GINZBURG-WEINSTEIN ISOMORPHISMS FOR PSEUDO-UNITARY GROUPS

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STATEMENT BY AUTHOR

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Abstract

Ginzburg and Weinstein proved in [GW92] that for a compact, semisimple Lie group K endowed with the Lu-Weinstein Poisson structure, there exists a Poisson diffeomorphism from the dual Poisson Lie group K^* to the dual $\mathfrak{k}^*$ of the Lie algebra of K endowed with the Lie-Poisson structure. We investigate the possibility of extending this result to the pseudo-unitary groups $SU(p,q)$, which are semisimple but not compact.

The main results presented here are the following.

- The Ginzburg-Weinstein proof hinges on the existence of a certain vector field X on $\mathfrak{k}^*$. We prove that for any p, q, the analogous vector field for the $SU(p,q)$ case exists on an open subset of $\mathfrak{k}^*$.

- Each generic dressing orbit $\Psi_{\vec{\lambda}}$ in the Poisson dual AN can be embedded in the complex flag manifold K/T. We show that for $SU(1,1)$ and $SU(1,2)$, the induced Poisson structure $\pi_{\vec{\lambda}}$ on $\Psi_{\vec{\lambda}}$ extends smoothly to the entire flag manifold.

- Finally, we prove the Ginzburg-Weinstein theorem for the $SU(1,1)$ case in two different ways: first, by constructing the vector field X in coordinates and proving that it satisfies the necessary properties, and second, by adapting the approach of [FR96] to the $SU(1,1)$ case.

1. Introduction

A Poisson structure on a manifold M is a Lie bracket on $C^\infty(M)$ which satisfies the Leibniz rule. For any symplectic manifold (M, ω), the symplectic structure ω induces a Poisson structure on M. On the other hand, any nondegenerate Poisson structure corresponds to a symplectic form. Thus, Poisson structures can be regarded as generalized symplectic structures. One of the simplest examples is the bracket

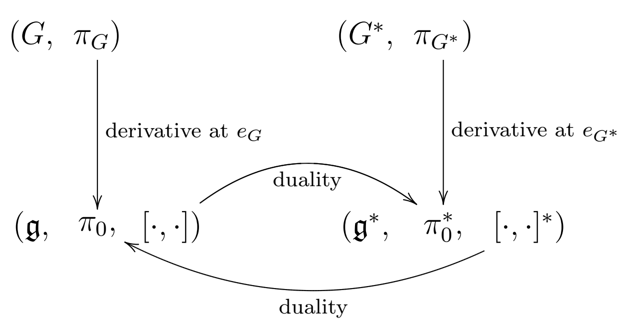
$$\{f, g\} := \sum_{i<j} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial x_j} \right)$$

on $\mathbb{R}^n$, which plays an important role in Hamiltonian mechanics. Over the past 40 years or so, connections have been found between Poisson geometry and many areas of pure and applied mathematics, including noncommutative algebra, classical and quantum mechanics, and infinite-dimensional Lie algebras, to name a few. The basic definitions and some of the fundamental results regarding Poisson manifolds are covered here. A detailed and thorough treatment is given in [Vai94], for example. A broad survey of more advanced topics in the field can be found in Alan Weinstein's article, [Wei98].

Poisson Lie theory lies at the intersection of Poisson geometry and Lie theory. The fundamental objects in this field are Poisson Lie groups–that is, Lie groups endowed with Poisson structures that are compatible with group multiplication. In other words, a Poisson Lie group is a space that is endowed with three compatible structures: a smooth structure, a group operation, and a Poisson bracket. They were originally introduced by Drinfel'd [Dri83] and Semenov-Tian-Shansky [STS85] in the 1980s. Poisson Lie groups have applications in statistical mechanics, quantum field theory, and integrable systems. This thesis, however, will be concerned with the theoretical side of Poisson Lie theory.

A fundamental property of Poisson Lie groups is that they come in dual pairs. Given a Poisson Lie group (G, π_G), the derivative of the Poisson structure π_G at the identity induces a Poisson structure π_0 on the Lie algebra $\mathfrak{g}$ of G. On the other hand, the Lie bracket on $\mathfrak{g}$ induces the Lie-Poisson structure π_0^* on the dual space $\mathfrak{g}^*$. There is a unique (connected, simply-connected) Poisson Lie group (G^*, π_G^*) such that the derivative at the identity of π_G^* is the Lie-Poisson structure π_0^*, and the Lie-Poisson bracket on $\mathfrak{g}$ induced by the Lie bracket on $\mathfrak{g}^*$ is the linear structure π_0. (See Figure 1.1.) Thus, to any Poisson Lie group (G, π_G) are associated two other, natural

Figure 1.1. Duality of Poisson Lie Groups



Poisson Lie groups: the dual $\mathfrak{g}^*$ of the Lie algebra of G endowed with the linear, Lie-Poisson structure (where $\mathfrak{g}^*$ is regarded as an abelian Lie group), and the dual Poisson Lie group G^* endowed with the corresponding nonlinear structure. Ginzburg and Weinstein proved in [GW92] that if G is compact and semisimple, then these two spaces are isomorphic as Poisson manifolds (but not as Poisson Lie groups). Boalch gave a new proof of this result in [Boa01]. A similar result for formal Poisson Lie groups was established in [EEM05].

For the special case $G = SU(2)$, Flaschka and Ratiu used the Gelfand-Tsetlin coordinates (which are given by eigenvalues of all principal $k \times k$ minors of an $n \times n$ matrix) to construct an explicit Ginzburg-Weinstein isomorphism ([FR96]). They conjectured, moreover, that for $G = SU(n)$, there exists a distinguished Ginzburg-Weinstein isomorphism which intertwines the Gelfand-Tsetlin coordinates on G^* and $\mathfrak{g}^*$. This conjecture was later proved by Alekseev and Meinrenken in [AM07].

The Ginzburg-Weinstein theorem establishes a correspondence between the nonlinear, Poisson-Lie setting and the linear, Lie-Poisson setting. Flaschka and Ratiu used this correspondence to convert the convexity theorem of Guillemin-Sternberg and Kirwan for Hamiltonian actions of compact Lie groups to a corresponding result for Poisson actions of compact, semisimple Poisson Lie groups ([FR96]). They remark, in fact, that "absolutely everything one does for Hamiltonian momentum mappings has a Poisson Lie counterpart."

Given a compact, simple Poisson Lie group K, a *Poisson K-space* is a symplectic manifold (M, ω) together with a Poisson action of K and a corresponding K^*-valued momentum map. A *Symplectic K-space* is a symplectic manifold (M, ω) together with a symplectic action of K and a corresponding $\mathfrak{k}^*$-valued momentum map. Alekseev established a one-to-one correspondence ([Ale97]) between the categories of Poisson K-spaces and symplectic K-spaces. This correspondence was further developed in [AMW01] and used to prove the Thompson Conjecture on singular values of products of complex matrices.

All of the results mentioned above (with the exception of [EEM05]) require compactness of the Poisson Lie group. A natural question to ask is whether some version of the Ginzburg-Weinstein theorem holds for noncompact groups. The pseudo-unitary groups $SU(p, q)$ are perhaps the best-behaved noncompact Lie groups, given their similarity to the unitary groups $SU(p+q)$. Moreover, for any p, q, $SU(p, q)$ admits a multiplicative (i.e., compatible with group multiplication) Poisson structure such that its dual Poisson group can be identified with the subgroup $AN \subset GL(p+q, \mathbb{C})$ of upper-triangular matrices with real, positive diagonal. The results presented in this thesis are related to the possibility of extending the Ginzburg-Weinstein theorem to the groups $SU(p, q)$. In particular, we show that much of the proof in [GW92] can be adapted to the $SU(p, q)$ case.

1.1. Outline

Chapter 2 covers basic results on Poisson geometry and Poisson Lie groups and outlines the proof of the Ginzburg-Weinstein theorem. Special emphasis is given to the ways in which compactness is used in the proof.

Chapter 3 concerns the $SU(p,q)$ case. The dressing action of G on its dual plays an important role in the Ginzburg-Weinstein proof. In the $G_0 = SU(p,q)$ case, the dressing action is not globally defined. We identify in Section 3.1, a subset of G_0^* on which the dressing action is defined for all elements of G_0. We then construct in Section 3.2 a G_0-equivariant, one-to-one map from this subset to $\mathfrak{g}_0^*$.

If G is compact and semisimple, G^* and $\mathfrak{g}^*$ are diffeomorphic. The nonlinear structure π on G^* and the linear structure π_0 on $\mathfrak{g}^*$ can therefore both be regarded as bivector fields on $\mathfrak{g}^*$. The proof in [GW92] hinges on the existence of a vector field X such that $[X, \pi] = \dot{\pi}$, where $\dot{\pi}$ is the derivative of a family of Poisson structures π_t leading from $\pi = \pi_1$ to π_0. The vector field X can be used to produce a flow ϕ_t which pushes π_0 forward along the family π_t. In particular, ϕ_1 is the desired Poisson diffeomorphism from $(\mathfrak{g}^*, \pi_0)$ to $(\mathfrak{g}^*, \pi_1)$. We prove in Section 3.3 that for $G_0 = SU(p,q)$, there exists a vector field X on an open submanifold of $\mathfrak{g}_0^*$ satisfying $[X, \pi] = \dot{\pi}$.

The existence of the flow ϕ_t is proved using compactness of the dressing orbits in G^*. In the $SU(p,q)$ case, the dressing orbits are not compact. However, the generic dressing orbits can be compactified in the following sense. Each generic orbit $\Psi_{\vec{\lambda}}$ can be identified with G_0/T, where T is the diagonal torus. Using the Iwasawa decomposition, the quotient G_0/T can then be embedded in the compact flag manifold K/T, where $K = SU(p+q)$. Under these identifications, the restriction of the Poisson structure π to the orbit $\Psi_{\vec{\lambda}}$ induces a Poisson structure $\pi_{\vec{\lambda}}$ on an open subset of K/T. In Section 3.4, we show that this Poisson structure extends smoothly to the entire flag manifold in the $SU(1,1)$ and $SU(1,2)$ cases.

The fact that $\pi_{\vec{\lambda}}$ extends to the flag manifold does not guarantee the existence of the flow ϕ_t, but it is an important step in that direction. Moreover, the result is interesting in its own right. Foth and Lu proved in [FL06] that there exists a Poisson structure on K/T such that the G_0-orbits in K/T are Poisson submanifolds. The results presented here are in some sense extensions of the Foth-Lu result for $SU(1,1)$ and $SU(1,2)$.

Finally, in Section 3.6, we prove the Ginzburg-Weinstein theorem (restricted to the subset of G_0^* on which the dressing action is globally defined) for the $SU(1,1)$ case. We do so in two different ways: first, by constructing the vector field X in coordinates and proving that it satisfies the necessary properties, and second, by adapting the approach of [FR96] for the $SU(2)$ case. The second approach is interesting because the resulting Poisson diffeomorphism intertwines the Gelfand-Tsetlin coordinates on G_0^* and $\mathfrak{g}_0^*$.

Background material on Lie theory, cohomology, and induced representations is included in the appendices for easy reference. The last appendix gives a formula for the Lu-Weinstein Poisson structure on AN in the 3×3 case.

2. Background Material

2.1. Poisson Geometry

In this section, we collect some basic facts and definitions regarding Poisson manifolds.

A Poisson structure can be thought of as a symplectic structure for which a sort of degeneracy is allowed. Recall that a symplectic manifold is a smooth manifold M endowed with a closed, non-degenerate 2-form ω. Non-degeneracy here means that at each point $p \in M$ and for each nonzero $X \in T_pM$, there exists $Y \in T_pM$ such that $\omega(X, Y) \neq 0$. More concretely, degeneracy of ω is equivalent to degeneracy of the matrix representing ω with respect to any choice of local coordinates. In particular, the map

$$\omega^\flat : \chi^1(M) \to \Omega^1(M) : X \mapsto \omega(\cdot, X)$$

(that is, contract X with ω in the second slot) is one-to-one, and its inverse defines a non-degenerate bivector field π on M. The bivector field π can be thought of as a bilinear operator on $C^\infty(M) \times C^\infty(M)$ by

$$\{f, g\}_\pi := \pi(df, dg).$$

(The subscript π will be omitted when there is no potential for confusion.) The fact that ω is closed is equivalent to the Jacobi identity for $\{\cdot, \cdot\}$: For $f, g, h \in C^\infty(M)$,

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0.$$

Thus, a symplectic structure can be thought of as a bivector field rather than a 2-form. As noted above, a bivector field π defined by the inverse of a symplectic form is non-degenerate by construction. Dropping the non-degeneracy requirement for π leads to the following definition.

Definition 2.1.1. A *Poisson manifold* is a manifold M endowed with a bilinear, antisymmetric bracket $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ which satisfies the Jacobi identity and the following Leibniz rule:

$$\{\phi_1\phi_2, \phi_3\} = \phi_1\{\phi_2, \phi_3\} + \phi_2\{\phi_1, \phi_3\}.$$

Since $\{\cdot,\cdot\}$ is a derivation in each argument, and since the tangent space at each point $x \in P$ can be identified with the space of derivations at x, it follows that there exists a $\pi \in \Gamma(\wedge^2 TM)$ such that $\{f, g\} = \pi(df, dg)$. From this point of view, the Jacobi identity for the Poisson bracket is equivalent to $[\pi, \pi] = 0$, where $[\cdot,\cdot]$ is the Schouten bracket, a generalization of the Lie bracket to multivector fields.

If a Poisson bivector (or Poisson *structure*) π is non-degenerate, then the map $\pi^\# : \Omega^1(M) \to \chi^1(M) : \eta \mapsto \pi(\cdot, \eta)$ is invertible, and its inverse defines a symplectic form. Poisson structures can therefore be thought of as generalized symplectic structures.

Definition 2.1.2. A *Poisson map* from a Poisson manifold P to a Poisson manifold Q (a morphism in the category of Poisson manifolds) is a smooth map $\phi : P \to Q$ such that

$$\phi^*\{f, g\}_Q = \{\phi^* f, \phi^* g\}_P.$$

Viewing the Poisson structures on P and Q as bivector fields π_P and π_Q, respectively, this is equivalent to $\phi_*(\pi_P) = \pi_Q$.

For $f \in C^\infty(P)$, $X_f := \pi^\#(df)$ is called the *Hamiltonian vector field* of f. In symplectic geometry, X_f is defined by the equation $\omega(\cdot, X_f) = df$. Thus, if $\pi = \omega^{-1}$, the two definitions coincide. The image of $\pi^\#$ defines an involutive distribution on P. The integral submanifolds of this distribution are called the *symplectic leaves* of the Poisson manifold P. The restriction of π to a symplectic leaf is non-degenerate, hence the name *symplectic*. Therefore, every Poisson manifold foliates as a union of even-dimensional symplectic leaves. (See [Vai94], Theorem 2.12.)

Alternatively, the symplectic leaves of a Poisson manifold (M, π) can be described as equivalence classes of points under the following equivalence relation: for $p, q \in M$, $p \sim q$ if and only if they both lie on an integral curve π_t of a Hamiltonian vector field X_f for some function f. The equivalence classes of M modulo $\sim$ are the symplectic leaves of (M, π).

2.1.1. Poisson Structures on $\mathbb{R}^3$

Define a Poisson structure on $\mathbb{R}^3$ by

$$\pi = P\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + Q\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + R\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

The condition $[\pi, \pi] = 0$ is equivalent to

$$P(R_y - Q_z) + Q(P_z - R_x) + R(Q_x - P_y) = 0.$$

If $\alpha = Adx + Bdy + Cdz$, then

$$\pi^{\#}(\alpha) = (QC - RB)\frac{\partial}{\partial x} + (RA - PC)\frac{\partial}{\partial y} + (PB - QA)\frac{\partial}{\partial z}.$$

Example 2.1.3. Consider the 2-tensor $\pi = x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$. It is easy to check that $[\pi, \pi] = 0$. Using the formula above, $\pi^{\#}(dx) = (zx)\frac{\partial}{\partial y} - (yx)\frac{\partial}{\partial z}$, which, for each point (x, y, z), can be identified with the vector $(0, zx, -yx)$ (based at the point (x, y, z)). Since $(0, zx, -yz) \cdot (x, y, z) = 0$, $\pi^{\#}(dx)$ can be thought of as a vector tangent to the sphere of radius $\sqrt{x^2 + y^2 + z^2}$. Similar computations show that $\pi^{\#}(dy)$ and $\pi^{\#}(dz)$ can also be identified with vectors tangent to the sphere. Thus, at each point p, $\pi^{\#}(\alpha)_p$ is tangent to a sphere for any 1-form α, and it is clear that any sphere centered at the origin is an integral submanifold of the involutive distribution defined by the image of $\pi^{\#}$. That is to say, spheres centered at the origin are symplectic leaves of the Poisson manifold $(\mathbb{R}^3, \pi)$. Furthermore, along with the single point $\{(0, 0, 0)\}$, these are all of the symplectic leaves of $\mathbb{R}^3$ with this Poisson structure.

2.1.2. Casimir Functions

Let (M, π) be a Poisson manifold with bracket $\{f, g\} = \pi(df, dg)$. Suppose f is a function such that $\{f, g\} = 0$ for all $g \in C^\infty(M)$. Such a function is called a *Casimir* for π.

Proposition 2.1.4. *A function f on M is a Casimir for π if and only if f is constant on all symplectic leaves in M.*

Proof. First, suppose f is a Casimir. Then $\pi^{\#}(df) = 0$, or, equivalently,

$$-\langle X_g, df\rangle = 0 \tag{2.1.1}$$

for all $g \in C^\infty(M)$. Let $p \in M$. For any $Y \in T_pM$, choose $g \in C^\infty(M)$ such that $X_g(p) = Y$. Let ϕ_t be the flow of X_g near p. Then by 2.1.1,

$$0 = X_g(f)(p) = \frac{d}{dt}|_{t=0} f(\phi_t(p))$$

It follows from single-variable calculus that f is constant along $\phi_t(p)$. Thus, we have shown that f is constant along any path through p which is tangent to the symplectic leaf Ψ_p through p, which implies that f is constant on Ψ_p.

Now suppose that f is constant on all symplectic leaves. Then for any $p \in M$, $g \in C^\infty(M)$, since X_g is tangent to Ψ_p,

$$0 = X_g(f)(p) = \langle \pi^{\#}(dg), df\rangle = \pi(dg, df)(p) = \{g, f\}(p),$$

which implies that $\{f, g\} = 0$. □

If f is a Casimir for (M, π), the level sets of regular values of f are unions of symplectic leaves, and therefore Poisson submanifolds.

Example 2.1.5. Let $\pi = x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$, as in 2.1.3. In terms of brackets, this is equivalent to

$$\{x, y\} = z, \{x, z\} = -y, \{y, z\} = x.$$

Since the Poisson bracket is a derivation in each slot, for two functions f and g on $\mathbb{R}^3$,

$$\{f,g\} = \frac{\partial f}{\partial x}\{x,g\} + \frac{\partial f}{\partial y}\{y,g\} + \frac{\partial f}{\partial z}\{z,g\}.$$

Using this formula, one can easily check that $C(x,y,z) = x^2 + y^2 + z^2$ commutes with the coordinate functions x, y, and z. This implies, by the Leibniz rule, that C commutes with all polynomials in these three variables. Using the density of polynomials in the space of smooth functions on (any compact subset of) $\mathbb{R}^3$, it follows that C is a Casimir for π. The level sets of C are therefore unions of symplectic leaves; in this case, each level set consists of a single symplectic leaf. Thus, the symplectic leaves are the spheres centered at the origin and the point $\{\vec{0}\}$, which agrees with the computation in 2.1.3.

Example 2.1.6. Let $\pi = x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} - z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$. In terms of brackets, this is equivalent to

$$\{x,y\} = -z, \{x,z\} = -y, \{y,z\} = x.$$

A Casimir for this Poisson structure is $C = x^2 + y^2 - z^2$. The symplectic leaves for π are then the connected components of the hyperboloids $C = k$ for $k \neq 0$, the connected components of the cone $z^2 = x^2 + y^2$ *without* the origin, and the single point at the origin.

Definition 2.1.7. Given two Poisson manifolds P and Q, the *product Poisson struc-*

ture on $P \times Q$ is defined by

$$\{f, g\} = \{f(\cdot, y), g(\cdot, y)\}_P(x) + \{f(x, \cdot), g(x, \cdot)\}_Q(y).$$

From the point of view of tensors, the product tensor $\pi_{P\times Q}$ is the sum $\pi_P + \pi_Q$.

2.2. Poisson Lie Groups

In this section, we will state some general facts about Poisson structures on Lie groups. Unless otherwise stated, proofs can be found in [LW90] and in [Lu].

Definition 2.2.1. A Lie group G is called a *Poisson Lie group* if G is a Poisson manifold and the multiplication map $m : G \times G \to G$ is a Poisson map, where $G \times G$ is endowed with the product Poisson structure. In this case, the Poisson tensor π is said to be *multiplicative*.

Remark 2.2.2. Note that a bivector field π on a Lie group G is multiplicative if and only if

$$\pi(gh) = (l_g)_*\pi(h) + (r_h)_*\pi(g)$$

for all $g, h \in G$, where l_g denotes left translation by g and r_h denotes right translation by h.

Example 2.2.3. The zero bivector field defines a multiplicative Poisson structure on any Lie group.

2.2.1. Lie-Poisson Structures

An important class of multiplicative Poisson structures are the *Lie-Poisson* structures induced by Lie algebras. Given a Lie algebra $\mathfrak{g}$ with Lie bracket $[\cdot,\cdot]$, a Poisson structure can be defined on the dual space $\mathfrak{g}^*$ as follows. The elements of $\mathfrak{g}$ can be thought of as linear functions on $\mathfrak{g}^*$. Define the bracket $\{\cdot,\cdot\}$ on the space of linear functions on $\mathfrak{g}^*$ by

$$\{X, Y\} := [X, Y], \text{ for } X, Y \in \mathfrak{g}.$$

By the Leibniz property of the Poisson bracket (which we declare is satisfied by $\{\cdot,\cdot\}$), and the Stone-Weierstrass theorem, this definition can be extended to all of $C^\infty(\mathfrak{g}^*)$. The result is a linear Poisson structure. That is, representing the bivector field corresponding to $\{\cdot,\cdot\}$ in any linear basis as $\pi = \sum_{i<j} c_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, the coefficients c_{ij} are linear. Furthermore, if $\mathfrak{g}^*$ is considered as the abelian Lie group $(\mathbb{R}^n, +)$, it follows easily from Remark 2.2.2 that the bivector field π is multiplicative.

A fundamental property of Lie-Poisson structures is given by the following theorem, which is proved in [KS97].

Theorem 2.2.4. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. The symplectic leaves of the Lie-Poisson structure on $\mathfrak{g}^*$ are the connected components of the coadjoint orbits.*

Example 2.2.5. Let $\mathfrak{g}_0$ be the Lie algebra $\mathfrak{su}(1,1)$. Then

$$\mathfrak{g} = \left\{ \begin{pmatrix} \mathbf{i}z & \eta \\ \overline{\eta} & -\mathbf{i}z \end{pmatrix} \ : \ z \in \mathbb{R}, \eta \in \mathbb{C} \right\}.$$

The Lie bracket is just the usual bracket of matrices:

$$[A,B] = AB - BA.$$

A convenient basis for $\mathfrak{g}$ consists of the elements X,Y, and H, where

$$\begin{aligned} X &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ Y &= \begin{pmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{pmatrix}, \\ H &= \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}. \end{aligned}$$

The Poisson bracket on $\mathfrak{g}_0^*$ is generated by the Lie bracket relations

$$[X,Y] = -2H \quad [X,H] = -2Y \quad [Y,H] = 2X.$$

Using the non-degenerate form

$$\langle X,Y\rangle := \Im \mathrm{Trace}(XY)$$

on $\mathfrak{sl}(2,\mathbb{C})$, the dual of $\mathfrak{g}_0$ can be identified with the Lie algebra

$$\mathfrak{a}+\mathfrak{n} := \left\{ \begin{pmatrix} \frac{z}{2} & x+\mathbf{i}y \\ 0 & -\frac{z}{2} \end{pmatrix} \ : \ x,y,z \in \mathbb{R} \right\}.$$

In coordinates (x,y,z) on $\mathfrak{a}+\mathfrak{n}$, the Poisson structure induced by the Lie bracket on $\mathfrak{su}(1,1)$ is then

$$\pi_0 = -z\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}.$$

2.2.2. r-matrices

Multiplicative Poisson structures can also be constructed using so-called *r-matrices*. Given a Lie group G with Lie algebra $\mathfrak{g}$ and an element $\Lambda \in \mathfrak{g} \wedge \mathfrak{g}$, define a bivector field π by

$$\pi(g) = (r_g)_*\Lambda - (l_g)_*\Lambda.$$

Using the definition of the Schouten bracket, it can be shown (see [Lu]) that $[\Lambda, \Lambda] \in \wedge^3\mathfrak{g}$ is Ad_G-invariant if and only if the bivector field π is Poisson, i.e., $[\pi, \pi] = 0$. Furthermore, it is easy to check that $\pi(gh) = (l_g)_*\pi(h) + (r_h)_*\pi(g)$ is for all $g, h \in G$. Thus, by Remark 2.2.2, π is multiplicative. The element $\Lambda \in \mathfrak{g} \wedge \mathfrak{g}$ is called an *r-matrix*.

Example 2.2.6. Let G be the pseudo-unitary group

$$SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \; : \; |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

Consider the matrices

$$\begin{aligned} Z_\alpha &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ W_\alpha &= \begin{pmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{pmatrix} \end{aligned}$$

in $\mathfrak{g}$. Let $\Lambda = \frac{1}{2} Z_\alpha \wedge W_\alpha$. Since $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ is 1-dimensional, Λ is Ad_G-invariant. Therefore, the bivector field π defined by 2.2.2 is a multiplicative Poisson structure on G. Using the tensor notation as described in [KS97], one can compute the Poisson brackets determined by π of the functions α, β, $\overline{\alpha}$ and $\overline{\beta}$ on G. Representing elements of $\mathfrak{g} \otimes \mathfrak{g}$ as 4×4 matrices,

$$\Lambda = \frac{1}{2} Z_\alpha \wedge W_\alpha = \frac{1}{2}\left(\begin{pmatrix} 0 & 0 & 0 & \mathbf{i} \\ 0 & 0 & -\mathbf{i} & 0 \\ 0 & \mathbf{i} & 0 & 0 \\ -\mathbf{i} & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & \mathbf{i} \\ 0 & 0 & \mathbf{i} & 0 \\ 0 & -\mathbf{i} & 0 & 0 \\ -\mathbf{i} & 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathbf{i} & 0 \\ 0 & \mathbf{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly, for $g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$, we have

$$g \otimes g = \begin{pmatrix} \alpha & \alpha\beta & \beta\alpha & \beta^2 \\ \alpha\overline{\beta} & |\alpha|^2 & |\beta|^2 & \beta\overline{\alpha} \\ \overline{\beta}\alpha & |\beta|^2 & |\alpha|^2 & \overline{\alpha}\beta \\ \overline{\beta}^2 & \overline{\beta}\overline{\alpha} & \overline{\alpha}\overline{\beta} & \overline{\alpha}^2 \end{pmatrix}.$$

Then setting

$$\{g \otimes, g\} := \begin{pmatrix} \{\alpha, \alpha\} & \{\alpha, \beta\} & \{\beta, \alpha\} & \{\beta, \beta\} \\ \{\alpha, \overline{\beta}\} & \{\alpha, \overline{\alpha}\} & \{\beta, \overline{\beta}\} & \{\beta, \overline{\alpha}\} \\ \{\overline{\beta}, \alpha\} & \{\overline{\beta}, \beta\} & \{\overline{\alpha}, \alpha\} & \{\overline{\alpha}, \beta\} \\ \{\overline{\beta}, \overline{\beta}\} & \{\overline{\beta}, \overline{\alpha}\} & \{\overline{\alpha}, \overline{\beta}\} & \{\overline{\alpha}, \overline{\alpha}\} \end{pmatrix},$$

the Poisson brackets of α, β, $\overline{\alpha}$ and $\overline{\beta}$ are determined by the *second Russian formula* (see [KS97]):

$$\{g \otimes, g\} = \Lambda \cdot g \otimes g - g \otimes g \cdot \Lambda = \begin{pmatrix} 0 & \mathbf{i}\alpha\beta & \mathbf{i}\alpha\beta & 0 \\ -\mathbf{i}\alpha\overline{\beta} & -2\mathbf{i}|\beta|^2 & 0 & -\mathbf{i}\beta\overline{\alpha} \\ \mathbf{i}\alpha\overline{\beta} & 0 & 2\mathbf{i}|\beta|^2 & \mathbf{i}\beta\overline{\alpha} \\ 0 & -\mathbf{i}\overline{\beta}\overline{\alpha} & \mathbf{i}\overline{\beta}\overline{\alpha} & 0 \end{pmatrix}.$$

This gives the brackets

$$\begin{array}{ccc} \{\alpha, \overline{\alpha}\} = -2\mathbf{i}|\beta|^2 & \{\alpha, \beta\} = -\mathbf{i}\alpha\beta & \{\alpha, \overline{\beta}\} = -\mathbf{i}\alpha\overline{\beta} \\ \{\overline{\alpha}, \beta\} = \mathbf{i}\beta\overline{\alpha} & \{\overline{\alpha}, \overline{\beta}\} = \mathbf{i}\overline{\beta}\overline{\alpha} & \{\beta, \overline{\beta}\} = 0. \end{array}$$

2.2.3. Lie Bialgebras

The Lie algebra associated to a Lie group can be thought of as a linearized version of the group. The linearization of a Poisson Lie group is a *Lie bialgebra*, which we define as follows.

Definition 2.2.7. Let $\mathfrak{g}$ be a Lie algebra with dual $\mathfrak{g}^*$. Then the pair $(\mathfrak{g}, \mathfrak{g}^*)$ is a *Lie bialgebra* if there is given a Lie algebra structure on $\mathfrak{g}^*$ such that the map $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ dual to the Lie bracket map $\mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ is a 1-cocycle on $\mathfrak{g}$ relative to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$.

Let G be a Lie group endowed with a bivector field π satisfying $\pi_e = 0$. Define a map $d_e\pi : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ by

$$X \mapsto \left(L_{\overline{X}}\pi\right)_e,$$

where $\overline{X}$ is any vector field on G with $\overline{X}_e = X$. We will call this map *the derivative of* π *at* e. Its dual, called the *linearization of* π *at* e, is a map

$$[\cdot, \cdot]_\pi : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}$$

given by

$$[\xi, \eta]_\pi = d_e(\pi(\overline{\xi}, \overline{\eta})),$$

where $\xi, \eta \in \mathfrak{g}$ and $\overline{\xi}$ and $\overline{\eta}$ are any 1-forms on G with $\overline{\xi}_e = \xi$ and $\overline{\eta}_e = \eta$. Lu and Weinstein have proved the following theorem [LW90], which gives criteria in terms of the maps $[\cdot, \cdot]_\pi$ and $d_e\pi$ for the bivector field π to be Poisson and/or multiplicative.

Theorem 2.2.8.

1. *If* π *is multiplicative, then* $d_e\pi$ *is a 1-cocycle relative to the adjoint representation of* $\mathfrak{g}$ *on* $\mathfrak{g} \wedge \mathfrak{g}$. *Conversely, if* G *is connected and simply-connected, then for any*

1-cocycle $\epsilon : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ *relative to the adjoint representation of* $\mathfrak{g}$ *on* $\mathfrak{g} \wedge \mathfrak{g}$*, there is a unique multiplicative bivector field* π *such that* $\epsilon = d_e \pi$*.*

2. *If* π *is a Poisson tensor, then the bracket* $[\cdot, \cdot]_\pi$ *on* $\mathfrak{g}^*$ *induced by* π *satisfies the Jacobi identity, i.e., it is a Lie bracket on* $\mathfrak{g}^*$*. Moreover, when* G *is connected, a multiplicative bivector field* π *is a Poisson tensor if and only if its derivative at* e *defines a Lie bracket* $[\cdot, \cdot]_\pi$ *on* $\mathfrak{g}^*$*.*

Therefore, we can associate to any Poisson Lie group G a particular Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ using $\delta = [\cdot, \cdot]_\pi$. This Lie bialgebra is called the *tangent Lie bialgebra* of the Poisson Lie group (G, π).

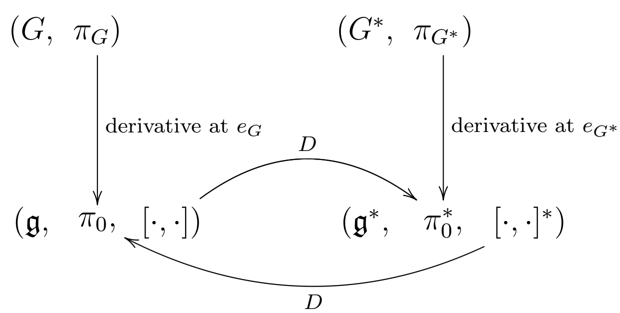
For connected and simply-connected Lie groups, there is a correspondence between tangent Lie bialgebras and multiplicative Poisson structures. More precisely,

Theorem 2.2.9. *If* (G, π) *is a Poisson Lie group, then the linearization of* π *at* e *defines a Lie algebra structure on* $\mathfrak{g}^*$ *such that* $(\mathfrak{g}, \mathfrak{g}^*)$ *form the tangent Lie bialgebra to* (G, π)*. Conversely, if* G *is connected and simply-connected, then for every Lie bialgebra* $(\mathfrak{g}, \mathfrak{g}^*)$ *there is a unique multiplicative Poisson structure* π *on* G *such that* $(\mathfrak{g}, \mathfrak{g}^*)$ *is the tangent Lie bialgebra to the Poisson Lie group* (G, π)*.*

Let G^* be the connected, simply-connected Lie group with Lie algebra $\mathfrak{g}^*$. Then by Theorem 2.2.9, there is a unique Poisson structure π' on G^* such that the tangent Lie bialgebra to the Poisson Lie group (G^*, π') is $(\mathfrak{g}^*, \mathfrak{g})$. The Poisson Lie group (G^*, π') is called the *dual* of (G, π).

The relationship between dual Poisson Lie groups can be thought of in the following way: the dual of (G, π_G) is a Poisson Lie group (G, π_{G^*}) such that the derivative of π_{G^*} at the identity in G^* is the linear structure π_0^* induced by the Lie bracket on $\mathfrak{g}$ and the derivative of π_G at the identity in G is the linear structure π_0 on $\mathfrak{g}$ induced by the Lie bracket on $\mathfrak{g}^*$. Figure 2.2.3 illustrates these relationships.

FIGURE 2.1. Duality of Poisson Lie Groups



Example 2.2.10. If G is a Lie group endowed with the zero Poisson structure (which is obviously multiplicative), then G^* can be identified with $(\mathfrak{g}^*, \pi_0)$, where π_0 is the Lie-Poisson structure. This can be seen by examining Figure 2.2.3.

We will need two more definitions.

Definition 2.2.11. A triple of Lie algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a *double Lie algebra* if $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Lie subalgebras of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as vector spaces.

Example 2.2.12. Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{g}_+ = \mathfrak{su}(n)$, and $\mathfrak{g}_- = \mathfrak{a} + \mathfrak{n}$. Then the triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a double Lie algebra.

Lu has proved in [Lu], Theorem 2.22, that given a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, it is always possible to define a Lie bracket on $\mathfrak{g} \oplus \mathfrak{g}^*$ so that $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ form a double Lie algebra. The vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ endowed with this bracket is often denoted by $\mathfrak{g} \bowtie \mathfrak{g}^*$.

There is a corresponding definition at the group level.

Definition 2.2.13. A triple of Lie groups (G, G_+, G_-) is a *double Lie group* if G_+ and G_- are both closed Lie subgroups of G such that the map $\alpha : G_+ \times G_- \to G$ defined by $(g_+, g_-) \mapsto g_+ g_-$ is a diffeomorphism.

Theorem 3.7 in [LW90] gives conditions under which a double Lie algebra can be used to produce a double Lie group. More specifically,

Theorem 2.2.14. *Suppose* $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ *is a double Lie algebra and* G *is the connected, simply connected Lie group with Lie algebra* $\mathfrak{g}$*. Let* G_+ *and* G_- *be the connected Lie subgroups of* G *with Lie algebras* $\mathfrak{g}_+$ *and* $\mathfrak{g}_-$*, respectively. If* G_+ *is compact and* G_- *is closed in* G*, then* (G, G_+, G_-) *is a double Lie group.*

It is sometimes useful to view Lie bialgebras as *Manin triples*, which are defined as follows.

Definition 2.2.15. A *Manin triple* is a triple of Lie algebras $(\mathfrak{g}_-, \mathfrak{g}_+, \mathfrak{g})$ together with an invariant, nondegenerate scalar product $\langle \cdot, \cdot \rangle$ such that $\mathfrak{g}_-$ and $\mathfrak{g}_+$ are isotropic with respect to $\langle \cdot, \cdot \rangle$, i.e, $\langle \mathfrak{g}_-, \mathfrak{g}_- \rangle = \langle \mathfrak{g}_+, \mathfrak{g}_+ \rangle = 0$.

As explained in [Lu], given a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ there is a natural scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g} \bowtie \mathfrak{g}^*$ such that $(\mathfrak{g}, \mathfrak{g}^*, \mathfrak{g} \bowtie \mathfrak{g}^*)$ together with $\langle \cdot, \cdot \rangle$ form a Manin triple. Conversely, given a Manin triple $(\mathfrak{g}_-, \mathfrak{g}_+, \mathfrak{g})$ with scalar product $\langle \cdot, \cdot \rangle$, $\mathfrak{g}_-$ can be identified with $\mathfrak{g}^+$ via $\langle \cdot, \cdot \rangle$. It follows from [Lu], Theorem 2.22, that $(\mathfrak{g}_+, \mathfrak{g}_+^*)$ is a Lie bialgebra. Thus, there is a one-to-one correspondence between Lie bialgebras and Manin triples.

Given a Lie bialgebra (g, g^*), one can give explicit formulas for the induced Poisson tensors on the corresponding Lie groups G and G^* (see [LW90], p521-522). Let $\mathfrak{d}$ be the double Lie algebra of (g, g^*), and let D be the connected, simply-connected Lie group with Lie algebra $\mathfrak{d}$. Let

$$\begin{aligned} p_{\mathfrak{g}} &: \mathfrak{d} \to \mathfrak{g} \text{ and} \\ p_{\mathfrak{g}^*} &: \mathfrak{d} \to \mathfrak{g}^* \end{aligned}$$

denote the natural projections, and let $G \to D : g \mapsto \overline{g}$ and $G^* \to D : u \mapsto \overline{u}$ be the Lie group homomorphisms obtained by respectively integrating the inclusion maps $\mathfrak{g} \hookrightarrow \mathfrak{d}$ and $\mathfrak{g}^* \hookrightarrow \mathfrak{d}$. Let Ad denote the adjoint action of D on its Lie algebra $\mathfrak{d}$. Then for $\alpha, \beta \in \mathfrak{g}^*$, $g \in G$, π_G is defined by

$$(r_{g^{-1}})_* \pi_G(\alpha, \beta) = \langle p_{\mathfrak{g}^*}(\mathrm{Ad}_{\overline{g}^{-1}}(\alpha), p_{\mathfrak{g}}(\mathrm{Ad}_{\overline{g}^{-1}}\beta) \rangle.$$

Similarly, for $X, Y \in \mathfrak{g}$, $u \in G^*$, π_{G^*} is defined by

$$(r_{u^{-1}})_* \pi_{G^*}(X, Y) = \langle p_{\mathfrak{g}}(\mathrm{Ad}_{\overline{u}^{-1}}(X), p_{\mathfrak{g}^*}(\mathrm{Ad}_{\overline{u}^{-1}}Y) \rangle.$$

2.2.4. The Lu-Weinstein Poisson Tensor

Lu and Weinstein showed in [LW90] that every connected, compact, semisimple Lie group admits a nontrivial, multiplicative Poisson tensor. If $\mathfrak{k}$ is a compact, semisimple Lie algebra, $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$ is its complexification, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is an Iwasawa decomposition, then $\mathfrak{k}^*$ can be identified with $\mathfrak{a} + \mathfrak{n}$ via the imaginary part $\Im\langle\cdot,\cdot\rangle$ of the Killing form. Moreover, $(\mathfrak{k}, \mathfrak{a} + \mathfrak{n}, \mathfrak{g})$ together with $\Im\langle\cdot,\cdot\rangle$ form a Manin triple. Therefore, if K and AN are the connected, simply-connected Lie groups associated to $\mathfrak{k}$ and $\mathfrak{a} + \mathfrak{n}$, there exist Poisson structures π_K and π_{AN} on K and AN, respectively such that (K, π_K), and (AN, π_{AN}) are dual Poisson Lie groups. The double Lie group in this case is $G = K^{\mathbb{C}} = KAN$. The Poisson structures π_K and π_{AN} are called the *Lu-Weinstein* Poisson structures (though Lu and Weinstein used the term *Bruhat-Poisson* structure for π_K).

According to [FL], this tensor can also be constructed in the following way. Since $\mathfrak{k}$ is a compact form of $\mathfrak{g}$, there is a corresponding Cartan involution, θ. Choose a Cartan subalgebra, $\mathfrak{t}$, of $\mathfrak{k}$ and let $\mathfrak{h} = \mathfrak{t}^{\mathbb{C}} \subset \mathfrak{g}$ be the complexification of $\mathfrak{t}$. Construct a root system, R, for $\mathfrak{h}$ via the usual eigenspace decomposition. Choose a set of positive roots, $R^+ \subset R$, and use the positive roots, $\alpha \in R^+$ to construct a Chevalley basis for $\mathfrak{h}$. This basis consists of elements E_α and $E_{-\alpha}$, $H_\alpha = [E_\alpha, E_{-\alpha}]$ for $\alpha \in R^+$. The Cartan subalgebra $\mathfrak{t}$ and set of positive roots, R^+, must satisfy $\theta(E_\alpha) = -E_{-\alpha}$ for all $\alpha \in R^+$. This guarantees that θ also fixes $X_\alpha = E_\alpha - E_{-\alpha}$, $Y_\alpha = i(E_\alpha + E_{-\alpha})$, and

iH_α for all $\alpha \in R^+$, i.e., that X_α, Y_α, and iH_α are in $\mathfrak{k}$. Then $\Lambda = \sum_{\alpha\in R^+} X_\alpha \wedge Y_\alpha$ generates a Poisson tensor for K by $\pi = r_g\Lambda - l_g\Lambda$.

Remark 2.2.16. If $K = SU(n)$, $K^* \cong AN$ is the space of upper triangular matrices in $GL(n,\mathbb{C})$ with real, positive diagonal. Although AN admits global coordinates in this case, computing an expression for π_{AN} in terms of these coordinates is nontrivial. The formula for the 2×2 case, using coordinates

$$\begin{pmatrix} e^{\frac{z}{2}} & x + \mathbf{i}y \\ 0 & e^{-\frac{z}{2}} \end{pmatrix} \leftrightarrow (x, y, z),$$

is

$$\pi_{AN} = (\sinh z)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.$$

The formula for π_{AN} in the 3×3 case is given in Appendix D.

2.2.5. The $SU(p,q)$ Case

The example of primary interest for this thesis involves the noncompact Lie algebra $\mathfrak{su}(p,q)$. Let $\mathfrak{a}+\mathfrak{n}$ denote the Lie subalgebra of $\mathfrak{sl}(n,\mathbb{C})$ consisting of upper-triangular matrices with real diagonal.

Lemma 2.2.17. *The subalgebras $\mathfrak{su}(p,q)$ and $\mathfrak{a}+\mathfrak{n}$ in $\mathfrak{sl}(n,\mathbb{C})$ are isotropic with respect to the imaginary part of the Killing form on $\mathfrak{sl}(n,\mathbb{C})$.*

Proof. The Killing form on $\mathfrak{sl}(n, \mathbb{C})$ is (a nonzero multiple of) the trace form. We can therefore take the imaginary part of the Killing form to be

$$\langle A, B\rangle := \Im(\mathrm{tr}(AB)). \tag{2.2.1}$$

Let $A, B \in \mathfrak{su}(p, q)$. Using (A.3), we have

$$\begin{aligned}
\mathrm{tr}(AB) &= \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}B_{ji} \\
&= \sum_{i\leq j}(A_{ij}B_{ji} + A_{ji}B_{ij}) \\
&= \begin{cases} -A_{ij}\overline{B_{ij}} - \overline{A_{ij}}B_{ij} & \text{if } i, j \leq p \text{ or } i, j > p \\ A_{ij}\overline{B_{ij}} + \overline{A_{ij}}B_{ij} & \text{otherwise.} \end{cases}
\end{aligned}$$

Since $A_{ij}\overline{B_{ij}} + \overline{A_{ij}}B_{ij}$ is real for any $A_{ij}, B_{ij} \in \mathbb{C}$, it follows that $\langle A, B\rangle = 0$.

Similarly, if $A, B \in \mathfrak{a} + \mathfrak{n}$, then

$$\mathrm{tr}(AB) = \sum_{i=1}^{n} A_{ii}B_{jj},$$

where A_{ii} and B_{ii} are real for all i. Thus, $\langle A, B\rangle = 0$ as desired. □

It follows from Lemma 2.2.17 that $(\mathfrak{su}(p, q), \mathfrak{a} + \mathfrak{n}, \mathfrak{sl}(n, \mathbb{C}))$ together with the form (2.2.1) is a Manin triple. By Theorem 2.2.9, there are Poisson structures π_{pq} on $SU(p, q)$ and π_{AN} on AN such that $(\mathfrak{su}(p, q), \mathfrak{a} + \mathfrak{n})$ is the tangent Lie bialgebra of $(SU(p, q), \pi_{pq})$ and $(\mathfrak{a} + \mathfrak{n}, \mathfrak{su}(p, q))$ is the tangent Lie bialgebra of (AN, π_{AN}). In this case, we will refer to π_{pq} and and π_{AN} as the Lu-Weinstein Poisson structures on their respective groups (with respect to the Manin triple $(\mathfrak{su}(p, q), \mathfrak{a} + \mathfrak{n}, \mathfrak{sl}(n, \mathbb{C}))$).

2.2.6. Poisson Homogeneous Spaces

Let M be a symplectic manifold with symplectic form ω, and let G be a Lie group acting on M. Recall that the action of G on M is called *symplectic* if it preserves ω, that is, if $g^*(\omega) = \omega$ for all $g \in G$. Any $X \in \mathfrak{g}$ induces a vector field, X_M on M by

$$(X_M)_m := \frac{d}{dt}\bigg|_{t=0} \exp(-tX) \cdot m$$

for any $m \in M$. (Here, $\cdot$ denotes the action of G on M.) Then the action of G on M preserves ω if and only if $\mathcal{L}_{X_M}\omega = 0$ for every $X \in \mathfrak{g}$. By Cartan's formula, $\mathcal{L}_{X_M}\omega = d(\iota_{X_M}\omega) + \iota_{X_M}(d\omega)$. Since ω is closed in this case, we have $\mathcal{L}_{X_M}\omega = d(\iota_{X_M}\omega)$. Therefore, the action of G on M is symplectic if and only if $\mathcal{L}_{X_M}\omega = 0$ if and only if $\iota_{X_M}\omega$ is closed for all $X \in \mathfrak{g}$.

Now suppose G is a Poisson Lie group with Poisson structure π_G and that M is a Poisson manifold with Poisson structure π_M.

Definition 2.2.18. An action $\mathcal{A} : G \times M \to M$ is *Poisson* if $\mathcal{A}$ is a Poisson map when $G \times M$ is endowed with the product Poisson structure, i.e., if $\mathcal{A}_*(\pi_G + \pi_M) = \pi_M$.

Note that if $\pi_G = 0$, then the action of G on M preserves π_M. If $\pi_G \neq 0$, however, then π_M may not be preserved by the G action.

Example 2.2.19. Let $(G, \pi_G) = (\mathbb{R}^3, x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z})$, and let $(M, \pi_M) = (\mathbb{R}^3, x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$. Suppose G acts on M by

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) := (x_2 + x_1, y_2 + x_1, z_2 + x_1).$$

The Jacobian for the action map $\mathcal{A} : G \times M \to M$ is given by

$$D\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that $\mathcal{A}_*(\pi_G + \pi_M) = 0 + \pi_M = \pi_M$. Therefore, this action is Poisson.

Definition 2.2.20. If (G, π_G) is a Poisson Lie group, a *Poisson homogeneous space* for (G, π) is a Poisson manifold (M, π_M) together with a transitive, Poisson action of (G, π).

2.2.7. Dressing Actions

Let G be a compact, connected Poisson Lie group, and let G^* be the dual. Then $\mathfrak{g}^*$ acts on G as follows: For $\xi \in \mathfrak{g}^*$, let ξ^l and ξ^r be respectively the left and right-invariant 1-forms on G with value ξ at e. Define maps

$$\lambda : \mathfrak{g}^* \to \chi(G) : \xi \mapsto \pi_G^{\#}(\xi^l),$$

and

$$\rho : \mathfrak{g}^* \to \chi(G) : \xi \mapsto -\pi_G^{\#}(\xi^r).$$

Then λ is a Lie algebra anti-homomorphism and ρ is a Lie algebra homomorphism.

For each $\xi \in \mathfrak{g}$, $\lambda(\xi)$ and $\rho(\xi)$ are called left and right *dressing vector fields*, respectively. Integrating λ gives rise to a local action of G^* on G as follows: Let $p \in G^*$ with $p = e^{t_0 X}$ for some $X \in \mathfrak{g}^*$, $t_0 \in \mathbb{R}$. For each $q \in G$, there is a 1-parameter group of diffeomorphisms, ϕ_t, generated by $\lambda(X)$ and defined on a neighborhood V of q. Define the action by

$$p \cdot q = \phi_{t_0}(q).$$

This (left) action is defined for p in some neighborhood U of the identity in G^* and for all $q \in G$. If exp maps $\mathfrak{g}^*$ onto G^* (if G^* is compact, for example), and if the dressing vector fields $\lambda(\xi)$ are complete, then we have a global action. Integrating ρ gives rise to a (right) action of G^* on G.

Similarly, if we define maps

$$\lambda' : \mathfrak{g} \to \chi(G^*) : \xi \mapsto \pi_{G^*}^{\#}(\xi^l),$$

and

$$\rho' : \mathfrak{g} \to \chi(G^*) : \xi \mapsto -\pi_{G^*}^{\#}(\xi^r),$$

integrating λ' and ρ' gives rise to left and right actions, respectively, of G on G^*. These actions of G and G^* on each other are called *dressing actions*.

In the above description of dressing actions, we started at the Lie algebra level (the infinitesimal version) and then integrated to get an action at the Lie group level. It is also possible to define the action at the Lie group level in the first place. Let d be the double Lie algebra of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, and let D be the corresponding connected, simply-connected Lie group. If the dressing vector fields are all complete, then D can be identified (as a manifold) with the product $G \times G^*$, so that the map

$$G \times G^* \to D : (g, h) \mapsto gh,$$

is a diffeomorphism onto. If this is the case, then D decomposes (uniquely) as

$$D = GG^* = G^*G.$$

Remark 2.2.21. According to [Lu], Proposition 2.43, if G is compact, then $D \cong GG^* \cong G \times G^*$.

For $k \in G$ and $l \in G^*$, we have $kl = l'k'$ for some $l' \in G^*$ and some $k' \in G$. Define the action of k on l (denoted by l^k) by $l^k = l'$. Similarly, define the action of l on k by $k^l = k'$. The the two actions are related by the equation

$$kl = l^k k^l, \tag{2.2.2}$$

A short computation shows that the action of G on G^* is a left action, and the action of G^* on G is a right action. Similarly, $lk = k'l'$ for some $k' \in G$ and some $l' \in G^*$. We can therefore define a right action of G on G^* and a left action of G^* on G by the same formulas. These actions are related by

$$lk = k^l l^k, \tag{2.2.3}$$

which is equivalent to (2.2.2).

Proposition 2.2.22. *The left and right dressing actions of G on G^* are Poisson.*

Example 2.2.23. In the case $K = SU(2)$, the dressing action of K on AN can be computed explicitly using the formulas from Example A.4 in Appendix A.

2.3. The Ginzburg-Weinstein Theorem

In this section, we give an outline of the Ginzburg-Weinstein argument and indicate how compactness is used. Let K be a compact, semisimple Lie group with Lie algebra $\mathfrak{k}$, and let π_K be the Lu-Weinstein Poisson structure. Let (K^*, π) be the dual Poisson Lie group to (K, π_K), and let $\mathfrak{k}^*$ be the dual of $\mathfrak{k}$ with Lie-Poisson structure π_0. Recall that in this case, $K^* = AN$, where $K(AN) = G$ is an Iwasawa decomposition for $G = K^{\mathbb{C}}$. The main steps in proving that $(\mathfrak{k}^*, \pi_0)$ and (K^*, π) are isomorphic as Poisson manifolds are as follows:

1. Construct a K-equivariant diffeomorphism $E : K^* \to \mathfrak{k}^*$, so that π and π_0 may both be regarded as bivector fields on $\mathfrak{k}^*$.

2. Construct a family of Poisson structures π_t with $\pi_1 = \pi$ and $\lim_{t\to 0} \pi_t = \pi_0$.

3. Prove the existence of a vector field X satisfying

   a. $[X, \pi] = \dot{\pi}$, where $\dot{\pi} := \frac{d}{dt}\,|_{t=1}\pi_t\,$,

   b. X has the zero linearization at the origin, and

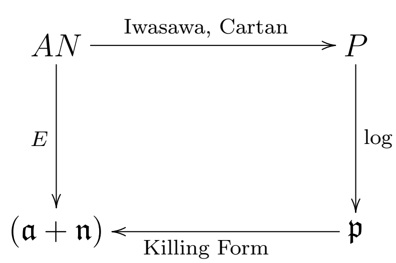
   c. X is tangent to the coadjoint orbits in $\mathfrak{k}^*$.

4. Construct a family of vector fields X_t, based on X, such that the flow ϕ_t of X_t pushes π_0 forward to π_t for $t \in [0, 1]$. Then $E^{-1} \circ \phi_1$ is the desired Poisson diffeomorphism from $(\mathfrak{k}^*, \pi_0)$ to (K^*, π).

Each step is discussed in turn below.

(1) Using the Iwasawa decomposition $G = K(AN)$ and the Cartan decomposition $G = KP$, $K^* = AN$ is naturally identified with P. At the Lie algebra level, $\mathfrak{a} + \mathfrak{n}$ is naturally identified with $\mathfrak{p}$, the (-1)-eigenspace for the Cartan involution, via the imaginary part $\langle \cdot, \cdot \rangle$ of the Killing form. Finally, the exponential map is a diffeomorphism from $\mathfrak{p}$ onto P. Combining all of these maps, we obtain E (see Figure 2.2).

FIGURE 2.2. Diagram of the Map E (Compact Case)



A somewhat more concrete description of the identification $AN \cong P$ is given in [FR96]. Let $\dagger : G \to G$ be the involution fixing P. Then $AN \cong P$ via

$$C \leftrightarrow C^\dagger C. \tag{2.3.1}$$

The corresponding map at the Lie algebra level is

$$\mathfrak{a} + \mathfrak{n} \leftrightarrow \mathfrak{p} : X \leftrightarrow X^\dagger + X.$$

Note that for $K = SU(n)$, P is the set of Hermitian matrices in $G = GL(n, \mathbb{C})$, and $\dagger$ is the conjugate transpose operation.

(2) Regarding π as a bivector field on $\mathfrak{k}^*$, define π_t by $\pi_t(\mathbf{x}) := \frac{\pi(t\mathbf{x})}{t}$, where the right-hand side is identified with an element of $\wedge^2 (T_{\mathbf{x}}\mathfrak{k}^*)$ by translation. Note that this definition makes implicit use of the fact that $\mathfrak{k}^*$ is a vector space. All of the Poisson structures π_t vanish only at the origin in $\mathfrak{k}^*$. Moreover, all of the structures π_t induce the same symplectic foliation as that of π_0.

(3) The majority of [GW92] is devoted to this step.

(3a) Finding a vector field Y such that $[Y, \pi] = \dot{\pi}$ amounts to solving a system of partial differential equations. Even in the case $K = SU(2)$, solving this system is not trivial. Ginzburg and Weinstein prove the existence of such an X as follows. They note, first, that $[\pi, \dot{\pi}] = 0$. That is, $\dot{\pi}$ is a Poisson cocycle with respect to π. Then in the language of Poisson cohomology, the goal is to show that $\dot{\pi}$ is a Poisson coboundary. Ginzburg and Weinstein accomplish this by proving that

$$H^2_\pi(\mathfrak{g}^*) = 0, \tag{2.3.2}$$

i.e., that *every* bivector field on $\mathfrak{g}^*$ that is a Poisson cocycle is a Poisson coboundary (a much stronger result).

(3b) Given Y such that $[Y, \pi] = \dot{\pi}$, Ginzburg and Weinstein construct another vector field ξ with the same linearization at the origin which also satisfies $[\xi, \pi] = \dot{\pi}$. Then for $X := Y - \xi$, $[X, \pi] = \dot{\pi}$ and X has the zero linearization at the origin. The argument uses the fact that every tensor π_t vanishes only at the origin, and the fact that all derivations of a semisimple Lie algebra are inner. In particular, compactness is not used here.

(3c) For any X such that $[X, \pi] = \dot{\pi}$, it can be shown that X is tangent to the coadjoint orbits. The argument in [GW92] makes use of the following results.

Proposition 2.3.1 ([GW92], Proposition 4.1)**.** *Suppose T is a torus acting on a compact manifold W. If ω_0 and ω_1 are T-invariant symplectic structures on W that admit moment maps J_0 and J_1, respectively, such that $J_0(x) = J_1(x)$ for every $x \in W^T$, then the cohomology classes of the symplectic structures coincide: $[\omega_0] = [\omega_1]$.*

Lemma 2.3.2 ([GW92], Lemma 5.1)**.** *On every coadjoint orbit $\theta \subset \mathfrak{k}^*$, all of the symplectic structures ω_t corresponding to the Poisson structures π_t lie in the same cohomology class.*

The proof of Lemma 2.3.2 makes use of Proposition 2.3.1, which applies because the coadjoint orbits are compact.

(4) With X as in (3), define a family of vector fields X_t by

$$X_t(\mathbf{x}) := \frac{X(t\mathbf{x})}{t^2},$$

where, as in the definition of π_t, the right-hand side is identified with a vector in $T_{\mathbf{x}}\mathfrak{k}^*$ by translation. Since X has the zero linearization at the origin, X_t is smooth for $t \in [0, 1]$. It follows from (3c) that all X_t are tangent to the coadjoint orbits. It then follows from the fact that the coadjoint orbits are compact that each X_t can be integrated to a family of flows ϕ_t. A straightforward computation based on (3a) shows that $\mathcal{L}_{X_t}\pi_t = [X_t, \pi_t] = \dot{\pi}_t$. This implies that $(\phi_t)_* \pi_0 = \pi_t$.

We now summarize the ways in which compactness is used in the argument above. In (1), compactness of K is required to obtain Iwasawa and Cartan decompositions. For the $SU(p,q)$ case, we do not have global Iwasawa and Cartan decompositions. However, an analogue of (2.3.1) can be used on a suitable open subset of AN (see Section 3.2).

The proof in (3a) of (2.3.2) given in [GW92] also uses compactness. An alternative proof is given in Section 3.3 as well as a proof of a slightly weaker result for the $SU(p,q)$ case.

Showing that X is tangent to the coadjoint orbits in (3c) relies on compactness of the coadjoint orbits in $\mathfrak{k}^*$. Compactness of the coadjoint orbits is used again in (4) in order to claim that the family of vector fields X_t has a global flow ϕ_t. In Section 3.4, we show that, for the $SU(1,1)$ and $SU(1,2)$ cases, each generic coadjoint orbit can be compactified in the sense that it can be embedded in a compact flag manifold $T\backslash K$ in such a way that the Poisson structure π extends smoothly to all of $T\backslash K$.

3. The $SU(p,q)$ Case

3.1. The Dressing Action

As noted in Section 2.2.7, for a compact Poisson Lie group K with dual group K^*, the double D can be identified as a manifold with $K \times K^*$, and the dressing vector fields give rise to global actions of K on K^*. For example, if $K = SU(n)$, the left dressing action of K on $K^* \cong AN$ is given by

$$(K \times K^*) \to K^* : (k,l) \mapsto p^l_{AN}(kl), \tag{3.1.1}$$

and the right dressing action is given by

$$(K^* \times K) \to K^* : (l,k) \mapsto p^r_{AN}(lk), \tag{3.1.2}$$

where the projections p^l_{AN} and p^r_{AN} are as in Appendix A. If K is replaced by the noncompact group $G_0 = SU(p,q)$, the dressing vector fields are not complete, and G_0 does not act globally on $G_0^* \cong AN$. However, we will show that the dressing action of G_0 is defined on the *admissible* elements of $A \subset AN$, which we now define.

Definition 3.1.1. An element $\vec{\lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_q) \in \mathfrak{a}$ is called *admissible* if $\lambda_i \geq \mu_j$ for all $1 \leq i \leq p$, $1 \leq j \leq q$. The set of admissible elements in $\mathfrak{a}$ will be denoted by $\mathfrak{a}_{adm}$. Admissible elements in A are defined in the same way, and the set of admissible elements in A will be denoted by A_{adm}.

Remark 3.1.2. The maximal compact subgroup $K_0 = S(U(p) \times U(q))$ in G_0 can be used to arrange $\lambda_1, \dots, \lambda_p$ and $\mu_1, \dots, \mu_q$ in any order. Therefore, we may assume that $\lambda_1 \geq \cdots \geq \lambda_p$ and $\mu_1 \geq \cdots \geq \mu_q$, and the admissibility condition becomes simply $\lambda_p > \mu_1$.

Proposition 3.1.3. *For any* $\exp(\vec{\lambda}) \in A_{adm}$*, the right dressing action, as given by (3.1.2) (with* K *replaced by* G_0*), is defined on* $\exp(\vec{\lambda})$ *for all* $g \in G_0$*.*

Proof. The following proof is due to Philip Foth. Suppose $\vec{\lambda} = (\lambda_1, \dots, \lambda_p, \mu_1, \dots, \mu_q)$ is admissible. Recall that G_0 is defined to be the subgroup of $GL(n, \mathbb{C})$ which preserves the following sesquilinear pairing on $\mathbb{C}^n$:

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{i=1}^{p} x_i \overline{y}_i - \sum_{i=p+1}^{n} x_i \overline{y}_i,$$

where $n = p + q$. Denote the corresponding pseudo-norm by $||\mathbf{x}||$. A vector $\mathbf{x} \in \mathbb{C}^n$ will be called *timelike* if its norm is real and positive and *spacelike* if is imaginary.

Let $g \in G_0$, and denote the columns of g by $w_1, \dots, w_n$. By definition, the columns of g are pseudo-orthonormal with respect to $\langle \cdot, \cdot \rangle$. The first p columns are timelike (with norm 1) and the last q columns are spacelike (with norm $\mathbf{i}$). Since the torus $T \subset G_0$ acts trivially on $\exp(\vec{\lambda})$ we will consider the case in which $g \notin T$.

Denote the columns of $\exp(\vec{\lambda})$ by $\mathbf{v}_1, \dots, \mathbf{v}_n$. The matrix $\exp(\vec{\lambda})$ is obtained from g by multiplying the i-th row by e^{λ_i} if $1 \leq i \leq p$, and by e^{μ_i} if $i > p$. Note that since the λ_is may be assumed to be in non-increasing order (see Remark 3.1.2), the first p columns of $\exp(\vec{\lambda})g$ will remain timelike.

Recall that for the compact group $K = SU(n)$, the decomposition $G = KAN$ is obtained by applying the Gram-Schmidt procedure to the columns of $g \in G$ and recording the operations in terms of multiplication by elementary matrices. If there is a decomposition $\exp(\vec{\lambda})g = sb$, then it can be obtained by the analogue of the Gram-Schmidt process with respect to the pseudo-metric $\langle \cdot, \cdot \rangle$. Proving the existence of such a decomposition amounts to showing that if we follow the Gram-Schmidt process, the first p columns of s will be timelike, the last q will be spacelike, and the diagonal entries of b will be positive real numbers. Note that it is sufficient to prove the first two assertions, since the diagonal entries of b represent rescalings.

Denote the columns of s by $\mathbf{u}_1, \ldots, \mathbf{u}_n$. We begin by showing that $\mathbf{u}_1, \ldots \mathbf{u}_p$ are timelike and that $r_1 \ldots r_p$ are real and positive.

Denote the diagonal entries of b by $(r_1, \ldots, r_n)$ and the off-diagonal entries by m_{ij}, where $m_{ij} = 0$ for $i > j$. Consider the first step in the Gram-Schmidt process, which involves simply normalizing $\mathbf{v}_1$. That is, we set $r_1 := ||\mathbf{v}_1||$ and $\mathbf{u}_1 := \frac{\mathbf{v}_1}{r_1}$. Since $\mathbf{v}_1$ is timelike, r_1 is real and positive, and $\mathbf{u}_1$ is also timelike.

For the second step (assuming that $p > 1$), we have

$$\mathbf{v}_2 = m_{12}\mathbf{u}_1 + r_2\mathbf{u}_2,$$

where $m_{12} = \langle \mathbf{v}_2, \mathbf{u}_1 \rangle$ and $r_2 = ||\mathbf{v}_2 - m_{12}\mathbf{u}_2||$. Therefore, we need to show that $\mathbf{v}_2 - m_{12}\mathbf{u}_1$ is timelike. Since

$$||\mathbf{v}_2 - m_{12}\vec{u}_1||^2 = ||\mathbf{v}_2||^2 - |\langle \mathbf{v}_2, \mathbf{u}_1 \rangle|^2,$$

and $||\mathbf{u}_1|| = 1$, this is equivalent to showing that

$$||\mathbf{v}_2||^2 \cdot ||\mathbf{u}_1||^2 > |\langle \mathbf{v}_2, \mathbf{u}_1 \rangle|^2,$$

or, after multiplying both sides by r_1^2, that

$$||\mathbf{v}_2||^2 \cdot ||\mathbf{v}_1||^2 > |\langle \mathbf{v}_2, \mathbf{v}_1 \rangle|^2. \tag{3.1.3}$$

Denote the coordinates of $\vec{w}_1$ by $(a_1, \ldots, a_p, b_1, \ldots, b_q)$ and the coordinates of $\mathbf{w}_2$ by $(c_1, \ldots, c_p, d_1, \ldots, d_q)$. Then

$$\sum_{i=1}^{p} |a_i|^2 - \sum_{j=1}^{q} |b_j|^2 = \sum_{i=1}^{p} |c_i|^2 - \sum_{j=1}^{q} |d_j|^2 = 1 \tag{3.1.4}$$

and

$$\sum_{i=1}^{p} a_i \bar{c}_i - \sum_{i=1}^{p} b_i \bar{d}_i = 0. \tag{3.1.5}$$

The coordinates of the vector $\mathbf{v}_1$ are given by

$$e^{\lambda_1} a_1, \ldots, e^{\lambda_p} a_p, e^{\mu_1} b_1, \ldots, e^{\mu_q} b_q),$$

and the coordinates of $\mathbf{v}_2$ by

$$e^{\lambda_1} c_1, \ldots, e^{\lambda_p} c_p, e^{\mu_1} d_1, \ldots, e^{\mu_q} d_q).$$

Then using (3.1.4), the right-hand side of Equation 3.1.3 becomes

$$\begin{aligned} & \left(\sum_{i=1}^{p} e^{2\lambda_i} |a_i|^2 - \sum_{j=1}^{p} e^{2\mu_j} |b_j|^2 \right) \cdot \left(\sum_{i=1}^{p} e^{2\lambda_i} |c_i|^2 - \sum_{j=1}^{p} e^{2\mu_j} |d_j|^2 \right) \\ = & \left(\sum_{i=1}^{p} \left(e^{2\lambda_i} - e^{2\mu_1}\right) |a_i|^2 + \sum_{i=1}^{p} \left(e^{2\mu_1} - e^{2\mu_j}\right) |b_i|^2 + e^{2\mu_1} \right) \\ & \cdot \left(\sum_{i=1}^{p} \left(e^{2\lambda_i} - e^{2\mu_1}\right) |c_i|^2 + \sum_{i=1}^{p} \left(e^{2\mu_1} - e^{2\mu_j}\right) |d_i|^2 + e^{2\mu_1} \right), \end{aligned}$$

which is strictly greater than

$$\left(\sum_{i=1}^{p}\left(e^{2\lambda_i}-e^{2\mu_1}\right)|a_i|^2+\sum_{i=1}^{p}\left(e^{2\mu_1}-e^{2\mu_j}\right)|b_i|^2\right)$$
$$\cdot\left(\sum_{i=1}^{p}\left(e^{2\lambda_i}-e^{2\mu_1}\right)|c_i|^2+\sum_{i=1}^{p}\left(e^{2\mu_1}-e^{2\mu_j}\right)|d_i|^2\right),$$

which, in turn, by the Cauchy-Schwarz Inequality, is greater than or equal to

$$\left|\sum_{i=1}^{p}\left(e^{2\lambda_i}-e^{2\mu_1}\right)a_i\overline{c}_i+\sum_{j=2}^{q}\left(e^{2\mu_1}-e^{2\mu_j}\right)b_j\overline{d}_j\right|^2, \tag{3.1.6}$$

which is exactly $|\langle \mathbf{v}_1, \mathbf{v}_2\rangle|$, if we take into account (3.1.5).

We will now similarly show that $\mathbf{u}_k$ is timelike for any $k \leq p$. The k-th step in the Gram-Schmidt algorithm gives

$$\mathbf{v}_k = \sum_{l=1}^{k-1} m_{lk}\mathbf{u}_l + r_k\mathbf{u}_k,$$

where $m_{lk} = \langle \mathbf{v}_k, \mathbf{u}_l\rangle$. Showing that $\mathbf{u}_k$ is timelike amounts to showing that

$$||\mathbf{v}_k||^2 > \sum_{l=1}^{k-1} |\langle \mathbf{v}_k, \mathbf{u}_l\rangle|^2. \tag{3.1.7}$$

The idea is to construct a vector $\mathbf{u}$ that plays the role of $\mathbf{u}_1$ in the $k = 2$ step. In particular, we need

$$\mathbf{v}_k = \langle \mathbf{v}_k, \mathbf{u}\rangle\mathbf{u} + r_k\mathbf{u}_k.$$

To this end, let $\mathbf{u}$ be the unique element of V defined by

$$\langle \mathbf{v}_k, \mathbf{u}\rangle \cdot \mathbf{u} = \sum_{l=1}^{k-1} m_{lk}\cdot\mathbf{u}_l = \sum_{l=1}^{k-1}\langle \mathbf{v}_k, \mathbf{u}_l\rangle \cdot \mathbf{u}_l,$$

The vector $\mathbf{u}$ is well-defined and unique because $\mathbf{u}_1, \ldots, \mathbf{u}_{k-1}$ and $\mathbf{v}_k$ are timelike (by admissibility of $\vec{\lambda}$), and $\langle \cdot, \cdot \rangle$ behaves like the usual dot product on the timelike cone. Note that $||\mathbf{u}||^2 = 1$.

Then (3.1.7) is equivalent to

$$||\mathbf{v}_k - \langle \mathbf{v}_k, \mathbf{u} \rangle \mathbf{u}||^2 = ||\mathbf{v}_k||^2 - ||\langle \mathbf{v}_k, \mathbf{u} \rangle||^2 > 0. \tag{3.1.8}$$

Consider the subspace $V \subset \mathbb{C}^n$ spanned by $\mathbf{u}_1, \ldots, \mathbf{u}_{k-1}$ (which is the same as the subspace spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$). It is isomorphic to the subspace W spanned by $\mathbf{w}_1, \ldots, \mathbf{w}_{k-1}$. The explicit isomorphism $\phi : V \to W$ is given by multiplying the first coordinate by $e^{-\lambda_1}$,...,the p-th by $e^{-\lambda_p}$,the $(p+1)$-st by $e^{-\mu_1}$,..., and the $(p+q)$-th by $e^{-\mu_q}$. Now let $\mathbf{w} = \frac{\phi(\mathbf{u})}{||\phi(\mathbf{u})||} \in W$, and let $\mathbf{v} = \phi(\mathbf{w}) = \frac{\mathbf{u}}{||\phi(\mathbf{u})||}$. Note that since $\mathbf{w} \in \text{span}\{\mathbf{w}_1, \ldots, \mathbf{w}_{k-1}\}$, $\langle \mathbf{w}_k, \mathbf{w} \rangle = 0$.

Since $||\mathbf{u}||^2 = 1$, (3.1.7) is equivalent to

$$||\mathbf{v}_k||^2 \cdot ||\mathbf{u}||^2 > \sum_{l=1}^{k-1} |\langle \mathbf{v}_k, \mathbf{u}_l \rangle|^2. \tag{3.1.9}$$

We now repeat the argument for the $k = 2$ case, replacing $\mathbf{v}_2$ with $\mathbf{v}_k$ and $\mathbf{w}_1$ with $\mathbf{w}$. Then $\mathbf{v_1}$ is also replaced by $\mathbf{v}$. We obtain

$$||\mathbf{v_k}||^2 ||\mathbf{v}||^2 > |\langle \mathbf{v}_k, \mathbf{v} \rangle|^2,$$

which, multiplying both sides by $||\phi(\mathbf{u})||^2$, is equivalent to (3.1.8).

Thus, we have established that the first p columns of s are timelike (and $r_1, \ldots, r_p$ are positive real numbers). Notice that the only way to complete a set of p timelike vectors to a pseudo-orthonormal basis with respect to $\langle\cdot,\cdot\rangle$ is to add q spacelike vectors. This is equivalent to the fact that the signature of $\langle\cdot,\cdot\rangle$ is basis-independent. Therefore, if we continue the Gram-Schmidt process, the last q columns of s will be spacelike, as required, and $r_{p+1},\ldots,r_{p+q}$ will be positive real numbers. This completes the proof. □

A similar result is true for the left dressing action.

Proposition 3.1.4. *For any* $\vec{\lambda}\in\mathfrak{a}_{adm}$*, the left dressing action, as given by (3.1.1) (with* K *replaced by* G_0*), is defined on* $\exp(-\vec{\lambda})$ *for all* $g\in G_0$*.*

Proof. This follows easily from Proposition 3.1.3 by taking inverses. □

Remark 3.1.5. By Proposition 3.1.3, each admissible orbit $\Psi_{\vec{\lambda}}$ is a (G_0,π_{pq})-homogeneous space. Moreover, since the diagonal torus $T\subset G_0$ stabilizes every point in A, the Poisson structure π_{AN} on AN_{adm} is T-invariant.

3.2. Mapping the Admissible Orbits from G_0^* to $\mathfrak{q}_0^*$

Define a Lie algebra anti-involution $\dagger:\mathfrak{g}\to\mathfrak{g}$ by

$$X^\dagger := J_{pq}X^*J_{pq},$$

where J_{pq} is the matrix (A.1). Then $X^\dagger=-X$ iff $X\in\mathfrak{g}_0$. We also define $\dagger:G\to G$ by the same formula. At the group level, we have $u^\dagger=u^{-1}$ iff $u\in G_0$. Let $Q:=\{u\in$

$G \,|\, u^\dagger = u\}$. Define a map $Sym : AN \to Q$ by

$$Sym(C) = C^\dagger C.$$

Proposition 3.2.1. *The Sym map is invertible on AN for any p, q.*

Before giving a formal proof, we demonstrate the strategy by way of an example. Consider the case $p = 1$, $q = 2$. Let

$$C := \begin{pmatrix} e^{z_1} & \alpha_{12} & \alpha_{13} \\ 0 & e^{z_2} & \alpha_{23} \\ 0 & 0 & e^{z_3} \end{pmatrix} \in AN.$$

Then

$$Sym(C) = \begin{pmatrix} e^{2z_1} & e^{z_1}\alpha_{12} & e^{z_1}\alpha_{13} \\ -e^{z_1}\overline{\alpha_{12}} & e^{2z_2} - |\alpha_{12}|^2 & e^{z_2}\alpha_{23} - \overline{\alpha_{12}}\alpha_{13} \\ -e^{z_1}\overline{\alpha_{13}} & e^{z_2}\overline{\alpha_{23}} - \alpha_{12}\overline{\alpha_{13}} & e^{2z_3} + |\alpha_{23}|^2 - |\alpha_{13}|^2. \end{pmatrix}.$$

Let $B := Sym(C)$. We will show that given B, we can deduce the values of all entries in C. We will proceed through the rows of B from top to bottom, looking only at entries on the diagonal and above. First, since $B_{11} = e^{2z_1}$, $z_1 = \frac{1}{2}(\ln B_{11})$. Knowing z_1 and B_{12}, we can deduce the value of α_{12}, and similarly for α_{13}. Now that α_{12} is known, it is easy to find z_2, and from here we can find α_{23}. Finally, the value of z_3 is determined by α_{23}, α_{13} and B_{33}. Thus, C is uniquely determined by B. The proof below formalizes this procedure.

Proof of Proposition 3.2.1. Let $C \in AN$. Then $C_{ii} = e^{z_i}$ for some $z_i \in \mathbb{R}$, and $C_{ij} = 0$ for $i > j$. We also know that for any i, j, $C^\dagger_{ij} = \pm C_{ji}$, which implies that $C^\dagger_{ij} = 0$ for $i < j$. Let $B := Sym(C) = C^\dagger C$. Assuming that the entries of

B are known, we will find the entries of C by working our way through the rows of B from top to bottom. That is, the proof will proceed by induction. In fact, it will suffice to consider only the entries B_{ij} with $i \leq j$. (This is clear from the definition of Q.) Beginning with the first row, since $B_{11} = e^{2z_1}$, $z_1 = \frac{1}{2}(\ln B_{11})$. Then $B_{1j} = e^{z_1} C_{1j} \Rightarrow C_{1j} = e^{-z_1} B_{1j}$ for $j = 2 \ldots n$. For the ith row of B, we have

$$\begin{aligned} B_{ij} &= \sum_{k=1}^{n} C^{\dagger}_{ik} C_{kj} \\ &= C_{ii} C_{ij} + \sum_{k<i} C^{\dagger}_{ik} C_{kj} + \sum_{k>i} C^{\dagger}_{ik} C_{kj} \\ &= e^{z_i} C_{ij} + \sum_{k<i} \pm \overline{C_{ki}} C_{kj} + 0. \end{aligned}$$

Assume that C_{km} is known for all $k < i$, $m \geq k$. Then for $j = i$, we obtain

$$z_i = \frac{1}{2} \ln \left(B_{ii} - \sum_{k<i} \pm |C_{ki}|^2 \right).$$

Now that z_i is known, we can solve for C_{ij}:

$$C_{ij} = -e^{-z_i} \left(B_{ij} - \sum_{k<i} \pm \overline{C_{ki}} C_{kj} \right). \tag{3.2.1}$$

It follows by induction on i that each entry in C is uniquely determined by B. □

Now define a map $sym : \mathfrak{a} + \mathfrak{n} \rightarrow \mathfrak{q}$ by

$$sym(X) = X^{\dagger} + X.$$

This map is the linearization of the Sym map. It is a linear isomorphism onto its image in $\mathfrak{q}$.

For $u \in G_0$, and $q \in Q$, let $\mathrm{Ad}^{-1}(q,u) := u^{-1}qu$. This defines a right action

$$\mathrm{Ad}^{-1} : Q \times G_0 \to Q. \tag{3.2.2}$$

The group G_0 acts on $\mathfrak{q} \cong \mathfrak{a} + \mathfrak{n}$ by the same formula. This is the (right) coadjoint action.

Proposition 3.2.2. *The right dressing action of $u \in G_0$ on AN defined in Section 2.2.7 corresponds under the Sym map to the* Ad^{-1} *action on Q.*

Proof. Let $u \in G_0$ and $l \in AN$. Then using 2.2.2 and 2.2.3,

$$\begin{aligned} u^{-1}\left(l^\dagger l\right)u &= u^\dagger l^\dagger u^l l^u \\ &= \left(u^l l^u\right)^\dagger u^l l^u \\ &= \left(l^u\right)^\dagger \left(u^l\right)^\dagger u^l l^u \\ &= \left(l^u\right)^\dagger l^u. \end{aligned}$$

□

Recall that a G_0-orbit in AN is called *admissible* if it passes through a point $\exp(\vec{\lambda}) \in A$ for some $\vec{\lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_q) \in \mathfrak{a}$ such that $\lambda_i \geq \mu_j$ for all $1 \leq i \leq p$, $1 \leq j \leq q$, and that $(AN)_{adm}$ denotes the set of admissible orbits in AN (see Section 3.1). Defining admissibility of G_0-orbits in Q and $\mathfrak{q}$ in the same way, denote by Q_{adm} and $(\mathfrak{a}+\mathfrak{n})_{adm}$ the unions of admissible orbits in Q and $\mathfrak{a}+\mathfrak{n}$, respectively. Using Propositions 3.2.1 and 3.2.2, it is easy to see that *Sym* is a bijection from $(AN)_{adm}$ onto Q_{adm} which is smooth on the interior of $(AN)_{adm}$.

Proposition 3.2.3. *Let Q_{el} and $\mathfrak{q}_{el}$ be the sets of elliptic orbits (orbits consisting of matrices with real eigenvalues) in Q and $\mathfrak{q}$, respectively. The restriction of the exponential map on $\mathfrak{q}$ to the cone $\mathfrak{q}_{el}$ of admissible orbits is a bijection onto Q_{el}. It is smooth on the interior of $\mathfrak{q}_{el}$*

Proof. Smoothness of the exponential map is well known. To show bijectivity, we follow the proof in [Che46] (p13) for the hermitian case. Let $\beta \in Q_{el}$. Then there exists $g \in G_0$ such that if we set $\mathbf{v}_i := g\mathbf{e}_i$ (where $\{\mathbf{e}_i\}$ are the standard basis vectors on $\mathbb{C}^n$), then for each $i = 1 \ldots n$, $\beta \mathbf{v}_i = \mu_i \mathbf{v}_i$ for some real, positive number μ_i. Now set $\lambda_i := \log \mu_i$ and define a matrix α by $\alpha \mathbf{v}_i := \lambda_i \mathbf{v}_i$ for each i. Then $g^{-1}\alpha g \mathbf{e}_i = \lambda_i \mathbf{e}_i$, which shows that $g^{-1}\alpha g$ is a real, diagonal matrix. Thus, $g^{-1}\alpha g \in Q_{el}$, which implies that $\alpha = g(g^{-1}\alpha g)g^{-1} \in Q_{el}$. Moreover, since $(\exp \alpha)\mathbf{v}_i = (\exp \lambda_i)\mathbf{v}_i = \mu_i \mathbf{v}_i$ for $i = 1 \ldots n$, it follows that $\exp \alpha = \beta$. Therefore, exp maps $\mathfrak{q}_{el}$ onto Q_{el}.

Now suppose that $\exp(\alpha') = \beta$ for some $\alpha' \in \mathfrak{q}_{el}$. We will show that $\alpha' = \alpha$, as constructed above. Let $\mathbf{v}' = \sum_i x_i \mathbf{v}$ be an eigenvector of α' with eigenvalue λ'. Then using [Che46], page 5, Proposition 2,

$$\beta \mathbf{v}' = (\exp \alpha')\mathbf{v}' = (\exp \lambda')\mathbf{v}' = \sum_i \mu_i x_i \mathbf{v}_i,$$

which implies that $x_i = 0$ if $\mu_i \neq \exp \lambda'$. Let i_0 be an index such that $x_{i_0} \neq 0$. Then $\mu_{i_0} = \exp \lambda' = \exp \lambda_{i_0}$. Since λ' and λ_{i_0} are both real, it follows that $\lambda' = \lambda_{i_0}$. On the other hand,

$$\alpha \mathbf{v} = \sum_i (x_i \log \mu_i)\mathbf{v}_i = \lambda' \mathbf{v}' = \alpha' \mathbf{v}'.$$

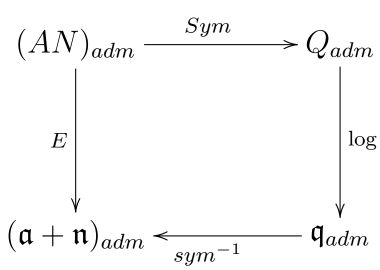
Thus, since the eigenvectors of α' form a basis for $\mathbb{C}^n$, we have shown that the linear transformations α and α' agree on a basis, and hence on all of $\mathbb{C}^n$. Therefore, $\alpha' = \alpha$, and the proof is complete. $\square$

Corollary 3.2.4. *The restriction of the exponential map on* $\mathfrak{q}$ *to the cone* $\mathfrak{q}_{adm}$ *of admissible orbits is a bijection onto* Q_{adm}. *It is smooth on the interior of* $\mathfrak{q}_{adm}$

Proposition 3.2.5. *The map* $E := sym^{-1} \circ \log \circ Sym$ *is a bijection from* $(AN)_{adm}$ *to* $(\mathfrak{a} + \mathfrak{n})_{adm}$*, it is smooth on the interior of* $(AN)_{adm}$*, and it is* G_0*-equivariant with respect to the right dressing action on* AN*, and the right coadjoint action on* $\mathfrak{a} + \mathfrak{n}$.

The construction of the map E is illustrated in Figure 3.1.

FIGURE 3.1. Diagram of the map E for $SU(p, q)$



Thus, if we restrict our attention to admissible orbits, the Poisson structures π_{AN} and π_0 can be thought of as bivector fields on the vector space $\mathfrak{a} + \mathfrak{n} \cong \mathfrak{q}$.

3.3. Poisson Cohomology

3.3.1. The Compact Case

Part of the Ginzburg-Weinstein argument involves proving the existence of a vector field X on $\mathfrak{g}^*$ such that $L_X \pi = \dot{\pi}$. This is done by combining several theorems which relate Poisson cohomology and Lie algebra cohomology. We first establish notation (see Appendix B for definitions of the various cohomology groups):

- $H^\star_\pi(G)$ is the Poisson cohomology of G.
- $H^\star_{top}(G)$ is the de Rham cohomology of G.
- $H^\star_{ct}(G, V, \rho)$ is *continuous (or, equivalently, differentiable)* group cohomology of G with respect to the representation ρ.
- $H^\star(\mathfrak{g}, V, \rho)$ is the Lie algebra cohomology of $\mathfrak{g}$ with respect to the representation ρ.
- $H^\star(\mathfrak{g}, \mathfrak{h}, V, \rho)$ is the relative Lie algebra cohomology of $\mathfrak{g}$ with respect to the subalgebra $\mathfrak{h}$ and the representation ρ.
- $H^\star(\mathfrak{g})$ is the Lie algebra cohomology of $\mathfrak{g}$ with respect to the trivial representation of $\mathfrak{g}$ on $\mathbb{R}$.
- Inv is the set of (smooth) functions on G^* that are invariant with respect to the dressing action of G on G^*.

If only one representation is present, $H^\star_{ct}(G,V,\rho)$ will be abbreviated by $H^\star_{ct}(G,V)$, and similarly for $H^\star(\mathfrak{g},V,\rho)$ and $H^\star(\mathfrak{g},\mathfrak{h},V,\rho)$. As in Section 2.3, since $[\pi,\dot{\pi}]=0$, $\dot{\pi}$ is a Poisson cocycle with respect to π. It therefore suffices to show that $\dot{\pi}$ is a Poisson coboundary. Ginzburg and Weinstein achieve this by proving that $H^2_\pi(G_0^*)=0$. They use the following two results:

Proposition 3.3.1. *If G is* any *Poisson Lie group, then $H^\star_\pi(G^*)=H^\star(\mathfrak{g},C^\infty(G^*))$.*

Proof. This is a special case of [LW90], Corollary 5.27. □

Remark 3.3.2. As noted in [GW92], it follows from [LW90], Proposition 5.26, that Proposition 3.3.1 also applies to any open submanifold $U\subset G^*$: $H^\star_\pi(U)=H^\star(\mathfrak{g},C^\infty(U))$. This fact will be useful for the $SU(p,q)$ case.

Theorem 3.3.3. *If G is a compact Poisson Lie group acting on a (not necessarily compact) manifold M, then $H^\star(\mathfrak{g},C^\infty(M))=H^\star(\mathfrak{g})\otimes\mathrm{Inv}$.*

A proof of this theorem will be given below. Combining Theorems 3.3.1 and 3.3.3, we obtain

Proposition 3.3.4. *If G is a compact Poisson Lie group, then $H^\star_\pi(G^*)\cong H^\star(\mathfrak{g})\otimes\mathrm{Inv}$.*

The necessary result now follows from Whitehead's Lemma, which implies that for a semisimple Lie algebra $\mathfrak{g}$, $H^1(\mathfrak{g})=H^2(\mathfrak{g})=0$ ([DAI98], Theorem 6.5.2).

Corollary 3.3.5. *If G is a compact Poisson Lie group, then $H^2_\pi(G^*) = 0$.*

Ginzburg and Weinstein give a proof of Theorem 3.3.3 which involves the Hodge decomposition and which seems to rely on compactness in a fundamental way. They remark, however, that an alternative proof can be given based on a theorem of Van Est. This approach, which we describe next, has the advantage that it can be adapted to the $SU(p,q)$ case in a relatively straightforward way.

Theorem 3.3.6 ([VE55], Theorem 2)**.** *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let ρ be a representation of G on a vector space V. We will also use ρ to denote the induced representation of $\mathfrak{g}$ on V. There exists a spectral sequence (E_t) with final term $E_\infty \cong H(\mathfrak{g}, V, \rho)$ (where $H(\mathfrak{g}, V, \rho)$ is given a convenient bigrading) such that $E_2^{r,s} \cong H^r_{top}(G) \otimes H^s_{ct}(G, V, \rho)$, where s is the degree of the filtration.*

Remark 3.3.7. The statement $E_\infty \cong H(\mathfrak{g}, V, \rho)$ means that $\bigoplus_{r+s=n} E_\infty^{r,s} \cong H^n(\mathfrak{g}, V, \rho)$ for all n.

Before giving the proof of Theorem 3.3.3, we record a preliminary result.

Proposition 3.3.8. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, let $\mathfrak{k}$ be a maximal, compact subalgebra of $\mathfrak{g}$, and let $\rho : G \to Aut(V)$ be a representation of G on a vector space V. Then*

$$H^p_{ct}(G, V, \rho) \cong H^p(\mathfrak{g}, \mathfrak{k}, V, \rho) \text{ for } p > 0.$$

In particular, if G is compact, then $H^p_{ct}(G, V, \rho) = 0$ (see Remark B.6).

Proof. This is a special case of [VE55], Theorem 3. □

Proof of Theorem 3.3.3. Let $V = C^\infty(M)$, and let ρ be the induced representation of G on V. Let (E_t) be the spectral sequence obtained by applying Theorem 3.3.6, to this action. For $s = 0$, it follows easily from the definition of group cohomology that

$$H^0_{ct}(G, V) \cong \text{Inv}.$$

Since $\mathfrak{g}$ is compact, it follows from Proposition 3.3.8 that $H^s_{ct}(G, V, \rho) = 0$ for $s > 0$. Thus, the second sheet E_2 has the form

$\vdots$	$\vdots$	$\vdots$	
$H^2_{top}(G) \otimes \text{Inv}$	0	0	$\ldots$
$H^1_{top}(G) \otimes \text{Inv}$	0	0	$\ldots$
$H^0_{top}(G) \otimes \text{Inv}$	0	0	$\ldots$

Since the only nonzero entries on E_2 are in the first column, the spectral sequence (E_t) stabilizes on the sheet $t = 2$. That is, $E^{r,s}_\infty = E^{r,s}_2$ for all r, s. Then (see Remark 3.3.7), for all $n \geq 0$,

$$H^n(\mathfrak{g}, V, \rho) \cong \bigoplus_{r+s=n} E^{r,s}_\infty = H^n_{top}(G) \otimes \text{Inv}. \tag{3.3.1}$$

Finally, according to Theorem B.8, since G is compact, $H^\star_{top}(G) \cong H^\star(\mathfrak{g})$, which implies that

$$H^\star(\mathfrak{g}, V, \rho) \cong H^\star(\mathfrak{g}) \otimes \text{Inv}.$$

□

3.3.2. The $SU(p,q)$ Case

We now consider the case in which the compact group G is replaced by the non-compact group $G_0 = SU(p,q)$. In this case, we are only interested in the set $AN_{adm} \subset AN = G_0^*$ of admissible elements. In order to use the Ginzburg-Weinstein argument, we need to show that there exists a vector field X on $(\mathfrak{a}+\mathfrak{n})_{adm} \cong AN_{adm}$ satisfying $[X,\pi] = \dot{\pi}$. What we would like to prove, then, is that $H^2_\pi(AN_{adm}) = 0$. While we have not yet achieved this result, we can prove a slightly weaker statement by restricting to the *generic* orbits in AN.

Definition 3.3.9. A dressing orbit $\Psi \subset AN$ will be called *generic* if $\mathrm{Stab}_{G_0}(p) \cong T$ for all $p \in \Psi$. The generic orbits in Q, $\mathfrak{a}+\mathfrak{n}$, and $\mathfrak{q}$ are defined in the same way for the Ad^{-1} action. The sets of generic orbits in each space are denoted by AN_{gen}, Q_{gen} $(\mathfrak{a}+\mathfrak{n})_{gen}$, and $\mathfrak{q}_{gen}$.

The generic orbits in AN are the orbits of maximal dimension. It is easy to check that an admissible orbit passing through $\vec{\lambda} = \mathrm{diag}(\lambda_1,\ldots,\lambda_n) \in \mathfrak{a}$ is generic if and only if $\lambda_i \neq \lambda_j$ for all $1 \leq i,j \leq n$, and that the stabilizer in G_0 of any generic $\vec{\lambda}$ is the torus T.

Let $C := AN_{adm} \bigcap AN_{gen}$. As in Remark 3.3.2, Proposition 3.3.1 applies to the open subset C of AN. Therefore, if Theorem 3.3.3 can be established for the $SU(p,q)$ case, then the analogue of Proposition 3.3.4 will follow for that case. That is, we need to prove

$$H^\star(\mathfrak{g}_0, C^\infty(C)) = H^\star(\mathfrak{g}_0) \otimes \mathrm{Inv}, \tag{3.3.2}$$

where Inv is the set of G_0-invariant functions on C.

Lemma 3.3.10. *With C as above,*

$$H^p_{ct}(G_0, C^\infty(C), \rho) = 0,$$

where ρ is the representation induced by the dressing action $\mathcal{D}$ of G_0 on C.

Remark 3.3.11. In the following proof, representations will be induced in two different ways. First, a right action $\mathcal{A} : M \times G \to M$ of a group G on a manifold M induces a representation of G on $C^\infty(M)$ by

$$(g.f)(x) := f(\mathcal{A}(x, g)). \tag{3.3.3}$$

Second, a representation of a subgroup $H \subset G$ on a vector space V induces a representation of G on a subspace of the functions from G to V, as described in Appendix C. It will be important to distinguish between these two notions.

Proof of Lemma 3.3.10. The right coadjoint action $\mathcal{D}$ of G_0 on C induces a representation $\delta : G_0 \to \mathrm{Aut}(C^\infty(C))$ defined by

$$(\phi(g)(f))(b) := f(\mathcal{D}\,(g)\,(b))$$

for $g \in G_0, b \in C$. The strategy in this proof will be to use Shapiro's Lemma to relate the group cohomology of G_0 to the cohomology of a compact subgroup. The first step is to show that δ is also induced–this time in the sense of Appendix C–by a representation τ of the torus T on a subset S of C.

Let $S = C \bigcap \mathfrak{a}$, so that $C \cong S \times T\backslash G_0$. Note that under this correspondence, the $\mathcal{D}$ action of G_0 becomes the following action on $S \times T\backslash G_0$: For $g \in G_0$, $s \in S$, $Tx \in T\backslash G_0$,

$$\mathcal{D}_g(s, Tx) = (\mathcal{D}_g(s), Txg). \tag{3.3.4}$$

Restricting the representation of G_0 on $C^\infty(C)$ to $T \subset G_0$ and to $S \subset C$, we obtain a trivial representation τ of T on $C^\infty(S)$. As in Appendix C, ρ induces a representation γ of G_0 on the vector space

$$\begin{aligned} R &= \operatorname{Ind}_H^{G_0}(C^\infty(S)) \\ &= \left\{ f : G_0 \to C^\infty(S) \,|\, f \text{ is smooth, } f(hg) = \rho(h)f(g) \ \forall h \in T, g \in G_0 \right\}. \end{aligned}$$

Recall that for $f \in R$, $g \in G_0$, γ is defined by

$$\gamma(f, g)(x) := f(xg) \text{ for } x \in G_0.$$

The condition $f(hg) = \tau(h)f(g) = f(g)$ for $h \in H$ and $g \in G_0$ means that the elements of R can be thought of as functions on the set of right cosets $T\backslash G_0$. Furthermore, using the fact that $C \cong S \times T\backslash G_0$, R can be identified with $C^\infty(C)$ as follows. For $f \in R$, define $\phi_f \in C^\infty(W) \cong C^\infty(S \times T\backslash G_0)$ by

$$\phi_f(s, Tx) = (f(Tx))(\mathcal{D}_x(s)).$$

It is easy to check that $R \to C^\infty(S \times T\backslash G_0) : f \mapsto \phi_f$ is a bijection. Then

$$\gamma(\phi_f, g)(s, Tx) = (f(Txg))(\mathcal{D}_g\mathcal{D}_x(s)) = \phi_f(\mathcal{D}_{xg}(s), Txg).$$

This shows that, taking into account (3.3.4), γ is just the representation of G_0 on $C^\infty(C)$ induced by the $\mathcal{D}$ action–i.e., $\gamma = \delta$. Thus, the right dressing representation δ of G_0 on $C^\infty(C)$ is induced (in the sense of Appendix C) by the trivial representation of T on $C^\infty(S)$.

Then by Shapiro's Lemma (Theorem C.9),

$$H^p_{ct}(G_0, C^\infty(C), \rho) \cong H^p_{ct}(G_0, R, \gamma) \cong H^p_{ct}(T, C^\infty(S), \gamma). \tag{3.3.5}$$

Since T is compact, by Proposition 3.3.8,

$$H^p_{ct}(G_0, C^\infty(C), \rho) \cong H^p_{ct}(T, C^\infty(S), \gamma) = 0. \tag{3.3.6}$$

□

We are now in a position to prove an analogue of Theorem 3.3.3 for the $SU(p, q)$ case.

Theorem 3.3.12. *With notation as above,* $H^\star(\mathfrak{g}_0, C^\infty(C)) = H^\star(\mathfrak{k}_0) \otimes \mathrm{Inv}$*, where* $\mathfrak{k}_0 = \mathfrak{s}(\mathfrak{u}(p) + \mathfrak{u}(q))$ *is the maximal compact subalgebra of* $\mathfrak{g}_0$*.*

Proof. Setting $V = C^\infty(C)$, replacing Proposition 3.3.8 with Lemma 3.3.10, and repeating the proof of Theorem 3.3.3, (3.3.1) becomes

$$H^\star(\mathfrak{g}_0, C^\infty(C)) \cong H^\star_{top}(G) \otimes \mathrm{Inv}. \tag{3.3.7}$$

Then by Theorem B.8,

$$H^\star(\mathfrak{g}_0, C^\infty(C)) \cong H^\star_{top}(K_0) \otimes \mathrm{Inv} \cong H^\star(\mathfrak{k}_0) \otimes \mathrm{Inv}.$$

□

Now the analogue of Proposition 3.3.4 follows from Theorem 3.3.12 and Proposition 3.3.1:

Theorem 3.3.13. *$H^\star_\pi(C) \cong H^\star(\mathfrak{k}_0) \otimes \mathrm{Inv}$.*

Corollary 3.3.14. *For any p, q, $H^2_\pi(C) = 0$.*

Proof. Since $\mathfrak{k}_0 = \mathfrak{s}(\mathfrak{u}(p) + \mathfrak{u}(q))$ is not semisimple, Whitehead's Lemma does not apply directly. However, $\mathfrak{s}(\mathfrak{u}(p) + \mathfrak{u}(q)) \cong \mathfrak{su}(p) \oplus \mathfrak{su}(q) \oplus \mathbb{R}$, where $\mathfrak{su}(p) \oplus \mathfrak{su}(q)$ is semisimple. Then (see [FF], Chapter 3, Section 1) using Whitehead's Lemma,

$$\begin{aligned} H^2(\mathfrak{k}_0) &\cong H^1(\mathfrak{su}(p) \oplus \mathfrak{su}(q)) \otimes H^1(\mathbb{R}) \\ &+ H^2(\mathfrak{su}(p) \oplus \mathfrak{su}(q)) \otimes H^0(\mathbb{R}) \\ &+ H^0(\mathfrak{su}(p) \oplus \mathfrak{su}(q)) \otimes H^2(\mathbb{R}) \\ &\cong 0 \otimes H^1(\mathbb{R}) + 0 \otimes H^0(\mathbb{R}) + H^0(\mathfrak{su}(p) \oplus \mathfrak{su}(q)) \otimes 0 \\ &= 0. \end{aligned}$$

□

Theorem 3.3.15. *With C as above, there exists a vector field X on C such that $[X, \pi] = \dot{\pi}$.*

If $p = q = 1$, then $\mathfrak{k}_0 = \mathfrak{t}$, the torus, which is not semisimple. In fact, $H^n(\mathfrak{t}) \neq 0$ for any n. However, the desired vector field for this case is constructed explicitly in Section 3.6.

3.4. Compactification of Orbits

In this section, we will embed each generic dressing orbit $\Psi_{\vec{\lambda}}$ into a compact flag manifold and compute an expression for the image of $\pi_{\vec{\lambda}}$, the restriction of π to $\Psi_{\vec{\lambda}}$, under this embedding. We will then show that for the $SU(1,1)$ and $SU(1,2)$ cases, the Poisson structure $\pi_{\vec{\lambda}}$ extends smoothly to the entire flag manifold.

3.4.1. A Family of Poisson Structures on G_0/T

Let $G_0 = SU(p,q)$, and let π denote the Poisson structure on $G_0^* \cong AN$ which is dual to the Lu-Weinstein Poisson structure on G_0. Each admissible dressing orbit in AN passes through a unique point $\exp(-\vec{\lambda}) \in A$, where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathfrak{a}$. Recall that the stabilizer subgroup in G_0 of any generic dressing orbit is isomorphic to the torus $T \subset G_0$. Let $\pi_{\vec{\lambda}}$ denote the restriction of π to the orbit $\Psi_{\vec{\lambda}}$ through the point $\exp(-\vec{\lambda}) \in A$. Then for a generic $\vec{\lambda}$, identifying $\Psi_{\vec{\lambda}}$ with G_0/T, $\pi_{\vec{\lambda}}$ can be thought of as a Poisson structure on G_0/T. In this subsection, we derive an expression for $\pi_{\vec{\lambda}}$ on G_0/T.

To simplify notation, let

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{ij} \tag{3.4.1}$$

denote an n matrix with an a in the iith slot, a b the ij-th slot, a c in the jith slot, a d in the jj-th slot, and zeros everywhere else. Using the expression

$$((r_{u^{-1}})_*\pi(u))\,(X,Y) = \langle p_{\mathfrak{g}_0}(\mathrm{Ad}_{u^{-1}}X), p_{\mathfrak{a}+\mathfrak{n}}(\mathrm{Ad}_{u^{-1}}Y)\rangle,$$

where $\langle,\rangle$ is the imaginary part of the trace form (the Killing form) and $p_{\mathfrak{g}_0}$ and $p_{\mathfrak{a}+\mathfrak{n}}$ are the natural projections, we will compute $\pi(\exp(-\vec{\lambda}))$.

Denote the standard roots for $\mathfrak{gl}(n,\mathbb{C})$ with respect to the usual, diagonal subalgebra $\mathfrak{h}$ by $\alpha_{ij} = e_i - e_j$. For a given p, q with $p+q = n$, a root α_{ij} (or the pair (i,j)) will be called *compact* if $i,j \leq p$ or $i,j \geq p$ and *noncompact* otherwise. Let

$$X_{ij} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}_{ij}, Y_{ij} = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}_{ij}, Z_{ij} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{ij}, W_{ij} = \begin{bmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}_{ij}.$$

For a given pair (i,j), The vectors X_{ij} and Y_{ij} are associated to the compact root α_{ij}. Similarly, Z_{ij} and W_{ij} are associated to a non-compact root α_{ij}.

The tangent space at $\exp(-\vec{\lambda})$ to $\Psi_{\vec{\lambda}}$ can be identified with $\mathfrak{n}$ via $r_{\exp(\vec{\lambda})}$. Thus, $T_{\exp(-\vec{\lambda})}\Psi_{\vec{\lambda}}$ can be identified with the space of strictly upper triangular complex matrices. For compact pairs (i,j), the projections to $\mathfrak{su}(p,q)$ and $\mathfrak{n}$ are given by

$$p_{\mathfrak{su}(p,q)}\left(\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}_{ij}\right) = \begin{bmatrix} 0 & -\bar{c} \\ c & 0 \end{bmatrix}_{ij},$$
$$p_{\mathfrak{n}}\left(\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}_{ij}\right) = \begin{bmatrix} 0 & b+\bar{c} \\ 0 & 0 \end{bmatrix}_{ij}.$$

Similarly, for noncompact (i,j), we have

$$p_{\mathfrak{su}(p,q)}\left(\begin{bmatrix}0 & b\\ c & 0\end{bmatrix}_{ij}\right) = \begin{bmatrix}0 & \bar{c}\\ c & 0\end{bmatrix}_{ij},$$
$$p_{\mathfrak{n}}\left(\begin{bmatrix}0 & b\\ c & 0\end{bmatrix}_{ij}\right) = \begin{bmatrix}0 & b-\bar{c}\\ 0 & 0\end{bmatrix}_{ij}.$$

A straightforward computation shows that

$$r_{\exp(\vec{\lambda})}(\pi)(X_{ij}, Y_{ij}) = r_{\exp(\vec{\lambda})}(\pi)(Z_{ij}, W_{ij}) = e^{2(\lambda_i - \lambda_j)} - 1, \tag{3.4.2}$$

and that all other terms are zero. It follows that

$$\pi(\exp(-\vec{\lambda})) = \sum_{(i,j)} e^{\lambda_i - \lambda_j} K_{ij} \wedge L_{ij}, \tag{3.4.3}$$

where $K_{ij} = \begin{bmatrix}0 & e^{\lambda_i} - \mathbf{i}e^{\lambda_j}\\ 0 & 0\end{bmatrix}_{ij}$ and $L_{ij} = \begin{bmatrix}0 & e^{-\lambda_i} - \mathbf{i}e^{-\lambda_j}\\ 0 & 0\end{bmatrix}_{ij}$.

Note that this is a sum over all pairs (i,j) corresponding to positive roots. Since $\Psi_{\vec{\lambda}}$ is G_0-Poisson homogeneous, $\pi_{\vec{\lambda}}$ is completely determined by 3.4.3.

The next step is to use the identification $G_0/T \cong \Psi_{\vec{\lambda}}$ and the corresponding identification of Lie algebras $\mathfrak{g}_0/\mathfrak{t} \cong T_{\exp(-\vec{\lambda})}\left(\Psi_{\vec{\lambda}}\right)$ to realize $\pi_{\vec{\lambda}}\left(\exp(-\vec{\lambda})\right)$ as an element of $\mathfrak{g}_0/\mathfrak{t}$. The map $\mathfrak{g}_0/\mathfrak{t} \to T_{\exp(-\vec{\lambda})}\left(\Psi_{\vec{\lambda}}\right)$ is given by

$$\phi : V \mapsto \frac{d}{dt}\exp(tV).\exp(-\vec{\lambda})\Big|_{t=0}.$$

Under this map,

$$X_{ij} \mapsto \begin{bmatrix} 0 & 1 - e^{2(\lambda_j - \lambda_i)} \\ 0 & 0 \end{bmatrix}_{ij}$$
$$Y_{ij} \mapsto \begin{bmatrix} 0 & 1 - e^{2(\lambda_j - \lambda_i)}\mathbf{i} \\ 0 & 0 \end{bmatrix}_{ij}$$
$$Z_{ij} \mapsto \begin{bmatrix} 0 & 1 - e^{2(\lambda_j - \lambda_i)} \\ 0 & 0 \end{bmatrix}_{ij}$$
$$W_{ij} \mapsto \begin{bmatrix} 0 & 1 - e^{2(\lambda_j - \lambda_i)}\mathbf{i} \\ 0 & 0 \end{bmatrix}_{ij}.$$

Remark 3.4.1. We are realizing $T_{\exp(-\vec{\lambda})}\left(\Psi_{\vec{\lambda}}\right)$ as the space of strictly upper-triangular matrices. This is done by right-translating to the identity in AN.

The inverse of ϕ is given by

$$\begin{bmatrix} 0 & a + \mathbf{i}b \\ 0 & 0 \end{bmatrix}_{ij} \mapsto \begin{cases} \frac{aX_{ij} + bY_{ij}}{1 - e^{2(\lambda_j - \lambda_i)}} & \text{for } (i,j) \text{ compact} \\ \frac{aZ_{ij} + bW_{ij}}{1 - e^{2(\lambda_j - \lambda_i)}} & \text{for } (i,j) \text{ noncompact.} \end{cases}$$

Pushing forward under ϕ^{-1}, we obtain an expression for $\left(\pi_{\vec{\lambda}}\right)_{eT}$ in $\mathfrak{g}_0/\mathfrak{t}$:

$$(\pi_{\vec{\lambda}})_{eT} = \sum_{\text{compact } (i,j)} \frac{1}{1 - e^{2(\lambda_i - \lambda_j)}} X_{ij} \wedge Y_{ij} + \sum_{\text{non-compact } (i,j)} \frac{1}{1 - e^{2(\lambda_i - \lambda_j)}} Z_{ij} \wedge W_{ij}.$$

This expression can also be written as

$$(\pi_{\vec{\lambda}})_{eT} = \sum_{\text{compact } \alpha_{ij}} \frac{1}{1 - e^{2\alpha_{ij}(\vec{\lambda})}} X_{\alpha_{ij}} \wedge Y_{\alpha_{ij}} + \sum_{\text{non-compact } \alpha_{ij}} \frac{1}{1 - e^{2\alpha_{ij}(\vec{\lambda})}} Z_{\alpha_{ij}} \wedge W_{\alpha_{ij}}.$$

Let π_{G_0} denote the Lu-Weinstein Poisson structure on G_0, and let $\mathcal{A} : G_0 \times G_0/T \to G_0/T$ denote the action of G_0 on G_0/T by left multiplication. (This action corresponds to the left dressing action on $\Psi_{\vec{\lambda}}$.)

Proposition 3.4.2. *Let* $\Gamma = \sum_{\alpha_{ij}>0} \frac{1}{1-e^{2\alpha_{ij}(\lambda)}} X_{\alpha_{ij}} \wedge Y_{\alpha_{ij}}$*, and let* V *be the left-invariant bivector field on* G_0/T *with value* Γ *at* eT*. The* (G_0, π_{G_0})*-homogeneous Poisson structure on* G_0/T *with value* Γ *at* eT *is*

$$\pi_{\vec{\lambda}} = \pi_{G_0} + V.$$

Proof. The Poisson structure π_λ is a (G_0, π_{G_0})-homogeneous if and only if

$$\mathcal{A}_*(\pi_{G_0} + \pi_{\vec{\lambda}}) = \pi_{\vec{\lambda}}. \tag{3.4.4}$$

Since π_{G_0} is multiplicative, we have

$$l_g(\pi_{G_0}(h)) + r_h(\pi_{G_0}(g)) = \pi_{G_0}(gh).$$

Then

$$\mathcal{A}_{(g,h)*}(\pi_{G_0}(g) + \pi_{\vec{\lambda}}(h)) = r_h(\pi_{G_0}(g)) + l_g(\pi_{G_0}(h)) + l_g(V(h)) = \pi_{G_0}(gh) + V(gh) = \pi_{\vec{\lambda}}(gh),$$

and 3.4.4 is satisfied. □

Remark 3.4.3. In the compact case $p = n$, $q = 0$, this formula agrees (up to a sign) with that of Evens and Lu in [EL99].

Remark 3.4.4. Similar computations for the right dressing action of G_0 on AN yield the same formula (up to a sign) for the Poisson structure $\pi_{\vec{\lambda}}$ on $T\backslash G_0$. In this case, π_λ is the restriction of π to the dressing orbit through $\exp(\vec{\lambda})$.

3.4.2. Embedding of Non-compact Orbits into Compact Orbits

Let $G = SL_n(\mathbb{C})$, $K = SU(n)$, $G_0 = SU(p,q)$, T =torus, and $B = TAN$ (the standard Borel subgroup). First, note that

$$G/B \to K/T : (kan)B \mapsto kT$$

is a diffeomorphism. Since G_0/T is embedded in G/B via $gT \mapsto gB$, we have an embedding

$$\phi : G_0/T \hookrightarrow K/T : gT \mapsto kT, \text{ where } g = kan.$$

We wish now to compute $\phi_*(\pi_{\vec{\lambda}})$. As noted above, in the 2×2 case, the $SU(2)$ portion of the Iwasawa decomposition of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\frac{1}{\sqrt{||a||^2 + ||c||^2}} \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix}. \tag{3.4.5}$$

Therefore, for $g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(1,1)$,

$$\phi(gT) = \frac{1}{\sqrt{||\alpha||^2 + ||\beta||^2}} \begin{pmatrix} \alpha & \bar{\beta} \\ -\beta & \bar{\alpha} \end{pmatrix} T. \tag{3.4.6}$$

For the higher-dimensional cases, the formulas are fairly horrible. However, the 2×2 formula will suffice for our purposes. Consider the curve $S_t := \exp(tZ_{ij})T$ in G_0/T. Let $R_t = \phi(S_t)$. Then $\phi_*(tZ_{ij}) = \frac{d}{dt}|_{t=0} R_t$ It is not difficult to compute $\exp(tZ_{ij})$, but it is simpler to substitute the first order approximation $S_t \approx I + tZ_{ij}$. For a matrix of this type, one can essentially use (3.4.5). At any rate, it is easy to check that

$$\phi(I + tZ_{ij}) = \frac{1}{\sqrt{1+t^2}}(I - tX_{ij}),$$

which implies that

$$\phi_*(Z_{ij}) = -X_{ij}.$$

A similar computation yields

$$\phi_*(W_{ij}) = Y_{ij}.$$

It follows that

$$\phi_*((\pi_{\vec{\lambda}})_{eT}) = \sum_{\text{compact } \alpha_{ij}} \frac{1}{1-e^{2\alpha_{ij}(\vec{\lambda})}} X_{\alpha_{ij}} \wedge Y_{\alpha_{ij}} - \sum_{\text{non-compact } \alpha_{ij}} \frac{1}{1-e^{2\alpha_{ij}(\vec{\lambda})}} X_{\alpha_{ij}} \wedge Y_{\alpha_{ij}}. \tag{3.4.7}$$

As described in [FL06], there exists a Poisson structure Π_v on K/T such that each G_0-orbit on K/T is a (G_0, π_{G_0})-homogeneous space–i.e., the action of (G_0, π_{G_0}) is Poisson.

Proposition 3.4.5. *The (G_0, π_{G_0})-homogeneous Poisson structure induced by $\pi_{\vec{\lambda}}$ on the open subset $\phi(G_0/T)$ of K/T is $V + \Pi_v$, where V is the left-invariant bivector field on K/T with value (3.4.7) at eT.*

Proof. The proof is essentially the same as the proof of Proposition 3.4.2. □

Remark 3.4.6. Similarly, the quotient $T\backslash G_0$ can be embedded in $T\backslash K$ via

$$Tg \mapsto Tp_K^r(g).$$

The resulting formula for the pushforward of $\pi_{\vec{\lambda}}$ under this embedding is again $V + \Pi_v$, but V is now the right-invariant bivector field with value (3.4.7) at Te.

3.5. G_0-orbits in $T\backslash K$

In this section, we will show that for $SU(1,1)$ and $SU(1,2)$, the Poisson structure $\pi_{\vec{\lambda}}$ induced by embedding any generic dressing orbit $\Psi_{\vec{\lambda}}$ into the flag manifold $T\backslash SU(n)$ extends smoothly to the entire flag manifold. The proofs in this section are computational in nature, and the computations are often both tedious and unenlightening. The calculations for the 3×3 case–most of which were done using a Maple program–are particularly gruesome. Therefore, some details have been omitted.

Let G be a complex, semisimple Lie group, let K and G_0 be respectively compact and non-compact real forms of G. As in Appendix A, let p^l_K and p^r_{AN} denote the projections from G to the factors K and AN, respectively, in the Iwasawa decomposition $G = K(AN)$. Also, let p^r_K and p^l_{AN} be the projection onto the factors in the decomposition $G = (AN)K$.

Define an action $\mathcal{A}_l : G \times K \to K$ of G on K by

$$g.k := \mathcal{A}_l(g,k) := p^l_K(gk) \tag{3.5.1}$$

for $g \in G$, $k \in K$. This action descends to the flag manifold K/T. Identifying K/T with G/B, where B is a Borel subgroup of G, the action of G on K/T is precisely the action of G on G/B by left multiplication. Similarly, we define a right action $\mathcal{A}_r : K \times G \to K$ by

$$g.k := \mathcal{A}_r(k,g) := p^r_K(kg). \tag{3.5.2}$$

As above, this action descends to the flag manifold $T\backslash K$. Under the identification

$T\backslash K \cong B\backslash G$, the $\mathcal{A}_r$ action on $T\backslash K$ corresponds to right-multiplication of G on $B\backslash G$. Wolf proved in [Wol69] that $T\backslash K$ (resp. K/T) has finitely-many G_0-orbits where G_0 acts by $\mathcal{A}_r$ (resp. $\mathcal{A}_l$), which implies that at least one orbit is open. He also proved that there is a unique, closed orbit and that this orbit is in the closure of all other orbits.

From this point on, we will consider the special case $G_0 = SU(p,q)$, $K = SU(n)$, where $n = p+q$. For $g \in G_0$ and $Q \in T\backslash K$, the action of g on Q will be denoted by $g.Q$. Using the notation (3.4.1), let

$$c_i := \exp\left(\frac{\pi}{4}\begin{bmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{bmatrix}_{i(n-i)}\right).$$

For each $0 \leq k \leq n$, let

$$C_k := \prod_{i=1}^{k} c_i,$$

and let

$$\hat{C}_k := \prod_{i=k+1}^{n} c_i.$$

In [FL], the G_0^s-orbits under $\mathcal{A}_r$ on $T\backslash K$ are studied, where

$$G_0^s = C_p^{-1} G_0 C_p.$$

For example, if $G_0 = SU(1,1)$, then $G_0^s = SL(2,\mathbb{R})$. Let K_0^s be the maximal compact subgroup of G_0^s. Then $K_0^s = C_p^{-1} K_0 C_p$, where $K_0 = S(U(p) \times U(q))$ is the maximal compact subgroup of G_0. Let W denote the Weyl group of $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$ with respect to the usual Cartan subalgebra $\mathfrak{h}$ of diagonal matrices. Then $W \cong S_n$. For each $1 \leq k \leq$

p, let W_p be the subgroup of W consisting of elements which have representatives in $C_k K_0^s C_k^{-1}$. (Note that the notation used here differs from that in [FL]. In particular, W_0 has a different meaning here.)

In this setting, by Theorem 5.18 of [FL] (see Remark 5.19), the G_0^s-orbits are parameterized by the points $T\dot{w}C_k$, where $\dot{w}$ is a representative in K for an element $w \in W/W_k$. In order to parameterize G_0-orbits rather than G_0^s-orbits, we note that since $G_0 = C_p G_0^s C_p^{-1}$, if $\mathrm{Stab}_{G_0^s}(Tk) = H$, then $\mathrm{Stab}_{G_0}(TkC_p^{-1}) = C_p H C_p^{-1}$. Therefore, the G_0-orbits on $T\backslash K$ can be parameterized by the points $T\dot{w}C_k C_p^{-1} = T\dot{w}\hat{C_k}^{-1}$, where $\dot{w}$ is a representative in K for an element $w \in W/W_k$.

3.5.1. Dimensions of Orbits

Proposition 3.5.1. *The dimension of the G_0^s orbit through the point TC_k is*

$$(p+q)^2-(p+q)+2k(p+q-k)-p(2q-1)-k = p^2+q^2-q+2k(p+q-k)-k. \quad (3.5.3)$$

Proof. Let d be the dimension, and let c be the codimension of the G_0^s orbit through TC_k. From example 6.11 in [FL], we have

$$k + l(w_0 w_b) - c = k + p(2(n-p)-1) - c = 2k(n-k).$$

(Note that we are considering the case $w = e$.) Solving for c gives

$$c = k - 2k(n-k) + p(2(n-p)-1) = k - 2k(p+q-k) + p(2q-1).$$

Therefore,

$$\begin{aligned} d &= (p+q)^2 - (p+q) - (k - 2k(p+q-k) + p(2q-1)) \\ &= p^2 + q^2 - q + 2k(p+q-k) - k, \end{aligned}$$

as desired. □

Corollary 3.5.2. *The dimension of the G_0 orbit through the point $T\hat{C}_k^{-1}$ is*

$$p^2 + q^2 - q + 2k(p+q-k) - k. \tag{3.5.4}$$

Note that the dimension of the orbit through $T\hat{C}_k^{-1}$ is an increasing function of k. When $k = p$, $(p+q)^2 - (p+q) + 2k(p+q-k) - p(2q-1) - k = (p+q)^2 - (p+q)$, and W_k is the Weyl group of K_0. Thus, $W_p = S_p \times S_q$. Therefore, the orbit through Te, which corresponds to $k = p$, is open. The closed orbit corresponds to the choice $k = 0$. It passes through the point $T\hat{C}_p^{-1}$ and has dimension $p^2 + q^2 - q$.

Example 3.5.3. In the case $p = q = 1$, there are two open, 2-dimensional orbits and one closed, 1-dimensional orbit. This example will be discussed in more detail in Section 3.5.2

Example 3.5.4. Consider the case $p = 2$, $q = 3$. The dimension of $T\backslash K$ is $(2+3)^2 - (2+3) = 20$. Setting $k = 0$ in (3.5.4), the dimension of the closed orbit is $2^2 + 3^2 - 3 = 10$. Setting $k = 1$, we find that there is also an intermediate orbit of dimension 17.

Remark 3.5.5. According to [FL], the dimension of the G_0^s-orbit through $T\dot{w}C_k$ (which is equal to the dimension of the G_0-orbit through $T\dot{w}\hat{C_k}^{-1}$) can be computed as follows. Let w_0 denote the longest Weyl group element, and let w_b denote the longest element in the subgroup of W generated by (the reflections corresponding to) the black dots in the Satake diagram for G_0. For each root $\alpha_{ij} = e_i - e_j$, let $s_{ij} \in W$ be the corresponding reflection. Now let

$$v_k = \prod_{i=1}^{k} s_{i(n-i)}.$$

Then the codimension of the G_0^s-orbit through $T\dot{w}C_k$ is the length of the Weyl group element $w \star (v_k w_0 w_b)$, where $\star$ denotes the *θ-twisted* conjugation

$$w \star w_1 := w w_1 \theta(w)^{-1}.$$

For the $SU(p,q)$ case, θ is just conjugation by J_{pq}, and twisted conjugation is the usual conjugation.

Example 3.5.6. Consider the case $p = 1$, $q = 2$. The dimension of $T\backslash K$ is $(1+2)^2 - (1+2) = 6$. Setting $k = 0$ in (3.5.4), the dimension of the closed orbit is $1^2 + 2^2 - 2 = 3$. Using Remark 3.5.5, one finds that there are also two 5-dimensional orbits. This example will be discussed in more detail in Section 3.5.3.

3.5.2. The $SU(1,1)$ Case

In the 2×2 case, it is relatively easy to carry out explicit computations. For $K = SU(2)$, $T\backslash K \cong S^2$. (This is the Hopf fibration.) The quotient map $\psi : K \to S^2$ is given by

$$T\begin{pmatrix}\alpha & \beta\\ -\bar\beta & \bar\alpha\end{pmatrix} \mapsto (2\Re(\alpha\bar\beta), 2\Im(\alpha\bar\beta), |\alpha|^2-|\beta|^2) = (x,y,z). \tag{3.5.5}$$

The orbits of $G_0 = SU(1,1)$ on $T\backslash K$ are parameterized by the points

$$Q_0 = T\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$
$$Q_1 = T\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$
$$Q_2 = T\frac{1}{\sqrt{2}}\begin{pmatrix}1 & \mathbf{i}\\ \mathbf{i} & 1\end{pmatrix}.$$

Denote the corresponding orbits by Ω_0, Ω_1, and Ω_2. The points Q_0 and Q_1 correspond to $k=1$; the point Q_2 corresponds to $k=0$. Under the quotient map π,

$$\begin{aligned} T\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} &\mapsto (0,0,1),\\ T\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} &\mapsto (0,0,-1), \text{ and}\\ T\frac{1}{\sqrt{2}}\begin{pmatrix}1 & \mathbf{i}\\ \mathbf{i} & 1\end{pmatrix} &\mapsto (0,-1,0).\end{aligned}$$

The orbits Ω_0 and Ω_1 are open (the open hemispheres of $T\backslash K \cong S^2$); the orbit Ω_2 is the unique closed orbit (the equator).

For the remainder of this section, let

$$c = \cosh t,$$

$$s = \sinh t.$$

Consider the 1-parameter subgroup

$$g_t = \begin{pmatrix}c & -\mathbf{i}s\\ \mathbf{i}s & c\end{pmatrix}$$

in G_0. Straightforward computations show that

$$\lim_{t\to\infty} g_t.Q_0 = \lim_{t\to\infty} g_t.Q_1 = Q_2.$$

From Section 3.4.2, for each $\vec{\lambda} = \lambda_1 > 0$, there is a Poisson structure $\pi_{\vec{\lambda}}$ that is defined on Ω_0. Since $\mathrm{Stab}_{G_0}(Q_0) = \mathrm{Stab}_{G_0}(Q_1) = T$, we have $\Omega_1 \cong G_0/T \cong \Omega_0$. Therefore, $\pi_{\vec{\lambda}}$ is also defined on Ω_1. We wish to show that $\pi_{\vec{\lambda}}$ extends smoothly to the entire flag manifold $T\backslash K$. For the $SU(1,1)$ case, it only remains to show that $\pi_{\vec{\lambda}}$ extends smoothly to the equator Ω_2. Recall that $\pi_{\vec{\lambda}}$ is defined on Ω_0 by its value at the point Q_0 and on Ω_1 by its value at Q_1. We have

$$\pi_{\vec{\lambda}}(Q_0) = -\frac{1}{1 - e^{2\alpha(\vec{\lambda})}} X_\alpha \wedge Y_\alpha,$$

where $\alpha := \alpha_{12}$ is the only positive root in this case. In order to extend $\pi_{\vec{\lambda}}$ to Ω_2, we will compute the limit of $\pi_{\vec{\lambda}}$ along the orbit $g_t.Q_0$. To make this computation concrete, identify the tangent space of $T\backslash K$ at each point Tk with $\mathfrak{k}/\mathfrak{t}$ via right translation by k^{-1}. Then $t \mapsto (g_t)_*(\pi_{\vec{\lambda}})$ is a path through $\wedge^2(\mathfrak{k}/\mathfrak{t})$. We will compute its limit as $t \to \infty$. More explicitly, for each t, let $k_t = p_K^r(g_t)$, and define a map $f_t : T\backslash K \to T\backslash K$ by

$$f_t(Tk) := Tp_K^r(kg_t)k_t^{-1} = Tp_K^r(kg_tk_t^{-1}) = Tp_K^r(kp_{AN}^l(g_t)).$$

Note that this map is well-defined because the torus "factors through" AN: For any $\tau \in T$, $\alpha \in AN$, there exists $\alpha' \in AN$ such that $\tau\alpha = \alpha'\tau$. Since f_t fixes the point Q_0, $(f_t)_{*Q_0}$ maps $\mathfrak{k}/\mathfrak{t}$ to itself. Moreover, $(f_t)_{Q_0*}$ is precisely $(g_t)_*$. Let $\tilde{f}_t : K \to K$ be the lift of f to K defined by

$$\tilde{f}_t(k) := p^r_K(kp^l_{AN}(g_t)).$$

Then $\tilde{f}$ maps e to e, so that $(\tilde{f}_t)_{*e}$ maps $\mathfrak{k}$ to $\mathfrak{k}$. For $V \in \mathfrak{k}$,

$$(\tilde{f}_t)_{*Q_0}(V) = \frac{d}{dr}\, \tilde{f}_t(\exp(rV))\Big|_{r=0}, \tag{3.5.6}$$

Using

$$\exp(rX_\alpha) = \begin{pmatrix} \cos r & \sin r \\ -\sin r & \cos r \end{pmatrix},$$
$$\exp(rY_\alpha) = \begin{pmatrix} \cos r & \mathbf{i}\sin r \\ \mathbf{i}\sin r & \cos r \end{pmatrix},$$

tedious but elementary computations give

$$\begin{aligned} (\tilde{f}_t)_{*e}(X_\alpha) &= \frac{1}{c^2+s^2} X_\alpha + csH_\alpha, \\ (\tilde{f}_t)_{*e}(Y_\alpha) &= -\frac{1}{c^2+s^2} Y_\alpha. \end{aligned}$$

Thus,

$$\begin{aligned} (f_t)_{*Q_0}(X_\alpha) &= \frac{1}{c^2+s^2} X_\alpha, \\ (f_t)_{*Q_0}(Y_\alpha) &= -\frac{1}{c^2+s^2} Y_\alpha. \end{aligned}$$

Finally, taking the limit as $t \to \infty$ yields

$$\begin{aligned} \lim_{t\to\infty} (f_t)_{*Q_0}(X_\alpha) &= 0, \\ \lim_{t\to\infty} (f_t)_{*Q_0}(Y_\alpha) &= 0. \end{aligned}$$

Therefore, $\lim_{t\to\infty}(f_t)_{*Q_0}\pi_{\vec{\lambda}} = 0$.

We will now compute the limit of $\pi_{\vec{\lambda}}$ along the orbit through Q_1 of the 1-parameter subgroup g_t. Again, the tangent space at each point Tk will be identified with $\mathfrak{k}/\mathfrak{t}$ via right multiplication by k^{-1}. As in the previous computation, $t \mapsto (g_t)_*(\pi_{\vec{\lambda}})$ can be thought of as a path through $\wedge^2(\mathfrak{k}/\mathfrak{t})$. First, let

$$C_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathbf{i} \\ \mathbf{i} & 1 \end{pmatrix}.$$

For each $t > 0$, let

$$k_t := p_K^r(C_1 g_t).$$

Define a map $h_t : T\backslash K \to T\backslash K$ by

$$h_t(Tk) := Tp_K^r(kC_1g_t)k_t^{-1} = Tp_K^r(kp_{AN}^l(C_1g_t)).$$

As above, h_t fixes the point Q_0, so that $(h_t)_{*Q_0}$ maps $\mathfrak{k}/\mathfrak{t}$ to itself; the lift $\tilde{h}$ of h to K fixes e, so that $(\tilde{h}_t)_{*Q_0}$ maps $\mathfrak{k}$ to $\mathfrak{k}$. Using (3.5.6), a computation gives

$$\begin{aligned} (\tilde{h}_t)_{*e}(X_\alpha) &= \frac{1}{c^2+s^2}X_\alpha, \\ (\tilde{h}_t)_{*e}(Y_\alpha) &= \frac{1}{c^2+s^2}Y_\alpha, \end{aligned}$$

which gives

$$\begin{aligned} (h_t)_{*Q_0}(X_\alpha) &= \frac{1}{(c+s)^2}X_\alpha, \\ (h_t)_{*Q_0}(Y_\alpha) &= \frac{1}{(c+s)^2}Y_\alpha. \end{aligned}$$

Taking the limit as $t \to \infty$ yields

$$\lim_{t\to\infty} (h_t)_{*Q_0}(X_\alpha) = 0,$$

$$\lim_{t\to\infty} (h_t)_{*Q_0}(Y_\alpha) = 0.$$

Therefore, $\lim_{t\to\infty}(h_t)_{*Q_0}\pi_{\vec{\lambda}} = 0$.

We now extend $\pi_{\vec{\lambda}}$ to all of $T\backslash K$ by setting

$$\pi_{\vec{\lambda}}(q) = \Pi_v \text{ for all } q \in Q_2.$$

By the computations above, $\pi_{\vec{\lambda}}$ is continuous at Q_2, and hence on all of Ω_2. We wish to show that it is smooth on Ω_2.

Lemma 3.5.7. *The union L of the curve $g_t.Q_0$, the curve $g_t.Q_1$, and the point $Q_2 = \lim_{t\to\infty} g_t.Q_0$ is a smooth, 1-dimensional submanifold of $T\backslash K$. Furthermore, it is transversal to Ω_2.*

Proof. In the coordinates given by (3.5.5), L is the half of the great circle $\{x = 0\}\bigcap S^2$ with $y < 0$. On the other hand, Ω_2 is the great circle $\{z = 0\}\bigcap S^2$. It is obvious that $T_{Q_2}\Omega_2 \oplus T_{Q_2}C = T_{Q_2}S^2$. □

Theorem 3.5.8. *The Poisson structure $\pi_{\vec{\lambda}}$ on the union of the G_0-orbits Ω_0 and Ω_1 in $T\backslash K$ extends smoothly to Ω_2, and thus to all of $T\backslash K$.*

Proof. Since the G_0-action is smooth, it suffices to show that $\pi_{\vec{\lambda}}$ is smooth at Q_2. By Lemma 3.5.7, there exist coordinates (x, y) in a neighborhood U of Q_2 such that

$\Omega_2 \bigcap U$ forms the x-axis and $C \bigcap U$ forms the y-axis. Suppose $\pi_{\vec{\lambda}} = f \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ in these coordinates. Since $\pi_{\vec{\lambda}}$ is defined on Ω_2 by its value at Q_2, by smoothness of the G_0-action on Ω_2, $\frac{\partial^n f}{dx^n}$ exists for all n. If we can show that $\pi_{\vec{\lambda}}$ varies smoothly along C, then $\frac{\partial^n f}{dy^n}$ exists for all n. It will follow, then, that $\pi_{\vec{\lambda}}$ is smooth on U, hence on Ω_2, hence on all of $T \backslash K$.

We first reparameterize so that the limit Q_2 corresponds to a finite value of t. Let

$$\phi(t) = \begin{cases} \frac{1}{(\cosh\left(\frac{1}{t}\right)+\sinh\left(\frac{1}{t}\right))^2} = e^{\frac{-1}{2t}} & \text{if } t < 0 \\ 0 & \text{if t=0} \\ \left(\frac{1}{\cosh^2\left(\frac{1}{t}\right)+\sinh^2\left(\frac{1}{t}\right)}\right)\left(\frac{1}{t}\right) = \frac{1}{\cosh\left(\frac{1}{t}\right)} & \text{if } t > 0. \end{cases} .$$

Then $c_1(t) := \phi(t) X_\alpha$ and $c_2(t) := \phi(t) Y_\alpha$ define smooth curves in $\mathfrak{k}/\mathfrak{t}$ which are smooth if and only if $\phi(t)$ is smooth. Moreover, if c_1 and c_2 are smooth, it follows that $\pi_{\vec{\lambda}}$ varies smoothly along C. Using induction and the definition of the derivative, it can be shown that $\phi(t)$ is smooth at $t = 0$, with $\phi^{(n)}(t) = 0$. (This implies that ϕ is not analytic). Therefore, $\phi(t)$ is smooth for all t, which implies that $\pi_{\vec{\lambda}}$ varies smoothly along C. □

In holomorphic coordinates

$$T\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \mapsto \left(\frac{\alpha}{\beta} , \frac{\overline{\alpha}}{\overline{\beta}}\right) =: (z , \overline{z}),$$

In these coordinates, we have

$$\Pi_v = \mathbf{i}(1 - |z^2|)|z|^2 \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \overline{z}},$$

and

$$\pi_{\vec{\lambda}} = \frac{\mathbf{i}}{2}(1 - |z|^2)\left(-\frac{1}{1 - e^{2\lambda}} + \left(\frac{1}{1 - e^{2\lambda}} + 2\right)|z|^2\right)\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \overline{z}},$$

which agrees with the formula in [KR93] for an arbitrary linear combination of Π_v and the G_0-invariant Poisson structure π_0 on $T\backslash K$.

3.5.3. The $SU(1,2)$ Case

The Weyl group of $SL(3,\mathbb{C})$ is $W \cong S_3$. As in Remark 3.5.5, let s_{ij} be the reflection corresponding to the root $\alpha_{ij} = e_i - e_j$. For example, the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a representative in K for $s_{12} \in W$.

In the $SU(1,2)$ case, $k = 0$ or $k = 1$. When $k = p = 1$, $C_k K_0^2 C_k^{-1} = K_0 = S(U(1) \times U(2))$. Therefore, W_1 is the subgroup generated by the reflection s_{23}. In order to compute W_0, we will use the fact that representatives of Weyl group elements in G are monomial matrices. Since $C_k = Id$, W_0 consists of Weyl group elements that have representatives in K_0^s. Note that if g is a representative in K_0^s for some $w \in W$, then $C_1 g C_1^{-1} \in K_0 \subset G_0$. Let us consider, for example, the Weyl group element s_{12}. Any representative in G for s_{12} is a monomial matrix of the form

$$g = \begin{pmatrix} 0 & x & 0 \\ y & 0 & 0 \\ 0 & 0 & z \end{pmatrix}. \tag{3.5.7}$$

If $C_1 g C_1^{-1} \in G_0$, then $(J(C_1 g C_1^{-1})^* J)(C_1 g C_1^{-1}) = Id$. However, using the expression (3.5.7),

$$(J(C_1 g C_1^{-1})^* J)(C_1 g C_1^{-1}) = \frac{1}{2} \begin{pmatrix} -|y|^2 & -\frac{\bar{z}x}{\sqrt{2}} & -\mathbf{i}|y|^2 \\ \frac{\bar{x}z}{\sqrt{2}} & 0 & -\frac{\bar{x}z}{\sqrt{2}} \\ -\mathbf{i}|y|^2 & \frac{\bar{z}x}{\sqrt{2}} & |y|^2 \end{pmatrix}.$$

Therefore, s_{12} does not have a representative in K_0^s. Hence, $s_{12} \notin W_0$. By a series of similar computations, one finds that W_0 is the 2-element subgroup generated by the reflection s_{13}. To simplify notation, let

$$D_{12} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathbf{i} & 0 \\ \mathbf{i} & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}, \quad D_{23} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & \mathbf{i} \\ 0 & \mathbf{i} & 1 \end{pmatrix}, \quad D_{13} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & \mathbf{i} \\ 0 & \sqrt{2} & 0 \\ \mathbf{i} & 0 & 1 \end{pmatrix}.$$

Then the $SU(1,2)$-orbits on $T\backslash K$ are parameterized by the points

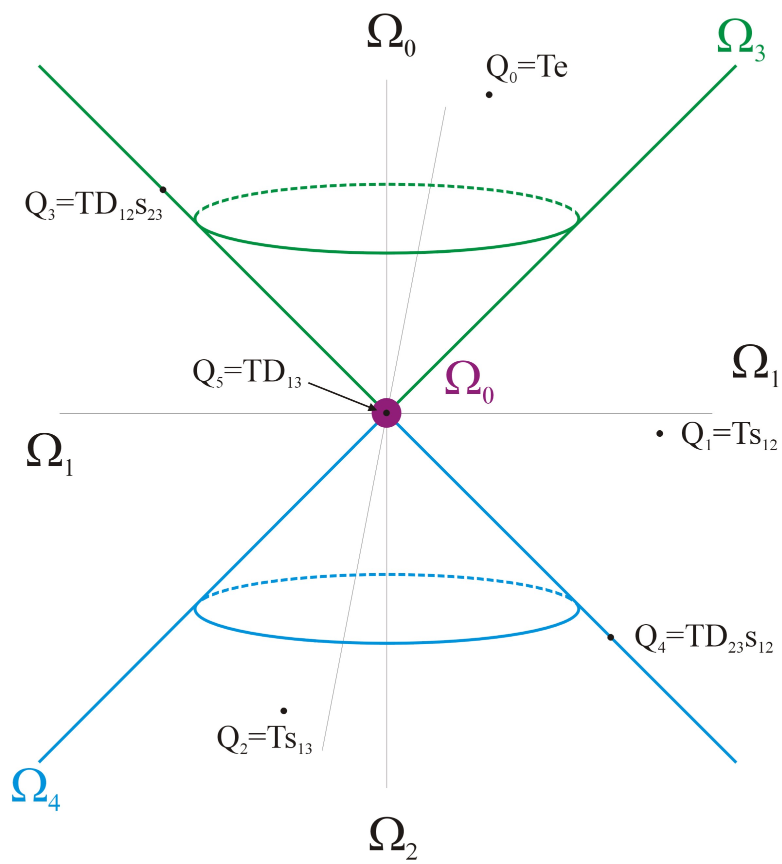
$$\begin{aligned} Q_0 &= Te\hat{C}_1^{-1} = Te \\ Q_1 &= Ts_{12}\hat{C}_1^{-1} = Ts_{12} \\ Q_2 &= Ts_{13}\hat{C}_1^{-1} = Ts_{13} \\ Q_3 &= Ts_{23}\hat{C}_0^{-1} = Ts_{23}D_{13} = TD_{12}s_{23} \\ Q_4 &= Ts_{12}\hat{C}_0^{-1} = Ts_{12}D_{13} = TD_{23}s_{12} \\ Q_5 &= Te\hat{C}_0^{-1} = TD_{13}. \end{aligned}$$

Let Ω_i denote the G_0-orbit through Q_i for $0 \leq i \leq 5$. The dimensions of these orbits can be computed using (3.5.5). In this case, $w_0 = s_{13}$, and $w_b = e$. Using the fact that the Satake diagram for $\mathfrak{su}(1,2)$ has no black dots (see [Bum04], for example), easy computations give

$$\begin{aligned} \dim(\Omega_0) &= \dim(\Omega_1) = \dim(\Omega_3) = 6 \\ \dim(\Omega_3) &= \dim(\Omega_4) = 5 \\ \dim(\Omega_5) &= 3. \end{aligned}$$

The decomposition of $T\backslash SU(3)$ into $SU(1,2)$-orbits is illustrated in Figure 3.2. Note that this representation is not completely accurate. It is only intended to lend geometric intuition to the argument that follows. The description in [Mat06] of the $SU(2,1)$-orbits on K/T was helpful in constructing this model.

FIGURE 3.2. $SU(1,2)$-Orbits on $T\backslash K$



By Section 3.4.2, for each admissible $\vec{\lambda}$, the Poisson structure $\pi_{\vec{\lambda}}$ on $T\backslash G_0$ can be pushed forward onto the orbit Ω_0 through Te. As in the $SU(1,1)$ case, we define $\pi_{\vec{\lambda}}$

on the other open orbits Ω_1 and Ω_3 by identification with Ω_0.

Theorem 3.5.9. *The Poisson structure $\pi_{\vec{\lambda}}$ on the union of the open G_0-orbits in $T\backslash K$ extends smoothly to all of $T\backslash K$.*

The proof of Theorem 3.5.9 will follow the same scheme as the proof for the $SU(1,1)$ case. Using a 1-parameter subgroup of G_0, we will construct smooth, 1-dimensional submanifolds which pass through points in each of the non-open orbits in $T\backslash K$. The Poisson structure $\pi_{\vec{\lambda}}$ will be defined on each of these orbits by its limit along the corresponding 1-dimensional submanifold. We will therefore need three 1-dimensional submanifolds L_3, L_4 and L_5 corresponding to the orbits Ω_3, Ω_4, and Ω_5, respectively. In the computations below, the tangent space to every point $Tk \in T\backslash K$ will be identified with $\mathfrak{k}/\mathfrak{t}$ via right-translation by k^{-1}. Consider the 1-parameter subgroup

$$g_t^{13} := \begin{pmatrix} c & 0 & -\mathbf{i}s \\ 0 & 1 & 0 \\ \mathbf{i}s & 0 & c \end{pmatrix}$$

in G_0. Let

$$Q_6 := Ts_{12}, \quad Q_7 := Ts_{12}s_{23}, \quad Q_8 := Ts_{23}s_{12}.$$

Note that $Q_6 \in \Omega_0$, $Q_7 \in \Omega_1$, and $Q_8 \in \Omega_2$. We have

$$\lim_{t\to\infty} g_t^{13}.Q_0 = \lim_{t\to\infty} g_t^{13}.Q_2 = Q_5$$

$$\lim_{t\to\infty} g_t^{13}.Q_1 = \lim_{t\to\infty} g_t^{13}.Q_8 = Q_4$$

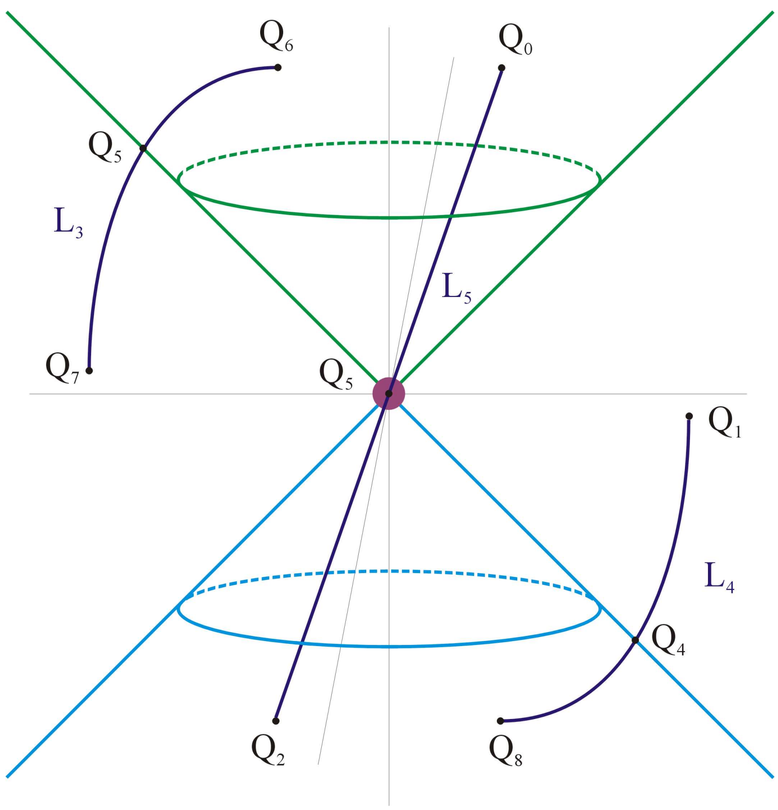
$$\lim_{t\to\infty} g_t^{13}.Q_6 = \lim_{t\to\infty} g_t^{13}.Q_7 = Q_3.$$

Let

$$\begin{aligned} L_3 &:= \left(\bigcup_{0\le t<\infty} g_t^{13}.Q_6\right)\bigcup Q_3 \bigcup \left(\bigcup_{0\le t<\infty} g_t^{13}.Q7\right)\\ L_4 &:= \left(\bigcup_{0\le t<\infty} g_t^{13}.Q_8\right)\bigcup Q_4 \bigcup \left(\bigcup_{0\le t<\infty} g_t^{13}.Q8\right)\\ L_5 &:= \left(\bigcup_{0\le t<\infty} g_t^{13}.Q_0\right)\bigcup Q_5 \bigcup \left(\bigcup_{0\le t<\infty} g_t^{13}.Q2\right). \end{aligned}$$

(See Figure 3.3.)

FIGURE 3.3. Paths through Lower-Dimensional Orbits



Lemma 3.5.10. *The sets L_3,L_4, and L_5 are smooth, 1-dimensional submanifolds of $T\backslash K$.*

Proof. Let

$$k_\theta^{13} = \begin{pmatrix} \cos\theta & 0 & \mathbf{i}\sin\theta \\ 0 & 1 & 0 \\ \mathbf{i}\sin\theta & 0 & \cos\theta \end{pmatrix},\ 0 \le \theta < 2\pi,$$

a 1-parameter subgroup in K. For each $0 \le t < \infty$, $p_K^r(g_t^{13}) = k_\theta$ for some $0 \le \theta < \frac{\pi}{4}$, and $p_K^r(g_t^{13}s_{13}) = k_\theta$ for some $\frac{\pi}{4} \le \theta < \frac{\pi}{2}$. Thus, L is the curve $\theta \mapsto Tk_\theta^{13}, 0 \le \theta < \frac{\pi}{2}$, which is obviously smooth.

Similarly, L_4 is the curve $\theta \mapsto Tk_\theta^{12}s_{23}$, and L_3 is the curve $\theta \mapsto Tk_\theta^{23}s_{12}$, where

$$k_\theta^{12} = \begin{pmatrix} \cos\theta & \mathbf{i}\sin\theta & 0 \\ \mathbf{i}\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},\ k_\theta^{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \mathbf{i}\sin\theta \\ 0 & \mathbf{i}\sin\theta & \cos\theta \end{pmatrix},$$

with $0 \le \theta < \frac{\pi}{2}$. □

Lemma 3.5.11. *The tangent space $T_{Q_i}L_i$ is not contained in $T_{Q_i}\Omega_i$, for $i = 3, 4, 5$.*

Proof. Straightforward computations show that the tangent space at Q_5 to the 3-dimensional G_0-orbit Ω_5 is spanned by the matrices

$$\left\{ \begin{pmatrix} 0 & -a-\mathbf{i}b & -c \\ a-\mathbf{i}b & 0 & b+\mathbf{i}a \\ c & -b+\mathbf{i}a & 0 \end{pmatrix} \ :\ a,b,c \in \mathbb{R} \right\}.$$

On the other hand,

$$T_{Q_5}L_5 = \left\{ \begin{pmatrix} 0 & 0 & \mathbf{i}x \\ 0 & 0 & 0 \\ \mathbf{i}x & 0 & 0 \end{pmatrix} \ :\ x \in \mathbb{R} \right\}.$$

It is clear that $T_{Q_5}L_5 \not\subseteq T_{Q_5}\Omega_5$.

For the case $i = 4$, the tangent space at Q_4 to the 3-dimensional G_0-orbit Ω_4 is spanned by the matrices

$$\left\{ \begin{pmatrix} 0 & \alpha & \beta \\ -\overline{\alpha} & 0 & x \\ \overline{\beta} & -x & 0 \end{pmatrix} \ : \ \alpha, \beta \in \mathbb{C}, \ x \in \mathbb{R} \right\}.$$

On the other hand,

$$T_{Q_4} L_4 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbf{i}x \\ 0 & \mathbf{i}x & 0 \end{pmatrix} \ : \ x \in \mathbb{R} \right\}.$$

It is clear that $T_{Q_4} L_4 \nsubseteq T_{Q_4} \Omega_4$.

The computations are similar for $i = 3$. □

Corollary 3.5.12. *Since Ω_3 and Ω_4 have codimension 1 in $T \backslash K$, Lemma 3.5.11 implies that L_3 and L_4 are transversal to Ω_3 and Ω_4, respectively.*

Proof of Theorem 3.5.9. A series of (extremely tedious) computations very similar to those in Section 3.5.2 shows that for $1 \leq i, j \leq n$, $0 \leq m \leq 8$,

$$\lim_{t \to \infty} (g_t^{13})_{*Q_m} (X_{ij}) = \lim_{t \to \infty} (g_t^{13})_{*Q_m} (Y_{ij}) = 0,$$

and that each X_{ij} and Y_{ij} varies smoothly along each L_m. Therefore, on Ω_3, Ω_4, and Ω_5, define $\pi_{\vec{\lambda}} = \pi_K$. Then as in the proof of Theorem 3.5.8, it follows from Corollary 3.5.12 and from the smoothness of the G_0 action that $\pi_{\vec{\lambda}}$ is smooth on $(T \backslash K) \setminus \Omega_5$.

To prove that $\pi_{\vec{\lambda}}$ is smooth on Ω_5, we will use L_5 to construct two more 1-dimensional submanifolds passing through Q_5 and thus obtain two more "directions" along which $\pi_{\vec{\lambda}}$ varies smoothly. First, note that

$$\mathrm{Stab}_{\mathfrak{g}_0} \left(g_t^{13} \right) = \left\{ \begin{pmatrix} \mathbf{i}z & 0 & \mathbf{i}r \\ 0 & -2\mathbf{i}z & 0 \\ -\mathbf{i}r & 0 & \mathbf{i}z \end{pmatrix} \ : \ r, z \in \mathbb{R} \right\}.$$

On the other hand,

$$\mathrm{Stab}_{\mathfrak{g}_0}(Q_5) = \left\{ \begin{pmatrix} \mathbf{i}(a+c) & \beta & c+\mathbf{i}b \\ \overline{\beta} & -2\mathbf{i}a & -\mathbf{i}\overline{\beta} \\ c-\mathbf{i}b & -\mathbf{i}\beta & \mathbf{i}(a-c) \end{pmatrix} : \ w, z \in \mathbb{R}, \ \eta, \beta \in \mathbb{C} \right\}.$$

Now choose elements $V_1, V_2 \in \mathrm{Stab}_{\mathfrak{g}_0}(Q_5) \setminus \mathrm{Stab}_{\mathfrak{g}_0}(g_t^{13})$. For example, we could choose

$$V_1 := \begin{pmatrix} \mathbf{i} & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -\mathbf{i} \end{pmatrix}, \quad V_2 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -\mathbf{i} \\ 0 & -\mathbf{i} & 0 \end{pmatrix}.$$

Then $L_5^1 := \exp(V_1).L_5$ and $L_5^1 := \exp(V_2).L_5$ are smooth, 1-dimensional submanifolds containing Q_5. Since $\exp(V_1)_{*Q_5}$ is an automorphism of $T_{Q_5}\Omega_5$, $T_{Q_5}L_5^1$ and $T_{Q_5}L_5^2$ are not contained in $T_{Q_5}\Omega_5$. Moreover, since $V_1, V_2 \notin \mathrm{Stab}_{\mathfrak{g}_0}(g_t^{13})$, $T_{Q_5}L_5^1$ and $T_{Q_5}L_5^2$ are not equal to $T_{Q_5}L_5$. Finally, the fact that $\pi_{\vec{\lambda}}$ varies smoothly along L_5 implies that it also varies smoothly along L_5^1 and L_5^2. It follows that there exist coordinates $x_1, \ldots, x_6$ in a neighborhood U of Q_5 such that $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \in T_{Q_5}\Omega_5$, $\frac{\partial}{\partial x_4} \in T_{Q_5}L_5$, $\frac{\partial}{\partial x_5} \in T_{Q_5}L_5^2$, $\frac{\partial}{\partial x_4} \in T_{Q_5}L_5^2$. By the smoothness of the G_0 action, the coefficient functions of $\pi_{\vec{\lambda}}$ on U are infinitely differentiable with respect to x_1, x_2, and x_3; since $\pi_{\vec{\lambda}}$ varies smoothly along L_5, L_5^1, and L_5^2, the coefficient functions are also infinitely differentiable with respect to x_3, x_4, and x_5. Thus $\pi_{\vec{\lambda}}$ is smooth at Q_5 and, again by the smoothness of the G_0 action, on all of Ω_5. □

3.6. The $SU(1,1)$ Case

In this section, we will show that the Ginzburg-Weinstein theorem (restricted to the admissible orbits) holds in the $SU(1,1)$ case. We will construct a Poisson isomorphism between AN_{adm}, endowed with the Lu-Weinstein Poisson structure, and an open subset of $(\mathfrak{a}+\mathfrak{n})_{adm}$, endowed with the Lie-Poisson structure induced by the Lie bracket on $\mathfrak{su}(1,1)$. In fact, we will do so in two different ways: first, by following the approach of [GW92], and second, by adapting the method of [FR96] for the $SU(2)$ case.

We begin by recording the relevant formulas for this case. Recall that the Lu-Weinstein Poisson tensor on AN corresponding to the Manin triple $(\mathfrak{su}, \mathfrak{a}+\mathfrak{n}, \mathfrak{sl}_2(C))$ is given by the following formula. For $u \in AN$, $X, Y \in T_e^*AN \cong \mathfrak{su}(1,1)$,

$$((r_{u^{-1}})_*\pi_{AN}(u))\,(X,Y) = \Im\langle p_{\mathfrak{su}(1,1)}(\mathrm{Ad}_{u^{-1}}X), p_{\mathfrak{a}+\mathfrak{n}}(\mathrm{Ad}_{u^{-1}}Y)\rangle,$$

where $\Im\langle\cdot,\cdot\rangle$ is the imaginary part of the trace form, and $p_{\mathfrak{su}(1,1)}$ and $p_{\mathfrak{a}+\mathfrak{k}}$ are the natural projections onto $\mathfrak{su}(1,1)$ and $\mathfrak{a}+\mathfrak{n}$, respectively. A somewhat laborious computation shows that, using global coordinates

$$u = \begin{pmatrix} e^{\frac{z}{2}} & x+iy \\ 0 & e^{-\frac{z}{2}} \end{pmatrix} \mapsto (x,y,z)$$

on AN, the Lu-Weinstein Poisson tensor is given by

$$\pi_{AN}(u) = (-\sinh z)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}.$$

A Casimir for this Poisson structure is

$$\xi(x,y,z) = x^2+y^2-2\cosh z,$$

and the symplectic leaves given by $x^2+y^2-2\cosh z = k$ are hyperboloid-like surfaces in 3-space. As shown in Example 2.2.5, using coordinates

$$\begin{pmatrix} \frac{z}{2} & x+\mathbf{i}y \\ 0 & -\frac{z}{2} \end{pmatrix} \leftrightarrow (x,y,z),$$

the linear Poisson structure on $\mathfrak{a}+\mathfrak{n}$ is

$$\pi_0 = -z\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}.$$

In the $p=q=1$ case, the fixed point set $\mathfrak{g}^\dagger$ is

$$\mathfrak{q} = \left\{ \begin{pmatrix} z & x+iy \\ -x+iy & -z \end{pmatrix} \mid x,y,z\in\mathbb{R} \right\},$$

with global coordinates (x,y,z). The admissible cone $\mathfrak{q}_{adm}$ in this case is determined by $z > \sqrt{x^2+y^2}$. The fixed point set of $\dagger$ at the group level is

$$Q = G^\dagger = \left\{ \begin{pmatrix} c & \beta \\ -\bar{\beta} & d \end{pmatrix} \mid c,d\in\mathbb{R},\ cd+|\beta|^2 = 1 \right\}.$$

For the open subset of Q where $c\neq 0$, one can define coordinates

$$(a,b,c) \leftrightarrow \begin{pmatrix} c & a+ib \\ -a+ib & \frac{1-a^2-b^2}{c} \end{pmatrix}.$$

In these coordinates, $Sym : AN \to Q$ is given by

$$\begin{array}{rcl} a &=& e^{\frac{z}{2}}x \\ b &=& e^{\frac{z}{2}}y \\ c &=& e^z \end{array} \qquad \begin{array}{rcl} x &=& \frac{a}{\sqrt{c}} \\ y &=& \frac{b}{\sqrt{c}} \\ z &=& \ln c. \end{array}$$

Using (a,b,c) coordinates, the Poisson tensor π_{AN} pushes forward under the Sym map to

$$\pi_Q = \frac{1}{2}(1-a^2-b^2-c^2)\frac{\partial}{\partial a}\wedge\frac{\partial}{\partial b} + bc\frac{\partial}{\partial c}\wedge\frac{\partial}{\partial a} + ac\frac{\partial}{\partial b}\wedge\frac{\partial}{\partial c},$$

with Casimir

$$F(a,b,c) = \frac{1+c^2-a^2-b^2}{c}.$$

For a constant k, the level set $F = k$ is the hyperboloid

$$\left(c + \frac{1}{2k}\right)^2 - a^2 - b^2 = \left(\frac{1}{2k}\right)^2 - 1.$$

The log map from Q_{adm} to $\mathfrak{q}_{adm}$ is given by

$$(a,b,c) \mapsto \left(\frac{\lambda}{\sinh(\lambda)}a, \frac{\lambda}{\sinh(\lambda)}b, \frac{\lambda c - \lambda\cosh(\lambda)}{\sinh(\lambda)}\right) = (x,y,z).$$

The image of the level set $F = k$ under the log map is the coadjoint orbit $\theta_{\vec{\lambda}} := \{(x,y,z) \mid \sqrt{z^2 - x^2 - y^2} = \lambda\}$, where $k = 2\cosh\lambda$. In (x,y,z) coordinates on $\mathfrak{q}$,

$$\begin{aligned} \pi := \log_*(Sym_*(\pi_{AN})) &= -z((\coth\lambda)\lambda + z)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \\ &\quad + y((\coth\lambda)\lambda + z)\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} \\ &\quad + x((\coth\lambda)\lambda + z)\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}. \end{aligned}$$

It is easy to see that

$$f(x,y,z) = z^2 - x^2 - y^2$$

is a Casimir for π.

For convenience, the computations in this section will be carried out in coordinates on $\mathfrak{q}_{adm}$ rather than on $(\mathfrak{a}+\mathfrak{n})_{adm}$. Under the map $sym : \mathfrak{a}+\mathfrak{n} \rightarrow: X^\dagger + X$, π_0 pushes forward to the tensor

$$-z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z},$$

which we also denote by π_0. Therefore, the symplectic leaves of π and π_0 are the same.

3.6.1. The Ginzburg-Weinstein Approach

Following the Ginzburg-Weinstein argument, define a bivector field π_t on $\mathfrak{q}$ by

$$\pi_t(\vec{v}) := \frac{\pi_t(t\vec{v})}{t},$$

where the expression on the right-hand side is identified with an element of $\wedge^2(T_{\vec{v}}\mathfrak{q})$ by translation. Now set

$$\dot{\pi}_t := \frac{d}{dt}\pi_t$$

and

$$\dot{\pi} := \frac{d}{dt}\,\pi_t\big|_{t=1}.$$

In coordinates,

$$\begin{aligned}\dot{\pi} \;=\; & -z\left(\lambda\coth\lambda + z - \frac{\lambda^2}{\sinh^2\lambda}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} \\ & y\left(\lambda\coth\lambda + z - \frac{\lambda^2}{\sinh^2\lambda}\right)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x} \\ & x\left(\lambda\coth\lambda + z - \frac{\lambda^2}{\sinh^2\lambda}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}.\end{aligned}$$

Proposition 3.6.1. *There exists a vector field X on $\mathfrak{q}_{adm} \cong (\mathfrak{a}+\mathfrak{n})_{adm}$ such that*

(1) $[X,\pi] = \dot{\pi}$,

(2) X has the zero linearization at the origin,

(3) X is tangent to the (generic) coadoint orbits in $\mathfrak{q}_{adm}$, and

(4) X is complete.

Note that in the compact case considered by Ginzburg and Weinstein, completeness follows from the fact that X is tangent to the symplectic leaves of π. For $SU(p,q)$, completeness must be shown by other means.

Proof. It will be convenient to convert to *hyperbolic coordinates* (λ, ϕ, s). The relations between rectangular and hyperbolic coordinates are

$$\begin{array}{rclcrcl} x &=& \lambda(\sinh s)(\cos\phi) & \qquad & \lambda &=& \sqrt{z^2 - x^2 - y^2} \\ y &=& \lambda(\sinh s)(\sin\phi) & & \phi &=& \arctan\left(\frac{y}{x}\right) \\ z &=& \lambda(\cosh s) & & s &=& \cosh^{-1}\left(\frac{z}{\lambda}\right). \end{array}$$

In these coordinates

$$\pi = \frac{1}{\sinh s}\left(\coth\lambda + \cosh s\right)\frac{\partial}{\partial\phi}\wedge\frac{\partial}{\partial s},$$

and

$$\dot{\pi} = \frac{1}{\sinh s}\left(\coth\lambda + \cosh s - \frac{\lambda}{\sinh^2\lambda}\right)\frac{\partial}{\partial\phi}\wedge\frac{\partial}{\partial s}.$$

Set

$$\begin{aligned} g(s) &:= \frac{1}{\sinh s}\left(\coth\lambda + \cosh s\right), \\ h(s) &:= \frac{1}{\sinh s}\left(\coth\lambda + \cosh s - \frac{\lambda}{\sinh^2\lambda}\right). \end{aligned}$$

The action of the torus $T \subset G_0$ on $\mathfrak{a} + \mathfrak{n}$ corresponds to rotation about the z-axis in $\mathfrak{q}$. Since π, and hence $\dot{\pi}$, are invariant under the torus action, we may assume that X

has the form $f(t)\frac{\partial}{\partial t}$, where $f(t)$ does not depend on ϕ. Then the equation $[X,\pi]=\dot{\pi}$ reduces to the ODE

$$f\frac{\partial g}{\partial s}-g\frac{\partial f}{\partial s}=h.$$

Rewriting the left-hand side using the quotient rule gives $-\frac{\partial}{\partial s}\left(\frac{f}{g}\right)\cdot g^2=h$, or, equivalently,

$$f=-g\cdot\int\frac{h}{g^2}\,ds. \tag{3.6.1}$$

Integrating by substitution, we obtain

$$\int\frac{h}{g^2}\,ds=\ln(\coth\lambda+\cosh s)+\frac{\lambda}{\sinh^2\lambda}\left(\frac{1}{\coth\lambda+\cosh s}\right)+C, \tag{3.6.2}$$

where C is constant with respect to s and ϕ. Setting $s=0$, (3.6.2) becomes

$$\ln(\coth\lambda+1)+\frac{\lambda}{\sinh^2\lambda}\left(\frac{1}{\coth\lambda+1}\right)+C.$$

Note that $g\to\infty$ as $t=0$. Therefore, to ensure smoothness when $s=0$, set $C=\ln(\coth\lambda+1)+\frac{\lambda}{\sinh^2\lambda}\left(\frac{1}{\coth\lambda+1}\right)$. Then the vector field

$$\begin{aligned}X &= -\left(\frac{1}{\sinh s}(\coth\lambda+\cosh s)\right)\cdot\left[\ln(\coth\lambda+\cosh s)+\frac{\lambda}{\sinh^2\lambda}\left(\frac{1}{\coth\lambda+\cosh s}\right)\right.\\ &\quad\left.-\left(\ln(\coth\lambda+1)+\frac{\lambda}{\sinh^2\lambda}\left(\frac{1}{\coth\lambda+1}\right)\right)\right]\frac{\partial}{\partial s}\\ &= -\left(\frac{\coth\lambda+\cosh s}{\sinh s}\right)\cdot\\ &\quad\left[\ln\left(\frac{\coth\lambda+\cosh s}{\coth\lambda+1}\right)+\frac{\lambda}{\sinh^2\lambda}\left(\frac{1}{\coth\lambda+\cosh s}-\frac{1}{\coth\lambda+1}\right)\right]\frac{\partial}{\partial s}\end{aligned}$$

extends smoothly to the positive z-axis (where it vanishes). This vector field is smooth on the open cone $z>\sqrt{x^2+y^2}$, extends continuously to the boundary $z=\sqrt{x^2+y^2}$, and satisfies $[X,\pi]=\dot{\pi}$.

We will now show that X has the zero linearization at the origin. Let

$$
\begin{aligned}
f \quad := \quad & -\left(\frac{\coth\lambda + \cosh s}{\sinh s}\right) \\
& \cdot \left[\ln\left(\frac{\coth\lambda + \cosh s}{\coth\lambda + 1}\right) + \frac{\lambda}{\sinh^2\lambda}\left(\frac{1}{\coth\lambda + \cosh s} - \frac{1}{\coth\lambda + 1}\right)\right],
\end{aligned}
$$

so that $X = f\frac{\partial}{\partial s}$. Since $\lambda = z = 0$ at the origin in $\mathbb{R}^3$, we can compute the linearization of f at $\vec{0}$ in terms of the variables z and λ. Note that

$$
\begin{aligned}
\cosh s &= \frac{z}{\lambda}, \\
\sinh s &= \frac{\sqrt{x^2+y^2}}{\lambda}.
\end{aligned}
$$

Also, for λ near 0, $\coth(\lambda) \sim \frac{1}{\lambda}$, and $\sinh(\lambda) \sim \lambda$. Then for λ, z near 0,

$$
\begin{aligned}
f &\sim \left(\frac{\frac{1}{\lambda} + \frac{z}{\lambda}}{\frac{\sqrt{x^2+y^2}}{\lambda}}\right) \cdot \left[\ln\left(\frac{\frac{1}{\lambda} + \frac{z}{\lambda}}{\frac{1}{\lambda} + 1}\right) + \frac{1}{\lambda}\left(\frac{1 - \frac{z}{\lambda}}{\left(\frac{1}{\lambda} + \frac{z}{\lambda}\right)\left(\frac{1}{\lambda} + 1\right)}\right)\right] \\
&\sim \frac{1+z}{\sqrt{x^2+y^2}}\left[\ln\left(\frac{1+z}{1+\lambda}\right) + \frac{\lambda - z}{(1+z)(1+\lambda)}\right] \\
&\sim \frac{1}{\sqrt{x^2+y^2}}\left[\left(z - \frac{z^2}{2} - \lambda + \frac{\lambda^2}{2}\right) + \left(\lambda - z + z^2 - \lambda^2\right)\right] \\
&= \frac{1}{\sqrt{x^2+y^2}}\left[\frac{z^2 - \lambda^2}{2}\right] \\
&= \frac{\sqrt{x^2+y^2}}{2}.
\end{aligned}
$$

Therefore, since

$$
\frac{\partial}{\partial s} = \frac{zx}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{zy}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y} + \sqrt{x^2+y^2}\frac{\partial}{\partial z},
$$

$$
X \sim zx\frac{\partial}{\partial x} + zy\frac{\partial}{\partial y} + \left(x^2+y^2\right)\frac{\partial}{\partial z}
$$

near $\vec{o}$. Thus, X has the zero linearization at the origin, as desired.

Finally, we wish to show that X is complete, so we will examine it's behavior for a fixed λ as $s \to \infty$. Note that as $s \to \infty$, $X \sim s\frac{\partial}{\partial s}$. We now project the upper sheet of the hyperboloid $\lambda^2 = z^2 - x^2 - y^2$ to the unit disk. This map is given by

$$\begin{aligned} r &= \sqrt{\tfrac{z-\lambda}{z+\lambda}} & x &= \left(\tfrac{2r\lambda}{1-r^2}\right)(\cos\theta) \\ \theta &= \arctan\left(\tfrac{y}{x}\right) & y &= \left(\tfrac{2r\lambda}{1-r^2}\right)(\sin\theta) \\ & & z &= \lambda\left(\tfrac{1+r^2}{1-r^2}\right). \end{aligned}$$

Note that $s \to \infty$ corresponds to $r \to 1$ in these coordinates. In rectangular coordinates,

$$s\frac{\partial}{\partial s} = \cosh^{-1}\left(\frac{z}{\sqrt{z^2-x^2-y^2}}\right)\left(\frac{zx}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{zy}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y} + \sqrt{x^2+y^2}\frac{\partial}{\partial z}\right),$$

and in (r,θ) coordinates

$$s\frac{\partial}{\partial s} = \left(\cosh^{-1}\left(\frac{1+r^2}{1-r^2}\right)\right)\left(\frac{1-r^2}{2}\right)\frac{\partial}{\partial r}.$$

Setting $u := (1-r^2)$, $v := \frac{1}{2u}$ and using L'Hopital's rule,

$$\begin{aligned} \lim_{r\to 1}\left(\cosh^{-1}\left(\frac{1+r^2}{1-r^2}\right)\right)\left(\frac{1-r^2}{2}\right) &= \lim_{u\to 0}\left(\frac{u}{2}\right)\cosh^{-1}\left(\frac{2}{u}\right) \\ &= \lim_{v\to\infty}\frac{\cosh^{-1}(v)}{v} \\ &= \lim_{v\to\infty}\frac{\frac{1}{\sqrt{v^2-1}}}{1} \\ &= 0. \end{aligned}$$

Therefore, the restriction X_λ of X to any hyperboloid $\lambda = \sqrt{z^2-x^2-y^2}$ extends continuously to the boundary, which we have identified with the unit circle in the plane. Since the closed unit disk is compact, it follows that X_λ is complete for every λ, which implies that X is complete. This completes the proof. $\square$

Given the vector field X from Proposition 3.6.1, the Ginzburg-Weinstein argument goes through as in the compact case (as described in Section 2.3). Defining X_t by

$$X_t(\vec{p}) := \frac{X(t\vec{p})}{t^2},$$

the corresponding flow φ_t pushes π_0 forward to π_t, and in particular, $(\varphi_1)_* (\pi_0) = \pi$, as desired.

3.6.2. The Flaschka-Ratiu Approach

In this subsection, we mimic the procedure used by Flaschka and Ratiu for the $SU(2)$ case. The idea is to use coordinate systems based on the Gelfand-Tsetlin coordinates, which are given by the eigenvalues of all $k \times k$ principal minors of an $n \times n$ matrix. Note that the Gelfand-Tsetlin coordinates are real on $\mathfrak{q}_{adm}$ and Q_{adm}.

The elements of $\mathfrak{q}$ with eigenvalues $\pm\lambda$ can be parameterized by the matrices

$$\begin{pmatrix} z & \sqrt{z^2 - \lambda^2} \cdot e^{\mathbf{i}\theta} \\ -\sqrt{z^2 - \lambda^2} \cdot e^{-\mathbf{i}\theta} & -z. \end{pmatrix}, \tag{3.6.3}$$

with $z > \lambda$ and $0 \leq \theta < 2\pi$. Define coordinates on $\mathfrak{q}$ by identifying (3.6.3) with $(z, \lambda, \theta) \in \mathbb{R}^3$. In these coordinates,

$$\pi_0 = \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial z}.$$

The symplectic structure induced by π_0 on *any* symplectic leaf $\theta_{\vec{\lambda}} = \sqrt{z^2 - y^2 - x^2} = \lambda$ is then

$$\omega_0 := d\theta \wedge dz.$$

Similarly, the elements of Q with eigenvalues $\exp(\pm\lambda)$ can be parameterized by the matrices

$$\begin{pmatrix} e^w & \sqrt{(e^w - e^\lambda)(e^w - e^{-\lambda})} \cdot e^{\mathbf{i}\theta} \\ -\sqrt{(e^w - e^\lambda)(e^w - e^{-\lambda})} \cdot e^{-\mathbf{i}\theta} & 2\cosh(\lambda) - e^w \end{pmatrix}. \tag{3.6.4}$$

Define coordinates on Q by identifying (3.6.3) with $(w, \lambda, \theta) \in \mathbb{R}^3$. In these coordinates,

$$\pi_Q = \frac{\partial}{\partial\theta} \wedge \frac{\partial}{\partial w}.$$

The symplectic structure induced by π_Q on *any* symplectic leaf $\Psi_{\vec{\lambda}} = \frac{1+c^2-a^2-b^2}{c} = 2(\cosh\lambda)$ is then

$$\omega_Q := d\theta \wedge dw.$$

Given these simple expressions for ω_0 and ω_1, for each λ, we can define a symplectomorphism from $\theta_{\vec{\lambda}}$ to $\Psi_{\vec{\lambda}}$ by identifying the matrix (3.6.3) with the matrix (3.6.4). A Poisson isomorphism f from $(\mathfrak{q}, \pi_0)$ to (Q, π_Q) is obtained by allowing λ to vary over the interval $(0, \infty)$. Equivalently, f sends (z, λ, θ) to (w, λ, θ). In terms of the coordinates

$$\begin{pmatrix} z & x + iy \\ -x + iy & -z \end{pmatrix} \leftrightarrow (x, y, z)$$

on $\mathfrak{q}$ and

$$\begin{pmatrix} c & a + ib \\ -a + ib & \frac{1-a^2-b^2}{c} \end{pmatrix} \leftrightarrow (a, b, c)$$

on Q, f is given by

$$\begin{aligned} a &= \sqrt{e^{2z} - e^{z}(2\cosh(\sqrt{z^2 - x^2 - y^2})) + 1}\left(\frac{x}{\sqrt{x^2 + y^2}}\right) \\ b &= \sqrt{e^{2z} - e^{z}(2\cosh(\sqrt{z^2 - x^2 - y^2})) + 1}\left(\frac{y}{\sqrt{x^2 + y^2}}\right) \\ c &= e^{z}. \end{aligned}$$

The map f is neither one-to-one nor onto, but it is one-to-one when restricted to $\mathfrak{q}_{adm}$.

Appendices

A. Lie Theory

A.3. The Pseudo-Unitary Groups

Any $n \times n$ matrix A defines a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^n$ by

$$\langle \mathbf{u}, \mathbf{v} \rangle := \mathbf{u}^t A \overline{\mathbf{v}}.$$

The unitary group $U(n)$ is defined to be the group of linear transformations on $\mathbb{C}^n$ which preserve the form determined by the identity matrix, i.e.,

$$\langle \mathbf{u}, \mathbf{v} \rangle := \mathbf{u}^t I_{n \times n} \overline{\mathbf{v}} = \mathbf{u} \cdot \mathbf{v},$$

the standard inner product on $\mathbb{C}^n$.

For each choice of nonnegative integers p, q with $p + q = n$ the *pseudo-unitary group* $U(p, q)$ is the Lie group of linear transformations which preserve the bilinear form determined by the matrix

$$J_{pq} := \begin{pmatrix} 1 & 0 & & \dots & & 0 \\ 0 & \ddots & & & & \\ & & 1 & \ddots & & \vdots \\ \vdots & & \ddots & -1 & & \\ & & & & \ddots & 0 \\ 0 & & \dots & & 0 & -1 \end{pmatrix} \tag{A.1}$$

which has p 1s followed by q -1s on the diagonal and 0s elsewhere. When there is no risk of confusion, J_{pq} will sometimes be abbreviated by J. Equivalently, $U(p, q)$

consists of the matrices A in $GL(n, \mathbb{C})$ which satisfy the equation

$$A^{-1} = JA^*J, \tag{A.2}$$

where A^* is the conjugate transpose of A. The Lie group $SU(p, q)$ is the subgroup of $U(p, q)$ of matrices with determinant 1.

Example A.2. The group $SU(1, 1)$ consists of all matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$$

with $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 - |\beta|^2 = 1$. Thus, $SU(1, 1)$ is topologically equivalent to a 3-dimensional hyperboloid in $\mathbb{R}^4$.

For the cases $p > 1$ or $q > 1$, it is not possible to describe the elements of $SU(p, q)$ as explicitly as in the $SU(1, 1)$ case. However, the Lie algebras of $U(p, q)$ and $SU(p, q)$ are easy to describe. The Lie algebra of $U(p, q)$ is

$$\mathfrak{u}(p, q) = \{X \in \mathfrak{gl}(p + q, \mathbb{C}) \mid JX^*J = -X\},$$

and the Lie algebra of $SU(p, q)$ is the subalgebra of $\mathfrak{u}(p, q)$ with trace 0.

The condition $JA^*J = -A$ means that

$$A_{ij} = \begin{cases} -\overline{A_{ji}} & \text{if } i, j \le p \text{ or } i, j \ge p \\ \overline{A_{ji}} & \text{otherwise.} \end{cases}. \tag{A.3}$$

In particular, when $i = j$, the first case is necessarily satisfied, so that $A_{ii} = -\overline{A_{ii}}$, which implies that the diagonal is imaginary. For example,

$$\mathfrak{u}(2,2) = \left\{ \begin{pmatrix} \mathbf{i}z_1 & a_{12} & a_{13} & a_{14} \\ -\overline{a_{12}} & \mathbf{i}z_2 & a_{23} & a_{24} \\ \overline{a_{13}} & \overline{a_{23}} & \mathbf{i}z_3 & a_{34} \\ \overline{a_{14}} & \overline{a_{24}} & -\overline{a_{34}} & \mathbf{i}z_4 \end{pmatrix} : \quad z_i \in \mathbb{R}, a_{ij} \in \mathbb{C} \right\}.$$

Given this concrete representation of $\mathfrak{su}(p,q)$, its dimension is easily seen to be $(p+q)^2 - 1$.

Each group $SU(p,q)$ is simple (see [Kna02], for instance). Like most of the classical groups, $SU(p,q)$ is connected for all p, q (see section 3.1 of [Ros02]).

Proposition A.3. *For every $p, q \neq 0$, the group $SU(p,q)$ is noncompact.*

Sketch of Proof. Since $SU(p,q)$ is simple, and hence semisimple, it will suffice to show that the Killing form on $\mathfrak{su}(p,q)$ is not negative-definite. This can be done using (A.3) and the fact that the Killing form on $\mathfrak{su}(p,q)$ is a multiple of the trace form (see [O'N83], for example). □

The maximal compact subgroup of $SU(p,q)$ is $S(U(p)\times U(q)) = SU(p,q) \bigcap SU(p+q)$. In block form, the corresponding Lie algebra is

$$\mathfrak{s}(\mathfrak{u}(p) \oplus \mathfrak{u}(q)) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \;:\; A \in \mathfrak{u}(p), B \in \mathfrak{u}(q), \mathrm{tr}(A+B) = 0 \right\}.$$

A.4. Iwasawa Decompositions

The following description is summarized from [Kna02].

Given a matrix $g \in SL(n, \mathbb{C})$, applying the Gram-Schmidt orthogonalization process to the columns of X yields a unique decomposition $g = kan$, where $k \in SU(n)$,

a is diagonal with real, positive entries, and n is upper-triangular with 1's on the diagonal.

Example A.4. For $n = 2$, the Iwasawa decomposition is easy to compute. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$. Then $g = kl$, where

$$k = \frac{1}{\sqrt{||a||^2 + ||c||^2}} \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix}$$

and

$$l = \begin{pmatrix} \sqrt{||a||^2 + ||c||^2} & \frac{\bar{a}b + \bar{c}d}{\sqrt{||a||^2+||c||^2}} \\ 0 & \frac{1}{\sqrt{||a||^2+||c||^2}} \end{pmatrix}.$$

Taking inverses yields the decomposition $G = (AN)K$. With g as above, we have $g = lk$, where

$$l = \begin{pmatrix} \frac{1}{\sqrt{||c||^2+||d||^2}} & \frac{\bar{c}a + \bar{d}b}{\sqrt{||c||^2+||d||^2}} \\ 0 & \sqrt{||c||^2 + ||d||^2} \end{pmatrix},$$

and

$$k = \frac{1}{\sqrt{||c||^2 + ||d||^2}} \begin{pmatrix} \bar{d} & -\bar{c} \\ c & d \end{pmatrix}$$

This decomposition generalizes to arbitrary semisimple Lie groups as follows. Let $\mathfrak{g}$ be a semisimple Lie algebra, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition. Since $\mathfrak{p}$ is finite-dimensional, there exists a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$. The set $\{\mathrm{ad}_H \mid H \in \mathfrak{a}\}$ is a commuting family of self-adjoint transformations of $\mathfrak{g}$. It follows that $\mathfrak{g}$ is an orthogonal direct sum of simultaneous eigenspaces for this family of transformations. The corresponding eigenvalues can be identified with elements of

$\mathfrak{a}^*$. For each $\lambda \in \mathfrak{a}^*$, let

$$g_\lambda = \{X \in \mathfrak{g} \,|\, [H, X] = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}.$$

The set

$$\Sigma := \{\lambda \in \mathfrak{a} \,|\, \lambda \neq 0,\ \mathfrak{g}_\lambda \neq 0\}$$

constitutes a root system for the vector space $\mathfrak{g}$. It can be shown that $\mathfrak{g}$ decomposes as the orthogonal direct sum

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda.$$

Choose a set of positive roots $\Sigma^+ \subset \Sigma$, and let

$$\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda.$$

Then $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$.

Proposition A.5 (Iwasawa Decomposition)**.** *The Lie algebra $\mathfrak{g}$ decomposes (as a vector space) as the direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$.*

If $\mathfrak{g}$ is the Lie algebra of a Lie group G, then exponentiating an Iwasawa decomposition of $\mathfrak{g}$ yields a corresponding Lie group decomposition,

$$G = KAN,$$

where K is compact, A is abelian, and N is unipotent. For each $g \in G$ there exist unique $k \in K$, $a \in A$, and $n \in N$ such that $g = kan$. Thus, $G \cong K \times A \times N$.

The projections onto the K and AN factors in these decompositions will be denoted as follows. If $g = k(an)$, then

$$p^l_K(g) = k, \tag{A.4}$$

$$p^r_{AN}(g) = an, \tag{A.5}$$

and if $g = (an)k$, then

$$p^r_K(g) = k, \tag{A.6}$$

$$p^l_{AN}(g) = an. \tag{A.7}$$

B. Cohomology

B.5. Group Cohomology

Suppose G is a group, A is a an abelian group, and

$$\rho : G \times A \to A : (g, a) \mapsto g \cdot a$$

is an action of G on A. let C^n denote the set of functions from G^n to A. (Note that G^0 can be taken to be a 1-point set.) Define differentials $d^n : C^n \to C^{n+1}$ by

$$\begin{aligned} d^n(\phi)(g_1, \ldots, g_{n+1}) \;&:=\; g_1 \cdot \phi(g_2, \ldots, g_{n+1}) \\ &+\; \sum_{i=1}^{n} (-1)^i \phi(g_1 \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2} \ldots, g_{n+1}) \\ &+\; (-1)^{n+1} \phi(g_1, \ldots, g_n). \end{aligned}$$

A computation shows that $d^{n+1} \circ d^n = 0$, so that

$$C^0 \xrightarrow{d^0} \cdots \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} \cdots$$

defines a cochain complex. The corresponding cohomology groups are defined (as usual) by $H^n(G, A, \rho) := \ker(d^n)/\mathrm{im}(d^{n-1})$.

If G and A are topological (resp. Lie) groups, then one can define *continuous (resp. smooth) group cohomology* by requiring that the elements of C^n in the complex above be continuous (resp. smooth). In fact, if G is a finite-dimensional Lie group, its continuous and smooth group cohomologies coincide. In this case, both cohomologies will be denoted by $H^n_{ct}(G, A, \rho)$.

B.6. Lie Algebra Cohomology

Representations of Lie groups are closely related to representations of their Lie algebras. We define Lie algebra cohomology as follows. Given a representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ of a Lie algebra $\mathfrak{g}$ on a vector space V, for each $0 \leq n \leq \dim(\mathfrak{g})$, define $C^k(\mathfrak{g}, V)$ to be the space of linear maps from $\wedge^n(\mathfrak{g})$ to V. Now define differentials $\delta^n : C^n(\mathfrak{g}, V) \to C^{n+1}(\mathfrak{g}, V)$ by

$$\begin{aligned}\delta^n(\phi)(X_1 \wedge \cdots \wedge X_{n+1}) \;&:=\; \sum_{0 \leq i \leq n} (-1)^n \rho(X_i)\phi(X_1, \ldots, \hat{X}_i, \ldots, X_n) \\ &+ \sum_{0 \leq i,j \leq n} (-1)^{i+j} \phi([X_i, X_j], \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots),\end{aligned}$$

for $\phi \in C^n(\mathfrak{g}, V)$, $X_1, \ldots, X_n \in \mathfrak{g}$.

It can be shown that $\delta^{n+1} \circ \delta^n = 0$. Therefore,

$$C^0 \xrightarrow{\delta^0} \cdots \xrightarrow{\delta^{n-1}} C^n \xrightarrow{\delta^n} C^{n+1} \xrightarrow{\delta^{n+1}} \cdots$$

defines a cochain complex. The corresponding cohomology groups are defined by $H^n(\mathfrak{g}, V, \rho) := \ker(\delta^n)/\mathrm{im}(\delta^{n-1})$.

If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, we define the Lie algebra cohomology of $\mathfrak{g}$ *relative* to $\mathfrak{h}$ (with coefficients in V) by defining $C^n(\mathfrak{g}, \mathfrak{h}, V, \rho)$ to be the set of cochains $\phi \in C^n(\mathfrak{g}, V, \rho)$ such that

$$\phi(X_1 \wedge \cdots \wedge X_n) = 0 \text{ for all } X_1 \in \mathfrak{h} \text{ and} \tag{B.1}$$

$$\delta^n(\phi)(X_1 \wedge \cdots \wedge X_{n+1}) = 0 \text{ for all } X_1 \in \mathfrak{h}. \tag{B.2}$$

Then δ, as defined above, is a differential on $\{C^n(\mathfrak{g}, \mathfrak{h}, V, \rho)\}$, and the corresponding cohomology groups of $\mathfrak{g}$ relative to $\mathfrak{h}$ are denoted by $H^n(\mathfrak{g}, \mathfrak{h}, V, \rho)$.

Remark B.6. Note that if $\mathfrak{h} = \mathfrak{g}$, then conditions (B.1) and (B.2) imply that $C^n(\mathfrak{g}, \mathfrak{h}, V, \rho) = 0$ for all n.

Remark B.7. It is common to write $H^n(\mathfrak{g})$ for the Lie algebra cohomology of $\mathfrak{g}$ with respect to the trivial representation of $\mathfrak{g}$ on $\mathbb{R}$.

B.7. Relations Between Lie Group and Lie Algebra Cohomologies

The formulas above suggest that Lie algebra cohomology is in some sense an infinitesimal version of Lie group cohomology. In fact, one of the primary reasons for studying Lie algebra cohomology is relations between the two complexes. For example,

Theorem B.8. *([FF], Section 3.1, Theorem 4) Let $\mathfrak{g}$ be a real, reductive Lie algebra with complexification $\mathfrak{g}^{\mathbb{C}}$. Then there exists a compact Lie group $\overline{G}$ such that the Lie algebra $\overline{\mathfrak{g}}$ of $\overline{G}$ is a compact real form of $\mathfrak{g}^{\mathbb{C}}$, and $H^\star(\mathfrak{g}) \cong H^\star_{top}(\overline{G})$.*

B.8. Poisson Cohomology

Poisson cohomology was first introduced by Lichnerowicz in [Lic77]. Given a Poisson manifold (M, π), for each $n \geq 0$, the bivector field π defines an operator $d^n_\pi : \chi^n \to$

χ^{n+1} by

$$X \mapsto [\pi, X].$$

The Jacobi identity $[\pi, \pi] = 0$ implies that $d_\pi^{n+1} \circ d^n = 0$. Thus, the pair (M, π) defines a cochain complex. The corresponding cohomology groups will be denoted by $H_\pi^n(M)$.

Recall that the bivector field π induces a map

$$\pi^\# : T^*M \to TM : \eta \mapsto \pi(\cdot, \eta).$$

Taking wedge powers of $\pi^\#$ yields a map from $\omega^n(M)$ to $\xi^n(M)$, which we also denote by $\pi^\#$. It can be shown (see [DZB$^+$05], section 2.1.3) that this map, called the *anchor map* of π, intertwines the de Rham operators d^n on the de Rham complex with the Lichnerowitz operators d_π^n on the complex described above. Thus, $\pi^\#$ induces a homomorphism from the de Rham cohomology groups $H^n(M)$ to the Poisson cohomology groups $H_\pi^n(M)$ of M. If π is symplectic, this homomorphism is an isomorphism ([DZB$^+$05], Theorem 2.1.4). Thus, for symplectic bivector fields, Poisson cohomology can often be computed.

In general, Poisson cohomology groups are usually very difficult to compute. In fact, according to [DZB$^+$05], $H_\pi^n(M)$ can be infinite dimensional even when M is finite-dimensional and compact.

C. Induced Representations

The theory of induced representations is well-studied. What follows is only a sketch. For more details, see, for example [Kir76]. Suppose G is a group, $H < G$ is a subgroup, and $\rho : H \to \mathrm{Aut}(V)$ of H on a vector space V. (For simplicity, all representations discussed here will be over $\mathbb{R}$ or $\mathbb{C}$.) Denote by $\mathrm{Ind}_H^G(V)$ the vector space E of functions from G to V satisfying the equation

$$f(hg) = \rho(h)f(g)$$

for all $h \in H, g \in G$. The group G acts on E by right translation: For $x, g \in G$, $f \in E$, set

$$(g.f)(x) := f(xg).$$

This representation of G on E is called the *induced* representation and will be denoted by $\mathrm{Ind}_H^G(\rho)$.

For Lie groups, one often imposes the additional requirement that (norms of) functions in E be L^2 with respect to Haar measure, so that E can be made into a Hilbert space. For this thesis, we will follow the approach of Borel and Wallach ([BW00]), which does not impose the L^2 condition.

The following is a specialization of Proposition 2.3 on page 175 of [BW00].

Theorem C.9 (Shapiro's Lemma)**.** *Let G be a Lie group, $H < G$ a closed subgroup,* *and* $\rho : H \to Aut(V)$ *a representation of* H *on a vector space* V*. Then*

$$H^{\star}_{ct}(G, \mathrm{Ind}^G_H(V), \mathrm{Ind}^G_H(\rho)) = H^{\star}_{ct}(H, V, \rho).$$

D. A Poisson Structure on AN

Using coordinates

$$\begin{pmatrix} d_1 & a_1+ib_1 & a_2+ib_2 \\ 0 & d_2 & a_3+ib_3 \\ 0 & 0 & d_3 \end{pmatrix},$$

the Poisson bracket on AN which makes it the dual of $U(3)$ with the Lu-Weinstein Poisson bracket is given below:

$$\begin{array}{rclrcl}
\{d_1,d_2\} &=& 0 & \{d_3,b_1\} &=& 0 \\
\{d_1,d_3\} &=& 0 & \{d_3,b_2\} &=& a_2d_3 \\
\{d_1,a_1\} &=& b_1d_1 & \{d_3,b_3\} &=& a_3d_3 \\
\{d_1,a_2\} &=& d_1b_2 & \{a_1,a_2\} &=& d_2b_3+a_1b_2 \\
\{d_1,a_3\} &=& 0 & \{a_1,a_3\} &=& d_2b_2 \\
\{d_1,b_1\} &=& -a_1d_1 & \{a_1,b_1\} &=& d_1^2-d_2^2 \\
\{d_1,b_2\} &=& -d_1a_2 & \{a_1,b_2\} &=& -(a_1a_2+d_2a_3) \\
\{d_1,b_3\} &=& 0 & \{a_1,b_3\} &=& -d_2a_2 \\
\{d_2,d_3\} &=& 0 & \{a_2,a_3\} &=& b_1d_2+a_3b_2 \\
\{d_2,a_1\} &=& -b_1d_2 & \{a_2,b_1\} &=& -(b_1b_2+d_2a_3) \\
\{d_2,a_2\} &=& 0 & \{a_2,b_2\} &=& d_1^2+a_1^2+b_1^2-a_3^2-b_3^2-d_3^2 \\
\{d_2,a_3\} &=& d_2b_3 & \{a_2,b_3\} &=& b_3b_2+a_1d_2 \\
\{d_2,b_1\} &=& a_1d_2 & \{a_3,b_1\} &=& d_2a_2 \\
\{d_2,b_2\} &=& 0 & \{a_3,b_2\} &=& a_1d_2+a_3a_2 \\
\{d_2,b_3\} &=& -d_2a_3 & \{a_3,b_3\} &=& d_2^2-d_3^2 \\
\{d_3,a_1\} &=& 0 & \{b_1,b_2\} &=& d_2b_3-b_1a_2 \\
\{d_3,a_2\} &=& -b_2d_3 & \{b_1,b_3\} &=& -d_2b_2 \\
\{d_3,a_3\} &=& -b_3d_3 & \{b_2,b_3\} &=& b_1d_2-b_3a_2.
\end{array}$$

Using coordinates

$$\begin{pmatrix} x_1 & u_1+iv_1 & u_2+iv_2 \\ u_1-iv_1 & x_2 & u_3+iv_3 \\ u_2-iv_2 & u_3-iv_3 & x_3 \end{pmatrix},$$

on P, the pushforward of this Poisson structure under the map $Sym : AN \to P : A \mapsto A^*A$ is

$$
\begin{array}{rclrcl}
\{x_1,x_2\} & = & 0 & \{x_3,v_1\} & = & 0 \\
\{x_1,x_3\} & = & 0 & \{x_3,v_2\} & = & 2x_1u_2+2u_1u_3-2v_1v_3 \\
\{x_1,u_1\} & = & 2x_1v_1 & \{x_3,v_3\} & = & 2u_3x_2+2u_1u_2+2v_1v_2 \\
\{x_1,u_2\} & = & 2x_1v_2 & \{u_1,u_2\} & = & v_2u_1+v_3x_1 \\
\{x_1,u_3\} & = & 0 & \{u_1,u_3\} & = & u_1v_3+v_2x_1 \\
\{x_1,v_1\} & = & -2u_1x_1 & \{u_1,v_1\} & = & x_1^2-x_1x_2 \\
\{x_1,v_2\} & = & -2x_1u_2 & \{u_1,v_2\} & = & -u_1u_2-u_3x_1 \\
\{x_1,v_3\} & = & 0 & \{u_1,v_3\} & = & -x_1u_2-u_1u_3 \\
\{x_2,x_3\} & = & 0 & \{u_2,u_3\} & = & x_1v_1+v_1x_2-v_1x_3-u_2v_3 \\
\{x_2,u_1\} & = & -2v_1x_1 & \{u_2,v_1\} & = & -v_1v_2-u_3x_1 \\
\{x_2,u_2\} & = & 2u_1v_3+2v_1u_3 & \{u_2,v_2\} & = & u_1^2+x_1^2-x_3x_1+v_1^2 \\
\{x_2,u_3\} & = & 2v_3x_2+2v_2u_1-2v_1u_2 & \{u_2,v_3\} & = & -u_1x_3+x_1u_1+u_1x_2+u_3u_2 \\
\{x_2,v_1\} & = & 2x_1u_1 & \{u_3,v_1\} & = & x_1u_2-v_1v_3 \\
\{x_2,v_2\} & = & -2u_1u_3+2v_1v_3 & \{u_3,v_2\} & = & x_1u_1-u_1x_3+u_1x_2+v_3v_2 \\
\{x_2,v_3\} & = & -2u_3x_2-2u_1u_2-2v_1v_2 & \{u_3,v_3\} & = & u_1^2+v_1^2-v_2^2+x_2^2-x_3x_2-u_2^2 \\
\{x_3,u_1\} & = & 0 & \{v_1,v_2\} & = & -v_1u_2+v_3x_1 \\
\{x_3,u_2\} & = & -2v_1u_3-2v_2x_1-2u_1v_3 & \{v_1,v_3\} & = & -v_2x_1-v_1u_3 \\
\{x_3,u_3\} & = & -2v_3x_2-2v_2u_1+2v_1u_2 & \{v_2,v_3\} & = & -v_1x_3+x_1v_1+v_1x_2+v_2u_3.
\end{array}
$$

These formulas were computed in Maple.

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