New Geometric Approaches to Finite Temperature String Theory

by

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As members of the Dissertation Committee, we certify that we have read the dissertation

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Abstract

In quantum field theory a system at finite temperature can equivalently be viewed as having a compactified dimension. This situation carries over into string theory and leads to thermal duality, which relates the physics of closed strings at temperature T to the physics at the inverse temperature $1/T$. Unfortunately, the classical definitions of thermodynamic quantities such as entropy and specific heat are not invariant under the thermal duality symmetry. We shall therefore pursue two different approaches. We shall investigate whether there might nevertheless exist special solutions for the string effective potential such that the duality symmetry will be preserved for *all* thermodynamic quantities. Imposing thermal duality covariance, we derive unique functional forms for the temperature-dependence of the string effective potentials.

The second approach is to investigate self-consistent modifications to the rules of ordinary thermodynamics such that thermal duality is preserved. After all, methods of calculation should not break fundamental symmetries. We therefore propose a modification of the traditional definitions of these quantities, yielding a manifestly duality-covariant thermodynamics. At low temperatures, these modifications produce "corrections" to the standard definitions of entropy and specific heat which are suppressed by powers of the string scale. These corrections may nevertheless be important for the full development of a consistent string thermodynamics.

One can also investigate the limitations of this geometric interpretation of temperature. Until recently, it appeared as though the temperature/geometry equivalence held in all string theories, but it appears to be broken for the heterotic string. We shall show this breaking by considering the $SO(32)$ heterotic string in ten dimensions.

The breaking of the geometric/finite temperature correspondence in the context

of the heterotic string, leads to two different philosophical approaches when examining string systems at finite temperature. One approach is to discard the geometrical interpretation of temperature and ignore the string consistency conditions to follow the standard rules of statistical mechanics. This approach does not seem to lead to self-consistent string models. The second approach is to take the string consistency conditions as fundamental and explore their implications for systems at finite temperature. We shall examine some of the consequences of this approach.

New Geometric Approaches to Finite Temperature String Theory

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Chapter 1
Introduction

An overarching theme of physics is that of unification. Increasingly disparate phenomena have been described by the same formalism. In the past century, our understanding of the universe has progressed on two very different frontiers. On the one hand, the development of the Standard Model has allowed for the testing of physics at length scales of $\mathcal{O}(10^{-18}\ m)$. Scientific progress on this frontier has led to the probing of physics at ever-smaller length scales. The physics of this length scale is dominated by the strong and electroweak forces. On the other hand, progress in our understanding of the force of gravity has progressed through the development of General Relativity. At this point, there does not appear to be any reason to expect that gravity should not unify with the other fundamental forces at some energy scale. String theory represents an attempt to unify the Standard Model and General Relativity and thereby unify all of the fundamental forces into one formalism.

One generally sees a plea for theoretical aesthetics offered as a justification for studying string theory. While it is true that string theory has some properties that make it appealing, (such as more finite amplitudes and fewer unfixed parameters) than a general field theory, this alone will not be our justification for studying it. One could also argue that studying string theory has led to advances in our understanding of gauge theories (the AdS/CFT correspondence is one example). These are both solid justifications for studying string theory. However, we shall propose another reason to study string theory.

Although humans have been scientifically studying gravity for a longer period than the other fundamental forces, there are still many aspects of it that are not very well understood. Much of this is due to the fact that the scale of quantum gravity is fourteen decades in energy above the energy scale of near-term experiments. This difference in scales means that any corrections to well-known processes due to the effects of quantum gravity will be very difficult to discern in any experiment. However, this does not mean that there do not exist unique processes due to quantum gravity which could be discovered. Thus, string theory as a theoretical framework for studying quantum gravity should not be ignored.

We shall use self-consistency conditions in string theory in an attempt to study the properties of string theory in settings that are well understood in the context of field theory. We will concentrate our efforts towards finding novel relationships between quantities that are unrelated in field theory. We shall primarily examine string theory at finite temperature. To this end, we shall first briefly describe string theory and then explain why one should expect that string theory might be different from field theory at finite temperature. However, string theory is a very broad field so this description will necessarily be short and lacking in detail. The interested reader should refer to any number of the textbooks written about string theory[1, 2].

1.1 What is String Theory?

At its core, string theory rests upon a deceptively simple idea. What if every currently known, seemingly fundamental point particle is actually an excitation of one fundamental object? This object, called a string, would be one-dimensional and have a characteristic size much smaller than currently experimentally probed, for scale the length of the string would be $\mathcal{O}(10^{-34}\ m)$ or fourteen decades in energy above the current experimental reach. Unfortunately, for technical reasons having to do with anomaly cancelations this requires that the string propagate in many more spacetime dimensions then are currently observed.

In this section, we shall broadly introduce many of the properties of string theory including the major classes of closed strings. We shall then examine the so-called "worldsheet picture" and some of the properties of the worldsheet. Finally, we shall quickly examine the spacetime spectra of different closed string models. Throughout this discussion, we shall be focusing only on closed strings at the perturbative level. We will therefore ignore many of the other interesting aspects of string theory such as open strings, D-branes, tachyon condensation, black holes, AdS/CFT, etc. Keeping this in mind, we shall now list and attempt to motivate some of the features of closed string theories.

- The fact that strings have a non-zero length that acts as a UV regulator and cancels many of the divergences present in general quantum field theories.
- All string models must contain a massless spin-2 excitation which has an equation of motion consistent with its interpretation as a graviton.
- Some string models contain the theoretical structures (*i.e.* gauge groups, particle representations, etc.) capable of reducing to the Standard Model in the appropriate limits.
- Many parameters which are, in principle, unfixed in general quantum field theories are generically dynamically generated in the context of string theory.

There are three main types of closed strings. There are bosonic strings, Type II strings, and heterotic strings. Bosonic strings only give rise to a particle spectrum consisting of spacetime bosons. As we shall see, at the perturbative level, these strings are unstable, because they give rise to a tachyonic mode. After this tachyon condenses, it is unclear what the form of this type of string is. Our main interest in this string will be based on its relative simplicity and the fact that many of the features found in the other two classes of strings can also be found in bosonic strings.

Type II strings and heterotic strings are within the class of so-called "superstrings". We shall not discuss the spectrum of the Type II string in great detail for the following reason. The gauge structures which can be found in perturbative Type II string models (*i.e.* those not coupled to D-branes[1]) in four spacetime dimensions are not large enough to contain the Standard Model. This makes this class of model relatively uninteresting from a phenomenological point of view.

Heterotic strings are the combination of the previous two classes of strings. These strings are made by "stitching" a Type II string to a bosonic string. We shall clarify this analogy later when we consider heterotic strings in detail. This stitching has some important consequences for the phenomenology of heterotic strings including a spacetime spectrum consisting not only of spacetime bosons *and* fermions, but also the potential for very large gauge symmetries. This class of string has been of primary interest for string phenomenologists for many years.

1.1.1 The Bosonic String

One can formulate all of classical mechanics in terms of the extremization of the worldlines of point-particles. For a free-particle propagating in d dimensions the action would be,

$$S = -m \int d\tau \, \left(-\frac{dX^\mu(\tau)}{d\tau} \frac{dX_\mu(\tau)}{d\tau} \right)^{1/2} \tag{1.1}$$

where τ is the proper time along the worldline, X^μ is the particle's position in the μ^{th} dimension. This action is merely the length of the worldline, thus the classical path would be the worldline with the minimum length. For a massless particle, this can be rewritten as,

$$S = \tfrac{1}{2} \int d\tau \, (-h_{\tau\tau}(\tau))^{-1/2} \, \frac{dX^\mu(\tau)}{d\tau} \frac{dX_\mu(\tau)}{d\tau} \tag{1.2}$$

where $h_{\tau\tau}$ is the metric on the worldline. The action in Eq. (1.2) is classically equivalent to the action in Eq. (1.1). The action listed in Eq. (1.2) is much more amenable

[1]One of the many realizations to come from the mid-90's was that Type II strings contain non-perturbative objects called D-branes. This realization has led to a re-examination of the phenomenology of Type II strings and there are currently many different ways to construct a semi-realistic embedding of the Standard Model within this new larger class of string model.

to a path integral analysis because it does not have the troublesome square root, but it also lacks the clear physical interpretation.

Before we proceed to strings there is one important point to note about this formulation of classical mechanics. The parameterization of the worldline does *not* affect the physics of the particle. Therefore, we could rewrite the action in terms of $X'^{\mu}(\tau'(\tau)) = X^{\mu}(\tau)$ and the exact same equations of motion must result. This invariance will be very important for the case of strings.

If we consider strings, instead of point particles, propagating through spacetime, instead of a one-dimensional worldline, a two-dimensional worldsheet results. This worldsheet can be parameterized through two variables (traditionally τ, σ as shown in Fig 1.1).

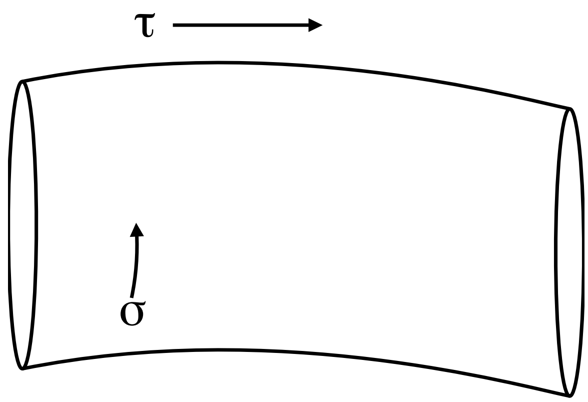

FIGURE 1.1. The string worldsheet with the worldsheet coordinates τ, σ.

The analogue of the action in Eq. (1.2) for the bosonic string is the Polyakov action[4],

$$S = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \, \sqrt{h} \, h^{\alpha\beta} \, g^{\mu\nu} \, \partial_\alpha X_\mu(\tau,\sigma) \partial_\beta X_\nu(\tau,\sigma) \tag{1.3}$$

where the X_μ function maps the worldsheet coordinates into spacetime coordinates, $h^{\alpha\beta}$ is the worldsheet metric, $g^{\mu\nu}$ is the spacetime metric, and the prefactor is the tension of the string. This action is classically equivalent to the Nambu-Goto action, which when extremized explicitly minimizes the area of the worldsheet.

String theory is a theory of quantum gravity. This means that, in principle, the background through which the string is propagating (*i.e.* spacetime) should actually be generated by the string itself. Spacetime is therefore a *derived* quantity. Keeping this fact in mind, we shall now re-interpret the action presented in Eq. (1.3).

We shall henceforth, consider the worldsheet to be a fundamental quantity. We shall consider the X^μ functions (recall these are the embedding functions which map worldsheet coordinates into spacetime coordinates) to be fields in this two dimensional space. From this perspective, the Lorentz symmetry is merely translated to an internal $SO(d-1,1)$ internal symmetry for the X^μ fields. Also from this perspective, the action written in Eq. (1.3), is for d non-interacting bosonic fields propagating in a two-dimensional space. We shall refer to this perspective as the worldsheet perspective. Henceforth, the term worldsheet theory will refer to this two dimensional non-interacting field theory.

The worldsheet has another very important property. Just as for the worldline, the coordinate system chosen for the worldsheet should *not* affect the spacetime physics. Therefore, there is another symmetry present on the worldsheet which the action written down in Eq. (1.3) respects. This is known as a conformal symmetry. This allows for arbitrary rescalings to occur with no changes to the physics. Therefore, the field theory we shall be considering will be a two-dimensional conformal field theory. It turns out these theories are very constrained and have been studied in depth[3].

Before we proceed to examining the fields present in this field theory, we should address one of the consequences of the conformal symmetry. Because the field theory is conformal, we can always locally rescale the worldsheet so that we are always working in a flat space (*i.e.* $h^{\alpha\beta} = \eta^{\alpha\beta}$). This corresponds to the equivalent of a gauge choice. We shall always choose to work in this "gauge", and therefore the worldsheet metric will not appear in any more actions that we examine. At this point, we shall also add the extra assumption that we are working in a flat spacetime. We shall discuss this assumption in more depth in Sect. 1.2.

We shall now examine the X^μ fields in detail. The X^μ fields are worldsheet bosons with the physical interpretation as embedding functions of the worldsheet into spacetime. This interpretation will lead to two important constraints. The X^μ fields represent real coordinates, so they must obey $[X^\mu(\tau,\sigma)]^\dagger = X^\mu(\tau,\sigma)$. The second constraint will be on the field's periodicity in the σ coordinate. The closed string has a finite length (typically chosen to be π), therefore the X^μ fields should obey the relation $X^\mu(\tau,\sigma+\pi) = X^\mu(\tau,\sigma)$ in order that they be single-valued for all values of σ. Both of these properties ultimately stem from the physical interpretation of the X^μ fields.

We shall rewrite the fields in terms of so-called "right-movers" and "left-movers". These can be written as,

$$X^\mu(\tau,\sigma) = X^\mu_L(\sigma+\tau) + X^\mu_R(\sigma-\tau). \tag{1.4}$$

We are free to perform this rewriting because of the linearity of the X^μ fields in the equations of motion resulting from Eq. 1.3. The motivation for this rewriting will become clear in Sect. 1.1.4.

After rewriting the X^μ fields in terms of left- and right-movers, the mode expansion is,

$$\begin{aligned}
X^\mu_L(\sigma+\tau) &= \tfrac{1}{2}x^\mu + \alpha' p^\mu(\sigma+\tau) + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-2in(\sigma+\tau)} \\
X^\mu_R(\sigma-\tau) &= \tfrac{1}{2}x^\mu - \alpha' p^\mu(\sigma-\tau) + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\tilde{\alpha}^\mu_n}{n}e^{-2in(\sigma-\tau)}
\end{aligned} \tag{1.5}$$

where the α' have been inserted on purely dimensional grounds to make the interpretation of x^μ, p^μ easier. The x^μ, p^μ terms physically represent the position and momentum of the string's center of mass. These terms come from the zero-mode excitation in the mode expansion.

The $\alpha, \tilde{\alpha}$ terms represent creation/annihilation operators for the left- and right-moving excitations respectively. These are the operators that will give rise to the spectrum of string excitations, which are the particles in spacetime. As the α operators are of primary interest we shall turn our attention to them. The reality condition placed upon the X fields requires that, $\alpha_n^\dagger = \alpha_{-n}$ with the same relation for the right-movers. The α operators are normalized so that they have canonical commutation relations (*i.e.* $[\alpha_j^\mu, \alpha_k^\nu] = j\ \delta_{j+k}\ g^{\mu\nu}$). From the mode expansion, we can see that this means that the α operators with a negative index are creation operators and those with a positive index are annihilation operators. The energy associated with these internal vibrational modes is,

$$\begin{aligned} H_L^{\text{int}} &= \sum_{n=1}^{\infty} \alpha_{-n}^{\mu} \alpha_{\mu n} \\ H_R^{\text{int}} &= \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^{\mu} \tilde{\alpha}_{\mu n}. \end{aligned} \tag{1.6}$$

These relations will be very important in Sect. 1.1.4. The excitations resulting from these operators are always spacetime bosons. Thus, we shall turn our attention to the Type II string.

1.1.2 The Type II String

It turns out that the excitations resulting from the bosonic string are always bosons in spacetime. For string theory to give rise to excitations which transform as fermions in spacetime, the action in Eq. (1.3) must be modified. One self-consistent way to modify the action is to enlarge the symmetry present. If the conformal symmetry is enlarged to a superconformal symmetry (by adding a worldsheet superpartner for each X^μ field) then the action for the Type II string can result.

The action for the Type II string is,

$$S = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau\ \partial_\alpha X^\mu \partial^\alpha X_\mu - i\overline{\psi}_\mu \rho^\alpha \partial_\alpha \psi^\mu) \tag{1.7}$$

where the ψ^μ fields are worldsheet fermions and the superpartners of the X^μ fields and the ρ^α are the two dimensional Dirac matrices. We stress that the field theory is still a two-dimensional non-interacting theory, only now the conformal symmetry present for the bosonic string has been enlarged to a superconformal symmetry.

The ψ fields are Majorana fermions in two dimensions. Although the ψ fields transform as fermions in the two dimensional field theory we are currently analyzing, they still possess an internal $SO(d-1,1)$ symmetry exactly like the X fields. This means that from the spacetime perspective the ψ fields actually transform as bosons! However, we shall continue our analysis of these fields and examine the mode expansions of them.

Analogous to the situation for the X fields, we shall divide the ψ fields into left- and right-moving fields. However, because the ψ fields are worldsheet spinors with two degrees of freedom, we can actually make a chirality choice so that we call the upper component, ψ_+, the left moving portion and the lower component, ψ_- the right-moving component.

The X^μ fields had to satisfy two requirements to maintain their interpretation as embedding functions. The ψ fields actually have no such interpretation, but still must satisfy two similar requirements. The ψ fields are Majorana fermions and therefore real. This means that the complex conjugate of the operators showing up in the mode expansion of the ψ field must satisfy a similar conjugation relation as those for the X fields.

The periodicity of the X fields was also constrained by their physical interpretation as embedding functions for the string in spacetime. The situation for the ψ fields is more complicated. The constraints for the boundary conditions of the ψ fields ultimately come from the supercurrents. The ψ fields may either be Ramond fermions (periodic)[5] or Neveu-Schwarz fermions (anti-periodic)[6] under $\sigma \to \sigma + \pi$.

The mode expansion for the ψ fields with Ramond boundary conditions is,

$$\begin{aligned}\psi_+^\mu(\sigma+\tau) &= \sum_{n\in\mathbb{Z}} b_n^\mu e^{-2in(\sigma+\tau)} \\ \psi_-^\mu(\sigma-\tau) &= \sum_{n\in\mathbb{Z}} \tilde{b}_n^\mu e^{+2in(\sigma-\tau)}.\end{aligned} \tag{1.8}$$

We note that unlike the case for the X fields, the zero-mode has no classical interpretation as the motion of the center-of-mass of the string. The mode expansion for the ψ fields with Neveu-Schwarz boundary conditions is identical if $n \to n+\frac{1}{2}$. This change to half-integers takes into account the anti-periodic nature of these boundary conditions.

At this level, the largest difference between the Ramond and the Neveu-Schwarz fields is the energy associated with each vibrational mode. The right-moving fermionic fields have energy,

$$\begin{aligned}H_R^{\text{int}} &= \sum_{n=0}^{\infty} n\ \tilde{b}^\mu_{-n}\tilde{b}_{\mu n} \quad \text{Ramond} \\ H_R^{\text{int}} &= \sum_{n=1/2}^{\infty} n\ \tilde{b}^\mu_{-n}\tilde{b}_{\mu n} \quad \text{Neveu}-\text{Schwarz}.\end{aligned} \tag{1.9}$$

A similar relation exists for the left-moving fields with $\tilde{b} \to b$. One important quality to note is that the zero-mode of the Ramond fermions has zero-energy associated with the excitation.

Of course, one cannot, in general, simply assign boundary conditions to world-sheet fermions with complete freedom. There is a limited number of sets of boundary condition choices that lead to completely self-consistent theories. A set of boundary conditions for the worldsheet fermions is known as a *sector*. In general, a self-consistent string theory will consist of the combination of many different sectors. We shall cover some of the subtleties of this in Sect. 1.1.4.

At this point, we have not addressed the issue of how many embedding fields the bosonic string or the Type II string must have. As the number of X fields present on the worldsheet is the number of spacetime dimensions this is a critical issue. It turns out that the symmetries of the action will determine how many spacetime dimensions the different strings are consistent in.

The bosonic string action has a conformal symmetry. There is an anomaly associated with this symmetry and to cancel this anomaly there must be twenty-six X^μ fields. Therefore, the bosonic string is consistent in twenty-six spacetime dimensions. The Type II string action has a superconformal symmetry. The anomaly associated with the superconformal symmetry is less severe than that of the conformal symmetry. The Type II string also has both X and ψ fields with which to cancel this anomaly. These two facts together allow the Type II string to be consistent in only ten spacetime dimensions.

1.1.3 The Heterotic String

The heterotic string has been the primary class of string of interest to phenomenologists for many years. Earlier, the heterotic string was described as the stitching together of the Type II string and the bosonic string. To understand what this statement could mean, one must consider from the worldsheet perspective how the right- and left-moving fields are related. From the worldsheet perspective, the left-movers and right-movers are two independent conformal field theories, which happen to have the same field content in the case of the bosonic string. This rather surprising fact allows for the right-moving fields to be the same as for Type II strings and the left-moving fields be the same as for the bosonic string. Thus, the action is[7],

$$S = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \ (\partial_- X^\mu_R)^2 - \psi^\mu_R \partial_- \psi_{R\mu} + (\partial_+ X^\mu_L)^2 + (\partial_+ X^i_L)^2 \tag{1.10}$$

where the X^i fields are fields which are purely internal degrees of freedom and the ∂_+, ∂_- correspond to derivatives with respect to $\sigma \pm \tau$ respectively.

As was discussed earlier, the number of embedding fields is different for Type II and bosonic strings. Thus, some of the left-moving X fields are interpreted as giving rise to internal symmetries (*i.e.* gauge symmetries). Of course, one cannot simply decide to change the indices of a field on the worldsheet. The internal degrees of freedom are compactified on a $\mathbb{Z}_2$ orbifold which has the property of giving the same degrees of freedom as a complex Majorana worldsheet fermion for each worldsheet boson. We shall denote this fermion as Ψ^i, and treat it as identical to two real fermions with correlated boundary conditions (*e.g.* the Ψ field is either Neveu-Schwarz or Ramond but not a mix of the two). With this rewriting the action becomes,

$$S = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau\ (\partial_- X^\mu_R)^2 - \psi^\mu_R \partial_- \psi_{R\mu} + (\partial_+ X^\mu_L)^2 + \Psi^i_L \partial_+ \Psi_{iL} \tag{1.11}$$

where the Ψ^i fields are the complex Majorana worldsheet fermions. The existence of the internal degrees of freedom has an interesting consequence for heterotic string models. The internal degrees of freedom generically give rise to internal symmetries (*e.g.* gauge symmetries). The gauge symmetries which can be realized for heterotic strings are what makes this class of string model so interesting for phenomenologists.

Before examining the spectrum of excitations which result from these various theories, we shall summarize the state of the different classes of closed string from the worldsheet perspective. From the worldsheet perspective, the most important difference between each of the different types of closed string comes from the fields present in their respective worldsheet theories. We have summarized this in Table 1.1.

1.1.4 String Spectra

Thus far, the analysis of string theory has been completely restricted to the worldsheet. We shall now quickly examine the spectrum of excitations which can result from string theory from a spacetime perspective. As before, there are a few facts, which in the interests of space we shall simply list. These facts have direct implications for the particle spectrum resulting from a string model.

Type of Closed String	Right-Movers		Left-Movers	
	Symmetry	Field Content	Symmetry	Field Content
Bosonic	conformal	26 X^μ	conformal	26 X^μ
Type II	superconformal	10 X^μ, 10 ψ^μ	superconformal	10 X^μ, 10 ψ^μ
Heterotic	superconformal	10 X^μ, 10 ψ^μ	conformal	10 X^μ, 16 X^i

TABLE 1.1. The field content and symmetries of the major types of closed strings. The index of the field denotes whether the field represents a purely internal degree of freedom or not. Greek indices represent spacetime degrees of freedom, whereas Roman indices represent purely internal degrees of freedom. The field content for each string is consistent with the strings propagating in their respective critical dimensionality.

- There is a vacuum energy associated with the right- and left-moving fields. This vacuum energy can not be neglected unlike the situation in quantum field theory. The vacuum energy for the right- and left-movers shall be denoted as a_R and a_L respectively.

- The vacuum energy is dependent on the field content of the worldsheet. Thus, the vacuum energy for each type of string is different. The contribution to vacuum energy for fermionic fields with Ramond boundary conditions is different from that of fermionic fields with Neveu-Schwarz boundary conditions. This means that the vacuum energy for each sector of a model is different.

- The mass-shell condition which a state must satisfy to be on-shell is $H_R^{\text{int}} + a_R = H_L^{\text{int}} + a_L$. Such a state is called level-matched.

- The mass of any level-matched string excitation is related to the vibrational energy of the string as follows: $M^2 = \frac{2}{\alpha'}(H_L^{\text{int}} + a_L + H_R^{\text{int}} + a_R)$, where M is the mass.

- The choice of sectors present for superstring models will impose *extra* conditions for a state to be in the string spectrum. These extra conditions are known as GSO constraints.

We shall begin with an examination of the particle spectrum coming from the bosonic string. The vacuum energy is the same for both the left- and right-movers because the field content is identical. It turns out the vacuum energy is $a_{L,R} = -1$ in units of the string scale. This vacuum energy is comparable to the energy of oscillator excitations with low mode number as can be seen in Eq. 1.6. This does make the task of examining the phenomenologically important excitations easier as only states with few excitations will need to be considered, as all other states will have masses comparable to the string scale.

We shall now begin our examination of the spectrum of excitations resulting from the bosonic string. The very first state we should examine is the vacuum, $H_L^{\text{int}} = H_R^{\text{int}} = 0$. It is clear that this state is level-matched, as $a_L = a_R$ and a quick examination of the mass reveals that it is actually tachyonic! This tachyon implies that the bosonic string vacuum is unstable. In spite of this fact, we shall examine one other state in the bosonic string spectrum and then we shall shift our attention to superstrings.

The next level-matched state would be the state corresponding to the lowest energy creation operator on both the left- and right-moving vacuum. This state would be,

$$\alpha_{-1}^{\mu}|0\rangle_L \times \tilde{\alpha}_{-1}^{\nu}|0\rangle_R. \tag{1.12}$$

This state has some interesting properties. The first property of interest is that this is a massless state as the vibrational energy exactly cancels the vacuum energy for both the right- and left-movers. This means that the state corresponds to a massless particle in spacetime. The second property of interest is that this state contains a spacetime spin-2 component (as it is the product of two operators which transform as vectors in spacetime). It turns out that these two properties together imply that the spin-2 portion of this state *must* have the equation of motion of the graviton! Thus, string theory is actually a theory of gravity at a very high energy scale. It is also interesting to note that this state has a spin-0 component, which corresponds to the dilaton. The vacuum expectation value of the dilaton controls the interaction strength for string vertices.

In general, analyzing the spacetime spectrum of superstrings is much more difficult than the bosonic string spectrum. The extra complications associated with the boundary conditions of the worldsheet fermions allow for much more variation within superstring theories (in their critical number of large spacetime dimensions) than for the bosonic string[1]. We shall first define what it means to consider a superstring model.

As mentioned in Sect. 1.1.2, a set of boundary conditions for worldsheet fermions is known as a sector. In general, a string model is composed of many different sectors. Many of the complications involved in string model building are related to the fact that string self-consistency conditions may forbid certain sectors from appearing in the same model or conversely require that extra sectors be included. As a further complication, string self-consistency conditions may alter the spacetime spectrum resulting from a certain sector of the model based on the presence or absence of other possible sectors. We shall therefore describe how it is possible that spacetime fermions result from superstring theories and then cover a few important properties of superstring spectra.

At this point, we have not shown how a spacetime fermion could result from string theory. At first glance, it might seem strange that string theory as currently formulated could give rise to a spacetime fermion as every field in the worldsheet theory transforms as a spacetime boson. However, recall that in Eq. (1.9) Ramond fermions have a zero-mode excitation which has no energy associated with it. This means that for theories with Ramond fermions, the vacuum is actually degenerate as

[1] In this entire chapter, we are focusing only on the closed string theories in their critical spacetime dimension (26 for bosonic, 10 for Type II and heterotic). If one compactifies some of these spacetime dimensions then the situation for the spectrum becomes significantly more complex.

there are two zero energy states, namely,

$$|V_0\rangle = |0\rangle \quad \text{and} \quad |V_1\rangle = b_0|0\rangle. \tag{1.13}$$

However, one can flip between these two states by applying the operator $\sqrt{2}b_0$. This operator has zero energy associated with it. This means that the actual vacuum for Ramond fermions is a two-component state in spacetime and this two-component state can be shown to be a spinor. Therefore, only models with fermions with Ramond boundary conditions can give rise to spacetime fermions.

The vacuum of the bosonic string gave rise to a spacetime tachyon. However, the vacuum of the heterotic string will *not* appear in the spacetime spectrum of a heterotic model. This does not occur for a very interesting reason. The vacuum energy for the lowest energy sector of a heterotic model is $a_L = -1, a_R = -1/2$. The difference in vacuum energies stems from the difference in the field content for the heterotic string. Therefore, the vacuum of the heterotic string will not appear in the spacetime spectrum of a heterotic model as an on-shell tachyon because it is not level-matched!

String models always give rise to an infinite number of on-shell particles in spacetime. For every phenomenologically interesting particle, there exists an infinite tower of particles with identical quantum numbers, but increasing mass. This is easy to see from the fact that there is no limit to the mode expansion, thus ever higher mass particles can always be produced.

As a final note about the spectrum of superstring models, we should consider spacetime supersymmetry. Superstring models have worldsheet supersymmetry by definition. However, this does not imply that all superstring models have spacetime supersymmetry as well! To have spacetime supersymmetry, there must be exactly as many spacetime bosons in the spectrum as spacetime fermions. However, this is not generically true for superstring models.

Clearly, this short introduction cannot properly do justice to string theory. However, we shall attempt to outline the important features. String amplitudes are calculated by considering different worldsheet topologies and the S-matrix can be calculated by summing over every possible worldsheet topology. Spacetime geometry is a derived quantity from the worldsheet. Therefore, worldsheet self-consistency is actually the most important quality a string model can have. Many of the seemingly strange spacetime symmetries of string theory ultimately stem from the fact that the worldsheet is fundamental. We stress that this new requirement of worldsheet self-consistency ultimately leads to many of the differences between field theory and string theory.

1.2 String Theory at Finite Temperature

The starting point for any discussion of string theory at finite temperature should be the method of calculating finite temperature amplitudes in string theory. Specifically, we shall focus on how to find the effective potential to one-loop for closed strings. We shall begin with a calculation of the effective potential to one-loop in field theory. We shall explicitly follow the standard prescription for transforming the field theoretic calculation into the string theoretic calculation. After we have accomplished this, we shall briefly describe the changes which must be made to accommodate fermions.

The effective potential in field theory in D spatial dimensions for an ideal gas of bosons at temperature T, is given by

$$\begin{aligned} F(T) &= \frac{T}{2}\sum_{n=0}^{\infty} g_n \int \frac{d^D k}{(2\pi)^D} \ln\,(k^2 + 4\pi^2 n^2 T^2) \\ &= \frac{T}{2}\sum_{n=0}^{\infty} g_n \int \frac{d^D k}{(2\pi)^D} \ln\,(k^2 + (M_n(T))^2), \end{aligned} \tag{1.14}$$

where $M_n \equiv 2\pi n T$ and is the "mass" of the Matsubara mode, and g_n represents the number of independent degrees of freedom at mass level n. For $D = 3$, this reduces to the expression well known in quantum field theory [9]. In order to rewrite this expression into the form commonly used in string theory we shall follow the standard prescription by rewriting this expression in terms of a Schwinger proper time t by using the identity

$$\log x \;=\; \int_1^x \frac{dy}{y} \;=\; \int_1^x dy \int_0^\infty dt\; e^{-yt} \;=\; -\int_0^\infty \frac{dt}{t}\; e^{-xt} \;+\; ...$$

where we have dropped an x-independent term. The x-independent term will not ultimately affect the final form of the effective potential and thus, dropping it will not adversely affect our result. We thus obtain

$$F(T) \;=\; -\frac{T}{2} \sum_n g_n \int \frac{d^D k}{(2\pi)^D} \int_0^\infty \frac{dt}{t} e^{-(k^2+(M_n(T))^2)t} \;. \tag{1.15}$$

It is interesting to note that in this form, the ultraviolet divergence associated with the $k^2 \to \infty$ region of integration has been mapped to the $t \to 0$ region of integration. By performing the momentum integration we arrive at

$$F(T) \;=\; -\frac{T}{2} \frac{1}{(4\pi)^{D/2}} \sum_n g_n \int_0^\infty \frac{dt}{t^{1+D/2}} e^{-(M_n(T))^2 t} \;. \tag{1.16}$$

We shall now change to a different set of variables. As was mentioned in Sect. 1.1.1, string theoretic amplitudes are modular invariant. Modular invariance will ultimately be responsible for removing the troublesome region of integration. In order to show how modular invariance can arise, we now shift our notation slightly by making the following substitutions. First, we define the *dimensionless* real parameter τ_2 as

$$\tau_2 \;=\; \frac{1}{4\pi}\, \mu^2\, t \tag{1.17}$$

where μ is an (as yet) unspecified mass scale. Second, we introduce an additional dimensionless real variable τ_1 by inserting

$$1 \;=\; \int_{-1/2}^{1/2} d\tau_1 \tag{1.18}$$

into our expressions. We then combine our two new parameters to form the complex variable

$$\tau \;\equiv\; \tau_1 + i\tau_2 \;, \tag{1.19}$$

thereby enabling us to rewrite our expression for $F(T)$ in the form

$$F(T) \;=\; -\frac{T}{2}\left(\frac{\mu}{4\pi}\right)^{D} \int_{\mathcal{S}} \frac{d^2\tau}{{\tau_2}^2}\, Z(\tau,T) \tag{1.20}$$

where $\mathcal{S}$ denotes the semi-infinite strip in the complex τ-plane

$$\mathcal{S} \;\equiv\; \left\{\tau \;:\; |\mathrm{Re}\,\tau| \le \tfrac{1}{2},\ \mathrm{Im}\,\tau \ge 0 \,\right\} \tag{1.21}$$

and where the integrand $Z(\tau)$ is

$$\begin{aligned} Z(\tau,T) \;&\equiv\; {\tau_2}^{1-D/2} \sum_n g_n\, \exp(-4\pi\tau_2 (M_n(T))^2/\mu^2) \\ &=\; {\tau_2}^{1-D/2} \sum_n g_n\, (\overline{q}q)^{(M_n(T))^2/\mu^2} \end{aligned} \tag{1.22}$$

with $q \equiv e^{2\pi i\tau}$.

We can now write $M_n(T)$ in terms of the canonical normalization for closed strings. This normalization sets the value for μ, namely $\mu = 2M_{\text{string}}$[2]. We can now add any term of the form

$${\tau_2}^{1-D/2} \sum_{\substack{m\neq n \\ m=n\,(\mathrm{mod}\,1)}} g_{mn}\; \overline{q}^{(M_m(T))^2/\mu^2}\; q^{(M_n(T))^2/\mu^2} \tag{1.23}$$

to $Z(\tau,T)$ without changing the effective potential. This is because any extra terms of the form (1.23) integrate to zero across the strip (1.21), regardless of the values of g_{mn}. We can therefore combine (1.22) and (1.23) in order to write our total integrand as

$$Z(\tau,T)) \;=\; {\tau_2}^{1-D/2} \sum_{m,n} g_{mn}\, \overline{q}^{(M_m(T))^2/\mu^2}\; q^{(M_n(T))^2/\mu^2}. \tag{1.24}$$

$Z(\tau,T)$ is commonly referred to as the string partition function.

We are not quite finished transforming the field theoretic expression for the effective potential at finite temperature (Eq. 1.14) into a form which is valid for string theoretic calculations. We have changed our variables of integration in order to utilize modular invariance, but before we fully utilize modular invariance we must note two properties of the string partition function. It turns out that string partition functions are always invariant under two symmetries, ($\tau \to \tau + 1$ and $\tau \to -1/\tau$). The first symmetry is clear simply from inspection, but the second requires a subtle cancelation in the g_{mn}.

These two transformations generate the group of modular transformations. As the integrand and the measure ($\frac{d^2\tau}{\tau_2^2}$) are both invariant under these transformations, but the region of integration ($\mathcal{S}$) is *not*, this integral is overcounting. The situation is similar to gauge symmetries in quantum field theory in that when calculating a scattering amplitude we must avoid overcounting by dividing out by the infinite symmetry volume factor; in other words, we must tally only those contributions which are inequivalent with respect to the symmetry. We must therefore truncate our strip region of integration so that the new (smaller) region of integration includes only one representative value of τ up to the combined modular transformations ($\tau \to \tau + 1$ and $\tau \to -1/\tau$). Such a region is given by

$$\mathcal{F} \;\equiv\; \left\{ \tau \;:\; |\mathrm{Re}\,\tau| \le \tfrac{1}{2},\, |\tau| \ge 1 \right\} \;, \tag{1.25}$$

and is commonly called the *fundamental domain of the modular group*. By excluding the real axis completely, modular invariance thus succeeds in canceling the ultraviolet divergence in the effective potential without introducing a new fundamental cutoff scale beyond μ; essentially the modular symmmetry renders the divergence spurious by enabling it to be reinterpreted as the infinite volume associated with a symmetry group. Dividing out by this volume, one thus obtains a new, manifestly finite expression for the cosmological constant:

$$F(T) \;=\; -\,\frac{T}{2}\,\left(\frac{\mu}{4\pi}\right)^D \int_{\mathcal{F}} \frac{d^2\tau}{{\tau_2}^2}\, Z(\tau,T) \;. \tag{1.26}$$

Indeed, our discussion thus far has merely described the standard "recipe" by which one calculates the one-loop effective potential in string theory [10].

We still have not addressed what the form of the string partition function should be. This is because the string partition function enumerates all of the excitations which are present in a specific string model. Because of this model dependence, we shall only discuss two aspects of the string partition function and then proceed to a brief discussion of some of the issues we shall attempt to address in finite temperature string theory.

As we shall be primarily interested in the temperature dependence of the effective potential, it is natural to focus on the contribution to the effective potential arising from the Matsubara modes. The contribution from the Matsubara modes fully determines the temperature dependence of the effective potential. Fortunately, for bosonic strings the string partition function factors into a particularly convenient form,

$$Z(\tau, T) = Z_{\text{model}}(\tau) \times Z_{\text{circ}}(\tau, T) \tag{1.27}$$

where Z_{circ} represents the contributions coming from the Matsubara modes and Z_{model} represents the contributions coming from the actual excitations of the string model. Thus, Z_{model} is actually the zero-temperature theory which is being examined at finite temperature. Z_{circ} therefore, represents the *extra* contribution to the energy coming from the sum over the Matsubara modes.

It turns out for bosonic string theories[8], Z_{circ} is,

$$Z_{\text{circ}}(\tau, T) \;=\; \sqrt{\tau_2} \sum_{m,n \in \mathbb{Z}} \overline{q}^{(ma-n/a)^2/4}\, q^{(ma+n/a)^2/4} \tag{1.28}$$

where $a = 2\pi T/M_{\text{string}}$. This expression looks very different from the equivalent expression in field theory. The expression in Eq. 1.28 has Matsubara modes whose contribution to the thermal partition function actually decreases with temperature. These Matsubara modes are thermal winding modes and are a generic feature for closed strings at finite temperature. They are actually required to be present for the theory to maintain modular invariance. As discussed earlier, modular invariance is a fundamental symmetry of string theory, thus these modes must also be present. We will discuss one of the interesting consequences of the existence of these winding modes in the future.

Up to this point, we have merely been discussing theories without spacetime fermions. Thus, the treatment up to now would be appropriate for bosonic string theories. However, superstrings generically produce spacetime fermions and thus some modification is necessary. As was mentioned earlier, the string partition function ($Z(\tau)$) encodes all of the particle excitations. It turns out that, the string partition function is actually a supertrace over the Fock spaces for the left- and right-moving CFTs. Thus, to incorporate fermions into the theory one must do two different things. Inside the model-dependent portion of the string partition function Z_{model}, fermionic excitations must contribute with a minus sign. Thus, we would expect that a general term in the string partition function would have the form, $g_{mn}\,(-1)^F\,\overline{q}^{(M_m/\mu)^2}q^{(M_n/\mu)^2}$ where F is the fermion number of excitation. The other modification required involves the Matsubara modes. The Matsubara modes for fermions are generically different from those of bosons (anti-periodic around the thermal circle instead of periodic), thus Z_{circ} is different. For a full discussion of this, please refer to Sect. 4.3.1.

An examination of Eq. 1.28 reveals an interesting feature. The functional form presented is invariant if the transformation $a \to 1/a$ is performed simultaneously with $m \leftrightarrow n$. As the only temperature dependent quantity is Z_{circ} this suggests a rather surprising relation. One can rewrite the transformation into the form $T \to T_c^2/T$ where $T_c \equiv M_{\text{string}}/2\pi$. This is the basis of thermal duality. We note that thermal duality is ultimately related to T-duality because of the remarkable correspondence between temperature and geometry. Thus, the algebraic basis of thermal duality lies in Eq. 1.28, but the physical basis is mu ch deeper and probably can be generalized to other strings.

The idea that the physics of one system at some temperature T should be related to the physics of a possibly different system at a temperature inversely related to T is not a new one. The Kramers-Wannier duality[11] is well-known in many different systems. The Kramers-Wannier duality has inspired a lot of study and has grown into a rather large field. In the interests of space, we refer the interested reader to a relatively recent review of systems exhibiting the Kramers-Wannier duality[12].

The relevant degrees of freedom for a system on one side of the Kramers-Wannier duality are not necessarily the same as the relevant degrees of freedom on the other side of the duality. When this is the case, we say that the system has undergone a Kosterlitz-Thouless phase transition[13]. This might seem to be identical to thermal duality, however it is actually very different. The classic example of a system undergoing a Kosterlitz-Thouless transition is the x-y model in two dimensions. In the x-y model, the relevant degrees of freedom change from quantum rotors to vortices. This is a change from a particle-like degree of freedom (quantum rotor) to a topological degree of freedom (vortex) and is characteristic of a Kosterlitz-Thouless transition. However, in thermal duality winding modes and momentum modes are interchanged and both classes of mode actually behave remarkably similarly as shown by the invariance of the thermal partition function under the thermal duality transformation. This difference ultimately leads to very different consequences for a system exhibiting thermal duality rather than Kramers-Wannier duality.

Given that we do not understand much of the non-perturbative structure of string theory, why would we expect that an analysis of what we do understand of string theory would have radically different behavior at finite temperature than a similar analysis performed in field theory? We have already seen that there is a potential for difference in the different way in which thermal duality is realized. However, we still have not addressed why we expect that there might be a difference.

One reason to expect that string theory at finite temperature should be different from field theory is the inclusion of gravity. Field theoretic analyses of finite temperature systems necessarily do not consider the effects of gravity. Quantum gravitational effects are expected to be very small in the energy regime generally under consideration in a field theoretic context. However, it is an interesting and very relevant question as to what the effects of quantum gravity are on the normal expectations of statistical mechanics. One reason why this is a difficult question to address is the fact that there is actually a tension between equilibrium statistical mechanics and gravity. Systems that are amenable to statistical mechanical treatment have a relatively large number of degrees of freedom implying that the larger a volume that can be analyzed the better. Gravitating systems suffer from a Jean's instability if too large a system is considered. The Jean's instability sets in at the length scale,

$$R_J \geq \left(\frac{\pi c}{G\rho}\right)^{\frac{1}{2}}. \tag{1.29}$$

Physically the Jean's instability is simply the statement that gravitating systems tend to collapse into black holes if they are too dense. This leads to the unconformable situation where a system with many degrees of freedom is preferable (*i.e.* very dense and very large), but such systems are not in equilibrium in the presence of gravity. At this point, it is actually very difficult to address the question of string theory in a non-trivial background at finite temperature. We shall therefore take the attitude that we shall ignore this problem and effectively attempt to find whether string theory in a flat background differs from the identical situation in field theory.

Another difference between sting theory and field theory cuts to the core of finite temperature examinations. The ability to identify finite temperature with compactified dimensions is a very useful tool for analyzing finite temperature systems. In field theory, there exists a large amount of freedom with respect to compactification. To a certain extent, one can take almost any field theory and examine it on any compact manifold, $\mathcal{M}$. However, in string theory the spacetime background is a *derived* quantity. As discussed earlier, worldsheet is the fundamental description of the physics and one finds the self-consistent spacetime structure by examining the worldsheet. This means that not every compact manifold, $\mathcal{M}$, can be generated from a self-consistent string model. This can potentially be a huge problem as identifying the radius of a compact dimension with inverse temperature requires two different things. First, the string model must allow for a manifold with a compact dimension, and second, the manifold must have a radius modulus.

One of the hallmarks of string thermodynamics is the Hagedorn transition. The Hagedorn transition is a phase transition that only occurs in theories where the degeneracy of states grows exponentially with mass[14]. The standard form for the thermal partition function is,

$$Z(T) = \sum_n g_n e^{-\beta E_n}$$

where g_n is the degeneracy of states with energy E_n. If we assume that $g_n \sim e^{\alpha E_n}$ then the partition function will diverge when $\alpha > \beta$. This is because the growth in the degeneracy of states overcomes the Boltzmann suppression present in the thermal partition function. In the context of string theory, one can reformulate this phenomena as the development of a physical tachyon at the Hagedorn temperature. In the course of this thesis, we shall usually refer to this other picture. This correspondence is fully developed in Sect. 5.5.1.

This dissertation explores different consequences of the temperature/geometry correspondence in the context of finite temperature string theory. It is outlined as follows: in chapter two, we shall discuss using thermal duality to constrain the form of different thermodynamic potentials for specific bosonic systems. In chapter three, we will explore the consequences of requiring that thermodynamic potentials respect thermal duality in a covariant manner. The last two chapters will consist of exploring to what extent the radius of compactification and temperature connection survives in string theory and examining a possible modification of this relation.

Chapter 2
Closed-Form Expressions for Thermodynamic Quantities from Thermal Duality

2.1 Introduction

Some of the most intriguing features of string theory have been the existence of numerous dualities which connect physics in what would otherwise appear to be vastly dissimilar regimes. Such dualities include strong/weak coupling duality (S-duality) as well as large/small compactification radius duality (T-duality), and together these form the bedrock upon which much of our understanding of the full, non-perturbative moduli space of string theory is based.

There is, however, an additional duality which has received far less scrutiny: this is *thermal duality*, which relates string theory at temperature T with string theory at the inverse temperature T_c^2/T where T_c is a critical (or self-dual) temperature related to the string scale. Thermal duality follows naturally from T-duality and Lorentz invariance, and thus has roots which are as deep as the dualities that occur at zero temperature. Given the importance of dualities of all sorts in extending our understanding of the unique features of non-perturbative string theory, we are led to ask what new insights can be gleaned from a study of thermal duality.

In this chapter, we shall focus on the first feature that immediately strikes any student of this subject: classical thermodynamics, as currently formulated, is not invariant (or covariant) under thermal duality. While certain thermodynamic quantities such as the free energy and the internal energy of an ideal closed string gas exhibit invariances (or covariances) under thermal duality transformations, other quantities such as entropy and specific heat do not.

In this chapter, we shall investigate whether thermal duality might nevertheless happen to be preserved for special choices of the effective potential. In other words, we shall investigate whether it is possible to construct an effective potential such that *all* corresponding physically relevant thermodynamic quantities will turn out to be duality covariant. Thus, in this way, we seek to exploit thermal duality in order to constrain the effective potential in a manner that transcends a direct order-by-order perturbative calculation.

Remarkably, we shall find that there exist a unique series of functional forms which have this property. Moreover, we shall demonstrate that these solutions successfully capture the leading temperature dependence of the one-loop effective potentials for a variety of finite-temperature string ground states involving time/temperature compactifications on S^1 (circles) and S^1/Z_2 (orbifolds) in all dimensions $D \geq 2$. The precision with which this occurs leads us to conjecture that our solutions might actually represent the *exact* solutions for the corresponding string effective potentials when results from all orders of perturbation theory are included.

Note that a preliminary summary of some of these results has appeared in Ref. [15]. Our goal here is to provide a more complete and self-contained discussion and derivation of these results. There are, however, numerous topics pertaining to string thermodynamics which we will not address in this chapter. These include the nature of the Hagedorn phase transition as well as the Jeans instability and general issues concerning the interplay between gravity and thermodynamics. It would be interesting to explore the extent to our results concerning thermal duality can shed light on these issues, and we hope to address these questions in future work.

2.2 Thermal duality and the rules of thermodynamics

Let us begin by quickly presenting some of the key ideas that will be relevant for our discussion. Our goal will be to highlight the manner in which the rules of standard thermodynamics generally tend to break thermal duality.

Just as in ordinary statistical mechanics, the fundamental quantity of interest in string thermodynamics is the one-loop thermal string partition function $Z_{\text{string}}(\tau, T)$. This partition function generally exhibits the symmetries of the underlying theory. For example, we shall assume that Z_{string} is invariant under modular transformations:

$$Z_{\text{string}}(\tau+1, T) \;=\; Z_{\text{string}}(-1/\tau, T) \;=\; Z_{\text{string}}(\tau, T)\;, \tag{2.1}$$

where τ is the complex modular parameter describing the shape of the one-loop (toroidal) worldsheet. Modular invariance is required for the consistency of the corresponding closed string model, and arises from the assumption of conformal invariance at the one-loop level.

More importantly, Z_{string} is invariant under thermal duality:

$$Z_{\text{string}}(\tau, T_c^2/T) \;=\; Z_{\text{string}}(\tau, T) \tag{2.2}$$

where T_c is the self-dual temperature. Thermal duality also has deep roots (for early papers, see Refs. [16, 17, 8, 18, 19, 1]). In general, finite-temperature effects can be incorporated into string theory [10] by compactifying an additional time dimension on a circle (or orbifold [20]) of radius $R_T = (2\pi T)^{-1}$. However, Lorentz invariance guarantees that the properties of this extra time dimension should be the same as those of the original space dimensions, and T-duality [21, 22, 23] tells us that closed string theory on a compactified space dimension of radius R is indistinguishable from that on a space of radius R_c^2/R where $R_c = \sqrt{\alpha'}$ is the self-dual radius. Together, these symmetries then imply thermal duality, with $T_c \equiv M_{\text{string}}/2\pi$. Note that the thermal duality symmetry holds to all orders in perturbation theory [18].

All thermodynamic quantities of interest are generated from Z_{string}. The finite-temperature vacuum amplitude $\mathcal{V}(T)$ is given by [10, 24, 16]

$$\mathcal{V}(T) \;\equiv\; -\tfrac{1}{2}\,\mathcal{M}^{D-1}\int_{\mathcal{F}}\frac{d^2\tau}{(\text{Im}\,\tau)^2} Z_{\text{string}}(\tau, T) \tag{2.3}$$

where $\mathcal{M} \equiv M_{\text{string}}/2\pi$ is the reduced string scale; D is the number of non-compact spacetime dimensions; and $\mathcal{F} \equiv \{\tau : |\text{Re}\,\tau| \leq \frac{1}{2}, \text{Im}\,\tau > 0, |\tau| \geq 1\}$ is the fundamental domain of the modular group. Note that $T_c = \mathcal{M}$. In general, $\mathcal{V}(T)$ plays the role usually taken by the logarithm of the statistical-mechanical partition function. Because of its role in governing the dynamics of the theory, we shall occasionally refer to the vacuum amplitude $\mathcal{V}(T)$ as the "effective potential" even though this terminology is often used instead to describe the free energy F. Given this definition for $\mathcal{V}$, the free energy F, internal energy U, entropy S, and specific heat c_V then follow from the standard thermodynamic definitions:

$$F = T\mathcal{V} \ , \qquad U = -T^2 \frac{d}{dT}\mathcal{V} \ , \qquad S = -\frac{d}{dT}F \ , \qquad c_V = \frac{d}{dT}U \ . \tag{2.4}$$

It is easy to see that the thermal duality invariance of Z_{string} is inherited by some of its descendants. Since $\mathcal{V}$ is just the modular integral of Z_{string}, $\mathcal{V}$ is also invariant under thermal duality transformations:

$$\mathcal{V}(T_c^2/T) \ = \ \mathcal{V}(T) \ . \tag{2.5}$$

Likewise, it is easy to verify that the free energy F and the internal energy U transform *covariantly* under thermal duality:

$$F(T_c^2/T) = \left(\frac{T_c}{T}\right)^2 F(T) \ , \qquad U(T_c^2/T) = -\left(\frac{T_c}{T}\right)^2 U(T) \ . \tag{2.6}$$

Thus, these quantities also respect the thermal duality symmetry; in fact, this symmetry sets a zero for the internal energy such that $U(T_c) = 0$.

Unfortunately, the entropy and specific heat fail to have any closed transformation properties under the thermal duality symmetry. Specifically, we find

$$\begin{aligned} S(T_c^2/T) \ &= \ -S(T) - 2F(T)/T \ , \\ c_V(T_c^2/T) \ &= \ c_V(T) - 2U(T)/T \ . \end{aligned} \tag{2.7}$$

This failure to transform covariantly suggests that entropy and specific heat are improperly defined from a string-theoretic standpoint. At best, they are not the proper "eigenquantities" which should correspond to physical observables.

It is easy to diagnose the source of this problem. In general, a function $f(T)$ will be called thermal duality covariant with weight k and sign $\gamma = \pm 1$ if, under the thermal duality transformation $T \to T_c^2/T$, we find

$$f(T) \;\to\; f(T_c^2/T) \;=\; \gamma\,(T_c/T)^k\, f(T)\;. \tag{2.8}$$

Thus, $\mathcal{V}$ has $(k,\gamma) = (0,1)$, while F and U have $(k,\gamma) = (2,1)$ and $(2,-1)$ respectively. Note that $\gamma = \pm 1$ are the only two possible choices consistent with the $\mathbb{Z}_2$ nature of the thermal duality transformation. In general, multiplication by T is a covariant operation, resulting in a function with weight $k+2$ and the same sign for γ. However, the temperature derivative d/dT generally breaks duality covariance. To see this, let us imagine that $f(T)$ has weight k and sign γ. Evaluating df/dT at temperature T_c^2/T, we then find

$$\begin{aligned}
\left[\frac{df}{dT}\right](T_c^2/T) \;&=\; \frac{d}{d(T_c^2/T)} f(T_c^2/T) \\
&=\; -\gamma \left(\frac{T}{T_c}\right)^2 \frac{d}{dT}\left[(T_c/T)^k f(T)\right] \\
&=\; -\gamma \left(\frac{T_c}{T}\right)^{k-2} \left(\frac{df}{dT} - \frac{kf}{T}\right)\;.
\end{aligned} \tag{2.9}$$

Thus, as a result of the second term above, we see that df/dT fails to transform covariantly under the thermal duality transformation unless f itself has $k = 0$. Since the vacuum amplitude $\mathcal{V}$ has $k = 0$, this explains why the internal energy U continues to be duality covariant (with $k = 2$) even though it involves a temperature derivative. However, since the free energy F and the internal energy U each already have $k = 2$, we see that subsequent derivatives yield quantities (such as the entropy S and specific heat c_V) which are no longer duality covariant.

2.3 Special solutions for string effective potentials

Let us now consider whether there might exist special finite-temperature vacuum amplitudes $\mathcal{V}(T)$ in which thermal duality covariance is preserved for all thermodynamic quantities. In other words, we shall seek special solutions for $\mathcal{V}(T)$ such that *all* of its thermodynamic descendants turn out to be duality covariant, even though the rules by which these quantities are calculated explicitly break this symmetry. We emphasize that in choosing this line of attack, we are necessarily losing generality; we are essentially limiting our attention to special, highly symmetric string ground states. Nevertheless, as we shall see, it is important to investigate this possibility.

2.3.1 General approach

In order to proceed along these lines, we first need to address a general mathematical question: from amongst all duality-covariant functions $f(T)$ of weight k and sign γ, are there any *special* functions $f(T)$ for which df/dT "accidentally" turns out to be covariant?

Given the derivative in Eq. (2.9), we see that there is only one way in which df/dT can possibly be thermal duality covariant: we must have

$$\frac{df}{dT} - \frac{kf(T)}{T} = -\,\delta\left(\frac{T_c}{T}\right)^{\ell}\frac{df}{dT} \tag{2.10}$$

for some sign δ and exponent ℓ. If Eq. (2.10) is satisfied, then we see from Eq. (2.9) that df/dT will indeed be covariant, with sign $\gamma\delta$ and weight $k+\ell-2$. Note that we must have $\delta = \pm 1$ in order to produce a consistent sign for df/dT. (The minus sign in front of δ has been inserted for future convenience.)

It is not difficult to find solutions for $f(T)$ in Eq. (2.10), since this is nothing but a linear first-order differential equation. For $\ell \neq 0$, we thus obtain the general solution

$$f \;\sim\; (T^{\ell} + \delta T_c^{\ell})^{k/\ell} \tag{2.11}$$

where we are disregarding an overall, arbitrary, T-independent normalization factor. However, in this derivation we assumed that f has weight k and sign γ. Checking the solution in Eq. (2.11), we find that this does not restrict the value of ℓ, but does require that $\delta^{k/\ell} = \gamma$.

By contrast, if $\ell = 0$ in Eq. (2.10), we obtain a non-zero solution for $f(T)$ only if $\gamma = 1$ and $\delta = +1$:

$$f \;\sim\; T^{k/2}\;. \tag{2.12}$$

As required, this also has weight k.

Thus, from amongst all possible covariant functions $f(T)$ with weight k and sign γ, we have found that only an extremely restrictive form for $f(T)$ guarantees that df/dT is also thermal duality covariant: either $f(T)$ must have the form given in Eq. (2.11) where $\ell \neq 0$ is arbitrary and where $\delta^{k/\ell} = \gamma$, with $\delta = \pm 1$; or $f(T)$ must have the form given in Eq. (2.12), which can occur only if $\gamma = 1$. Of course, overall multiplicative factors of T_c can always be introduced in either expression as needed on dimensional grounds.

2.3.2 Preserving duality covariance for entropy and specific heat: A thermal duality "bootstrap"

Using this, let us now reconsider our original thermodynamic problem. We begin with a vacuum amplitude $\mathcal{V}$, which is invariant under thermal duality transformations. Thus, $\mathcal{V}$ necessarily has $k = 0$ and $\gamma = 1$. From this, we proceed to derive F and U. Once again, these quantities are also automatically duality covariant; they each have weight $k = 2$ and their signs are $\gamma = +1$ and -1 respectively. Up to this point, the functional forms for $\mathcal{V}$, F, and U are completely arbitrary (subject to the above constraints on their weights and signs). However, it is in calculating S and c_V that potential difficulties arise, for we must demand that S and c_V be simultaneously covariant as well. This then provides two new non-trivial constraints on the forms of F and U, as discussed above. Working backwards, this then provides a very restrictive set of possibilities for the vacuum amplitudes $\mathcal{V}$ from which both F and U are derived. In other words, we will have essentially used a "bootstrap" formed by demanding the covariance of S and c_V to deduce a particular form (or set of forms) of the vacuum amplitude $\mathcal{V}$.

Carrying out this calculation is relatively straightforward. We first focus on the entropy S. In order for S to be thermal duality covariant, the free energy F (which must have weight $k = 2$ and sign $\gamma = 1$) is required to take the form

$$F(T) \sim -\frac{(T^\ell + \delta T_c^\ell)^{2/\ell}}{T_c} \tag{2.13}$$

where

$$\delta^{2/\ell} = 1 \, . \tag{2.14}$$

Note that the factor of T_c in the denominator of Eq. (2.13) has been inserted on dimensional grounds (where we implicitly express our thermodynamic quantities in units of $\mathcal{M}^{D-1}$); likewise, we have also inserted an overall minus sign for future convenience. Also note that Eq. (2.14) restricts us to $\delta = +1$ for even ℓ, but allows $\delta = \pm 1$ for odd ℓ. This form for F guarantees that S, which takes the form

$$S(T) \sim 2\,\frac{T^{\ell-1}}{T_c}\,(T^\ell + \delta T_c^\ell)^{2/\ell - 1} \, , \tag{2.15}$$

is covariant with weight ℓ and sign δ.

We are of course deliberately disregarding the $\ell = 0$ possibility, stemming from Eq. (2.12), that $F(T) \sim T$. We reject this possibility not only because this would make $F(T)$ independent of T_c (which is unexpected from a string calculation), but also because it leads to an entropy which is completely temperature-independent and hence unphysical.

Given $F(T)$ in Eq. (2.13), we immediately determine that $\mathcal{V}(T)$ must take the general form

$$\mathcal{V}(T) \sim -\frac{(T^\ell + \delta T_c^\ell)^{2/\ell}}{T T_c} \, . \tag{2.16}$$

Note that this is indeed invariant under thermal duality transformations, as required. This in turn implies that $U(T)$ must have the general form

$$U(T) \sim \frac{1}{T_c}\,(T^\ell + \delta T_c^\ell)^{2/\ell - 1}\,(T^\ell - \delta T_c^\ell) \, , \tag{2.17}$$

which is of course consistent with our requirement that U have weight 2 and sign -1. Thus, up to this point, we have found that the entropy will be thermal duality covariant (along with the effective potential, the free energy, and the internal energy) if and only if $\mathcal{V}(T)$ takes the form (2.16).

We now impose our requirement that c_V also be thermal duality covariant. As we shall see, this will provide a constraint on the value of ℓ. Since $U(T)$ is given in Eq. (2.17), we can immediately calculate the specific heat, obtaining

$$c_V(T) \;\sim\; 2\,\frac{T^{\ell-1}}{T_c}\,(T^\ell+\delta T_c^\ell)^{2/\ell-2}\,\left[T^\ell+(\ell-1)\delta T_c^\ell\right]\;. \tag{2.18}$$

Clearly, this quantity fails to be duality covariant unless the final factor in square brackets takes the form $T^\ell \pm \delta T_c^\ell$ with $\delta=\pm 1$, or unless this factor takes the form T^ℓ (in which case this factor joins with the overall $T^{\ell-1}$ prefactor to modify the duality weight of c_V). These two options occur only for $\ell=2$ or $\ell=1$ respectively.

Note that the $\ell=1,2$ cases provide maximal duality symmetry for our solutions. Indeed, in these cases, our solution for $U(T)$ also simultaneously takes the form

$$U(T) \;\sim\; \frac{(T^m+\epsilon T_c^m)^{2/m}}{T_c} \tag{2.19}$$

for some m and sign $\epsilon=\pm 1$ (with $\epsilon^{2/m}=-1$), as required from Eq. (2.11) in order to yield a covariant specific heat $c_V=dU/dT$. Moreover, since the specific heat is also given by the relation $c_V=TdS/dT$, our solution for $S(T)$ also takes this same special form in these cases.

Thus, summarizing, we see that our requirement of preserving general covariance for our thermodynamic quantities forces them to have a particular form:

$$\begin{aligned}
\mathcal{V}^{(\ell)}(T) &\sim -(T^\ell+\delta T_c^\ell)^{2/\ell}/TT_c\\
F^{(\ell)}(T) &\sim -(T^\ell+\delta T_c^\ell)^{2/\ell}/T_c\\
U^{(\ell)}(T) &\sim \quad (T^\ell+\delta T_c^\ell)^{2/\ell-1}(T^\ell-\delta T_c^\ell)/T_c\\
S^{(\ell)}(T) &\sim 2\,T^{\ell-1}(T^\ell+\delta T_c^\ell)^{2/\ell-1}/T_c\\
c_V^{(\ell)}(T) &\sim 2\,T^{\ell-1}(T^\ell+\delta T_c^\ell)^{2/\ell-2}\left[T^\ell+(\ell-1)\delta T_c^\ell\right]/T_c
\end{aligned} \tag{2.20}$$

where

$$\delta \;=\; \begin{cases} +1 & \ell \text{ even} \\ \pm 1 & \ell \text{ odd} \end{cases} \quad . \tag{2.21}$$

These solutions are plotted in Fig. 2.1, and ensure that $\mathcal{V}$, F, U, and S are all thermal duality covariant for any value of ℓ. While $\mathcal{V}$, F, and U have duality weights $(k,\gamma) = (0,1)$, $(2,1)$, and $(2,-1)$ respectively, the entropy S has duality weight and sign $(k,\gamma) = (\ell,\delta)$. Observe that the traditional relation $U = F + TS$ continues to hold for all ℓ.

However, c_V will also be thermal duality covariant if and only if $\ell = 1$ or $\ell = 2$. The explicit solutions in these cases reduce to

$$\begin{array}{lrcl} \ell = 2: & \mathcal{V}^{(2)}(T) & = & -(T^2 + T_c^2)/(TT_c) \\ & F^{(2)}(T) & = & -(T^2 + T_c^2)/T_c \\ & U^{(2)}(T) & = & \;\;\,(T^2 - T_c^2)/T_c \\ & S^{(2)}(T) & = & 2T/T_c \\ & c_V^{(2)}(T) & = & 2T/T_c \end{array} \tag{2.22}$$

and

$$\begin{array}{lrcl} \ell = 1: & \mathcal{V}^{(1)}(T) & = & -(T + \delta T_c)^2/(TT_c) \\ & F^{(1)}(T) & = & -(T + \delta T_c)^2/T_c \\ & U^{(1)}(T) & = & \;\;\,(T^2 - T_c^2)/T_c \\ & S^{(1)}(T) & = & 2(T/T_c + \delta) \\ & c_V^{(1)}(T) & = & 2T/T_c \end{array} \tag{2.23}$$

where $\delta = \pm 1$. Note that c_V has weight $k_c = 2$ and sign $\gamma_c = 1$ for both the $\ell = 2$ and $\ell = 1$ solutions.

Clearly, the $\ell = 1$ and $\ell = 2$ solutions are closely related. They share the same expressions for U and c_V, yet their expressions for $\mathcal{V}$, F, and S are shifted by constants

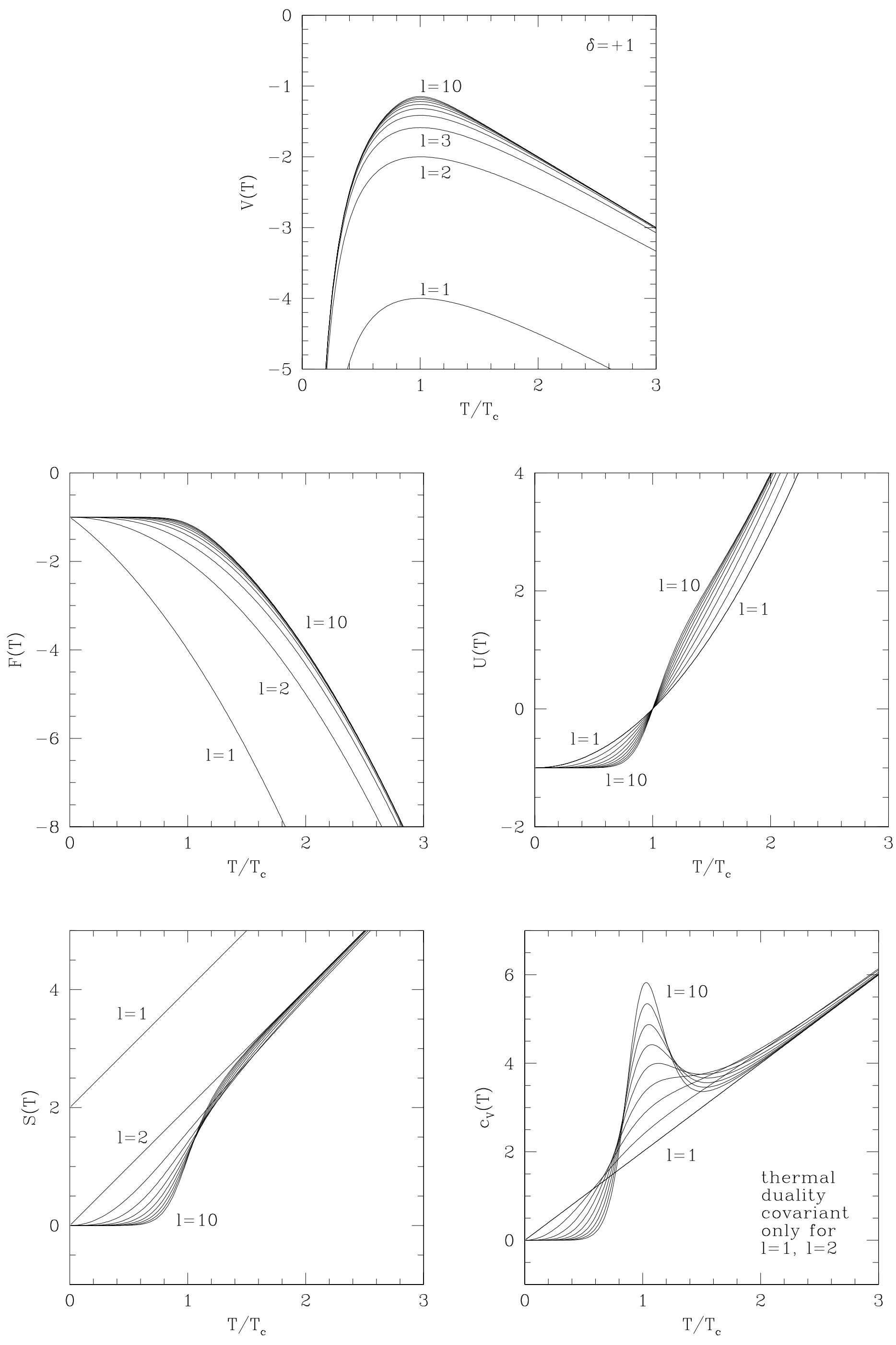

FIGURE 2.1. The thermodynamic quantities $\mathcal{V}$, F, U, S, and c_V in Eq. (2.20), plotted as functions of T for $1 \leq \ell \leq 10$ and $\delta = +1$, in units of $\mathcal{M} \equiv M_{\text{string}}/2\pi = T_c$. All quantities except for c_V are thermal duality covariant for all ℓ, while c_V is covariant only for $\ell = 1, 2$. For these values of ℓ, the entropy and specific heat are exactly linear functions of T. Note that c_V develops a discontinuity as $\ell \to \infty$, suggesting the emergence of a second-order phase transition in this limit.

or extra linear terms:

$$\begin{aligned}
\mathcal{V}^{(\ell=1)} &= \mathcal{V}^{(\ell=2)} - 2\delta \\
F^{(\ell=1)} &= F^{(\ell=2)} - 2\delta T \\
S^{(\ell=1)} &= S^{(\ell=2)} + 2\delta \ .
\end{aligned} \tag{2.24}$$

This shift symmetry will be important in the following.

These $\ell = 1, 2$ solutions also exhibit other intriguing symmetries. For example, since $F(T) \sim -(T^2 + T_c^2)/T_c$ and $U(T) \sim (T^2 - T_c^2)/T_c$ for $\ell = 2$, we see that $F(iT) = U(T)$ and $U(iT) = F(T)$. In other words, we have the formal symmetry

$$T \;\rightarrow\; iT : \qquad F \;\longleftrightarrow\; U \ . \tag{2.25}$$

Since $F = T\mathcal{V}$ and $U = -T^2 d\mathcal{V}/dT$, this immediately leads to a symmetry for $\mathcal{V}(T)$:

$$iT\,\mathcal{V}(iT) \;=\; -\,T^2 \frac{d\mathcal{V}}{T} \ , \tag{2.26}$$

or equivalently

$$\frac{d\mathcal{V}}{dT} \;=\; \frac{\mathcal{V}(iT)}{iT} \ . \tag{2.27}$$

This symmetry is remarkable because it relates the troublesome temperature derivative $d\mathcal{V}/dT$ to $\mathcal{V}$ itself. Since $\mathcal{V}(T)$ is defined through a modular integral as in Eq. (2.3), this implies that quantities involving the temperature derivative of $\mathcal{V}$ can now be written as

$$\frac{d\mathcal{V}}{dT} \;=\; -\,\tfrac{1}{2}\,\mathcal{M}^{D-1}\,\frac{1}{iT}\int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} Z_{\mathrm{string}}(iT) \ . \tag{2.28}$$

Moreover, it is easy to show that just as the symmetry (2.25) leads to the symmetry (2.27), it also leads to a symmetry for the second derivative:

$$\frac{d^2\mathcal{V}}{dT^2} \;=\; \frac{\mathcal{V}(T)}{T^2} - \frac{1}{T}\frac{d\mathcal{V}}{dT} \;=\; \frac{1}{T^2}\left[\mathcal{V}(T) + i\mathcal{V}(iT)\right] \ . \tag{2.29}$$

It is, in fact, this identity that enforces $S = c_V$ for our $\ell = 2$ solutions. Similar symmetries also hold for the $\ell > 2$ solutions.

2.4 Comparison with explicit one-loop calculations: Temperature dependence of effective potentials

We now seek to determine the extent to which our closed-form solutions match the results of explicit one-loop modular integrations of the sort that can emerge from actual finite-temperature string ground states. Such comparisons are extremely important because our derivation of the functional forms given in Sect. 2.3 was "top-down", based entirely on thermal duality symmetries, and did not make reference to any perturbative, order-by-order calculation. Moreover, our discussion was completely model-independent.

Nevertheless, as we shall now discuss, our expressions successfully capture the leading temperature dependence of the one-loop effective potentials for a variety of modular integrals involving time/temperature compactifications on S^1 (circles) and S^1/Z_2 (orbifolds). Moreover, this will occur for all spacetime dimensions $D \geq 2$. As we shall see, the precision with which this occurs will ultimately lead us to conjecture that our solutions actually represent the *exact* solutions for the corresponding string effective potentials when results from all orders of perturbation theory (and perhaps even non-perturbative effects) are included.

2.4.1 Calculating the one-loop effective potential

Let us first recall the calculation of the one-loop effective potential for a finite-temperature string ground state in which the time/temperature direction is compactified on a circle. This is appropriate, *e.g.*, for compactifications of the bosonic string, and we shall consider such circle compactifications for most of what follows. In D spacetime dimensions, the one-loop effective potential for such compactifications takes the form in Eq. (2.3), where

$$Z_{\text{string}}(\tau, T) \;\equiv\; Z_{\text{model}}(\tau)\, Z_{\text{circ}}(\tau, T) \;. \tag{2.30}$$

Here $Z_{\rm model}$ represents the trace over the Fock space of states (*i.e.*, the partition function) of the string model in question, formulated at zero temperature. For example, in the case of the bosonic string compactified to D spacetime dimensions, $Z_{\rm model}$ takes the general form

$$Z_{\rm model} ~=~ ({\rm Im}\,\tau)^{1-D/2}\, \frac{\overline{\Theta}^{26-D}\Theta^{26-D}}{\overline{\eta}^{24}\eta^{24}} \qquad (2.31)$$

where the numerator $\overline{\Theta}^{26-D}\Theta^{26-D}$ schematically represents a sum over the $2(26-D)$-dimensional compactification lattice for left- and right-movers. Note that in general, $Z_{\rm model}$ is the quantity which appears in the calculation of the one-loop cosmological constant (vacuum energy density) of the model:

$$\Lambda ~\equiv~ -\tfrac{1}{2}\,\mathcal{M}^D \int_{\mathcal{F}} \frac{d^2\tau}{({\rm Im}\,\tau)^2} Z_{\rm model} ~. \qquad (2.32)$$

By contrast, the remaining factor $Z_{\rm string}$ represents the sum over Matsubara frequencies. For extended objects such as strings, this includes not only "momentum" Matsubara states but also "winding" Matsubara states. For time/temperature circle compactifications, $Z_{\rm string}$ is given by[1]

$$Z_{\rm circ}(\tau,T) ~=~ \sqrt{{\rm Im}\,\tau}\, \sum_{m,n\in\mathbb{Z}} \overline{q}^{(ma-n/a)^2/4}\, q^{(ma+n/a)^2/4} \qquad (2.33)$$

Here the double sum tallies both the Matsubara momentum and winding states, with $q\equiv \exp(2\pi i\tau)$ and $a\equiv 2\pi T/M_{\rm string} = T/T_c$ where $T_c\equiv M_{\rm string}/2\pi = \mathcal{M}$. Thus, thermal duality symmetry is nothing but the symmetry $(a\leftrightarrow 1/a, m\leftrightarrow n)$ in Eq. (2.33).

It is important to emphasize that a factorization of the form given in Eq. (2.30) holds only for the simplest finite-temperature string constructions (such as for the

[1] Since we are defining $Z_{\rm circ}$ to represent the sum over Matsubara frequencies, we do not include the Dedekind η-function denominators which would traditionally be required in order to interpret $Z_{\rm circ}$ as the partition function of a boson compactified on a circle of radius $R_T\equiv(2\pi T)^{-1}$. This does not represent a violation of modular invariance, since the extra factor of $\sqrt{{\rm Im}\,\tau}$ in Eq. (2.33) compensates for their absence. Note that this factor offsets the similar factors in $Z_{\rm model}$ (just as the summation in $Z_{\rm circ}$ combines with the lattice sums in $Z_{\rm model}$), thereby effectively reducing by one the dimensionality of the resulting finite-temperature string model compared with the dimensionality of the original string model at zero temperature.

bosonic string). In more realistic setups, simple factorizations such as this are not possible, and one typically has more complicated configurations (see, *e.g.*, Refs. [25, 17, 26, 8, 27]). In this section, however, we shall confine our attention to this simplest case because it is the situation in which thermal duality is most directly manifest.

2.4.2 Asymptotic behavior for low and high temperatures

Given the form of these partition functions, it is straightforward to deduce the leading behavior in the $T \to 0$ and $T \to \infty$ limits, and verify that this behavior matches the corresponding behavior of our solutions in Sect. 2.3. Taking the $T \to 0$ limit of $Z_{\rm circ}$, we find

$$Z_{\rm circ} \;\to\; \frac{1}{a} \qquad \text{as} \;\; a \to 0 \; . \tag{2.34}$$

This implies the limiting behavior

$$\mathcal{V}(T) \;\sim\; \frac{\Lambda}{T} \qquad \text{as} \;\; T/T_c \to 0 \; , \tag{2.35}$$

where Λ is the one-loop cosmological constant in Eq. (2.32). This in turn implies that $F(T) \to \Lambda$ as $T/T_c \to 0$.

However, this leading behavior for $\mathcal{V}(T)$ and $F(T)$ coincides exactly with the $T \to 0$ temperature dependence of the solutions found in Sect. 2.3 for arbitrary ℓ. In fact, this agreement allows us to go one step further and deduce the overall normalization of our solutions for arbitrary ℓ with $\delta = +1$:

$$\mathcal{V}^{(\ell)}(T) \;=\; \frac{\Lambda}{T_c}\,\frac{(T^\ell + T_c^\ell)^{2/\ell}}{T T_c} \; . \tag{2.36}$$

We can also consider the opposite, high-temperature limit $T \to \infty$ in Eq. (2.36), obtaining [8, 19, 1]

$$\mathcal{V}^{(\ell)}(T) \;\sim\; \frac{\Lambda}{T_c}\,\frac{T}{T_c} \qquad \text{as} \;\; T \to \infty \; . \tag{2.37}$$

This implies that $F^{(\ell)}(T) \sim T^2$ as $T \to \infty$, correctly reproducing the celebrated high-temperature behavior which signals the reduced number of degrees of freedom in finite-temperature string theory relative to field theory [8]. Note that these correct limiting behaviors are obtained for all values of ℓ.

Having thus verified that our solutions $\mathcal{V}^{(\ell)}(T)$ in Sect. 2.3 correctly reproduce the expected, leading $T \to 0$ and $T \to \infty$ behaviors for all ℓ, we now turn to a more detailed study of this scaling behavior as a function of temperature. It turns out that this will enable us to understand the role played by the free parameter ℓ.

In ordinary quantum field theory, the free energy $F(T)$ at large temperatures typically scales like T^D where D is the spacetime dimension. This in turn implies that the entropy S should scale like T^{D-1}. However, as already noted above, in string theory we have $F(T) \sim T^2$ as $T \to \infty$, implying that $S(T) \sim T$ as $T \to \infty$. Thus, string theory behaves asymptotically as though it has an effective dimensionality $D_{\text{eff}} = 2$.

Of course, the field-theory limit of string theory is expected to occur for $T \ll T_c$. Given this, it is interesting to examine the effective dimensionality (*i.e.*, the effective scaling exponent) of our solutions as a function of temperature. In general, it is easiest to define this effective dimensionality $D_{\text{eff}}(T)$ by considering the entropy: since $S^{(\ell)}(T)$ is a monotonically increasing function of T, we can define $D_{\text{eff}}(T)$ as the effective scaling exponent at temperature T, setting $S^{(\ell)}(T) \sim T^{D_{\text{eff}}-1}$. We thus have, as a general definition,

$$D_{\text{eff}} \equiv 1 + \frac{d\ln S}{d\ln T} = 1 + \frac{T}{S}\frac{dS}{dT} = 1 + \frac{c_V}{S} , \tag{2.38}$$

where the last equality follows from the thermodynamic identity $c_V = T dS/dT$.

These results for $D_{\text{eff}}(T)$ are plotted in Fig. 2.2. As we see, each of our solutions successfully interpolates between $D_{\text{eff}} = \ell$ for $T \ll T_c$ and $D_{\text{eff}} = 2$ for $T \gg T_c$. Indeed, only the $\ell = 2$ solution has $D_{\text{eff}} = 2$ for all T.

Given this observation, it is now possible to interpret our solutions physically. For small temperatures $T \ll T_c$, the entropy behaves as we expect on the basis of field theory, growing according to the power-law $S^{(\ell)}(T) \sim T^{\ell-1}$. Indeed, this is nothing

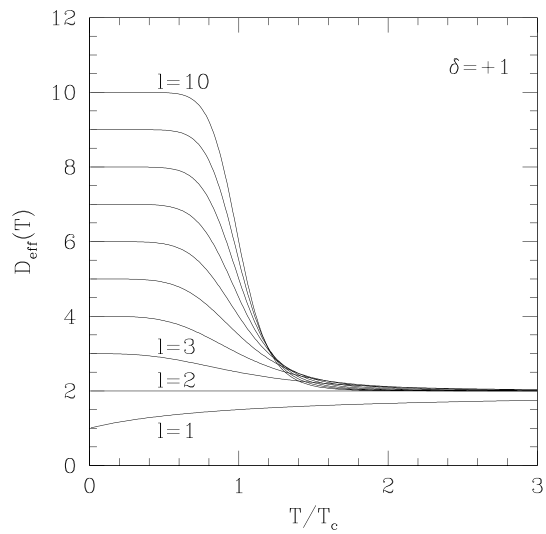

FIGURE 2.2. The effective dimensionalities D_{eff} of our thermodynamic solutions, plotted as functions of T for $1 \leq \ell \leq 10$ and $\delta = +1$. All of our solutions successfully interpolate between $D_{\text{eff}} = \ell$ for $T \ll T_c$ and $D_{\text{eff}} = 2$ for $T \gg T_c$. Only the $\ell = 2$ solution has $D_{\text{eff}} = 2$ for all T.

but the high-temperature limit of the low-energy effective *field* theory, which leads us to interpret ℓ as the spacetime dimension D. However, as T approaches the reduced string scale T_c, we see that this asymptotic behavior begins to change, with the $T^{\ell-1}$ growth in the entropy ultimately turning into the expected *linear* growth for $T \gg T_c$. This is then the asymptotic *string* limit.

Of course, our identification of ℓ as the spacetime dimension D is subject to one important caveat. Since D can be defined only through the *high*-temperature limit of the underlying *field* theory, our identification of ℓ with D assumes that we can properly identify the *high*-temperature field-theory limit with the *low*-temperature string-theory limit for which $S^{(\ell)}(T) \sim T^{\ell-1}$. In other words, this identification of ℓ with D is sensitive to the manner in which the high-temperature limit of field theory matches onto what ultimately becomes the low-temperature limit of string theory. However, we see from Fig. 2.2 that in all our solutions, D_{eff} remains very close to ℓ

for almost all of the temperature range up to T_c. Thus, we expect our association of ℓ with D to be reasonably accurate. Moreover, in the special case with $\ell = 2$, we know that $D_{\text{eff}} = 2$ for all T. We thus expect that this case should correspond to $D = 2$ exactly.

If we consider the same issue from the perspective of the free energy, we can also immediately see the origin of this difference between the high-temperature scaling behaviors in field theory and in string theory. Note that our solution for the free energy can be written as

$$F^{(\ell)}(T) \;=\; \Lambda\left[1+\left(\frac{T}{T_c}\right)^{\ell}\right]^{2/\ell} \tag{2.39}$$

where we have inserted the normalization factor Λ determined above. Expanding this solution for small temperatures, we find

$$F^{(\ell)}(T) \;\sim\; \Lambda \;+\; \frac{2\Lambda}{\ell}\left(\frac{T}{T_c}\right)^{\ell} \;+\; \ldots \qquad \text{for}\;\; T \ll T_c\;. \tag{2.40}$$

Thus, as already observed above, $F^{(\ell)}(T)$ begins with a *constant term* Λ; the field-theoretic power-law scaling T^{ℓ} appears only at subleading order. However, it is precisely this constant term which ultimately determines the high-temperature scaling behavior in string theory. Recall that if f is a general weight-k covariant function scaling as $f(T) \sim T^p$ at small temperatures, then f must scale as $f(T) \sim T^{k-p}$ at high temperatures. Thus, the unusual string-theoretic scaling behavior $F(T) \sim T^2$ at high temperatures can ultimately be attributed to the fact that $F(T)$ leads with a constant term Λ at small temperatures.

Many of these facts are already well known as general statements in the string literature (see, *e.g.*, Ref. [1]). What we are observing here, however, is that our functional forms correctly exhibit all of these properties simultaneously.

2.4.3 Direct comparison for all temperatures

Since we have already determined that our solutions exhibit the expected low- and high-temperature scaling behaviors for all ℓ, the question now boils down to whether these solutions correctly match the expected temperature dependence at *intermediate* temperatures where $T \approx T_c$. In other words, we now wish to do a direct comparison at all temperatures.

For simplicity, we begin in $D = 2$ by considering model-independent situations in which we set[2] $Z_{\rm model}$ to 1. Since $Z_{\rm model}$ does not contain any temperature dependence of its own, this simplification enables us to focus directly on the temperature dependence arising from $Z_{\rm circ}$. Our expression for $\mathcal{V}(T)$ from Eqs. (2.3), (2.30), and (2.33) then reduces to

$$\mathcal{V}^{(D=2)}(T) \;=\; -\tfrac{1}{2}\,\mathcal{M}\int_{\mathcal{F}} \frac{d^2\tau}{({\rm Im}\,\tau)^{3/2}} \sum_{m,n\in\mathbb{Z}} \overline{q}^{(ma-n/a)^2/4}\, q^{(ma+n/a)^2/4}~, \tag{2.41}$$

with a corresponding "cosmological constant" given by

$$\Lambda \;=\; -\tfrac{1}{2}\,\mathcal{M}^2 \int_{\mathcal{F}} \frac{d^2\tau}{({\rm Im}\,\tau)^2} \;=\; -\frac{\pi}{6}\,\mathcal{M}^2~. \tag{2.42}$$

Since $D = 2$ in this case, we expect that our expression for $\mathcal{V}^{(D=2)}(T)$ should directly match onto our $\ell = 2$ solution. Remarkably, this is exactly what occurs: $\mathcal{V}^{(D=2)}(T)$ is *exactly equal* to our $\ell = 2$ solution $\mathcal{V}^{(\ell=2)}(T)$ with $\delta = +1$:

$$\mathcal{V}^{(D=2)}(T) \;=\; -\frac{\pi}{6}\,\frac{T^2+T_c^2}{T}~, \tag{2.43}$$

where we have used the fact that $T_c = \mathcal{M}$. Note that Eq. (2.43) holds for *all* temperatures T. Thus, our closed-form $\ell = 2$ solution from Sect. 2.3 exactly reproduces the complete temperature dependence corresponding to the $D = 2$ circle compactification in Eq. (2.41)!

[2] Setting $Z_{\rm model} = 1$ does not violate the form given in Eq. (2.31) since we can equivalently write $Z_{\rm model} = |\vartheta_2\vartheta_3\vartheta_4|^{16}/(2^{16}|\eta|^{48})$ where ϑ_i are the Jacobi theta functions satisfying $\vartheta_2\vartheta_3\vartheta_4 = 2\eta^3$.

Mathematically, this is a rather surprising result. In Eq. (2.41), the temperature dependence of $\mathcal{V}^{(D=2)}(T)$ enters only through the quantity $a \equiv T/T_c$ which appears in the exponents of q and $\overline{q}$; this temperature dependence, taking the form of a sum of τ-dependent exponentials, is then integrated over the fundamental domain of the modular group. Nevertheless, we find that the net result of this integration is to produce the simple, closed-form result given in Eq. (2.43). Moreover, as we have already seen in Sect. 2.3, this temperature dependence is given by precisely the functional form which is necessary in order to ensure that *all* thermodynamic quantities, including *both* the entropy and the specific heat, are thermal duality covariant.

This agreement provides an important link between the "top-down" analysis of Sect. 2.3 and our direct "bottom-up" string calculation. This agreement is especially illuminating, given that our "top-down" derivation made use of a powerful, non-perturbative duality symmetry, while our "bottom-up" string calculation represents only a one-loop result. Taking this agreement seriously, we are tempted to view the one-loop result for this $D = 2$ example as "exact", receiving no further contributions at higher loops. Of course, in the absence of an actual string model underlying the expression in Eq. (2.41), this statement is only meant to be suggestive.

Before leaving the $D = 2$ special case, we remark that time/temperature circle compactifications are not the only possibility in the construction of finite-temperature string ground states. Another choice (perhaps even a preferred choice phenomenologically [20]) is to compactify on an $S^1/\mathbb{Z}_2$ orbifold, *i.e.*, a line segment. Indeed, under our factorization assumption in Eq. (2.30), these two choices represent the only two consistent geometries on which a finite-temperature string ground state may be formulated [28]. In the case of an orbifold compactification, we simply replace $Z_{\rm circ}$ in Eq. (2.33) with [28]

$$Z_{\rm orb}(\tau, T) \;=\; \tfrac{1}{2}\, Z_{\rm circ}(\tau, T) \;+\; Z_{\rm circ}(\tau, T_c) \;-\; \tfrac{1}{2}\, Z_{\rm circ}(\tau, T_c/2)~. \qquad (2.44)$$

In this expression, the first term represents the contributions from the untwisted states, while the remaining terms are *temperature-independent* (*i.e.*, they are evaluated at fixed specified temperatures which are independent of T) and represent the contributions from the twisted states. Since we already know the complete temperature dependence arising from Z_{circ} in Eq. (2.43), we immediately find that the effective potential in the orbifold case has the exact closed-form solution

$$\mathcal{V}_{\text{orb}}^{(D=2)} \;=\; -\frac{\pi}{12}\left[\frac{T^2+T_c^2}{T} \;+\; \frac{3}{2}\right] . \tag{2.45}$$

Of course, this is nothing but our Z_{circ} solution, rescaled and shifted by an additive constant. However, recall from Eq. (2.24) that the $\ell = 1$ solution differs from the $\ell = 2$ solution merely through such an additive shift. Since the circle solution corresponds to $\ell = 2$, this suggests that our orbifold solution in Eq. (2.45) can be expressed exactly as a linear combination of the $\ell = 2$ and $\ell = 1$ solutions in Eq. (2.36), and this is indeed the case:

$$\mathcal{V}_{\text{orb}}^{(D=2)} \;=\; \frac{3}{4}\,\mathcal{V}^{(\ell=1)} \;+\; \frac{1}{4}\,\mathcal{V}^{(\ell=2)} \tag{2.46}$$

where we have taken $\delta = +1$ in the $\ell = 1$ solution. Once again, we stress that this is an *exact* representation for the complete temperature dependence of the $D = 2$ orbifold case. Note that in writing this expression, we have identified the normalization constant $\Lambda = \Lambda_{\text{circ}}/2$; this follows from the low-temperature limit of Eq. (2.44), even though this Λ is no longer the cosmological constant of the original zero-temperature model. Also note that even though the $\ell = 1, 2$ solutions are shifted relative to each other by an additive constant, we cannot write $\mathcal{V}_{\text{orb}}$ purely in terms of either of the $\ell = 1$ solutions (with $\delta = \pm 1$) because the additive shifts in these $\ell = 1$ solutions are ± 2 relative to the $\ell = 2$ solution. According to Eq. (2.45), however, our shift constant is $3/2$ relative to the $\ell = 2$ solution. This fact has some important consequences which we shall discuss in Sect. 2.6.

Given the exact orbifold solution in Eq. (2.45), we can immediately see the thermodynamic effects of compactifying the time/temperature dimension on an orbifold rather than a circle. While the internal energy and specific heat are unaffected by this choice, the free energy picks up an additional linear term and the entropy picks up an additive constant. The latter has been called a “fixed-point” entropy [20] since it arises from the fixed points of the $S^1/\mathbb{Z}_2$ orbifold and survives even in the $T \to 0$ limit; in the present case this fixed-point entropy is given exactly as

$$S_{\rm fixed-point} \;=\; \pi/8 \;. \tag{2.47}$$

Let us now proceed to consider the case in higher dimensions $D > 2$. As might be expected, things are more complicated. For arbitrary D, the expression in Eq. (2.41) now generalizes to

$$\mathcal{V}^{(D)}(T) \;=\; -\,\tfrac{1}{2}\,\mathcal{M}^{D-1} \int_{\mathcal{F}} \frac{d^2\tau}{({\rm Im}\,\tau)^{(D+1)/2}} \sum_{m,n\in\mathbb{Z}} \overline{q}^{(ma-n/a)^2/4}\, q^{(ma+n/a)^2/4} \;, \tag{2.48}$$

where we incorporate the D-dependent prefactor $({\rm Im}\,\tau)^{1-D/2}$ from $Z_{\rm model}$ but continue to disregard the rest of this function for simplicity. While Eq. (2.48) is not modular invariant, it captures the dominant T- and D-dependence that we wish to explore.

As evident from Eq. (2.48), the net effect of altering the spacetime dimension is to change the power of the $({\rm Im}\,\tau)$ factor that appears in the measure of the integral. If we view the $Z_{\rm circ}$ integrand as a power series in q and $\overline{q}$, with each term separately integrated and then summed to produce the effective potential $\mathcal{V}^{(D)}$, we see that the dominant effect of changing the spacetime dimension is to *reweight* the contributions from each term in the $Z_{\rm circ}$ power series because they are now being integrating over the modular-group fundamental domain with an altered measure. Thus, it is not immediately apparent how the temperature dependence found in the $\ell = 2$ case should change.

Nevertheless, we find that our functions $\mathcal{V}^{(\ell)}(T)$ from Sect. 2.3 continue to successfully capture the dominant temperature dependence of the resulting integrals, with $\ell = D$. Unlike the case with $D = \ell = 2$, this agreement is only approximate rather than exact. Nevertheless, we find that this agreement holds to within one or two percent over the entire temperature range $0 \leq T \leq \infty$. Indeed, if we were to superimpose a plot of $\mathcal{V}^{(D)}(T)$ over the plot of $\mathcal{V}^{(\ell)}(T)$ in Fig. 2.1, taking $\ell = D$, we would not be able to discern the difference at the level of magnification in Fig. 2.1.

Once again, this is a rather striking result, suggesting that our functional forms continue to capture the dominant temperature dependence, even in higher dimensions. Of course, for $D > 2$, our solutions and the above one-loop results do not agree exactly. However, given the significant role played by thermal duality in constraining the form of the effective potential to the specific functional forms that we have found in Sect. 2.3, and given the precision with which the above one-loop results appear to match these functional forms, it is natural to attribute the failure to obtain an exact agreement for $D > 2$ to the fact that $\mathcal{V}^{(D)}(T)$ in Eq. (2.48) is itself only a one-loop approximation. We thus are led to conjecture that our functional forms $\mathcal{V}^{(\ell)}(T)$ indeed represent the exact solutions for the finite-temperature effective potentials, even in higher dimensions, and that these solutions will emerge only when the contributions from all orders in perturbation theory are included. Viewed from this perspective, it is perhaps all the more remarkable that we found an exact agreement for $D = 2$, suggesting that the one-loop result is already exact in this special case, with no further renormalization.

Let us now consider what happens if we do not make the simplification that $Z_{\text{model}} = 1$ [or $Z_{\text{model}} = (\text{Im}\,\tau)^{1-D/2}$]. Of course, in order to select an appropriate Z_{model}, we must actually construct a bona-fide string model (*e.g.*, a specific bosonic string compactification); moreover, this model must be tachyon-free if our effective potential is to be finite. These constraints force Z_{model} to take the form $Z \sim 1 + \sum_{mn} a_{mn}\overline{q}^m q^n$ where $a_{mn} = 0$ if $m = n < 0$ (no physical tachyons). The presence of the leading constant term in the power expansion means that the leading temperature dependence of $\mathcal{V}^{(D)}$ will continue to be the same as we had when we merely set $Z_{\text{model}} = 1$. Indeed, the contributions from the higher terms in Z_{model} are exponentially suppressed relative to the leading term, which means that the net effect of the extra, model-dependent terms in Z_{model} is to provide an exponentially suppressed reweighting of the contributions from the different terms in the power-series expansion of $Z_{\rm circ}$. Thus, the net effect of inserting a non-trivial $Z_{\rm model}$ into $\mathcal{V}^{(D)}$ is merely to change the *subleading* temperature dependence in a model-dependent way. Thus, we conclude that the leading temperature dependence continues to be captured by our solutions $\mathcal{V}^{(\ell)}(T)$ even when $Z_{\rm model} \neq 1$; indeed, this is the universal, model-independent contribution. Moreover, if our conjecture is correct, then we expect these subleading model-dependent contributions to be washed out as higher-order contributions are included in the perturbation sum. Just as for the $D = 2$ special case, similar remarks apply if we replace the thermal compactification geometry from a circle to an orbifold.

Finally, let us briefly comment on the most general cases of all, namely those in which the finite-temperature partition functions do *not* factorize as in Eq. (2.30). Such cases include compactifications with temperature-dependent Wilson lines, and are expected to emerge in heterotic or Type II theories where non-trivial phases must be introduced in the combined thermal partition function (ultimately due to presence of spacetime fermions). For example, non-factorized thermal partition functions emerge for finite-temperature string theories whose zero-temperature limits are spacetime supersymmetric; these theories necessarily have thermal partition functions in which the cancellations inherent in supersymmetry are non-trivially mixed with the Matsubara sums (see, *e.g.*, the examples in Refs. [25, 17, 26, 8, 27]). In such cases, however, the effective potentials do not generally exhibit thermal duality — indeed, such theories may be considered to be finite-temperature string ground states in which thermal duality is spontaneously broken. Such theories are therefore beyond the scope of this chapter.

2.5 Effective scaling dimensionalities: Connection to holography?

In Eq. (2.38), we defined the notion of an effective dimensionality D_{eff} which governs the scaling behavior of the entropy $S(T)$, with $S(T) \sim T^{D_{\text{eff}}-1}$. As we have seen, this scaling coefficient generally ranges from $D_{\text{eff}} = D$ as $T \to 0$ to $D_{\text{eff}} = 2$ as $T \to \infty$. The limiting behavior as $T \to 0$ is precisely as expected on the basis of ordinary quantum field theory, while the opposite limiting behavior as $T \to \infty$ is precisely as required by thermal duality.

This reduction in the effective dimensionality of the system at high temperatures is extremely reminiscent of holography (such an interpretation can also be found, *e.g.*, in Refs. [20]). Indeed, the scaling of our thermodynamic quantities departs from the ordinary D-dimensional scaling that would be expected on the basis of quantum field theory, and begins to behave as though the number of accessible degrees of freedom populates not the full D-dimensional spacetime, but rather a subspace of smaller dimensionality. Of course, an analysis formulated in flat space (such as ours) cannot address questions pertaining to the geometry of this subspace, and thus cannot determine whether the surviving degrees of freedom are really to be associated with a subspace or boundary of the original geometry. However, from the restricted perspective emerging from a mere counting of states, we see our scaling behavior differs significantly from field-theoretic expectations, suggesting some sort of reduction in the effective dimensionality associated with thermally accessible degrees of freedom as $T \to \infty$.

Of course, taking the $T \to \infty$ limit is merely of formal interest. In a theory with thermal duality, there is no difference between the range $T > T_c$ and the range $T < T_c$ since these ranges capture the same physics and are thus indistinguishable. Or, phrased another way, thermal duality tells us that there is a "maximum" temperature in the same sense that T-duality tells us there is a minimum radius. This is also consistent with our expectation that there should be a Hagedorn-type phase transition at or near T_c, with the theory ultimately entering a new phase marked by new degrees of freedom. Thus, we should really only consider the range $0 \leq T \leq T_c$.

Given this, let us consider the value of D_{eff} not as $T \to 0$ or $T \to \infty$, but as $T \to T_c$. As discussed above, this is truly the "high-temperature" limit of string theory. With our specific closed-form solutions $\mathcal{V}^{(\ell)}(T)$ in Eq. (2.20), the general definition in Eq. (2.38) yields

$$D_{\text{eff}}(T) \;=\; \frac{2T^D + DT_c^D}{T^D + T_c^D} \tag{2.49}$$

where we have identified $\ell = D$. We thus obtain

$$D_{\text{eff}}(T_c) \;=\; \tfrac{1}{2}\,(2 + D)\ . \tag{2.50}$$

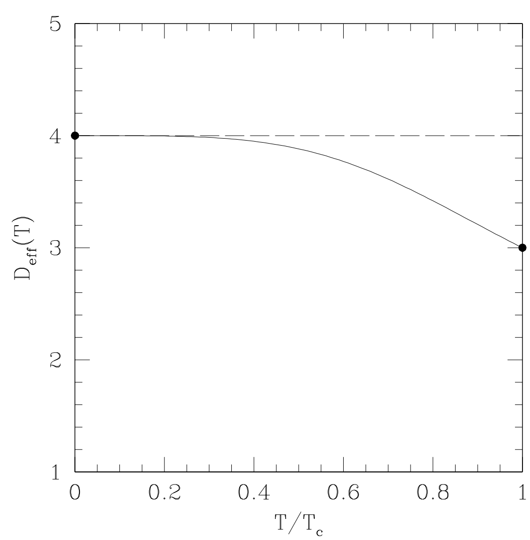

FIGURE 2.3. The effective dimensionality D_{eff} of our four-dimensional thermodynamic solutions, plotted as a function of T. These solutions behave "holographically" in the range $0 \leq T \leq T_c$, with the effective scaling dimensionality falling exactly from $D_{\text{eff}} = 4$ to $D_{\text{eff}} = 3$. The dotted line indicates the behavior that would be expected within quantum field theory.

This result indicates that $D_{\text{eff}}(T_c) < D$ for all $D > 2$. In other words, for all

$D > 2$, the effective scaling of the number of degrees of freedom at high temperatures is reduced compared with our field-theoretic expectations at low temperatures. However, taking the predictions of holography seriously, we can ask when this reduction in D_{eff} is truly "holographic" in the sense that D_{eff} is reduced by exactly one unit as $T \to T_c$, dropping from D to $D-1$. This would be analogous, for example, to what occurs for black holes, where quantities such as entropy scale not with the three-volume of the black hole, but with its area. Remarkably, demanding that D_{eff} drop by precisely one unit yields

$$D_{\text{eff}}(T_c) = D - 1 \qquad \Longrightarrow \qquad D = 4 \ . \tag{2.51}$$

Thus, we see that it is precisely in four dimensions that our solutions behave "holographically" in the range $0 \leq T \leq T_c$, with the effective scaling dimensionality falling exactly by one unit from $D_{\text{eff}} = 4$ to $D_{\text{eff}} = 3$. This behavior is plotted in Fig. 2.3.

While it is tempting to interpret this reduction in D_{eff} as a holographic effect, we again caution that our setup (based on a flat-space calculation) is incapable of yielding the additional geometric information that this claim would require. Such an analysis is beyond the scope of this chapter, and would require reformulating the predictions of thermal duality for string theories in non-trivial D-dimensional backgrounds, and then determining whether we could formulate a map between degrees of freedom in the bulk of the D-dimensional volume and those on the $(D-1)$-dimensional boundary of this volume. Nevertheless, we find this reduction in D_{eff} to be an extremely intriguing phenomenon, especially since our exact solutions lead to a strictly "holographic" reduction in D_{eff} for $D = 4$. We thus believe that this approach towards understanding the relation between thermal duality and holography is worthy of further investigation.

2.6 Discussion

In this chapter, we set out to address a very simple issue: even though thermal duality is an apparent fundamental property of string theory, emerging as a consequence of Lorentz invariance and T-duality, the rules of classical thermodynamics do not appear to respect this symmetry. Even when the vacuum amplitude $\mathcal{V}(T)$ exhibits thermal duality, thermodynamic quantities such as entropy and specific heat do not. Given this situation, we sought to determine whether special string ground states might exist such that thermal duality will nevertheless be exhibited by all of the usual thermodynamic quantities of interest.

We began by deriving specific solutions $\mathcal{V}(T)$ such that thermal duality is preserved not only for the free and internal energies, but also for the entropy and specific heat. The complete set of such solutions is itemized in Eqs. (2.20), (2.22), and (2.23). While the solutions for general ℓ preserve thermal duality for all thermodynamic quantities except the specific heat, the $\ell = 1, 2$ solutions preserve thermal duality for *all* of the thermodynamic quantities.

We then investigated the extent to which these solutions might emerge from modular integrals of the sort that would be expected in one-loop calculations from actual string ground states. Remarkably, we found that our $\ell = 1, 2$ closed-form solutions provide *exact* representations for $D = 2$ modular integrals corresponding to time/temperature compactifications on circles and orbifolds. This agreement is particularly encouraging from a mathematical standpoint, since our derivation of these functional forms is entirely "top-down", proceeding only from thermal duality symmetry principles, and has nothing to do with specific "bottom-up" constructions involving specific one-loop modular integrations. The fact that these two approaches agree exactly, yielding the same results even in highly simplified cases, suggests that thermal duality is likely to play an important role governing self-consistent string ground states. Indeed, as we saw in Sect. 2.5, these $\ell = 1, 2$ solutions also ensure that modular invariance is also preserved for all relevant thermodynamic quantities.

By contrast, our remaining $\ell > 2$ closed-form solutions do not serve as exact representations of appropriate $D > 2$ modular integrals. Nevertheless, we found that they provide extremely accurate *approximations* to such integrals in a wide variety of cases. This led us to conjecture that our $\ell > 2$ functional forms may indeed provide *exact* solutions for the effective potentials corresponding to wide classes of finite-temperature string ground states once the contributions from all orders of perturbation theory (and perhaps even non-perturbative effects) are included. After all, our method of deriving these solutions rests solely on the requirement of thermal duality, a symmetry which (like the T-duality from which it is derived) holds to all orders in perturbation theory, and even non-perturbatively. Thus, if this conjecture is correct, it is perhaps not surprising that our $\ell > 2$ solutions transcend the results of intrinsically one-loop calculations.

In this connection, it is important to stress that the free energy $F(T)$ exhibits thermal duality order by order in string theory. Our conjecture does not alter this behavior. What we are conjecturing, however, is that the sum of these order-by-order perturbative functions $F(T)$ actually exhibits an additional symmetry, one which guarantees that the entropy $S(T)$ is also duality covariant. Thus, while thermal duality is indeed preserved order by order for the string free energy, we are conjecturing that the entropy, which normally fails to exhibit this symmetry at any order, actually will exhibit this symmetry when all of these separate order-by-order contributions are summed together.

Of course, this conjecture requires not only a special temperature behavior at each order, but also a specific value of the string coupling κ. To see this, recall that the full free energy $F(\kappa, T)$ depends not only on the temperature T but also on the string coupling κ. Specifically, if $F_g(T)$ is the genus-g contribution to the total free energy

$F(\kappa, T)$, then

$$F(\kappa, T) \;=\; \sum_{g=1}^{\infty} \kappa^{2(g-1)} F_g(T) \;. \tag{2.52}$$

In general, the genus-g free energy transforms as a weight-$2g$ duality-covariant function,

$$F_g(T_c^2/T) \;=\; (T_c/T)^{2g}\, F_g(T) \;, \tag{2.53}$$

which is why the total free energy, like its genus-one contribution, transforms as a weight-two duality-covariant function:

$$F(\kappa, T_c^2/T) \;=\; (T_c/T)^2\, F(\kappa T_c/T, T) \;. \tag{2.54}$$

The corresponding shift in the string coupling is precisely analogous to what occurs in T-duality. However, since the string coupling κ parametrizes the relative weightings of the contributions from each genus, any new symmetry which appears only in the sum over all genera must hold only for a specific value of the string coupling. Our conjecture, which claims that the full free energy $F(\kappa, T)$ must have the exact temperature dependence given by $F^{(\ell)}(T)$ with $\ell = D$, must therefore hold only for a specific value of the string coupling which in turn must presumably be fixed by other, non-perturbative effects.

While these are exciting speculations, we are nevertheless left with our original question as to whether there exist special finite-temperature string ground states for which *all* relevant thermodynamic quantities exhibit thermal duality. For $D > 2$, it seems that such states do *not* exist: even if the above conjecture is correct and the exact effective potentials of such string models match our $\ell > 2$ functional forms, these functional forms do not preserve thermal duality covariance for the specific heat. Only the $\ell = 1, 2$ solutions have this property. However, for $D = 2$, the answer to this question may be somewhat more positive, for the case of time/temperature circle compactifications with $Z_{\text{model}} = 1$ appears to yield exactly what we require. Thus, even when we take $Z_{\text{model}} \neq 1$, our above conjecture suggests that the corrections that are induced by the non-trivial Z_{model} might ultimately disappear when contributions from all orders are included. Indeed, in this way, our conjecture would lead to a model-independent universal form for the effective potentials corresponding to such compactifications. However, it is important to realize that even if the circle case leads to a duality-covariant entropy and specific heat, the corresponding orbifold case certainly does not. Since the additive shift in the effective potential that accrues in passing from the circle to the orbifold is given by $3/2$ rather than $\delta = \pm 1$, the orbifold case corresponds not to the $\ell = 1$ solution but rather to a *linear combination* of the $\ell = 1$ and $\ell = 2$ solutions, as indicated in Eq. (2.46). The resulting entropy is thus a linear combination of two terms with different duality weights, and fails to be covariant at all. Of course, the specific heat continues to be covariant, since the specific heat is unaffected by the contributions from the orbifold fixed points.

What then are we to conclude from this analysis? Clearly, if string theory is to resurrect thermal duality for quantities such as entropy and specific heat, the miracle is not likely to lie in the clever choice of a string ground state. Rather, the miracle is more likely to lie in the structure of thermodynamics itself, as a possible string-theoretic modification of the usual rules of classical thermodynamics according to which quantities such as entropy and specific heat are calculated. Indeed, as we shall see in Ref. [29], such an approach is capable of restoring thermal duality to *all* thermodynamic quantities — regardless of the specific ground state — and leads to a new, manifestly duality-covariant string thermodynamics. The development of such a theory will be explored in Ref. [29].

Chapter 3
Thermal Duality Covariant Thermodynamics

3.1 Introduction

In the previous chapter, we observed that the rules of ordinary thermodynamics generally fail to respect the thermal duality symmetry of string theory under which the physics at temperature T is related to the physics at temperature T_c^2/T, where T_c is a critical (or self-dual) temperature related to the string scale. The reason for this failure is simple: even though the string vacuum amplitude $\mathcal{V}(T)$ might exhibit an invariance under this symmetry, with $\mathcal{V}(T) = \mathcal{V}(T_c^2/T)$, the subsequent temperature derivatives d/dT that are needed in order to calculate other thermodynamic quantities generally destroy this symmetry. This then results in quantities such as entropy and specific heat which fail to exhibit thermal duality symmetries.

It is, of course, entirely possible that thermal duality should be viewed only as an "accidental" symmetry of the string vacuum amplitude; we thus would have no problem with the loss of this symmetry when calculating other thermodynamic quantities. However, given that the thermal duality symmetry of string theory follows immediately from T-duality and Lorentz invariance, this symmetry appears to be every bit as deep as the dualities that occur at zero temperature. Thus, it seems more natural to consider thermal duality as a fundamental property of a consistent string theory, and demand that this symmetry hold for *all* physically relevant thermodynamic quantities.

This problem was considered in Ref. [30], where it was shown that there exist certain special vacuum amplitudes $\mathcal{V}(T)$ for which all thermodynamic quantities exhibit the thermal duality symmetry. Moreover, it was shown that these solutions for $\mathcal{V}(T)$ correspond to highly symmetric string modular integrals in which the time/temperature direction is compactified on S^1 (a circle) or $S^1/\mathbb{Z}_2$ (a line segment). Thus, for such constructions, there is no loss of thermal duality. In fact, as discussed in Ref. [30], the constraint of thermal duality in such cases may be of sufficient strength to enable an exact, closed-form evaluation of the relevant thermodynamic quantities.

However, this method of restoring thermal duality is less than satisfactory. Because this approach applies only for certain selected ground states, it lacks the generality that should apply to a fundamental symmetry. If thermal duality is to be considered an intrinsic property of finite-temperature string theory (akin to T-duality), then the formulation of the theory itself — including its rules of calculation — should respect this symmetry regardless of the specific ground state.

This argument should apply even if the specific string ground state in question does not exhibit thermal duality (such as may occur in finite-temperature string constructions utilizing temperature-dependent Wilson lines). Indeed, even when thermal duality is "spontaneously broken" in this way, the theoretical definitions of all relevant physical thermodynamic quantities should still reflect this duality symmetry. After all, it is certainly acceptable if the entropy or specific heat fail to exhibit thermal duality because the ground state fails to yield a duality-symmetric vacuum amplitude $\mathcal{V}(T)$. However, it is not acceptable if this failure arises because the *definitions* of the entropy or specific heat in terms of $\mathcal{V}(T)$ are themselves not duality covariant.

For this reason, we are motivated to develop an alternative, fully covariant string thermodynamics in which thermal duality is manifest. This is the goal of the present chapter. As we shall see, this new duality-covariant framework will preserve the definitions of free energy and internal energy, but will lead naturally to modifications in the usual thermodynamic definitions for other quantities such as entropy and specific heat. At low temperatures, these modifications produce "corrections" to the standard definitions of entropy and specific heat which are suppressed by powers of the string scale. These corrections are therefore unanticipated from the low-energy (low-temperature) point of view. At higher temperatures, however, these modifications are significant, and may be important for a full understanding of string thermodynamics at or near the self-dual temperature T_c. In fact, we shall find that our new, string-corrected entropy is often smaller than the usual entropy, with the suppression becoming increasingly severe as the temperature approaches the string scale. This suggests an intriguing connection with the holographic principle, and leads to some novel speculations concerning the physics near the critical temperature.

3.2 Thermal duality and traditional thermodynamics

Thermal duality [16, 17, 8, 18, 19, 1] is a symmetry which relates string thermodynamics at temperature T to string thermodynamics at the inverse temperature T_c^2/T. Here T_c is a critical, self-dual temperature which is ultimately set by the string scale. It is easy to see how thermal duality emerges. In string theory (just as in ordinary quantum field theory), finite-temperature effects can be incorporated [10, 1] by compactifying an additional time dimension on a circle of radius $R_T = (2\pi T)^{-1}$. However, Lorentz invariance guarantees that the properties of this extra time dimension should be the same as those of the original space dimensions, and T-duality [21, 22, 23] tells us that closed string theory on a compactified space dimension of radius R is indistinguishable from that on a space of radius R_c^2/R where $R_c \equiv \sqrt{\alpha'} = M_{\rm string}^{-1}$ is a critical, self-dual radius. Together, these two symmetries thus imply a thermal duality symmetry under which the physics (specifically the one-loop string partition function $Z_{\rm string}$) should be invariant with respect to the thermal duality transformation $T \to T_c^2/T$:

$$Z_{\rm string}(T_c^2/T) \;=\; Z_{\rm string}(T) \tag{3.1}$$

where $T_c \equiv M_{\rm string}/2\pi$. Note that this symmetry holds to all orders in perturbation theory [18].

As was discussed in Sect. 2.2, all thermodynamic quantities of interest are generated from this partition function. The one-loop vacuum amplitude $\mathcal{V}(T)$ is given by [10, 24, 16]

$$\mathcal{V}(T) \;\equiv\; -\,\tfrac{1}{2}\,\mathcal{M}^{D-1}\int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} Z_{\rm string}(T) \tag{3.2}$$

where $\mathcal{M} \equiv M_{\rm string}/2\pi$ is the reduced string scale; D is the spacetime dimension; τ is the complex modular parameter describing the shape of the one-loop toroidal worldsheet; and $\mathcal{F} \equiv \{\tau : |\mathrm{Re}\,\tau| \le \frac{1}{2}, \mathrm{Im}\,\tau > 0, |\tau| \ge 1\}$ is the fundamental domain of the modular group. In general, $\mathcal{V}(T)$ plays the role normally assumed by $-\ln Z$ where Z is the usual thermodynamic partition function in the canonical ensemble. Given this definition for $\mathcal{V}$, the free energy F, internal energy U, entropy S, and specific heat c_V follow from the usual thermodynamic definitions:

$$F = T\mathcal{V}~, \qquad U = -T^2\frac{d}{dT}\mathcal{V}~, \qquad S = -\frac{d}{dT}F~, \qquad c_V = \frac{d}{dT}U~. \tag{3.3}$$

It follows directly from these definitions that $U = F + TS$ and that $c_V = TdS/dT$. Note that $\Lambda \equiv \lim_{T\to 0} F(T)$ is the usual one-loop zero-temperature cosmological constant.

It is straightforward to determine the extent to which the thermal duality exhibited by $Z_{\rm string}$ in Eq. (3.1) is inherited by its descendants in Eqs. (3.2) and (2.4). Since $\mathcal{V}$ is nothing but the modular integral of $Z_{\rm string}$, the invariance of $Z_{\rm string}$ under the thermal duality transformation immediately implies the invariance of $\mathcal{V}$:

$$\mathcal{V}(T_c^2/T) \;=\; \mathcal{V}(T)~. \tag{3.4}$$

Similarly, from its definition in Eq. (2.4), we find that F transforms *covariantly* under thermal duality:

$$F(T_c^2/T) \;=\; (T_c/T)^2\,F(T)~. \tag{3.5}$$

Thus, F also respects the thermal duality symmetry. Finally, it is easy to verify that the internal energy U also transforms covariantly under thermal duality:

$$U(T_c^2/T) \;=\; -\,(T_c/T)^2\,U(T)~. \tag{3.6}$$

The overall minus sign in this duality transformation has the net effect of fixing a zero for the internal energy such that it vanishes at the self-dual temperature, $U(T_c) = 0$. Since $dU/dT > 0$ for all $T < T_c$, this zero of energy requires $U(T) < 0$ for $T < T_c$.

Unfortunately, the entropy and specific heat fail to transform either invariantly or covariantly under the duality transformation. In other words, these quantities fail to transform as *bona fide* representations of the thermal duality symmetry. If thermal duality is indeed a fundamental property of string theory, the failure of the entropy and specific heat to transform covariantly under thermal duality suggests that these quantities are improperly defined from a string-theoretic standpoint. At best, they are not the proper "eigenquantities" which should correspond to physical observables.

It is straightforward to determine the source of the difficulty. Even though Z_{string} and $\mathcal{V}$ are thermal duality invariant, the passage to the remaining thermodynamic quantities involves the mathematical operations of multiplication by, and differentiation with respect to, the temperature T. While multiplication by T preserves covariance under the thermal duality symmetry, *differentiation with respect to T generally does not*. Indeed, although the derivative in the definition for $U(T)$ happens to preserve thermal duality covariance, this covariance is broken by the subsequent differentiations which are needed to construct the entropy and specific heat.

The problem of a derivative failing to preserve a symmetry is an old one in physics; *the solution is to construct the analogue of a covariant derivative.* This procedure is well known in gauge theories, where the need to construct a covariant derivative respecting the local gauge symmetry requires the introduction of an entirely new degree of freedom, namely the gauge field. Fortunately, in the present case of the thermal duality, the situation is far simpler.

3.3 Modular invariance and threshold corrections: An analogy, and some history

As a digression, let us first consider an analogous case involving modular invariance. This case will be mathematically similar to the case of thermal duality transformations. In general, a modular-covariant function $f(\tau)$ is one for which

$$f\left(\frac{a\tau+b}{c\tau+d}\right) \;=\; (c\tau+d)^k f(\tau) \tag{3.7}$$

for all $a,b,c,d\in\mathbb{Z}$ with $ad-bc\in\mathbb{Z}$. The quantity k is called the modular weight of f. Note that the special case with $(a,b,c,d)=(0,-1,1,0)$ yields the modular transformation $\tau\to-1/\tau$, which is very similar to the thermal duality transformation. However, if f is modular covariant with weight k, it is easy to verify that $df/d\tau$ is not modular covariant; in other words, $d/d\tau$ is not a modular-covariant derivative. Instead, the appropriate modular-covariant derivative acting on a modular function of weight k is

$$D_\tau \;\equiv\; \frac{d}{d\tau} \;-\; \frac{ik}{2\,\mathrm{Im}\,\tau}~. \tag{3.8}$$

This ensures that if f is a modular-covariant function of weight k, then $D_\tau f$ is also modular covariant, with weight $k+2$.

The existence of this modular-covariant derivative is not merely a mathematical nicety: it turns out to play an important role in calculating string threshold corrections to low-energy gauge couplings [31]. Recall that in string theory, the partition function $Z_{\text{string}}(\tau)$ is a modular-invariant trace over all states in the string Fock space, where τ is the complex parameter describing the shape of the torus (one-loop diagram); the final result can generally be written as a sum of products of modular-covariant functions $f(\tau)$ and their complex conjugates. However, in order to calculate threshold corrections to the running of the low-energy gauge couplings due to the infinite towers of massive string states, the rules of ordinary quantum field theory instruct us to calculate a slightly different trace over the Fock space in which the contribution from each state is now multiplied by its squared gauge charge [32]. However, it turns out that multiplication by the squared gauge charge in the trace is mathematically equivalent to replacing certain occurrences of f in the final partition function with the derivative $df/d\tau$, thereby breaking the underlying modular invariance of the theory. Thus, it appears that the usual calculations inherited from quantum field theory lead to results which fail to respect the underlying string symmetries.

This state of affairs persisted for almost a decade until it was found [33] that a full *string* calculation performed in the presence of a suitable infrared regulator introduces additional unexpected contributions to the threshold corrections. Remarkably, these extra contributions correspond to adding the second term in Eq. (3.8) to each occurence of $df/d\tau$, thereby elevating the non-covariant derivative $d/d\tau$ into the full covariant derivative D_τ. These extra contributions are intrinsically gravitational in origin, arising from spacetime curvature backreactions and dilaton tadpoles, and thus would not have been anticipated from a straightforward field-theoretic derivation. However, these extra contributions are precisely what are needed to restore modular invariance to the full string threshold calculation, as expected from the string perspective. A review of this situation can be found in Ref. [31].

The lesson from this example is clear: although field-theoretic considerations may suggest the existence of certain derivatives in the definitions of physically relevant quantities, a full string calculation of these quantities should only involve those covariant forms of these derivatives which respect the underlying string symmetries. What we are proposing, then, is to follow this example in the case of the temperature derivatives appearing in traditional string thermodynamics, using thermal duality covariance as our guide.

3.4 Thermal duality covariant derivatives

We shall now proceed to construct our thermal duality covariant derivatives. We begin with a mathematical definition: if a function $f(T)$ has the duality transformation

$$f(T_c^2/T) \;=\; \gamma\,(T_c/T)^k\, f(T) \tag{3.9}$$

with $\gamma = \pm 1$, we shall say that $f(T)$ is a thermal duality covariant function with "weight" k and sign ± 1 ("even" or "odd"). Note that $\gamma = \pm 1$ are the most general coefficients which preserve the $\mathbb{Z}_2$ nature of the thermal duality transformation.

It is easy to verify that multiplication by T is a covariant operation, resulting in a function with weight $k+2$ and the same sign for γ. Our goal, however, is to construct a thermal duality covariant derivative. Towards this end, let us imagine that this derivative takes the general form

$$D_T \;=\; \frac{d}{dT} \;+\; \frac{g(T)}{T} \tag{3.10}$$

where $g(T)$ is a function of T and T_c. Explicitly evaluating $[D_T f](T_c^2/T)$ using Eqs. (3.9) and (3.10), we then find that $D_T f$ will be duality covariant with weight $k-2$ and sign $-\gamma$, *i.e.*,

$$[D_T f](T_c^2/T) \;=\; -\,\gamma\,(T_c/T)^{k-2}\,[D_T f](T)\;, \tag{3.11}$$

only if $g(T)$ satisfies the constraint

$$g(T) \;+\; g(T_c^2/T) \;=\; -\,k\;. \tag{3.12}$$

Note that $g(T) = 0$ is *not* a solution if $k \neq 0$; in other words, for non-zero k, we *must* make an additional contribution to the ordinary temperature derivative in order to preserve duality covariance. Indeed, since Eq. (3.12) must hold for all T, the function $g(T)$ must be proportional to the weight k. Our task is then to find a suitable function $g(T)$.

In principle, there may be many functions $g(T)$ which satisfy Eq. (3.12). However, again taking duality covariance as our guide, let us suppose that $g(T)$ is itself a duality-covariant function with weight α and sign γ_g:

$$g(T_c^2/T) \;=\; \gamma_g \,(T_c/T)^\alpha \, g(T)~. \tag{3.13}$$

Substituting this into Eq. (3.12), we then obtain a solution for $g(T)$:

$$g(T) \;=\; -\,\frac{k}{1+\gamma_g (T_c/T)^\alpha} \;=\; -\,\frac{kT^\alpha}{T^\alpha + \gamma_g T_c^\alpha}~. \tag{3.14}$$

Thus, this solution for $g(T)$ ensures a duality-covariant derivative for all α and γ_g.

Thus far, the values of α and γ_g are unfixed. In certain circumstances, however, we can impose various physical constraints in order to narrow the range of possibilities. For example, we might wish to demand that our covariant derivative reduce to the usual derivative as $T/T_c \to 0$, with only small corrections suppressed by inverse powers of T_c. In other words, we wish to demand

$$\frac{g(T)f}{T} \;\ll\; \frac{df}{dT} \qquad \text{as}~~ T/T_c \to 0~. \tag{3.15}$$

With $g(T)$ given by Eq. (3.14), this generally restricts us to the cases with $\alpha > 1$, although this constraint can be evaded or strengthened depending on the specific function f. Likewise, if we wish to retain the usual symmetry under which the temperature derivative is odd under $T \to -T$, we should require $\alpha \in 2\mathbb{Z}$, although once again this constraint is not mandatory. Finally, we would like our covariant derivatives to remain finite as $T \to T_c$. Thus, we shall restrict our attention to the cases with $\gamma_g = +1$, deferring our discussion of the $\gamma_g = -1$ case to Sect. 3.7. We shall, however, leave α as a free (positive) parameter.

Thus, combining our results and taking $\gamma_g = +1$, we obtain a thermal duality covariant derivative given by

$$D_T \;=\; \frac{d}{dT} \;-\; \frac{k}{T}\,\frac{T^\alpha}{T^\alpha + T_c^\alpha}~. \tag{3.16}$$

In this derivative, the second term functions as a "correction" term which is suppressed when $T \ll T_c$, but which grows large as the temperature approaches the string scale. Indeed, α essentially governs the *rate* at which our correction term becomes significant as $T \to T_c$. Of course, the presence of this correction term is critical, ensuring that if f is covariant with weight k and sign γ, then $D_T f$ is also covariant, with weight $k-2$ and sign $-\gamma$. Note that unlike the case with modular transformations, there is no thermal duality covariant derivative which preserves the sign of γ.

It may seem strange that our covariant derivative depends on k, which is a property of the function upon which the derivative operates. However, this is completely analogous to the situation we have just discussed for modular invariance in Sect. 3.3. Indeed, even in gauge theory, the gauge-covariant derivative depends on the gauge charge of the state on which it operates. In this analogy, k functions as the duality "charge" of the function $f(T)$, and the remaining factor $T^\alpha/(T^\alpha + T_c^\alpha)$ functions as the duality "gauge field" (*i.e.*, as the connection).

In principle, the value of α is unconstrained as long as $\alpha > 1$. We note, however, that in the limit as $T \to T_c$, the covariant derivative in Eq. (3.16) takes the limiting form

$$D_T \;\to\; \frac{d}{dT} \;-\; \frac{k}{2T}\;. \tag{3.17}$$

This is the direct analogue of Eq. (3.8), and is equivalent to the general derivative in Eq. (3.16) with $\alpha = 0$. Thus, the $\alpha = 0$ case will continue to have relevance at the critical temperature T_c. Moreover, as we shall see in Sect. 3.7, this derivative has another important property as well.

We stress that this form for the covariant derivative is not unique. In principle, any function $g(T)$ satisfying Eq. (3.12) could serve in the construction of a covariant derivative. Of course, physically sensible solutions for $g(T)$ must have the property that $g(T)/T \to 0$ as $T/T_c \to 0$, so that our "corrections" vanish at small temperatures and traditional thermodynamics is restored. Likewise, at the other extreme, we see directly from Eq. (3.12) that there are only two possibilities as $T \to T_c$: either $g(T)$ remains finite, in which case we must have $g(T) \to -k/2$, or $g(T)$ diverges, in which case we must have $g(T) \to \pm\infty$ as $T \to T_c^{\mp}$. In the former case, we necessarily obtain the covariant derivative (3.17) as $T \to T_c$, regardless of the specific solution for $g(T)$. The specific solution for $g(T)$ therefore serves only to interpolate between the fixed $T \to 0$ and $T \to T_c$ limits.

Presumably, the specific form of $g(T)$ [and if $g(T)$ is covariant, the specific value of α] can be determined through a full string calculation including gravitational back-reactions (analogous to the calculation performed in Ref. [33]) in which this covariant derivative is obtained from first principles. However, the important point from our analysis is that there is *necessarily* a string-suppressed "correction" term which must be added to the usual temperature derivative, and that its form is already significantly constrained, especially in the $T \to T_c$ limit. Thus, we shall continue to use the covariant derivative (3.16) in the following, even though we must bear in mind that other solutions for $g(T)$ may exist.

3.5 A duality-covariant string thermodynamics

Given this covariant derivative, we can now construct a manifestly covariant thermodynamics: our procedure is simply to replace all derivatives in Eq. (2.4) with the duality-covariant derivative in Eq. (3.16). We thus obtain

$$\tilde{F} = T\mathcal{V}\ , \quad \tilde{U} = -T^2 D_T \mathcal{V}\ , \quad \tilde{S} = -D_T \tilde{F}\ , \quad \tilde{c}_V = D_T \tilde{U}\ . \tag{3.18}$$

The tildes are inserted to emphasize that the new quantities we are defining need not, *a priori*, be the same as their traditional counterparts.

Let us now determine the implications of these definitions. Of course, since $\mathcal{V}$ is duality *invariant* (*i.e.*, $\mathcal{V}$ has $k = 0$ with $\gamma = +1$), we see that $\tilde{F}$ continues to be covariant with $k = 2$ and $\gamma = +1$. Thus the free energy $F(T)$ is unaltered: $\tilde{F} = F$. This is expected, since we saw in Eq. (3.5) that F is already thermal duality covariant.

A similar situation exists for the internal energy $\tilde{U}(T)$. Since $\mathcal{V}$ is covariant with $k = 0$, we see that the covariant derivative D_T in this special case is exactly the same as the usual derivative d/dT. Thus, the internal energy is also unaffected: $\tilde{U} = U$. Of course, this also makes sense, since $U(T)$ was already seen to be covariant in Eq. (3.6), with $k = 2$ and $\gamma = -1$. However, this example illustrates that any duality-covariant quantity can (and should) be expressed in terms of covariant derivatives. Thus, the internal energy U continues to fit into our overall framework involving only those derivatives.

We now turn our attention to $\tilde{S}$ and $\tilde{c}_V$. It is in these cases that new features arise. Since $\tilde{F} = F$ is already covariant with $k = 2$ and $\gamma = +1$, we find that

$$\tilde{S} \;=\; -\,D_T F \;=\; S \;+\; \frac{2\,T^{\alpha-1}\,F}{T^\alpha + T_c^\alpha} \;=\; S \;+\; \frac{2\,T^\alpha}{T^\alpha + T_c^\alpha}\,\mathcal{V}\;. \tag{3.19}$$

Thus, we see that the "corrected" entropy $\tilde{S}$ differs from the usual entropy S by the addition of an extra string-suppressed term proportional to the free energy F. Indeed, it is this corrected entropy $\tilde{S}$ which is thermal duality covariant, transforming with $k = 0$ and $\gamma = -1$. Interestingly, since $\tilde{S}$ is finite and odd, we see that the corrected entropy has a zero at the critical temperature: $\tilde{S}(T_c) = 0$. This resembles the situation with the internal energy U, which also vanishes at $T = T_c$; in fact, we find

$$\tilde{S} \;\to\; S + \frac{F}{T} = \frac{U}{T} \qquad \text{as} \;\; T \to T_c\;. \tag{3.20}$$

Of course, both of these properties differ significantly from our usual expectations.

Since our corrections to the entropy are suppressed by powers of the string scale, we see that $\tilde{S}$ continues to obey the third law of thermodynamics, with $\tilde{S} \to 0$ as $T \to 0$ in situations with a massless unique ground state. As discussed in Sect. 3.4, this is the result of requiring $\alpha > 1$. However, imposing our general condition in

Eq. (3.15), we find that we must actually restrict ourselves to values of α for which

$$\frac{2\,T^{\alpha-1}\,F}{T^\alpha + T_c^\alpha} \ll S \qquad \text{as} \;\; T/T_c \to 0 \; . \tag{3.21}$$

In general, depending on the particular thermodynamic system under study, this can yield constraints which are stronger than $\alpha > 1$.

Finally, the corrected specific heat is given by

$$\tilde{c}_V \;=\; D_T U \;=\; c_V \;-\; \frac{2\,T^{\alpha-1}\,U}{T^\alpha + T_c^\alpha} \;=\; c_V \;+\; \frac{2\,T^{\alpha+1}}{T^\alpha + T_c^\alpha}\,\frac{d\mathcal{V}}{dT} \; . \tag{3.22}$$

Thus, the difference between the uncorrected and corrected specific heats is a string-suppressed term proportional to the internal energy U. Since U vanishes at $T = T_c$ as a result of thermal duality, the corrected specific heat $\tilde{c}_V$ approaches the uncorrected specific heat c_V both as $T \to 0$ *and* as $T \to T_c$; indeed, in the latter limit, we find

$$\tilde{c}_V \;\to\; c_V - \frac{U}{T} = c_V - \tilde{S} \qquad \text{as} \;\; T \to T_c \; . \tag{3.23}$$

Note that in general, $\tilde{c}_V$ is duality invariant and even, just like $\mathcal{V}$. Once again, for a consistent low-temperature limit which reproduces ordinary thermodynamics, we must choose α such that

$$\frac{2\,T^{\alpha-1}\,U}{T^\alpha + T_c^\alpha} \ll c_V \qquad \text{as} \;\; T/T_c \to 0 \; . \tag{3.24}$$

This constraint typically yields bounds on α which are the same as those stemming from Eq. (3.21).

These new, corrected definitions for entropy and specific heat restore a certain similarity between the pairs of thermodynamic quantities $(\tilde{F}, \tilde{U})$ and $(\tilde{S}, \tilde{c}_V)$. Members of each pair share the same duality weight k and have opposite signs for γ. Of course, the first pair has weight $k = 2$ while the second pair has $k = 0$.

Given these results, we can also see explicitly why the usual uncorrected entropy S and specific heat c_V fail to transform correctly under the thermal duality transformations. From Eqs. (3.19) and (3.22), we see that S and c_V can each be re-expressed as admixtures of $k = 0$ quantities which have opposite parities (even or odd) under the thermal duality transformation. For example, S is a linear combination of $\tilde{S}$ (which is odd) and $T^{\alpha-1}F/(T^\alpha + T_c^\alpha)$ (a small correction term which is even). Only the proper "corrected" linear combinations inherent in $(\tilde{S}, \tilde{c}_V)$ disentangle this behavior.

Thus, we conclude that the "natural" duality weights and signs for the entropy and specific heat are $k = 0$ and $\gamma = \mp 1$ respectively, with the corrections in Eqs. (3.19) and (3.22) having the net effect of restoring these properties to an otherwise non-covariant S and c_V. Moreover, as we have seen, these transformation properties also make sense from the standpoint of the usual thermodynamic identities. Of course, these conclusions hold only to the extent that our functions are considered to be completely general. For example, as discussed in Ref. [30], it is possible to construct special vacuum amplitudes $\mathcal{V}(T)$ such that the *uncorrected* entropy $S(T)$ turns out to be "accidentally" covariant with a non-zero weight. However, even these functions are unsatisfactory because they are the results of definitions which fail to respect the thermal duality symmetry. Thus, such "accidentally" covariant entropies should still be corrected in the manner described here, thereby restoring the proper weights and signs to these thermodynamic quantities. We shall see explicit examples of this below.

In most realistic examples, the free energy F and the uncorrected entropy S have opposite overall signs. Thus, our string-theoretic corrections to the entropy in Eq. (3.19) generally tend to *decrease* the entropy,

$$\tilde{S} \;\leq\; S \;, \tag{3.25}$$

with the suppression becoming increasingly severe as $T \to T_c$. Likewise, in the range $T < T_c$, the internal energy U and the specific heat c_V also typically have opposite signs. [Recall that $U < 0$ for $T < T_c$, as discussed below Eq. (3.6).] We therefore find that

$$\tilde{c}_V \;\geq\; c_V \;, \tag{3.26}$$

with the bound saturating both at $T = 0$ [where $g(T) = 0$] and at $T = T_c$ [where

$U(T) = 0$]. As we shall see, these inequalities will be extremely important in the following.

3.6 An explicit duality-covariant example

In this section, we shall calculate the string-corrected entropy and specific heat within the context of a specific example displaying thermal duality. Towards this end, let us consider the vacuum amplitude [30]

$$\mathcal{V}^{(D)}(T) \;=\; -\,\frac{(T^D + T_c^D)^{2/D}}{T T_c} \tag{3.27}$$

where $D \geq 1$ is an arbitrary exponent. (Here and henceforth, we shall express all thermodynamic quantities in units of the reduced string scale $\mathcal{M} \equiv M_{\rm string}/2\pi$.) As discussed in Ref. [30], this function $\mathcal{V}(T)$ emerges as the vacuum amplitude corresponding (either exactly, or approximately and highly accurately) to finite-temperature string constructions in which the time/temperature dimension is compactified on a circle. The parameter D is the spacetime dimension of the zero-temperature string model. Note that this functional form for $\mathcal{V}$ has the property that the resulting entropy S is "accidentally" covariant with weight D and sign $+1$; however, this will play no role in the following. Indeed, the corresponding specific heat is non-covariant for all $D > 2$.

3.6.1 String-corrected entropy

Given this functional form for $\mathcal{V}(T)$, it is straightforward to calculate both the traditional entropy $S(T)$ and the corrected entropy $\tilde{S}(T)$ as functions of temperature, obtaining

$$S^{(D)}(T) \;=\; 2\,\frac{T^{D-1}}{T_c}\,(T^D + T_c^D)^{2/D-1} \tag{3.28}$$

and

$$\tilde{S}^{(D)}(T) \;=\; 2\,\frac{(T^D + T_c^D)^{2/D-1}}{T T_c\,(T^\alpha + T_c^\alpha)}\,(T^D T_c^\alpha - T^\alpha T_c^D)\;. \tag{3.29}$$

Note, in particular, that the relative sizes of the string corrections to the entropy are not small in this example unless $\alpha \geq D+1$. This is the strengthened bound on α which emerges from Eq. (3.21) for this system.

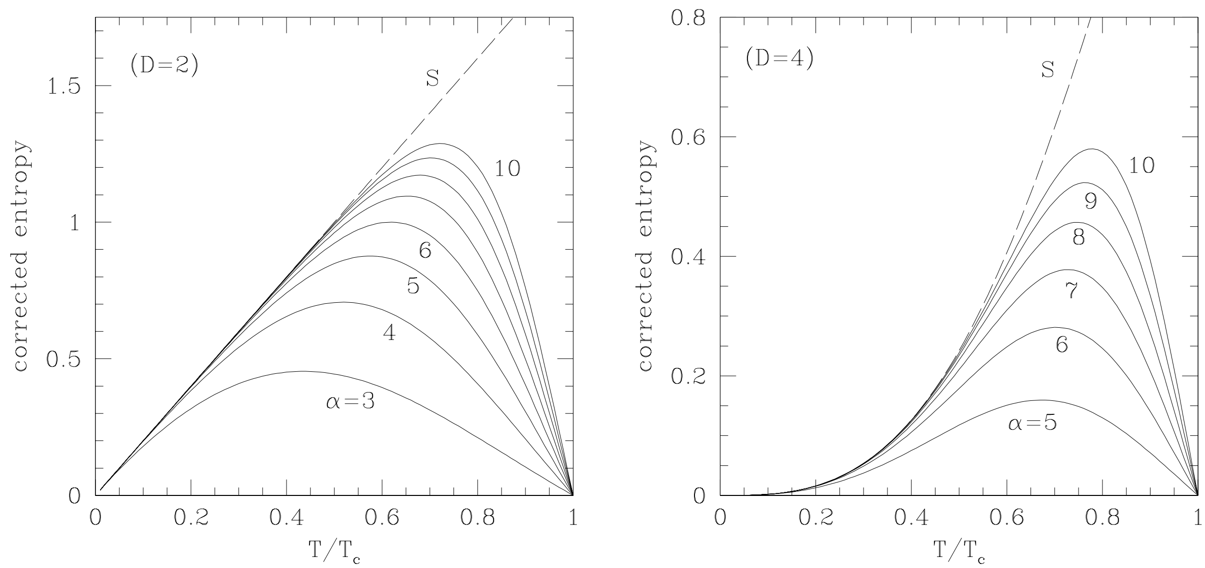

FIGURE 3.1. The string-corrected entropies $\tilde{S}^{(D)}$ given in Eq. (3.29), plotted as functions of T for $D=2$ (left plot) and $D=4$ (right plot). In each case, we have plotted the string-corrected entropies for $D+1 \leq \alpha \leq 10$, while the uncorrected entropy is indicated with a dashed line. In all cases, the corrected entropies are smaller than the traditional entropies, and vanish as $T \to T_c$. The parameter α governs the relative size of the string corrections and thus the rate with which the corrected entropy begins to separate from the uncorrected entropy.

These functions for $S^{(D)}(T)$ and $\tilde{S}^{(D)}(T)$ are plotted in Fig. 3.1 for the special cases with $D=2$ and $D=4$. We can immediately see the behavior of $\tilde{S}^{(D)}(T)$ as a function of T. At low temperatures $T \ll T_c$, we see that $\tilde{S}$ follows S quite closely; indeed the "corrections" to the traditional entropy [*e.g.*, as measured by the ratio $(S-\tilde{S})/S$] remain small as long as $T \ll T_c$. At higher temperatures, however, $\tilde{S}$ is increasingly suppressed relative to S, and ultimately vanishes as $T \to T_c$. This is required by the fact that $\tilde{S}$ must be an odd function under $T \to T_c^2/T$ for all α.

For sufficiently small temperatures, our corrected entropies resemble the traditional entropy and grow with increasing temperature, with $d\tilde{S}^{(D)}/dT > 0$ all D and α. This conforms to our standard notions of entropy as a measure of disorder. However, as T approaches the critical temperature, we see that $d\tilde{S}^{(D)}/dT$ ultimately changes sign. At first glance, this might appear to signal an inconsistency in our string-corrected thermodynamics. However, as is well known in string thermodynamics (see, *e.g.*, Refs. [14, 34, 35, 36, 37, 38, 39, 8, 40, 41, 42, 43, 44]), we expect that a phase transition or other Hagedorn-related event should occur at large temperatures at or near T_c. Thus, rather than interpret $d\tilde{S}^{(D)}/dT < 0$ as a loss of disorder, it is tempting to interpret this sign change as the beginning of a possible phase transition and the conversion of the system into new degrees of freedom. Thus, as the temperature increases towards the critical temperature, fewer and fewer of the original degrees of freedom remain in the system, and thus the entropy associated with these original degrees of freedom begins to decrease.

Of course, verifying this speculation would require a more complete understanding of the nature of the string physics near the critical temperature. Our point here, however, is that a fully covariant treatment of entropy necessarily requires the introduction of corrections which, in this case, ultimately drive the corrected entropy to zero at the critical temperature.

3.6.2 String-corrected specific heat

We can also perform a similar analysis for the specific heat. Once again starting from Eq. (3.27), we obtain

$$c_V^{(D)}(T) \;=\; 2\,\frac{T^{D-1}}{T_c}\,(T^D + T_c^D)^{2/D-2}\left[T^D + (D-1)\,T_c^D\right] \qquad (3.30)$$

and

$$\begin{aligned}\tilde{c}_V^{(D)}(T) \;=\; 2\,\frac{(T^D + T_c^D)^{2/D-2}}{T T_c\,(T^\alpha + T_c^\alpha)}\Big[&T^{2D}T_c^\alpha + T^\alpha T_c^{2D}\\ &+ (D-1)\,(T^D T_c^{\alpha+D} + T^{\alpha+D}T_c^D)\Big]\,. \qquad (3.31)\end{aligned}$$

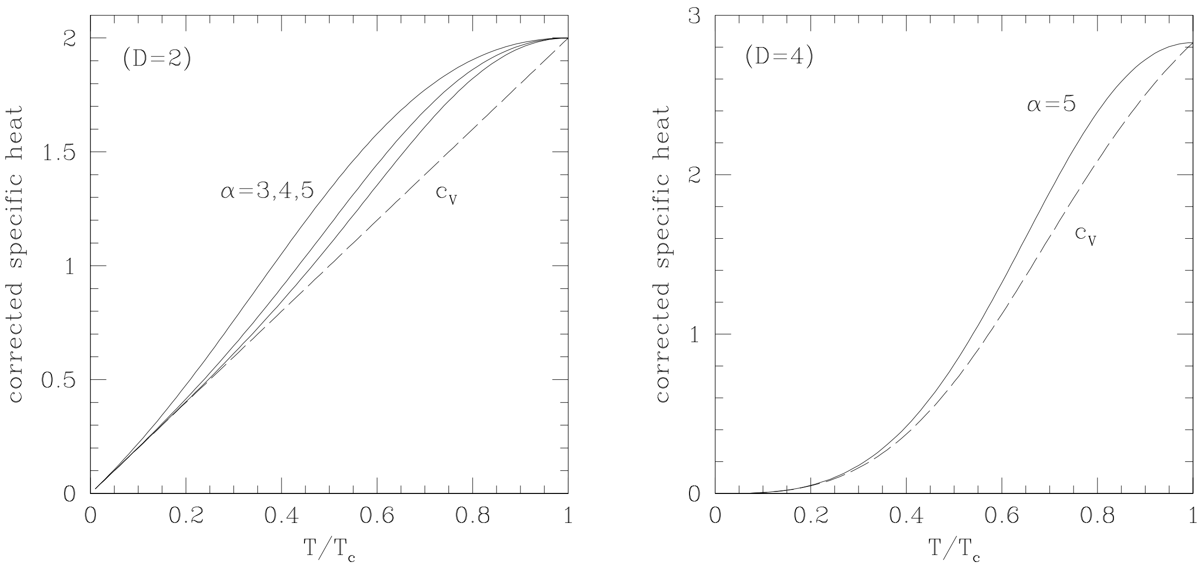

FIGURE 3.2. The string-corrected specific heats $\tilde{c}_V^{(D)}$ given in Eq. (3.31), plotted as functions of T for $D = 2$ (left plot) and $D = 4$ (right plot). For the sake of clarity, we have illustrated only the cases with the largest relative corrections: we have taken $\alpha = 3, 4, 5$ for $D = 2$, and $\alpha = 5$ for $D = 4$. Note that in all cases, the corrected specific heats exceed the traditional specific heat, agreeing with the traditional heat only at $T = 0$ and $T = T_c$. Moreover, as a consequence of thermal duality, we have $d\tilde{c}_V^{(D)}/dT = 0$ at $T = T_c$ for all α.

These functions are plotted in Fig. 3.2 for the special cases with $D = 2$ and $D = 4$. Once again, we observe that $\tilde{c}_V^{(D)} \geq c_V^{(D)}$, with the bound saturating only at $T = 0$ and $T = T_c$. Note that $d\tilde{c}_V^{(D)}/dT = 0$ at $T = T_c$ for all α. This is a direct consequence of thermal duality, and indicates that the corrected specific heat loses all temperature sensitivity at T_c.

We conclude this section with two further comments. First, throughout this section, we have focused exclusively on the behavior for $T \leq T_c$. As mentioned above, we have done this in the expectation that a phase transition or other Hagedorn-related event should occur at large temperatures at or near T_c. However, from a purely mathematical perspective, we could easily have continued our analysis beyond T_c, since our string-corrected entropies and specific heats are (by construction) thermal duality invariant. For example, since the specific heat is necessarily an even (invariant) function under $T \to T_c^2/T$, we see that $\tilde{c}_V^{(D)}$ continues to remain positive for all T and ultimately declines beyond T_c. This is in sharp contrast to the uncorrected specific heat, which continues to rise indefinitely. On the other hand, the string-corrected entropy $\tilde{S}^{(D)}$ is necessarily an odd function under $T \to T_c^2/T$. Thus, $\tilde{S}^{(D)}$ becomes *negative* beyond T_c. This provides dramatic illustration of the fact that, as already anticipated from other considerations, new physics must intercede at or near the string scale.

Our second comment concerns the duality weights of the entropy and specific heat. As already mentioned at the beginning of this section (and as explained more fully in Ref. [30]), the *uncorrected* entropy $S^{(D)}$ in this example is actually already covariant, with duality weight D and sign $+1$. Thus, it may seem that no further corrections are necessary in this case. However, as we have seen in Sect. 3.5, the proper duality weight and sign for the entropy are $k = 0$ and $\gamma = -1$ respectively. Thus, the net effect of our corrections in this case is to "convert" an even entropy function of weight D into an odd entropy function of weight zero. Of course, these corrections also simultaneously restore duality invariance to the specific heat, where it was otherwise lacking.

3.6.3 Effective dimensionalities and holography

Finally, we now investigate the *scaling* behavior of our corrected thermodynamic quantities as functions of temperature. As we shall see, this will enable us to provide a possible physical interpretation to our string-theoretic corrections.

In ordinary quantum field theory, the free energy $F(T)$ at large temperatures typically scales like T^D where D is the spacetime dimension. This in turn implies that the entropy S should scale like T^{D-1}. However, in string theory we have $F(T) \sim T^2$ as $T \to \infty$, implying that $S(T) \sim T$ as $T \to \infty$. Thus, string theory behaves asymptotically as though it has an effective dimensionality $D_{\text{eff}} = 2$.

At first glance, these two sets of results might not appear to be in conflict since they apply to different theories. However, the field-theory limit of string theory is expected to occur for $T \ll T_c$, and thus the field-theory behavior must be embedded within the larger string-theory behavior. It is therefore interesting to examine the effective dimensionality (*i.e.*, the effective scaling exponent) of our thermodynamic quantities as a function of temperature. As discussed in Ref. [30], it is easiest to define this effective dimensionality $D_{\text{eff}}(T)$ by considering the entropy: since $S(T)$ is a monotonically increasing function of T, we can define $D_{\text{eff}}(T)$ as the effective scaling exponent at temperature T, setting $S(T) \sim T^{D_{\text{eff}}-1}$. We thus obtain

$$D_{\text{eff}} \;\equiv\; 1 + \frac{d\ln S}{d\ln T} \;=\; 1 + \frac{T}{S}\frac{dS}{dT} \;=\; 1 + \frac{c_V}{S} \;, \tag{3.32}$$

where the last equality follows from the thermodynamic identity $c_V = TdS/dT$.

Given the entropy $S^{(D)}$ in Eq. (3.28) and the specific heat $c_V^{(D)}$ in Eq. (3.30), it is straightforward to calculate $D_{\text{eff}}(T)$ as a function of temperature T. This calculation was originally performed in Ref. [30], where a plot of $D_{\text{eff}}(T)$ is given. In each case, it is found that D_{eff} *interpolates* between $D_{\text{eff}} = D$ for $T \ll T_c$ and $D_{\text{eff}} = 2$ for $T \gg T_c$. It is, of course, easy to interpret this result. At small temperatures $T \ll T_c$, the entropy behaves as we expect on the basis of field theory, growing according to the power-law $S^{(D)}(T) \sim T^{D-1}$. Indeed, this low-temperature limit of string theory can be identified as the high-temperature limit of the low-energy effective field theory. However, as T approaches the string scale T_c, we see that this scaling behavior begins to change, with the T^{D-1} growth in the entropy ultimately becoming the expected *linear* growth for $T \gg T_c$. This is then the asymptotic *string* limit.

These observations originally appeared in Ref. [30]. However, given these observations, let us now proceed to determine the effective dimensionalities $\tilde{D}_{\text{eff}}$ of our *string-corrected* entropies. In complete analogy with D_{eff}, these corrected effective dimensionalities $\tilde{D}_{\text{eff}}$ may be defined as

$$\tilde{D}_{\text{eff}} \;\equiv\; 1 + \frac{d \ln \tilde{S}}{d \ln T} \;=\; 1 + \frac{T}{\tilde{S}} \frac{d\tilde{S}}{dT} \;. \tag{3.33}$$

Note that since $\tilde{c}_V \neq T d\tilde{S}/dT$, we cannot write Eq. (3.33) easily in terms of $\tilde{c}_V$.

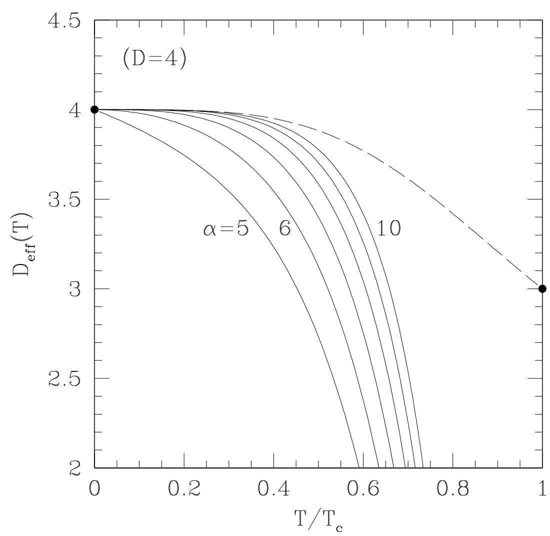

FIGURE 3.3. The effective dimensionalities $\tilde{D}_{\text{eff}}$ of the four-dimensional string-corrected entropies $\tilde{S}^{(4)}$, plotted as functions of T for $5 \leq \alpha \leq 10$. The effective dimensionality of the uncorrected entropy $S^{(4)}$ is also shown (dashed line).

The results are plotted in Fig. 3.3 for $D = 4$. As expected, all of our corrected entropies exhibit an initial scaling with $\tilde{D}_{\text{eff}} = D = 4$ as $T/T_c \to 0$; this is guaranteed by our original requirement that $\alpha \geq D+1$. This implies that none of our string corrections disturb the expected field-theoretic behavior at low temperatures. However, as T becomes larger and approaches the string scale, we see that the net effect of our string corrections is *to reduce the effective scaling dimensionality of the entropy even more rapidly than in the uncorrected case.*

Thus, combining our results from Figs. 3.1 and 3.3, we see that our string corrections have two net effects on the entropy as $T \to T_c$: they reduce its overall magnitude, and they also reduce its scaling exponent (effective dimensionality) as a function of temperature. It is important to stress that these are, in principle, uncorrelated effects: the first relates to the overall size of a function, while the second has to do with its rate of growth. As a stark example of this point, observe that if the scaling behavior of the corrected entropy had been $(T/T_c)^3$ rather than $(T/T_c)^4$ for all $T \leq T_c$, this *decrease* in the scaling exponent would have resulted in an *increase* in the entropy, not a decrease. This would have been interpreted as the appearance of more degrees of freedom at low temperatures, not fewer.

Of course, there is a natural interpretation for an effect which simultaneously decreases not only the entropy but also the effective dimensionality that governs its scaling: such an effect is holographic. Thus, we see that our duality-inspired corrections to the laws of thermodynamics are holographic in nature, enhancing the tendency towards holography that already exists in traditional string thermodynamics. Indeed, as originally observed in Ref. [30], we see from Fig. 3.3 that the *uncorrected* effective dimensionality already shows a holographic decline from $D_{\text{eff}} = 4$ at $T/T_c \ll 1$ to $D_{\text{eff}} = 3$ at $T \to T_c$. Our corrections thus enhance this effect, introducing this holographic behavior even more strikingly and at lower temperatures.

Of course, as discussed more fully in Ref. [30], there are a number of outstanding issues that need to be addressed before we can truly identify this phenomenon with holography. In particular, an analysis formulated in flat space (such as ours) cannot address questions pertaining to the geometry of holography, and thus cannot determine whether the modified scaling behavior and the implied reduction in the number of associated degrees of freedom are really to be associated with a lower-dimensional subspace (or boundary) of the original geometry. Indeed, such an analysis is beyond the scope of this chapter, and would require reformulating the predictions of thermal duality for string theories in non-trivial D-dimensional backgrounds, and then developing a map between degrees of freedom in the bulk of the D-dimensional volume and those on a lower-dimensional section of this volume. Thus, as indicated in Ref. [30], the possible connection between thermal duality and holography remains to be explored further.

3.7 Alternative formulations for a duality-covariant thermodynamics

In this section, we shall investigate other possible formulations for a duality-covariant thermodynamics. As we shall see, a wide set of possibilities exists: some of these lead to drastically different phenomenologies, while others have drastically different theoretical underpinnings.

3.7.1 Alternative covariant derivatives

First, as stressed in Sect. 3.4, our thermal duality covariant derivative in Eq. (3.16) is not unique: *any* function $g(T)$ satisfying Eq. (3.12) can be exploited in the construction of a covariant derivative as in Eq. (3.10). As an example, let us again remain within the class of covariant functions $g(T)$ given in Eq. (3.14) and consider the physics that results if we choose $\gamma_g = -1$ rather than $\gamma_g = +1$. Our covariant derivative in Eq. (3.16) then becomes

$$D_T^{(-)} = \frac{d}{dT} + \frac{k}{T}\frac{T^\alpha}{T_c^\alpha - T^\alpha} , \tag{3.34}$$

leading to the definitions

$$\begin{aligned} \tilde{S} &\equiv -D_T^{(-)}F = S - \frac{2\,T^{\alpha-1}\,F}{T_c^\alpha - T^\alpha} = S - \frac{2\,T^\alpha}{T_c^\alpha - T^\alpha}\mathcal{V} , \\ \tilde{c}_V &\equiv D_T^{(-)}U = c_V + \frac{2\,T^{\alpha-1}\,U}{T_c^\alpha - T^\alpha} = c_V - \frac{2\,T^{\alpha+1}}{T_c^\alpha - T^\alpha}\frac{d\mathcal{V}}{dT} . \end{aligned} \tag{3.35}$$

Note that unlike the case with $\gamma_g = +1$, we now have $\tilde{S} \geq S$ and $\tilde{c}_V \leq c_V$. However, as required, we still find that our string-corrected quantities $\tilde{S}$ and $\tilde{c}_V$ are duality covariant with weight $k = 0$ and signs ∓ 1 respectively. Moreover, in the case of the covariant example given in Eq. (3.27), a proper low-temperature (field-theory) limit is guaranteed for all $\alpha \geq D$.

At first glance, the definitions in Eq. (3.35) might appear to be unacceptable because of the apparent divergences in $\tilde{S}$ and $\tilde{c}_V$ as $T \to T_c$. For example, since the corrections to the entropy are positive and the corrections to the specific heat are negative, we might worry that the definitions in Eq. (3.35) would result in the asymptotic behavior $\tilde{S} \to \infty$ and $\tilde{c}_V \to -\infty$ as $T \to T_c$. While a positively divergent entropy leads to no specific difficulty (and might be interpreted as a Hagedorn-like phenomenon), a negative specific heat necessarily results in an inconsistent thermodynamics in which thermal fluctuations grow without bound and ultimately destabilize the system.

However, these concerns are ultimately spurious. Because the internal energy U vanishes at $T = T_c$ as a result of thermal duality, the specific heat actually remains finite and positive as $T \to T_c$. Indeed, the divergence in the definition of the covariant derivative cancels against the vanishing of the internal energy, resulting in a string-corrected specific heat which takes the finite asymptotic value

$$\tilde{c}_V^{(D)} \;\to\; 2^{2/D-1}\, D\,(1-2/\alpha) \qquad \text{as} \;\; T \to T_c \; . \tag{3.36}$$

Note that this quantity is positive for all $\alpha > 2$.

The resulting string-corrected entropies and specific heats are plotted in Fig. 3.4 for $D = 4$. As expected, we see that $\tilde{c}_V^{(4)}$ remains positive in all cases, while $\tilde{S}^{(4)}$ is now monotonically increasing as a function of temperature for all $T \leq T_c$. Clearly, the effect of these corrections is no longer "holographic" as it was for $\gamma_g = 1$. However, this possibility also results in a fully consistent, duality-covariant string thermodynamics.

At present, we have no physical basis on which to prefer one version of the covariant thermodynamics over another. Even though they lead to drastically different phenomenologies, they are each internally self-consistent and have the same low-temperature (field-theoretic) limits. However, our main point in this chapter is that *some* string-theoretic correction is necessary in order to restore thermal duality covariance to the usual rules of thermodynamics, and that it is possible to introduce

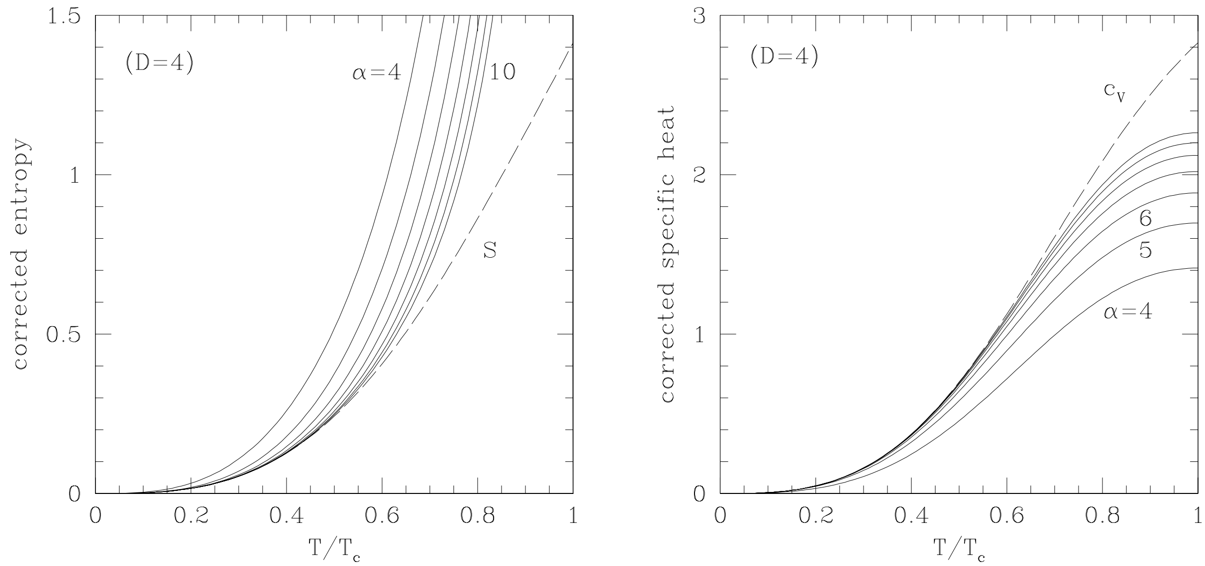

FIGURE 3.4. The string-corrected four-dimensional entropies $\tilde{S}^{(4)}$ and specific heats $\tilde{c}_V^{(4)}$ given in Eq. (3.35), plotted as functions of T. In each case, we have plotted these thermodynamic quantities for $4 \leq \alpha \leq 10$, while the uncorrected quantities are indicated with a dashed line. Note that $\tilde{c}_V^{(D)}$ remains positive in all cases, while $\tilde{S}^{(D)}$ is now monotonically increasing as a function of temperature for all $T \leq T_c$.

such corrections without disturbing the usual low-temperature physics associated with traditional thermodynamics. The decision as to the preferred specific form of the covariant derivative awaits a full string calculation, perhaps along the lines discussed in Sect. 3.3.

3.7.2 Alternative thermodynamic structures

Changing the specific form of the covariant derivative is not the only way in which we might approach the construction of an alternative thermodynamics. Indeed, even within the context of a fixed covariant derivative, there are other structural options that can be explored.

In order to understand these other options, let us first recall the structure of the traditional thermodynamics. This structure is defined through the definitions in

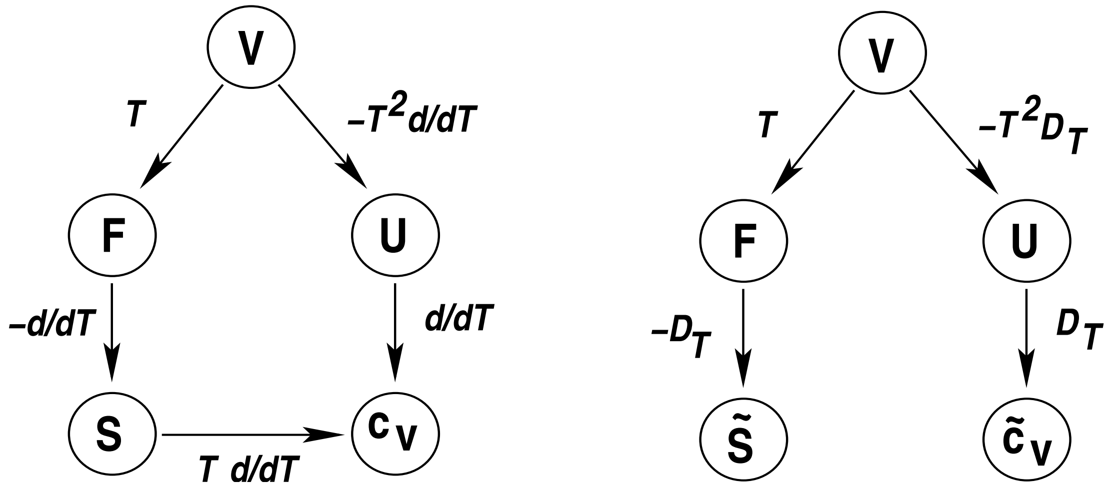

FIGURE 3.5. Relations between thermodynamic quantities. (a) Traditional thermodynamics: All thermodynamic quantities are related to each other through temperature multiplications and differentiations. (b) Our string-corrected thermodynamics: we replace the usual temperature derivatives by duality-covariant derivatives, maintaining the definitions of $\tilde{S}$ and $\tilde{c}_V$ in terms of their respective thermodynamic potentials F and U. However, $\tilde{c}_V$ is no longer related to $\tilde{S}$ through either type of temperature derivative.

Eq. (2.4), and is illustrated in Fig. 3.5(a). Note that the thermodynamic quantities are related to each other through temperature multiplications and differentiations, forming a closed self-consistent set of definitions. Of course, the temperature derivatives involved in these definitions do not respect thermal duality, which is why we were motivated to construct a thermal duality covariant temperature derivative. Using this, we then developed a manifestly duality-covariant thermodynamics by replacing all ordinary temperature derivatives with duality-covariant derivatives. This resulted in a version of thermodynamics whose structure is illustrated in Fig. 5(b). Indeed, as evident in Eq. (3.18), our new quantities $\tilde{S}$ and $\tilde{c}_V$ are defined as covariant derivatives of their respective thermodynamic potentials F and U.

However, this replacement of d/dT by D_T does not preserve the entire structure of the traditional thermodynamics: the final direct "link" between the entropy and specific heat is broken. In the traditional thermodynamics, these two quantities are related by the identity

$$c_V \;=\; T\frac{d}{dT}S\;, \tag{3.37}$$

yet $\tilde{c}_V$ and $\tilde{S}$ are not related in this way through either Td/dT or TD_T. (Note that since $\tilde{S}$ has zero weight, d/dT and D_T are actually the same operator when acting on $\tilde{S}$.) Indeed, the fact that $\tilde{c}_V \neq Td\tilde{S}/dT$ is immediately apparent upon comparing Figs. 3.1, 3.2, and 3.4.

Of course, one might argue that preserving Eq. (3.37) is not as critical as preserving the identifications of the entropy and specific heat as derivatives of their respective potentials. However, in traditional thermodynamics, the identity (3.37) is critical for interpreting entropy in terms of heat transfer,

$$dS \;=\; \frac{dQ}{T}\;. \tag{3.38}$$

To see this, recall that a heat transfer dQ induces a change in internal energy $dU = dQ$ (where we are not distinguishing between exact and inexact differentials and where we have set $dW = 0$). However, since $dU = c_V dT$, we see that Eq. (3.38) cannot hold unless S and c_V are related through Eq. (3.37).

There are various ways in which this situation can be addressed. One option, of course, is to regard the relation (3.37) as more fundamental than the separate relations between either the entropy or specific heat and their respective thermodynamic potentials. We could then establish a covariant thermodynamics by replacing our previous definition for the corrected specific heat with a new definition stemming directly from the corrected entropy:

$$\tilde{S} \;=\; -\,D_T F\;, \qquad\qquad \tilde{c}_V' \;=\; TD_T\tilde{S}\;. \tag{3.39}$$

This option is illustrated in Fig. 3.6(a). Alternatively, we could retain the previous corrected specific heat $\tilde{c}_V$, and implicitly define a new corrected entropy (up to an overall additive constant) relative to this specific heat:

$$\tilde{c}_V \;=\; D_T U\;, \qquad\qquad \tilde{c}_V \;=\; TD_T\tilde{S}'\;. \tag{3.40}$$

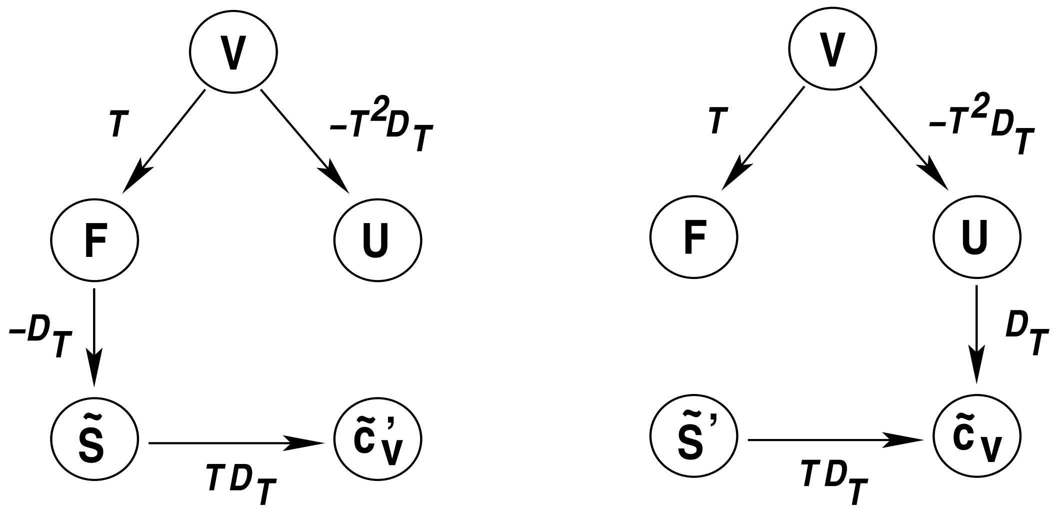

FIGURE 3.6. Relations between thermodynamic quantities in alternative formulations of duality-covariant thermodynamics. (a) In this version based on Eq. (3.39), the corrected entropy $\tilde{S}$ is defined through the free energy, but the corrected specific heat $\tilde{c}_V$ is defined through the corrected entropy. (b) In this version based on Eq. (3.40), the corrected specific heat is defined through the internal energy, and the corrected entropy is defined implicitly through the corrected specific heat.

This option is illustrated in Fig. 3.6(b).

Despite their differences, each of these options results in a fully consistent, duality covariant thermodynamics. The primary difference between them, of course, is in the interpretation given to their corrected entropies. The corrected entropy $\tilde{S}$, which appears in Eq. (3.18) and Eq. (3.39), is derived from the free energy which in turn is derived directly from the partition function $\mathcal{V}$. This entropy should thus retain its interpretation as a counting of states (*i.e.*, as a measure of disorder). The corrected entropy $\tilde{S}'$, by contrast, is defined implicitly through Eq. (3.40). This entropy should thus retain its interpretation pertaining to heat transfer.

Given these observations, the question then arises as to whether there exist any special covariant derivatives D_T for which *all* of the "links" in these diagrams are generalized and continue to hold. As we shall now prove, only one such derivative exists.

To see this, we first observe that the diagram in Fig. 3.5(a) "closes" for the usual thermodynamics as a result of the operator identity

$$T\frac{d^2}{dT^2}T \;=\; \frac{d}{dT}T^2\frac{d}{dT}\;. \tag{3.41}$$

This in turn holds as a result of the commutation relation

$$[T, d/dT] \;=\; -1\;. \tag{3.42}$$

Indeed, when acting on $\mathcal{V}$, each side of Eq. (3.41) provides a different route to the second derivative c_V: the left side passes through F and S, while the right side passes through U.

We now seek to duplicate this success for our covariant derivative D_T. It is straightforward to demonstrate that

$$[T, D_T^{(k)}] \;=\; -1 \qquad\qquad \text{for all}\;\; k \tag{3.43}$$

where k is the weight coefficient within D_T; indeed, Eq. (3.42) is nothing but the $k = 0$ special case of Eq. (3.43). However, in order to have our diagrams "close" for arbitrary vacuum amplitudes $\mathcal{V}$, our covariant derivatives must now satisfy the generalized relation

$$T D_T^{(0)} D_T^{(2)} T \;=\; D_T^{(2)} T^2 D_T^{(0)}\;. \tag{3.44}$$

Without loss of generality, let us write $D_T^{(0)} = d/dT$ and $D_T^{(2)} = d/dT + g(T)/T$, as in Eq. (3.10). We then find

$$[D_T^{(2)}, D_T^{(0)}] \;=\; -\frac{d}{dT}\left(\frac{g}{T}\right)\;, \tag{3.45}$$

which, along with the commutation relation in Eq. (3.43), enables us to reduce Eq. (3.44) to the differential equation $dh/dT = -h/T$ where $h \equiv g/T$. The only solution to this equation has $h \sim T^{-1}$, or g equal to a constant.[1] However, according

[1]This conclusion can also be reached directly by observing that $\tilde{c}_V$ and $\tilde{S}$ are related through the modified identity $\tilde{c}_V = T d\tilde{S}/dT + (dg/dT)F$. This reduces to the traditional identity only if $g(T)$ is a constant. However, the above derivation involving the commutation relations of our covariant derivatives exposes the underlying algebraic structure behind the failure of the traditional identity when $g(T)$ is not a constant.

to Eq. (3.12), this constant must be equal to $-k/2$. We thus find that $g(T) = -k/2$ is the unique solution which preserves all of our thermodynamic identities, resulting in the unique covariant derivative

$$D_T \;=\; \frac{d}{dT} \;-\; \frac{k}{2T}\;. \tag{3.46}$$

Remarkably, this is precisely the derivative that we already found in Eq. (3.17). However, we see that we would now have to take this as our covariant derivative for *all* values of T in order to preserve all of the "links" in our covariant thermodynamics. In other words, following Eq. (3.20), we would have to define $\tilde{S} \equiv U/T$ for all T.

It is easy to see how this corrected entropy manages to retain both of its interpretations pertaining to heat transfer and state counting. By defining $\tilde{S} \equiv U/T$, we are providing a direct relation between the corrected entropy and the internal energy, which in turn can be directly related to heat. A similar argument applies to counting states. Recall that the traditional entropy S is special in that it depends on the temperature only though the normalized Boltzmann probabilities $P_i \equiv p_i/Z$ where $p_i \equiv \exp(-E_i/T)$ and $Z \equiv \sum_i p_i$:

$$S \;=\; -\sum_i P_i \,\ln\, P_i\;. \tag{3.47}$$

It is this expression which enables us to associate the emergence of order with the vanishing of S: as $T \to 0$, we find $P_i = 0$ for all excited states and $P_i = 1$ for the ground state. However, if we now take $\tilde{S} \equiv U/T = S + F/T$, we find

$$\begin{aligned}
\tilde{S} \;=\; S + F/T \;&=\; -\sum_i P_i \ln\, P_i - \ln Z \\
&=\; -\sum_i P_i\,(\ln\, p_i - \ln\, Z) - \ln Z \\
&=\; -\sum_i \,P_i\, \ln\, p_i
\end{aligned} \tag{3.48}$$

where in the first line we have identified $F = -T \ln Z$ (as appropriate for the usual canonical ensemble). We thus see that $\tilde{S} \equiv U/T$ is given by an expression which is similar to Eq. (3.47) but in which the final normalized Boltzmann probability P_i is simply replaced by the *unnormalized* Boltzmann probability p_i. Indeed, with this definition, $\tilde{S}$ is sensitive to the distribution of *individual* Boltzmann probabilities in precisely the same way as the usual entropy S, and differs only in its dependence on their combined sum. Thus $\tilde{S}$ can also be taken as a direct measure of disorder.

Unfortunately, these definitions fail a crucial test: they do not have a smooth limit as $T/T_c \to 0$ in which traditional thermodynamics is restored. Rather, this solution for D_T exists only as a special point, a unique alternative thermodynamics which does not connect smoothly back to the traditional case. We are therefore forced to disregard this possibility. We see, then, that it is not generally possible to construct a thermal duality covariant thermodynamics which simultaneously preserves *all* of the traditional relations between our thermodynamic quantities.

3.8 Discussion

In the previous sections, we have advanced a proposal to modify traditional thermodynamics. Needless to say, this raises a number of important issues. In this section, we shall address some of these issues and provide some possible interpretations and resolutions.

First, one might wonder whether it is truly necessary to take the drastic step of modifying the rules of thermodynamics. After all, even in condensed-matter physics, there are systems (such as arrays of Josephson junctions) which exhibit temperature-inversion symmetries which are analogues of thermal duality. One does not modify the laws of thermodynamics when analyzing these systems; one merely accepts the fact that their free energies and their entropies may exhibit the underlying symmetry to different degrees. It is natural to wonder, then, whether the laws of thermodynamics should really vary with the system. Should not the laws of thermodynamics *transcend* the system under study?

The critical difference, however, is that this proposal is not about a particular system or configuration of matter. Rather, this is a proposal whose inspiration is string theory, a purported theory of matter itself at the most fundamental energy scales. Thus, the proposed string-theoretic corrections are to be interpreted as universal, valid for all systems regardless of their underlying symmetries. The analogy with gauge invariance is apt. Classical electromagnetism is the theory underlying all electromagnetic phenomena, and it exhibits gauge invariance at its most fundamental level. Within the framework of this theory, regardless of the particular system or charge distribution under study, we do not ascribe physical reality to quantities which are not gauge invariant; likewise we would not tolerate a calculational methodology which explicitly breaks gauge invariance in a way that does not lead to gauge-invariant results. If thermal duality is truly a fundamental string symmetry, then the same should be true here. Just as gauge invariance is used as a guide when performing calculations and extending our models into new domains, thermal duality is similarly being exploited to determine the forms of possible string-theoretic corrections to the laws of thermodynamics. We know that such corrections are necessary because the traditional laws explicitly break a symmetry which we are holding to be fundamental.

Likewise, one might wonder whether we must take thermal duality as a fundamental symmetry. After all, thermal duality might simply be an accident of certain compactifications. However, thermal duality is intimately related to T-duality and Lorentz invariance, and both of these are certainly fundamental symmetries in string theory. Indeed, T-duality is often taken as evidence that strings "feel" the spacetime in which they propagate in a way that does not distinguish between large and small. Symmetries such as these are not considered accidents; rather, they are taken as clues, evidence for the need to reinterpret the nature of time and space at the string scale. Since the roots of thermal duality are firmly embedded in T-duality, it would seem that the implications of thermal duality should be taken just as seriously. Thus, if

T-duality tells us that our understanding of space itself may require modification at the string scale, the correspondence between compactified zero-temperature theories and uncompactified finite-temperature theories suggests that the same must be true of our understanding of thermodynamics. It is then completely natural that the laws of thermodynamics would require modification.

There are, of course, closed string compactifications which fail to exhibit thermal duality, just as there are closed string compactifications which fail to be self-dual under T-duality transformations. Indeed, the analogy is exact at a mathematical level: such compactifications have certain orbifold twists which mix into the compactification and spontaneously break the underlying symmetry. However, the important point is that these are only *spontaneous* breakings of the fundamental symmetry; as with all Scherk-Schwarz breakings, the symmetry-breaking effects scale with the inverse volume of the compactification and disappear in the infinite-volume limit. The existence of compactifications in which these symmetries are spontaneously broken does not alter the primary point that these are still fundamental symmetries in string theory, and we should not expect the *rules* of the theory itself to violate them. As stated in the Introduction, it is acceptable if the entropy $S(T)$ turns out to be non-covariant because the underlying vacuum amplitude $\mathcal{V}(T)$ is non-covariant for a particular twisted string ground state. It is not acceptable, however, if the covariance of $S(T)$ is lost only because this quantity is defined in a way that fails to respect the underlying symmetry.

Another important issue concerns the possible interpretations of the quantities $\tilde{S}$ and $\tilde{c}_V$. For example, a conservative interpretation would be to assert that these new quantities are merely the proper "eigenquantities" with respect to thermal duality transformations, and that the entropy and specific heat are not eigenquantities but rather linear combinations of these eigenquantities. In this way, one would not need to impose the further interpretation that $\tilde{S}$ is itself the actual entropy, that $\tilde{c}_V$ is itself the actual specific heat. In other words, one would then avoid the need to interpret the extra string-suppressed terms in the definitions of $\tilde{S}$ and $\tilde{c}_V$ as *corrections*.

However, while such an interpretation is logically consistent, we would then be placed in the somewhat awkward position of associating physical observables such as entropy and specific heat with mathematical quantities that fail to exhibit our fundamental symmetries. If we believe fully that entropy and specific heat are physical observables, we are motivated to associate them with mathematical quantities such as $\tilde{S}$ and $\tilde{c}_V$ which are consistent with these symmetries. While this is indeed a more ambitious interpretation of the results, such an interpretation seems especially natural in light of the fact that the extra terms involved are, as noted, suppressed by powers of the string scale and hence are unobservable at low temperatures.

But this in turn raises another important issue. Given this stronger interpretation, a quantity such as entropy now has an extra contribution in its definition, one which depends on an energy scale T_c which is in turn related to the string scale. How can this be justified, given our expectation that entropy is merely a counting of states? Indeed, it seems that entropy should be a pure number without reference to any physical scale.

There are two potential answers to this question. First, the covariant derivative with $\alpha = 0$ actually does not introduce any new scale T_c. Moreover, this is the unique derivative which restores thermal duality while simultaneously managing to close all of the "links" in the thermodynamics diagrams in Sect. 7. Of course, this derivative does not admit traditional thermodynamics as a low-temperature limit, thus requiring that it be interpreted only as strongly as the above "conservative" approach would permit.

The second answer, however, is the more relevant one. It is certainly true that one is, in general, introducing a physical scale into the definition of entropy; this was hardly to be avoided, since the symmetry one is attempting to restore by doing so also contains a physical scale. However, this is not just any scale: this is the fundamental scale of string theory, the scale which one expects to govern the relative sizes of string-related phenomena associated with quantum gravity and a possible breakdown of our usual notions of spacetime geometry. It is not too much to imagine that this profound alteration should also affect the very meaning of degrees of freedom and counting of states. Indeed, it is natural to suspect a connection with holography in this regard.

Of course, such a state of affairs would seem completely natural if the number of degrees of freedom in the theory were to change as one approaches the string scale due to some hitherto unknown gravitational or string-induced effect. This would indeed be in the spirit of holography. However, at first glance, it migth appear that our proposed modification to the laws of thermodynamics does not appear to be changing physical quantities such as the degrees of freedom of the theory; by redefining entropy, it may instead appear that our proposal merely changes the probabilistic rules by which they are counted.

At a deeper level, however, it is not readily evident how to distinguish between the two situations. Even with the usual definition of entropy, we count all states equally because we assume that each microstate of the system is equally likely to occur, that a given system explores all of its energetically allowed states with equal probability. This assumption is ultimately the bedrock of standard thermodynamics, but it is possible that this assumption is violated at the string scale. After all, we already know that this assumption is violated in purely classical (deterministic) systems, which must obey the Poincaré recurrence theorem and hence cannot truly explore the space of states completely randomly. In such systems, the validity of such an assumption becomes a question of timescales, and these ultimately depend on the relevant physical parameters of the system. Even in a quantum-mechanical system, this assumption is justified only in a rough statistical sense, thanks to quantum-mechanical uncertainties in specifying our states; once again, the validity of the assumption depends on the physical parameters of the system. It is therefore not too much to expect that near the string scale, new quantum-gravitational or string-induced effects may also ultimately distort the manner in which the system explores all of its energetically allowed states. If so, the string-corrected entropy may be precisely what accounts for this phenomenon, providing a recipe for computing an "effective" number of degrees of freedom after all gravitational or string-induced effects are included. Indeed, as long as the final corrected entropy exhibits thermal duality along with the other thermodynamic quantities, it may not be possible to determine whether the true change is in the number of degrees of freedom or in the manner by which they are counted. Only the final count is important.

Clearly, such discussions ultimately tend in a philosophical direction and do not lead to simple answers. However, the important point is that the proposed thermodynamics differs from the standard thermodynamics only through effects which are unmeasurably small at temperatures much below the string scale. Given that physics is an experimental science, we cannot prove or disprove this proposal except through recourse to aesthetics. In this case, aesthetics means symmetry. The proposed modifications to thermodynamics restore one symmetry, namely thermal duality, but imply profound changes to our understanding of entropy. Thus, it is natural that our understanding of quantities such as entropy would require profound alteration as we approach the fundamental scale of quantum gravity and string theory.

3.9 Conclusions and Open Questions

In this chapter, we have addressed a fundamental issue: is it possible to construct a thermodynamics which is manifestly covariant with respect to the thermal duality symmetry of string theory?

In one sense, this approach was successful. We were able to construct a manifestly covariant derivative, and through this derivative we were able to constuct a manifestly covariant thermodynamics which not only reduces to the standard thermodynamics at low temperatures, but which leads to corrections that become significant only near the string scale. As pointed out by Sagredo, this alone guarantees that such a theory is experimentally viable as an extension to the standard rules of thermodynamics. Given that this theory restores a fundamental duality symmetry where it was otherwise lacking, we believe that such extensions to the rules of thermodynamics are worthy of further exploration.

Adopting this attitude, we are then led to a number of outstanding questions. First, of course, there are several theoretical issues. Most importantly, we needed to make an assumption for the form of the function $g(T)$ in our covariant derivative. While many of our main conclusions are independent of the specific form of $g(T)$, it still remains to calculate this function from first principles through a string calculation analogous to that discussed in Sect. 3.3. This would, we believe, place our proposal on firmer theoretical footing. Another theoretical issue concerns the possible relation, if any, between our results and holography. Given that we are changing the rules by which entropy is to be calculated — indeed changing the very definition of entropy itself — it is important to study whether and how the effects of these string corrections can be interpreted in a holographic context. We have already seen, for example, that in many cases these string corrections tend to profoundly alter the scaling behavior of the entropy with temperature, thereby decreasing the effective spacetime dimensionality associated with the entropy. However, as discussed earlier, interpreting this effect as truly "holographic" would also require a geometric understanding of how the degrees of freedom contributing to $\tilde{S}$ may be mapped from a volume to the boundary of a volume. This issue cannot be addressed in our formulation which is thus far based on strings in flat (infinite-volume) backgrounds.

There are also many phenomenological issues that are prompted by our approach. For example, how do our results extend to theories in which thermal duality is spontaneously broken (see, *e.g.*, Refs. [25, 17, 26, 8, 27, 20]), as well as to open strings and branes? The answers to these questions could have important implications for recent brane-world scenarios. Likewise, it is interesting to consider the possible applications of our results to early-universe cosmology, particularly regarding the issues of Hagedorn-like phase transitions and entropy generation.

In another sense, however, our investigations have perhaps raised more questions than they have answered.

The structure of thermodynamics is so tightly constrained, and the underpinnings of thermodynamics rest on such elementary axioms of probability and state-counting, that it would seem to be an extremely risky undertaking to attempt any alteration or generalization of these principles. We have already seen in Sect. 3.7, for example, that there are several possible generalizations of the traditional rules of thermodynamics, yet none of these approaches simultaneously preserves all of the different shades of interpretation that are normally ascribed to quantities such as entropy.

Many of these theoretical issues could perhaps be resolved (or at least placed on firmer footing) if we were to develop a formulation of our generalized thermodynamics based on the *microcanonical* ensemble. Yet we can immediately see the difficulties in doing so. By its very nature, thermal duality is a symmetry with respect to transformations in temperature; clearly temperature is the independent variable. In order to develop an equivalent microcanonical formulation, however, we require the internal energy U to be the independent variable. We would thus need to express thermal duality as a symmetry under transformations of U. We would then attempt to take our string-corrected entropy $\tilde{S}$ as the fundamental quantity (*i.e.*, the string-corrected counting of states), and demonstrate that $d\tilde{S}/dU$ (or even a covariant derivative $D_U\tilde{S}$) is equivalent to the inverse of our original temperature $1/T$. However, it is easy to verify that this microcanonical approach does not generally lead to results which are consistent with those of the canonical ensemble. Indeed, we believe that the fundamental difficulty in this approach rests on the need to find a microcanonical-ensemble equivalent of thermal duality — *i.e.*, a formulation of this symmetry which does not take T as the independent parameter. As long as our approach to string thermodynamics rests on the canonical ensemble and string partition functions, this formulation is likely to elude us. Similar issues concerning the relation between the microcanonical and canonical ensembles are well known to exist in attempting to understand the Hagedorn transition, and may also play a role in generic problems concerning the interplay between gravity and thermodynamics, such as the Jeans instability.

What are we to make of these results? On the one hand, we could be content with the observation that there exist special solutions for $\mathcal{V}(T)$, as discussed in Ref. [30], for which the traditional entropy S (and occasionally even the specific heat c_V) turn out to be duality covariant. Indeed, in Ref. [30], we conjectured that these special solutions $\mathcal{V}(T)$ may represent the exact results of actual string calculations when the contributions from all orders in string perturbation theory are included. However, we continue to remain sympathetic to the original motivation of this chapter, namely that the rules of thermodynamics should themselves respect this symmetry in a manifest fashion. Indeed, it is by thrashing out how this can occur that we continue to hope to gain insight into the possible nature of temperature, state counting, and thermodynamics near the string scale. After all, if thermal effects can truly be associated with spacetime compactification through the Matsubara/Kaluza-Klein correspondence, then our expectations of an unusual "quantum geometry" near the string scale — one which does not distinguish between "large" and "small" — should simultaneously lead to expectations of an equally unusual thermodynamics near the string scale which does not distinguish between "hot" and "cold" in the traditional sense. Thermal duality should then serve as a tool towards deducing the nature of these new effects. We thus consider the investigation in this chapter to be an initial, and hopefully provocative, attempt in this direction.

Chapter 4
Breakdown of the Radius/Temperature Correspondence

4.1 Introduction

In this chapter, we shall examine the relationship between temperature and radius of compactification for closed strings. In field theory, these two quantities are intimately related and it has always been assumed that the same is true in all of string theory. This relation in field theory allows one to calculate finite temperature effects simply by working with specific compactified geometries. The geometry/temperature relation was shown to hold for bosonic strings [10] and was always assumed to hold for all other closed strings. We shall examine the relation more closely for the Type II string and the Heterotic string. We shall explicitly show that the temperature/geometry relation *must* be very different if it holds at all for the $SO(32)$ heterotic string. We shall then comment on some potential implications of this fact.

This chapter is organized as follows. First, in Sect 4.2, we shall examine the connections between systems at finite temperature and systems with compact dimensions. Then, in Sect. 4.3, we present the general formalism by which finite-temperature string models may be constructed. We shall explicitly construct string partition functions by doing the Boltzmann sums and then by using the radius/temperature relation. From this, we will derive which $\mathbb{Z}_2$ orbifold the radius method requires to agree with the Boltzmann sum. After we have identified the critical $\mathbb{Z}_2$ orbifold, in Sects. 4.4, 4.5 we will examine whether the $\mathbb{Z}_2$ orbifold is a valid one for specific models.

4.2 Preliminaries: The "geometry" of temperature

The connections between compact dimensions and temperature run deep. As we shall see, examining a system at finite temperature is nearly identical to examining a system with one compact dimension. The journey from compact dimensions to statistical mechanics will actually begin with some useful mathematical identities which when recast correctly will show the way for the connection. The mathematical identities in question are merely the summations of the hyperbolic trigonometric functions namely,

$$\begin{aligned} \sinh(x) &= x \prod_{n=1}^{\infty} \left(\frac{x^2}{\pi^2 n^2} + 1 \right) \\ \cosh(x) &= \prod_{n=0}^{\infty} \left(\frac{x^2}{\pi^2 (n+\frac{1}{2})^2} + 1 \right), \end{aligned} \tag{4.1}$$

but we can make these identities somewhat more suggestive with the substitution, $x = \frac{E}{2T}$ along with taking the natural log of both sides. These steps lead to,

$$\begin{aligned} \ln(1 - e^{-E/T}) &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \ln\left[E^2 + 4\pi^2 n^2 T^2\right] - \frac{E}{2T} + ... \\ \ln(1 + e^{-E/T}) &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \ln\left[E^2 + 4\pi^2 (n+\frac{1}{2})^2 T^2\right] - \frac{E}{2T} + ... \,. \end{aligned} \tag{4.2}$$

At this point, it is appropriate to interpret this result. If E is interpreted as a particle's energy then the summation over n would represent the extra contribution to the mass energy coming from a compactified dimension. The contribution from the sum over n is appropriate for a particle propagating on a circle of radius $R_T = (2\pi T)^{-1}$ with periodic boundary conditions for the top half of the equation and anti-periodic boundary conditions for the bottom half. Therefore, the left-hand side of the equation can be rewritten so that a geometric interpretation is appropriate.

As we now have a geometric interpretation for the right-hand side of Eq. 4.2, our attention turns to the left-hand side. If we consider free bosons of mass, m. Their thermal partition function is,

$$\begin{aligned} Z_{\mathrm{b}}(T) &\sim \prod_{\vec{p}}(1+e^{-\frac{E_{\vec{p}}}{T}}+e^{-2\frac{E_{\vec{p}}}{T}}+...) \\ &\sim \prod_{\vec{p}}\frac{1}{1-e^{-\frac{E_{\vec{p}}}{T}}}, \end{aligned} \tag{4.3}$$

where $E_{\vec{p}}^2 = |\vec{p}|^2 + m^2$. This leads to a free energy of,

$$\begin{aligned} F_b(T) &= -T\ \ln\ Z_b(T) \\ &= T\int\frac{d^3\vec{p}}{(2\pi)^3}\ln(1-e^{-\frac{E_{\vec{p}}}{T}}). \end{aligned} \tag{4.4}$$

Whereas, for a free fermion of mass m a similar expression for the free energy is obtained, namely,

$$F_f(T) = -T\int\frac{d^3\vec{p}}{(2\pi)^3}\ln(1+e^{-\frac{E_{\vec{p}}}{T}}). \tag{4.5}$$

However, both of the integrands are precisely what we obtained on the left side in Eq. 4.2. We therefore now have a connection between temperature and geometry. Namely, the free energy of (fermions) bosons can be calculated using Kaluza-Klein towers with (half) integer modings. This is known as the Matsubara formulation. The modes propagating around the "thermal circle" are known as Matsubara modes and they physically act identically to Kaluza-Klein modes. In the Matsubara formulation the free energies are,

$$\begin{aligned} F_b(T) &= \frac{T}{2}\int\frac{d^3\vec{p}}{(2\pi)^3}\sum_{n\in\mathbb{Z}}\ln\left[E_p^2+(\frac{n}{R_T})^2\right] \\ F_f(T) &= -\frac{T}{2}\int\frac{d^3\vec{p}}{(2\pi)^3}\sum_{n\in\mathbb{Z}}\ln\left[E_p^2+(\frac{n+1/2}{R_T})^2\right]. \end{aligned} \tag{4.6}$$

These expressions are well known in the quantum field theory literature [9]. There are a few technical details which have been glossed over in the treatment above which we shall now address. Technically, the calculations were performed in a Euclidean space without a time dimension. The lack of a time dimension is the reason why this calculation must be performed when the system is in equilibrium. However, this relation does allow for the examination of finite temperature systems using the techniques developed for compact extra-dimensions. This reformulation of finite temperature effects to geometric effects is precisely what we shall examine in the context of closed strings for the remainder of the chapter.

4.3 String Model Building with Compact Dimensions and Temperature

As discussed earlier, we may use the relation between geometry and temperature to calculate thermodynamic potentials in two very different ways. One may actually calculate the thermal partition function with Boltzmann sums and then follow the standard progression to all of the thermodynamic potentials. The other approach is to compactify the time dimension on a circle with radius proportional to inverse temperature and then using the resulting Matsubara modes find the thermodynamic potentials. The question of whether this is valid in all string theoretic contexts is very important because strings with compact dimensions are much more well understood than strings at finite temperature.

In investigating this question, we shall first identify what the quantities of interest are for the comparison. Then we shall explicitly outline the procedure to find the Boltzmann sum for a zero temperature string model. After this, we shall also outline the procedure to find the finite radius extrapolation of a string model. Once these two procedures are fully developed, we shall see what constraints can be placed on the zero temperature model such that the two procedures agree.

4.3.1 String Partition Function and Matsubara Modes

A method of comparing the finite temperature extrapolations achieved using both the geometric and the traditional approach is necessary. We shall use the string partition function which results from both methods. We note that one could simply require that the same vacuum amplitude is arrived at, but given that the temperature/geometry correspondence is assumed to be fundamental simply requiring the same thermodynamics is not sufficient. However, the first step is to review the standard method utilized to proceed from the string partition function to the thermal partition function. In general, once we have determined the correct finite-temperature partition function $Z_{\rm string}(\tau, T)$ for a given zero-temperature string model, the one-loop thermal vacuum amplitude $\mathcal{V}(T)$ (the analogue of the logarithm of the statistical-mechanical partition function) is given by a modular integral of the form [10]

$$\mathcal{V}(T) \equiv -\tfrac{1}{2}\,\mathcal{M}^{D-1} \int_{\mathcal{F}} \frac{d^2\tau}{({\rm Im}\,\tau)^2}\, Z_{\rm string}(\tau, T) \tag{4.7}$$

where $\mathcal{M} \equiv M_{\rm string}/(2\pi)$ is the reduced string scale; D is the number of non-compact spacetime dimensions; and where

$$\mathcal{F} \equiv \{\tau : \ |{\rm Re}\,\tau| \le \tfrac{1}{2}, {\rm Im}\,\tau > 0, |\tau| \ge 1\} \tag{4.8}$$

is the fundamental domain of the modular group. We shall often abbreviate $\tau_1 \equiv {\rm Re}\,\tau$ and $\tau_2 \equiv {\rm Im}\,\tau$.

The string partition function for any string model at zero temperature *must* be a modular invariant. It also turns out that at finite temperature the string partition function must still be a modular invariant [50]. This severely limits the form of the string partition function, it also forces the introduction of new types of Matsubara modes for closed strings[1]. Until this point, a closed string model at finite temperature could be treated almost exactly like a field theory at finite temperature where the particle content of the field theory is merely the spectrum of excitations of the string. However, the requirement of modular invariance forces the inclusion of excitations of the string which one would not *a prior* have thought to include from strictly field

[1]Of course, we could formulate this entire chapter in terms of the strip, $\mathcal{S} \equiv \{\tau : \ |{\rm Re}\,\tau| \le \frac{1}{2}, |{\rm Im}|\,\tau > 0\}$. This would require many modular transformations on the partition function which ultimately hide the winding modes, but does not change the physics. We feel it is easier to see the physics leaving everything in the manifestly modular invariant form.

theoretic expectations. Modular invariance forces the inclusion of winding Matsubara modes which differ from momentum Matsubara modes in a few significant ways. The typical Kaluza-Klein Matsubara mode has a mass which scales like $M_{\rm KK} \sim mT$ where m is the momentum number and T is the temperature. However the winding Matsubara modes have masses which scale like $M_{\rm winding} \sim w/T$ where w is the winding number. Modular invariance also forces the winding Matsubara modes to only be integer moded, simply because the KK Matsubara modes are (half-)integer moded.

So now the question is how to incorporate the Matsubara modes into the partition function. In the partition function these modes will appear as,

$$Z_{\rm Matsubara}(\tau, T) \;=\; \sqrt{\tau_2} \sum_{m,w} \overline{q}^{(mT-w/T)^2/4}\, q^{(mT+w/T)^2/4} \tag{4.9}$$

where τ is the complex toroidal modular parameter, $q \equiv e^{2\pi i\tau}$ and m, w represent the momentum and winding numbers respectively. The only remaining issue then is what the restrictions on the summation variables m, w should be. From our expectations from field theory we know that there should be at least two different groups of Matsubara modes namely,

$$\begin{aligned} \mathcal{M}_{1/2} &= \sum_{m,w} (m \in \mathbb{Z} + \frac{1}{2}, w \in \mathbb{Z})(\tau, T) \\ \mathcal{M}_0 &= \sum_{m,w} (m \in \mathbb{Z}, w \in \mathbb{Z})(\tau, T) \end{aligned} \tag{4.10}$$

where the top sum is for the modes which are anti-symmetric around the thermal circle (fermions from our field theoretic expectations and the bottom sum is for the modes which are symmetric around the thermal circle (bosons from our field theoretic expectations. Modular transformations, however suggest that a better grouping would be that originally written down by Rohm in a different context [46],

$$\begin{aligned} \mathcal{E}_0 &= \{m \in \mathbb{Z},\ w \text{ even}\} \\ \mathcal{O}_0 &= \{m \in \mathbb{Z},\ w \text{ odd}\} \\ \mathcal{E}_{1/2} &= \{m \in \mathbb{Z} + \tfrac{1}{2},\ w \text{ even}\} \\ \mathcal{O}_{1/2} &= \{m \in \mathbb{Z} + \tfrac{1}{2},\ w \text{ odd}\}\ . \end{aligned} \tag{4.11}$$

One would expect that modular transformations will mix these functions. In fact, $S : \tau \to -1/\tau$ mixes the functions in the following manner,

$$\begin{pmatrix} \mathcal{E}_0 \\ \mathcal{E}_{1/2} \\ \mathcal{O}_0 \\ \mathcal{O}_{1/2} \end{pmatrix} (-1/\tau) = \tfrac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{E}_0 \\ \mathcal{E}_{1/2} \\ \mathcal{O}_0 \\ \mathcal{O}_{1/2} \end{pmatrix} (\tau)\ . \tag{4.12}$$

Whereas under $T : \tau \to \tau + 1$, $\mathcal{E}_0, \mathcal{E}_{1/2}, \mathcal{O}_0$ are all invariant while $\mathcal{O}_{1/2}$ picks up a minus sign.

Now as the Matsubara modes capture the full temperature dependence of a string partition function, we are in a position to write the general form of the string partition function at finite temperature. It is,

$$\begin{aligned} Z_{\text{therm}}(\tau, T) &= Z^{(1)}(\tau)\ \mathcal{E}_0(\tau, T) + Z^{(2)}(\tau)\ \mathcal{E}_{1/2}(\tau, T) \\ &\quad + Z^{(3)}(\tau)\ \mathcal{O}_0(\tau, T) + Z^{(4)}(\tau)\ \mathcal{O}_{1/2}(\tau, T) \end{aligned} \tag{4.13}$$

this is the most general manifestly modular invariant form of the partition function. Any other form must be a series of modular transformations away from this form. Henceforth, this will be the starting place for all string model constructions.

4.3.2 Boltzmann Sums

As we saw in Sect. 4.3.1, modular invariance restricts the string partition function at finite temperature to the form in Eq. 4.13. The only remaining freedom is therefore the contents of $Z^{(i)}$. This also determines what sorts of excitations the zero-temperature states accrue when the model is examined at finite temperature so it is the most important question for this chapter.

The first step in determining the contents of $Z^{(i)}$ is to look at the $T \to 0$ limit of the $\mathcal{E}/\mathcal{O}$ functions. The temperature controls the mass of all of the Matsubara modes and as the mass of the winding modes goes as $M_w \sim \frac{w}{T}$, in the $T \to 0$ limit only the $w = 0$ modes survive. This means that the $\mathcal{O}_{0,1/2}$ terms will disappear and, the remaining Matsubara mode sums, $(\mathcal{E}_{0,1/2})$ will become continuous and equal. Thus $Z^{(1)} + Z^{(2)}$ must be the zero-temperature string model. Once, $Z^{(1)}$ and $Z^{(2)}$ are determined $Z^{(3)}$ and $Z^{(4)}$ will be fixed by modular invariance considerations. Therefore, using modular invariance, we have reduced the entire question about how a zero-temperature string model looks at finite temperature to the way in which we divide the states in the zero-temperature model into $Z^{(1)}$ and $Z^{(2)}$.

If we start with a zero temperature string model of the form,

$$Z(\tau) = Z_{\text{bosons}}(\tau) + Z_{\text{fermions}}(\tau) \tag{4.14}$$

this would immediately lead to the division,

$$\begin{aligned} Z^{(1)}(\tau) &= Z_{\text{bosons}}(\tau) \\ Z^{(2)}(\tau) &= Z_{\text{fermions}}(\tau), \end{aligned} \tag{4.15}$$

from demanding that the moding for the Matsubara modes be consistent with our usual field theory expectations. Requiring that Z_{therm} be modular invariant leads to,

$$\begin{aligned} Z^{(3)}(\tau) &= \frac{1}{2}\left(Z^{(1)}(\frac{-1}{\tau}) - Z^{(2)}(\frac{-1}{\tau}) + Z^{(1)}(\frac{\tau-1}{\tau}) - Z^{(2)}(\frac{\tau-1}{\tau})\right) \\ Z^{(4)}(\tau) &= \frac{1}{2}\left(Z^{(1)}(\frac{-1}{\tau}) - Z^{(2)}(\frac{-1}{\tau}) - Z^{(1)}(\frac{\tau-1}{\tau}) + Z^{(2)}(\frac{\tau-1}{\tau})\right). \end{aligned} \tag{4.16}$$

Thus, we have done the Boltzmann sums for the zero-temperature model. The particle spectrum at finite temperature is what one would have expected in field theory (as long as the field theory also necessitated the winding modes).

Now the question remains, can this same partition function be arrived at in the geometric formalism. Of course, the geometric method is necessarily more general as many different possible compactifications can exist. However, this just means that we should be able to find *which* compactification corresponds to this Boltzmann sum and then examine whether the compactification is actually a valid one for all zero temperature models.

4.3.3 Construction of String Models with a compactified dimensions

As we have already constructed a modular invariant partition function using the strict temperature approach, we need to now see how to construct a model with a compact dimension. The following discussion follows the mathematical treatment in Ref. [46], suitably T-dualized in order to apply to inverse radius rather than geometric radius [8]. This is done to more easily make contact with the partition function resulting from the traditional formalism. Let us suppose that we begin with a D-dimensional infinite radius closed string model whose one-loop partition function is given by $Z(\tau)$.

The first step in the construction is to compactify this theory on circle of radius R_T. At this stage, we then have a string partition function $Z_{\text{circ}}(\tau, a)$ of the form

$$Z_{\text{circ}}(\tau, a) \;\equiv\; Z(\tau)\, Z_{\text{circle}}(\tau, a) \tag{4.17}$$

where $a \equiv (R_T M_{\text{string}})^{-1}$ and the extra factor Z_{circle} represents a double summation over integer momentum and winding modes similar to that in Eq. 4.9 except for the sum over m, w is only over integers.

However, at this stage in the construction, we see that each of the states within $Z(\tau)$ is multiplied by the same spectrum of integer momentum and winding modes within Z_{circle}. The next step, therefore, is to break this degeneracy, ensuring that some of the states within $Z(\tau)$ continue to have integer modes, while others have *half-integer* modings (so that they are anti-periodic around the compactified dimension).

In string theory, the only way to accomplish this in a self-consistent manner is by twisting or orbifolding the compactified theory in Eq. (4.17). We shall generally let K denote such an operator. However, we will also need to couple K with an operator that can distinguish between between integer and half-integer momenta. As we shall see, such an operator is given by $\mathcal{T} : y \to y + \pi R_T$, where y is the (T-dual) coordinate along the compactified dimension. This is nothing but a shift around half the circumference of the (dualized) circle, so that the states which are invariant under $\mathcal{T}$ are those with even winding numbers. This will then necessarily re-introduce states with odd winding numbers in the twisted sectors, along with states having half-integer momentum numbers.

Given these operators, the final step in our procedure is to orbifold the circle-compactified theory in Eq. (4.17) by the $\mathbb{Z}_2$ product operator $\mathcal{T}K$. What does this do to our partition function? While K acts on the original component $Z(\tau)$, the operator $\mathcal{T}$ acts on the sum $Z_{\text{circle}}(\tau, a)$. Since states contributing to Z_{circle} with even (odd) values of n are even (odd) under $\mathcal{T}$, let us distinguish the specific values of m and n by using the four functions introduced earlier,$\mathcal{E}_{0,1/2}$ and $\mathcal{O}_{0,1/2}$.

Note that $Z_{\text{circle}} = \mathcal{E}_0 + \mathcal{O}_0$. Given this, our original (untwisted) partition function in Eq. (4.17) can be rewritten as

$$Z^{+}_{\text{circ},+} \;=\; Z^{+}_{+}\,(\mathcal{E}_0 + \mathcal{O}_0) \tag{4.18}$$

where $Z^{+}_{+}(\tau) \equiv Z(\tau)$. Therefore, in order to project onto the states invariant under $\mathcal{T}K$, we add to Eq. (4.18) the contributions from the projection sector

$$Z^{-}_{\text{circ},+} \;=\; Z^{-}_{+}\,(\mathcal{E}_0 - \mathcal{O}_0) \tag{4.19}$$

where Z^{-}_{+} is the K-projection sector for the non-thermal contribution Z^{+}_{+}.

To ensure that the partition function is a modular invariant it must contain the S-transform of the projection sector,

$$Z^{+}_{\text{circ},-} \;=\; Z^{+}_{-}\,(\mathcal{E}_{1/2} + \mathcal{O}_{1/2}) \tag{4.20}$$

as well as its corresponding T-transformation

$$Z^{-}_{\text{circ},-} \;=\; Z^{-}_{-}\left(\mathcal{E}_{1/2} - \mathcal{O}_{1/2}\right) \;. \tag{4.21}$$

The net result of the orbifold, then, is a $(D-1)$-dimensional string model with total partition function

$$\begin{aligned}
Z_{\text{string}}(\tau, a) \;&=\; \frac{1}{2}\left(Z^{+}_{\text{circ},+} + Z^{-}_{\text{circ},+} + Z^{+}_{\text{circ},-} + Z^{-}_{\text{circ},-}\right) \\
&=\; \frac{1}{2}\Big\{\mathcal{E}_0\left(Z^{+}_{+} + Z^{-}_{+}\right) \;+\; \mathcal{E}_{1/2}\left(Z^{+}_{-} + Z^{-}_{-}\right) \\
&\qquad +\; \mathcal{O}_0\left(Z^{+}_{+} - Z^{-}_{+}\right) \;+\; \mathcal{O}_{1/2}\left(Z^{+}_{-} - Z^{-}_{-}\right)\Big\} \;.
\end{aligned} \tag{4.22}$$

It is straightforward to interpret the physics of this model. As $a \to \infty$, we find that $\mathcal{E}_{1/2}$ and $\mathcal{O}_{1/2}$ each vanish while $\mathcal{E}_0$ and $\mathcal{O}_0$ become equal; thus the partition function of our model reduces to

$$Z_{\text{string}}(\tau, a) \;\to\; Z^{+}_{+} \equiv Z \qquad \text{as} \;\; a \to \infty \;. \tag{4.23}$$

In other words, we see that the original D-dimensional model with which we started can now be interpreted as the $a \to \infty$ limit of the $(D-1)$-dimensional model we have constructed. By contrast, as $a \to 0$, we find that $\mathcal{O}_0$ and $\mathcal{O}_{1/2}$ each vanish while $\mathcal{E}_0$ and $\mathcal{E}_{1/2}$ become equal. Thus

$$Z_{\text{string}}(\tau, a) \;\to\; \tfrac{1}{2}\left(Z^{+}_{+} + Z^{-}_{+} + Z^{+}_{-} + Z^{-}_{-}\right) \qquad \text{as} \;\; a \to 0 \;. \tag{4.24}$$

However, this is nothing but the K-orbifold of the original D-dimensional model with which we began. Of course, since K is a $\mathbb{Z}_2$ operator, we know that $K^2 = \mathbf{1}$. We may therefore change our perspective and equivalently view our D-dimensional $a \to \infty$ model as the orbifold of our D-dimensional $a \to 0$ model. A note on naming conventions here, in general we will refer to the orbifold as K if the untwisted sector in question is the $a \to \infty$ model and Q if the untwisted sector is the $a \to 0$ model. This means that although K and Q are the *same* orbifold their actions might be characterized differently because of the fact that they have different untwisted sectors.

Thus, to summarize, we see that all string models with a compact dimension must have partition functions of the modular-invariant form [46, 8, 47]

$$\begin{aligned} Z_{\text{therm}}(\tau,T) \;&=\; Z^{(1)}(\tau)\;\mathcal{E}_0(\tau,T) \;+\; Z^{(2)}(\tau)\;\mathcal{E}_{1/2}(\tau,T) \\ &\quad +\; Z^{(3)}(\tau)\;\mathcal{O}_0(\tau,T) \;+\; Z^{(4)}(\tau)\;\mathcal{O}_{1/2}(\tau,T)\;, \end{aligned} \tag{4.25}$$

where $Z^{(i)}$ represent general, model-specific, non-radius dependent contributions to the total partition function Z_{string}. In the $a \to 0$ limit, we obtain a partition function of the form

$$Z_{\text{model}} \;=\; Z^{(1)} + Z^{(2)}\;, \tag{4.26}$$

and thus we may interpret Eq. (4.25) as describing the finite radius extrapolation of the model described in Eq. (5.13). By contrast, the opposite $a \to \infty$ limit yields

$$\tilde{Z}_{\text{model}} \;=\; Z^{(1)} + Z^{(3)}\;, \tag{4.27}$$

which corresponds to a different D-dimensional string model. Thus, the partition function in Eq. (4.25) can be viewed as mathematically *interpolating* between one string model at $a = 0$ [whose partition function is given in Eq. (4.26)] and a *different* string model as $a \to \infty$ [whose partition function is given in Eq. (4.27)]. These two models are related directly in D dimensions through the action of the $\mathbb{Z}_2$ orbifold operator Q. In thermal contexts, we expect that the orbifold Q will contain a $(-1)^F$, where F represents spacetime fermion number, factor guaranteeing that finite-temperature effects will break whatever supersymmetry might have existed at zero temperature.

This is a general result, so it bears repeating: *All D-dimensional models with a compact dimension are $(D-1)$- dimensional interpolating models, with the inverse radius a serving as an interpolating parameter. As $a \to 0$, we obtain a D-dimensional string model M_1; this is identified as the infinite radius string model whose finite radius extrapolation we have constructed. By contrast, as $a \to \infty$, we obtain a different D-dimensional string model M_2 which must be a $\mathbb{Z}_2$ orbifold of M_1.*

A comment on semantics is in order here. Strictly speaking, in the $a \to \infty$ limit we obtain a $(D-1)$-dimensional degenerate (*i.e.*, zero-radius) model M_2 which is actually only T-dual to a D-dimensional model. Thus, if M_2 is the $a \to \infty$ limit of our $(D-1)$-dimensional thermal interpolating model, then we should more correctly state that our $(D-1)$-dimensional thermal model interpolates between the D-dimensional models M_1 and $\tilde{M}_2$, where $\tilde{M}_2$ is the T-dual of M_2. In some sense, this distinction is only a matter of semantics, having to do with the naming of the $a \to \infty$ endpoint of the interpolation; moreover, for closed strings we should properly regard both M_2 and $\tilde{M}_2$ as being D-dimensional since they each have a continuous spectrum of states associated with the formerly compactified dimension. For simplicity, therefore, we shall continue to refer to such an interpolating model as connecting M_1 and M_2 in the remainder of this chapter. However, it is important to note that it is M_2 (and not $\tilde{M}_2$) which must be the Q-orbifold of M_1.

Note this construction method is the most general method of constructing a string model with one compact dimension. This leads to the situation that for each specified D-dimensional string model M_1, there will in general exist many $(D-1)$-dimensional string models which extrapolate away from it. This depends on the choice of the $\mathbb{Z}_2$ orbifold Q, or equivalently on the choice of the second model M_2 to which one interpolates. In other words, the requirement of modular invariance alone is not sufficient to determine a unique interpolation, and is therefore not sufficient to determine a unique extrapolation. The determination of which orbifold Q is the correct thermal orbifold will depend on extra inputs from our expectations from thermal physics.

4.3.4 Implications for $T \sim 1/R$ identification

At this point, we have two different methods of finding the finite temperature extrapolation of any zero temperature string model. In principle, the two methods should be able to agree. The geometric orbifold construction outlined in Sect. (4.3.3) is more general than the traditional method outlined in Sect. (4.3.2). However, the partition function resulting from the geometric method is guaranteed to come from a *bona-fide* world-sheet, whereas the partition function resulting from the traditional method is predicated on the assumption that string theory at finite temperature is very similiar to quantum field theory at finite temperature.

Leaving the relative merits of the two different methods of extrapolating a zero-temperature model to finite temperature aside, the real question is when do the two methods produce the same model. The crux of the traditional method was the division of the zero-temperature model into the parts $Z^{(1)}$ and $Z^{(2)}$ on the basis of their thermodynamic properties. However, the geometric construction method utilized a $\mathbb{Z}_2$ orbifold. This means that the question of whether or not the two methods agree comes down to whether the orbifold which would divide the zero-temperature spectrum in the same manner as the field theoretic expectations exists for that specific zero-temperature model.

We shall therefore examine different classes of closed-string models. The first class we will examine will be the Type II closed string models in ten dimensions. We shall show that indeed an orbifold exists, namely $(-1)^F$ where F is spacetime fermion number, so both methods agree and the thermodynamic method is guaranteed to be self-consistent. However, when we examine heterotic strings in ten dimensions we shall see that $(-1)^F$ fails to be a consistent orbifold.

4.4 Type II strings

Let us begin the comparison between the two construction methods by considering the case of the ten-dimensional Type II superstrings. For concreteness, we shall focus on the (chiral) Type IIB string; the case of the (non-chiral) Type IIA string proceeds in exactly the same manner. The Type IIB string at zero temperature has the partition function

$$Z_{\rm IIB} \;=\; Z^{(8)}_{\rm boson}\,(\overline{\chi}_V - \overline{\chi}_S)\,(\chi_V - \chi_S) \tag{4.28}$$

where the contribution from the worldsheet bosons is given in terms of the Dedekind η-function as

$$Z^{(n)}_{\rm boson} \;\equiv\; {\tau_2}^{-n/2}\,(\overline{\eta}\eta)^{-n}~, \tag{4.29}$$

and where the contributions from the left-moving (right-moving) worldsheet fermions are written in terms of the unbarred (barred) characters χ_i ($\overline{\chi}_i$) of the transverse $SO(8)$ Lorentz group. In general, the subscripts I, V, S, and C refer to the identify, vector, spinor, and conjugate spinor representations for any $SO(2n)$ affine Lie group; these representations have conformal dimensions $\{h_I, h_V, h_S, h_C\} = \{0, \frac{1}{2}, \frac{n}{8}, \frac{n}{8}\}$, and have corresponding characters which can be expressed in terms of Jacobi ϑ-functions as

$$\begin{aligned}
\chi_I &= \tfrac{1}{2}\,({\vartheta_3}^n + {\vartheta_4}^n)/\eta^n &=& \; q^{h_I - c/24}\,(1 + n(2n-1)\,q + ...)\\
\chi_V &= \tfrac{1}{2}\,({\vartheta_3}^n - {\vartheta_4}^n)/\eta^n &=& \; q^{h_V - c/24}\,(2n + ...)\\
\chi_S &= \tfrac{1}{2}\,({\vartheta_2}^n + {\vartheta_1}^n)/\eta^n &=& \; q^{h_S - c/24}\,(2^{n-1} + ...)\\
\chi_C &= \tfrac{1}{2}\,({\vartheta_2}^n - {\vartheta_1}^n)/\eta^n &=& \; q^{h_C - c/24}\,(2^{n-1} + ...)
\end{aligned} \tag{4.30}$$

where the central charge is $c = n$ at affine level one. For the ten-dimensional transverse Lorentz group $SO(8)$, the distinction between S and C is equivalent to relative spacetime chirality. Note that the $SO(8)$ transverse Lorentz group has a triality symmetry under which the vector and spinor representations are indistinguishable. Thus $\chi_V = \chi_S$ and $\overline{\chi}_V = \overline{\chi}_S$, resulting in a (vanishing) supersymmetric partition function in Eq. (4.28). The presence of two such factors in Eq. (4.28) reflects the $\mathcal{N} = 2$ supersymmetry of this model at zero temperature.

Let us now consider the extrapolation of this theory to finite temperature. If we apply the the standard thermodynamic prescription in Ref. [8], we obtain a nine-dimensional extrapolation with partition function

$$
\begin{aligned}
Z_{\rm string}(\tau,T) \;=\; Z^{(7)}_{\rm boson} \times \Big\{ \quad & \mathcal{E}_0 \quad [\overline{\chi}_V\chi_V + \overline{\chi}_S\chi_S] \\
- \quad & \mathcal{E}_{1/2} \quad [\overline{\chi}_V\chi_S + \overline{\chi}_S\chi_V] \\
+ \quad & \mathcal{O}_0 \quad [\overline{\chi}_I\chi_I + \overline{\chi}_C\chi_C] \\
- \quad & \mathcal{O}_{1/2} \quad [\overline{\chi}_I\chi_C + \overline{\chi}_C\chi_I] \;\Big\} \;.
\end{aligned}
\tag{4.31}
$$

It is easy to interpret this partition function in the language of our construction in Sect. 4.3, and in the process verify that this result is self-consistent. First, we see that the $T \to 0$ limit of this expression reproduces the Type IIB partition function in Eq. (4.28), while the $T \to \infty$ limit of this expression yields the partition function of the non-supersymmetric ten-dimensional Type 0B superstring:

$$
Z_{\rm 0B} \;=\; Z^{(8)}_{\rm boson}\, (\overline{\chi}_I\chi_I + \overline{\chi}_V\chi_V + \overline{\chi}_S\chi_S + \overline{\chi}_C\chi_C) \;. \tag{4.32}
$$

Thus, the nine-dimensional thermal model in Eq. (4.31) interpolates between the Type IIB string and the Type 0B string, thereby breaking supersymmetry for all $T > 0$. (The extra factor of $Z^{(1)}_{\rm boson}$ required in these limits emerges as the limit of the $\mathcal{E}/\mathcal{O}$ functions.)

The Type IIB closed string in ten dimensions is particularly simple from the perspective of geometric model building. There is only *one* self-consistent $\mathbb{Z}_2$ orbifold for and it is simply $Q = (-1)^F$. This Q changes the Type IIB model to the Type 0B model, exactly the same result as the thermal extrapolation!

We conclude, then, that the standard thermodynamic prescription agrees completely with the geometric method of construction for the case of the Type IIB string. The case of the Type IIA string is almost exactly the same; we can simply replace $\chi_S \leftrightarrow \chi_C$ for the left-moving characters throughout the above expressions. The Type IIA thermal extrapolation is therefore one which interpolates between the Type IIA string at $T = 0$ and the Type 0A string as $T \to \infty$.

4.5 Heterotic Strings

Let us now turn to the case of the ten-dimensional $SO(32)$ heterotic string. It is here that we shall find an important difference relative to the usual thermodynamic prescription. The ten-dimensional $SO(32)$ heterotic string has the zero-temperature partition function

$$Z_{SO(32)} = Z^{(8)}_{\text{boson}} \, (\overline{\chi}_V - \overline{\chi}_S) \, (\chi_I + \chi_S) \ . \tag{4.33}$$

As with the Type II string, the contribution from the worldsheet bosons is given in Eq. (4.29) and the contributions from the right-moving worldsheet fermions are written in terms of the barred characters $\overline{\chi}_i$ of the transverse $SO(8)$ Lorentz group. The major new notational difference in the heterotic case is that the contributions from the left-moving (internal) worldsheet fermions are now written as products of the unbarred characters χ_i of an internal $SO(32)$ gauge group.

Let us now consider the extrapolation of this theory to finite temperature. If we were to apply the standard prescriptions, we would obtain a nine-dimensional extrapolation with partition function

$$\begin{aligned} Z = Z^{(7)}_{\text{boson}} \times \Big\{ \quad & \mathcal{E}_0 \quad \overline{\chi}_V \ (\chi_I + \chi_S) \\ - \ & \mathcal{E}_{1/2} \quad \overline{\chi}_S \ (\chi_I + \chi_S) \\ - \ & \mathcal{O}_0 \quad \overline{\chi}_C \ (\chi_I + \chi_S) \\ + \ & \mathcal{O}_{1/2} \quad \overline{\chi}_I \ (\chi_I + \chi_S) \ \Big\} \ . \end{aligned} \tag{4.34}$$

This is indeed modular invariant, and incorporates proper thermal spin-statistics relations for states with zero thermal windings (*i.e.*, states multiplying the $\mathcal{E}_0$ and $\mathcal{E}_{1/2}$ thermal functions). Despite these successes, it is easy to demonstrate that Eq. (4.34) cannot represent a self-consistent thermal extrapolation according to the construction procedure we laid out in Sect. 4.3 — *i.e.*, that this cannot be the partition function of a *bona-fide* self-consistent nine-dimensional string model.

Before giving a definitive argument to this effect, let us begin by noting that there are certain immediate clues that all is not well. First, we observe that Eq. (4.34) would appear to represent a non-supersymmetric interpolation between two *supersymmetric* limits, one at $T = 0$ and the other at $T \to \infty$, both of which represent the same $SO(32)$ heterotic string model but with opposite spacetime chiralities! Indeed, while the $T \to 0$ limit of Eq. (4.34) reproduces Eq. (4.33), the $T \to \infty$ limit yields

$$Z'_{SO(32)} \;=\; Z^{(8)}_{\text{boson}}\,(\overline{\chi}_V - \overline{\chi}_C)\,(\chi_I + \chi_S)\;. \tag{4.35}$$

This is precisely the same model with which we started at zero temperature, only now involving spacetime spinors of opposite chirality. This is quite unlike the case of the Type II string, where the $T \to \infty$ endpoint model was the non-supersymmetric Type 0A or 0B theory. It is also quite unexpected for interpolations that are thermal in nature as this requires that spacetime supersymmetry be broken at all temperatures besides 0 and ∞!

Second, we observe that in Eq. (4.34), the string worldsheet CFT ground state character $\overline{\chi}_I\chi_I^2$ appears within the sector multiplied by $\mathcal{O}_{1/2}$. To see why this is a problem, let us first observe that it follows from modular invariance that this is the only place such a term could possibly have appeared: since the ground state of the heterotic string is not level-matched, having worldsheet energies $(H_R, H_L) = (-1/2, -1)$, invariance under $\tau \to \tau + 1$ forces such a term to be multiplied by the function $\mathcal{O}_{1/2}$, which also fails to be level-matched. In other words, modular invariance requires a term such as $\overline{\chi}_I\chi_I^2$, if it exists, to appear multiplied by $\mathcal{O}_{1/2}$ rather than by any of the other thermal functions. However, this is a problem because the $\mathcal{O}_{1/2}$ sector must be interpreted as completely *twisted* with respect to the $\mathbb{Z}_2$ thermal orbifold. Indeed, no matter whether we run our interpolations from $T \to 0$ to $T \to \infty$ or backwards from $T \to \infty$ to $T \to 0$, the contributions multiplying $\mathcal{O}_{1/2}$ can only correspond to twisted sectors. However, we do not expect to see the ground state of a self-consistent conformal field theory emerging from a twisted sector. Equivalently stated, we expect a term of the form $\overline{\chi}_I \chi_I^2$ to appear multiplied by $\mathcal{E}_0$, $\mathcal{E}_{1/2}$, or $\mathcal{O}_0$, but never $\mathcal{O}_{1/2}$. Thus, combining these observations, we see that a self-consistent heterotic thermal extrapolation should not have a term of the form $\overline{\chi}_I \chi_I^2$ appearing *anywhere* in its partition function. Yet, this term appears within Eq. (4.34).

In order to diagnose the source of the problem, let us return to our original observation that the nine-dimensional "model" in Eq. (4.34) seems to interpolate between to supersymmetric endpoints: the $SO(32)$ heterotic string at zero temperature, and the chirality-flipped $SO(32)$ heterotic string at infinite temperature. However, according to our discussion in Sect. 4.3, this can represent a consistent nine-dimensional interpolation only if the chirality-flipped $SO(32)$ model can be viewed as a $\mathbb{Z}_2$ orbifold of the unflipped $SO(32)$ model. We shall now demonstrate that there is no such $\mathbb{Z}_2$ orbifold which can accomplish this transformation.

To see this, let us consider the worldsheet sector giving rise to the gravitino of the original supersymmetric $SO(32)$ model. Recall that in the heterotic string, the gravitino state is realized in the Ramond sector as the spin-3/2 component within the tensor product

$$\text{gravitino:} \qquad \tilde{g}^{\alpha\nu} \;\subset\; \{\tilde{b}_0\}^\alpha |0\rangle_R \;\otimes\; \alpha^\nu_{-1}|0\rangle_L \;. \tag{4.36}$$

Here α^ν_{-1} denotes the lowest excitation of the left-moving worldsheet coordinate boson X^ν, with its Lorentz vector index ν, while $\{\tilde{b}_0\}^\alpha$ schematically indicates the Ramond zero-mode combinations which collectively give rise to the spacetime Lorentz spinor index α, as required for the spin-3/2 gravitino state. Note that in order for such a state to be massless and level-matched, it must emerge from a sector in which the left-moving (conformal) side of the heterotic string is in the completely Neveu-Schwarz ground state, while the right-moving (superconformal) side of the heterotic string is in the completely Ramond ground state. Note that in ten dimensions, this is the *only* sector which can ever give rise to spacetime gravitinos, and as such this sector is unique.

So what must our desired orbifold do? In general, orbifolds project certain states out of the spectrum from untwisted sectors, but then introduce new twisted sectors from which additional states may emerge. In our case, the desired orbifold must project out the gravitino that previously emerged from the gravitino sector. This is because the Q-orbifold must break spacetime supersymmetry (or alternatively, because the gravitino has the wrong chirality). However, our orbifolding procedure must also somehow restore an opposite-chirality gravitino from a twisted sector. This is necessary so that the net result of the orbifolding procedure can be the chirality-flipped supersymmetric $SO(32)$ model.

Ordinarily, there are many instances in which a given state might be projected out of the spectrum from an untwisted sector only to re-emerge from a twisted sector. However, as we have noted above, the gravitino sector is *unique* within the context of ten-dimensional heterotic strings — this is the only sector which can provide gravitinos of either chirality. It has a unique worldsheet construction, and thus no other sector can re-introduce the needed opposite-chirality gravitino once the original gravitino has been projected out of the original untwisted sector. We conclude, then, that there is no self-consistent $\mathbb{Z}_2$ orbifold Q which can possibly transform the ten-dimensional supersymmetric $SO(32)$ heterotic string into a chirality-flipped version of itself. It then follows from the construction presented in Sect. 4.3 that there is no self-consistent nine-dimensional interpolation between these two ten-dimensional endpoints.

Note that we are not saying that an orbifold cannot project out certain states from an untwisted sector, only to have them re-emerge (even with chirality flips) from a twisted sector. This indeed happens quite often. Rather, we are claiming that for the gravitino in ten dimensions, the actual *worldsheet sector* from which such a state can arise is unique, and thus such a sector cannot be both untwisted and twisted with respect to the same orbifold. In other words, each sector can contribute only once in a given string model.

Note that in this discussion, we are identifying our endpoint models M_1 and M_2 as the supersymmetric $SO(32)$ heterotic string models with opposite chiralities, in accordance with the results of Sect. 4.3. Since $M_1 = \tilde{M}_2$ in this case (where $\tilde{M}_2$ is the T-dual of M_2), one might instead try to form the desired thermal model by interpolating between M_1 and itself, *i.e.*, between M_1 and $\tilde{M}_2$, thereby avoiding the chirality flip. However, the chirality flip is not the real issue here. Indeed, the original $SO(32)$ heterotic gravitino must always be projected out of the spectrum by the Q-orbifold; otherwise, supersymmetry would not be broken by thermal effects in such an interpolation. A new gravitino cannot then be re-introduced from a twisted sector, regardless of its chirality.

We conclude, then, that Eq. (4.34), although modular invariant, fails to represent a self-consistent thermal extrapolation of the ten-dimensional $SO(32)$ heterotic string. In particular, it cannot correspond to a self-consistent nine-dimensional interpolating string model at the worldsheet level. Identical arguments apply as well to the $E_8 \times E_8$ heterotic string. This is the only way that the finite temperature extrapolation of any zero temperature model could actually fail. The fundamental reason for this failure stems from the fact that when dividing the particle spectrum of the zero temperature model into $Z^{(1)}$ and $Z^{(2)}$ the typical method of doing this is not to explicitly orbifold the model. The division is always done with the expectations of low energy physics and it happens for closed heterotic strings in ten dimensions that the orbifold necessary to accomplish this does *not* exist!

4.6 Conclusions

In this chapter, we have reviewed how to explicitly build models with a compact dimension and at finite temperature. Physics at energy scales much below the string scale suggests that it should be possible to get the *same* model from both of these methods as there is ultimately a connection between radius of compactification and finite temperatures. When examining Type II strings this connection seems to hold, however for the Heterotic string the connection seems to be broken. This conflict suggests that the Heterotic string at finite temperatures may behave differently from the typical expectations. The conflict also suggests that a reexamination of the implications that finite temperature field theory has for finite temperature string theory may be in order.

The breaking of the connection between radius of compactification and finite temperature also raises many questions about string thermodynamics in general. The self-consistency of the typical $SO(32)$ finite temperature extrapolation is very much in doubt. It could very well be the case that the partition function normally associated with this model may not come from *any* worldsheet construction. The situation of a partition function seeming to be self-consistent but not coming from a legitimate string model has of course occurred before, namely the so-called “Atkin-Lehner” symmetry. If indeed, the typical thermal extrapolation of the $SO(32)$ string is inconsistent then a new series of tests for whether a string model is the correct finite temperature extrapolation must be developed. Though, perhaps it should not be surprising that the finite temperature behavior of fundamentally extended objects be different from the finite temperature behavior of point particles, just as quantum statistical mechanics differs greatly from classical statistical mechanics.

Chapter 5
Criteria for constructing finite-temperature string models

5.1 Introduction

In the previous chapter, we showed that for heterotic strings in ten dimensions the well-known radius/temperature correspondence breaks down. This was done by deriving, at the level of string partition functions, the specific requirements for the radius of a compactified dimension to be interpretable as temperature. We then showed, that the required $\mathbb{Z}_2$ orbifold is actually not a valid orbifold. So now one is left with the question, what determines the correct finite temperature extrapolation for a zero-temperature string model?

In this chapter, we shall take the attitude that string theory at finite temperature *is* different from field theory at finite temperature so expectations from field theory should only apply in the correct limits. We shall then ask, given this what properties should we expect that the finite temperature extrapolation of a zero temperature model have? We will then examine the implications that these new requirements would have on the hagedorn transition. We would like to take this opportunity to stress that *most* closed string models still exhibit the temperature/radius correspondence and that all of the requirements that we place on the finite temperature extrapolation would lead one to the same model as the field theory expectations suggest. However, we would also like to point out that new "stringy" symmetries could radically change the thermal behavior of any system when the energy gets to the string scale. In this spirit, we would like to examine the possibilities.

5.2 Building Finite Temperature Extrapolations using $\mathbb{Z}_2$ Orbifolds

One can find the finite temperature extrapolations of zero temperature string models using a geometric or Boltzmann sum approach. It has long been suspected that these two approaches give the same model, but as was shown in, Ref [52] for heterotic strings these two approaches can *not* give the same extrapolation. This forces one to make a choice as to which approach should be pursued. We shall explore the consequences of favoring the geometric technique over the traditional sum technique.

It will be important for us to see how to construct a model with a compact dimension at the level of explicit model-building along with the associated thermal partition functions. The following discussion follows the mathematical treatment in Ref. [46], suitably T-dualized in order to apply to inverse radius rather than geometric radius [8] in order to more easily make contact with thermal models. Let us suppose that we begin with a D-dimensional infinite radius closed string model whose one-loop partition function is given by $Z(\tau)$, where τ is the complex toroidal modular parameter. The first step in the construction is to compactify this theory on circle of radius R_T. At this stage, we then have a string partition function $Z_{\text{circ}}(\tau, a)$ of the form

$$Z_{\text{circ}}(\tau, a) \;\equiv\; Z(\tau)\, Z_{\text{circle}}(\tau, a) \tag{5.1}$$

where $a \equiv (R_T M_{\text{string}})^{-1}$ and the extra factor Z_{circle} represents a double summation over integer momentum and winding modes:

$$Z_{\text{circle}}(\tau, a) \;=\; \sqrt{\tau_2} \sum_{m,n\in\mathbb{Z}} \overline{q}^{(ma-n/a)^2/4}\, q^{(ma+n/a)^2/4} \tag{5.2}$$

with $\tau_2 \equiv \text{Im}\,\tau$. Here (m, n) represent the momentum and winding numbers, respectively. However, at this stage in the construction, we see that each of the states within $Z(\tau)$ is multiplied by the same spectrum of integer momentum and winding modes within Z_{circle}. The next step, therefore, is to break this degeneracy, ensuring that some of the states within $Z(\tau)$ continue to have integer modes, while others have *half-integer* modings (so that they are anti-periodic around the compactified dimension).

In string theory, the only way to accomplish this in a self-consistent manner is by twisting or orbifolding the compactified theory in Eq. (5.1). We shall generally let K denote such an operator. However, we will also need to couple K with an operator that can distinguish between between integer and half-integer momenta. As we shall see, such an operator is given by $\mathcal{T} : y \to y + \pi R_T$, where y is the (T-dual) coordinate along the compactified dimension. This is nothing but a shift around half the circumference of the (dualized) circle, so that the states which are invariant under $\mathcal{T}$ are those with even winding numbers. This will then necessarily re-introduce states with odd winding numbers in the twisted sectors, along with states having half-integer momentum numbers.

Given these operators, the final step in our procedure is to orbifold the circle-compactified theory in Eq. (5.1) by the $\mathbb{Z}_2$ product operator $\mathcal{T}K$. What does this do to our partition function? While K acts on the original component $Z(\tau)$, the operator $\mathcal{T}$ acts on the sum $Z_{\text{circle}}(\tau, a)$. Since states contributing to Z_{circle} with even (odd) values of n are even (odd) under $\mathcal{T}$, let us distinguish the specific values of m and n by introducing [46] four new functions $\mathcal{E}_{0,1/2}$ and $\mathcal{O}_{0,1/2}$ which are the same as Z_{circle} in Eq. (5.2) except for the following restrictions on their summation variables:

$$\begin{aligned} \mathcal{E}_0 &= \{m \in \mathbb{Z},\ n \text{ even}\} \\ \mathcal{O}_0 &= \{m \in \mathbb{Z},\ n \text{ odd}\} \\ \mathcal{E}_{1/2} &= \{m \in \mathbb{Z} + \tfrac{1}{2},\ n \text{ even}\} \\ \mathcal{O}_{1/2} &= \{m \in \mathbb{Z} + \tfrac{1}{2},\ n \text{ odd}\}\ . \end{aligned} \tag{5.3}$$

Note that $Z_{\text{circle}} = \mathcal{E}_0 + \mathcal{O}_0$. Given this, our original (untwisted) partition function in Eq. (5.1) can be rewritten as

$$Z^+_{\text{circ},+} \;=\; Z^+_+ \,(\mathcal{E}_0 + \mathcal{O}_0) \tag{5.4}$$

where $Z_+^+(\tau) \equiv Z(\tau)$. Therefore, in order to project onto the states invariant under $\mathcal{T}K$, we add to Eq. (5.4) the contributions from the projection sector

$$Z^-_{\mathrm{circ},+} \;=\; Z^-_+ \left(\mathcal{E}_0 - \mathcal{O}_0\right) \tag{5.5}$$

where Z_+^- is the K-projection sector for the non-thermal contribution Z_+^+. In the usual fashion, modular invariance then fully determines the rest of the partition function. To find the rest of the partition function we must first examine how the $\mathcal{E}/\mathcal{O}$ functions transform under $S : \tau \to -1/\tau$ and $T : \tau \to \tau + 1$. Under S the $\mathcal{E}/\mathcal{O}$ functions transform as,

$$\begin{pmatrix} \mathcal{E}_0 \\ \mathcal{E}_{1/2} \\ \mathcal{O}_0 \\ \mathcal{O}_{1/2} \end{pmatrix} (-1/\tau) \;=\; \tfrac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{E}_0 \\ \mathcal{E}_{1/2} \\ \mathcal{O}_0 \\ \mathcal{O}_{1/2} \end{pmatrix} (\tau) \ . \tag{5.6}$$

Whereas under T, $\mathcal{E}_0, \mathcal{E}_{1/2}, \mathcal{O}_0$ are all invariant while $\mathcal{O}_{1/2}$ picks up a minus sign. To ensure that the partition function is a modular invariant it must contain the S-transform of the projection sector,

$$Z^+_{\mathrm{circ},-} \;=\; Z^+_- \left(\mathcal{E}_{1/2} + \mathcal{O}_{1/2}\right) \tag{5.7}$$

as well as its corresponding T-transformation

$$Z^-_{\mathrm{circ},-} \;=\; Z^-_- \left(\mathcal{E}_{1/2} - \mathcal{O}_{1/2}\right) \ . \tag{5.8}$$

The net result of the orbifold, then, is a $(D-1)$-dimensional string model with total partition function

$$\begin{aligned} Z_{\mathrm{string}}(\tau, a) \;&=\; \frac{1}{2} \left(Z^+_{\mathrm{circ},+} + Z^-_{\mathrm{circ},+} + Z^+_{\mathrm{circ},-} + Z^-_{\mathrm{circ},-}\right) \\ &=\; \frac{1}{2} \Big\{ \mathcal{E}_0 \left(Z^+_+ + Z^-_+\right) \;+\; \mathcal{E}_{1/2} \left(Z^+_- + Z^-_-\right) \\ &\qquad +\; \mathcal{O}_0 \left(Z^+_+ - Z^-_+\right) \;+\; \mathcal{O}_{1/2} \left(Z^+_- - Z^-_-\right) \Big\} \ . \end{aligned} \tag{5.9}$$

It is straightforward to interpret the physics of this model. As $a \to \infty$, we find that $\mathcal{E}_{1/2}$ and $\mathcal{O}_{1/2}$ each vanish while $\mathcal{E}_0$ and $\mathcal{O}_0$ become equal; thus the partition function of our model reduces to

$$Z_{\text{string}}(\tau, a) \;\to\; Z_+^+ \equiv Z \qquad \text{as} \;\; a \to \infty \; . \tag{5.10}$$

In other words, we see that the original D-dimensional model with which we started can now be interpreted as the $a \to \infty$ limit of the $(D-1)$-dimensional model we have constructed. By contrast, as $a \to 0$, we find that $\mathcal{O}_0$ and $\mathcal{O}_{1/2}$ each vanish while $\mathcal{E}_0$ and $\mathcal{E}_{1/2}$ become equal. Thus

$$Z_{\text{string}}(\tau, a) \;\to\; \tfrac{1}{2}\left(Z_+^+ + Z_+^- + Z_-^+ + Z_-^-\right) \qquad \text{as} \;\; a \to 0 \; . \tag{5.11}$$

However, this is nothing but the K-orbifold of the original D-dimensional model with which we began. Of course, since K is a $\mathbb{Z}_2$ operator, we know that $K^2 = \mathbf{1}$. We may therefore change our perspective and equivalently view our D-dimensional $a \to \infty$ model as the orbifold of our D-dimensional $a \to 0$ model. A note on naming conventions here, in general we will refer to the orbifold as K if the untwisted sector in question is the $a \to \infty$ model and Q if the untwisted sector is the $a \to 0$ model. This means that although K and Q are the *same* orbifold their actions might be characterized differently because of the fact that they have different untwisted sectors.

Thus, to summarize, we see that all string models with a compact dimension must have partition functions of the modular-invariant form [46, 8, 47]

$$\begin{aligned} Z_{\text{string}}(\tau, a) \;=&\; Z^{(1)}(\tau)\; \mathcal{E}_0(\tau, a) \;+\; Z^{(2)}(\tau)\; \mathcal{E}_{1/2}(\tau, a) \\ &+\; Z^{(3)}(\tau)\; \mathcal{O}_0(\tau, a) \;+\; Z^{(4)}(\tau)\; \mathcal{O}_{1/2}(\tau, a) \; , \end{aligned} \tag{5.12}$$

where $Z^{(i)}$ represent general, model-specific, non-radius dependent contributions to the total partition function Z_{string}. In the $a \to 0$ limit, we obtain a partition function of the form

$$Z_{\text{model}} \;=\; Z^{(1)} + Z^{(2)} \; , \tag{5.13}$$

and thus we may interpret Eq. (5.12) as describing the finite radius extrapolation of the model described in Eq. (5.13). By contrast, the opposite $a \to \infty$ limit yields

$$\tilde{Z}_{\rm model} \;=\; Z^{(1)} + Z^{(3)} \;, \tag{5.14}$$

which corresponds to a different D-dimensional string model. Thus, the partition function in Eq. (5.12) can be viewed as mathematically *interpolating* between one string model at $a = 0$ [whose partition function is given in Eq. (5.13)] and a *different* string model as $a \to \infty$ [whose partition function is given in Eq. (5.14)]. These two models are related directly in D dimensions through the action of the $\mathbb{Z}_2$ orbifold operator Q. In thermal contexts, we expect that the orbifold Q will contain a $(-1)^F$, where F represents spacetime fermion number, factor guaranteeing that finite-temperature effects will break whatever supersymmetry might have existed at zero temperature.

This is a general result, so it bears repeating: *All D-dimensional models with a compact dimension are $(D-1)$- dimensional interpolating models, with the inverse radius a serving as an interpolating parameter. As $a \to 0$, we obtain a D-dimensional string model M_1; this is identified as the infinite radius string model whose finite radius extrapolation we have constructed. By contrast, as $a \to \infty$, we obtain a different D-dimensional string model M_2 which must be a $\mathbb{Z}_2$ orbifold of M_1.*

A comment on semantics is in order here. Strictly speaking, in the $a \to \infty$ limit we obtain a $(D-1)$-dimensional degenerate (*i.e.*, zero-radius) model M_2 which is actually only T-dual to a D-dimensional model. Thus, if M_2 is the $a \to \infty$ limit of our $(D-1)$-dimensional thermal interpolating model, then we should more correctly state that our $(D-1)$-dimensional thermal model interpolates between the D-dimensional models M_1 and $\tilde{M}_2$, where $\tilde{M}_2$ is the T-dual of M_2. In some sense, this distinction is only a matter of semantics, having to do with the naming of the $a \to \infty$ endpoint of the interpolation; moreover, for closed strings we should properly regard both M_2 and $\tilde{M}_2$ as being D-dimensional since they each have a continuous spectrum of states associated with the formerly compactified dimension. For simplicity, therefore, we shall continue to refer to such an interpolating model as connecting M_1 and M_2 in the remainder of this chapter. However, it is important to note that it is M_2 (and not $\tilde{M}_2$) which must be the Q-orbifold of M_1.

Note this construction method is the most general method of constructing a string model with one compact dimension. This leads to the situation that for each specified D-dimensional string model M_1, there will in general exist many $(D-1)$-dimensional string models which extrapolate away from it. This depends on the choice of the $\mathbb{Z}_2$ orbifold Q, or equivalently on the choice of the second model M_2 to which one interpolates. In other words, the requirement of modular invariance alone is not sufficient to determine a unique interpolation, and is therefore not sufficient to determine a unique extrapolation. The determination of which orbifold Q is the correct thermal orbifold will depend on extra inputs from our expectations from thermal physics.

5.3 What defines a self-consistent finite-temperature extrapolation?

As discussed in Sect 5.1, for certain classes of closed string models namely heterotic strings the well known temperature/radius relation is broken. This leaves the situation of heterotic strings at finite temperature in a very tricky position. On the one hand, there is the well-known method of calculating Boltzmann sums for particles so that their thermal behavior may be examined. However, there is no guarantee, as there is with other classes of strings, that this will actually be anomaly-free theory which ultimately comes from the world-sheet. In particular, the very fact that there is *no* self-consistent $\mathbb{Z}_2$ orbifold which reproduces the typical Boltzmann sum suggests that even if a world-sheet successfully reproduced a seemingly thermal spectrum that this thermal spectrum would only be valid at one temperature (*i.e.*there would be no temperature modulus). On the other hand, if Boltzmann summing is not utilized the question of what actually determines a self-consistent finite temperature extrapolation must be asked. This is the question that we will attempt to answer below.

Let us now enumerate what we believe are the weakest possible conditions that can be imposed in order to have a self-consistent finite-temperature extrapolation of a given D-dimensional zero-temperature string model. Throughout, our goal is to impose only the most conservative requirements for self-consistency. The conditions we shall impose are as follows:

- First, the finite-temperature extrapolation should represent a self-consistent $(D-1)$-dimensional string model in its own right. In other words, it should satisfy all necessary worldsheet constraints such as conformal and superconformal invariance, self-consistent GSO projections, zero-temperature spin-statistics relations, *etc.*

- Second, this model should have an identifiable radius modulus corresponding to a *bona-fide* geometric compactification circle.

- Finally, this compactification circle should be interpretable *thermally* in the field-theory limit. This means that all states which survive in the field-theory limit should satisfy field-theoretic *thermal* spin-statistics relations.

In practice, this last requirement means that all massless spacetime bosons (fermions) with zero thermal windings in the theory should have thermal momentum excitations which are periodic (anti-periodic) around the thermal circle. Note, in particular, that we do *not* make any demands on the states with non-zero thermal winding modes; such stringy states have no field-theoretic limits, and are beyond our usual expectations. Likewise, by restricting our attention to only the *massless* states, we are again focusing on only the light states which can emerge in an appropriate low-energy field-theoretic limit. We stress that such *thermal* spin-statistics relations should be contrasted with the *zero-temperature* spin-statistics relations mentioned in the first of our requirements, which demand only that spacetime bosons (fermions) contribute with a positive (negative) sign to the overall partition function.

As was shown in the Ref [52], the geometric approach to building models involved orbifolding. The above constraints will allow for the elimination of most of the normally allowed orbifolds. Given these conditions, we can now examine how well our construction in Sect. I fares. Clearly, our first condition is automatically satisfied for *any* orbifold Q, since the construction of our thermal model in Sect. I proceeded by legitimate string-theoretic steps such as compactification and orbifolding. As long as our original zero-temperature model is self-consistent and as long as the orbifold Q is a legitimate allowed orbifold for this model, we are guaranteed that the resulting thermal model satisfies the first constraint. Similarly, the second constraint is also satisfied, again by construction.

However, the third constraint is more subtle. At first glance, it might seem that we have also satisfied our third constraint when we assumed that Q contains a $(-1)^F$ factor and coupled it with the half-shift $\mathcal{T}$ when constructing our thermal orbifold. However, there are two reasons why this may fail to be the case. First, our orbifold Q may generally contain other factors beyond $(-1)^F$ (such as gauge-group Wilson lines). Thus, in such cases, the thermal circle periodicities would be correlated not with spacetime fermion number alone, but with a combination of spacetime fermion number and Wilson-line eigenvalues. Of course, one might attempt to avoid this by taking $Q = (-1)^F$ directly, with no additional Wilson-line factors. In cases when this can be done, this issue will not arise, but we shall see shortly that this cannot always be done. The second reason, however, is more general. Recall that in our construction in Sect. I, we began with a D-dimensional string model which turned out to be the $T \to \infty$ limit of the thermal model we eventually constructed. Indeed, it was only the Q-orbifold of this model which emerged in the zero-temperature limit. Therefore, when we implemented the orbifold factor $(-1)^F$ in our construction, this acted on the bosons/fermions of the $T \to \infty$ model, but not necessarily those of the

$T \to 0$ model. In other words, if a model interpolating from M_1 to M_2 is the proper thermal extrapolation for the zero-temperature model M_1 (obeying proper thermal spin-statistics relations for the bosons and fermions of M_1), there is no guarantee that the T-dual model, which interpolates from M_2 to M_1, will be the appropriate thermal extrapolation for the model M_2. Thus, the construction we outlined in Sect. I — although completely general — is not by itself capable of guaranteeing that we have successfully maintained proper thermal spin-statistics relations when the final $(D-1)$-dimensional interpolation is constructed. Note that this remains true even in cases when $Q = (-1)^F$.

It is therefore this third requirement involving proper thermal spin-statistics relations which can be used to select the proper orbifold Q, and with it the correct thermal extrapolation for a given string model. We shall give two explicit examples of this procedure below.

5.4 Correct finite-temperature extrapolation for the $SO(32)$ heterotic string

In order to derive the correct finite-temperature extrapolation for the $SO(32)$ heterotic string, we follow our procedure in Sect. I. Specifically, we must choose an appropriate orbifold Q of the $SO(32)$ string, and then develop the corresponding nine-dimensional interpolating model.

What are the possible self-consistent $\mathbb{Z}_2$ orbifolds of the $SO(32)$ heterotic string? Clearly, this question boils down to the question of identifying possible $T \to \infty$ endpoints in ten dimensions for our corresponding nine-dimensional interpolation. Fortunately, all heterotic strings in ten dimensions have been classified [48]. In addition to the supersymmetric $SO(32)$ and $E_8 \times E_8$ heterotic strings, there are seven additional non-supersymmetric strings. These are the tachyon-free $SO(16) \times SO(16)$ string model as well as six tachyonic string models with gauge groups $SO(32)$, $SO(8)\times$

$SO(24)$, $U(16)$, $SO(16)\times E_8$, $(E_7)^2\times SU(2)^2$, and E_8. The tachyons in the latter six models all have worldsheet energies $(H_R, H_L) = (-1/2, -1/2)$. However, not all of these models can be realized as $\mathbb{Z}_2$ orbifolds of the supersymmetric $SO(32)$ model, and even in the remaining cases, not all of the resulting interpolating models will have a radius of compactification that can be interpreted thermally in the field-theory limit, as required by our third condition in Sect. II.

Fortunately, one interpolation meets all of our requirements. Perhaps not surprisingly, this is the interpolation between the supersymmetric $SO(32)$ string and the non-supersymmetric $SO(32)$ string. Note that the non-supersymmetric $SO(32)$ heterotic string has partition function

$$\begin{aligned} Z \;=\; Z^{(8)}_{\rm boson} \;\times \Big\{ & \overline{\chi}_I\,(\chi_I\chi_V+\chi_V\chi_I) \;+\; \overline{\chi}_V\,(\chi_I^2+\chi_V^2) \\ & -\,\overline{\chi}_S\,(\chi_S^2+\chi_C^2) \;-\; \overline{\chi}_C\,(\chi_S\chi_C+\chi_C\chi_S) \Big\}\;. \end{aligned} \tag{5.15}$$

Following the procedure outlined in Sect. I, we then find that the corresponding nine-dimensional interpolating model has the partition function [49]

$$\begin{aligned} Z_{\rm string}(\tau,T) \;=\; Z^{(7)}_{\rm boson}\times \Big\{ \quad & \mathcal{E}_0 \quad [\overline{\chi}_V\,(\chi_I^2+\chi_V^2) \;-\; \overline{\chi}_S\,(\chi_S^2+\chi_C^2)] \\ +\; & \mathcal{E}_{1/2} \quad [\overline{\chi}_V\,(\chi_S^2+\chi_C^2) \;-\; \overline{\chi}_S\,(\chi_I^2+\chi_V^2)] \\ +\; & \mathcal{O}_0 \quad [\overline{\chi}_I\,(\chi_I\chi_V+\chi_V\chi_I) \;-\; \overline{\chi}_C\,(\chi_S\chi_C+\chi_C\chi_S)] \\ +\; & \mathcal{O}_{1/2} \quad [\overline{\chi}_I\,(\chi_S\chi_C+\chi_C\chi_S) \;-\; \overline{\chi}_C\,(\chi_I\chi_V+\chi_V\chi_I)] \Big\} \end{aligned} \tag{5.16}$$

Note, in particular, that this correctly reproduces Eq. (4.33) in the $T\to 0$ limit as well as Eq. (5.15) in the $T\to\infty$ limit. Moreover, it satisfies thermal spin-statistics relations for the massless states with zero string windings: all massless states multiplying $\mathcal{E}_0$ are spacetime bosons, while all massless states multiplying $\mathcal{E}_{1/2}$ are spacetime fermions. (Note in this context there are no massless states which contribute to terms of the forms $\overline{\chi}_{S,V}\chi^2_{S,C}$, since $\chi^2_{S,C}$ has conformal dimension $h=2$.)

We stress that this is the unique thermal extrapolation which satisfies the conditions we put forth in Sect. II. Indeed, only this extrapolation corresponds to a self-consistent nine-dimensional interpolating model with an identifiable thermal radius of compactification with proper thermal spin-statistics relations in the field-theory limit. However, there are some unique features involved in such an interpolation. While it was perhaps already expected from Ref. [8] that states with non-trivial thermal winding modes might behave in a counter-intuitive fashion, violating finite-temperature spin-statistics relations in the $\mathcal{O}_0$ and $\mathcal{O}_{1/2}$ sectors, we now see that our interpolations necessarily have apparent Planck-scale thermal spin-statistics violations even for the states with *zero* windings, *i.e.*, states with conformal dimensions $h > 1$ which appear in the $\mathcal{E}_0$ and $\mathcal{E}_{1/2}$ sectors. Planck-scale violations of this sort appear to be unavoidable, even for zero-winding states, and (as we shall argue below) are required by modular invariance in the context of self-consistent interpolating models. It would be interesting to understand the thermal implications of these states as far as Planck-scale physics is concerned.

Note, however, that although these Planck-scale states appear to violate *thermal* spin-statistics relations, they still obey zero-temperature spin-statistics relations, as required. In other words, all spacetime bosons contribute positively to the partition function, while all spacetime fermions contribute negatively, with minus signs in front of their corresponding characters.

5.5 Implications for the Hagedorn transition

Let us now discuss the implications of following the geometric approach for generating finite temperature extrapolations for the Hagedorn transition [14, 1, 8]. Our focus here will be on the tachyons and temperature associated with the Hagedorn transition; for a more complete set of references concerning the history and possible interpretations and implications of the Hagedorn transition, see Ref. [45].

5.5.1 The Hagedorn transition: UV versus IR

We begin with several preliminary remarks concerning the UV/IR nature of the Hagedorn transition. In general, once we have determined the correct finite-temperature partition function $Z_{\rm string}(\tau, T)$ for a given zero-temperature string model, the one-loop thermal vacuum amplitude $\mathcal{V}(T)$ (the analogue of the logarithm of the statistical-mechanical partition function) is given by a modular integral of the form [10]

$$\mathcal{V}(T) \equiv -\tfrac{1}{2}\,\mathcal{M}^{D-1}\int_{\mathcal{F}} \frac{d^2\tau}{({\rm Im}\,\tau)^2}\, Z_{\rm string}(\tau, T) \tag{5.17}$$

where $\mathcal{M} \equiv M_{\rm string}/(2\pi)$ is the reduced string scale; D is the number of non-compact spacetime dimensions; and where

$$\mathcal{F} \equiv \{\tau :\ |{\rm Re}\,\tau| \le \tfrac{1}{2}, {\rm Im}\,\tau > 0, |\tau| \ge 1\} \tag{5.18}$$

is the fundamental domain of the modular group. We shall often abbreviate $\tau_1 \equiv {\rm Re}\,\tau$ and $\tau_2 \equiv {\rm Im}\,\tau$. Given this definition for $\mathcal{V}$, the free energy F, internal energy U, entropy S, and specific heat c_V then follow from the standard thermodynamic definitions $F \equiv T\mathcal{V}$, $U \equiv -T^2 d\mathcal{V}/dT$, $S \equiv -dF/dT$, and $c_V \equiv dU/dT$.

In string theory, the Hagedorn transition is usually associated with a divergence or other discontinuity in the vacuum amplitude $\mathcal{V}(T)$ as a function of temperature. There are two ways in which such a divergence may arise. First, of course, there may be an ultraviolet divergence due to the well-known exponential rise in the degeneracy of string states. However, such an ultraviolet divergence would normally be associated with the $\tau_2 \to 0$ region of integration in Eq. (2.3), and we see from Eq. (5.18) that this region of integration has been eliminated from the integral as a result of modular invariance — *i.e.*, as the result of the truncation of the region of integration to the modular group fundamental domain in Eq. (5.18). Thus, strictly speaking, there can be no UV divergence contributing to $\mathcal{V}(T)$. On the other hand, there may be purely infrared divergences coming from on-shell physical tachyons within $Z_{\rm string}(\tau, T)$; such states would lead to divergences in the infrared region $\tau_2 \to \infty$. Thus, a study of the Hagedorn transition in string theory essentially reduces to a study of the possible tachyonic structure of $Z_{\rm string}(\tau, T)$ as a function of temperature.

Before proceeding further, we caution that we reach this conclusion only because we have chosen to work in the so-called $\mathcal{F}$-representation for $\mathcal{V}(T)$ given in Eq. (5.17). By contrast, utilizing Poisson resummations and modular transformations [50], we can always rewrite $\mathcal{V}(T)$ as the integration of a different integrand $Z'_{\rm string}(\tau, T)$ over the strip

$$\mathcal{S} \equiv \{\tau : \; |\mathrm{Re}\,\tau| \leq \tfrac{1}{2}, \mathrm{Im}\,\tau > 0\} \; . \tag{5.19}$$

In such an $\mathcal{S}$-representation, the former IR divergence as $\tau_2 \to \infty$ is transformed into a UV divergence as $\tau_2 \to 0$. This formulation thus has the advantage of relating the Hagedorn transformation directly to a UV phenomenon such as the exponential rise in the degeneracy of states. However, both formulations are mathematically equivalent; indeed, modular invariance provides a tight relation between the tachyonic structure of a given partition function and the rate of exponential growth in its asymptotic degeneracy of states [51, 53, 54]. In the following, therefore, we shall utilize the $\mathcal{F}$-representation for $\mathcal{V}(T)$ and focus on only the tachyonic structure of $Z_{\rm string}(\tau, T)$, but we shall comment on the connection to the asymptotic degeneracy of states in Sect. V C.

5.5.2 A new Hagedorn temperature for heterotic strings

So what then are the potential tachyonic states within $Z_{\rm string}(\tau, T)$, and at what temperature T_H do they first arise? In other words, at what critical temperature T_H do new massless states emerge within $Z_{\rm string}$, on their way to becoming tachyonic? Note that we are focusing on *thermally massless* states, *i.e.*, states whose masses depend on temperature, states for which masslessness is achieved at a critical temperature T_H as the result of a balance between a tachyonic non-thermal mass and an additional positive non-zero thermal mass contribution. It is sufficient to focus on such massless states since their emergence is the signal of the long-range order normally associated with a phase transition. These are the states which then presumably become tachyonic beyond T_H, leading to the instabilities normally associated with a phase transition.

Given our results for $Z_{\text{string}}(\tau, T)$ in Sects. III and IV, it is straightforward to obtain the corresponding Hagedorn temperatures. Let us begin by considering the case of the Type II strings, for which the appropriate thermal function is given in Eq. (4.31). Recalling the conformal dimensions associated with the different characters in Eq. (4.31), we see that the only potentially tachyonic contributions in this expression arise from the term $\overline{\chi}_I \chi_I \mathcal{O}_0$. Thus, only this sector has the potential to give rise to thermally massless level-matched states. Solving the conditions for masslessness, we find that the $(m, n) = (0, \pm 1)$ thermal excitations of the $(H_R, H_L) = (-1/2, -1/2)$ tachyons within $\overline{\chi}_I \chi_I$ will become thermally massless at the temperature $T_H = \mathcal{M}/\sqrt{2}$. These thermal states are massive for $T < T_H$, and tachyonic for $T > T_H$. We thus identify $T_H = \mathcal{M}/\sqrt{2}$ as the Hagedorn temperature for Type II strings. Note that this analysis is in complete agreement with the standard derivations [1, 8] of the Hagedorn temperature for Type II strings.

However, the main difference occurs in the case of the heterotic string. Performing exactly the same analysis for the thermal partition function given in Eq. (5.16), we find that only the term $\overline{\chi}_I(\chi_I \chi_V + \chi_V \chi_I)\mathcal{O}_0$ is capable of giving rise to thermally massless level-matched states. Indeed, the $SO(16) \times SO(16)$ character $(\chi_I \chi_V + \chi_V \chi_I)$ gives rise to the 32 on-shell $(H_R, H_L) = (-1/2, -1/2)$ tachyons of non-supersymmetric $SO(32)$ string which serves as the $T \to \infty$ endpoint of the interpolation, and we find that the $(m, n) = (0, \pm 1)$ thermal excitations of these states are massless at $T_H = \mathcal{M}/\sqrt{2}$, massive below this temperature, and tachyonic above it. This is exactly the same as for the Type II string, and there are no other tachyonic sectors within Eq. (5.16) which could give rise to other phase transitions at lower temperatures.

We thus conclude that the ten-dimensional supersymmetric $SO(32)$ heterotic string has a Hagedorn temperature given by $T_H = \mathcal{M}/\sqrt{2}$, which is exactly the same as the Hagedorn temperature for the Type II string. A similar analysis with the same result also applies for the $E_8 \times E_8$ heterotic string as well as the non-supersymmetric tachyon-free $SO(16) \times SO(16)$ string. We thus find that

$$T_H \;=\; \frac{\mathcal{M}}{\sqrt{2}} \;=\; \frac{M_{\rm string}}{2\sqrt{2}\pi} \qquad \text{for } \textit{all} \text{ tachyon-free closed strings in } D=10 \;, \tag{5.20}$$

both Type II and heterotic! In other words, by carefully constructing self-consistent interpolating models with their required GSO projections, we have uncovered a universal Hagedorn temperature for all closed tachyon-free strings in ten dimensions.

This is clearly a major difference relative to our usual expectations. Indeed, if we had performed the same analysis using the (inconsistent) expression in Eq. (4.34), we would have found that the $(H_R, H_L) = (-1/2, -1)$ off-shell tachyon within the sector $\overline{\chi}_I \chi_I^2 \mathcal{O}_{1/2}$ contains thermal excitations $(m, n) = \pm(1/2, 1)$ which become thermally massless at the expected (heterotic) Hagedorn temperature $T_H = 2\mathcal{M}/(2+\sqrt{2}) = (2-\sqrt{2})\mathcal{M}$. However, as we discussed in Sect. IV, this state is actually GSO-projected out of the spectrum when we construct the proper thermal interpolating model in Eq. (5.16). We thus find that the Hagedorn temperature for the heterotic string is altered.

It is not surprising, perhaps, that the Type II and heterotic strings share a common Hagedorn temperature once the correct thermal extrapolations are taken into account. In the case of the Type II string, the ground state is a tachyon with worldsheet energies $(H_R, H_L) = (-1/2, -1/2)$. This state is level-matched, and survives into the corresponding thermal extrapolating function in Eq. (4.31). In the case of the heterotic string, by contrast, the ground state has vacuum energies $(H_R, H_L) = (-1/2, -1)$. Although this would naively appear to change the associated Hagedorn temperature, the fact that this state is not level-matched, together with modular invariance, ends up forcing this state to appear within the thermally twisted sector $\mathcal{O}_{1/2}$ where its appearance would be inconsistent. Thus, all thermal contributions from this state are projected out, and only the "next-deepest" tachyon, again with $(H_R, H_L) = (-1/2, -1/2)$, survives to affect the resulting thermodynamics. Since this surviving heterotic tachyon has exactly the same worldsheet energies as the Type II ground state, the heterotic and Type II theories have exactly the same Hagedorn temperatures.

5.5.3 Reconciling the new Hagedorn temperature with the asymptotic degeneracy of states

The above arguments are clearly based on an IR analysis of the tachyonic structure of our thermal interpolating models. One may therefore wonder how it is possible to find a Hagedorn temperature $T_H = \mathcal{M}/\sqrt{2}$ for a heterotic string such as the $SO(32)$ string, given that its zero-temperature bosonic and fermionic densities of states nevertheless continue to exhibit an exponential rate of growth normally associated with the usual heterotic Hagedorn temperature $T_H = (2-\sqrt{2})\mathcal{M}$. This is a very important question which we shall now address.

We shall develop our response in several layers. First, let us recall the usual direct connection between the asymptotic degeneracy of states and the corresponding Hagedorn temperature [14]: if g_M denotes the number of states with mass M, then the thermal partition function is given by $Z(T) = \sum g_M e^{-M/T}$. However, if $g_M \sim e^{\alpha M}$ as $M \to \infty$, then $Z(T)$ diverges for $T \geq T_H \equiv 1/\alpha$. This appears to provide a firm link between the Hagedorn temperature and the asymptotic degeneracy of states.

Of course, one might argue that this kind of partition function $Z(T) = \sum g_M e^{-M/T}$ is not a proper string-theoretic partition function; it assumes that the string is nothing but a collection of the states to which its excitations give rise. Indeed, we must perform a proper string-theoretic vacuum-amplitude calculation as outlined in Eq. (5.17), using a string partition function $Z_{\rm string}(\tau, T)$ which depends not only on the temperature T but also a torus parameter τ. We must then integrate over τ over a restricted fundamental domain.

However, the same basic argument connecting the asymptotic degeneracy of states with the Hagedorn transition continues to apply, even for the proper string-theory calculation. After all, we may easily transform to the $\mathcal{S}$-representation for $\mathcal{V}(T)$, as discussed in Sect. V A; in this representation, the connection between the asymptotic rise in the degeneracy of states and the UV divergence as $\tau_2 \to 0$ becomes manifest in the $\tau_2 \to 0$ region. How then can we interpret the increase in the Hagedorn temperature from the traditional heterotic value $T_H = (2-\sqrt{2})\mathcal{M}$ to the new, slightly higher value $T_H = \mathcal{M}/\sqrt{2}$? It seems that this would require a corresponding *decrease* in the exponential rate of growth in the asymptotic density of string states.

To answer this question, we must first recognize that the transition from the $\mathcal{F}$-representation to the $\mathcal{S}$-representation is highly non-trivial in the case of string theories containing spacetime fermions (such as the Type II and heterotic strings). In the case of the bosonic string, for example, the thermal partition function necessarily takes the factorized form given in Eq. (5.1); the absence of spacetime fermions implies that no subsequent thermal $\mathbb{Z}_2$ orbifolding is required. Such a partition function is particularly easy to transform to the $\mathcal{S}$-representation where the connection between the degeneracy of states and the Hagedorn temperature is immediate and apparent in the $\tau_2 \to 0$ region (which is why we do not find a change in the Hagedorn temperature for bosonic strings, assuming they could be made stable at zero temperature). However, as we have seen above, for Type II and heterotic strings the thermal partition function necessarily takes the more complicated form given in Eq. (5.12). The failure of this form to factorize — indeed, the presence of the half-integer shifts in the thermal momenta for the $\mathcal{E}_{1/2}$ and $\mathcal{O}_{1/2}$ sectors — is the direct consequence of the $\mathbb{Z}_2$ thermal orbifold. However, when we take modular transformations of this partition function in order to shift to the $\mathcal{S}$-representation for $\mathcal{V}(T)$, as described in Ref. [50], this half-integer shift is transformed into non-trivial $\mathbb{Z}_2$ *phases* (*i.e.*, minus signs) in the corresponding asymptotic degeneracies of states. These minus signs act to cancel the dominant exponential divergences in the degeneracies of states, allowing *subleading* exponential divergences to dominate. (Such subleading exponential terms are discussed fully in Refs. [51, 53, 54].) This subleading, reduced asymptotic growth then correlates directly with the increase in the Hagedorn temperature that we have found.

Still, one may argue on general conformal-field-theory (CFT) grounds that such a change in the Hagedorn temperature should not be possible. After all, the heterotic string has a worldsheet CFT with central charges $(c_R, c_L) = (12, 24)$ in light-cone gauge; this is implicit in the fact that the heterotic string ground state has worldsheet vacuum energies $(H_R, H_L) = (-c_R/24, -c_L/24) = (-1/2, -1)$. The usual arguments for the Hagedorn transition then lead directly to a Hagedorn temperature given in terms of these central charges as

$$T_H = \left(\sqrt{\frac{c_L}{24}} + \sqrt{\frac{c_R}{24}}\right)^{-1} \mathcal{M} , \tag{5.21}$$

and we have seen that this argument certainly holds in the case of the ten-dimensional Type II strings. However, this argument fails in the case of the heterotic strings because the heterotic string ground state, encapsulated within the CFT character $\overline{\chi}_I \chi_I^2$, has been *GSO-projected* out of the thermal spectrum in Eq. (5.16). Indeed, as stressed above, this is the primary difference between Eq. (4.34) and Eq. (5.16). Or, to phrase this point slightly differently, the GSO-projections inherent in Eq. (5.16) have deformed the worldsheet CFT of the theory so that it effectively has a new string ground state consisting of the $(H_R, H_L) = (-1/2, -1/2)$ tachyons encapsulated within the surviving term $\overline{\chi}_I(\chi_I \chi_V + \chi_V \chi_I)$ in Eq. (5.16). Since this effective ground state has depth $(H_R, H_L) = (-1/2, -1/2)$, we can identify the "effective" central charge of this deformed left/right worldsheet theory after the GSO projections as $(c_R, c_L) = (12, 12)$, just as for the Type II string.

It is also easy to understand how this deformation manages to affect the asymptotic exponential growth in the degeneracy of states within Z_{therm}. Recall that in a given CFT with central charge c, each character $\chi_i(\tau)$ represents a trace over the Verma module associated with the corresponding primary field ϕ_i, and has a q-expansion of the form [51, 53, 54]

$$\chi_i(\tau) \;=\; q^{h_i - c/24} \sum_{p=0}^{\infty} a_p^{(i)}\, q^p \qquad \text{with } a_p^{(i)} \;\sim\; \exp\left(4\pi\sqrt{\frac{cp}{24}}\right) \quad \text{as } p \to \infty\;. \tag{5.22}$$

Here h_i is the conformal weight of the primary field ϕ_i. In deriving this result via the standard contour-integral methods [51, 53, 54], one uses the fact that each character χ_i is connected to the identity character χ_I of the CFT ground state through a $\tau \to -1/\tau$ modular transformation. (The existence of this connection is guaranteed from the CFT fusion rules and the Verlinde formula [55].) It is for this reason that the q-expansion of each character χ_i of the CFT *individually* has coefficients which exhibit an exponential growth rate related to the underlying central charge of the worldsheet CFT. Likewise, this is why *individual* products of left- and right-moving characters yield asymptotic growth rates consistent with the traditional Hagedorn temperature.

However, in our thermal partition function in Eq. (5.16), we see that we essentially have four *combinations* of left/right characters which multiply our $\mathcal{E}/\mathcal{O}$ functions. Explicitly, these combinations are given by

$$\begin{aligned}
Z^{(1)} &= \overline{\chi}_V\,(\chi_I^2 + \chi_V^2) \;-\; \overline{\chi}_S\,(\chi_S^2 + \chi_C^2) \\
Z^{(2)} &= \overline{\chi}_V\,(\chi_S^2 + \chi_C^2) \;-\; \overline{\chi}_S\,(\chi_I^2 + \chi_V^2) \\
Z^{(3)} &= \overline{\chi}_I\,(\chi_I\chi_V + \chi_V\chi_I) \;-\; \overline{\chi}_C\,(\chi_S\chi_C + \chi_C\chi_S) \\
Z^{(4)} &= \overline{\chi}_I\,(\chi_S\chi_C + \chi_C\chi_S) \;-\; \overline{\chi}_C\,(\chi_I\chi_V + \chi_V\chi_I)\;.
\end{aligned} \tag{5.23}$$

These four left/right character combinations close amongst themselves under modular transformations, and thus function as a new "effective" set of characters for our "effective" (deformed) left/right worldsheet CFT. However, this effective character set does not contain the heterotic CFT ground-state $\overline{\chi}_I\chi_I^2$; indeed, the most tachyonic term that survives in this character set is the term $\overline{\chi}_I(\chi_I\chi_V + \chi_V\chi_I)$ within $Z^{(3)}$. We thus see that this effective left/right CFT has a reduced "effective" central charge compared with the original left/right CFT prior to GSO projections, as discussed above.

This is also directly evident from the $(q,\overline{q})$-expansions of the left/right character combinations $Z^{(1)}$ and $Z^{(2)}$. For example, looking at $Z^{(1)}$, we find that the first term $\overline{\chi}_V(\chi_I^2 + \chi_V^2)$ individually has a $(q,\overline{q})$-expansion with coefficients (mass-level degeneracies) exhibiting an exponential growth rate consistent with the traditional Hagedorn temperature. However, the same is also true of the second term within $Z^{(1)}$, namely $\overline{\chi}_S(\chi_S^2 + \chi_C^2)$, and the fact that these terms are *subtracted* in forming $Z^{(1)}$ actually ends up *cancelling* this leading exponential behavior. What remains is only a subleading exponential growth rate consistent with our re-identified Hagedorn temperature. A similar cancellation also holds for $Z^{(2)}$; note that cancellations of these sorts are discussed in detail in Ref. [54]. Thus, at the partition-function level, this reduction in the Hagedorn temperature is a direct result of the minus signs within the combinations $Z^{(1)}$ and $Z^{(2)}$ in Eq. (5.23), which in turn are a direct result of the non-standard thermal spin-statistics relations that we have already observed at the Planck scale.

In some sense, this entire argument may be summarized as follows. Let us look again at the original partition function of the $SO(32)$ heterotic string model in Eq. (4.33). As a result of spacetime supersymmetry, this partition function vanishes identically — *i.e.*, all of its level-degeneracy coefficients are identically zero. There is no exponential growth here at all. But one does not examine the *whole* partition function in order to derive a Hagedorn temperature; one instead looks at its separate bosonic and fermionic contributions. Ordinarily, these contributions would be identified as

$$
\begin{aligned}
Z^{(\text{bosonic})}_{SO(32)} &= Z^{(8)}_{\text{boson}}\ \overline{\chi}_V\,(\chi_I^2+\chi_V^2+\chi_S^2+\chi_C^2)\ ,\\
Z^{(\text{fermionic})}_{SO(32)} &= Z^{(8)}_{\text{boson}}\ \overline{\chi}_S\,(\chi_I^2+\chi_V^2+\chi_S^2+\chi_C^2)\ ,
\end{aligned}
\tag{5.24}
$$

and indeed each of these expressions separately exhibits an exponential rise in the degeneracy of states which is consistent with the traditional heterotic Hagedorn temperature. But what do we really mean by "bosonic" and "fermionic"? Taking a thermodynamic definition, we are forced to identify such states according to their periodicities around the thermal circle. Therefore, given our $(D-1)$-dimensional interpolating-model analysis, we now see that for the heterotic string, Eq. (5.24) is not the correct way to separate the total partition function into its separate thermal components. Instead, for thermal purposes, we now see that the proper separation is into the different components

$$
\begin{aligned}
Z^{(\text{"bosonic"})}_{SO(32)} &= Z^{(8)}_{\text{boson}}\ \left[\overline{\chi}_V\,(\chi_I^2+\chi_V^2)\ -\ \overline{\chi}_S\,(\chi_S^2+\chi_C^2)\right]\ ,\\
Z^{(\text{"fermionic"})}_{SO(32)} &= Z^{(8)}_{\text{boson}}\ \left[\overline{\chi}_S\,(\chi_I^2+\chi_V^2)\ -\ \overline{\chi}_V\,(\chi_S^2+\chi_C^2)\right]\ ,
\end{aligned}
\tag{5.25}
$$

since these are the components that appear in the $\mathcal{E}_0$ and $\mathcal{E}_{1/2}$ sectors when the proper thermal extrapolation is constructed. It is therefore *these* components which function as the "bosonic" and "fermionic" contributions as far as thermal effects are concerned, and indeed these are the components which exhibit the slower exponential growth associated with our re-identified Hagedorn temperature. We stress that both Eq. (5.24) and Eq. (5.25) correctly separate those massless bosonic and fermionic states which survive in the field-theory limit. Their only difference is in their treatment of the stringy Planck-scale states which have no field-theoretic limits. In other words, Eq. (5.25) correctly identifies bosons and fermions according to their thermal behaviors; what is unusual is the connection between this behavior and the spacetime Lorentz spins of the Planck-scale states.

We thus conclude that all tachyon-free closed strings in ten dimensions share a universal Hagedorn temperature. Although the heterotic string would naively appear to have a slightly lower Hagedorn temperature than the Type II string due to its non-level-matched ground state, self-consistency also requires a set of non-trivial GSO projections which compensate for the non-level-matched ground state by inducing a cancellation in the asymptotic degeneracies of states, thereby pushing the associated Hagedorn temperature back to the Type II value.

5.6 Beyond ten dimensions: Additional general observations

In ten dimensions, we found that each of the closed string theories which are tachyon-free at zero temperature has a Hagedorn transition associated with a tachyon that emerges in its thermal extrapolation, with worldsheet energies $(H_R, H_L) = (-\frac{1}{2}, -\frac{1}{2})$. For the supersymmetric Type II strings, we have seen that this tachyon always emerges in the thermal extrapolation because the only possible $T \to \infty$ endpoints for the corresponding interpolating models are the Type 0 strings, which necessarily contain these tachyons. In the case of the $SO(32)$ and $E_8 \times E_8$ heterotic strings, we have seen that we must also identify the appropriate non-supersymmetric heterotic string models to serve as suitable $T \to \infty$ endpoints. While the tachyon-free $SO(16) \times SO(16)$ string could have logically served as this endpoint, it turns out that this choice would violate thermal spin-statistics constraints in both the $SO(32)$ and $E_8 \times E_8$ cases [56, 49]. Thus, in each case, our $T \to \infty$ endpoint model must be one of the remaining non-supersymmetric string models [*e.g.*, for the $SO(32)$ string, we found that the endpoint was the non-supersymmetric $SO(32)$ string]. Since each of these remaining non-supersymmetric models has tachyons with $(H_R, H_L) = (-1/2, -1/2)$, we again have a Hagedorn transition, albeit with a re-identified Hagedorn temperature.

Given these results, two obvious questions arise. First, is it a general property that *all* heterotic strings will have new, re-identified Hagedorn temperatures? Our belief is that this is indeed the case, regardless of the spacetime dimension. As we have argued in Sect. V, the usual Hagedorn transition in the heterotic case requires the existence of the term $\overline{\chi}_I \chi_I^2 \mathcal{O}_{1/2}$ (or more generally, the ground-state character multiplied by $\mathcal{O}_{1/2}$) in the corresponding thermal interpolating partition function, yet the thermally twisted nature of the $\mathcal{O}_{1/2}$ sector precludes this from happening. We believe that this is a general argument which transcends the particular gravitino-based orbifold argument which was also provided in Sect. III.

A second, equally important issue concerns whether it might be possible to eliminate the Hagedorn phase transition completely by finding a zero-temperature string model for which the appropriate thermal extrapolation involves a $T \to \infty$ endpoint model which is *non-supersymmetric but tachyon-free*. While this did not occur in ten dimensions, this remains a logical possibility in lower dimensions where many such non-supersymmetric tachyon-free string models exist, both of the superstring and heterotic string variety.

While we do not know the answer to this question in the case of the superstring, we can prove that it is impossible to evade such a phase transition entirely in the cases of heterotic strings which are supersymmetric at zero temperature. Our proof runs as follows [45]. Even if the appropriate $T \to \infty$ endpoint theory happens to be tachyon-free, there will always exist another state in the thermal extrapolation whose thermal excitations will trigger a non-trivial phase transition. This is the so-called "proto-gravitino" state:

$$\text{proto-gravitino:} \qquad \tilde{\phi}^{\alpha} \;\equiv\; \{\tilde{b}_0\}^{\alpha}|0\rangle_R \;\otimes\; |0\rangle_L \;. \tag{5.26}$$

Note that this state is constructed exactly as the gravitino in Eq. (4.36), but without the left-moving worldsheet coordinate excitation. However, it is important to realize that *GSO projections are completely insensitive to the presence or absence of excitations of the worldsheet coordinate bosonic fields*. Thus, if our zero-temperature heterotic string model is supersymmetric and the gravitino is therefore present in the original zero-temperature theory, then the proto-gravitino must also always be present in the original zero-temperature theory.

Since this state emerges, like the gravitino itself, from the untwisted (fermionic) gravity sector of the original $T \to 0$ model, its contributions must appear multiplied by $\mathcal{E}_{1/2}$ within any self-consistent heterotic thermal extrapolation away from that model. [For example, in the case of the $SO(32)$ heterotic string interpolation in Eq. (5.16), the proto-gravitino contribution was hiding within the term $-\overline{\chi}_S \chi_I^2 \mathcal{E}_{1/2}$.] However, this proto-gravitino state is then necessarily endowed with a thermal excitation of $(m,n) = (1/2, 2)$ which is massive for all $T \neq T_*$ but exactly massless at the single temperature $T = T_*$ where $T_* \equiv 2\mathcal{M}$. (Note that since this state is fermionic, Lorentz invariance prevents it from becoming tachyonic at any temperature.) The sudden appearance of a new massless state at $T = T_*$ signals the emergence of long-range order in the thermal theory, and can again be associated with a Hagedorn-like phase transition. However, since this state hits masslessness only once as a function of temperature and never becomes tachyonic, this turns out [45] to be a very weak, p^{th}-order phase transition, where p is related to the spacetime dimension D:

$$p = \begin{cases} D & \text{for } D \text{ even} \\ D-1 & \text{for } D \text{ odd}\ . \end{cases} \tag{5.27}$$

Thus, we conclude that for supersymmetric heterotic strings, it is never possible to completely evade a Hagedorn-like phase transition. However, the phase transition associated with the proto-gravitino state appears only at the relatively high temperature $T_* \equiv 2\mathcal{M}$, and thus will be completely irrelevant if tachyon-induced Hagedorn transitions appear at lower temperatures.

5.7 Conclusions

In this chapter, we investigated the manner in which a given zero-temperature string model may be extrapolated to finite temperature. Following relatively conservative conditions for self-consistency, we nevertheless found that the traditional Hagedorn transition does not exist for heterotic strings but is instead replaced by a new, "re-identified" Hagedorn transition which emerges at the somewhat higher temperature normally associated with Type II strings. This allowed us to uncover a universal Hagedorn temperature for all tachyon-free closed string theories in ten dimensions. We also showed that these results are not in conflict with the exponential rise in the degeneracy of string states in these models.

Clearly, many outstanding questions remain. Perhaps the two most critical are the issues of the *existence* and *uniqueness* of thermal extrapolations satisfying the general criteria we put forth in Sect. II. In other words, it is important to demonstrate that, for any given D-dimensional zero-temperature string model, there always exists one and only one suitable corresponding $T \to \infty$ endpoint D-dimensional string model such that the corresponding $(D-1)$-dimensional interpolation is thermally consistent according to our general criteria, including proper spin-statistics relations. In ten dimensions, we have already seen that such extrapolations exist and are unique. However, neither property has been proven in lower dimensions. This is clearly an important issue that requires further study.

Another interesting question concerns the thermal fate of string models which are non-supersymmetric but tachyon-free at zero temperature: is it ever possible that such a non-supersymmetric model will have a thermal extrapolation whose $T \to \infty$ limit is *supersymmetric*? If so, this would be an example of a situation in which the zero-temperature theory is non-supersymmetric, but in which thermal effects compensate for this inequity between bosons and fermions and thereby *introduce* (rather than break) supersymmetry as $T \to \infty$. In other words, such thermal effects would be "SUSY-making" rather than SUSY-breaking, with SUSY-breaking occurring at *lower* temperatures. This phenomenon would be intrinsically string-theoretic, since only for closed strings does the $T \to \infty$ limit yield a theory of the same dimensionality as the original zero-temperature theory. No examples exhibiting this phenomenon exist in ten dimensions, but it would be interesting to explore whether such examples might exist in lower dimensions.

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