SNAPSHOT IMAGING POLARIMETERS

USING SPATIAL MODULATION

by

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The recent demonstration of a novel snapshot imaging polarimeter using the fringe modulation technique shows a promise in building a compact and moving-parts-free device. As just demonstrated in principle, this technique has not been adequately studied. In the effort of advancing this technique, we build a complete theory framework that can address the key issues regarding the polarization aberrations caused by using the functional elements. With this model, we can have the necessary knowledge in designing, analyzing and optimizing the systems. Also, we propose a broader technique that uses arbitrary modulation instead of sinusoidal fringes, which can give us more engineering freedom and can be the solution of achromatizing the system. In the hardware aspect, several important progresses are made. We extend the polarimeter technique from visible to middle wavelength infrared by using the yttrium vanadate crystals. Also, we incorporate a Savart Plate polarimeter into a fundus camera to measure the human eye’s retinal retardance, useful information for glaucoma diagnosis. Thirdly, a world-smallest imaging polarimeter is proposed and demonstrated, which may open many applications in security, remote sensing and bioscience.
CHAPTER 1

INTRODUCTION

The polarization of light provides extra dimensions of information about the light or the object that the light interacts with. Historically, the light polarization has been characterized by the Stokes vector $\vec{S}$ and the Mueller matrix $M$. The Stokes vector describes a unique state of polarization that can be located on the Poincare sphere, and the Mueller matrix represents the polarization behavior of the object in terms of changing the incident $\vec{S}$. It turns out that these terms are powerful for people to study physical or chemical properties of an interested object. For example, by taking an object’s Stokes images, a higher contrast as well as other information (e.g., the degree of polarization, ellipticity angle, etc) that a photograph camera does not provide can be obtained, largely helping identify the object, understand its properties and discover new phenomena. This technique, namely the imaging polarimetry, is used in remote sensing, military surveillance, astronomy observation and optical testing. A Stokes imaging polarimeter is such an instrument that can capture the Stokes of objects.

A conventional method to determine $\vec{S}$ is to use several rotating polarization elements to obtain the complete $\vec{S}$ (S0 through S3) in a series of measurements. This approach suffers from mechanical complexity, vibration noise, heat generation and other unwanted issues. The incapability of measuring a non-static objects or making real-time detection is also a limitation. Under many circumstances, a snapshot imaging polarimeter (SIP) that measures the complete Stokes vector simultaneously is needed. For example, a satellite polarimetry camera can scan the target area in a high
speed, and a pulse-like chemical reaction can be caught by the camera at the same instant. There are several schemes having been developed to make an SIP: the division of aperture (DoA), the division of amplitude (DoAM), and the division of focal-plane array (DoFPA), all implementing separate optical paths or sensor pixels to detect individual Stokes parameters; and all have advantages and disadvantages [1]. Compared to the rotational polarimeter, an SIP eliminates the moving parts, and is immune to issues like the vibration and heat generation. However, majority of these systems still have considerable complexity in layout, for example, the DoA- and DoM-type polarimeters utilizing four distinct imaging paths for individual Stokes parameters.

Recently, an innovative scheme was developed by Oka, *et al.*, in which four birefringent prisms [2] or two Savart Plates (SP) [3, 4] were used to form a complex interference pattern to encode the four Stokes parameters within a single image. Both methods use a common imaging path to implement the detection. In a prism polarimeter, four calcite prisms in sequence are placed in the image plane to create linear retardance in certain directions, while in a Savart Plate polarimeter, Savart Plates are placed in the collimated space of a 4-f imaging system to produce wave-front shear to form similar linear retardance at the image plane. A Fourier transformation (FT) based algorithm is used in both system to recover the four Stokes images. Compared to other snapshot strategies, the prism polarimeter shows potential in building a compact device, while the SP polarimeter can be more easily assembled, fabricated and cost-effective. Both, nevertheless, are limited to monochromatic or
quasi-monochromatic illumination, which can be a drawback when a high signal is needed in a passive application.

As a novel technique, this type of SIP has not been extensively studied. Many important issues are not addressed so far. This dissertation is motivated by the attempts to flesh out these issues and also to improve and extend this technique in order to achieve a high performance system for several challenging applications.

The first problem in the prism polarimeter system is that, due to the inclined surface, the birefringent prisms produce unwanted aberrations that may seriously degrade the device performance. The principle of a prism polarimeter was initially formulated through a Mueller-matrix theory [2] which did not count for the beam split effects happened inside the prisms. A more versatile model is required. Similar demands are seen in the SP polarimeters. The SPs produce less aberration, but their physical asymmetry raises concerns on the linear retardance produced at the image plane. Secondly, the 4f system in the SP polarimeter seems to be a non-optimized layout (Chapter 3) that may limit the device capabilities when a compact device is needed. It turns out that a geometric-modeling on the SP system can provide clues to make a significant system reduction (Chapter 4). Moreover, a closer look at the carrier frequency generated by the SPs enlightens us to scale down the system to a miniature size (Chapter 4). Thirdly, the mathematical nature of the modulation pattern gives an inspiration to think of extending the sinusoidal-modulated system to an arbitrary-modulated system which may introduce more engineering freedom for system designers and also help solve the bandwidth issue (Chapter V). Under this new
concept, a universal reconstruction and calibration algorithm is required to replace the Fourier method that works on the sinusoidal functions exclusively. For the fourth point, there is a strong demand to apply the device into the infrared range for detecting black-body emissions which dominates the signal in many applications. This spurs a search for a highly birefringent material in the infrared because the calcite crystal has very weak birefringence in the infrared. Importantly, such demanded material is useful for not only the prism but also the Savart Plate SIPs. Finally, we wish to apply the SP SIP for glaucoma diagnosis in a human eye. The human eye possesses optical retardation in the retina nerve fiber layer due to the micro-tube structures lying inside the layer. The clinical information of the retardance distribution of the retina nerve fiber layer has been shown to be proportional to the nerve fiber layer thickness and an early indicator of glaucoma.

Chapter 2 introduces a fringe decomposition model to facilitate the raytracing analysis of the prism SIP. Detailed results from numerical simulation and experiments are presented. The yttrium vanadate (YVO$_4$) is founded to be an ideal material for making a middle wavelength infrared (MWIR) SIP. The measurement of its birefringence in the MWIR is presented. A MWIR YVO$_4$ prism SIP is built and the preliminary testing results are presented.

Chapter 3 derives a rigorous wave model for the SP SIP. The proof of reducing the system size is provided. Detailed raytracing analysis is given to evaluate the aberration and phase linearity under the reduced system. Then, experimental results are presented as the demonstration of the design.
Chapter 4 shows a system evolution of the SP polarimeter for the purpose of realizing a miniature SIP. A geometric model is developed to validate the reduced system. Then, numerical and experimental results in an enlarged system are presented as the evidence to show the successful principle of a miniature SP SIP.

Chapter 5 develops a theory framework to include the polarization-dependent aberrations in an SIP. As an example, this model is applied to the Savart Plate SIP with numerical results presented to illustrate the quantitative strength of this.

Chapter 6 extends the sinusoidal-modulated SIP to an arbitrary-modulated SIP through proposing a new reconstruction and calibration algorithms. A Stokes imaging model is simplified to help understand the physical insight of the system design and reconstruction accuracy. Detailed simulation results are then presented for theoretical demonstration, in which a broad band system is designed and proved.

Chapter 7 introduces a fundus SIP camera that integrates the SP SIP with a common fundus to measure the retina retardance. The basic layout and preliminary experimental results are presented. The illumination challenge is addressed and some future works are suggested.
CHAPTER 2
PRISM SNAPSHOT IMAGING POLARIMETERS

2.1 Introduction

In a conventional polarimeter, several measurements are needed by rotating the polarization elements to solve the four Stokes parameters of an object. As simple in concept, this method suffers from unwanted vibration noise, device volume and mechanical complexity. Oka and Kaneko (2003) developed a snapshot imaging polarimetry technique which replaces the traditional polarization optics with four sequential birefringent prisms (Fig. 2.1) [2]. If we treat the wavefront incident on the prisms as planewaves and consider the prisms as being thin enough to ignore the beam splitting (BS) and beam deviation (BD) effects induced on the wavefronts (Fig. 2.2 illustrates these effects), then a straightforward application of Mueller calculus gives the image intensity $I(x,y)$ as

$$
I(x,y) = \frac{1}{2} S_0(x,y) + \frac{1}{2} S_1(x,y) \cos(2\pi U x) + \frac{1}{4} S_2(x,y) \left[ \cos[2\pi U(x - y)] - \cos[2\pi U(x + y)] \right] - \frac{1}{4} S_3(x,y) \left[ \sin[2\pi U(x - y)] + \sin[2\pi U(x + y)] \right] \tag{2.1}
$$

where $S_{0-3}(x,y)$ represents the spatially-dependent $\vec{S}$ of the incident beam, and $U$ denotes the fringe spatial frequency. If we take the Fourier transform of $I(x,y)$, we find that the four Stokes parameters are separated into individual channels in the frequency domain, and thus can be reconstructed simultaneously from a single measured image.
However, a closer look at this approach shows that some assumptions can frustrate optimal design of the instrument, limiting its performance. In particular, there are 3 effects ignored in the above approach which are important for understanding the limitations of the instrument. (1) The presence of the objective lens means that each pixel on the detector measures the light integrated across a range of angles of incidence onto the prisms. In a geometric model, this can be approximated by sampling the pupil and tracing rays from the pupil plane to the image plane. (2) A planewave entering a
A birefringent plate is sheared such that the ordinary wave (o-wave) and extraordinary wave (e-wave) propagate at slightly different angles. When they exit the plate, the o- and e-wave are displaced relative to one another, producing a beam-splitting (BS) effect. (3)

Since the prisms are not plane parallel plates, a ray exits the prism in a direction which is not parallel to the ray incident on the prism. This deflection angle is dependent on the index of refraction of the prism, and therefore differs for the o- and e-wave, producing the beam deviation (BD) effect (Figure 2.2). Further complicating this effect, depending on the angle of incidence and the angle of the crystal axis orientation, the deflection of the e-wave varies as the angle of incidence varies. The BS and BD effects both degrade the image, the first by enlarging the spot size (reducing resolution and fringe contrast), and the latter by introducing distortion. Both are amplified in systems with a small F-number.
To address these problems, a more complete imaging model is needed. A complete model based on vectorial diffraction theory may solve the problem, because all three effects are incorporated into the intensity pattern and the interferogram. However, the deformation of the wavefront in the birefringent prisms, and the tilted interfaces render the calculation complicated. The geometric raytracing method is a good compromise for most imaging systems, offering easy access for aberration analysis as well as describing the ray path. Algorithms for raytracing in a uniaxial material have been greatly developed in recent years [5-8]. The majority of them are based on Huygen’s principles and feature considerable complexity. A relatively simple approach, developed by Liang [9] by making use of phase matching conditions requires no additional assumptions and is the method used here.

In the first part (Sections 2.2-2.5) of this chapter, we build up a geometric imaging model, based on fringe decomposition, capable of analyzing the off-axis rays passing through the birefringent prisms as well as the prism BS and BD effects. We then simulate a system of calcite prisms and illustrate the BS and BD effects in the presence of a lens, showing that for prism apex angles less than 2° the Mueller calculus is sufficient, but for larger apex angles the geometric model provides a way to optimizing the system design without introducing unnecessary error. With the necessary background provided, we outline a procedure for designing and optimizing a prismatic polarimeter.

The calcite system, however, is mainly effective in visible since calcite’s birefringence disappears outside of that spectral range. This spurs a search for a highly birefringent crystal in the middle wavelength infrared (MWIR):3-5µm, for the purpose of implementing the device to this important wavelength band for commercial or military
applications. For a normal MWIR application, the birefringence ($\Delta n = n_e - n_o$) of the prism material is required to be larger than 0.2 (Section 2.6). Other traditional birefringent crystals, e.g., quartz, and LiNbO$_3$ are not satisfactory, because of both their low $\Delta n$ in the MWIR region and limited transmission. The well-known rutile (TiO$_2$) crystal is a possible candidate since it transmits to 5µm and possesses a high birefringence of over 0.2 [10]. Very recently, the yttrium vanadate (YVO$_4$) crystal, mostly used as a laser host crystal, has drawn much attention [11,12] due to its comparatively high birefringence ($\Delta n \sim 0.2$) in the visible and near infrared (NIR) and its superior mechanical and physical properties which lead the material to the fabrication of telecommunication elements. In comparison to rutile, surface processing is easier with YVO$_4$ which greatly reduces the manufacture cost. These properties are very important, since a prism apex angle of less than 2º is used in the polarimeter, which is a big challenge for mechanical fabrication [2]. However, few references have been found with measurements on the birefringence of YVO$_4$ in MWIR. DeShazer, et.al., (2002) [13] presented experimental results of YVO$_4$ refractive indices using the minimum deviation method. But no measurement details were discussed there and the wavelength was limited to less than 3µm. One may also question the accuracy of the minimum deviation method since the extraordinary wave does not comply with Snell’s law in a birefringent prism. Casix Inc. reported the birefringence of YVO$_4$ in visible and NIR up to 1.60µm, but this is not sufficient for the MWIR. Therefore, we needed to make a complete and accurate measurement of the birefringence of this material in the MWIR. In the second part (Section 2.5-2.7) of this chapter, we will use the channel spectra technique [14, 15] as the method of birefringence measurement in order to explore the potential of the YVO$_4$ crystal for MWIR applications. Also, a system
was fabricated in the MWIR with YVO₄ is reported at the end of this chapter with preliminary experimental results presented.

2.2 Analysis model

2.2.1 The BS and BD of birefringent prisms

To consider a beam incident on a uniaxial medium, we can first split the beam into two components, one whose electric (E) field is normal to the optic axis of the crystal (the ordinary light) and one whose E field lies in the plane containing both the optic axis and the wavevector \( \vec{k}_e \) (the extraordinary light). The refractive index of o-light is uniform across the medium, whereas that of the e-light obeys [9]

\[
n_e(\theta) = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2 \theta + n_e^2 \cos^2 \theta}} ,
\]

where \( \theta \) is the angle between the e wavevector \( \vec{k}_e \) and the optic axis and \( n_o \) and \( n_e \) are the refractive indices along the principle axes of the crystal. Note that \( \theta \) is dependent on the angle of incidence. The difficulty in treating the e-wave results from the fact that the wavevector \( \vec{k}_e \) and Poynting vector \( \vec{s}_e \) are not parallel. \( \vec{k}_e \) complies with Snell’s Law at the interface whereas \( \vec{s}_e \), the detected signal, does not. The direction cosines \( \xi_x, \xi_y, \) and \( \xi_z \) of the Poynting vector \( \vec{s}_e \), with respect to the coordinate system \( (x, y, z) \) defined by the air-crystal plane interface are:
\[
\xi_x = \cos \alpha \cos \theta_e + \frac{\sin \theta_e \sin \alpha (\sigma_x \sin \theta_e - \sigma_y \cos \theta_e)}{\sqrt{\sigma_x^2 + (\sigma_y \sin \theta_e - \sigma_y \cos \theta_e)^2}} \\
\xi_y = \cos \alpha \sin \theta_e - \frac{\cos \theta_e \sin \alpha (\sigma_x \sin \theta_e - \sigma_y \cos \theta_e)}{\sqrt{\sigma_x^2 + (\sigma_y \sin \theta_e - \sigma_y \cos \theta_e)^2}} \\
\xi_z = \frac{\sigma_z \sin \alpha}{\sqrt{\sigma_x^2 + (\sigma_y \sin \theta_e - \sigma_y \cos \theta_e)^2}} 
\]

(2.3)

where \(\alpha\) is the dispersion angle, \(\sigma_x, \sigma_y, \sigma_z\) are the direction cosines of the optic axis (also with respect to the air-crystal plane interface), and \(\theta_e\) is the angle between \(\ell_e\) and the x-axis [9]. Used together with the Snell’s Law, these formulae make possible the tracing of e-rays through the birefringent materials.

However, the sequential arrangement of prisms greatly complicates the situation compared to a single plane interface. Fig. 2.1(b) illustrates the prism layout. PR1 and PR2 are mounted oppositely with their acute dihedral edges parallel to the x-axis. This is arranged to create a retardance which varies linearly along x. Similarly, PR3 and PR4 have their acute dihedral edge parallel to the x-axis, creating a retardance varying linearly along the x-axis. The fast axes of the four prisms are oriented at 0\(^{\circ}\), 90\(^{\circ}\), 45\(^{\circ}\), and -45\(^{\circ}\) respectively (relative to the x-axis). A ray incident on PR1 is split into o/e rays, and due to the orthogonal orientation of the optic axes of PR1 and PR2, upon entering prism PR2 the two rays exchange their o/e status. Since the optic axis of PR3 is oriented at 45\(^{\circ}\) relative to that of PR2, each o- and e-ray incident onto the PR2-PR3 interface is split into equal o- and e-rays in the new medium, and finally switch their o/e status upon entering PR4. Table 2.1 summarizes the process of tracing these four rays through the prisms, denoted by Rays 1 to 4.
Each of the four rays can be considered as forming one of four independent images, summed together at the focal plane array (FPA). Displacement between the four images leads to a loss of resolution due to BS and BD effects. A second problem is a loss of phase purity, which we call “phase hybridization”. The expansion and non-uniform shift of each image spot results in a phase mixture within adjacent image points, and thus reducing the visibility of fringes and introducing errors in the image reconstruction. Both problems are amplified if the lens is not properly designed.

### 2.2.2 Fringe decomposition

Besides the regular imaging process, the formation of well-shaped fringes is also vital to system operation. Mueller calculus only predicts the overall intensity pattern when the BS and BD effects are not present (Eq.2.1) and provides a poor guide to exploring how these effects modify the measurement. To modify the setup for raytracing purposes, the fringe pattern has to be decomposed. Table 2.1 indicates that the optical path differences

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>PR1</td>
<td>o</td>
<td>E_y(t)</td>
<td>o</td>
<td>E_x(t)</td>
<td>e</td>
<td>E_x(t)</td>
</tr>
<tr>
<td>PR2</td>
<td>e</td>
<td>E_y(t)</td>
<td>e</td>
<td>E_x(t)</td>
<td>o</td>
<td>E_x(t)</td>
</tr>
<tr>
<td>PR3</td>
<td>e</td>
<td>E_y(t)/\sqrt{2}</td>
<td>o</td>
<td>E_x(t)/\sqrt{2}</td>
<td>e</td>
<td>E_x(t)/\sqrt{2}</td>
</tr>
<tr>
<td>PR4</td>
<td>o</td>
<td>E_y(t)/\sqrt{2}</td>
<td>e</td>
<td>E_x(t)/\sqrt{2}</td>
<td>o</td>
<td>E_x(t)/\sqrt{2}</td>
</tr>
</tbody>
</table>

| Field @ Z | \(\frac{1}{2}E_y(t)e^{-i\phi_{y}(x,y)}\) | \(\frac{1}{2}E_y(t)e^{-i\phi_{y}(x,y)}\) | \(\frac{1}{2}E_x(t)e^{-i\phi_{x}(x,y)}\) | \(\frac{1}{2}E_x(t)e^{-i\phi_{x}(x,y)}\) |
(OPDs) between each pair of ray configurations may contribute to the fringes. For instance, Ray1 and Ray2 have the same o-light path in PR1 and PR2 but disjunct optical paths in PR3 and PR4, so the OPD between them should vary along the x-axis. So should their interference. Similarly, ray configurations 1&3, 1&4, 2&3, 2&4, and 3&4 may produce five independent sets of fringes, along y, x-y, x+y, y, and x directions respectively. To see how these come about, we start from an incident ray with its electric field decomposed sequentially through the prisms in terms of the optic axis orientation inside each prism, and then we get four linearly polarized rays (i.e., Ray1 to Ray4). The components of the fields along the axis of the analyzer are transmitted. The details of each ray configuration is tabulated in Table 2.1, where $E_x(t)$ and $E_y(t)$ represent the time-dependent complex amplitudes of the incident light field along the x and y axes, respectively. $\phi_1$, $\phi_2$, $\phi_3$, and $\phi_4$ denote the cumulative change in phase experienced by Ray1 to Ray4 as they pass through the prisms, and the differences $(\phi_i-\phi_j)$ between each pair of them are the OPDs. The total irradiance on the image plane, $I = |E|^2$, is therefore

$$I = \left\langle \left[ \frac{1}{2} E_y(t)e^{i\phi_1} - \frac{1}{2} E_y(t)e^{i\phi_2} + \frac{1}{2} E_x(t)e^{i\phi_3} + \frac{1}{2} E_x(t)e^{i\phi_4} \right]^2 \right\rangle$$

$$= \frac{1}{4} \left[ \left\langle |E_x|^2 \right\rangle + \left\langle |E_y|^2 \right\rangle - \left\langle |E_y|^2 \right\rangle e^{i(\phi_2-\phi_1)} + c.c. \right] + \left\langle |E_x|^2 \right\rangle e^{i(\phi_4-\phi_3)} + c.c. \right]$$

$$+ \left[ \left\langle E_x^* E_y \right\rangle e^{i(\phi_3-\phi_1)} + c.c. \right] - \left[ \left\langle E_x^* E_y \right\rangle e^{i(\phi_4-\phi_1)} + c.c. \right] + \left[ \left\langle E_x^* E_x \right\rangle e^{i(\phi_4-\phi_2)} + c.c. \right] - \left[ \left\langle E_y^* E_y \right\rangle e^{i(\phi_3-\phi_2)} + c.c. \right]$$

$$= \left[ \left\langle E_x^* E_x \right\rangle e^{i(\phi_4-\phi_2)} + c.c. \right] - \left[ \left\langle E_y^* E_y \right\rangle e^{i(\phi_3-\phi_2)} + c.c. \right]$$

(2.4)

where the angular brackets are time averages, the asterisk denotes the complex conjugate and c.c. stands for the complex conjugate of the preceding term. Each bracketed term except the first (the intensity background) forms a fringe due to the spatial variation of...
the phase terms. Recalling the definition of Stokes parameters [16] and making the substitutions

\[
\langle |E_x| \rangle + \langle |E_y| \rangle = S_0, \quad \langle |E_z| \rangle = \frac{1}{2}(S_0 + S_i), \quad \langle E_x E_y \rangle = \frac{1}{2}(S_0 - S_i), \quad \langle E_x^* E_z \rangle = \frac{1}{2}(S_z + iS_i)
\]

into Eq.2.4, we get

\[
I = \frac{1}{2} S_0 - \frac{1}{4} (S_0 - S_i) \cos(\varphi_2 - \varphi_1) + \frac{1}{4} (S_0 + S_i) \cos(\varphi_4 - \varphi_3)
\]

\[
+ \frac{1}{4} \Re \left\{ (S_2 + iS_1) \left[ e^{i(\varphi_1 - \varphi_2)} - e^{i(\varphi_1 - \varphi_3)} + e^{i(\varphi_4 - \varphi_1)} - e^{i(\varphi_4 - \varphi_2)} \right] \right\}
\]

(2.5)

where \(\Re \{ \cdot \}\) takes the real part of a complex number. If we temporarily go ahead and adopt the on-axis plane-wave assumption, then the phase relations between \(\varphi_1\) through \(\varphi_4\) can be written as

\[
\varphi_2(x, y) - \varphi_1(x, y) = \varphi_4(x, y) - \varphi_3(x, y) = 2\pi U x,
\]

\[
\varphi_1(x, y) - \varphi_3(x, y) = \varphi_2(x, y) - \varphi_4(x, y) = 2\pi U y,
\]

\[
\varphi_4(x, y) - \varphi_1(x, y) = 2\pi U (x - y), \varphi_2(x, y) - \varphi_3(x, y) = 2\pi U (x + y),
\]

(2.6)

where \(U = 2 \cdot (n_0 - n_e) \cdot \tan(\theta)/\lambda\). Plugging Eq.2.6 into Eq.2.5, the intensity pattern simplifies to the form given in Eq.2.1, verifying the assumptions taken in the Mueller calculus approach. In reality, however, the OPDs are not exactly linear due to the reasons discussed in Sec.2.2.2. Considerable deviations from linearity are possible, in which case the cancellation of \(S_0\) between the second and third terms in Eq.2.5 cannot occur, and errors will leak into the \(S_1\) component if we continue to use Eq.1. We must really calculate \(\varphi_1\) to \(\varphi_4\) to know the practical OPDs.

The advantages of the model used here are: (a) the entire image pattern is determined by the four individual ray configurations, so that it is now possible to use ray-tracing to evaluate such effects as BS, BD, phase hybridization, etc.; (b) complicated imaging systems can be modeled with little increase in difficulty for the analysis; and (c) the use
of raytracing tools provides methods for other high-level optimization, such as maximizing energy transmission and achromatizing for broadband applications.

2.3 Simulation and Optimization

2.3.1 Optimum focus

Since multiple parameters (e.g., pupil size, field coordinates) are involved in the discussion, we will first investigate the system with a given pupil for the on-axis field, and then find a proper pupil size and apply the optimum condition to a large field of view. Throughout this paper, we assume the scene is at infinity and model the lens as being aberration-free, 50 mm focal length, and 15 mm in diameter so that the diffraction limited spot on the FPA is \( \sim 2.5\mu \text{m} \) in radius. The simulation model contains calcite prisms, each with an apex angle \( \theta \) of 1.5º, a clear aperture of 12 mm \( \times \) 12 mm, and a thickness of 500 \( \mu \text{m} \) at the base. The operating wavelength is taken to be 0.633\( \mu \text{m} \). The distance \( z' \) from the lens to the back surface of PR4 is defined as the image distance. Unless stated otherwise, all the following discussions will refer to the image plane. The on-axis field \((h_x, h_y) = (0,0)\) is selected as the initial setup for optimization and the r.m.s. spot radius versus \( z' \) is defined as the merit function. The objective for imaging optimization is to minimize the image spot size below the detector resolution. This is also necessary to diminish the phase hybridization.

Figure 2.3(a) shows three typical spot diagrams at various \( z' \) of 50.18 mm, 50.2491 mm and 50.33 mm, respectively. Because of the BS effect, one image spot is made up of four independent spots, colored by red, yellow, blue and green for Ray 1 to
Figure 2.3 (a) The spot diagrams (square grid) at $z' = 50.18\text{ mm}$, $z' = 50.2491\text{ mm}$, $z' = 50.33\text{ mm}$, respectively. (b) The merit function of r.m.s. image spot radius versus $z'$ for Ray 1 to Ray 4, respectively. The inset is the enlarged part at the bottom. For comparison, a similar curve has been plotted for the case of replacing the set of 4 calcite prisms with a glass plate of equal thickness and index of refraction equal to the ordinary index of calcite.
Ray 4 respectively. These four image spots are shifted relative to each other. Figure 2.3 also shows an obvious asymmetry in the image spot shapes, incurred by the physical asymmetry of the prisms. However the dominant factor in determining the image spot size and its shape is the prism position $z'$, as illustrated in Fig. 2.3(b). At $z'=50.2491\,\text{mm}$, the r.m.s. spot size reaches its smallest radius of $1.8\,\mu\text{m}$, which is well below the detector resolution for most detector arrays. A lucky feature of Fig. 2.3(b) is that all four ray configurations show very similar changes as a function of $z'$ (the same minimum and same slopes). Hence, a trade-off in optimizing the four images is avoided. Additionally, the centroid of the image spots at the best focus is slightly shifted (by less than $2\,\mu\text{m}$) from the paraxial image point, indicating a small BD effect for the on-axis field.

2.3.2 Pupil size dependence

The pupil size which defines the marginal ray angle is another key factor to characterize the system. It is anticipated that the edge rays through the pupil will suffer more BS and BD inside the prisms than the central rays. By fixing the other system parameters, we can obtain the pupil size dependence on the merit function, the result of which is shown in Fig.2.4 (for an on-axis field and $z'=50.2491\,\text{mm}$). Figure 2.4 (a) shows the spot diagrams for the pupil diameters 6mm, 20mm, and 30mm. If diffraction effects are ignored, then a smaller image spot corresponds to a smaller pupil. The spot asymmetry is also worsened with increasing pupil size, whereas the image shift is not significantly changed. If diffraction effects are included in the simulation, then competition happens between the diffraction
Figure 2.4 (a) The spot diagrams with the pupil diameter $d_p$ equal 6mm, 20mm and 30mm, respectively, with diffraction effects ignored. (b) The r.m.s. image spot radius versus the pupil diameter $d_p$. The system is at the optimum focus for the on-axis field.
limit and the geometric image spot size; a bigger pupil reduces the size of Airy disk but increases the BS and BD effects. We use $r_d = 0.61\lambda/\text{NA}$ to approximate the diffraction limit (though real diffraction should consider the prism geometry) and take the maximum of $r_d$ and the raytracing results. Figure 2.4(b) plots the final spot radius versus the pupil diameter $d_p$, the optimum image spot size occurring for a pupil diameter of 8-15mm (we keep using $d_p=15mm$ in the following calculation for a maximum throughput). Note that the optimum focus position will vary with different choices of pupil diameter.

2.3.3 Field dependence

Due to the lens, different field angles will experience different BS and BD inside the prisms. Object points at larger field angles are imaged farther off-axis, where more BS and BD occur in the prisms. Retaining the specifications used in Fig.2.4, Figure 2.5(a) illustrates the sensitivity of the spot size on the field angles. Looking at the first row of Fig. 2.5(a), we see that the image spot becomes larger as the field angle increases in the x-direction. The next row of three examples gives results for field angles symmetrical about the z-axis. The difference in spot shape can be attributed to the asymmetry of the prisms again. Compared to the on-axis field, the off-axis fields have comatic image spots. The spot radius increases to a maximum of about 10µm at the maximum field angle given by $h_x=7^\circ$, $h_y=0^\circ$. This value is still reasonable for typical applications. Fig. 2.5(b) illustrates the variation in image spot size with focus position (as in Fig. 2.3(b)), for all of the six field angles listed, and for both of ray configurations, 1&3 (Ray 2&4 are not shown because they correspond very closely to 1&3). Again, we see that all configurations share the same optimum focus position, eliminating any competition.
Figure 2.5 (a). The spot diagrams for different field points (at the optimum focus), \(i.e.,\) 
\((0,0), (3^\circ,0),(7^\circ,0),(-7^\circ,0),(0,7^\circ) \) and \((0,-7^\circ)\) for \(z' = 50.2491 \text{ mm} \) and \(d_p = 15 \text{ mm}\). (b) 
The merit function of spot radius versus \(z'\) at different fields (by different line shapes), 
\((0,0), (7^\circ,0), (-7^\circ,0), (0,7^\circ) \) and \((0,-7^\circ)\), for Ray1 and Ray3.
among different angles inside the FOV. In other words, it is only necessary to consider the on-axis field position to locate the optimum focus.

In addition to blurring, image distortion is also a concern. Due to the non-uniform BS and BD effects across the FOV, the image spots of different field coordinates will be unequally shifted. To estimate the distortion, we average the image centroids of Ray1 to Ray 4, with an equal weight for each, to obtain the real image position. The intersection point of the chief ray (the chief ray of the optical system with no prisms present) with the image plane is used as the reference image point. The shift between the real image position and the reference point indicates the distortion. At the optimum focus ($z' = 50.2491$ mm), with $d_p=15$ mm and across a FOV of $\pm7^\circ$, the maximum distortion is less than $4.5\mu$m (occurring at the edge of the FOV) which is not a problem for the polarimeter design considered here.

Finally, we can note that in each of the spot diagrams of Fig. 2.3(a), 2.4(a) and 2.5(a), Ray1 and Ray2 are always closer to each other than are Ray3 and Ray4. Since Rays 1&2 originate from the same ray path in PR$_1$ and PR$_2$, we can conclude that PR$_1$ and PR$_2$ generate the majority of the BS and BD effects seen at the image. Thus, when optimizing the system, we find that minimizing the apex angle used in prisms PR$_1$ and PR$_2$ is more important than minimizing the angle used in PR$_3$ and PR$_4$. This information is especially important in the middle/long infrared, where it is necessary to use a much larger prism angle.

2.3.4 OPD mappings
As discussed in Sec. 2.2.1, the spatial linearity of the phase is essential for the device operation, whereas the BS and BD effects and the off-axis rays will introduce errors in the phase mappings, eventually causing errors in the reconstructed polarimetric images. To evaluate the phase errors, we trace through all the ray configurations to obtain $\phi_1$ to $\phi_4$. In the simulation, we uniformly sample the FOV by a square grid, and for each field point, we average the phase $\phi_i$ ($i=1,2,3,4$ for Ray 1, 2, 3, 4) across the pupil, resulting in the mean phase $\overline{\phi}$. The optimum focus is again used, since only if the image spot is confined within a single detector can the phase mixture from adjacent image spots be ignored. Also, equal weight is assumed across the pupil and other specifications are kept as those in Fig. 2.5(b) with $z' = 50.2491$ mm. The subtraction between each pair of $\overline{\phi}_i$ and $\overline{\phi}_j$ gives the real OPDs, which are normalized by $\lambda$ and shown in Fig. 2.6.

<table>
<thead>
<tr>
<th>Value mm$^{-1}$</th>
<th>$U_{21}$</th>
<th>$U_{43}$</th>
<th>$U_{41}$</th>
<th>$U_{23}$</th>
<th>$U_{13}$</th>
<th>$U_{24}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deviation</td>
<td>0.4</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Table 2.2. The fitted slopes and deviation percentages of different OPD mappings in Fig. 5

It is seen that, under the modeled conditions, the phase linearity in all six mappings generally remains. Only a small curvature occurs at the edges of the image simply because of the enhanced BS and BD effects at larger field angles. Linearity guarantees the validity of Eq. 1 and the reconstruction algorithm. However, small variations are observed unpredictably distributed across the image, which slightly weakens the linearity uniformity. These variations can introduce small errors in the reconstructed Stokes images. To compare with the ideal linear phase, we select the central lines in each plot, and fit the data with an approximate slope $U_{ij}$, where the subscripts denotes the pair of
Figure 2.6 The OPD mappings (normalized by \( \lambda \)) of the fringe components between all six ray configuration pairs.
ray configurations. The theoretical value is $U_{\text{theory}} = 2(n_n - n_i)\tan(\theta)/\lambda = 14.065\text{mm}^3$ for this system. The phase deviation percentage defined by $\text{Deviation}\% = \left|U_i - U_{\text{theory}}\right|/U_{\text{theory}} \times 100\%$ is calculated and tabulated in Table 2.2 which shows small values in the deviation for all six mappings. Importantly, since $U_{21}$ and $U_{43}$ are almost equal, then $(\varphi_2 - \varphi_1)$ and $(\varphi_4 - \varphi_3)$ are also equal, so that the $S_0$ component of the second and third terms in Eq.2.5 will very nearly cancel, which is one of the condition necessary to go from Eq.2.5 to back to Eq.2.1 (i.e., validating Mueller theory results). The difference between $U_{13}$ and $U_{24}$, no matter how great will not affect measurement because the $y$-dependent modulation is not used in the image reconstruction (it is automatically filtered out in the reconstruction) [2]. As for $U_{41}$ and $U_{23}$, a small phase deviation (as defined above) in them can be compensated-for in the calibration, which is an approach that will be reported elsewhere.

2.3.5 Large apex angle

The apex angles $\theta_{1,2}$ and $\theta_{3,4}$ of the prism are parameters for the system optimization. Since the spatial frequency $U = 2(n_n - n_i)\tan(\theta)/\lambda$ of fringes characterizes the bandwidth of each signal channel (Eq.2.1), increasing the apex angle of prism will widen the bandwidth. A large apex angle can also compensate for small values of birefringence in the prisms. However, all the above results imply that the increase in the apex angle will be limited by the aberrations introduced due to these prisms. A larger apex angle causing more refraction will generate more BS and BD effects. Figure 2.7 illustrates such effect by making a simple comparison between apex angle values of 1.5° and 4.5° in a calcite system (both for on-axis field and at optimized focus). The minimum image spot, $>20\mu\text{m}$
in the latter system of $\theta = 4.5^\circ$, would result in a significant loss of resolution for a detector with a pixel pitch of less than $\sim 20\mu m$.

![Figure 2.7](image)

Figure 2.7 The spot diagrams of different apex angles of a calcite system, both for on-axis field and at optimized foci. (a) $\alpha = 1.5^\circ$ (b) $\alpha = 4.5^\circ$, with r.m.s spot size $> 20\mu m$.

### 2.3.6 Approach for designing a prismatic imaging polarimeter

The previous sections have discussed how the resolution at the image plane depends on the system parameters for the polarimeter. The geometric model makes clear that polarimetry errors in the instrument will be dominated by the increase in the spot size rather than in distortion. This permits a straightforward approach to using the geometric model to design the polarimeter:

1) The basic system requirement is a minimum spatial resolution in the reconstructed Stokes images, defining the required bandwidth $U$ for the Stokes images.

2) Select a detector and lens to achieve a desired FOV and pupil diameter.
3) Select a birefringent material and use \( \theta = \arctan\left( \frac{U\lambda}{2\Delta n} \right) \) to obtain the prisms’ apex angles.

4) Using the geometric model, estimate the blurring at the image to locate the cutoff frequency in the measurement (where the MTF falls below the noise spectrum).

5) If the cutoff frequency is less than 1.5U, then the Stokes images will suffer significant error, requiring a modification of the system parameters. When this happens, we can select a material with a higher birefringence, reduce the FOV, or increase the system f/#.

When trading off FOV vs. f/#, it is useful to consider that for typical systems the design is much more sensitive to f/# than FOV.

6) Finally, it is also necessary to verify that the distortion present in the system is small enough to prevent excessive aliasing artifacts due to deviation from phase linearity.

2.3.7 Conclusions

In order to study the BS and BD effects of the birefringent prisms on the performance of a prismatic imaging polarimeter, a useful approach is to construct a geometric imaging model within an optical design software package. This enables a detailed simulation and analysis of the imaging performance. Previous publication of the results from the prismatic polarimeter [2] showed reconstruction artifacts, which raises the question of whether the BS and BD effects of the prisms prevented accurate system performance. We have been able to show that for a standard imaging system the use of calcite birefringent prisms will produce little degradation in the reconstructed Stokes images. However, an
imaging polarimeter requiring larger apex angles may experience serious reconstruction
error, for which the geometric model provides a convenient means of re-designing the
system to produce Stokes images at the desired resolution.

2.4 An MWIR Snapshot Imaging Polarimeter Using YVO₄

2.4.1 Material issue

The birefringence of the prism material is a key parameter to determine the prism
design and the device performance. The relation for deciding the required
birefringence is derived by the prism geometry and sampling theory as,

\[
\frac{2 \cdot \Delta n \cdot \tan \alpha}{\lambda} = \frac{1}{N \cdot dp}
\]  

(2.7)

where \( \Delta n \) is the birefringence, \( \alpha \) is the prism apex angle, \( \lambda \) is the wavelength, \( N \) is
the CCD pixel number per fringe, and \( dp \) is the pixel spacing. When \( N \) increases,
the signal bandwidth is reduced and thus for the resolution. \( N=3 \) is the best choice,
4-6 are also good. Figure 2.8 illustrates the required birefringence as a function of
prism angle with different \( N \). In the MWIR, to achieve a good sampling (spatial
resolution) with a small \( N \), one can either use large \( \Delta n \) or a big apex angle. But a
birefringence over 0.2 is approximately the minimum in a decent design. One
cannot increase the apex angle too much, since the image spot divergence due to
the refraction by the prisms will make the image spot size unacceptable for a
normal CCD array (Fig.2.7). Therefore, for the MWIR, a high birefringence
material is strongly demanded. Traditional birefringent crystals, e.g., calcite,
quartz, and LiNbO₃ are not satisfactory, because of both their low $\Delta n$ in the MWIR region and limited transmission (Table 2.3).

![Graph](image)

Figure 2.8 The required birefringence $\Delta n$ versus the prism angle $\alpha$ at $\lambda=4\mu$m with $N=3$, 4, 6 and 10.

Table 2.3 Traditional birefringent crystals in MWIR

<table>
<thead>
<tr>
<th>Materials</th>
<th>Birefringence</th>
<th>Transmission</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calcite</td>
<td>&lt;0.17</td>
<td>0.35-2.3µm</td>
</tr>
<tr>
<td>Quartz</td>
<td>&lt;0.015</td>
<td>0.3-3.5µm</td>
</tr>
<tr>
<td>LiNO₃</td>
<td>&lt;0.1</td>
<td>0.4-5.2µm</td>
</tr>
<tr>
<td>Rutile</td>
<td>&gt;0.22</td>
<td>0.4-5µm</td>
</tr>
</tbody>
</table>

The rutile ($\text{TiO}_2$) crystal is a possible candidate because it transmits out to 5µm and posses a high birefringence of over 0.22 in MWIR. Recently, the yttrium
vanadate (YVO₄) crystals have drawn much attention due to its comparatively high birefringence (Δn~0.2) in the visible and near infrared (NIR). We carried out a complete and accurate measurement of the birefringence of this material in the MWIR.

### 2.4.2 Birefringence of YVO₄

Figure 2.9 (a) shows the schematic of the experimental setup, i.e., the so called channel spectra measurement, which consists of a white light source, two polarizers (P1, P2) with their transmission axes parallel, and a Fourier Transform Infrared spectrometer. Fig.2.9(b) is the picture of the laboratory setup. The use of FTIR allows a one-time measurement over a wide spectral range, with high resolution. The test sample, a retarding plate, is placed between P1 and P2 as shown in Fig.2.9. P1 and P2 are both BaF₂ wire grid polarizers from Thorlabs Inc, with transmission of 0.4-12µm. The parallel alignment of P1 and P2 is made by rotating 90° from the zero transmission configuration without the sample. The detector is an MCT detector with a high SNR at 2.5-17µm.

Starting from the Mueller calculus, the spectral intensity can be written as

$$I(\lambda) = A \cos(2\pi \cdot \Delta n \cdot d / \lambda) + B$$

which is an intensity spectrally modulated. A is the constant amplitude of modulation, B is a constant, d is the sample thickness and \(\lambda\) is the wavelength. \(\Delta n\) may have dispersion. The constant A reaches maximum when the optic axis of the sample is rotated 45° with respect to the transmission axes of the polarizers. In reality, we first align the optic axis of the sample with the transmission axes of P1 and P2 to obtain the minimum signal, and then rotate the sample 45° to maximize spectral modulation. In the wave number
dependence, the product of $\Delta n$ and $d$ determines the fringe period of a local spectral region. The relationship of satisfying the peaks and valleys of the spectrum is

$$\Delta n \cdot d = M \cdot \lambda,$$

$$\Delta n \cdot d = (M + 1/2) \cdot \lambda.$$

(2.9)
where $M$ is an integer. Knowing the positions of peaks and valleys and $M$ enables us to obtain the $\Delta n$.

Three samples (plane parallel plates) of YVO$_4$ single crystal (zircon tetragonal symmetry), grown with the Czochralski method and fabricated by Casix Inc, are tested at the room temperature. The optic axis (also slow axis) of each sample is in the surface plane. The front and back surfaces are polished with no AR coating so no external birefringence is introduced, and this results in observable parallel plate interference at a low SNR. A micrometer with a precision of 1$\mu$m is used to measure the thickness of the YVO$_4$ plates, which gives $d_1=0.737\text{mm} \pm 1\mu$m for the sample No.1, $d_2=0.735\text{mm} \pm 1\mu$m for the sample No.2, and $d_3=0.737\text{mm} \pm 1\mu$m for the sample No.3. As the basic property, the
transmission of the material was measured beforehand. The thicknesses of the plate samples No.1 through No.3 are approximately the same value of a real prism block being used in a polarimeter. Figure 2.10 shows the typical transmission curve measured on the sample No.1. Up to 5.3 µm a high transparent (transmission>60%) band which ensures the use of YVO₄ prisms in the MWIR is seen. The loss of transmission in the 6-8µm region is not considered.

Figure 2.11 The channel spectrum of the YVO₄ sample No.1, with the thickness of 737µm.
The channeled spectrum of the YVO₄ plate No.1 based on the configuration of Fig.2.9 is shown in Fig.2.11. The spectrum is normalized by a background measurement made without the sample. We can see obvious fringes from 2.15\(\mu\)m to 7\(\mu\)m, which convincingly shows the existence of birefringence. The optimal range with well-defined fringes falls in the MWIR (middle figure), enabling a high accuracy of the measured data. Strong oscillations occur below 2.5\(\mu\)m due to the low SNR of the detector and the parallel plate interference effect. Here, we determine the peak/valley positions using interpolation method to find the most central points of a peak/valley. The fact that the fringes die at 5\(\mu\)m is consistent with the absorption feature in Fig.2.10. Interestingly, beyond 6\(\mu\)m, the fringes revive within 1 \(\mu\)m. In order to fit the integer \(M\) accurately, we choose the optimal MWIR range, to count fringe period in the wave number dependence. The fringe period is compared between adjacent peaks, which indicates that around 0.25\(\mu\)m\(^{-1}\), \(i.e., \lambda\) = 4\(\mu\)m, the spectrum has a constant period. This indicates a non-dispersion region around \(\lambda\) = 4\(\mu\)m and gives a \(\Delta n\) of 0.21956 by the relation \(\Delta n = 1/[d(v_1 - v_2)]\) at 3.95938\(\mu\)m (\(v_1\) and \(v_2\) are the wave numbers of two adjacent peaks). Plugging this value into Eq.2.9 results in \(M = 41\) at 3.95938\(\mu\)m as denoted in Fig.2.11, and consequently other order numbers on the rest of the peaks and valleys. The complete results of the birefringence measurements are tabulated in Table 2.5. Similar procedures were implemented on the sample No.2 and No.3 and all the birefringence results of the three samples are plotted in Fig.2.12. It is seen that a high birefringence over 0.21 appears throughout MWIR for the samples No.1-3. This high birefringence proves that YVO₄ is an excellent material for the prisms in the MWIR imaging polarimeter. It is also believed that many relevant MWIR applications which involve
polarization optics will benefit from this material as well. Moreover, the measured $\Delta n$ in the MWIR is approximately $0.01\sim0.02$ higher than the predicted values by the Sellmeier equation which is based on the measurement from visible to $1.60\mu$m [11]. The difference suggests new equations for the refractive indices of $\text{YVO}_4$ in MWIR. It is seen that the dispersion of the birefringence is small which implies that $\text{YVO}_4$ may also be a good material candidate for the MWIR channeled spectral polarimeters [17].

![Figure 2.12 The birefringence of the $\text{YVO}_4$ sample No.1 through No.3. The solid circles, empty circles and solid triangles represent the samples No.1, No.2 and No.3, respectively.](image)

There are three aspects in the measurement which might introduce errors: the tilting of the sample which changes the effective optical path length, the non-dispersion
treatment around 4µm, and the peak/valley position reading error. We checked the measurement within a tilting angle of 10º of the sample. The channel spectrum does not show observable difference. This is because of the slight difference of cosine function with small angles. The non-dispersion treatment works accurately and only brings a birefringence error in order of ~10⁻⁵. One can assume a dispersion term Δ²n in Eq.2.9 and estimate Δ²n by solving Eq.2.9 with the data around 4µm and M=41, which results in this order. As for the reading error of the peak position, one unit change in the integer M allows an error of 0.25µm⁻¹ in reading the fringe period in the wave number space. But in reality, the maximum reading error in the well-defined fringe region (3-5µm) is surely within 0.02µm⁻¹, which is more than ten times smaller than the tolerance. The interpolation method for reading peaks/valleys works well since the birefringence curve in Fig.2.12 shows a smooth continuity even in the high oscillating wavelength range, but the accuracy in this region is lower than that in the MWIR. The good consistency of the measured results between the samples No.1 through No.3 in Fig.2.12 further indicates the measurement reliability, so the results can be trustable and be as precise to 3 decimal places.

In conclusion, we have for the first time measured the high birefringence of the YVO₄ crystal in the MWIR. The use of a FTIR together with the channel spectra technique provides a quick and accurate measurement. This, along with the wide transmission range, YVO₄ is recommended as an ideal prism material for MWIR polarimeters and other possible applications in the MWIR.
2.4.3 Demonstration of a YVO$_4$ SIP

A real system using the YVO$_4$ crystal is built and tested [18]. Figure 2.13 shows a view of the experimental setup, where a set of 3.1° YVO$_4$ prisms are tested in a relay system with a 320x240 InSb focal plane array (FPA). In reality, we rotate the prisms group by approximately 21°. The object is a spherical light bulb painted with black Krylon high temperature spray paint. The raw image of the blackbody radiation at 4.5µm is shown in Fig. 2.14, where clear fringes are seen across the object. Figure 2.15(a) shows the $\tilde{S}$

Table 2.4 The birefringence of YVO$_4$ sample No.1 at the room temperature

<table>
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<tr>
<th>$M$</th>
<th>$\lambda$ (µm)</th>
<th>$\Delta n$</th>
<th>$M$</th>
<th>$\lambda$ (µm)</th>
<th>$\Delta n$</th>
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</tbody>
</table>

$d = 737 \pm 1$µm
reconstruction, illustrating the featured pattern of a spherical surface. For a rationally symmetric curved surface, its blackbody emission turns to be S-polarizing dominant and the degree of polarization (DoP) increases from on-axis to off-axis areas since the refracting angle increases. Due to the Cartesian definition of coordinate in the Stokes vector, the Stokes images of the bulb are not rotational symmetric but bi-axial symmetric (Fig.2.15(a)). The $S_3$ component is close to zero since the black body emission is mostly linear polarizing. The DoLP image in Fig.2.15 (b) reveals the rational symmetry of the object.

Figure 2.13 Image of the experimental setup for the MWIR YVO$_4$ SIP.

Fig. 2.14 Raw image of a spherical light bulb.
Figure 2.15 (a) Reconstructed $\tilde{S}$. (b) Degree of linear polarization of the image in Fig. 2.14.
CHAPTER 3

SAVART PLATE SNAPSHOT IMAGING POLARIMETERS

3.1 Introduction

The Savart Plate (SP) SIP provides an alternative way to create the required linear retardance to validate Eq.2.1. The birefringent prisms form the linear retardance at the image plane while the SP can produce the same linear retardance in the collimated space of a 4f imaging system [3,4]. The 4f imaging system provides a pair of 2D optical Fourier transformations taking place at the back focal planes of the first and second lenses. By placing the SPs at the first Fourier plane, a small shift is introduced to the wave-front that propagates to this plane. The second Fourier transformation converts the shift to a linear phase term attached to the final complex field. Compared to the prism SIP, the SP system is easier to mount, align and fabricate, while sacrificing the device volume compared to the prism system. We want to know if the SP system can be reduced in size to be a viable candidate for many applications that favor a compact or portable device. Also, we hope to implement raytracing analysis for system evaluation.

In this chapter, we develop a wave optics model based on which the idea that the collimated lens of the 4f system can be removed. Then, we apply raytracing analysis to the evaluate system that has a minimum size. A comparison between the SP and prism SIPs is made. Finally, we experimentally demonstrate the reduced SP SIP by fabricating a system for evaluation.
Figure 3.1 The optical layout of a SP SIP. The insert I illustrates the detailed configuration of the SP group. The colored solid lines represent the wave-vector of the split wave-fronts inside the SP group from a common incident wave. The double arrowed solid lines, each tilting at 45º with respect to the edge of the surfaces, depict the optic axes of the calcite plates. The insert II shows the four emerging waves at the back surface of SP 2. The “e” (extraordinary) and “o” (ordinary) letters denote the polarization consequences through the system.
3.2 Wave-Optics Model

Figure 3.1 schematically shows the optical layout of a modified SP SIP, which contains two lenses (L1 and L2) sandwiching two SPs (SP1, SP2), a half-wave plate (HWP) and an analyzer (A) (we refer the structure SP1/HWP/SP2 as the SP group). Instead of placing the SP group at the back focal plane of L1, we assume an unfixed distance $d_1$ between L1 and SP1 and so for $d_2$ between L2 and SP2. For simplicity, we ignore the thickness of the SP group. The HWP orientated at 22.5º rotates electric field vector existing SP1 by 45º. The object plane is at the front focal plane of L1 and the image plane is at the rear focal plane of L2.

For convenience, we temporally remove the SP1/HWP/SP2/A group to formulate a general model for the imaging process. We assume an incoherent monochromatic case with the center wavelength $\lambda$. The Frensnel diffraction equation is used to derive the image intensity at the image plane. Then, we obtain the complex amplitude of the electric field at the rear focal plane of L2 as (Appendix A),

$$U_i(x_i, y_i) = \frac{e^{(jkd_2)}}{j\lambda f_2} \frac{e^{jkd_2}}{FT\{U_s(x_s, y_s)\}} \frac{s_{x_i y_i}}{\lambda f_2, \lambda f_2}$$

(3.1)

where $FT\{}$ denotes the 2D forward Fourier transformation (FT), $U$ represents the complex amplitude, $i$ denotes the image plane, $s$ in the subscript represents the plane where the SP group is cloated, and $x$ and $y$ are the coordinates. We see that the
complex amplitude at the image plane is the FT of that at the S plane times a quadratic phase term and constant. Also, we have (Appendix A)

\[
FT\{U_s(x_s,y_s)\}_{f_x,f_y} = FT\{U_i(x_i,y_i)\}_{f_x,f_y} \cdot H(f_x,f_y) = FT\{U_i(x_i,y_i)\}_{f_x,f_y} \{e^{jkd} \exp[-j\pi\lambda d_i(f_x^2 + f_y^2)]\}
\]

(3.2)

where \(f_x\) and \(f_y\) present the spatial frequency along \(x\) and \(y\) axes, \(t\) denotes the intermediate plane after \(L_1\), and \(H(f_x,f_y)\) is the response function in free space. \(U_i\) can be written as (Appendix A),

\[
U_i(x_i,y_i) = \frac{e^{j(kd_i)}e^{j(\frac{\pi}{\lambda^2}(x_i^2 + y_i^2))}}{j\lambda f_i} \cdot FT\{U_o(x_o,y_o)e^{j\pi\lambda d_i(x_o^2 + y_o^2)}\}_{x_o,y_o,\frac{\lambda_x f_i}{\lambda f_i},\frac{\lambda_y f_i}{\lambda f_i}} \cdot \exp[-j\frac{\pi}{\lambda f_i}(x_i^2 + y_i^2)]
\]

\[
= e^{j(kd_i)} \cdot \frac{FT\{U_o(x_o,y_o)e^{j\pi\lambda d_i(x_o^2 + y_o^2)}\}_{x_o,y_o,\frac{\lambda_x f_i}{\lambda f_i},\frac{\lambda_y f_i}{\lambda f_i}}}{j\lambda f_i} \cdot \exp[-j\frac{\pi}{\lambda f_i}(x_i^2 + y_i^2)]
\]

(3.3)

where \(o\) denotes the object plane. Plugging Eq.3.3 into Eq.3.2 gives the total image complex field as,

\[
FT\{U_s(x_s,y_s)\}_{x_s,y_s,\frac{\lambda_x f_i}{\lambda f_i},\frac{\lambda_y f_i}{\lambda f_i}} = FT\{U_i(x_i,y_i)\}_{x_i,y_i,\frac{\lambda_x f_i}{\lambda f_i},\frac{\lambda_y f_i}{\lambda f_i}} \cdot H(x_i,\frac{\lambda_x f_i}{\lambda f_i},y_i,\frac{\lambda_y f_i}{\lambda f_i}) = FT\left(\frac{e^{j(kd_i)}}{j\lambda f_i} \cdot \frac{FT\{U_o(x_o,y_o)e^{j\pi\lambda d_i(x_o^2 + y_o^2)}\}_{x_o,y_o,\frac{\lambda_x f_i}{\lambda f_i},\frac{\lambda_y f_i}{\lambda f_i}}}{\cdot \exp[-j\frac{\pi}{\lambda f_i}(x_i^2 + y_i^2)]}\right)
\]

\[
= e^{j(kd_i)} U_o(-\frac{f_i}{f_x},\frac{f_i}{f_y}) e^{j\frac{\pi}{\lambda f_i}\left(\frac{f_i}{f_x}\right)^2} \cdot \exp[-j\pi\lambda d_i\left(\frac{x_i}{\lambda f_x} + \left(\frac{y_i}{\lambda f_y}\right)^2\right)]
\]

\[
= e^{j(kd_i)} U_o(-\frac{f_i}{f_x},\frac{f_i}{f_y}) e^{j\frac{\pi}{\lambda f_i}\left(\frac{f_i}{f_x}\right)^2} \cdot \exp[-j\pi\lambda d_i\left(\frac{x_i}{\lambda f_x} + \left(\frac{y_i}{\lambda f_y}\right)^2\right)]
\]
where we can see that the final image is a reversed object magnified by the ratio of the two focal lengths and multiplied by a phase term that only vanishes when \(d_1 = f_1\) and \(d_2 = f_2\).

Now we need to add the SP group to include the operation of polarization. This step gives similar fringe decomposition as the prism SIP. Essentially, a SP is a wavefront shearing element, in which two orthogonal polarization components the ordinary (o-) and extraordinary (e-) waves experience splitting at the incident interface and then after propagating through the SP a shearing from each other [19]. The specific orientations of SP1/HWP/SP2 shown in insert I of Fig. 3.1, are designed to produce four split wave-fronts with equal and diagonal shearing distances as illustrated in insert II of Fig. 3.1. Note that the HWP at 22.5° rotates the electric field vector by 45°, enabling the second SP to split and to shear the wave-fronts exiting the first SP. The insertion of an analyzer at 45° is to display the resultant polarization interferogram. Mathematically, we can write down the complex electric field after the analyzer as the following,

\[
\tilde{U}_s(x_s, y_s; \Delta) = \cos^2 45^\circ [U_{sx}(x_s - 2\Delta, y_s) - U_{sy}(x_s - \Delta, y_s - \Delta)] \\
+ \cos^2 45^\circ [U_{sx}(x_s - \Delta, y_s + \Delta) + U_{sy}(x_s, y_s)] (\hat{e}_x + \hat{e}_y)
\]

(3.5)

where \(\Delta\) is the shearing unit associated with the SP, \(\hat{e}_x\) and \(\hat{e}_y\) are the unit vectors along \(x\) and \(y\) axes. \((\hat{e}_x + \hat{e}_y)\) represents the transmission axis of the analyzer. Eq. 3.5 describes the SP group effect exerting on the incident wave that four diagonally
shearing wavefronts are formed. The cosine square terms in Eq.3.5 comes from the
twice projection of the incident electric field vector to the analyzer’s transmission axis.

Plugging Eq.3.5 into Eq.4.4, we have,

\[
\begin{align*}
&\text{FT} \{ \hat{U}_x(x_y, y, \Delta) \} \left\{ \frac{x_y}{y_0} \frac{x}{y_0} \right\} \\
&= (\bar{e}_x + \bar{e}_y) \cdot \cos^2 45^\circ \cdot \text{FT} \{ U_{0x} \} \cdot (e^{j2\pi \frac{x}{y_0}} + e^{j2\pi \frac{y}{y_0}}) + \text{FT} \{ U_{0y} \} \cdot (1 - e^{j2\pi \frac{x}{y_0}}) \\
&= \frac{\bar{e}_x + \bar{e}_y}{\text{LinearPol}} \cdot \cos^2 45^\circ \cdot \text{FT} \{ U_{0x} \} \left( -\frac{f_1}{f_2} x - \frac{f_1}{f_2} y \right) e^{j2\pi \frac{x^2 + y^2}{y_0^2} (f_1 - d_2)} \left(1 - e^{j2\pi \frac{x}{y_0}} \right) \\
&= \frac{(\bar{e}_x + \bar{e}_y)}{\text{LinearPol}} \cdot \cos^2 45^\circ \cdot \text{FT} \{ U_{0x} \} \left( -\frac{f_1}{f_2} x - \frac{f_1}{f_2} y \right) e^{j2\pi \frac{x^2 + y^2}{y_0^2} (f_1 - d_2)} \left(1 - e^{j2\pi \frac{x}{y_0}} \right) \\
&= \left(U_{0x} \left( -\frac{f_1}{f_2} x - \frac{f_1}{f_2} y \right) e^{j2\pi \frac{x}{y_0}} + e^{j2\pi \frac{y}{y_0}} \right) \left(1 - e^{j2\pi \frac{x}{y_0}} \right) \\
\end{align*}
\]

(3.6)

Finally, by plugging Eq.3.6 into Eq.3.1, we obtain the complex electric field at the
image plane as,

\[
\begin{align*}
U_i(x, y) &= (\bar{e}_x + \bar{e}_y) e^{j2\pi \frac{x}{y_0} + d_x} e^{j2\pi \frac{y}{y_0}} \\
&= \frac{\bar{e}_x + \bar{e}_y}{\text{LinearPol}} \cdot \cos^2 45^\circ \cdot \text{FT} \{ U_{0x} \} \left( -\frac{f_1}{f_2} x - \frac{f_1}{f_2} y \right) e^{j2\pi \frac{x}{y_0}} + e^{j2\pi \frac{y}{y_0}} \left(1 - e^{j2\pi \frac{x}{y_0}} \right) \\
\end{align*}
\]

(3.7)

where each term in the bracket represents a split wave component. Note that the linear
phase terms attached to each component results from the second Fourier
transformation of the system. Importantly, we find that \(d_1\) and \(d_2\) do not have to sum
to equal \(f_1 + f_2\), because the net quadratic phase term vanishes when we calculate
the total intensity. This means: 1) we can eliminate $d_1$ and $d_2$ to minimize the device that just accommodate the SP group 2) the location of SP group is not as critical as the original design; rather it can be at any place between L1 and L2. For these reasons, we can reduce the system by a maximum factor of 2 of the overall size.

We now derive the total image intensity. We ignore the coherent terms that represent speckle noise as,

$$I_i(x_i, y_i) = U_i(x_i, y_i) U_i^*(x_i, y_i)$$

$$= \frac{1}{4\lambda^2 f_1 f_2} \left\{ U_{a_{\alpha}}^* \left( e^{i2\pi \Delta \eta \frac{A}{M_1}} + e^{i2\pi \Delta \eta \frac{A}{M_2}} \right) + U_{a_{\alpha}}^* \left( 1 - e^{i2\pi \Delta \eta \frac{A}{M_1}} \right) \right\} \left\{ U_{a_{\alpha}}^* \left( e^{i2\pi \Delta \eta \frac{A}{M_2}} + e^{i2\pi \Delta \eta \frac{A}{M_2}} \right) + U_{a_{\alpha}}^* \left( 1 - e^{i2\pi \Delta \eta \frac{A}{M_1}} \right) \right\}$$

$$= \frac{1}{4\lambda^2 f_1 f_2} \left( 2 \left| U_{a_{\alpha}} \right|^2 + \left| U_{a_{\alpha}} \right|^2 \right)$$

$$+ (U_{a_{\alpha}}^* U_{a_{\alpha}} e^{i2\pi \Delta \eta A}) + (-U_{a_{\alpha}}^* U_{a_{\alpha}} e^{i2\pi \Delta \eta A})$$

$$+ (U_{a_{\alpha}}^* U_{a_{\alpha}} e^{i2\pi \Delta \eta A}) + (-U_{a_{\alpha}}^* U_{a_{\alpha}} e^{i2\pi \Delta \eta A})$$

$$+ \left\{ 2S_0 + 2\cos(2\pi\Omega(x_i + y_i)) \right\} \left( \left| U_{a_{\alpha}} \right|^2 - \left| U_{a_{\alpha}} \right|^2 \right)$$

$$+ \frac{1}{4\lambda^2 f_1 f_2} \left( 2S_0 + 2\cos(2\pi\Omega(x_i + y_i)) \right) S_i + \frac{1}{2} \left( S_2 + iS_3 \right) 2(\cos(2\pi\Omega x_i) - \cos(2\pi\Omega y_i))$$

$$= \frac{1}{4\lambda^2 f_1 f_2} \left( \frac{1}{2} S_0 + 2\cos(2\pi\Omega(x_i + y_i)) \right) S_i$$

$$+ \frac{1}{4} \left| S_{23} \right| \cos[2\pi(2\Omega) x_i + \arg(\Phi_{23})] - \frac{1}{4} \left| S_{23} \right| \cos[2\pi(2\Omega) y_i - \arg(\Phi_{23})]$$

$$= \left( S_2 + iS_3 \right)$$

(3.8)

where

$$\Omega = \frac{\Delta}{\lambda f_2}, \quad S_{23} = S_2 + iS_3$$

In the derivation, we can see that each Stokes parameter is modulated by a various fringe component just as the prism SIP. The spectrum of (3.7) should look like Fig.3.2
(a), where there is a 45° rotation from the prism SIP’s spectrum.

![Diagram showing the spectrum of a SP SIP](image)

Figure 3.2 The spectrum of a SP SIP

Note that in Eq.3.8 (Fig.3.2(a)), the modulation magnitude of each Stokes parameter is different. For $S_{23}$, the spectral peak is half of the $S_1$ peak, leading to lower reconstruction accuracy of $S_{23}$ at the same level of noise. We can adjust the analyzer transmission axis to force an equal modulation magnitude through $S_1$ to $S_{23}$. Assume we rotate the analyzer to $\theta^\circ$ such that Eq. 3.8 becomes,

$$I_i(x_i, y_i) = \frac{1}{4\lambda^2 f_1 f_2} \left[ \left| (P_1 \ast e^{j2\pi \alpha_1 \frac{x_i}{\lambda}} + P_2 \ast e^{j2\pi \alpha_2 (x_i - y_i)}) \right|^2 + \left| (P_2 \ast -1 \ast e^{j2\pi \alpha_2 \frac{x_i}{\lambda}}) \right|^2 \right]$$

$$\ast \left[ (P_1 \ast e^{j2\pi \alpha_1 \frac{y_i}{\lambda}} + P_2 \ast e^{j2\pi \alpha_2 (y_i - x_i)}) \right] + \left| (P_2 \ast -1 \ast e^{j2\pi \alpha_2 \frac{y_i}{\lambda}}) \right|^2 \right]$$

$$+ (P_1 \ast P_2 \left| U_{ox} \right|^2 e^{j2\pi \Omega (x_i - x_o)} + c.c.) + (-P_1 \ast P_2 \left| U_{oy} \right|^2 e^{j2\pi \Omega (y_i - y_o)} + c.c.)$$

$$+ (P_1 \ast P_2 U_{ox} e^{j2\pi \Omega x_i} + c.c.) + (-P_1 \ast P_2 U_{oy} e^{j2\pi \Omega (y_i - x_i)} + c.c.)$$

$$+ (P_2 \ast U_{ox} e^{j2\pi \Omega (x_i - y_o)} + c.c.) + (-P_1 \ast P_2 U_{oy} e^{j2\pi \Omega (y_i - y_o)} + c.c.)$$

(3.9)
where $P_1$ and $P_2$ represents the amplitude transmittance of different polarization components illustrated in Eq.3.5 upon passing the analyzer. Without repeating the derivation as Eq.3.8, we can directly plot each term in Eq.3.9 in the frequency domain as shown in Fig.3.2(b). We force the all the $S_1$ and $S_{23}$ peaks to have the same height by making,

$$P_2^2 - P_1^2 = P_1 \cdot P_2$$  \hspace{1cm} \text{(3.10)}

which gives $P_2 = 2P_1$ under the conservation condition $P_1^2 + P_2^2 = 1$. Since $\cos(\theta^*) = \sqrt{P_1}$, we obtain that $\theta = 63.4^\circ$, which is $18^\circ$ more than the original $45^\circ$ setup. A finer tuning can be done by taking the system MTF into account, under which all Stokes parameters’ peaks except for the $S_0$ will have the same peak heights.

### 3.3 Raytracing Analysis

Similar to the raytracing simulation for the prism SIP, we apply this technique to a SP system. We design a system with the following parameters: $f_1 = 500\text{mm}, f_2 = 75\text{mm}$, the CCD pixel size = $4.65\ \mu\text{m}$, the carrier frequency $\Omega = 6$ pixels/fringe and $\lambda = 633\text{nm}$. 

Figure 3.3 The Spot diagrams of a SP SIP. The unit is micron.
Due to the relations, $\Omega = \frac{\Lambda}{\lambda f_2}$ and $\sqrt{2\Delta} = 0.075t$ [19], where $t$ is the thickness of the SP, we can obtain the required thickness $t$ for each SP, which is 2.67 cm in the current design. We model $L_1$ and $L_2$ both as paraxial lenses in order to isolate the effects from them. Also, we use a minimum cavity length between $L_1$ and $L_2$ to accommodate the SP group. Figure 3.3 shows the image spot diagrams and Figure 3.4 shows the ray shearing diagrams at the back surface of the analyzer. Two different fields $(0^\circ, 0^\circ)$ $(20^\circ, 20^\circ)$ are sampled and the object is at the front focal plane of $L_1$. It is not surprising that the image is aberration free in both on-axis and off-axis fields since the SP is a plane parallel plate (PPP). Consequently, any practical aberration should mainly result from $L_1$ and $L_2$. This fact makes the SP SIP superior over the prism SIP because the prism SIP inherently generates aberration. The ray shearing diagrams in Fig.3.4 illustrate the simulated wavefront shearing after passing the SP group as insert II of Fig.3.1. The on-axis ray shearing diagrams are consistent with theoretical calculation, i.e., four rays are sheared equally and diagonally with values of $\sqrt{2\Delta} = 1.725mm$. Conversely in the off-axis case, the four rays are not sheared ideally and their shearing distances are not equal to each other. Both issues are due to the asymmetry of the SP itself. This deviation worsens from the on-axis to off-axis and it may seriously affect the sinusoidal nature of the fringes, which produces errors in the reconstruction. Fortunately, our simulation tells us that such deviation increase slowly from on-axis to off-axis and turns out to be sufficiently calibrated by a reference image (see experimental results in Section 3.4). Table 3.1 gives a comparison between the SP SIP and the prism SIP. Overall, the SP is better than the
prism system in terms of aberration sensitivity and cost-effectiveness.

![Diagram of object fields](image)

**Figure 3.4** Ray shearing diagrams of a SP SIP at the back surface of the SP2 for the (0°, 0°) and (20°, 20°) field points respectively. For vision purpose, one incident ray is shown. The dotted rectangle is drawn to illustrate the diagonal shearing between the four rays.

<table>
<thead>
<tr>
<th>Advantage</th>
<th>Savart Plate</th>
<th>Prism</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>higher resolution, aberration free, easy fabrication, lower cost, easier alignment, larger FOV and DOF</td>
<td>compact size</td>
</tr>
<tr>
<td>Disadvantage</td>
<td>larger in size, larger vignetting</td>
<td>lower resolution, BS and BD effects, limited FOV and DOF, uneasy fabrication and mounting</td>
</tr>
</tbody>
</table>

**Table 3.1** Comparison between the SP and prism SIPS

### 3.4 Experimental Results

To demonstrate the above design, we fabricate a SP polarimeter with commercially available elements, all of which are one-inch in diameter and mounted separately. The SPs are manufactured by Karl Lambrecht Inc. The thickness of each SP is about 2.67
Figure 3.5 (a) The raw image of a LP at 98°. (b) The spectrum of image.

Figure 3.6 (a) Reconstructed Stokes of the image in Fig.3.5(a). (b) The ellipticity angle. (c) The azimuth angle.
cm calculated for forming the required shearing distance. Figure 3.5 (a) shows a raw image of a standard object, a linear polarizer (LP) orientated at 98°. Its spectrum is shown in Fig.3.5 (b), which is consistent to the theoretical spectrum in Fig.3.2. In the reconstruction, we use a uniform polarized image formed by a LP at 22.5º (\(S_1 = 0.707, S_2 =0.707, S_3 = 0\)) to be the reference for calibrating the system. The reconstruction results are shown in Fig.3.6 (a), where uniform and flat Stokes images are seen. We calculate the average values of each Stokes image in Fig.3.6(a) as \(S_1 = -0.9016, S_2=-0.2574, S_3 =-0.0101\). Ideally, a LP at 98º gives \(S_1 =-0.9612, S_2=-0.2756, S_3 =0\). The offset is within ±0.05 of each Stokes parameter’s absolute value. We also calculate the standard deviation of each Stokes parameter in Fig.3.6 (a), which are as small as \(\Delta S_1 = 0.0021, \Delta S_2=0.0067, \Delta S_3 =0.0081\) in absolute values. The errors may result from many sources: 1) the reference LP is not exactly aligned at 22.5º. 2) The CCD detector is not at the best focus and thus leads to defocus errors. 3) L1 and L2 may produce additional aberrations locally that degrade the fringe quality. 4) The phase non-linearity errors may be not fully calibrated. 5) The noise can also decrease the accuracy. With the \(\vec{S}\) extracted, we can calculate other quantities such as the ellipticity angle \(\varepsilon\) and the azimuth angle \(\theta\) based on them as

\[
\varepsilon(x,y) = \frac{1}{2} \tan^{-1} \left[ \frac{S_1(x,y)}{\sqrt{S_1(x,y)^2 + S_2(x,y)^2}} \right] \\
\theta(x,y) = \frac{1}{2} \tan^{-1} \left[ \frac{S_1(x,y)}{S_2(x,y)} \right]
\]

(3.11)

Figure 3.6 (b) and (c) show the calculated \(\varepsilon\) and \(\theta\) both in degrees. Since the object is a LP, the \(\varepsilon\) should ideally be zero degree. Our results in Figure 3.6 (b) are consistent to
Figure 3.7 (a) The raw image of a plastic film curved as a cylindrical surface. (b) Reconstructed $\tilde{S}$ of the image in (a).

Figure 3.8 (a) The ellipticity angle calculated from the Stokes in Fig.3.7 (b). (b) The azimuth angle calculated from the Stokes in Fig.3.7 (b).
expected values with errors mostly within ±5°. Also, the azimuth angle should ideally be 98° (the orientation of the LP); Fig.3.6(c) shows an agreement to the expected values.

Figure 3.7(a) shows an unknown object, a transparent plastic film that lumps in the center in front of a LP. We first make the film naturally curved like a cylindrical surface as the insert of Fig.3.7(a). The raw image of the object is shown in Fig.3.7(a), where we leave the upper space for the LP to compare with the film. By the raw image, it is not obvious that this is a film has significant curvature. However, the Stokes images can characterize this feature. Since the film has retardance, it generates circular polarization, i.e., $S_3$. Also, the curved surface of film leads to a gradual change in the state of polarization (SOP). The reconstruction of the $\vec{S}$, in Fig.3.7 (b) shows both these features, the occurring of $S_3$ and the gradually varying patterns. The periodic pattern is due to the fast changing of the film surface and the period decreases from the central area to the side area since the film is flat at the center and becomes curved to the sides forming a parabola shape. The upper space has a constant background, since the LP produces a uniform SOP. This can be verified from either the ellipcity angle or the azimuth angle image. As we discussed above, $\varepsilon$ is nearly 0° and $\theta$ represents the LP transmission axis which is about 22° in Fig.3.8(b).

We next make the film flat but inclined at an angle as illustrated in the right insert of Fig.3.9 (a). The raw image is shown in Fig.3.9(a). We can hardly tell the difference between the object in Fig.3.7(a) and that in Fig.3.9(a), but the reconstructed $\vec{S}$ clearly shows the difference. Since the film has a uniform slope, the change of SOP
has a nearly constant period as in the reconstructed $\tilde{S}$ of Fig.3.8(b). We shape the film back to a cylindrical surface but with a lower curvature. The raw image is shown in 3.10(a). The reconstructed Stokes images shown in Fig.3.10(b) illustrates a relaxed periodic pattern since the film has a reduced curvature. Finally, we move the LP to the front of the film which should block all non-linear polarization. Figure 3.10 (a) and (b) show the raw image and the reconstruction results, in which $S_3$ disappears while $S_1$ and $S_2$ become uniform images as similar as Fig.3.6(a).

### 3.5 Conclusions

We develop a wave optics model for the Savart Plate system. This model helps shorten the system size by a maximum factor of 2. The raytracing analysis based on the fringe decomposition provides numerical evidence to confirm the design and the theory. Compared to the prism SIP, the Savart Plate SIP shows advantages in reducing polarization aberrations. The experimental results demonstrate the device validity, and also showed a powerful capability of detection in identifying objects’ polarization signatures.
Figure 3.9 (a) The raw image of a flat plastic film tilted at an angle. (b) The reconstructed Stokes of the image in (a).
Figure 3.10 (a) The raw image of a plastic film with smaller curvature than Fig.3.7(a). (b) The reconstructed Stokes of the image in (a).
Figure 3.11 (a) The raw image of a flat plastic film behind a LP. (b) The reconstructed Stokes of the image in (a).
CHAPTER 4

A MINIATURE SNAPSHOT IMAGING POLARIMETER

In remote sensing, bioscience or other scientific areas, a miniature polarimetry camera (MPC) that can measure the state of polarization (SOP) of an object would be very powerful. For example, a surveillance MPC would be valuable on small unmanned air vehicles. Likewise, an endoscopic MPC could be easily inserted into the body cavity. These are challenging tasks for conventional polarimeters that use rotating polarization elements as discussed in previous chapters. In this chapter, we propose and demonstrate a Savart-Plate-based MPC that evolves from the previous SP SIP in Chapter 3. Also, we develop a geometrical ray model to demonstrate that this system sustains the working principle for the device. More importantly, it significantly reduces the system size, fabrication and alignment requirements.

4.1 Layout Reduction

Chapter 3 has proved that the 4f imaging system can be reduced in half if the SP group thickness ignored. We further note that in the SP system, \( f_1 \) does not needs to equal \( f_2 \), depending on the magnification used. When imaging a distant object, the front lens \( L_1 \) has a focal length equal to the object distance. If the object moves to infinity, the front lens’s power will approach zero and be a plane-parallel plate (Fig.4.1(a)). At this extreme case, the front lens can be eliminated. An interesting question is whether the front lens is required when imaging a finite-conjugate object
as Fig.4.1(b). We will show that both theoretical and experimental results will support this modification, that it is not only valid and but also valuable for achieving a MPC.

We use a geometric model to re-interpret the system. In Chapter 3, the SP produces lateral shift between two orthogonal polarization wave-fronts in the Fourier plane in order to create linear phase terms in the 2nd FT; while the geometric interpretation gives a straightforward picture. In the ray model, the SP produces linear optical path difference (OPD) between orthogonal polarization rays directly. Figure 4.2 illustrates the formation of OPD between two orthogonal polarization rays passing through a SP. Note that the OPD is not formed inside the SP but created out of the SP.

Figure 4.1 (a) Illustration of increasing the object distance to infinity in a two-lens SP SIP.  (b) Layout of a reduced SP SIP.
Assume that $\theta_0$ is the incident angle of the ray, we can write the OPD as:

\[
OPD = \sqrt{2}\Delta \sin \theta_0 \tag{4.1}
\]

where $\sqrt{2}\Delta$ is the shearing distance generated by the SP. The ray paths inside the SP will not produce OPD. By Snell’s law, we know that the refracted angles of the two split rays denoted as e/o ray and o/e ray in Fig.4.2 complies with the following equation,

\[
e/o : \sin \theta_0 = n_e \sin \theta_{1e} = n_o \sin \theta_{1o} \\
o/e : \sin \theta_0 = n_o \sin \theta_{2o} = n_o \sin \theta_{2e} \tag{4.2}
\]

where “e” and “o” denote the polarization status inside the SP, “1” and “2” represent the 1st and 2nd layers of the SP.

Figure 4.2 OPD formation in a SP

Then, we obtain the optical path length (OPL) of each ray as,
where $d$ is the thickness of one layer of a SP. We see that the OPLs of the two rays inside the SP are equal.

We can use an analogous derivation as that in Chapter 2 to derive the total intensity. As shown in Fig.3.1(a), the total irradiance at the image plane is a sum of four rays and their mutual interference terms, as

$$I = \left\langle \left| \frac{1}{2} E_1(t) e^{-i\varphi_1} - \frac{1}{2} E_2(t) e^{-i\varphi_2} + \frac{1}{2} E_3(t) e^{-i\varphi_3} + \frac{1}{2} E_4(t) e^{-i\varphi_4} \right|^2 \right\rangle$$

(4.4)

where the bracket stands for the time average and each term in the bracket represents the electric field of a ray after the analyzer. $\varphi_1$ through $\varphi_4$ denote the accumulative phase of each ray path. With the imaging lens, $\varphi_1$ through $\varphi_4$ can be directly interpreted by the coordinates at the image plane as:

$$\varphi_1(x_i, y_i) = 0; \quad \varphi_2(x_i, y_i) = 2\pi \frac{\Delta}{\lambda_f} (x_i + y_i);$$

$$\varphi_3(x_i, y_i) = 2\pi \frac{2\Delta}{\lambda_f} x_i; \quad \varphi_4(x, y) = 2\pi \frac{\Delta}{\lambda_f} (x_i - y_i)$$

(4.5)

where $x_i$ and $y_i$ are the coordinates at the image plane, $\lambda$ is the wavelength and $\sqrt{2\Delta}$ represents the shearing distance generated by a SP. Using the definition of the Stokes parameters [16], we finally arrive at the intensity pattern as
\[
I(x_i, y_i) = \frac{1}{2} S_0 + \frac{1}{2} S_1 \cos(2\pi\Omega(x_i + y_i)) \\
\quad + \frac{1}{4} |S_{23}| \cos[2\pi(2\Omega)x_i + \arg(S_{23})] - \frac{1}{4} |S_{23}| \cos[2\pi(2\Omega)y_i + \arg(S_{23})]; \\
S_{23} = S_2 + i S_3; \quad \Omega = \frac{\Delta}{\lambda f}; \\
\]

which is similar as the prisms SIP. This pattern principally supports the new scheme with the reduced imaging system. However, in the derivation, we ignore two issues: 1) the plane-parallel feature of the SP can cause unwanted aberration; 2) the OPDs on each image spot is averaged cross the numerical aperture (NA) when \(\theta_0\) varies in the range of NA for every pupil coordinates. To evaluate these effects, we need to apply the raytracing analysis and also to experimentally test this reduced scheme.

4.2 Miniaturization

The elimination of the front lens gives us a benefits to make a compact SIP. To further decrease the camera to a miniature SIP, we need to find more tricks. The fringe frequencies in Eq. 4.6 which we call the carrier frequency (CF) is the key enabler for the miniaturizing. Note that the CF is proportional to \(\Omega = \frac{\Delta}{\lambda f}\). For a calcite SP, the shearing distance \(\sqrt{2}\Delta = 0.075t\) (\(\mu m\)), where \(t\) is the thickness in mm of the SP [19].

At a fixed wavelength and CFs, \(t\) is proportional to \(f\), the focal length of the imaging lens. This becomes important in that the total length of the polarimeter can be scaled by a common factor. If a miniature lens implemented, then the whole device can be scaled down significantly. We take a cell phone camera as an example, with \(f = 5\) mm.
and pixel spacing 4.75 μm, \( \lambda = 0.55 \mu m \), and \( \Omega = 4 \) pixels/fringe. The required thickness \( t \) then equals 1.54 mm and the total length of the polarimeter is shorter than 1.5 cm, which to our best knowledge is the smallest imaging polarimeter to date. If changing the calcite to other higher birefringence materials, we can further shorten the length. Compared to other polarization elements, the plane-parallel feature of the SPs also makes it more cost-effective in fabrication and assembly.

4.3 Raytracing Results

As a theoretical demonstration, the above mentioned parameters are inserted into Zemax with a 5 mm diameter aperture. The image spot diagrams and ray shearing diagrams from this simulation can be seen in Fig.4.3. Two different field points are sampled and the object is 1 m distant. The image quality looks impressive for both the on-axis and off-axis field points, with the aberrated spot size close to the diffraction limit. This result diminishes the concern that the SPs may introduce excessive aberrations when imaging a finite conjugate object. In Fig.4.3(b), the on-axis ray shearing diagrams are consistent with expectation, i.e., four rays are sheared equally and diagonally with values of \( \sqrt{2}\Delta = 113 \mu m \). Conversely in the off-axis case, the four rays are not sheared ideally and their shearing distances are not equal to each other. Both issues are due to the asymmetry of the SP itself. This deviation worsens form on-axis to off-axis and it may seriously affect the sinusoidal nature of the fringes, which produces errors in the reconstruction. Fortunately, our simulation tells that such deviation worsens slowly from on-axis to off-axis and it turns out that errors can be
well calibrated by a reference image (see results below). Moreover, we find through
the simulation that the MPC works well in a large range of depth of focus (DOP) and
a wide field of view (FOV).

Figure 4.3 (a) The simulated image spot diagrams of an MPC at a best focus. We
model the imaging lens as a paraxial lens. The object is at 1 meter away and two
field points (0°, 0°) and (25°, 25°) are sampled. The off-axis spot diagram displays
anisotropy. (b) The ray shearing diagrams at the back surface of the SP 2 for the
(0°, 0°) and (25°, 25°) field points respectively. For vision purpose, one incident ray
is shown. The dotted rectangle is drawn to illustrate the diagonal shearing between
the four rays.
4.4 Experimental Demonstration

For a proof-of-concept demonstration, we make use of the enlarged system that is fabricated in Chapter 3, by simply removing the front lens L1. The 75mm focal-length back lens is remained which extends the total length by a factor of 15 from a MPC proposed above. A photo of the actual polarimeter is shown in Fig.4.4, in which we see a portable size of the polarimetry camera.

Figure 4.4 The demonstrated polarimetry camera with a scaling factor of 15 from a MPC. The CCD camera has a pixel spacing of 4.75µm.

Figure 4.5(a) shows a raw image obtained by the polarimetry camera and Fig.4.5(b) shows the reconstructed Stokes images. Again, a uniform polarized image is formed with a LP at 22.5° (S₁ = 0.707, S₂ = 0.707, S₃ = 0) to be a reference image to calibrate the system. The camera’s viewing angle is about 60 degrees with respect to the ground and the object, a car, is about 30 m away. In Fig.4.5(a), the clear fringes seen across the image indicates the existence of polarization signals. This is due to the
Figure 4.5 (a) The raw image obtained by the polarimetry camera. A 3nm bandwidth filter is used in front of the polarimeter. (b) The reconstructed Stokes images
Figure 4.6 (a) The degree of polarization of the object in Fig. 4.5(a). (b) The azimuth angle of the same object.
skewed sun-object-camera angle and less scattering surfaces that are inside the scene. Moreover, there is a change in the fringe pattern on various surfaces (e.g., the window) of the car, indicating that different SOPs are reflected. The reconstructed Stokes images in Fig.4.5(b) also demonstrate this by clearly identifying the car in different Stokes images. We can see that the circular polarization (in $S_3$) is weaker than the linear polarization ($S_1$ and $S_2$). The region around the broken ground (below the car in the image) looks noisy, because it contains higher frequency that the processing algorithm can fully recover [2]. Figure 4.6 shows the degree of polarization (DoP) and the azimuth angle of the object. The car’s window shows a higher DoP since the fringe contrast is higher than other surfaces on the car. The back window of the car has nearly zero DoP since there is very little light reflected from this area. The azimuth angle is also interesting in that the mirrored image on the top of the car shows a higher contrast in Fig.4.6(b) over the rest of the car. This is because the mirrored image is formed by a double reflection (sun-to-object and object-to-car) and the SOP defers drastically from the light directly reflected from the car. In other parts of the car, the azimuth angle change mildly, since the difference mainly results from the surface’s reflecting angle.

Figure 4.7(a) shows another picture of a stadium building. We see that the sky is heavily polarized with high contrast fringes behind the building. The flat and reflective surfaces of the building are well identified in the reconstructed $\tilde{S}$ in Fig.4.7(b) due to strong reflection. However, the poles and frames show wield patterns in $\tilde{S}$, which is suspicious. This can be attributed to an insufficient
Figure 4.7 (a) Raw image. (b) Reconstructed $\tilde{S}$. (c) Degree of Polarization of the object.
sampling of the fringe across small structures. The degree of polarization in Fig.4.7(c) shows that the saturated places on the top of the building cannot be measured since the DoP is zero. Figure 4.8 shows a scene of another car. In the $\tilde{S}$ reconstruction, the $S_1$ signal becomes weak while the S2 and S3 images show a clear shape of the car. Evidently, the sun-car-viewer angle changes from that in Fig.4.5(a).

It is worth mentioning that, compared to a MPC, the enlarged system suffers from
larger aberrations. Aberrations degrade the camera performance in two ways: 1) introducing errors in the reconstruction, especially for off-axis fields; 2) washing out the fringe contrast. With the current encouraging results, it is expected that a better performance can be achieved with a MPC.

4.5 Conclusions

In conclusion, we have proposed and demonstrated a novel snapshot polarimetry camera by forming a complex interference pattern at the image plane. This approach has a unique advantage of building a MPC that may open many applications where polarization information is requested. The simple configuration, easy alignment, and cost-effectiveness also make this polarimeter very competitive. The proof-of-concept device has been demonstrated by both the numerical simulation and real experimental results.
CHAPTER 5
ADVANCED MODEL FOR SIMULATING POLAIRZTION ABERRATION

5.1 Introduction
In Chapter 2-4, detailed raytracing analyses are made in different systems. The raytracing can quickly reveal physical imperfections inside a system; however, it cannot provide a direct answer in terms of how much error the system will suffer from when bearing a certain level of aberrations. An advanced simulation model is required to precisely count for the polarization-dependent effects. In this chapter, we will first develop a theory framework to include the polarization-dependent aberrations and then apply this algorithm to the Savart Plate SIP. The simulation results will be presented to illustrate the quantitative power of this model for numerical simulation of a prism or Savart Plate SIP.

5.2 Theory Framework
In a polarization-dependent imaging system, a polarization-spread matrix (PSM) can be used to characterize the system, as written as [20],

$$\hat{S}_{img}(\vec{r}) = \int_{\mathbb{R}^2} M(\vec{r}; \vec{r}') \hat{S}_{obj}(\vec{r}') d^2\vec{r}'$$  \hspace{1cm} (5.1)

The PSM can be obtained by,

$$M(\vec{r}; \vec{r}') = S[h \odot h^*(\vec{r}; \vec{r}')] S^{-1}$$  \hspace{1cm} (5.2)

where the transformation matrix

$$S = \begin{bmatrix}
1 & 0 & 0 & 1 \\
1 & 0 & 0 & -1 \\
0 & 1 & 1 & 0 \\
0 & -i & i & 0
\end{bmatrix}$$
, \( h(\vec{r}; \vec{r'}) \) is the so-called amplitude response matrix (ARM) and \( \otimes \) denotes the Kronecker product. The ARM can be written as,

\[
h(\vec{r}; \vec{r'}) = c \cdot \exp[ik\psi(\vec{r}; \vec{r'})] \begin{bmatrix}
F\{j_{11}(\vec{\rho}; \vec{r'})\}_{\vec{r}} & F\{j_{12}(\vec{\rho}; \vec{r'})\}_{\vec{r}} \\
F\{j_{13}(\vec{\rho}; \vec{r'})\}_{\vec{r}} & F\{j_{14}(\vec{\rho}; \vec{r'})\}_{\vec{r}}
\end{bmatrix} \begin{bmatrix}
J_{11}(\vec{r}; \vec{r'}) & J_{12}(\vec{r}; \vec{r'}) \\
J_{21}(\vec{r}; \vec{r'}) & J_{22}(\vec{r}; \vec{r'})
\end{bmatrix}
\]

(5.3)

where \( \vec{\rho} \) is the position vector in pupil coordinate, \( c \) a complex constant, \( \psi(\vec{r}; \vec{r'}) \) a quadratic phase term, \( F\{\} \) denotes the Fourier transformation of the bracketed function and \( j_{ij}(\vec{r}; \vec{r'}) \) represents for the system Jones matrix element that is spatial-variant. The key is to obtain the system Jones matrix. The raytracing technique can help us find the system Jones matrix. Assume we can decompose the light field vector along two orthogonal directions subscripted as \( E_1 \) and \( E_2 \), and the light propagation from the entrance pupil (EP) to exit pupil (XP) can be thus be described as,

\[
\begin{bmatrix}
E_{1,xp}(\vec{\rho}) \\
E_{2,xp}(\vec{\rho})
\end{bmatrix} = \begin{bmatrix}
E_{1,ap}(\vec{\rho}) \\
E_{2,ap}(\vec{\rho})
\end{bmatrix}
\]

(5.4)

where

\[
J = \begin{bmatrix}
j_{11}(\vec{\rho}; \vec{r'}) & j_{12}(\vec{\rho}; \vec{r'}) \\
j_{21}(\vec{\rho}; \vec{r'}) & j_{22}(\vec{\rho}; \vec{r'})
\end{bmatrix}
\]

\( J \) is the system Jones matrix and also called the polarization aberration matrix (PAM). Note that the \( \vec{r}' \) dependence in \( J \) is suppressed. In general, PAM is not symmetric or Hermitian when cascaded elements are used. Therefore, we do not have enough equations to solve \( J \) if we only know the output Jones vector. Nevertheless, by aligning one of the field vector decomposition axes to the analyzer’s transmission axis, we can null two
elements in $J$ as,

$$\begin{bmatrix} E_{1,\text{ep}}(\hat{\rho}) \\ E_{2,\text{ep}}(\hat{\rho}) \end{bmatrix} = \begin{bmatrix} j_{11}(\hat{\rho};\hat{r}') & j_{12}(\hat{\rho};\hat{r}') \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_{1,\text{ep}}(\hat{\rho}) \\ E_{2,\text{ep}}(\hat{\rho}) \end{bmatrix}$$  \hspace{1cm} (5.5)$$

As we introduced in Chapter 2 and Chapter 3, the final image field can be usually decomposed into several components; each component originates from the input fields either of $E_{1,\text{ep}}(\hat{\rho})$ or $E_{2,\text{ep}}(\hat{\rho})$. We can always formulate the following equation,

$$\begin{bmatrix} j_{11}(\hat{\rho};\hat{r}') & j_{12}(\hat{\rho};\hat{r}') \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} j_{11}(\hat{\rho};\hat{r}') & j_{12}(\hat{\rho};\hat{r}') \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_{1,\text{ep}}(\hat{\rho}) \\ E_{2,\text{ep}}(\hat{\rho}) \end{bmatrix}$$

$$\begin{bmatrix} E_{1,\text{ep}}(\hat{\rho}) \sum_{l=1}^{N_1} C_l \exp\{ikW_l(\hat{\rho};\hat{r}')\} + E_{2,\text{ep}}(\hat{\rho}) \sum_{q=1}^{N_2} C_q \exp\{ikW_q(\hat{\rho};\hat{r}')\} \\ 0 \end{bmatrix}$$

where $C_l$ and $C_q$ are the amplitude transmittance, $N_1$ and $N_2$ are the total number of the decomposed wave components (of $E_1$ or $E_2$) from a common incident ray, and $W_l(\hat{\rho};\hat{r}')$ and $W_q(\hat{\rho};\hat{r}')$ are the aberration terms obtained by raytracing analysis. The obvious solution to Eq.5.6 is,

$$j_{11}(\hat{\rho};\hat{r}') = \sum_{l=1}^{N_1} C_l \exp\{ikW_l(\hat{\rho};\hat{r}')\}$$

$$j_{12}(\hat{\rho};\hat{r}') = \sum_{q=1}^{N_2} C_q \exp\{ikW_q(\hat{\rho};\hat{r}')\}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$  \hspace{1cm} (5.7)$$

which provides an access for inputting raytracing data. Moreover, if we assume the system is approximately shift-invariant, we can ignore the $\hat{r}'$ dependence in Eq.5.7. By Eq.5.2 and Eq.5.5, we can obtain the PSM as,
With the PAM, we can compute the PSM using Eq. 5.6 to implement a more rigorous simulation.

### 5.3 Example

To test the above theory, we take the SP SIP as a demonstrative example. In general, the Jones matrix of the SP SIP can be written as,

$$
\begin{bmatrix}
J_{11}J_{11}' + J_{12}J_{12}' & J_{11}J_{11}' - J_{12}J_{12}' & J_{11}J_{12}' + J_{12}J_{11}' & i(J_{11}J_{12}' - J_{12}J_{11}')
\end{bmatrix}
$$

(5.8)

$$M(\vec{r}; \vec{r}') = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

To test the above theory, we take the SP SIP as a demonstrative example. In general, the Jones matrix of the SP SIP can be written as,

$$j_{11}(\tilde{\rho}; \tilde{r}') = P_{x1}(\tilde{\rho}; \tilde{r}') \mathcal{O}_1 [\rho_s - 2\Delta_{x}(\tilde{\rho}; \tilde{r}'), \rho_s + \Delta_{y}(\tilde{\rho}; \tilde{r}')] + P_{y1}(\tilde{\rho}; \tilde{r}') \mathcal{O}_2 [\rho_s - \Delta_{y}(\tilde{\rho}; \tilde{r}'), \rho_s + \Delta_{y}(\tilde{\rho}; \tilde{r}')]$$

$$j_{12}(\tilde{\rho}; \tilde{r}') = -P_{x2}(\tilde{\rho}; \tilde{r}') \mathcal{O}_3 [\rho_s - \Delta_{y}(\tilde{\rho}; \tilde{r}'), \rho_s - \Delta_{y}(\tilde{\rho}; \tilde{r}')] + P_{y2}(\tilde{\rho}; \tilde{r}') \mathcal{O}_4 [\rho_s + \Delta_{x}(\tilde{\rho}; \tilde{r}'), \rho_s + \Delta_{x}(\tilde{\rho}; \tilde{r}')]$$

(5.9)

where $P_{x,y,1,2}(\tilde{\rho}; \tilde{r}')$ represents the pupil function (including aberration) for each decomposed component and $\mathcal{O}_i$ is the shifting operator which is defined as,

$$\mathcal{O}_i [\rho_s + \Delta_{x}(\tilde{\rho}; \tilde{r}'), \rho_s + \Delta_{y}(\tilde{\rho}; \tilde{r}')] w(\tilde{\rho}; \tilde{r}') = w(\rho_s + \Delta_{x}(\tilde{\rho}; \tilde{r}'), \rho_s + \Delta_{y}(\tilde{\rho}; \tilde{r}'); \tilde{r}')$$

(5.10)

where $\Delta_{x,y,1,2}(\tilde{\rho}; \tilde{r}')$ is the physics wave-front shift rendered by the SP group. There are both $\tilde{\rho}$ and $\tilde{r}'$ dependence in the shift $\Delta_{x,y,1,2}(\tilde{\rho}; \tilde{r}')$ because of the asymmetric nature inherent in the SP group (illustrated in Chapter 2 and 3). In ideal systems, we can assume that
\[ \Delta_{x'} = \Delta_{x''} = \Delta_{y'} = \Delta_{y''} = \Delta \]
\[ \Delta_{y'} = \Delta_{x''} = \Delta_{y'} = 0 \]

(5.11)

where \( \Delta \) is a constant. This treatment ignores the error caused by spatial variance of \( \Delta \) but can allow the derivation proceed as,

\[ j_{11}(\hat{p}; \vec{r}) = P_{x,1}(\hat{p}; \vec{r}) \hat{O}(\vec{r}, -2\Delta, \rho_{x}) + P_{x,2}(\hat{p}; \vec{r}) \hat{O}(\vec{r}, -\Delta, \rho_{x} + \Delta) \]
\[ j_{12}(\hat{p}; \vec{r}) = -P_{y,1}(\hat{p}; \vec{r}) \hat{O}(\vec{r}, -\Delta, \rho_{y} - \Delta) + P_{y,2}(\hat{p}; \vec{r}) \hat{O}(\vec{r}, \rho_{y}) \]

(5.12)

By implementing Fourier transformation on Eq.5.12, we obtain,

\[ J_{11}(\vec{r}; \vec{r}') = [p_{x,1}(\vec{r}; \vec{r}') e^{2\Delta \rho_{x}} + p_{x,2}(\vec{r}; \vec{r}') e^{4\Delta \rho_{x}}] \]
\[ J_{12}(\vec{r}; \vec{r}') = [p_{x,1}(\vec{r}; \vec{r}') - p_{y,2}(\vec{r}; \vec{r}') e^{4\Delta \rho_{x}}] \]

(5.13)

where \( x_0 \) and \( y_0 \) are the Cartesian axes of \( \vec{r}' \), and \( p_{x/y,1/2}(\vec{r}; \vec{r}') = F\{p_{x/y,1/2}(\hat{p}; \vec{r})\}_{r} \) the polarization-dependent amplitude PSF. By Eq.5.13, we have:

\[ J_{11}' = p_{x,1}p_{x,1}^* + p_{x,2}p_{x,2}^* + p_{x,1}p_{x,2}^* e^{2\Delta \rho_{x}} + p_{x,2}p_{y,2}^* e^{4\Delta \rho_{x}} + p_{x,1}p_{x,2}^* e^{2\Delta \rho_{x}} e^{2\Delta \rho_{x}} + p_{x,2}p_{y,2}^* e^{4\Delta \rho_{x}} \]
\[ J_{12}' = p_{x,1}p_{x,1}^* + p_{y,2}p_{y,2}^* - p_{x,1}p_{y,2}^* e^{2\Delta \rho_{x}} - p_{y,2}p_{y,2}^* e^{4\Delta \rho_{x}} \]
\[ J_{11}' = p_{x,1}p_{x,1}^* e^{2\Delta \rho_{x}} - p_{y,2}p_{y,2}^* e^{4\Delta \rho_{x}} + p_{x,2}p_{y,2}^* e^{4\Delta \rho_{x}} - p_{y,2}p_{y,2}^* e^{2\Delta \rho_{x}} \]

(5.14)

Then, plugging Eq.5.14 into Eq.5.10, we can obtain,

\[ m_{00}(\vec{r}; \vec{r}') = Q + e^{2\Delta \rho_{x}} R + c.c. \]
\[ m_{01}(\vec{r}; \vec{r}') = P + e^{2\Delta \rho_{x}} T + c.c. \]
\[ m_{02}(\vec{r}; \vec{r}') = U e^{2\Delta \rho_{x}} + c.c. - V e^{4\Delta \rho_{x}} + c.c. + W e^{2\Delta \rho_{x}} + c.c. - Z e^{4\Delta \rho_{x}} + c.c. \]
\[ m_{03}(\vec{r}; \vec{r}') = i(U e^{2\Delta \rho_{x}} - c.c.) - i(V e^{4\Delta \rho_{x}} - c.c.) + i(W e^{2\Delta \rho_{x}} - c.c.) - i(Z e^{4\Delta \rho_{x}} - c.c.) \]
where we define

\[
Q = p_{x,1}p_{x,1}^* + p_{x,2}p_{x,2}^* + p_{y,1}p_{y,1}^* + p_{y,2}p_{y,2}^*
\]

\[
R = p_{x,1}p_{x,2}^* - p_{y,1}p_{y,2}^*
\]

\[
P = p_{x,1}p_{x,1}^* + p_{x,2}p_{x,2}^* - p_{y,1}p_{y,1}^* - p_{y,2}p_{y,2}^*
\]

\[
T = p_{x,1}p_{x,2}^* + p_{y,1}p_{y,2}^*
\]

\[
U = p_{x,1}p_{y,1}^*
\]

\[
V = p_{x,1}p_{y,2}^*
\]

\[
W = p_{x,2}p_{y,1}^*
\]

\[
Z = p_{x,2}p_{y,2}^*
\]

as the “deteriorative” functions. Note that due to the polarization-dependence in the amplitude PSFs, \( R, P \) and \( T \) are not zero in general. Now, the Stokes imaging Eq.5.1 becomes,

\[
\tilde{S}_{\text{img}}(\vec{r}) = \int_{\mathbb{R}^2} \begin{bmatrix}
    m_{00}(\vec{r}; \vec{r}') \\
    m_{01}(\vec{r}; \vec{r}') \\
    m_{02}(\vec{r}; \vec{r}') \\
    m_{03}(\vec{r}; \vec{r}')
\end{bmatrix} \cdot \tilde{S}_{\text{obj}}(\vec{r}')d^2\vec{r}'.
\]

(5.16)

By plugging Eq.5.15 into Eq.5.16, we can obtain for the \( S_0 \) term as,

\[
S_{0,\text{img}}(\vec{r}) = \int_{\mathbb{R}^2} \left[ \sum_{i=0}^{3} m_{0i} \otimes S_{i,\text{obj}} \right] d\vec{r}' = \sum_{i=0}^{3} \left[ \int_{\mathbb{R}^2} m_{0i} \otimes S_{i,\text{obj}} \right] d\vec{r}'.
\]

\[
= Q \otimes S_{0,\text{obj}} + R \otimes \left[ e^{i\Delta(\chi_{x}+\chi_{y})} S_{0,\text{obj}} \right] + R^* \otimes \left[ e^{-i\Delta(\chi_{x}+\chi_{y})} S_{0,\text{obj}} \right] \\
+ P \otimes S_{1,\text{obj}} + T \otimes \left[ e^{i\Delta(\chi_{x}+\chi_{y})} S_{1,\text{obj}} \right] + T^* \otimes \left[ e^{-i\Delta(\chi_{x}+\chi_{y})} S_{1,\text{obj}} \right] \\
+ U \otimes \left[ e^{i2\Delta\chi_{x}} S_{23,\text{obj}}^+ \right] + U^* \otimes \left[ e^{-i2\Delta\chi_{x}} S_{23,\text{obj}}^- \right] \\
+(W-V) \otimes \left[ e^{i2\Delta(\chi_{x}-\chi_{y})} S_{23,\text{obj}}^+ \right] -(W-V)^* \otimes \left[ e^{-i2\Delta(\chi_{x}-\chi_{y})} S_{23,\text{obj}}^- \right] \\
+ Z \otimes \left[ e^{-i2\Delta\chi_{y}} S_{23,\text{obj}}^+ \right] + Z^* \otimes \left[ e^{+i2\Delta\chi_{y}} S_{23,\text{obj}}^- \right]
\]
where we define
\[ S_{23, \text{obj}} = S_2 + iS_3 \]
\[ S_{23, \text{obj}} = S_2 - iS_3 \]
\[ f \otimes g = \int \mathbb{R} f(\vec{r}; \vec{r}') \cdot g(\vec{r}') d\vec{r}' \]

In a shift-variant system, \( \otimes \) becomes a convolution operator. The above equation provides several important insights: 1) each term in the sum stands for a Stokes channel carried by a spatial frequency; 2) each channel is “blurred” by a unique deteriorative function determined by the polarization aberration; 3) various channels mixes due to the presence of polarization PSF. For example, \( S_1 \) and \( S_0 \) are mixed when \( R \) and \( P \) are not negligible; 4) Two conjugate terms cannot be combined to form a net sinusoidal modulation since the polarization PSFs usually have a phase that stops the \( \tilde{S} \) term from being factored out of the \( \otimes \) operation. For example, \( U^* \otimes e^{-i2\Delta \psi} \) and \( U \otimes e^{i2\Delta \psi} \) cannot be combined; 5) In an approximately shift-invariant system, the Fourier approach will still work in generally, since the deteriorative function only reduces the channel peak in the Fourier domain and its additional phase can be canceled by a reference image. To ensure the above theory is consistent to previous modeling of the SP SIP, we assume an aberration- and diffraction-free system, such that \( p_{x/y,z/\pm 1/2}(\vec{r}; \vec{r}') = \delta_{x/y,z/\pm 1/2}(\vec{r} - \vec{r}') \). In this case, Eq.5.17 becomes
\[ S_{0, \text{img}}(\vec{r}) = 8 \left\{ \frac{1}{2} S_{0, \text{obj}}(\vec{r}) + \frac{1}{4} \cos[\Delta(x_u + y_v)] S_{1, \text{obj}}(\vec{r}) \right. \]
\[ + \left. \frac{1}{4} S_{23, \text{obj}} \cdot \cos \left[ 2\Delta_x + \arg(S_{23, \text{obj}}) \right] (\vec{r}) - \frac{1}{4} S_{25, \text{obj}} \cdot \cos \left[ 2\Delta_y - \arg(S_{25, \text{obj}}) \right] (\vec{r}) \right\} \]
\[ (5.18) \]

which is exactly the same as we know before.
In a practical system, assuming the polarization PSF has minimized asymmetry and the aberration has also been optimized, such that \( \arg(p_{x,y,z}^* p_{x,y,z}) \approx 0 \) and \( \|p_{x,y,z}\|^2 \approx |p|^2 \).

In this case, Eq.5.17 becomes

\[
S_{0,\text{img}}(\vec{r}) \approx 8|p|^2 (\vec{r}) \ast \left\{ \frac{1}{2} S_{0,\text{obj}}(\vec{r}) + \frac{1}{2} \cos(\Delta x_{,0} + y_{,0}) \cdot S_{1,\text{obj}}(\vec{r}) \right. \\
+ \frac{1}{4} S_{23,\text{obj}}(\vec{r}) \cdot \cos\left(2\Delta x_{,0} + \arg S_{23,\text{obj}}(\vec{r})\right) \\
\left. - \frac{1}{4} S_{23,\text{obj}} \cdot \cos\left(2\Delta y_{,0} - \arg S_{23,\text{obj}}(\vec{r})\right) \right\}
\]

(5.19).

We can see that except for the blurring effect the system maintains an effective modulation format as supposed. This fact promises an acceptable usage of a well optimized SIP system. The above theory framework also works for the prism SIPs.

### 5.4 Simulation Results

We now input the raytracing results of the SP system in Chapter 4 into Eq.5.17 to simulate the intensity pattern. In all the calculations below, we consider that the system is shift-invariant so that the operator \( \otimes \) represents the convolution. By multiplying the spectra of the deteriorative functions and Stokes object, we can obtain the convolution by inverse Fourier transforming the product. To demonstrate the above theory is correct, we first assume that the system is aberration free. We input an ideal Stokes object as Fig.5.1(a). Fig.5.1(b) shows the simulated image, a familiar pattern as seen in Chapter 4. The spectrum in Fig.5.1(c) of this image also proves that the simulated image contains the expected carrier frequencies. Note that in Fig.5.1(d), the cross section of the image shows that the fringe has a high contrast for the aberration-free system. The \( \pm(\Delta, -\Delta) \) channels vanishes due to the perfect cancellation between \( W \) and \( V \). Fig.5.1(e) shows the
reconstruction of the $\tilde{S}$. An comparison between the reconstruction and the original object is illustrated in Fig.5.1(d). From the well reconstructed $\tilde{S}$, we can be convinced that Eq.5.17 is reliable for further simulations.

Figure 5.1 (a) Stokes object. (b) Simulated image of an aberration-free system. (c) Spectrum. (d) Cross-section of the image (e) Reconstruction. (f) Cross-section of the reconstruction. Blue is the object and green is the reconnected values.
By implementing the raytracing program, we can obtain an authentic view of the ray bundles. Fig.5.2(a) and (b) shows the side and top views of the actual ray bundle spanning across the focus region. We can see that due to the inherent isotropy of the SP group, the four ray-components are split and focus separately in the optimal image plane denoted at \( z' \). We will first simulate at this position to see how the split effects change the device performance. The spot diagram (for on-axis field) at this position is shown in Fig.5.2(c). At this position, the tilt aberration is dominant over other aberrations as four tiny image spots are formed but split diagonally. By inputting the aberration coefficients (here, we use 1-37 Zernike coefficients as in Appendix B), we can obtain the simulated image and the Stokes reconstruction, as in Fig.5.3.
Figure 5.3 (a) Image at $z'$. (b) Cross-section of the image. (c) Reconstruction (c) Cross-section of the reconstruction. Blue is the object and green is the reconnected values.

Due to the various tilt aberrations that each ray-component suffers from, multiple images occur in the simulated image, even though the optimal spot diagram for each ray-component is achieved at $z'$. Compared to the aberration-free image in Fig.5.1, the fringe contrast is severely destroyed as the mutual-shift between the four images breaks the one-one interference relation. Figure 5.3(c) shows the forced reconstruction of $\tilde{S}$, which are obviously unacceptable.

To restore the system to be effective, we can defocus the system a bit so that the spot diagrams of the four ray-components can overlap each other and interfere again. We pick up the image location that is 50µm to the left of $z'$ as denoted as $z''$. The spot diagram at this location is shown in Fig.5.2(d). We can see that the defocus aberration becomes dominant at this location and it helps the four split ray-components to overlap again for
Figure 5.4 (a) Image at z''. (b) Cross-section of the image. (c) Reconstruction (c) Cross-section of the reconstruction. Blue is the object and green is the reconnected values.

interference. We insert the first 37 Zernike coefficients to the simulation program and obtain the image as shown in Fig.5.4(a) in which decent contrast fringes are recover. The reconstructed $\hat{S}$ of this image are shown in Fig.5.4(c) with the cross-section of the reconstruction shown in (d). We can see that the proper defocus decrease the reconstruction errors drastically. The $S_0$ reconstruction deviates the biggest amount in value from the original object because the defocus-dominant aberration redistributes (blurs) the intensity significantly in the image plane. Also, the $S_1$ leaks into the $S_0$, causing parts of the errors. The same leakage of $S_0$ into the $S_1$ channel also produces extra errors in the $S_1$ reconstruction (Fig.5.4(b)). To verify this leakage effect, we make the object un-polarizing with $S_1$, $S_2$ and $S_3$ all being zero. The simulated image in
Fig. 5.5(a) (cross-section in Fig. 5.5(b)) demonstrates the residual fringe effect as low contrast diagonal fringes are still seen for this un-polarizing object. Fig. 5.5 (c) and (d) shows the corrected reconstruction of Fig. 5.4 (a) that subtracts the residual $S_1$ of Fig. 5.5(b) and (c). The correction works well in the side region, but not for the central area. This is due to the defocus aberration acts as a low-pass filter that blurs the high frequency necessary to recover the correct values.

Figure 5.5 (a) Image of an un-polarizing object at $z''$. (b) Cross-section of the image. (c) Reconstruction (c) Cross-section of the reconstruction. Blue is the object and green is the reconnected values.
Finally, it worth mentioning that under this framework, virtually any polarization aberrations can be numerical studied.

5.4 Conclusions

We have developed a theory framework for simulating the polarization-dependent aberrations that are present in the SP or prisms SIPs. Based on this model, the actual image that a specific design of system will generate can be foreseen. The measurement accuracy can be also simulated. Moreover, some side effects, such as the Stokes channels mixture, the disturbance from the polarization-dependent PSFs on the Stokes are revealed. This information will be important for future improvement in the optical design and imaging processing algorithms.
CHAPTER 6

GENERALIZATION TO ARBITRARY MODULATION

6.1 Introduction

As introduced in Chapter 1-4, the prism and Savart Plate polarimeters represent a unique technique that uses a complex fringe pattern to modulate $\bar{S}$ for a simultaneous detection of them. For convenience, we revisit Eq.2.1 as the typical modulation pattern of such device, as,

$$I(x,y) = \frac{1}{2} S_0(x,y) + \frac{1}{2} S_1(x,y) \cos(2\pi Ux)$$

$$+ \frac{1}{4} S_2(x,y) \left\{ \cos\left[2\pi U(x-y)\right] - \cos\left[2\pi U(x+y)\right] \right\}$$

$$- \frac{1}{4} S_3(x,y) \left\{ \sin\left[2\pi U(x-y)\right] + \sin\left[2\pi U(x+y)\right] \right\}$$

(6.1)

Eq.6.1, as carefully formulated, requires strict configurations of each element (e.g., a specific shape and orientation), which not only restricts design flexibility but also increases alignment difficulty.

However, the linear nature of Eq.6.1 inspires one to relax the design restrictions. Note that Eq.6.1 is essentially a linear combination of the Stokes parameters, with each Stokes parameter modulated by a 2D function. An interesting question comes that whether arbitrary functions other than the sinusoidal functions would be useful? If yes, what are the requirements and benefits of doing so?

In this paper, we study the universal case that arbitrary functions are used to modulate the Stokes parameters in an image. To validate this extended approach, a new reconstruction and calibration technique that replaces the original Fourier
transformation is proposed. Then a Stokes imaging model is developed to help understand the device principles regarding design and optimization. The major considerations for choosing effective elements are discussed. After that, we demonstrate the whole theory by simulating a new modulation pattern, and illustrate a powerful application that achromatizes the polarimeter to a broad bandwidth. In the end, we propose a simulation route that can include polarization aberration produced by a specific design.

6.2 Reconstruction and Calibration

We assume that the total intensity \( I(\vec{r}) \) of the image has the following form,

\[
I(\vec{r}) = ac_0(\vec{r}) \cdot S_0(\vec{r}) + ac_1(\vec{r}) \cdot S_1(\vec{r}) + ac_2(\vec{r}) \cdot S_2(\vec{r}) + ac_3(\vec{r}) \cdot S_3(\vec{r}) + n(\vec{r})
\]

(6.2)

where \( \vec{r} \) is the position vector in the image plane, \( S_{0-4}(\vec{r}) \) are the spatially dependent Stokes parameters, \( n(\vec{r}) \) is the noise and \( ac_{0-4}(\vec{r}) \) represents any effective modulation functions. Since the Fourier method is only useful for sinusoidal fringes, a universally applicable reconstruction technique is demanded to work with arbitrary functions \( ac_{0-4}(\vec{r}) \). To solve the Stokes parameters from Eq.6.2, we apply a multiplexing method by assuming all the Stokes parameters change little within a tiny area and by picking up four adjacent points to formulate a group equation as,

\[
I(\vec{r} + \Delta_i) = ac_0(\vec{r} + \Delta_i) \cdot S_0(\vec{r}) + ac_1(\vec{r} + \Delta_i) \cdot S_1(\vec{r}) + ac_2(\vec{r} + \Delta_i) \cdot S_2(\vec{r}) + ac_3(\vec{r} + \Delta_i) \cdot S_3(\vec{r}) + n(\vec{r} + \Delta_i)
\]

\[
I(\vec{r} + \Delta_i) = ac_0(\vec{r} + \Delta_i) \cdot S_0(\vec{r}) + ac_1(\vec{r} + \Delta_i) \cdot S_1(\vec{r}) + ac_2(\vec{r} + \Delta_i) \cdot S_2(\vec{r}) + ac_3(\vec{r} + \Delta_i) \cdot S_3(\vec{r}) + n(\vec{r} + \Delta_i)
\]

\[
I(\vec{r} + \Delta_i) = ac_0(\vec{r} + \Delta_i) \cdot S_0(\vec{r}) + ac_1(\vec{r} + \Delta_i) \cdot S_1(\vec{r}) + ac_2(\vec{r} + \Delta_i) \cdot S_2(\vec{r}) + ac_3(\vec{r} + \Delta_i) \cdot S_3(\vec{r}) + n(\vec{r} + \Delta_i)
\]

\[
I(\vec{r} + \Delta_i) = ac_0(\vec{r} + \Delta_i) \cdot S_0(\vec{r}) + ac_1(\vec{r} + \Delta_i) \cdot S_1(\vec{r}) + ac_2(\vec{r} + \Delta_i) \cdot S_2(\vec{r}) + ac_3(\vec{r} + \Delta_i) \cdot S_3(\vec{r}) + n(\vec{r} + \Delta_i)
\]

(6.3)
where $\Delta_{1-4}$ represents the small distances shifted from $\vec{r}$ as illustrated in Fig.1. We can transform Eq.6.3 into a matrix format as

$$\vec{I} = A\vec{S} + \vec{n} \tag{6.4}$$

where

$$\vec{I} = \begin{bmatrix} I(\vec{r} + \Delta_1) \\ I(\vec{r} + \Delta_2) \\ I(\vec{r} + \Delta_3) \\ I(\vec{r} + \Delta_4) \end{bmatrix}$$

and

$$A = \begin{bmatrix} ac_1(\vec{r} + \Delta_1) & ac_2(\vec{r} + \Delta_1) & ac_3(\vec{r} + \Delta_1) & ac_4(\vec{r} + \Delta_1) \\ ac_1(\vec{r} + \Delta_2) & ac_2(\vec{r} + \Delta_2) & ac_3(\vec{r} + \Delta_2) & ac_4(\vec{r} + \Delta_2) \\ ac_1(\vec{r} + \Delta_3) & ac_2(\vec{r} + \Delta_3) & ac_3(\vec{r} + \Delta_3) & ac_4(\vec{r} + \Delta_3) \\ ac_1(\vec{r} + \Delta_4) & ac_2(\vec{r} + \Delta_4) & ac_3(\vec{r} + \Delta_4) & ac_4(\vec{r} + \Delta_4) \end{bmatrix}$$

$$\vec{S} = \begin{bmatrix} s_0(\vec{r}) \\ s_1(\vec{r}) \\ s_2(\vec{r}) \\ s_3(\vec{r}) \end{bmatrix}$$

$$\vec{n} = \begin{bmatrix} n(\vec{r} + \Delta_1) \\ n(\vec{r} + \Delta_2) \\ n(\vec{r} + \Delta_3) \\ n(\vec{r} + \Delta_4) \end{bmatrix}$$

In reality, $\Delta_{1-4}$ corresponds to the spacing between adjacent pixels. Four points are the minimum number for recovering the complete Stokes vector. More points can be used in trading off with the resolution. Eq.6.4 is a classic linear equation. Given the matrix $A$, $\vec{S}$ can be estimated. A good estimator is the Moore-Penrose pseudoinverse $A^\dagger$ that gives the least norm square errors as,

$$\vec{S}_{est} = A^\dagger\vec{I} \tag{6.5}$$
In Section 6.4, we will use this estimator for the reconstruction.

It convinces by the above formulation that the sinusoidal fringes are not the unique choice for encoding \( \tilde{S} \); rather, any function is possible. This feature may provide more flexibility for the design, the optimization and the fabrication of the device. In fact, the DoFPA polarimeter represents a special case in which a tessellated modulation pattern is used by masking a micro-polarizer or -retarder array on the focal plane [21, 22].

To obtain \( A \), the knowledge of \( ac_{0-4}(\vec{r}) \) functions is needed. We can send enough reference Stokes vectors into the system to measure \( ac_{0-4}(\vec{r}) \), which refers to the “calibration” procedure. It is virtually a time-domain multiplexing method as illustrated in the following; assuming that \( N \) sets of \( \tilde{S} \) are sent into the polarimeter, and the \( n^{th} \) (denoted in the superscript) measurement of the image intensity is as,

\[
I^n(\vec{r}) = ac_0(\vec{r}) \cdot S^n_0(\vec{r}) + ac_1(\vec{r}) \cdot S^n_1(\vec{r}) + ac_2(\vec{r}) \cdot S^n_2(\vec{r}) + ac_3(\vec{r}) \cdot S^n_3(\vec{r}) + n^n(\vec{r})
\]

Again, we formulate the total \( N \) equations into a matrix format,

\[
\overline{I} = S \overline{ac} + \overline{N}
\]

(6.6)

where

\[
\overline{I} = \begin{bmatrix} I^1(\vec{r}) \\ I^2(\vec{r}) \\ \cdots \\ I^n(\vec{r}) \end{bmatrix} \quad \overline{ac} = \begin{bmatrix} ac_0(\vec{r}) \\ ac_1(\vec{r}) \\ ac_2(\vec{r}) \\ ac_3(\vec{r}) \end{bmatrix} \quad S = \begin{bmatrix} s^1_0(\vec{r}) & s^1_1(\vec{r}) & s^1_2(\vec{r}) & s^1_3(\vec{r}) \\ s^2_0(\vec{r}) & s^2_1(\vec{r}) & s^2_2(\vec{r}) & s^2_3(\vec{r}) \\ \cdots \\ s^n_0(\vec{r}) & s^n_1(\vec{r}) & s^n_2(\vec{r}) & s^n_3(\vec{r}) \end{bmatrix} \quad \overline{N} = \begin{bmatrix} n^1(\vec{r}) \\ n^2(\vec{r}) \\ \cdots \\ n^n(\vec{r}) \end{bmatrix}
\]

This also provides a pseudo-inverse solution for the \( \overline{ac} \),

\[
\overline{ac}_{\overline{ac}}(\vec{r}) = S^\dagger \overline{I}(\vec{r})
\]

(6.8)

where \( S^\dagger \) represents the pseudo-inverse. Note that since the \( S \) matrix must have a rank equal or over 4 to avoid singular solutions, each input \( \tilde{S} \) must be a non-zero vector.
This can be achieved by rotating a linear retarder, e.g., a quarter wave plate (QWP) at a series angles with a linear polarizer behind it.

### 6.3 Stokes imaging model

Eq. 6.4 assumes a perfect one-one mapping from the object to the image. However, this ideal linear equation is not ensured for a system with practical imperfections, e.g., polarization aberration and misalignment. The beam split and beam deviation effects produced by birefringent prisms have been proved to affect the validity of Eq. 2.1 in a prism polarimeter [23]. Imperfections vary case to case. We need to have a basic model that illustrates the common principles of the device. In a polarization-dependent imaging system, a polarization-spread matrix (PSM) can be used to characterize the system, as written as [20],

\[
\tilde{S}_{\text{img}}(\vec{r}) = \int_{B_2} M(\vec{r}; \vec{r}') \tilde{S}_{\text{obj}}(\vec{r}') d^2 \vec{r}'
\]  
(6.9)

where we assume a unit magnification and incoherent light and

\[
\tilde{S}_{\text{img}}(\vec{r}) = \begin{bmatrix} s_0(\vec{r}') \\ s_1(\vec{r}') \\ s_2(\vec{r}') \\ s_3(\vec{r}') \end{bmatrix}
\]

\[
\tilde{S}_{\text{obj}}(\vec{r}) = \begin{bmatrix} s_0(\vec{r}) \\ s_1(\vec{r}) \\ s_2(\vec{r}) \\ s_3(\vec{r}) \end{bmatrix}
\]

\[
M(\vec{r}; \vec{r}') = \begin{bmatrix} m_{00}(\vec{r}; \vec{r}') & m_{01}(\vec{r}; \vec{r}') & m_{02}(\vec{r}; \vec{r}') & m_{03}(\vec{r}; \vec{r}') \\ m_{10}(\vec{r}; \vec{r}') & m_{11}(\vec{r}; \vec{r}') & m_{12}(\vec{r}; \vec{r}') & m_{13}(\vec{r}; \vec{r}') \\ m_{20}(\vec{r}; \vec{r}') & m_{21}(\vec{r}; \vec{r}') & m_{22}(\vec{r}; \vec{r}') & m_{23}(\vec{r}; \vec{r}') \\ m_{30}(\vec{r}; \vec{r}') & m_{31}(\vec{r}; \vec{r}') & m_{32}(\vec{r}; \vec{r}') & m_{33}(\vec{r}; \vec{r}') \end{bmatrix}
\]

, \( \vec{r}' \) is the position vector in the object plane, \( \tilde{S}_{\text{obj/obj}}(\vec{r}) \) are the \( \tilde{S} \) in the object or image plane. \( M \) represents the system Mueller matrix with a spatial-variant point-spread function dependent on \( \vec{r}' \) and \( \vec{r} \) in each element. Any term \( m_y(\vec{r}; \vec{r}') \) in \( M \) can be computed by the characterizing the system’s Jones Matrix. Due to the generalization, we do not specifically know the \( M \) matrix. But, we can further proceed from Eq.6.9 by
making a rough approximation as,

\[
M(\vec{r}; \vec{r}') = \begin{bmatrix} 
  m_{00}(\vec{r}') & m_{01}(\vec{r}') & m_{02}(\vec{r}') & m_{03}(\vec{r}') \\
  m_{10}(\vec{r}') & m_{11}(\vec{r}') & m_{12}(\vec{r}') & m_{13}(\vec{r}') \\
  m_{20}(\vec{r}') & m_{21}(\vec{r}') & m_{22}(\vec{r}') & m_{23}(\vec{r}') \\
  m_{30}(\vec{r}') & m_{31}(\vec{r}') & m_{32}(\vec{r}') & m_{33}(\vec{r}') 
\end{bmatrix} P(\vec{r}; \vec{r}') 
\]  

(6.10)

in which we treat each matrix element as a product of a chief polarization operation function \( m_{ij}(\vec{r}') \) (a real function) that is determined by the core polarization elements and a universal point spread function \( P(\vec{r}; \vec{r}') \) (PSF) that has dependence on both \( \vec{r}' \) and \( \vec{r} \). By saying the approximation rough, we ignore the difference between different polarization-dependent PSFs. Practically, it is acceptable when the polarization-dependent aberration does not defer drastically from state to state, as the case illustrated in the spot diagrams in Fig.4.3. In general, \( P(\vec{r}; \vec{r}') \) is a boundary-constrained function with a limited width. We only derive for \( S_0 \), since the other Stokes parameters are immeasurable,

\[
S_0(\vec{r}) = I(\vec{r}) - n(\vec{r}) \\
= \int_{R^2} \left\{ \sum_i m_{0i}(\vec{r}') \cdot S_i(\vec{r}') \right\} \cdot P(\vec{r}; \vec{r}') \cdot d^2 \vec{r}' \\
= \sum_i \left\{ \int_{R^2} m_{0i}(\vec{r}') \cdot S_i(\vec{r}') \cdot P(\vec{r}; \vec{r}') \cdot d^2 \vec{r}' \right\} 
\]

(6.11)

Rearrange Eq.6.11 to,

\[
I(\vec{r}) = \sum_i \left\{ \int_{R^2} m_{0i}(\vec{r}') \cdot S_i(\vec{r}') \cdot P(\vec{r}; \vec{r}') \cdot d^2 \vec{r}' \right\} + n(\vec{r}) 
\]

(6.12)

For simplicity, we treat the integrals above into two parts, one inside the width of the PSF, and one outside them with a nearly zero response. Further, assume that the inside area has a nearly constant response denoted by \( p(\vec{r}) \) which further simplifies Eq.6.12 as,
\[ I(\vec{r}) = \sum_{i}^{4} \left\{ \int_{\Omega R_2} m_{0_i}(\vec{r}') \cdot S_i(\vec{r}') \cdot 0 \, d^2 \vec{r}' \right\} \]
\[ + \sum_{i}^{4} \left\{ \int_{\Omega R_2} m_{0_i}(\vec{r}') \cdot S_i(\vec{r}') \cdot P(\vec{r}) \cdot d^2 \vec{r}' \right\} + n(\vec{r}) \]
\[ \approx P(\vec{r}) \sum_{i}^{4} \left\{ \int_{\Omega R_2} m_{0_i}(\vec{r}') \cdot S_i(\vec{r}') \cdot d^2 \vec{r}' \right\} + n(\vec{r}) \]  

(6.13)

where \( \int_{\Omega R_2} \) and \( \int_{\Omega R_2} \) represents the integral outside and inside the width of the response function. As seen in Eq.6.13, a linear equation is not necessarily formed as the Stokes parameters are bounded with \( m_{0_i}(\vec{r}') \) in the integral. To make Eq.6.13 linear, either \( m_{0_i}(\vec{r}') \) or \( S_i(\vec{r}') \) or both need to be factored out of the integral. This requires the associated integrated function to be slowly varying across the widths of the PSF, as we sort in three general cases in the following.

**Figure 6.2** The 1D illustration of a “fast” or “slow” function.

- **Case 1:** \( m_{0_i}(\vec{r}') , i = 0-3, \) all vary fast

We define “fast” and “slow” as the variation speed in comparison with the PSF, as sketched in Fig.6.2. In this case, any fast \( S_i(\vec{r}') \) is wrapped with \( m_{0_i}(\vec{r}') \) in the integral. In general, Eq.6.13 can be written as,

\[ I(\vec{r}) = P \left\{ \sum_{i}^{4} S_i(\vec{r}') \cdot m_{0_i}(\vec{r}') \right\} + \sum_{k=1}^{4} S_i(\vec{r}') \cdot \overline{m}_{0_i} + n(\vec{r}) \]
where
\[
S_i(\vec{r}')\cdot m_{0i}(\vec{r}') = \int_{\mathbb{R}^2} S_i(\vec{r}')\cdot m_{0i}(\vec{r}')d\vec{r}
\]
\[
m_{0i} = \int_{\mathbb{R}^2} m_{0i}(\vec{r}')d\vec{r}
\]

and \(k\) denotes the number of fast Stokes parameters and \(S_i(\vec{r}')\) represents the average value of the slow Stokes parameters (note that only at \(\vec{r}' = \vec{r}\) \(S_i(\vec{r}')\) has a value). When \(k\) is not zero, Eq.6.14 is not yet a linear equation of \(S\), which corresponds to two possibilities: 1) the detected region contains the fast changing \(\vec{S}\) (e.g., a sharp edge) that cannot be resolved; 2) the system is severely aberrated with a broad PSF that forces the \(m_{0i}(\vec{r}')\) to be fast.

When \(k=0\), Eq.6.14 becomes an effective linear equation if some requirements on the \(m_{0i}\) terms are met. First, they need to be spatially varying from pixel to pixel to form a modulation. Second, their variations need to be strong in magnitude so that the \(A\) matrix is not a flat matrix. A flat matrix has low singular values and is sensitive to noise amplification in the singular value decomposition (SVD) [24]. This limits the highest frequency of \(m_{0i}(\vec{r}')\) to avoid an over-sampling modulation that a pixel cannot resolve. Thirdly, if constant \(m_{0i}\) terms have to appear like the \(S_0\) term in Eq.6.1, the number of such terms should not exceed one, otherwise the Stokes parameters will be easily mixed between any two flatly modulated terms. Also, zero value is not allowed for this constant term or the associated Stokes parameter will be lost. These requirements are necessary to meet for making a rank of 4 matrix \(A\).

- **Case 2:** \(m_{0i}(\vec{r}')\), \(i = 0-3\), all vary slowly

In this case, Eq.6.13 becomes a linear equation with \(m_{0i}(\vec{r}')\) coming out of the
integral as

$$I(\vec{r}) = P(\vec{r}) \sum_{i} \sum_{j} S_{ij} \cdot m_{ij}(\vec{r}) + n(\vec{r})$$

(6.15)

where $m_{ij}(\vec{r})$ represents the average value of the $m_{ij}(\vec{r})$ function inside the integral region. Since $m_{ij}(\vec{r})$ changes slowly, the $A$ matrix will be more sensitive to noise than the Case 1 with $k = 0$.

- **Case 3: $m_{ij}(\vec{r})$, part vary fast and part vary slowly**

This is a complex case that can lead to many possibilities. In principle, to make Eq.6.13 linear, the Stokes parameter has to be slow if the bounded $m_{ij}(\vec{r})$ is fast; or no requirement is needed for the Stokes parameter if the bounded $m_{ij}(\vec{r})$ is slow.

Eq.6.16 gives both situations as,

$$I(\vec{r}) = P(\vec{r}) \left( \sum_{i} \sum_{j} S_{ij} \cdot m_{ij}(\vec{r}) + \sum_{i} S_{i} \cdot m_{i0}(\vec{r}) \right) + n(\vec{r})$$

(6.16)

where the first term in the bracket represents that $m_{ij}(\vec{r})$ is fast and second term represents the opposite. Due to the same reason in Case 2, the number of constant $m_{ij}(\vec{r})$ functions should not exceed one.

In summary, a linear equation of Stokes parameters is not unconditionally true in a SIP that uses spatial modulations. Some conditions are required to meet to ensure a 4-rank $A$ matrix. The modulation is not necessary limited to sinusoidal functions; any pattern as long as it matches the above conditions is acceptable.

### 6.4 Selecting Elements

Eq.6.13 gives useful guidance for choosing effective elements to produce spatial
modulation. As seen, the first row, \textit{i.e.}, the analyze vector of the $M$ matrix needs to be free of zero. In reality, most polarization elements can be divided into two categories: retarders and ditenuators. One can try different numbers, shapes and configurations of these two elements to obtain a zero-free analyze vector. Mathematically, it is obtained by plugging each element’s Mueller matrix in the order of its physical consequence to a total multiplication of all the elements. The following tips help find an effective design.

1) Only retarders cannot eliminate the zero terms, since the first row or column of a retarder’s Mueller matrix has three zeros as,

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & m_{11} & m_{12} & m_{13} \\
0 & m_{21} & m_{22} & m_{23} \\
0 & m_{31} & m_{32} & m_{33}
\end{bmatrix}
\]  
(6.17)

Also, in reality, linear retarders (LR) are more commonly available than circular retarders (CRs).

2) A ditenuator has less zeros in the Mueller matrix and linear detonators (LD) are more common than circular detonators (CD). Let us look at a LD’s Mueller matrix as,

\[
\begin{bmatrix}
1 & m_{01} & m_{02} & 0 \\
m_{01} & m_{11} & m_{12} & m_{13} \\
m_{02} & m_{21} & m_{22} & m_{23} \\
0 & m_{31} & m_{32} & m_{33}
\end{bmatrix}
\]  
(6.18)

This implies that a combination of LDs and LDs is able to form a zero-free analyzer vector. Unlike the prism polarimeter or Savart Plate polarimeter, in which special orientation angles (\textit{e.g.}, 0º, 90º, 45º or 135º) of each element have to be used, the present approach does not have rigorous requirements on the element’s orientation (see Section 6.5).
3) Table 6.1 provides some basic combinations of LRs and LDs, with the letter “Y” indicating an effective combination and “N” for ineffective. As seen, two LRs and one LD (LR|LR|LD) or two LDs and one LR (LD|LR|LD) are the minimum combinations to produce a zero-free analyze vector. More complex combinations are feasible but no obvious benefits are seen to add more elements in.

4) To make the analyze vector spatially varying, there are three strategies that can be considered in the place of the image plane. The easiest way is varying the retarder’s thickness spatially, like the prisms that have a linear thickness distribution along a direction. An alternative is to make the LR’s birefringence spatially varying, which is not easy for common techniques. Thirdly we can make the transmittance of LDs spatially varying – also a challenge to today’s technique. It should be noted that the Savart Plate (SP) polarimeter represents an exceptional case in which linear retardance is formed by positioning the SPs at the pupil plane to act as beam shearing elements.

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Table 6.1: Basic combinations of LDs and LRs

6.5 Application: Achromatization

One limitation in the prism and SP polarimeters is the narrow bandwidth due to the
fringe contrast reduction under a broadband. In many passive applications, a broadband system is significant for gaining a high signal-to-noise ratio (SNR). Approximately, the maximum bandwidth $\Delta \lambda_{\text{max}} = \frac{\lambda^2}{L_z}$, where $\lambda$ is the center wavelength and $L_z$ is the maximum retardence generated by the element. To expand the bandwidth, $L_z$ needs to be reduced. One solution is making a periodic retarder array in which each period remains the modulation whereas its maximum retardance is confined to a small value.

This concept is illustrated by integrating Eq.6.15 through a spectral band $\Delta \lambda$ as,

$$I(\bar{r}, \Delta \lambda) = \int_{\Delta \lambda} I(\bar{r}, \lambda) d\lambda$$

$$= \int_{\Delta \lambda} \left\{ P(\bar{r}) \sum_i S_i(\bar{r}, \lambda) \bar{m}_i(\lambda) \right\} d\lambda + n(\bar{r})$$

$$= P(\bar{r}) \sum_i \bar{S}_i(\bar{r}) \int_{\Delta \lambda} m_i(\lambda) d\lambda + n(\bar{r})$$

(6.19)

We assume the $\bar{S}_i$ are spectrally flat and their average values are denoted by $\bar{S}_i(\bar{r})$. In the area where the retardance is close to zero, the $m_i(\lambda)$ term is nearly wavelength independent (as represented by $\bar{m}_i(\lambda)$) and can be factored out leaving Eq.6.19 to be,

$$I(\bar{r}, \Delta \lambda) = P(\bar{r}) \sum_i \bar{S}_i(\bar{r}) \bar{m}_i(\lambda) \Delta \lambda + n(\bar{r})$$

(6.20)

Thus, the original modulation of a monochromatic system can be remained.

### 6.6 Numerical Demonstration of the Concepts

For proof-of-concept, we numerically demonstrate the SIP by performing the calibration and reconstruction procedures. For simplicity, we assume that the system has been optimized with ignorable aberration. Chapter 5 introduces an advanced model for computer simulation that takes practical polarization aberration into account.
Instead of using Eq.6.1, we design a different system that uses two LRs and one linear polarizer (LP). The modulation functions \( m_{bi}(\vec{r}') \) of this system are tuned to have strong variations as described by Eq.6.14 with \( k=1 \), by adjusting the LRs’ retardance functions and orientation angles. Specifically, we assign the two LRs (LR1 and LR2) as the following:

\[
R_{LR1}(22.5^\circ, x, y) \propto \sqrt{(x - w_x)^2 + (y - w_y)^2} \\
R_{LR1}(75^\circ, x, y) \propto (x - w_x) + (y - w_y)
\]  

(6.21)

where \( R \) denotes the retardance, \( x \) and \( y \) are coordinates in the image plane, and the degree values in the brackets represent the element’s orientation. \( w_x \) and \( w_y \) are used to decentre the retardance distribution to form a semicircle like pattern. Figure 6.3 plots the retardance of LR1 and LR2. Unlike the prism polarimeter, this system has a non-linear retardance in LR1, and unusual orientations in both LR1 and LR2. The LP is orientated horizontally. Besides, one can design other patterns by differing the retardances and orientations. Under certain circumstances, other designs may provide more advantage for improving the system or reducing the cost.

The resulted coefficients \( ac_{0-4}(\vec{r}) \) are shown in Fig.6.4. Compared to a prism or Savart polarimeter, our system forms a totally different modulation – no straight line fringes and no specifically orientated patterns. The Fourier algorithm is not applicable to this pattern any more. We then simulate to reconstruct the \( ac_{0-4}(\vec{r}) \). The coefficients \( ac_{0-4}(\vec{r}) \) are obtained by inputting 180 Stokes vectors into the system through rotating a QWP in front of a LP(0°). Gaussian noise is added to each of the reference image. The
reconstructed $a_{c_{0-4}}(\tilde{r})$ are shown in Fig.6.5 (a), which looks nearly identical to Fig.6.4. The reconstruction errors are shown in Fig.6.5(b), proving that most reconstruction errors are close to zero.

Figure 6.3 Retardance of (a) LR1 (b) LR2. The unit is wave.

Figure 6.4 Ideal $a_{c_{0-4}}(\tilde{r})$ of the new system in Fig. 6.3
Figure 6.5 (a) The reconstructed modulation functions $ac_{0-3}(\vec{r})$. (b) The reconstruction errors of $ac_{0-3}(\vec{r})$, where $\Delta ac_i = ac_{real} - ac_{reconstructed}$ ($i = 0, 1, 2, 3$).
To demonstrate the reconstruction procedure, we assume a Stokes object as shown in Fig.6.6, which contains both steep-edged objects and slowly-changing objects. An average SNR of 20 is simulated in the image as shown in Fig.6.7 (a), which looks strange because of the above complex modulation. We plot the spectrum of the raw image in Fig.6.7(b), in which we see a drastically different spectrum from the prism or SP SIPs. Also note that the spectrum contains high frequency content which is due to the fast modulation functions as shown in Fig.6.6.

The reconstructed \( \hat{S} \) are shown in Fig.6.8(a). As seen, the Stokes objects are well identified in each Stokes image in both shapes and values. Since the sharp-edge areas cannot be precisely reconstructed (as discussed in Section 6.3), these areas looks spiky. We use a smoothing filter to reduce the errors. Fig.6.8(b) shows the absolute errors of the recovered \( \hat{S} \), which is comparable to the experimental results obtained by a prism or SP polarimeter. In addition, the noise is a dominant error source.
There are three strategies that help reduce the noise: 1) using frame averaging, *e.g.*, 5 frame per image; 2) using more adjacent pixels with a sacrifice of camera resource; 3) using an interactive reconstruction algorithm that often requires a complex

Figure 6.7 (a) The simulated raw image of an LR/LR/LD system with the input Stokes vector in Fig.6.6. (b) The spectrum of the image in (a).
Figure 6.8 (a) The reconstructed Stokes vectors from the image of Fig.6.7(a).
(b) The reconstruction errors of $S_{0-3}$, where $\Delta S_i = S_{\text{real}} - S_{\text{reconstructed}}$ ($i = 0, 1, 2, 3$).
Figure 6.9 (a) The simulated image with slower modulation functions. (b) The spectrum of the image in (a). (c) The reconstructed Stokes of the image in (a).
measurement on the correlation matrix [24].

Figure 6.9 shows a system with slower modulation compared to that in Fig.6.7. The general shapes of LR1 and LR2 are remained but the change of retardance is slowed down. In Fig.6.9(a), the modulation has wider spacing than that in Fig.6.7. This can also be verified in the spectrum of Fig.6.9 (b) that lower spatial frequency content dominates the image. As we analyzed in Section 6.3, slower modulation leads to a slow $A$ matrix and thus bigger reconstruction errors. This can be seen in Fig.6.9(c) as the accuracy decreases from that in Fig.6.8.

Finally, we simulate the achromatization of system as proposed in Section 6.4. We design a periodic retardance function in LR1 and LR2 as the following,

$$R_{LR1}(0^\circ, x, y) \propto \sin(\alpha y); \quad R_{LR2}(45^\circ, x, y) \propto \cos(\beta x)$$

(6.22)

where $\alpha$ and $\beta$ are adjustable scaling parameters. The 2D plot of the periodic retardance is shown in Fig.6.10. We test a bandwidth of $\pm 250$nm centered at 550nm, which covers
Figure 6.11 (a) The simulated image of a broadband system. (b) The spectrum of the image in (a). (c) The reconstructed Stokes of the image in (a).
the majority of visible, and assume that the $\tilde{S}$ are spectrally flat in the bandwidth. Fig.6.11(a) shows the simulated image, displaying a maintained contrast compared to the quasi-monochromatic image in Fig.6.7(a). The spectrum of the raw image is illustrated in Fig.6.11 (b), which shows a grid array of frequency peaks. This is because the sinusoidal retardance produces high order harmonic frequencies in the image. Fig.6.11 (c) shows the reconstructed $\tilde{S}$, with a little decrease in the accuracy compared to Fig.6.8. However, the bandwidth is increased by a factor of 200.

It should be noted that we have not considered the polarization aberration effects inside these above systems. We can estimate the aberration level in the new designs by comparing the geometry of them with that of a prism SIP. Since the retarders that are used here do not have considerably larger slope or curvature in their shape compared to the prisms in Chapter 2, the aberration should be close to that prism SIP.

6.7 General Approach for SIP’s Design

In summary, we can design and test a system following the below steps:

1. Decide a specific SIP design (e.g., one of the demonstrated examples in Section 6.6), based on the rules discussed in Section 6.4

2. Demonstrate it in an ideal simulation as Section 6.6 that excludes polarization aberrations.

3. Tune the system to be optimized in terms of measurement accuracy and resolution.

4. Input the elements’ optical and geometric parameters into Zemax to obtain the
aberration coefficients that are associated with various ray components as demonstrated in Chapter 2-4.

(5) Form the PAM and PSM and then simulated the modulated image again by assuming a linear-shift-invariant system as discussed in Chapter 5.

(6) Test the calibration and reconstruction on the aberration-present image to verify if the system works in expected performance.

(7) According to (6), make further optimization in the design and go to (4) again.

6.8 Conclusions

We generalize the framework for a novel type of SIPs that make use of spatial modulations for encoding the $\tilde{S}$. Our theory provides a universal reconstruction and calibration techniques that allow for wider freedom in system design. The numerical simulation convinces that our new approach is competitive compared to the original Fourier approach. More than that, it shows potentials for further improving the device, with an example of achromatizing the system to a broadband. Moreover, the Stokes imaging model provides an essential understanding of the system.
One application of the imaging polarimetry is for glaucoma diagnosis. A healthy human eye has dense micro-tube structures in the retina nerve fiber layer (RNFL). These micro tubes acts as long chain molecules, producing different refractive indices along parallel and perpendicular axes with respect to the tube orientation, and thus optical retardance results. Glaucoma causes a reduction in the RNFL thickness; the fiber density can be significantly reduced and the retardance in RNFL can be destroyed. By measuring the retardance distribution of the patient’s retina, doctors may compare them with healthy people to judge whether there is an ongoing trend of the patient to suffer from the glaucoma. This approach provides a quick and non-invasive way for diagnosing glaucoma at an early stage, leading to earlier intervention and treatment of disease.

7.1 Optical Layout

Figure 7.1 shows the schematic layout of a fundus SIP. The idea is straightforward in that the SP SIP is integrated with a standard fundus camera that can image the eye’s retina. The fundus camera mainly has two parts, the illumination path and the image path. The illumination path is designed to transfer the energy of a light source into a donut pattern that is cast on the pupil of people’s eye. This donut pattern is important in that it not only forms a uniform illumination pattern onto the retina but also avoids
Figure 7.1 (a) Layout of the Fundus SIP. (b) The real system of (a).
the central part reflection from the cornea to enter into the imaging path. The image
path is designed to form a conjugate of the retina at an intermediate image plane.
Then the SP SIP will relay that intermediate image through the polarimeter. We build
the entire system with commercially available elements. All the elements has 1 inch
diameter as shown in Figure 7.1 (b).

7.2 Simulation
For a theoretical demonstration, we presume a retardance distribution of retina as
shown in Fig.7.2(a) for simulation purpose. The retina retardance is like a bow tie
shape centered about the nerve head as the thickness of the RNFL around the nerve
head is larger than other regions. We also assume that the retardance has a fast axis
along 0° and the incident light is a 22.5° linear polarized light with S₁ = 0.707,
S₂=0.707 and S₃ = 0. The simulated image is shown in Fig.7.2(b), in which Gaussian
noise is added. Fig.7.2(c) shows the reconstructed $\tilde{S}$ based on the same algorithm in
Chapter 2 and 3. Since the retardance is oriented along x axis, the S₁ component is not
affected, as consistently demonstrated in Fig.7.2(c). However, the S₂ and S₃ are
changed by an amount that depends on the retardance. As we can see, such effects
provide a higher-contrast image of the retina thickness, and thus will be valuable for
the physician to diagnose the human’s eye.

7.3 Experiments and Discussion
We try several preliminary tests with a human subject. Figure 7.3(a) shows the raw
Figure 7.2 (a) Simulated retina retardance. (b) The real system of (a). (c) The reconstructed Stokes.
image of the subject’s retina at $\lambda=633\text{nm}$. We can see the fringe pattern is superimposed on the retina image. However, the signal from the retina is weak, even though a maximum gain of 24 and maximum exposure time of 65 ms are used. This is mainly because of the low reflectance of retina which is less than 5% at the working wavelength. The reconstruction of the $\vec{S}$ is given in Fig.7.3(b). As we see, the blood vessels are revealed not only in the intensity $S_0$ but also in $S_3$. $S_1$ and $S_2$ show a suspicious bowtie shape pattern around the nerve head, which is the location expected. The camera’s FOV is not optimized to show the entire retina. Figure 7.4 shows the calculated Degree of Polarization (DoP), ellipticity angle and azimuth angle of the image in Fig.7.3(a) and Fig.7.3(b). The nerve head shape is revealed in the DoP image while the ellipticity angle illustrates the blood vessel as it retardance is different from other parts. In the azimuth angle, the suspicious bowtie pattern is seen, which may be an evidence of the RNFL’s retardance distribution. These results can be repeated as shown in Fig.7.5 through Fig.7.8. Figure 7.5(a) shows the 2$^{nd}$ measurement of the same human subject. In the reconstruction in Fig.7.5(b) and Fig.7.7(c), similar patterns are seen as those in the 1$^{st}$ measurement. However, the values between the two measurements are not well repeated. This can be attributed to 1) the low SNR causes errors; 2) each time the cornea reflection contributes different polarization signals to contaminate the measured image. Figure 7.7 and 7.8 show the 3$^{rd}$ measurement of the human subject, in which familiar results are seen as the previous figures.

Overall, the results obtained so far are of low accuracy and low contrast. They are
not satisfactory for practical use. As mentioned above, the low signal level is the chief reason for low accuracy. A combination of several deficiencies lead to the low signal: 1) the low retina’s reflectance which has been reported with a value less than 10% throughout the visible and near infrared wavelength range; 2) the limited bandwidth of the SIP, only 3nm in this case; 3) the non-optimized illumination optics, for example, the source needs to be reshaped to produce optimized radiance pattern, and the illumination path needs to be shortened in length for reducing vignetting; 4) the motion of the human eye limits the exposure time into ms scale; 5) the imaging path is long also causing vignetting; 6) the donut pattern inherently scarifies a big amount of the light. Among these reasons, points 1, 4 and 6 are hard to eliminate while points 2, 3, 5 can be considered as future works for further improvement. Especially for point 2, the signal level can be increased without taking pains to alter illumination and imaging optics, by implementing a broadband system (such as the proposed method in Section 5.3).

Other than the low signal, other problems can also reduce the reconstruction accuracy. Firstly, the aberration of the imaging optics can reduce the fringe contrast and thus increase the errors. As illustrated in Fig.7.3 through Fig.7.8, the reconstruction near the edge of the scene looks noisier than the central area, which is not as observed as previous chapters. This increased error is due to the fringe contrast loss on the edge as the off-axis aberration increases quickly from on-axis to off-axis. Secondly, the ghost reflection from the objective lens produces extra polarization signals to mix into the raw image. This error can be subtracted by taking a
background measurement as we did in the above figures. Thirdly, the reflection from cornea produces an equal or stronger signal over the signal from the retina itself. Due to the air/cornea interface, the reflectance of cornea is nearly three times that of the retina. As we mentioned, this error is not static and needs a more dynamic approach to remove. Finally, the illumination optics that includes the fold-mirror depolarizes the incident light by a factor of 0.3 in terms of DoP and this also leads to a low contrast of the fringes.

7.4 Summary and Future Works

We fabricated a Fundus SIP that can measure the retardance property of human retina. Preliminary results show that the basic structures of the retina can be revealed in the Stokes mappings. However, the current performance has not been optimized to achieve a high SNR and a high sensitivity. For this Fundus SIP to be practically applicable, more engineering are desired. The following aspects can be considered as the most important work for the future.

1) Find a way to achromatize the system. This is the most effective approach to realize a high signal.

2) Optimize the illumination optics. A customer-designed lamp shape can be considered, for example, a parabola-shaped LED array. Also, a multi-reflection structure can be used to compensate the 50% loss by the polarizer that is in front of the source. Similar technique is very successful in the LCD displays.

3) A common-path illumination system can be used by placing the source (e.g.,
LEDs) inside the imaging tube to avoid depolarization caused by the fold mirror.

4) Both the illumination and imaging optics can be more customer-designed to achieve a uniform illumination on the retina and less aberration in the ultimate image. Currently, we leave the central area of the retina insufficiently illuminated, even though this is already the best setup we can make. The image quality also has more room for improvement by making customer lenses.

5) Find an intelligent way in hardware or software to isolate the reflection from the cornea surface.

6) With the above efforts paid, the cornea retardance is the next concern that has to be addressed in the final reconstruction.
Figure 7.3 (a) Raw image (b) Reconstructed Stokes
Figure 7.4 (a) Degree of polarization. (b) Ellipticity angle (c) Azimuth angle based on Fig. 7.3. The units are all in degree.
Figure 7.5 (a) Raw image (b) Reconstructed Stokes
Figure 7.6 (a) Degree of polarization. (b) Ellipticity angle (c) Azimuth angle based on Fig. 7.5. The units are all in degree.
Figure 7.7 (a) Raw image (b) Reconstructed Stokes
Figure 7.8 (a) Degree of polarization. (b) Ellipcity angle (c) Azimuth angle based on Fig.7.7. The units are all in degree.
Appendix A

**Diffraction through a Thin Lens**

### A.1. Basic diffraction formula

In a free space, there are two ways to obtain the complex amplitude of wave-front at a distance of z from the origin. First is to use the free-space transfer function as,

\[ h(x', y') = \frac{\exp(ikz)}{i\lambda z} \exp\left(\frac{ik(x'^2 + y'^2)}{2z}\right) \]  

(A.1)

where \(x\) and \(y\) denote the lateral coordinates, \(z\) in the subscript represents the plane at \(z\) distance, \(k\) is the wave vector along \(z\) and \(\lambda\) is the wavelength. The Fourier transformation of the transfer function can be approximated as,

\[ H(f_x, f_y) = \exp(ikz) \exp\left[-i\pi\lambda z(f_x^2 + f_y^2)\right] \]  

(A.2)

where \(f_x\) and \(f_y\) represent the spatial frequencies along \(x\) and \(y\) axes. Therefore, at \(z\) distance, the following relation is formed,

\[ F_z(f_x, f_y) = F_o(f_x, f_y)H(f_x, f_y) \]  

(A.3)

where \(F_z(f_x, f_y)\) is the Fourier transformation of the complex amplitude at \(z\), \(F_o(f_x, f_y)\) is the FT of the complex amplitude at origin.

The second approach is to use the Huygens formula. Since most likely a mm-scale focal-length lens is used, we can use the Frensel version of the Huygens diffraction formula as,
\[ U_z(x_z, y_z) = \frac{e^{(ikz)}}{i\lambda z} \iint [U_0(x_0, y_0)e^{i\frac{k}{2z}(x_0^2 + y_0^2)}] \exp[-i2\pi\left(\frac{x_z}{\lambda z}x_0 + \frac{y_z}{\lambda z}y_0\right)]dx_0dy_0 \]

\[ = \frac{e^{(ikz)}}{i\lambda z} \operatorname{FT}\{U_0(x_0, y_0)e^{i\frac{k}{2z}(x_0^2 + y_0^2)}\}\bigg|_{x_0, y_0}^{x_z, y_z} \]

\[ (A.4) \]

![Figure A.1 Diffraction by a thin lens](image)

**A.2 Fourier Transformation by a Thin Lens**

Now we review the FT by a thin lens as illustrated in Fig. A.1. At rear focal plane, the complex amplitude is

\[ U_f(x_f, y_f) = \]

\[ \frac{e^{(ikf)}}{i\lambda f} \iint [U_i(x_i, y_i)e^{i\frac{k}{2f}(x_i^2 + y_i^2)}] \exp[-i2\pi\left(\frac{x_f}{\lambda f}x_i + \frac{y_f}{\lambda f}y_i\right)]dx_idy_i \]

\[ (A.5) \]
where \( t \) in the subscript denotes the plane immediately after the lens and \( f \) is the focal length of the lens. We know that

\[
U_t(x_t, y_t) = U_l(x_l, y_l)e^{-\frac{i \phi}{2f}(x_t^2 + y_t^2)} \quad (A.6)
\]

where \( l \) in the subscript denotes the plane immediately before the lens. Plugging A.6 into Eq.A.5, we obtain,

\[
U_f(x_f, y_f) =
\]

\[
\frac{e^{(ikf)}}{i \lambda f} \iint [U_l(x_l, y_l)e^{-\frac{i \phi}{2f}(x_t^2 + y_t^2)} e^{\frac{i \phi}{2f}(x_l^2 + y_l^2)}] \exp[-i2\pi(\frac{x_f}{\lambda f} x_t + \frac{y_f}{\lambda f} y_t)]dx_td_y_t
\]

\[
= \frac{e^{(ikf)}}{i \lambda f} \text{FT}\{U_l(x_l, y_l)\}_{\frac{x_f}{\lambda f}, \frac{y_f}{\lambda f}} \quad (A.7)
\]

With Eq. A.2, we can have,

\[
F_t(f_x, f_y) = F_o(f_x, f_y) \ast H(f_x, f_y)
\]

\[
= F_o(f_x, f_y) \{\exp(ikd_o) \exp[-i\pi \lambda d_o(f_x^2 + f_y^2)]\}_{\frac{x_f}{\lambda f}, \frac{y_f}{\lambda f}}
\]

\[
= F_o(f_x, f_y) \{\exp(ikd_o) \exp[-i \frac{k}{\lambda} (x_f^2 + y_f^2)(\frac{d_o}{f^2})] \}
\]

\[
(A.8)
\]

where “o” in the subscript represents the object plane. Plugging Eq.A.8 into Eq.A.7, we obtain the image plane complex amplitude as,
\[ U_f(x_f, y_f) = \]
\[ = \frac{e^{i k f} e^{i k d_0} e^{(\frac{1}{2} f (x_f^2 + y_f^2 - 1 \cdot d_0))}}{i \lambda f} \left\{ F_0 \{ U_0(x_0, y_0) \} \right\}_{x_f \ y_f}^{x_f \ y_f} \]

\[(A.9)\]

When \( d_0 = f \), Eq. A.9 becomes a pure FT of the object as

\[ U_f(x_f, y_f) = \]
\[ = \frac{e^{i 2 k f}}{i \lambda f} \left\{ FT \{ U_0(x_0, y_0) \} \right\}_{x_f \ y_f}^{x_f \ y_f} \]

\[(A.10)\]
## Appendix B

**ZERNIKE POLYNOMIAL COEFFICIENTS FOR SECTION 5.4**

### B.1 Zernike Polynomials

Table B.1 Zernike Polynomials

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### B.2 Zernike Coefficients Used in Section 5.7.4

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Table B.3: Zernike Polynomial Coefficients at \(z''\) in wave (\(\lambda = 0.633\mu m\))
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REFERENCES


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