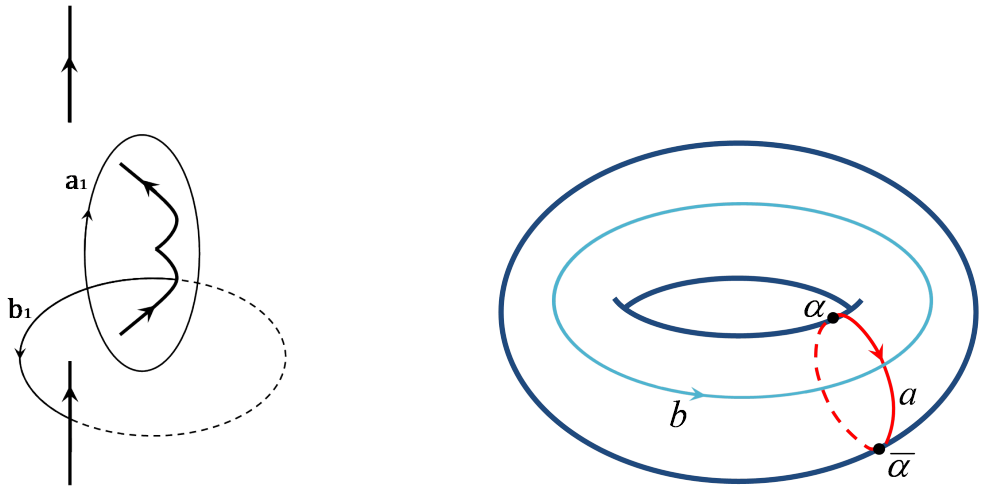

FIGURE 6.10. Our choice of homology basis on Σ. *Left:* Homology basis represented on the cut plane. Bold lines indicate the cuts; the elements of the homology are given by solid and dashed curves. Solid arcs lie on sheet one, while dashed arcs lie on sheet two. *Right:* Representation of the homology basis on the torus.

Using the notation set up in Appendix B, we have on Σ one normalized holomorphic differential,

$$\nu(P) = \frac{k_\nu}{\widehat{\mathcal{R}}_\Sigma(P)} \mathrm{d}z(P), \qquad k_\nu = \left(\frac{1}{2\pi i} \oint_{a_1} \frac{1}{\widehat{\mathcal{R}}_\Sigma(P)} \mathrm{d}z(P) \right)^{-1}. \tag{6.75}$$

where we recall that $\widehat{\mathcal{R}}_\Sigma$ is the lifting of $\widehat{\mathcal{R}}$ to the Riemann surface as defined by equation (B.5).

The associated period matrix, in genus one, a scalar, is

$$H = \oint_{b_1} \nu, \tag{6.76}$$

with which we define the genus one Θ-function,

$$\Theta(z) = \sum_{n \in \mathbb{Z}} \exp\left(\frac{1}{2} H n^2 + z n \right). \tag{6.77}$$

The Riemann constant for our Θ-function takes the simple form,

$$K = i\pi + \frac{1}{2} H. \tag{6.78}$$

Finally, we consider the Abel map $A : \Sigma \to \mathrm{Jac}(\Sigma)$ defined for general compact Riemann surfaces by (B.12). On Σ the Abel map takes the form

$$A(P) = \int_{-iq}^{P} \nu(P'). \tag{6.79}$$

Observe that we have fixed the base point of our Abel map at $P_0 = -iq$. This is a choice of convenience that will simplify subsequent calculations.

We need one final ingredient to construct our Baker-Akhiezer functions. We need an abelian differential of the second kind, $\hat{\omega}$, that satisfies the following properties:

$$\begin{aligned} &1. \quad \hat{\omega} \text{ is holomorphic on } \Sigma\backslash\{\infty_1, \infty_2\}. \\ &2. \quad \text{Near } \infty_1,\ \hat{\omega} = \mathrm{d}p + \mathcal{O}\left(\tfrac{dz(P)}{z^2(P)}\right). \\ &3. \quad \text{Near } \infty_2,\ \hat{\omega} = -\mathrm{d}p + \mathcal{O}\left(\tfrac{dz(P)}{z^2(P)}\right). \\ &4. \quad \oint_{a_1} \hat{\omega} = 0 \end{aligned} \tag{6.80}$$

where $\mathrm{d}p$ is the exterior derivative of the polynomial $p(z)$ given by (6.50).

Claim 4. There is a unique differential $\hat{\omega}$ satisfying the conditions in (6.80) given by the formula

$$\hat{\omega} = p_1 \left(\frac{z^2 - \mathrm{Re}(\alpha) z}{\widehat{\mathcal{R}}_\Sigma(P)} dz(P) + k_{\hat{\omega}} \nu \right), \tag{6.81}$$

where,

$$k_{\hat{\omega}} = -\frac{1}{2\pi i} \oint_{a_1} \frac{z^2 - \mathrm{Re}(\alpha) z}{\widehat{\mathcal{R}}_\Sigma(P)} dz(P)$$

Proof: Properties 1 and 4 are clearly satisfied by (6.81). To verify properties 2 and 3 one simply expands $\mathcal{R}_\Sigma(P)$ for large z using (6.48) and (B.5). To prove uniqeness, suppose $\check{\omega}$ is another differential on Σ satisfying these properties. The difference $\hat{\omega} - \check{\omega}$ is holomorphic on Σ because the principal parts of each differential cancel each other. Finally, property 4 guarantees that $\oint_{a_1} \hat{\omega} - \check{\omega} = 0$, but a holomorphic differential that vanishes over all a-cycles is necessarily the zero differential. Thus $\hat{\omega} = \check{\omega}$ establishing uniqueness. □

Remark 6. The hat on $\hat{\omega}$ is used to indicate that $\hat{\omega}$ is built using the function $\widehat{\mathcal{R}}(z)$ cut on the extended bands Γ_g and $\widehat{\Gamma}$ defined in equation (6.62). Later when we descend our formula back to the cut complex plane, it will be useful to write things in terms of the function $\mathcal{R}(z)$ defined by (6.45) which is cut along the bands Γ_b. Comparing definitions, we have the equivalence

$$\omega = \begin{cases} \hat{\omega} & : \ z \text{ to the right of } \Gamma_{ext} \\ -\hat{\omega} & : \ z \text{ to the left of } \Gamma_{ext} \end{cases},$$

where,

$$\omega = p_1 \frac{z^2 - \mathrm{Re}(\alpha) z + k_{\hat{\omega}} k_\nu}{\mathcal{R}_\Sigma(P)} dz(P).$$

We are now in a position to describe the Baker-Akhiezer functions $\xi_1(P)$ and $\xi_2(P)$. Recall the pole divisors of these functions, $\mathcal{D}_1 = P_1$ and $\mathcal{D}_2 = P_2$, which we previously have shown are non-special. For $m = 1, 2$ fix vectors $V_m \in \mathbb{C}$ such that $V_m = A(\mathcal{D}_m) + K \in \mathrm{Jac}(\Sigma)$. For $m = 1, 2$ consider the functions

$$F_m(P) = \frac{\Theta(A(P) - V_m - iU + \frac{i}{\epsilon}\Omega)}{\Theta(A(P) - V_m)} \exp\left(-i \int_{-iq}^{P} \hat{\omega}\right) \tag{6.82}$$

where the path of integration in the exponential is the same as in the Abel map $A(P)$ and the constant U is the value of the b-period of $\hat{\omega}$,

$$U = \oint_{b_1} \hat{\omega}. \tag{6.83}$$

Strictly speaking, $F_m(P)$ is not well defined, depending on the path of integration the value of $F_m(P)$ can change by a multiplicative factor. If we let $P + a_1$ and $P + b_1$ denote the addition of a a or b-cycle to a fixed path from $-iq$ to P then using (B.8) and (6.80) we have the following relations,

$$\begin{aligned} F_m(P + a_1) &= F_m(P) \\ F_m(P + b_1) &= F_m(P) e^{-\frac{i}{\epsilon}\Omega}. \end{aligned} \tag{6.84}$$

To build a single valued function, we need to restrict the allowable paths of integration in the Abel map (and the exponential factor). We do so by disallowing augmentation of a path by b-cycles, specifically define

$$u(P) = \int_{-iq}^{P} \nu, \quad \text{path lies entirely on the same sheet as } P. \tag{6.85}$$

Under this restriction $u(P)$ is still not single valued, properly speaking $u(P)$ is a map into $\mathbb{C}/2\pi i\mathbb{Z}$. However, because the a-periods leave Θ invariant, the composition of Θ with $u(P)$ is a single valued function on each cut sheet of the Riemann surface. Define

$$y_m(P) = \frac{\Theta(u(P) - V_m - iU + \frac{i}{\epsilon}\Omega)}{\Theta(u(P) - V_m)} \exp\left(-i \int_{-iq}^{P} \hat{\omega}\right). \tag{6.86}$$

By using the restricted Abel map $y_m(P)$ is a single valued function on each cut sheet of the Riemann surface. From Corollary 32 in Appendix B, we know that y_m has a simple pole at the point P_m and is elsewhere holomorphic on each sheet. Next, consider the behavior of the boundary values $(y_m(P))_\pm$ on each sheet as P approaches the cuts on each sheet. From the antisymmetry $\mathcal{R}_\Sigma(P^1(z)) = -\mathcal{R}_\Sigma(P^2(z))$ it follows that for $P \in \Gamma_{ext}$ we have $u_+(P) = u_-(P) \pmod{2\pi i\mathbb{Z}}$ and thus y_m is holomorphic across these cuts. For $z \in \Gamma_g$ we

have, by examining the homology basis (Figure 6.10),

$$u_+(P) = u_-(P - b_1) \pmod{2\pi i\mathbb{Z}}, \qquad P \in \Gamma_g \tag{6.87}$$

Using these boundary relations in (6.84), it follows that $y_m(P)$ satisfies the desired jump relation,

$$y_+(P) = y_-(P)e^{\frac{i}{\epsilon}\Omega}, \qquad P \in \Gamma_g$$

The above properties indicate that y_m is nearly our desired Baker-Akhiezer function $\xi_m(P)$. From (6.80) the asymptotic behavior of y_m differs from the desired behavior for $\xi_m(P)$ by a multiplicative constant. Setting

$$N_1 = \lim_{P\to\infty_1} y_1(P)e^{ip(z)} \tag{6.88}$$

$$N_2 = \lim_{P\to\infty_2} y_2(P)e^{-ip(z)}$$

we find that our Baker-Akhiezer functions $\xi_m(P)$ are given by the formulae:

$$\xi_m(P) = \frac{1}{N_m} y_m(P). \tag{6.89}$$

Remark 7. In general, one needs to show that the function y_m we have constructed is not identically zero. For general genus G this event may be characterized by the existence of a special divisor of G points, $\mathcal{D} = P_1 + \ldots + P_G$, on Σ such that $A(\mathcal{D}) = V_m + U - \frac{i}{\epsilon}\Omega$. If the genus $G > 1$ then such a divisor may exist and one must work to show that this does not occur. In our case $G = 1$ and in this simplified setting every differential of the form $\mathcal{D} = P$ is necessarily non-special and thus y_m cannot identically vanish.

Representation of the formulae on the cut plane

The Baker-Akhiezer functions (6.89) we have built solve the hyperelliptic Riemann Hilbert problem for $Q(z)$ given in section 6.3.1. We now want to reduce the above constructions on the Riemann surface Σ to concrete formulae on the cut plane.

We say that a differential η on Σ is *antisymmetric* provided that given a z in the cut plane that is not a branch point, that $\eta(P^1(z)) = -\eta(P^2(z))$ where $P^1(z)$ and $P^2(z)$ are the two preimages of z in Σ. Among the antisymmetric differentials are the differentials ν and $\hat{\omega}$ associated with our solution of the hyperelliptic problem. Using the antisymmetry we reduce integrals of our differentials over the elements of our homology basis (see Figure 6.10) to integrals on the first sheet of Σ. By collapsing the loops onto the cuts we get:

$$\oint_{a_1} \eta = 2\int_{\Gamma_g} \eta_+(z)\,\mathrm{d}z \qquad\qquad \oint_{b_1} \eta = -2\int_{\gamma_b^*} \eta(z)\,\mathrm{d}z. \tag{6.90}$$

Applying this identification to (6.76) and (6.83) we get the following explicit formulae

$$H = -2\pi i \left(\int_{\gamma_b^*} \frac{1}{\widehat{\mathcal{R}}(z)} \mathrm{d}z \right) \Big/ \left(\int_{\Gamma_g} \frac{1}{\widehat{\mathcal{R}}_+(z)} \mathrm{d}z \right) \tag{6.91}$$

$$U = -2p_1 \int_{\gamma_b^*} \frac{z^2 - \mathrm{Re}(\alpha) z + k_\nu k_{\hat{\omega}}}{\widehat{\mathcal{R}}(z)} \mathrm{d}z. \tag{6.92}$$

Exploiting the reality of the coefficients of $p(z)$ and the complex conjugation symmetry of the square root one easily shows that H is purely real; U, however, is in general fully complex. For the Riemann constant K we use (6.91) and fix K exactly by the formula

$$K = i\pi + \frac{1}{2} H, \tag{6.93}$$

where the equality is understood to be exact, no longer modulo a member of the period lattice.

We can also use the antisymmetry to descend the restricted Abel map $u(P)$ to the cut plane. Let

$$u(z) = \int_{-iq}^{P^1(z)} \nu \tag{6.94}$$

then up to the addition of a period of the Θ function we have

$$u(z) = u(P^1(z)) = -u(P^2(z)).$$

In terms of the restricted Abel map $u(z)$ and our explicit choice (6.93) of the Riemann constant K, the constants V_1 and V_2 take the form,

$$\begin{aligned} V_1 &= i\pi + \frac{1}{2} H + u(z_f), \\ V_2 &= i\pi + \frac{1}{2} H - u(z_f), \end{aligned} \tag{6.95}$$

where we recall that z_f, defined by (6.71), is the projection of the finite zero of the functions $f_1(P)$ and $f_2(P)$ onto the complex plane.

Finally, let

$$\begin{aligned} \chi_1 &= \lim_{P \to \infty_1} \left(\textstyle\int_{-iq}^{P} \hat{\omega} - p(z) \right) = \textstyle\int_{-iq}^{\infty_1} (\hat{\omega} - \mathrm{d}p) - p(-iq) \\ &= \textstyle\int_{-iq}^{\infty} (\omega - \mathrm{d}p) - p(-iq), \\ \chi_2 &= \lim_{P \to \infty_2} \left(\textstyle\int_{-iq}^{P} \hat{\omega} + p(z) \right) = \textstyle\int_{-iq}^{\infty_2} (\omega + \mathrm{d}p) + p(-iq) \\ &= - \textstyle\int_{-iq}^{\infty} (\omega - \mathrm{d}p) + p(-iq). \end{aligned} \tag{6.96}$$

Here we have used Remark 6 and the antisymmetry of $\hat{\omega}$ to simplify each expression. In the

above integrals the paths lie entirely on a single sheet, and we note the difference between ∞_k, the two infinities on the Riemann surface Σ, and ∞ the single complex infinity in the plane. Comparing the expressions in (6.96) we see that the two numbers differ only by a sign, so we drop the subscripts and write simply,

$$\chi = \chi_1 = -\chi_2. \tag{6.97}$$

With these exact choices of representation, the functions y_m become

$$y_m(P) = \begin{cases} \frac{\Theta(u(z)-V_m-iU+i\Omega/\epsilon)}{\Theta(u(z)-V_m)} e^{-i\int_{-iq}^{z}\hat{\omega}} & : \quad P \text{ in sheet 1} \\ \frac{\Theta(-u(z)-V_m-iU+i\Omega/\epsilon)}{\Theta(u(z)+V_m)} e^{i\int_{-iq}^{z}\hat{\omega}} & : \quad P \text{ in sheet 2} \end{cases}, \tag{6.98}$$

where now every integration is understood to lie on the first sheet. From these equations and (6.97) we recover the normalizing constants

$$N_1 = \frac{\Theta(u(+\infty) - V_1 - iU + i\Omega/\epsilon)}{\Theta(u(+\infty) - V_1)} e^{-i\chi}, \tag{6.99}$$

$$N_2 = \frac{\Theta(-u(-\infty) - V_2 - iU + i\Omega/\epsilon)}{\Theta(u(-\infty) + V_2)} e^{i\chi}. \tag{6.100}$$

which allow us to recover the full outer solution $O(z)$ using (6.59) and (6.63) from $f(P)$ and $\xi(P)$.

Define the functions $b^{\pm}(z)$ by

$$b^{\pm}(z) = \frac{1}{2}\left[1 \pm \left(\frac{z-\alpha}{z-iq}\right)^{1/2}\left(\frac{z-\bar{\alpha}}{z+iq}\right)^{1/2}\right], \tag{6.101}$$

where the square roots are branched on Γ_b and normalized to approach unity as $z \to \infty$. The outer solution $O(z)$ is then recovered from $f(P)$ and $\xi(P)$ through (6.72), (6.63), and (6.59):

$$O(z) = \mathrm{diag}\left(\frac{\Theta(u(+\infty) - V_1)}{\Theta(u(+\infty) - V_1 - iU + i\Omega/\epsilon)}, \frac{\Theta(u(-\infty) + V_2)}{\Theta(-u(-\infty) - V_2 - iU + i\Omega/\epsilon)}\right) \times$$
$$e^{i\chi\sigma_3}\mathcal{T}(z)\left(e^{-i\int_{-iq}^{z}\omega}\right)^{\sigma_3} \tag{6.102}$$

where $\mathcal{T}(z)$ is the matrix built from the projections of $f_m(P)$ and $z_m(P)$ onto the cut plane,

$$\mathcal{T} = \begin{pmatrix} b^+(z)\frac{\Theta(u(z)-V_1-iU+i\Omega/\epsilon)}{\Theta(u(z)-V_1)} & b^-(z)\frac{\Theta(-u(z)-V_1-iU+i\Omega/\epsilon)}{\Theta(u(z)+V_1)} \\ b^-(z)\frac{\Theta(u(z)-V_2-iU+i\Omega/\epsilon)}{\Theta(u(z)-V_2)} & b^+(z)\frac{\Theta(-u(z)-V_2-iU+i\Omega/\epsilon)}{\Theta(u(z)+V_2)} \end{pmatrix} \quad : \ z \text{ right of } \Gamma_{ext}\cup\Gamma_b\cup\Gamma_g$$

$$\begin{pmatrix} b^+(z)\frac{\Theta(-u(z)-V_1-iU+i\Omega/\epsilon)}{\Theta(u(z)+V_1)} & b^-(z)\frac{\Theta(u(z)-V_1-iU+i\Omega/\epsilon)}{\Theta(u(z)-V_1)} \\ b^-(z)\frac{\Theta(-u(z)-V_2-iU+i\Omega/\epsilon)}{\Theta(u(z)+V_2)} & b^+(z)\frac{\Theta(u(z)-V_2-iU+i\Omega/\epsilon)}{\Theta(u(z)-V_2)} \end{pmatrix} \quad : \ z \text{ left of } \Gamma_{ext}\cup\Gamma_b\cup\Gamma_g \tag{6.103}$$

Note that in the formula for $O(z)$ we have used Remark 6 to replace the f_k with $b^\pm$ and $\hat{\omega}$ with ω. The only term which retains explicit dependence on $\widehat{\mathcal{R}}$ is the restricted Abel map $u(z)$.

Now that we have an explicit formula for the outer model $O(z)$, we make the following observations. By construction, $O(z)$ has no poles; the ratios of Θ-functions appearing in $\mathcal{T}$ do have simple poles, but these are exactly canceled by the zeros of the multiplicative pre-factors $b^\pm(z)$. $O(z)$ does have singularities, as required by the RHP for $O(z)$, at $z=\pm iq$ each entry has a square-root singularity it inherits from the factor $b^\pm(z)$. Finally, though the formula for the outer solution is complicated, its dependence on ϵ is very simple; ϵ appears only as part of the parameter $i\Omega/\epsilon$ appearing in the argument of the Θ-functions. Since Ω is purely real (cf. (6.28)), changing the value of ϵ adds an imaginary perturbation to the argument of $\Theta(\,\cdot\,)$ which we known to be periodic in the imaginary direction. Thus, we conclude that away from $z=\pm iq$, $O(z)$ is bounded as $\epsilon\to 0$. We record this fact because we will need later to estimate the size of the jumps in the global error RHP.

6.4 The Inner Approximations and Global Error Estimation

6.4.1 The model problem near α and $\bar{\alpha}$: the Airy Parametrix.

Near the points α and $\bar{\alpha}$ the outer model $O(z)$ ceases to be a uniformly valid approximation. The contours Γ_3 and γ_g (Γ_3^* and γ_g^*), whose perturbation from identity is controlled by the function $\varphi(z)$, terminate at $z=\alpha$ ($z=\bar{\alpha}$). The local behavior of $\varphi(z)$ near these points is given by $\varphi(z)=c+k(z-p)^{3/2}+h.o.t.$, where p is either point α or $\bar{\alpha}$ and the constants c and k depend on the choice of p. The local behavior of the phase and the local structure of the jump matrices, see Figure 6.11, are completely different than those near the stationary phase points $z=z_1$ and $z=z_0$. We cannot expect that the local model near α or $\bar{\alpha}$ be built using the parabolic cylinder RHP used near the stationary phase points. As we show below, the appropriate local model near these points is related instead to the Airy RHP.

The Airy RHP:

Find a function $\Psi(\zeta)$ such that:

- Ψ is analytic in $\mathbb{C}\backslash\Sigma,\ \Sigma=\{\zeta \ : \ \arg\zeta=0,\pi,\pm 2\pi/3\}$.

- Ψ takes continuous boundary values on Σ which satisfy the jump relations $\Psi_+(\zeta) = \Psi_-(\zeta)v(\zeta)$, where

$$v(\zeta) = \begin{cases} \begin{pmatrix} 1 & e^{-\frac{4}{3}\zeta^{3/2}} \\ 0 & 1 \end{pmatrix} & : \quad \arg(\zeta) = 0 \\ \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix} & : \quad |\arg(\zeta)| = 2\pi/3 \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & : \quad \arg(\zeta) = \pi. \end{cases} \tag{6.104}$$

- For ζ large, Ψ has the following asymptotic behavior:

$$\Psi(\zeta) = \frac{e^{\frac{i\pi}{12}}}{2\sqrt{\pi}} \zeta^{-\sigma_3/4} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{i\pi}{4}\sigma_3} \left[I + \mathcal{O}(\zeta^{-3/2}) \right] \tag{6.105}$$

This problem, like the parabolic cylinder RHP, is one of the standard local problems that emerge in the study of integrable waves and random matrix theory. The solution $\Psi(\zeta)$ to this RHP is constructed from Airy functions. Let $\omega = e^{\frac{2\pi i}{3}}$, and define $\widetilde{\Psi}$ to be

$$\widetilde{\Psi}(\zeta) = \begin{cases} \begin{pmatrix} \mathrm{Ai}(\zeta) & \mathrm{Ai}(\omega^2\zeta) \\ \mathrm{Ai}'(\zeta) & \omega^2 \mathrm{Ai}'(\omega^2\zeta) \end{pmatrix} & : \quad \zeta \in (0, \pi) \\ \begin{pmatrix} \mathrm{Ai}(\zeta) & -\omega^2 \mathrm{Ai}(\omega\zeta) \\ \mathrm{Ai}'(\zeta) & -\mathrm{Ai}'(\omega\zeta) \end{pmatrix} & : \quad \zeta \in (-\pi, 0) \end{cases},$$

then the solution, which we will denote by $\Psi_{\mathrm{Ai}}(\zeta)$, to the RHP given by (6.105) and (6.104) is

$$\Psi_{\mathrm{Ai}}(\zeta) = \begin{cases} \widetilde{\Psi}(\zeta) e^{(\frac{2}{3}\zeta^{3/2} - \frac{\pi i}{6})\sigma_3} & : \quad \arg(\zeta) \in (0, \frac{2\pi}{3}) \\ \widetilde{\Psi}(\zeta) e^{(\frac{2}{3}\zeta^{3/2} - \frac{\pi i}{6})\sigma_3} \begin{pmatrix} 1 & 0 \\ -e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix} & : \quad \arg(\zeta) \in (\frac{2\pi}{3}, \pi) \\ \widetilde{\Psi}(\zeta) e^{(\frac{2}{3}\zeta^{3/2} - \frac{\pi i}{6})\sigma_3} \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix} & : \quad \arg(\zeta) \in (-\pi, -\frac{2\pi}{3}) \\ \widetilde{\Psi}(\zeta) e^{(\frac{2}{3}\zeta^{3/2} - \frac{\pi i}{6})\sigma_3} & : \quad \arg(\zeta) \in (-\frac{2\pi}{3}, 0) \end{cases}. \tag{6.106}$$

Clearly, this function satisfies the jump conditions for $|\arg(\zeta)| = 2\pi/3$. The normalization condition follows from the asymptotic behavior of the Airy functions ([AS64] p.448) and to verify the jump condition on $\mathbb{R}$ one needs the Wronskians of the various Airy functions

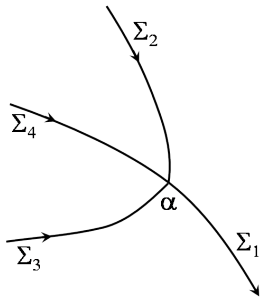

FIGURE 6.11. The jump matrices of $Q(z)$ near $z = \alpha$, note that the contours have been relabeled and reoriented.

([AS64] p.446) and the fundamental identity

$$\mathrm{Ai}(\zeta) + \omega\,\mathrm{Ai}(\omega\zeta) + \omega^2\,\mathrm{Ai}(\omega^2\zeta) = 0.$$

One can also derive the solution directly by first reducing the problem to constant jump form and then transforming it into a differential equation. We will not pursue this here.

As we will now show, by explicit transformations the local jumps near $z = \alpha$ and $z = \bar{\alpha}$ of the full problem for $Q(z)$ can be mapped exactly onto the jumps of the Airy RHP. We will work mainly with the problem near $z = \alpha$ and then quickly cover the changes necessary to handle the problem near $z = \bar{\alpha}$.

Near $z = \alpha$ let us introduce a local relabeling and reorientation of the contours which we depict in Figure 6.11. From (6.55) and the equivalences observed in Remark 2, the local jump matrices on these relabeled contours are

$$V_Q(z) = \begin{cases} \begin{pmatrix} e^{-\frac{i}{\epsilon}\Omega} & 0 \\ \frac{2\nu_+}{iq}\delta^{-2}(z)s^{-2}(z)e^{\frac{i}{\epsilon}(\varphi_+ + \Omega)} & e^{\frac{i}{\epsilon}\Omega} \end{pmatrix} & : \quad z \in \Sigma_1 \\ \begin{pmatrix} 1 & \frac{iq}{2\nu}\delta^2(z)s^2(z)e^{-\frac{i}{\epsilon}\varphi} \\ 0 & 1 \end{pmatrix} & : \quad z \in \Sigma_2 \\ \begin{pmatrix} 1 & -\frac{iq}{2\nu}\delta^2(z)s^2(z)e^{-\frac{i}{\epsilon}\varphi} \\ 0 & 1 \end{pmatrix} & : \quad z \in \Sigma_3 \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & : \quad z \in \Sigma_4 \end{cases}.$$

These formulae simplify remarkably when we recall the local behavior of $s(z)$ near $z = \alpha$; using (6.52), we have for $z \in \mathbb{C}^+$,

$$s^2(z) = \frac{2\nu}{iq}\delta^{-2}(z)e^{\mathcal{R}(z)\left(f(z) + \frac{i\pi}{2}\widehat{\mathcal{R}}^{-1}(z)\right)}$$

where $f(z)$ is the function defined by (6.53) analytic in any sufficiently small neighborhood of α; $\mathcal{R}(z)$ and $\widehat{\mathcal{R}}(z)$ are the two different branchings of the square-root

$$\sqrt{(z-iq)(z-\alpha)(z+iq)(z-\bar{\alpha})}$$

defined by (6.45) and (6.51). Inside any small neighborhood $\mathcal{U}_\alpha$ of α or $\mathcal{U}_{\bar{\alpha}}$ of $\bar{\alpha}$ the ratio of these branchings of the square root, which we denote by $c(z)$, satisfies,

$$c(z) = \mathcal{R}(z)/\widehat{\mathcal{R}}(z) = \begin{cases} 1 & : \quad z \in (\mathcal{U}_\alpha)_- \cup (\mathcal{U}_{\bar{\alpha}})_- \\ -1 & : \quad z \in (\mathcal{U}_\alpha)_+ \cup (\mathcal{U}_{\bar{\alpha}})_+ \end{cases}, \tag{6.107}$$

where the sets $(\mathcal{U}_\alpha)_\pm$ are the halves of $\mathcal{U}_\alpha$ to the left and right of γ respectively. Inserting these observations into the jump matrices we see that the exact jump matrices of $Q(z)$ near $z = \alpha$ take the simple form:

$$V_Q(z) = \begin{cases} \begin{pmatrix} e^{-\frac{i}{\epsilon}\Omega} & 0 \\ -ie^{-\mathcal{R}(z)f(z)}e^{\frac{i}{\epsilon}(\varphi_+ + \Omega)} & e^{\frac{i}{\epsilon}\Omega} \end{pmatrix} & : \quad z \in \Sigma_1 \\ \begin{pmatrix} 1 & ie^{\mathcal{R}(z)f(z)}e^{-\frac{i}{\epsilon}\varphi} \\ 0 & 1 \end{pmatrix} & : \quad z \in \Sigma_2 \\ \begin{pmatrix} 1 & ie^{\mathcal{R}(z)f(z)}e^{-\frac{i}{\epsilon}\varphi} \\ 0 & 1 \end{pmatrix} & : \quad z \in \Sigma_3 \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & : \quad z \in \Sigma_4 \end{cases}.$$

We now want to build a model which solves the above jump relations exactly. Let us begin by introducing a local change of variables. The phase function $\varphi(z)$, originally defined in (6.24), can be written for $z \in \mathcal{U}_\alpha$ using the formula

$$\varphi(z) = \begin{cases} 4t\int_\alpha^z S_2(\lambda)[\lambda + \alpha_r - 2\mu]\ \mathrm{d}\lambda - \Omega & : \quad z \in (\mathcal{U}_\alpha)_+ \\ 4t\int_\alpha^z S_2(\lambda)[\lambda + \alpha_r - 2\mu]\ \mathrm{d}\lambda + \Omega & : \quad z \in (\mathcal{U}_\alpha)_- \end{cases}, \tag{6.108}$$

where the path of integration lies in $\mathcal{U}_\alpha$ and is understood to avoid the contour γ. We introduce a local variable defined by

$$\frac{4}{3}\zeta_\alpha^{3/2} = \begin{cases} -\frac{i}{\epsilon}\ (\varphi(z) + \Omega) & : \quad z \in (\mathcal{U}_\alpha)_+ \\ -\frac{i}{\epsilon}(\varphi(z) - \Omega) & : \quad z \in (\mathcal{U}_\alpha)_- \end{cases}. \tag{6.109}$$

Combining (6.108) and (6.109) it follows that $\zeta^{3/2}$ is analytic in $\mathcal{U}_\alpha \backslash \gamma_b$. Equivalently, we can define ζ using the formula

$$\zeta_\alpha = \left(-\frac{3it}{\epsilon}\int_\alpha^z S_2(\lambda)[\lambda + \alpha_r - 2\mu]\ \mathrm{d}\lambda\right)^{2/3}, \tag{6.110}$$

where the path of integration stays inside $\mathcal{U}_\alpha$ and avoid the band γ_b but may pass through the gap γ_g. Because the integral term vanishes at $z = \alpha$ like a $3/2-$root it follows that (6.110) defines a locally analytic and invertible change of variables on some suitably small, but ϵ-independent, neighborhood of α. We now fix $\mathcal{U}_\alpha$ to be such a neighborhood. Inside $\mathcal{U}_\alpha$ we choose the contours Σ_i, $i = 1, \ldots, 4$, such that their images in the ζ-plane lie on the straight line rays $\arg(\zeta) = 0,\ 2\pi/3,\ -2\pi/3,\ \pi$.

Let us then write our model $A_\alpha(z)$ in the following form:

$$K_\alpha(z)\Psi^{(1)}(\zeta_\alpha(z))e^{-\frac{1}{2}\mathcal{R}(z)f(z)\sigma_3}e^{\frac{ic(z)}{2\epsilon}\sigma_3} \tag{6.111}$$

the resulting jumps for $\Psi^{(1)}(\zeta_\alpha)$ are given by

$$V_{\Psi^{(1)}}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -ie^{-\frac{4}{3}\zeta_\alpha^{3/2}(z)} & 1 \end{pmatrix} & : \quad z \in \Sigma_1 \\ \begin{pmatrix} 1 & ie^{\frac{4}{3}\zeta_\alpha^{3/2}(z)} \\ 0 & 1 \end{pmatrix} & : \quad z \in \Sigma_2 \\ \begin{pmatrix} 1 & ie^{\frac{4}{3}\zeta_\alpha^{3/2}(z)} \\ 0 & 1 \end{pmatrix} & : \quad z \in \Sigma_3 \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & : \quad z \in \Sigma_4 \end{cases}, \tag{6.112}$$

which are nearly the jump matrices of the Airy RHP given by (6.104). To map ourselves exactly onto those jumps we reintroduce the spin matrix $\sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ and the scalar function $\tau(z)$ defined by

$$\tau(z) = \left(\frac{z - iq}{z - \alpha}\right)\left(\frac{z - \bar{\alpha}}{z + iq}\right), \tag{6.113}$$

branched on $\gamma_b \cup \gamma_b^*$, and normalized to approach unity at ∞. On both cuts $\tau(z)$ satisfies the jump relation $\tau_+(z) = i\tau_-(z)$. Using these we make the change of variables,

$$\Psi^{(1)}(\zeta) = \Psi(\zeta)\tau(z)\sigma_2 e^{\frac{i\pi}{4}\sigma_3}. \tag{6.114}$$

Using the jump relations for $\Psi^{(1)}$ given by (6.112), we see that the new unknown $\Psi(\zeta)$ has jumps which have been reduced to exactly the Airy RHP whose solution, $\Psi_{\mathrm{Ai}}(\zeta)$, we have already described above. Using (6.114), (6.111), and (6.106) our model inside $\mathcal{U}_\alpha$ is given by

$$A_\alpha(z) = K_\alpha(z)\tau(z)\Psi_{\mathrm{Ai}}(\zeta_\alpha(z))\sigma_2 e^{\left(\frac{ic(z)}{2\epsilon}\Omega + \frac{i\pi}{4} - \frac{1}{2}\mathcal{R}(z)f(z)\right)\sigma_3}. \tag{6.115}$$

Matching condition for selecting $K_\alpha(z)$:
To complete our description of $A_\alpha(z)$ we need to choose the function $K_\alpha(z)$ such that K_α

is analytic in $\mathcal{U}_\alpha$ and on the boundary $\partial\mathcal{U}_\alpha$ we have,

$$O^{-1}(z)A_\alpha(z) = I + \mathcal{O}(\epsilon^{-\eta}), \tag{6.116}$$

for some $\eta > 0$ to be determined. Due to the re-scaling in the mapping $\zeta = \zeta_\alpha(z)$, the z on the boundary $\partial\mathcal{U}_\alpha$ get mapped to very large values of ζ, $\zeta_\alpha(z) = \mathcal{O}(\epsilon^{-2/3})$. Thus we need to understand the large ζ behavior of $\Psi_{\rm Ai}(\zeta)$. One can work out that uniformly in the ζ-plane, the large ζ behavior of $\Psi_{\rm Ai}(\zeta)$ is given by

$$\Psi_{\rm Ai}(\zeta) = \frac{e^{\frac{i\pi}{12}}}{2\sqrt{\pi}}\zeta^{-\sigma_3/4}\sum_{k=0}^{\infty}\begin{pmatrix}(-1)^k s_k & s_k\\ -(-1)^k t_k & t_k\end{pmatrix}e^{-\frac{i\pi}{4}\sigma_3}\left(\frac{2}{3}\zeta^{3/2}\right)^{-k}, \tag{6.117}$$

where

$$s_0 = t_0 = 1, \quad s_k = \frac{\Gamma(3k+\frac{1}{2})}{54^k k!\Gamma(k+\frac{1}{2})}, \quad t_k = -\frac{6k+1}{6k-1}s_k, \quad \text{for } k \geq 1,$$

see [DKM$^+$99] for more details. If we then set

$$K_\alpha(z) = \frac{O(z)}{\tau(z)}e^{-\left(\frac{ic(z)}{2\epsilon}\Omega + \frac{i\pi}{4} - \frac{1}{2}\mathcal{R}(z)f(z)\right)\sigma_3}\sigma_2\left[\frac{e^{\frac{i\pi}{12}}}{2\sqrt{\pi}}\zeta_\alpha^{-\frac{\sigma_3}{4}}(z)\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}e^{-\frac{i\pi}{4}\sigma_3}\right]^{-1}, \tag{6.118}$$

then pairing (6.117) and (6.110), uniformly on $\partial\mathcal{U}_\alpha$ we have

$$O^{-1}(z)A_\alpha(z) = I + \mathcal{O}(\epsilon). \tag{6.119}$$

This definition of $K_\alpha(z)$ clearly accomplishes the desired matching condition, but is it also analytic in $\mathcal{U}_\alpha$? Each term in (6.118) is clearly analytic in $\mathcal{U}_\alpha\backslash\gamma$, we need to check the jump relations for $K_\alpha(z)$ on γ. For $z \in \gamma_g \cap \mathcal{U}_\alpha$ the only factors of K_α which are not analytic are $O(z)$ and $c(z)$; their behavior on γ_g are given by (6.58) and (6.107) respectively. Using these jump relations we can calculate for $z \in \gamma_g$ the jump of K_α:

$$V_K = (K_\alpha(z))_-^{-1}(K_\alpha(z))_+ = F^{-1}(z)e^{\frac{i}{2\epsilon}\Omega\sigma_3}\begin{pmatrix}e^{-\frac{i}{\epsilon}\Omega} & 0\\ 0 & e^{\frac{i}{\epsilon}\Omega}\end{pmatrix}e^{-\frac{i}{2\epsilon}\Omega\sigma_3}F(z) = I,$$

which tells us that $K_\alpha(z)$ is analytic for $z \in \gamma_g$. Here we have used $F(z)$ to collect the nonessential analytic factors in K_α which do not contribute to the calculation of the jump matrix. For $z \in \gamma_b$, we can repeat the calculation observing that now, in addition to $O(z)$ and $c(z)$, we need to pay attention to the branch cuts of $R(z)$, $\tau(z)$, and $\zeta_\alpha(z)$. For $z \in \gamma_b$,

$$\frac{\tau_-(z)}{\tau_+(z)}\sigma_2 e^{\left(\frac{ic_-(z)}{2\epsilon}\Omega + \frac{i\pi}{4} - \frac{1}{2}\mathcal{R}_-(z)f(z)\right)\sigma_3}O_-^{-1}(z)\times$$
$$O_+(z)e^{-\left(\frac{ic_+(z)}{2\epsilon}\Omega + \frac{i\pi}{4} - \frac{1}{2}\mathcal{R}_+(z)f(z)\right)\sigma_3}\sigma_2 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

where we have again used (6.58) to evaluate $O_-^{-1}O_+$ for $z \in \gamma_b$. Using this intermediate

calculation, the jump of $K_\alpha(z)$ across γ_b is given by

$$V_K(z) = \frac{1}{2}\zeta_{\alpha-}^{-\sigma_3/4}(z)\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\zeta_{\alpha+}^{\sigma_3/4}(z) = I,$$

which shows that $K(z)$ is analytic across γ_b. Putting these calculations together, $K_\alpha(z)$ is analytic in $\mathcal{U}_\alpha\backslash\{\alpha\}$. However, examining (6.118), at worst K_α has a 1/4-root singularity at α, thus the singularity at α is removable and $K_\alpha(z)$ is analytic in $\mathcal{U}_\alpha$.

The model problem near $z = \bar{\alpha}$

The process of building a uniformly valid model near $z = \bar{\alpha}$ is nearly identical to the problem near $z = \alpha$. For the sake of brevity we will just summarize the steps, omitting the details. Let $\mathcal{U}_{\bar{\alpha}}$ be a sufficiently small disc centered at $\bar{\alpha}$ (setting $\mathcal{U}_{\bar{\alpha}} = \mathcal{U}_\alpha^*$ works). Using the local expansion formula for $s(z)$ near $z = \bar{\alpha}$, formula (6.52), the exact jump matrices for $Q(z)$ inside $\mathcal{U}_{\bar{\alpha}}$ are,

$$V_Q(z) = \begin{cases} \begin{pmatrix} e^{-\frac{i}{\epsilon}\Omega} & -ie^{\mathcal{R}(z)f(z)}e^{-\frac{i}{\epsilon}(\varphi_+ + \Omega)} \\ 0 & e^{\frac{i}{\epsilon}\Omega}\end{pmatrix} & : \quad z \in \Sigma_1^* \\ \begin{pmatrix} 1 & 0 \\ ie^{-\mathcal{R}(z)f(z)}e^{\frac{i}{\epsilon}\varphi} & 1\end{pmatrix} & : \quad z \in \Sigma_2^* \\ \begin{pmatrix} 1 & 0 \\ ie^{-\mathcal{R}(z)f(z)}e^{\frac{i}{\epsilon}\varphi} & 1\end{pmatrix} & : \quad z \in \Sigma_3^* \\ \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} & : \quad z \in \Sigma_4^* \end{cases}.$$

We then introduce a local change of variables near $z = \bar{\alpha}$ given by

$$\frac{4}{3}\zeta_{\bar{\alpha}}(z) = \begin{cases} \frac{i}{\epsilon}(\varphi + \Omega) & : \quad z \in (\mathcal{U}_{\bar{\alpha}})_+ \\ \frac{i}{\epsilon}(\varphi - \Omega) & : \quad z \in (\mathcal{U}_{\bar{\alpha}})_- \end{cases}, \tag{6.120}$$

or equivalently,

$$\zeta_{\bar{\alpha}}(z) = \left(\frac{3it}{\epsilon}\int_{\bar{\alpha}}^z S_2(\lambda)(\lambda + \alpha_r - 2\mu)\,\mathrm{d}\lambda\right)^{2/3}. \tag{6.121}$$

This change of variables is locally analytic and invertible due to the 3/2-root behavior of φ near $z = \bar{\alpha}$. We use the change of variables to select the contours Σ_i^*; we choose each contour so that $\arg(\zeta_{\bar{\alpha}}(\Sigma_k \cap \mathcal{U}_{\bar{\alpha}})) = 0,\ 2\pi/3,\ -2\pi/3$, and π for $k = 1, \ldots, 4$. The local jumps can be mapped exactly onto the Airy RHP by the appropriate transformation, which is identical to the transformations used near $z = \alpha$ with the twist matrix σ_2 omitted; at the end of the day our local model $A_{\bar{\alpha}}(z)$ is given by

$$A_{\bar{\alpha}}(z) = K_{\bar{\alpha}}(z)\Psi_{\mathrm{Ai}}(\zeta_{\bar{\alpha}}(z))\tau(z)e^{\left(\frac{ic(z)}{2\epsilon}\Omega + \frac{i\pi}{4} - \frac{1}{2}\mathcal{R}(z)f(z)\right)\sigma_3}. \tag{6.122}$$

The matching factor $K_{\bar{\alpha}}(z)$ is selected by the same mechanism as its counterpart $K_\alpha(z)$. Omitting the details, essentially repeating the calculations determining $K_\alpha(z)$ shows that

$$K_{\bar{\alpha}}(z) = \frac{O(z)}{\tau(z)} e^{-\left(\frac{ic(z)}{2\epsilon}\Omega + \frac{i\pi}{4} - \frac{1}{2}\mathcal{R}(z)f(z)\right)\sigma_3} \left[\frac{e^{\frac{i\pi}{12}}}{2\sqrt{\pi}} \zeta_{\bar{\alpha}}^{-\frac{\sigma_3}{4}}(z) \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{i\pi}{4}\sigma_3} \right]^{-1} \tag{6.123}$$

is an analytic function in $\mathcal{U}_{\bar{\alpha}}$ and that uniformly on $\partial\mathcal{U}_{\bar{\alpha}}$ we have

$$O^{-1}(z)A_{\bar{\alpha}}(z) = I + \mathcal{O}(\epsilon). \tag{6.124}$$

6.4.2 The Local Models near $z = z_1$ and $z = z_0$

To complete our global parametrix we need to introduce local models at the two real stationary phase points z_1 and z_0 where the contours Γ_i and their complex conjugates return to the real axis. By now we are very familiar with these local models, they are the same models introduced in the previous two chapters, both for $x > L$ outside the support and $x < L$ inside the support prior to the first breaking time. There are a few small differences of course; the phase functions are now those modulated by the twice-cut g-function, $\varphi(z) = \theta(z) - 2g_2(z)$ and $\varphi_1(z) = \varphi(z) + 4L\nu(z)$; and the corresponding location of their respective stationary phase point z_0 and z_1 are now defined by (6.29) and Prop. 21 respectively. These changes though are mainly notational the local quadratic behavior of the phase functions near z_0 and z_1 is unchanged. Making the appropriate approximation to the jumps near these points we can reduce everything to the parabolic cylinder RHP (cf. (4.33) and (4.34)) by explicit transformations. Because we have seen these steps twice already, we will be very brief here. We will state the model to be solved and give its solution. The reader should refer to Sections 4.2.2, 4.2.3 and 5.4.2 for more details.

Local model near $z = z_0$
Using the approximation formula (4.28) for $\delta(z)$ near z_0 and the jump matrices (6.55) of the full problem for $Q(z)$, we introduce the following local model jumps for $z \in \mathcal{U}_0$, a small fixed-size neighborhood of z_0:

$$V_0 = \begin{cases} \begin{pmatrix} e^{-\frac{i}{\epsilon}\Omega} & 0 \\ r_{0+}(z_0)s^{-2}(z_0)\delta_0^{-2}(z-z_0)^{-2i\kappa_0}e^{\frac{i}{\epsilon}(\varphi_+ + \Omega)} & e^{\frac{i}{\epsilon}\Omega} \end{pmatrix} & : \quad z \in \Gamma_0 \cap \mathcal{U}_0 \\ \begin{pmatrix} 1 & \frac{\bar{r}_0(z_0)}{1+|r_0(z_0)|^2}s^2(z_0)\delta_0^2(z-z_0)^{2i\kappa_0}e^{-\frac{i}{\epsilon}\varphi} \\ 0 & 1 \end{pmatrix} & : \quad z \in \Gamma_3 \cap \mathcal{U}_0 \\ \begin{pmatrix} 1 & 0 \\ \frac{r_0(z_0)}{1+|r_0(z_0)|^2}s^{-2}(z_0)\delta_0^{-2}(z-z_0)^{-2i\kappa_0}e^{\frac{i}{\epsilon}\varphi} & 1 \end{pmatrix} & : \quad z \in \Gamma_3^* \cap \mathcal{U}_0 \\ \begin{pmatrix} e^{-\frac{i}{\epsilon}\Omega} & -\bar{r}_{0+}(z_0)s^2(z_0)\delta_0^2(z-z_0)^{2i\kappa_0}e^{-\frac{i}{\epsilon}(\varphi_+ + \Omega)} \\ 0 & e^{\frac{i}{\epsilon}\Omega} \end{pmatrix} & : \quad z \in \Gamma_0^* \end{cases} \tag{6.125}$$

To reduce these jumps to exactly the parabolic cylinder case (4.34), we introduce the

following local change of variables. Define $\zeta_0(z)$ by the relationship

$$\frac{1}{2}\zeta_0^2(z) = \frac{4ti}{\epsilon}\int_{z_0}^{z} S_{2+}(\lambda)(\lambda - z_0)\,\mathrm{d}\lambda. \tag{6.126}$$

In terms of ζ_0, we have the equality,

$$\frac{i}{\epsilon}\Big(\varphi_+(z) - \varphi_+(z_0)\Big) = \frac{1}{2}\zeta_0^2(z).$$

The left hand side of (6.126) is clearly quadratic near $z = z_0$; this implies that the mapping $\zeta = \zeta_0(z)$ is a locally analytic and invertible change of variables mapping small neighborhoods of $z = z_0$ to expanding neighborhoods of $\zeta = 0$. The disc $\mathcal{U}_0$ is selected such that $\zeta = \zeta_0(z)$ defines an analytic and invertible change of variables on $\mathcal{U}_0$. We use our freedom to deform the contours Γ_3 and γ_g (and their complex conjugates) so that the image contours lie on the straight line rays of steepest decent: $\arg(\zeta_0(\gamma_g \cap \mathcal{U}_0)) = \pi/4$ and $\arg(\zeta_0(\Gamma_3 \cap \mathcal{U}_0)) = 3\pi/4$.

The local model $A_0(z)$ which satisfies these jumps is given by

$$A_0(z) = K_0(z)\Psi_{P.C.}(\zeta_0(z); r_0(z_0))\times \\ s(z_0)^{-\sigma_3}\delta_0^{-\sigma_3}\left(\frac{\epsilon}{\varphi_+''(z_0)}\right)^{-i\kappa_0\sigma_3/2} e^{\frac{i}{2\epsilon}(\varphi_+(z_0)+\Omega+c(z)\Omega)\sigma_3}, \tag{6.127}$$

where we have recycled the function $c(z)$ defined by (6.107). To see this formula is true, one checks that if we replace $\Psi_{P.C.}(\zeta_0(z))$ with an unknown $\Psi(\zeta_0(z))$ the resulting jumps for $\Psi(\zeta)$ are precisely those of the parabolic cylinder RHP given by (4.34).

Using the asymptotic expansion, (4.33), of $\Psi_{P.C}$, by defining the matching function $K_0(z)$ by the formula

$$K_0(z) = O(z)s(z_0)^{\sigma_3}\delta_0^{\sigma_3}\left(\frac{\epsilon}{\varphi_+''(z_0)}\right)^{i\kappa_0\sigma_3/2} e^{-\frac{i}{2\epsilon}(\varphi_+(z_0)+\Omega+c(z)\Omega)\sigma_3}, \tag{6.128}$$

the outer and inner models match to leading order uniformly on the boundary $\partial\mathcal{U}_0$, that is

$$O^{-1}(z)A_0(z) = I + \mathcal{O}(\epsilon^{1/2}). \tag{6.129}$$

Each term in $K_0(z)$ is bounded in $\mathcal{U}_0$ and analytic away from the gap contour, Γ_G. Using the jump relations (6.58) and formula (6.107) for $c(z)$ its easy to check that $K_0(z)$ has no jump across Γ_G, thus $K_0(z)$ is an analytic function for $z \in \mathcal{U}_0$.

Local Model near $z = z_1$

Near $z = z_1$ we introduce, as before, a two level model built to account for the presence of phase factor $e^{4Li\nu/\epsilon}$ in addition to the stationary function $e^{i\varphi_1/\epsilon}$. Let us introduce a sufficiently small, fixed size, neighborhood $\mathcal{U}_1$ and a smaller shrinking disk $\mathcal{D} = \{z : |z - z_1| < \epsilon^\beta\}$ where $\beta = 1 - \eta$, $\eta \ll 1/2$. Let $\mathcal{A}$ denote the annular region $\mathcal{U}_0 \backslash \mathcal{D}$. We then seek

a local model $A_1(z)$ in the form

$$A_1(z) = \begin{cases} B_{\mathcal{A}}(z) & : \quad z \in \mathcal{A} \\ B_{\mathcal{D}}(z) & : \quad z \in \mathcal{D} \end{cases} .$$

In the annular region we ask that $B_{\mathcal{A}}(z)$ satisfy the jump relationship $B_{\mathcal{A}+}(z) = B_{\mathcal{A}-}(z)V_{\mathcal{A}}(z)$, where

$$V_{\mathcal{A}} = \begin{cases} \begin{pmatrix} 1 & 0 \\ -r_0(z_1)s^{-2}(z_1)(z-z_1)^{-2i\kappa_1}(c_\delta)^{-2}e^{\frac{i}{\epsilon}\varphi_1(z)} & 1 \end{pmatrix} & : \quad z \in \Gamma_1 \cap \mathcal{U}_1 \\ \begin{pmatrix} 1 & \frac{-\bar{r}_0(z_1)}{1+|r_0(z_1)|^2}s^2(z_1)(z-z_1)^{2i\kappa_1}c_\delta^2 e^{-\frac{i}{\epsilon}\varphi_1(z)} \\ 0 & 1 \end{pmatrix} & : \quad z \in \Gamma_2 \cap \mathcal{U}_1 \\ \begin{pmatrix} 1 & -\bar{r}_0(z_1)s^2(z_1)(z-z_1)^{2i\kappa_1}c_\delta^2 e^{-\frac{i}{\epsilon}\varphi_1(z)} \\ 0 & 1 \end{pmatrix} & : \quad z \in \Gamma_1^* \cap \mathcal{U}_1 \\ \begin{pmatrix} 1 & 0 \\ \frac{-r_0(z_1)}{1+|r_0(z_1)|^2}s^{-2}(z_1)(z-z_1)^{-2i\kappa_0}c_\delta^{-2}e^{\frac{i}{\epsilon}\varphi_1(z)} & 1 \end{pmatrix} & : \quad z \in \Gamma_2^* \cap \mathcal{U}_1 \end{cases} . \tag{6.130}$$

Here we have made use of the constant c_δ previously defined by (4.44) and $\kappa_1 = -\frac{1}{2\pi}\log(1+|r_0(z_1)|^2)$.

This problem is directly mapped onto the parabolic cylinder RHP defined by (4.33) and (4.34) by the introduction of the appropriate local variable and factoring off constants. To wit, define the map $\zeta = \zeta_1(z)$ by the relation

$$\frac{1}{2}\zeta_1^2(z) = \frac{1}{\epsilon}\left(\varphi_1(z) - \varphi_1(z_1)\right). \tag{6.131}$$

Using Prop. 21 it is easy to verify that this is an analytic invertible map from any suitably small, but ϵ independent, neighborhood of z_1 to an expanding neighborhood of the origin in the ζ-plane. Using our ability to deform our lensing contours, we choose the contours Γ_1 and Γ_2 such that, restricted to $\mathcal{U}_1$, their images under $\zeta_1(z)$ line on the straight line rays $\arg(\zeta) = i\pi/4$ and $3i\pi/4$, respectively. The annular model $B_{\mathcal{A}}(z)$ is then given by

$$B_{\mathcal{A}}(z) = K_{\mathcal{A}}(z)\Psi_{P.C.}(\zeta_1(z); -r_1(z_1))\left(\frac{\epsilon}{\varphi_1''(z_1)}\right)^{-i\kappa_1\sigma_3/2} c_\delta^{-\sigma_3} s(z_1)^{-\sigma_3} e^{\frac{i}{2\epsilon}\varphi_1(z_1)\sigma_3}, \tag{6.132}$$

The pre-factor $K_{\mathcal{A}}(z)$ is an analytic matching function. Its calculation is identical to the previous calculation determining $K_0(z)$. Defining

$$K_{\mathcal{A}}(z) = O(z)\left(\frac{\epsilon}{\varphi_1''(z_1)}\right)^{i\kappa_1\sigma_3/2} c_\delta^{\sigma_3} s(z_1)^{\sigma_3} e^{-\frac{i}{2\epsilon}\varphi_1(z_1)\sigma_3}, \tag{6.133}$$

we see that uniformly on the boundary $\partial\mathcal{U}_1$,

$$O^{-1}(z)B_{\mathcal{A}}(z) = I + \mathcal{O}(\epsilon^{1/2}). \tag{6.134}$$

Inside the shrinking disc $\mathcal{D}$ the annular model $B_{\mathcal{A}}(z)$ is no longer a good approximation of the true jumps. Inside $\mathcal{D}$, the linear oscillations generated by $\exp(4Li\nu/\epsilon)$ are no longer negligible while the quadratic phase $\varphi_1(z)$ is nearly constant; we need a model that accounts for these oscillations. We seek our model $B_{\mathcal{D}}$ in the form

$$B_{\mathcal{D}}(z) = K_{\mathcal{D}}(z)L(z)e^{\frac{i}{2\epsilon}\varphi_1(z)}, \tag{6.135}$$

where, as always, the prefactor $K_{\mathcal{D}}(z)$ is an analytic function, here chosen to match the inner disc model to the annular model on the boundary $\partial\mathcal{D}$. The function $L(z)$ we choose to satisfies the jump relation $L_+(z) = L_-(z)V_L$, where

$$V_L = [s(z_1)(z-z_1)^{i\kappa_1}C_\delta(z)]^{\operatorname{ad}\sigma_3}\widehat{V}_L, \tag{6.136}$$

where we recall $\kappa_1 = -\frac{1}{2\pi}\log(1+|r_0(z_1)|^2)$, C_δ is the piecewise constant given by (4.52). The matrix $\widehat{V}_L$ is given on each contour by

$$\widehat{V}_L(z) = \begin{cases} \begin{pmatrix} 1 & \frac{\bar{r}_0(z_1)}{1+|r_0(z_1)|^2}e^{\frac{4Li\nu}{\epsilon}} \\ 0 & 1 \end{pmatrix} & : \quad z \in \Sigma_1 \\ \begin{pmatrix} 1 & 0 \\ -r_0(z_1)(1+|r_0(z_1)|^2) & 1 \end{pmatrix} & : \quad z \in \Sigma_2 \\ \begin{pmatrix} 1-r_0^2(z_1)e^{\frac{4Li\nu}{\epsilon}} & \frac{\bar{r}_0(z_1)(1-e^{\frac{4Li\nu}{\epsilon}})}{(1+|r_0(z_1)|^2)^2} \\ 0 & (1-r_0^2(z_1)e^{\frac{4Li\nu}{\epsilon}})^{-1} \end{pmatrix} & : \quad z \in \Sigma_3 \\ \begin{pmatrix} 1 & 0 \\ \frac{r_0(z_1)}{1+|r_0(z_1)|^2}e^{-\frac{4Li\nu}{\epsilon}} & 1 \end{pmatrix} & : \quad z \in \Sigma_1^* \\ \begin{pmatrix} 1 & -\bar{r}_0(z_1)(1+|r_0(z_1)|^2) \\ 0 & 1 \end{pmatrix} & : \quad z \in \Sigma_2^* \\ \begin{pmatrix} 1-\bar{r}_0^2(z_1)e^{-\frac{4Li\nu}{\epsilon}} & 0 \\ \frac{r_0(z_1)(1-e^{-\frac{4Li\nu}{\epsilon}})}{(1+|r_0(z_1)|^2)^2} & (1-\bar{r}_0^2(z_1)e^{-\frac{4Li\nu}{\epsilon}})^{-1} \end{pmatrix} & : \quad z \in \Sigma_3^* \end{cases}$$

To simplify the notation we have relabeled and reoriented the contours inside $\mathcal{D}$, see Figure 4.11 in Section 4.2.3 for the labeling and orientation of the contours Σ_i and Σ_i^*.

To complete our model in $\mathcal{D}$ we need to specify the function $L(z)$ satisfying the above

jump conditions and the matching function $K_{\mathcal{D}}(z)$. These functions are given by

$$L(z) = \begin{cases} (z-z_1)^{-i\kappa_1\sigma_3} & : \quad z \in \mathcal{D}_0 \\ L_{k-1}(z)\left[(z-z_1)^{i\kappa_1}C_\delta(z)\right]^{\mathrm{ad}\,\sigma_3} V_k(z) & : \quad z \in \mathcal{D}_k, \quad k=1,2,3 \\ L^*_{k-1}(z)\left[(z-z_1)^{i\kappa_1}C_\delta(z)\right]^{\mathrm{ad}\,\sigma_3} V_k^{-*}(z) & : \quad z \in \mathcal{D}_k^*, \quad k=1,2,3. \end{cases} \tag{6.137}$$

and

$$K_{\mathcal{D}}(z) = K_{\mathcal{A}}(z) P_+(0; -r_0(z_1)) \begin{pmatrix} 1 & 0 \\ r_0(z_1) & 1 \end{pmatrix} c_\delta^{-\sigma_3} s(z_1)^{-\sigma_3}. \tag{6.138}$$

Here we recall that the function $P(\zeta; a)$, defined by (4.36), is part of construction of the solution $\Psi_{P.C.}$ to the parabolic cylinder RHP. The proof that $L(z)$ and $K_{\mathcal{D}}(z)$ satisfy the jump conditions and the matching function $K_{\mathcal{D}}$ are virtually identical to case for $x > L$ which is given in Prop. 9. In the proof we show that the matching condition on the boundary $\partial\mathcal{D}$ satisfies:

$$B_{\mathcal{A}}^{-1}(z) B_{\mathcal{D}}(z) = I + \mathcal{O}\left(\epsilon^{1/2-\beta}\right). \tag{6.139}$$

6.4.3 Global Error Riemann Hilbert Problem

We now have a complete description of the global parametrix $A(z)$,

$$A(z) = \begin{cases} A_0(z) & : \quad z \in \mathcal{U}_0 \\ B_{\mathcal{A}}(z) & : \quad z \in \mathcal{A} \\ B_{\mathcal{D}}(z) & : \quad z \in \mathcal{D} \\ A_\alpha(z) & : \quad z \in \mathcal{U}_\alpha \\ A_{\bar\alpha}(z) & : \quad z \in \mathcal{U}_{\bar\alpha} \\ O(z) & : \quad \text{elsewhere} \end{cases}. \tag{6.140}$$

The outer solution $O(z)$ was defined in Section 6.3.2 by (6.102); the parabolic cylinder models $A_0(z)$ and $B_{\mathcal{A}}(z)$ defined by (6.127) and (5.66) respectively, and the disc model $B_{\mathcal{D}}$ defined by (6.137) were described in Section 6.4.2. The new wrinkle added by our two cut g-function is the new local models $A_\alpha(z)$, defined by (6.115), and $A_{\bar\alpha}(z)$, defined by (6.122), described in terms of the solution of the Airy RHP as described in Section 6.4.1.

We want to prove that our parametrix provides a uniform asymptotic approximation of the true solution $Q(z)$ to the RHP defined by (6.54) and (6.55). In doing so we will recover a valid asymptotic expansion of the solution $m(z)$ to the original RHP for NLS and thus an asymptotic description of the solution $q(x,t)$ to NLS for square barrier initial data in the space-time set $\mathcal{S}_2$, given by (6.144) below, where we have verified that the two-cut g-function ansatz controls the inverse spectral analysis.

Consider the error matrix, $E(z)$, defined by

$$E(z) = Q(z) A^{-1}(z). \tag{6.141}$$

Using the RHP for $Q(z)$ and the properties of our global parametrix, this definition generates a RHP for the error matrix, $E(z)$. Let Γ_E consist of the contours Γ_Q together with the

counter-clockwise oriented circles $\partial\mathcal{U}_0$, ∂U_1, $\partial\mathcal{U}_\alpha$, $\partial\mathcal{U}_{\bar{\alpha}}$, and $\partial\mathcal{D}$ where the global parametrix is patched together, see Figure 6.12.

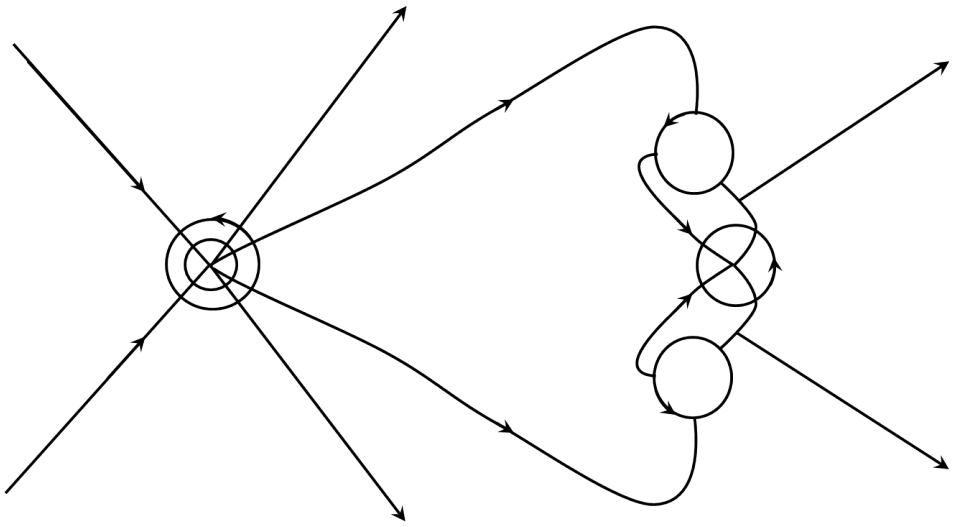

FIGURE 6.12. The collection of nontrivial jump contours Γ_E for the Error RHP. In principle Γ_E should also contain the band contours $\gamma_b \cup \gamma_b^*$ and the rays inside the two disks $\mathcal{U}_\alpha$ and $\mathcal{U}_{\bar{\alpha}}$. However, the jumps of the outer model $O(z)$ and the Airy models $A_\alpha(z)$ and $A_{\bar{\alpha}}(z)$ exactly match those of the full problem $Q(z)$ on these contours and thus the error, $E(z) = Q(z)A^{-1}(z)$, is analytic across these contours.

RHP for E(z):

Find a 2×2 matrix $E(z)$ which satisfies the following properties.

- $E(z)$ is analytic in $\mathbb{C}\backslash\Gamma_E$.
- As $z \to \infty$,
$$E(z) = I + \mathcal{O}(1/z). \tag{6.142}$$
- For $z \in \Gamma_E$, $E(z)$ assumes continuous boundary values $E_\pm(z)$ which satisfy the jump relation $E_+(z) = E_-(z)V_E(z)$, where
$$V_E = \begin{cases} A_-(z)V_Q(z)V_A^{-1}A_-^{-1} & : \quad z \in \Gamma_Q \\ O(z)A_\alpha^{-1}(z) & : \quad z \in \partial\mathcal{U}_\alpha \\ O(z)A_{\bar{\alpha}}^{-1}(z) & : \quad z \in \partial\mathcal{U}_{\bar{\alpha}} \\ O(z)A_0^{-1}(z) & : \quad z \in \partial\mathcal{U}_0 \\ O(z)B_{\mathcal{A}}^{-1}(z) & : \quad z \in \partial\mathcal{U}_1 \\ B_{\mathcal{A}}(z)B_{\mathcal{D}}^{-1}(z) & : \quad z \in \partial\mathcal{D} \end{cases} \tag{6.143}$$

Remark 8. Though we have naively stated that $E(z)$ has jumps given by (6.143) this is misleading. In the outer model, the jumps of the function $O(z)$ exactly match those of the full problem $Q(z)$ on the bands $\gamma_b \cup \gamma_b^*$, compare (6.58) with (6.55). Thus the jump of $E(z)$ along the bands is identity, that is, $E(z)$ is analytic across the band contours. Similarly, looking back at the derivation of the Airy models in Section 6.4.1, the local jumps of $Q(z)$ were not approximated in the model, A_α exactly matches the jump relations of $Q(z)$ inside $\mathcal{U}_\alpha$, thus $E(z)$ is analytic inside the sets $\mathcal{U}_\alpha$ and $\mathcal{U}_{\bar{\alpha}}$.

We solve the RHP for $E(z)$ by using the theory of small norm RHPs developed in Section 4.3.1. The key estimate we need is contained in the following lemma. Recall, the space-time set

$$\mathcal{S}_2 = \{(x,t) \,:\, 0 \le x < L,\ T_1(x) < t < T_2(x)\}\,, \tag{6.144}$$

where we recall the breaking times $T_1(x)$ and $T_2(x)$ have been previously defined in (6.1) and Prop. 21 respectively. For $(x,t) \in \mathcal{S}_2$ we have shown that the two band ansatz leads to a g-function which stabilized the RHP. The lemma estimating the perturbation of the error jumps from identity is as follows:

Lemma 22. *Let $K \subset \mathcal{S}_S$ be compact and $\ell \in \mathbb{N}_0$. Then there exists a constant $M_K^\ell < \infty$ such that for any $(x,t) \in K$ and $\rho \ll 1/2$, the jump matrix $V_E(z)$ defined by* (5.77) *satisfies the inequality,*

$$\left\| s^\ell (V_E - I) \right\|_{L^p(\Gamma_E)} \le M_K^\ell \epsilon^{1/2-\rho}, \tag{6.145}$$

for $p = 1, 2,$ or ∞ for every sufficiently small value of ϵ.

Proof. The proof of this lemma is identical to the proofs of the analogous lemmas in the previous chapters, see Lemmas 12 and 17. We make the following observations. First, the parametrix $A(z)$ is uniformly bounded in ϵ along every jump contour of $E(z)$. The only place it fails to be bounded is inside the disks centered at α and $\bar{\alpha}$, this because the Airy model RHP grows with ϵ inside the disk, however, $E(z)$ has no jumps inside these disks. One may also worry that the outer solution $O(z)$ now described as the ration of θ functions, is unbounded for fixed z as $\epsilon \to 0$. This is not the case, in the remarks following (6.103) we show that the outer model $O(z)$ is everywhere bounded in ϵ. Second, we must also consider the new error on the disk boundaries $\partial\mathcal{U}_\alpha$ and $\partial\mathcal{U}_{\bar{\alpha}}$, however using (6.119) and (6.124) the error on these boundaries is linear in ϵ and is thus a lower order error which does not change the estimate from the previous proofs. We omit the remaining calculations, the reader can easily produce a proof by mimicking the previous proofs. □

Using Lemma 22 and the small-norm theory presented in Section 4.3.1 we can write down the solution $E(z)$ to the error RHP defined by (6.142) and (6.143) as the unique solution of a singular integral equation.

Theorem 23. *For each $(x,t) \in \mathcal{S}_2$, defined by* (6.144)*, the global error Riemann-Hilbert problem defined by* (6.142) *and* (6.143) *has a unique solution given for $z \notin \Gamma_E$ by the formula*

$$E(z) = I + \frac{1}{2\pi i} \int_{\Gamma_E} \frac{(\eta(s) + I)(V_E(s) - I)}{s - z} ds,$$

where $\eta(s)$ is the unique solution of the equation

$$\eta - \mathcal{C}_-(\eta(V_E - I)) = \mathcal{C}_-(V_E - I).$$

In addition as $z \to \infty$ away from Γ_E, $E(z)$ has a valid asymptotic expansion,

$$E(z) = I + \frac{E^{(1)}}{z} + \dots,$$

where each moment $E^{(k)}$ is bounded, and in particular,

$$\left\| E^{(1)} \right\| \leq M \epsilon^{1/2-\rho}$$

for any $\rho > 0$. The bound is uniform on any compact $K \subset \mathcal{S}_2$.

Proof. Let $w_-(s) = V_E - I$ and $w_+(s) = 0$ and choose $K \subset \mathcal{S}_2$, compact. Then for each $(x,t) \in K$, the proof is a direct consequence of Lemma 22 controlling the size of $V_E - I$ and Lemmas 10 and 11 which relate the Cauchy transform to the solution of small-norm RHPs. □

Now that we have our hands on the solution $E(z)$ to the error RHP, we can use the series of explicit transformation used to transform the original RHP for $m(z)$ into $E(z)$:

$$m \mapsto M \mapsto N \mapsto P \mapsto Q \mapsto E,$$

to recover a formula for the solution $m(z)$ of the NLS RHP we original sought. What's more, because the error is asymptotically near identity, we get an asymptotic description of the solution $m(z)$ from which an asymptotic expression for the solution $q(x,t)$ of NLS follows by (3.17). These results are the content of Theorem 19 which we began this chapter by stating. We reproduce the theorem here and now give the culmination of its proof.

Theorem. *Define the set $\mathcal{S}_2$ by,*

$$\mathcal{S}_1 = \{(x,t) \ : \ 0 \leq |x| < L, \ \ T_1(x) < t < T_2(x)\},$$

for the functions $T_1(x)$ and $T_2(x)$ defined by (6.1) *and* (6.32) *respectively. Then for each value $(x,t) \in \mathcal{S}_2$ the solution $q(x,t)$ to the semiclassically focusing NLS equation*

$$i\epsilon q_t + \frac{\epsilon^2}{2} q_{xx} + |q|^2 q = 0, \qquad q(x,0) = q(x),$$

where $q(x)$ is the square barrier initial data (1.2)*, has the following asymptotically valid*

description as $\epsilon \to 0^+$*:*

$$q(x,t) = q_0(x,t) + \mathcal{O}(\epsilon^{1/2-\rho}) \tag{6.146}$$

for any number $\rho > 0$*. Additionally, the convergence is uniform on any compact subset of* $\mathcal{S}_2$*. The function* $q_0(x,t)$ *is described in terms of genus-one theta functions:*

$$q_0(x,t) = i\,\mathrm{Re}(\alpha(x,t))\frac{\Theta(\mathbf{Z})}{\Theta(\mathbf{Y})}\frac{\Theta(\mathbf{Y}+iU-i\Omega/\epsilon)}{\Theta(\mathbf{Z}-iU+i\Omega/\epsilon)}e^{-\frac{2i}{\epsilon}(g(\infty)-\epsilon\chi)} + \mathcal{O}(\epsilon^{1/2-\rho}), \tag{6.147}$$

where, $\mathbf{Z} = u(\infty) - V_1$ *and* $\mathbf{Y} = u(\infty) + V_1$*; the function* $u(z) = \int_{-iq}^{z}\nu$ *is defined by* (6.94)*; the constants* V_1*,* U*, and* χ *are defined by* (6.95)*,* (6.92)*, and* (6.97)*, respectively; the function* $\alpha(x,t)$ *is defined in Prop. 20, and the constants* Ω *and* $g(\infty)$ *are defined by* (6.28) *and* (6.25)*.*

Proof. Let $(x,t) \in K \subset \mathcal{S}_2$, for K compact. Using the explicit transformation formulae (6.4), (6.6), (6.34), and (6.41), together with the global parametrix $A(z)$ summarized in (6.140), the solution $m(z)$ to the original NLS RHP is given for suitably large z along the positive imaginary axis by the formula

$$m(z) = e^{-\frac{i}{\epsilon}g(\infty)\sigma_3}\beta(z)E(z)O(z)s^{\sigma_3}(z)\delta^{\sigma_3}(z)e^{\frac{i}{\epsilon}g(z)\sigma_3}.$$

the result follows from (3.17) and the asymptotic expansion of $O_{12}(z)$, which from (6.103) is given by

$$O_{12}(z) = \frac{\mathrm{Re}(\alpha(x,t))}{2z}\frac{\Theta(\mathbf{Z})}{\Theta(\mathbf{Y})}\frac{\Theta(\mathbf{Y}+iU-i\Omega/\epsilon)}{\Theta(\mathbf{Z}-iU+i\Omega/\epsilon)}e^{i(p(z)+2\chi)} + \mathcal{O}(1/z^2),$$

and the asymptotic expansions of $\delta(z)$, $s(z)$, and $\beta(z)$ which are given by (6.38), (6.49), and (6.42). The asymptotic correction to the leading behavior and uniform convergence on K are a direct consequence of Theorem 23. □

Appendix A

Analytic extension of the Jost solutions

In the discussion of the forward scattering problem the first difficulty we encountered was how to find solutions to the ZS problem,

$$i\epsilon\sigma_3\psi_x - \begin{pmatrix} 0 & q(x) \\ q(x) & 0 \end{pmatrix}\psi = z\psi, \tag{A.1}$$

for $z \notin \mathbb{R}$. As a first step we found integral representations for the Jost solutions to (A.1) for z restricted to the real axis. Recall that the Jost solutions are the unique solutions Ψ and Φ of (A.1) with the normalizations:

$$\begin{aligned} N &= \Psi \exp(\frac{i}{\epsilon} zx\sigma_3) \to I \qquad as \qquad x \to \infty \\ O &= \Phi \exp(\frac{i}{\epsilon} zx\sigma_3) \to I \qquad as \qquad x \to -\infty. \end{aligned}$$

We showed that the Jost solutions to (A.1) have integral representations given by,

$$\begin{aligned} N(x,0;z) &= I + \frac{1}{\epsilon}\int_{\infty}^{x} e^{\frac{i}{\epsilon}z(s-x)\sigma_3}\begin{pmatrix} 0 & q(s) \\ -q(s) & 0 \end{pmatrix} N(s,0)e^{-\frac{i}{\epsilon}z(s-x)\sigma_3}\,\mathrm{d}s, \\ O(x,0;z) &= I + \frac{1}{\epsilon}\int_{-\infty}^{x} e^{\frac{i}{\epsilon}z(s-x)\sigma_3}\begin{pmatrix} 0 & q(s) \\ -q(s) & 0 \end{pmatrix} O(s,0)e^{-\frac{i}{\epsilon}z(s-x)\sigma_3}\,\mathrm{d}s. \end{aligned}$$

These formulae were derived for $z \in \mathbb{R}$ where the exponentials are oscillatory. If $z \notin \mathbb{R}$ then the situation becomes more delicate. By examining the integral equations for the resulting column vectors we are able to show that each column exists and is analytic in at least a half-plane. The regions of analyticity depend upon the rate at which the initial condition $q(x)$ decays. These results are the subject of the following theorem.

Theorem (Analytic Extensions)**.** *For $N = (N_1, N_2)$ and $O = (O_1, O_2)$ as given above the column vectors exists and have analytic extensions under the following conditions on $q(x)$:*

1. *If $q(x) \in L^1(\mathbb{R})$, then N_1 and O_2 extend to $\mathbb{C}^-$ while N_2 and O_1 extend to $\mathbb{C}^+$.*
2. *If $\exists\ A, b$ such that $|q(x)| < Ae^{-bx}$ then N_1 and O_2 extend to $\left\{z|\ Im(z) < \frac{\epsilon}{2}b\right\}$ while N_2 and O_1 extend to $\left\{z|\ Im(z) > -\frac{\epsilon}{2}b\right\}$.*
3. *If $q(x)$ is compactly supported than both N and O are entire functions of z.*

Proof. Our proof is based on Nuemann series and is nearly identical for each column, so we only work through the details for $N_2(x;z)$. For vectors v, let $||v||(x) = \sum_i |v_i(x)|$, and for matrices $A(x)$, let $||A||(x) = \sup_i \sum_j |a_{ij}(x)|$. These are the usual pointwise vector and matrix 1-norms respectively.

The integral equation for N_2 is given by

$$N_2(x;z) = e_2 + \frac{1}{\epsilon}\int_\infty^x \begin{pmatrix} 0 & q(s)e^{\frac{2i}{\epsilon}z(s-x)} \\ -\overline{q(s)} & 0 \end{pmatrix} N_2(s;z)\,\mathrm{d}s \tag{A.2}$$

Let $K(s,x;z)$ denote the kernel of this integral equation. Consider the Nuemann iterates, denoted $N_2^k(x)$, defined by $N_2^0(x) = 0$ and

$$N_2^k(x;z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{\epsilon}\int_\infty^x K(x,s;z)N_2^{k-1}(s)\,\mathrm{d}s, \qquad k \geq 1.$$

Clearly $||N_2^1 - N_2^0|| = 1$, and for $k \geq 1$ we have the upper bound,

$$\begin{aligned} ||N_2^{k+1} - N_2^k||(x) &= \left\| \frac{1}{\epsilon}\int_\infty^x K(x,s;z)(N_2^k - N_2^{k-1})\,\mathrm{d}s \right\| \\ &\leq \frac{1}{\epsilon}\int_x^\infty ||K||(x,s;z)\ ||N_2^k - N_2^{k-1}||\,\mathrm{d}s. \end{aligned} \tag{A.3}$$

The matrix norm of $K(s,x;z)$ depends on the location of z in the complex plane:

$$\|K\|(x,s;z) = \begin{cases} |q(s)| & : \quad \mathrm{Im}(z) > 0 \\ |q(s)|e^{-\frac{2}{\epsilon}\mathrm{Im}(z)(s-x)} & : \quad \mathrm{Im}(z) < 0 \end{cases}. \tag{A.4}$$

We now consider each of the conditions on $q(x)$.

Case (1), $q \in L^1(\mathbb{R})$.
Without stronger knowledge of how q decays at infinity we cannot control $\|K\|(x,s;z)$ for $\mathrm{Im}(z) < 0$ so we restrict ourselves to $\mathrm{Im}(z) > 0$ for which $\|K\|(x,s;z) = |q(s)|$.

Claim 5. If $z \in \mathbb{C}^+$, then for all $k \geq 0$,

$$\|N_2^{k+1}(x;z) - N_2^k(x;z)\| \ \leq\ \frac{1}{k!\epsilon^k}\left(\int_x^\infty |q(s)|\ ds\right)^k.$$

Proof of Claim 5. The base case for $k = 0$ was shown above. In the inductive step (A.3)

and (A.4) imply that

$$\begin{aligned}
\|N_2^{k+1}(x;z) - N_2^k(x;z)\| &\leq \frac{1}{\epsilon}\int_x^\infty |q(s)|\ \|N_2^k(s;z) - N_2^{k-1}(s;z)\|\ ds \\
&\leq \frac{1}{\epsilon}\int_x^\infty |q(s)| \frac{1}{(k-1)!\epsilon^{k-1}} \left(\int_s^\infty |q(t)|\ dt\right)^{k-1} ds \\
&= \frac{1}{k!\epsilon^k}\int_x^\infty k|q(s)| \left(\int_s^\infty |q(t)|\ dt\right)^{k-1} ds \\
&= \frac{1}{k!\epsilon^k}\int_x^\infty -\frac{d}{ds}\left(\int_s^\infty |q(t)|\ dt\right)^{k} ds \\
&= \frac{1}{k!\epsilon^k}\left(\int_x^\infty |q(s)|\ ds\right)^{k}.
\end{aligned}$$

□

It follows easily that the solution of (A.2), $N_2(x;z) = \sum_{k=0}^\infty N_2^{k+1}(x;z) - N_2^k(x;z)$ converges absolutely for all $z \in \mathbb{C}^+$ since

$$\begin{aligned}
\|N_2(x;z)\| &\leq \sum_{k=0}^\infty \|N_2^{k+1}(x;z) - N_2^k(x;z)\| \leq \sum_{k=0}^\infty \frac{1}{k!\epsilon^k}\left(\int_x^\infty |q(s)|\ ds\right)^k \\
&= \exp\left(\frac{1}{\epsilon}\int_x^\infty |q(s)|\, \mathrm{d}s\right) < \infty,
\end{aligned}$$

since by assumption $q \in L^1(\mathbb{R})$. The absolute convergence of the Nuemann iterates implies both the existence and analyticity of the solution for each $z \in \mathbb{C}^+$.

Case (2), $|q(x)| \leq Ae^{-b|x|}$.

This condition guarantees that $q \in L^1(\mathbb{R})$ so the argument for case (1) shows that $N_2(x;z)$ exists for all $z \in \mathbb{C}^+$. The stronger requirements of case (2) allow $N_2(x;z)$ to be extended into a small band of $\mathbb{C}^-$. We make the following

Claim 6. For $z \in \mathbb{C}^-$ and $|q(x)| \leq Ae^{-b|x|}$, we have for all $k \geq 0$

$$\|N_2^{k+1}(x;z) - N_2^k(x;z)\| \ \leq\ \frac{1}{k!}\left(\frac{A}{\epsilon}\right)^k e^{-\frac{2}{\epsilon}k\Im(z)} \left(\int_x^\infty e^{-b|t|+2\Im(z)t}\ dt\right)^k.$$

Proof of Claim 6. As before, the base case, $k = 0$, is obvious. At the inductive step we

use (A.3) and (A.4) to get

$$\begin{aligned}
\|N_2^{k+1}(x;z) - N_2^k(x;z)\| &\leq \frac{1}{\epsilon}\int_x^\infty |q(s)| e^{-\frac{2}{\epsilon}\operatorname{Im}(z)(s-x)} \, \|N_2^k(s;z) - N_2^{k-1}(s;z)\| \, ds \\
&\leq \frac{1}{k!}\left(\frac{A}{\epsilon}\right)^k e^{\frac{2}{\epsilon}\operatorname{Im}(z)x} \int_x^\infty k e^{-b|s| - \frac{2}{\epsilon}k\operatorname{Im}(z)s} \left(\int_s^\infty e^{-b|t| - \frac{2}{\epsilon}\operatorname{Im}(z)t} \, dt\right)^{k-1} ds \\
&\leq \frac{1}{k!}\left(\frac{A}{\epsilon}\right)^k e^{\frac{2}{\epsilon}k\operatorname{Im}(z)x} \int_x^\infty k e^{-b|s| - \frac{2}{\epsilon}\operatorname{Im}(z)s} \left(\int_s^\infty e^{-b|t| - \frac{2}{\epsilon}\operatorname{Im}(z)t} \, dt\right)^{k-1} ds \\
&= \frac{1}{k!}\left(\frac{A}{\epsilon}\right)^k e^{\frac{2}{\epsilon}k\operatorname{Im}(z)x} \int_x^\infty -\frac{d}{ds}\left(\int_s^\infty e^{-b|t| - \frac{2}{\epsilon}\operatorname{Im}(z)t} \, dt\right)^k ds \\
&= \frac{1}{k!}\left(\frac{A}{\epsilon}\right)^k e^{\frac{2}{\epsilon}k\operatorname{Im}(z)x} \left(\int_x^\infty e^{-b|t| - \frac{2}{\epsilon}\operatorname{Im}(z)t} \, dt\right)^k.
\end{aligned}$$

□

Finally, we want to show that the solution vector $N_2(x;z) = \sum_{k=0}^\infty N_2^{k+1}(x;z) - N_2^k(x;z)$ will converge in a small band of $\mathbb{C}^-$. Proceeding as in the first case, we examine the absolute convergence of the telescoping sum

$$\begin{aligned}
\|N_2(x;z)\| &\leq \sum_{k=0}^\infty \|N_2^{k+1}(x;z) - N_2^k(x;z)\| \\
&\leq \sum_{k=0}^\infty \frac{1}{k!}\left(\frac{A}{\epsilon}\right)^k e^{-\frac{2}{\epsilon}k\operatorname{Im}(z)} \left(\int_x^\infty e^{-b|t| - \frac{2}{\epsilon}\operatorname{Im}(z)t} \, dt\right)^k \\
&= \exp\left(\frac{A}{\epsilon} e^{\frac{2}{\epsilon}\operatorname{Im}(z)x} \int_x^\infty e^{-b|t| - \frac{2}{\epsilon}\operatorname{Im}(z)t} \, dt\right).
\end{aligned}$$

This sum converges absolutely for every x, provided that the integrand is $L^1(\mathbb{R})$; this happens precisely for $\left\{z \mid \operatorname{Im}(z) > -\frac{\epsilon}{2}b\right\}$. As before, absolute uniform convergence of the telescoping sum immediately establishes existence and z-analyticity of the column vector $N_2(x,0;z)$.

Case (3), Compact support.

We wish to show that for compactly supported potentials q that N_2 is an entire function. This follows immediately from Case 2; for any value of b, we can always choose A_q such that $q(x) < A_q e^{-b|x|}$. Specifically, suppose that $\operatorname{supp}(q) \subset [-L, L]$. Then for a given b setting $A_q = \|q\|_\infty e^{bL}$ puts us in Case 2. By taking b arbitrarily large we get that $N_2(x;z)$ is analytic in the entire complex plane.

□

Appendix B

Riemann Surfaces, Divisors, and Theta functions

Here we collect facts concerning various aspects of meromorphic functions on Riemann surfaces which will help us describe the solution $v(P)$ to the hyperelliptic problem posed above. We present these facts without proof, our purpose being to fix notation and to collect the relevant classical results without diving head first into a review of function theory on Riemann surfaces. These facts are taken from [Sie88b], [Sie88a], and [Dub81] and the reader is referred to these sources for the details we have omitted.

Homology, differentials, and definition of Θ

Let us consider a Riemann surface Σ of genus G. We will always assume that Σ is compact. Associated with Σ is the *first homology group*, denoted $H_1(\Sigma, \mathbb{Z})$. The elements of $H_1(\Sigma, \mathbb{Z})$ are equivalence classes of closed oriented contours on Σ which cannot be contracted to a single point. Two contours a_1 and a_2 are of the same class, or *homologous*, provided that the contour $a_1 a_2^{-1}$—the contour produced by following a_1 in the positive and then a_2 in the reverse direction—is an oriented boundary on the surface Σ. A basic result of topology states that the dimension of the first homology group is $2g$. A *homology basis* is a collection of $2g$ contours $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ which generate $H_1(\Sigma, \mathbb{Z})$ and have the intersection properties

$$a_j \circ b_k = \delta_{j,k}, \qquad a_j \circ a_k = 0, \qquad b_j \circ b_k = 0, \qquad j, k = 1, \ldots, g \tag{B.1}$$

where $C_1 \circ C_2$ counts the number of times that C_2 crosses C_1 from right-to-left (with respect to the orientation of C_1) minus the number of crossings from left-to-right. The choice of basis in not unique. Given any homology basis $(\vec{a}, \vec{b})$ and any symplectic linear transformation $\mathbf{S} \in Sp(2G, \mathbb{Z})$, the map $(\vec{a}, \vec{b}) \mapsto (\vec{a}, \vec{b})\mathbf{S}$ produces another homology basis; in fact, every homology basis may be generated in this way. The particular homology basis chosen for any application should be selected to simplify calculations.

Given any closed contour C on Σ and a closed differential ω the quantity $\oint_C \omega$ is called a period of the differential ω. The following results has several useful consequences which we will use in what follows;

Theorem 24 (Riemann's period relations)**.** *Suppose ω and ν are closed C^1 meromorphic differentials on Σ. Then*

$$\iint \omega \wedge \nu = \sum_{j=1}^{G} \left(\left(\oint_{a_j} \omega \right) \left(\oint_{b_j} \nu \right) - \left(\oint_{b_j} \omega \right) \left(\oint_{a_j} \nu \right) \right) \tag{B.2}$$

The proof of Theorem 24 follows from the application of Stokes' theorem on a canonical dissection of Σ. Two immediate consequences of this result are (1) that all the a_k periods

of a nonzero holomorphic differential cannot vanish and (2) that the space of holomorphic differentials is $G-$dimensional.

Given a homology basis, one can define the normalized basis of holomorphic differentials (sometimes called the Abelian differentials of the first kind) on Σ. The normalized basis $\{\nu_j\}_{j=1}^G$, for the given homology basis, satisfies

$$\oint_{a_k} \nu_j = 2\pi i \delta_{j,k}, \qquad j,k = 1,\ldots,G. \tag{B.3}$$

In the hyperelliptic case where Σ is the Riemann surface associated with the function $R^2(z) = \prod_{k=0}^G (z - z_k)(z - z_k^*)$ the holomorphic differentials can be represented on each sheet by the formula

$$\nu_k = \frac{\sum_{j=0}^{G-1} c_{jk} z(P)^j}{R_\Sigma(P)} \mathrm{d}z(P) \tag{B.4}$$

where R_Σ is the lifting of R to Σ,

$$R_\Sigma(P) = \begin{cases} R(z(P)) & : \quad P \text{ in sheet 1} \\ -R(z(P)) & : \quad P \text{ in sheet 2} \end{cases}. \tag{B.5}$$

The normalization conditions (B.3) become in this case a linear system of G^2 equations in G^2 unknowns which uniquely determines the c_{jk}'s.

Equipped with the normalized holomorphic differentials the *period matrix* $\mathbf{H}$ is defined componentwise by the formulae

$$\mathbf{H}_{jk} = \oint_{b_j} \nu_k. \tag{B.6}$$

It follows from Riemann's period relations (B.2) that $\mathbf{H}$ is a symmetric matrix with negative definite real part. It should be clear that the period matrix inherits from the normalized differentials ν_k dependence on the choice of homology basis. Using $\mathbf{H}$ one can define the associated Riemann theta function, $\Theta : \mathbb{C}^G \to \mathbb{C}$:

$$\Theta(\mathbf{w}|\mathbf{H}) := \sum_{\mathbf{n}\in\mathbb{Z}^G} \exp\left(\frac{1}{2}\mathbf{n}^T\mathbf{H}\mathbf{n} - \mathbf{n}^T\mathbf{w}\right). \tag{B.7}$$

Because $\mathbf{H}$ has negative definite real part the sum converges absolutely and uniformly on any compact subset of $\mathbb{C}^g$, from which it follows that Θ is entire. A fundamental property of Θ is its periodicity; Θ has G periods and G quasi-periods given by the relation

$$\Theta(\mathbf{w} + 2\pi i\mathbf{n} + \mathbf{H}\mathbf{m}) = \exp\left(-\frac{1}{2}\mathbf{m}^T\mathbf{H}\mathbf{m} - \mathbf{m}^T\mathbf{w}\right)\Theta(w), \qquad \mathbf{n},\mathbf{m} \in \mathbb{Z}^G. \tag{B.8}$$

Often, one works with a fixed surface and a given homology basis. In these cases the dependence of the Θ function on the period matrix $\mathbf{H}$ is suppressed in the notation, and we write simply $\Theta(\mathbf{w})$ in place of $\Theta(\mathbf{w}|\mathbf{H})$.

Divisors and the Riemann-Roch Theorem

A *divisor* on the compact Riemann surface Σ is a map $\mathcal{D} : \Sigma \to \mathbb{Z}$ which is nonzero at only finitely many points $P \in \Sigma$. We denote the set of all divisors on Σ by $\mathrm{Div}(\Sigma)$ and represent each divisor as the formal sum

$$\mathcal{D} = \sum_{P \in \Sigma} \mathcal{D}(P)P.$$

Using this representation, $\mathrm{Div}(\Sigma)$ is clearly an additive Abelian group.

On $\mathrm{Div}(\Sigma)$ we place a partial ordering; we say that $\mathcal{D} \geq \mathcal{D}'$ if $\mathcal{D}(P) \geq \mathcal{D}'(P)$ for all $P \in \Sigma$. A divisor $\mathcal{D}$ is called *effective* if

$$\mathcal{D} \geq 0,$$

where 0 is the zero divisor, *i.e.* $0(P) = 0$ for all $P \in \Sigma$. We say that $\mathcal{D}'$ is a *multiple* of $\mathcal{D}$ (or equivalently that $\mathcal{D}$ divides $\mathcal{D}'$) if

$$\mathcal{D}' \geq \mathcal{D}.$$

The *degree* of a divisor, denoted $\deg(\mathcal{D})$ is defined to be

$$\deg(\mathcal{D}) = \sum_{P \in \Sigma} \mathcal{D}(P).$$

Let $\mathcal{M}(\Sigma)$ and $\mathcal{M}^1(\Sigma)$ denote, respectively, the spaces of meromorphic functions and differentials on Σ. We can associate with each nonzero element of these sets a divisor as follows:

- For $f \in \mathcal{M}(\Sigma)\backslash\{0\}$, the divisor (f) is defined by

$$(f)(P) = \begin{cases} m & : \ P \text{ zero of order } m \\ -m & : \ P \text{ pole of order } m \\ 0 & : \ \text{otherwise} \end{cases} \tag{B.9}$$

- For $\omega \in \mathcal{M}^1(\Sigma)\backslash\{0\}$, let $z(P)$ be a local coordinate near P, then ω has the local representation $\omega = f(z(p))\mathrm{d}z$ for $f \in \mathcal{M}(\Sigma)\backslash 0$. Define $(\omega)(P) = (f)(P)$ where $(f)(P)$ is given by (B.9).

We have the immediate consquences, f is holomorphic $\leftrightarrow$ $(f) \geq 0$, $f \in \mathcal{M}(\Sigma) \to$ $\deg((f)) = 0$, and $\omega \in \mathcal{M}^1(\Sigma) \to$ $\deg((\omega)) = 2G - 2$.

A divisor is called *principal* if it can be represented as the divisor of a meromorphic function and *cannonical* if it can be represented as the divisor of a meromorphic differential. Two differentials $\mathcal{D}$ and $\mathcal{D}'$ are in the same *divisor class* if there exists a $f \in \mathcal{M}(\Sigma)$ such that $\mathcal{D} - \mathcal{D}' = (f)$.

Next we consider two important vector spaces associated with a given divisor $\mathcal{D}$. Define

$$\mathcal{L}(\mathcal{D}) = \{f \in \mathcal{M}(\Sigma) \mid (f) \geq \mathcal{D}\},$$
$$\mathcal{L}^1(\mathcal{D}) = \{\omega \in \mathcal{M}^1(\Sigma) \mid (\omega) \geq \mathcal{D}\}.$$

We denote the dimension of each vector space over $\mathbb{C}$ by

$$r(\mathcal{D}) = \dim \mathcal{L}(\mathcal{D}),$$
$$i(\mathcal{D}) = \dim \mathcal{L}^1(\mathcal{D}).$$

We say that a divisor $\mathcal{D}$ is *special* provided $i(\mathcal{D}) > 0$ otherwise we say that $\mathcal{D}$ is *nonspecial*. Though the spaces $\mathcal{L}(\mathcal{D})$ and $\mathcal{L}^1(\mathcal{D})$ depend on the particular divisor it is easy to show that the dimensions of these spaces, $r(\mathcal{D})$ and $i(\mathcal{D})$, depend only on the divisor class. The following important theorem gives us a tool to get a handle on the size of $r(\mathcal{D})$ and $i(\mathcal{D})$, particularly in the case when $\mathcal{D}$ is nonspecial.

Theorem 25 (**Riemann-Roch**). *Let Σ be a compact Riemann surface of genus G and $\mathcal{D} \in \mathrm{Div}(\Sigma)$. Then, $r(-\mathcal{D})$ and $i(\mathcal{D})$ are finite and*

$$r(-D) - i(\mathcal{D}) = \deg(\mathcal{D}) - G + 1. \tag{B.10}$$

Abel Maps and Zeros of the Θ function

Recall that for a particular choice of homology basis $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ we have an associated set of normalized holomorphic differentials ν_k satisfying the condition (B.3) and a period matrix $\mathbf{H}$ whose columns $\mathbf{h}_k$ are the b_j-periods of the differential ν_k. Define the lattice of points

$$\Lambda = \{z \in \mathbb{C}^G \mid z = 2\mathbf{n}\pi i + \mathbf{H}\mathbf{m},\ \mathbf{n}, \mathbf{m} \in \mathbb{Z}^G\}. \tag{B.11}$$

The *Jacobi variety* of Σ, is the torus

$$\mathrm{Jac}(\Sigma) = \mathbb{C}^G/\Lambda.$$

Now fix a base point $P_0 \in \Sigma$, the *Abel map*, $\mathbf{A} : \Sigma \to \mathrm{Jac}(\Sigma)$ is defined componentwise by the formula,

$$A_k(P) = \int_{P_0}^{P} \nu_k \qquad \mod \Lambda. \tag{B.12}$$

The map is well defined because the difference of integrating from P_0 to P along different paths is precisely an integer sum of the periods of the normalized differentials, that is the difference is a point in the lattice Λ. The Abel map can be naturally extended to divisors. For $\mathcal{D} = \sum n_k P_k$ we define

$$\mathbf{A}(\mathcal{D}) = \sum n_k \mathbf{A}(P_k).$$

Though we have suppressed the dependence in our notation, the Abel map $\mathbf{A}$ also depends on our choice of base point P_0 in general. However, for divisors $\mathcal{D}$ with $\deg(\mathcal{D}) = 0$, the value of $\mathbf{A}(\mathcal{D})$ is independent of our choice of P_0. The Abel map characterizes the principal divisors:

Theorem 26 (**Abel's Theorem**). *A divisor* $\mathcal{D} \in \mathrm{Div}(\Sigma)$ *is principal if and only if*

$$\deg(\mathcal{D}) = 0 \quad \textit{and} \quad \boldsymbol{A}(\mathcal{D}) = 0 \tag{B.13}$$

Corollary 27. *Two divisors* $\mathcal{D}$ *and* $\mathcal{D}'$ *are equivalent if and only if*

$$\deg(\mathcal{D}) = \deg(\mathcal{D}') \quad \textit{and} \quad \boldsymbol{A}(\mathcal{D}) = \boldsymbol{A}(\mathcal{D}').$$

The Abel map is closely connected with the *Jacobi inversion problem* on the Riemann surface Σ. Let $\mathrm{Sym}^G(\Sigma)$, called the *G-th symmetric power* of Σ, be the collection of all unordered sets of G points from Σ. The Jacobi inversion problem ask one to invert the restriction of the Abel map to $\mathrm{Sym}^G(\Sigma)$, that is given $\zeta \in \mathrm{Jac}(\Sigma)$ find G points $P_1, \ldots, P_G$ such that

$$\sum_{j=1}^{G} \mathbf{A}(P_j) = \zeta \qquad \mod \Lambda. \tag{B.14}$$

The solution of this problem is related to the Θ-functions introduced earlier.

By composing the Θ-function with the Abel map we generate Θ-functions defined on the Riemann surface Σ. Specifically, fix a $\xi \in \mathbb{C}^G$ and define $F : \Sigma \to \mathbb{C}$ by

$$F(P) = \Theta(\mathbf{A}(P) - \xi).$$

If we let $\widehat{\Sigma}$ denote the canonical dissection of Σ along the homology basis into a $4G$-gon, then $F(P)$ is single valued and analytic on $\widehat{\Sigma}$. However, on the connected Riemann surface Σ, $F(P)$ is, in general, multivalued. Given different choices of the representative for $\mathbf{A}(P) \in \mathrm{Jac}(\Sigma)$ the difference is an element of the lattice Λ for which the fundamental period relations (B.8) relate the multiple values of $F(P)$ by the multiplicative formula

$$F(P_2) = \exp\left(-\frac{1}{2}\mathbf{m}^T\mathbf{H}\mathbf{m} - \mathbf{m}^T\mathbf{w}\right) F(P_1), \qquad P_1 = P_2 \mod \Lambda \tag{B.15}$$

So though it may not make sense to speak of the value of F at a point P in general, it does make sense to ask about the zeros of F as the multiplicative factor in (B.15) is strictly nonzero.

In the remainder we enumerate a collection of properties concerning the zeros of the function $F(P)$ which together allow one to construct the solution to the hyperelliptic Riemann Hilbert problem for $Q(z)$ given in section 6.3.1.

Lemma 28. *If* $F(P)$ *is not identically zero, then* $F(P)$ *has exactly* G *zeros on* $\widehat{\Sigma}$ *counting multiplicities.*

Lemma 29. *Suppose* $F(P)$ *is not identically zero, let* $\mathcal{D} = P_1 + \ldots + P_G$ *be the divisor of zeros of* F *on* Σ. *Then on* $\mathrm{Jac}(\Sigma)$,

$$\mathbf{A}(\mathcal{D}) = \xi - \mathbf{K} \tag{B.16}$$

where $\mathbf{K} \in \mathbb{C}^G$ *is the vector of Riemann Constants given componentwise by the formula*

$$K_j = i\pi + \frac{1}{2}H_{jj} - \frac{1}{2\pi i}\sum_{\substack{l=1 \\ l\neq j}}^{G} \oint_{a_j} (\nu_l(P)A_j(P))\,, \qquad j = 1, \ldots, G. \tag{B.17}$$

To prove these lemmas one integrates $\frac{1}{2\pi i}\log F(P)$ and $\frac{1}{2\pi i}F(P)\mathrm{d}\log F(P)$, respectively, over the boundary of the canonical dissection of the Riemann surface.

Theorem 30. *The function* $F(P) = \Theta(\mathbf{A}(P) - \xi)$ *is identically zero on* Σ *if and only if the point* ξ *can be written in the form*

$$\xi = \mathbf{A}(\mathcal{D}_Q) + \mathbf{K} \tag{B.18}$$

where $Q \in \mathrm{Sym}^G(\Sigma)$ *and the divisor* $\mathcal{D}_Q = Q_1 + \ldots + Q_G$ *is special.*

Theorem 31. *Let* $\zeta \in \mathbb{C}^G$ *be such that* $F(P) = \Theta(\mathbf{A}(P) - \zeta - \mathbf{K})$ *does not vanish identically. Then*

1. *$F(P)$ has G zeros $P_1, \ldots, P_G$ on Σ, which give a solution of the Jacobi inversion problem*

$$A_j(P_1) + \ldots + A_j(P_G) = \sum_{k=1}^{G} \int_{P_0}^{P_k} \nu_j = \zeta_j \qquad \mod \Lambda \qquad j = 1, \ldots, G \tag{B.19}$$

2. *The divisor $\mathcal{D} = P_1 + \ldots + P_G$ is non-special.*

3. *The points $P_1, \ldots, P_G$ are uniquely determined from the system* (B.19) *up to permutation.*

An immediate and very useful corollary of Theorem 31 allows us to build a Theta function on Σ with a specific set of G zeros.

Corollary 32. *Let* $\{P_1, \ldots, P_G\} \in \mathrm{Sym}^G(\Sigma)$ *and* $\mathcal{D}_P = P_1 + \ldots + P_G$ *the associated divisor. If* $\mathcal{D}_P$ *is non-special then the function* $F(P) = \Theta(\mathbf{A}(P) - A(\mathcal{D}_P) - \mathbf{K})$ *has exactly G zeros* $P = P_1, \ldots, P = P_G$ *on* Σ.

References

[AS64] Milton Abramowitz and Irene A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[DKM$^+$99] P. Deift, T. Kriecherbauer, K. T-R McLaughlin, S. Venakides, and X. Zhou. Strong asymptotics of orthogonal polynomials with respect to exponential weights. *Comm. Pure Appl. Math.*, 52(12):1491–1552, 1999.

[Dub81] B. A. Dubrovin. Theta-functions and nonlinear equations. *Uspekhi Mat. Nauk*, 36(2(218)):11–80, 1981. With an appendix by I. M. Krichever.

[DZ94] Percy Deift and Xin Zhou. Long-time behavior of the non-focusing nonlinear schrödinger equation: A case study. *New Series: Lectures in Mathematical Sciences*, (5), 1994.

[DZ02] Percy Deift and Xin Zhou. A priori L^p-estimates for solutions of Riemann-Hilbert problems. *Int. Math. Res. Not.*, (40):2121–2154, 2002.

[KMM03] Spyridon Kamvissis, Kenneth D. T.-R. McLaughlin, and Peter D. Miller. *Semiclassical soliton ensembles for the focusing nonlinear Schrödinger equation*, volume 154 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2003.

[KS02] M. Klaus and J. K. Shaw. Purely imaginary eigenvalues of Zakharov-Shabat systems. *Phys. Rev. E (3)*, 65(3):036607, 5, 2002.

[KS03] M. Klaus and J. K. Shaw. On the eigenvalues of Zakharov-Shabat systems. *SIAM J. Math. Anal.*, 34(4):759–773 (electronic), 2003.

[Mil02] Peter D. Miller. Asymptotics of semiclassical soliton ensembles: rigorous justification of the WKB approximation. *Int. Math. Res. Not.*, (8):383–454, 2002.

[Sie88a] C. L. Siegel. *Topics in complex function theory. Vol. I.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1988. Elliptic functions and uniformization theory, Translated from the German by A. Shenitzer and D. Solitar, With a preface by Wilhelm Magnus, Reprint of the 1969 edition, A Wiley-Interscience Publication.

[Sie88b] C. L. Siegel. *Topics in complex function theory. Vol. II.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1988. Automorphic functions and abelian integrals, Translated from the German by A. Shenitzer and M. Tretkoff, With a preface by Wilhelm Magnus, Reprint of the 1971 edition, A Wiley-Interscience Publication.

[SY74] Junkichi Satsuma and Nobuo Yajima. Initial value problems of one-dimensional self-modulation of nonlinear waves in dispersive media. *Progr. Theoret. Phys. Suppl. No. 55*, pages 284–306, 1974.

[TV00] Alexander Tovbis and Stephanos Venakides. The eigenvalue problem for the focusing nonlinear Schrödinger equation: new solvable cases. *Phys. D*, 146(1-4):150–164, 2000.

[TVZ04] Alexander Tovbis, Stephanos Venakides, and Xin Zhou. On semiclassical (zero dispersion limit) solutions of the focusing nonlinear Schrödinger equation. *Comm. Pure Appl. Math.*, 57(7):877–985, 2004.

[TVZ06] Alexander Tovbis, Stephanos Venakides, and Xin Zhou. On the long-time limit of semiclassical (zero dispersion limit) solutions of the focusing nonlinear Schrödinger equation: pure radiation case. *Comm. Pure Appl. Math.*, 59(10):1379–1432, 2006.

[Zho98] Xin Zhou. L^2-Sobolev space bijectivity of the scattering and inverse scattering transforms. *Comm. Pure Appl. Math.*, 51(7):697–731, 1998.

[ZS71] V. E. Zakharov and A. B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Ž. Èksper. Teoret. Fiz.*, 61(1):118–134, 1971.