Numerical Investigation of the Nonlinear Transition Regime in Supersonic Boundary Layers

by

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A Dissertation Submitted to the Faculty of the Department of Aerospace and Mechanical Engineering
In Partial Fulfillment of the Requirements For the Degree of
Doctor of Philosophy
With a Major in Aerospace Engineering
In the Graduate College
The University of Arizona

2009
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Acknowledgments

This dissertation would have not been possible without the guidance and financial support of my advisor Dr. Hermann F. Fasel. I am especially grateful for his patience and openness towards my personal research interests, which did not always match the proposed research for the funding agencies.

The initial part of this work was greatly dependent on the cooperation of Dr. Alexander D. Kosinov since he provided his experimental data for comparison. Although I unfortunately never had the chance to meet him personally during my time as a graduate student, I am indebted to his valuable advices and I highly appreciated all discussions with him by email.

Many special thanks to Dr. Anatoli Tumin who was not only a member of my dissertation committee, but also my minor advisor. During many valuable and excellent discussions about eigenvalue spectra, receptivity and resonance triads, Dr. Tumin helped me to gain a deeper understanding of the transition physics.

Furthermore, I want to thank my committee members, Drs. Edward Kerschen, Jeffrey Jacobs and Juan Restrepo for their valuable comments and input. I wish to give my special appreciations to all my colleagues, office-mates and friends who made my research work more enjoyable. In particular, I am grateful for the many discussions and all the help I have received from Frank Husmeier, Clay Koevary, Andreas Laible, Jayahar Sivasubramanian and Neil Terwilliger, who have also worked on supersonic or hypersonic transition research during my stay at the AME. My day-to-day life at the AME and at the University of Arizona was made easier by many people. Here, I want to especially thank Aurora Rau and Barbara Heefner for all their administrative work.

My studies in the United States were only possible with the moral and financial support of my entire family and the financial help of the Erich-Becker-Stiftung and this should also be gratefully acknowledged. Computer time was mainly provided by the CCIT at the University of Arizona and by NASA Ames.
Dedication

To my wonderful wife Patricia.
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Abbreviations

2D, 3D Two-/three-dimensional
CFD Computational Fluid Dynamics
DNS Direct Numerical Simulation
EXP Experiments
FFT Fast Fourier Transformation
IC Initial Condition
LST Linear Stability Theory
NSCC Navier-Stokes Compressible in C
PSE Parabolic Stability Equations
TDNS Temporal Direct Numerical Simulation
TPS Thermal Protection System

Greek

\( \alpha \) Complex streamwise wavenumber
\( \alpha_i \) Streamwise amplification rate
\( \alpha_r, \beta \) Streamwise, spanwise/azimuthal wavenumber
\( \gamma \) Ratio of specific heats
\( \delta \) Boundary layer thickness
\( \delta_c \) Boundary layer thickness from Reynolds-averaged streamw. velocity
\( \delta_1 \) Displacement thickness
\( \Delta t \) Step size in time
\( \Delta x, \Delta y, \Delta z \) Grid spacing in streamw., spanw. and spanw./azimuthal direction
\( \eta \) Factor that determines grid size close to the wall
\( \theta \) Phase
\( \theta_c \) Cone half angle
\( \Theta \) Momentum thickness
List of Symbols—Continued

\( \kappa \)  
Von Kármán constant

\( \lambda_x, \lambda_z \)  
Streamwise and spanwise/azimuthal wave length

\( \mu \)  
Dynamic viscosity

\( \nu \)  
Kinematic viscosity

\( \rho \)  
Density

\( \sigma \)  
Faktor to determine degree of upwinding

\( \tau_{ij} \)  
Stress tensor

\( \psi \)  
Wave angle

\( \omega \)  
Angular frequency

\( \omega_x, \omega_z \)  
Streamwise, spanw./azimuthal vorticity

Roman

\( a \)  
Speed of sound

\( A \)  
Disturbance amplitude

\( c_f \)  
Skin-friction coefficient

\( c_{f,i} \)  
Incompressible reference skin-friction coefficient

\( c_p, c_v \)  
Specific heats at constant pressure and volume, respectively

\( c_{ph} \)  
Phase speed in propagation direction

\( e \)  
Disturbance voltage from hot-wire measurements

\( E \)  
Mean voltage from hot-wire measurements

\( E_c, F_c, G_c \)  
Convective flux vectors

\( E_d, F_d, G_d \)  
Dissipative flux vectors

\( E_t \)  
Total energy

\( E_{\alpha \alpha} \)  
One-dimensional energy spectrum for quantity \( \alpha \)

\( f \)  
Frequency

\( F \)  
Normalized frequency

\( F_c \)  
Skin-friction transformation function
List of Symbols—Continued

$F_x$  Reynolds number transformation function
$H$  Source term
$H_{12}$  Shape factor
$i_e$  Internal energy
$k$  Thermal conductivity/Fourier mode
$k$  Wave vector
$k_c$  Azimuthal modenumber
$K$  Number of spectral modes for the spanw./azimuthal direction
$L$  Arbitrary reference length
$L_e$  Viscous length scale used for linear stability theory
$M$  Mach number
$M'$  Fluctuation Mach number
$M_t$  Turbulent Mach number
$n_x, n_y, n_z$  Grid points in streamw., wall-normal, spanw./azimuthal direction
$N$  Normalized amplitude
$p$  Pressure
$Pr$  Prandtl number
$q$  Heat-flux vector
$Q$  Q-criterion (Vortex identification)
$r$  Radius of the conical coordinate system
$r_{nose}$  Cone nose radius
$r_{turb}$  Turbulent recovery factor
$R$  Specific gas constant for air
$Re$  Reynolds number
$Re_x$  Local Reynolds number
$Re_{x,i}$  Incompressible reference local Reynolds number
**List of Symbols—Continued**

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<tr>
<td>$R_x$</td>
<td>Square root of local Reynolds number</td>
</tr>
<tr>
<td>$Re_\theta$</td>
<td>Reynolds number based on momentum thickness</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
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<tr>
<td>$t_1$</td>
<td>Pulse duration for the generation of a wave packet</td>
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<tr>
<td>$t_{sim}$</td>
<td>Duration of a simulation</td>
</tr>
<tr>
<td>$T$</td>
<td>Temperature/Disturbance period</td>
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<tr>
<td>$T_{forcing}$</td>
<td>Forcing period</td>
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<tr>
<td>$u, v, w$</td>
<td>Streamwise, wall-normal and spanwise velocity component</td>
</tr>
<tr>
<td>$U, V$</td>
<td>Streamwise, wall-normal mean velocity</td>
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<td>$U$</td>
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<td>Streamw. velocity distribution over the disturbance hole/slot</td>
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<td>Domain height</td>
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<td>$z_W$</td>
<td>Domain width</td>
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**Subscripts**

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<th>Description</th>
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<td>$\infty$</td>
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<td>Maximum</td>
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**List of Symbols—Continued**

- \( ref \)  
  Reference

- \( sec \)  
  Secondary

- \( w \)  
  Wall value

**Superscripts**

- \( c \)  
  Asymmetric (cosine mode)

- \( s \)  
  Symmetric (sine mode)

- \( - \)  
  Reynolds-averaged flow quantity

- \( ' \)  
  Disturbance quantity/fluctuation about Reynolds average

- \( '' \)  
  Fluctuation about Favre average

- \( + \)  
  In near-wall units

- \( \ast \)  
  Dimensional value

- \( \sim \)  
  Variable in Fourier space/Rotated coordinate system
Abstract

The nonlinear transition regime of supersonic boundary layers at low to moderate supersonic Mach numbers (Mach 2-3.5) under wind-tunnel conditions is studied using linear stability theory (LST) and direct numerical simulations (DNS). Two main flow configurations are chosen, a flat-plate boundary layer and a cone boundary layer.

Previous investigations of the early nonlinear transition regime have mainly focused on two nonlinear mechanisms, the so-called “oblique breakdown” mechanism and “asymmetric subharmonic resonance”. The first mechanism has only been investigated numerically while the second mechanism was first observed in experiments. This dissertation discusses three open questions related to both mechanisms: (i) Can oblique breakdown be identified in old wind-tunnel experiments published in the literature, (ii) what is the most dominant breakdown mechanism for a supersonic boundary layer, oblique breakdown or asymmetric subharmonic resonance, and (iii) does oblique breakdown lead to a fully developed turbulent boundary layer?

By adopting the flow conditions and the disturbance generation of a specific experiment from the literature, in which asymmetric subharmonic resonance in a wave train was studied, it was possible to show that oblique breakdown might also have been present in the experiment, although oblique breakdown was not reported by the experimentalists. Hence, this experiment might provide the first experimental evidence of oblique breakdown for a supersonic boundary layer. The second question was addressed by performing DNS of a wave packet. A wave packet is typically used as a model of a broadband disturbance environment. If a nonlinear mechanism is dominant, it should leave strong imprints in the disturbance spectrum of the wave packet. In the DNS of the wave packet, oblique breakdown was visible in the disturbance spectrum while subharmonic resonance played only a minor role. To study the last question, a set of DNS of the entire transition path from the linear regime to the turbulence stage was conducted. Some of these simulations demonstrated that the flow reached turbulence near the downstream end of the computational domain.
1. Introduction

“Imagine taking off from any U.S. airport and landing anywhere in the world in less than two hours. Or making a quick hop to the International Space Station and back. Sound far-fetched? Not anymore.”


In 2002, NASA first presented mockups of the proposed Hyper-X series (figure 1.1) at the 50th annual Experimental Aircraft Association’s AirVenture Air Show in Oshkosh, Wisconsin. The announcement on their website started with the phrase quoted above. In the next 20 years, the U.S. intent to develop new supersonic and hypersonic flight technology in order to launch a future fleet of government and commercial hypersonic flight vehicles. The Hyper-X series will serve as flight demonstrator and test platform for these new air-/spacecrafts.

Figure 1.1 Illustration of the X-43A aircraft.

Whether these plans really succeed, will strongly dependent on the understanding of the flow around those aircrafts. Vehicles, cruising at speeds higher than the speed of sound, experience high heating rates at their surface. For large Reynolds numbers, these aerothermal loads are even further increased (by a factor of four or more when compared to a laminar flow) due to the transition process of a laminar high-speed boundary layer to turbulence. Therefore, boundary-layer transition has important design implications for any thermal protection system (TPS) needed to protect a high-speed vehicle and its crew (e.g. re-entry vehicles). During the design process of the Space Shuttle Orbiter, for example, boundary layer transition was recognized to be a significant aerothermodynamic challenge (Berry et al., 2006).
In the past and more recently for the Crew Exploration Vehicle (CEV), engineers used a rather conservative approach for the design of a TPS. In this approach the turbulent boundary layer was assumed to be present over the entire TPS (Berry & Horvarth, 2007). For the design of future high-speed vehicles, however, the design margins will need to be reduced to enhance payload capabilities. To reach this goal, the transition process of a high-speed boundary layer (at supersonic and hypersonic speeds) has to be better understood in order to provide the design community accurate physical models for the prediction of transition (Reshotko, 2007).

Boundary-layer transition has been studied for various configurations. The simplest configurations are a flat-plate boundary layer and a boundary layer on a cone. Both configurations are commonly studied since they represent “generic” cases. Earlier research efforts for compressible boundary layers mainly focused on boundary-layer receptivity (the process by which disturbances generate instability waves in the boundary layer) and the linear development of instability waves. The recent publications by Fedorov (2003) and Tumin (2007) provide a summary over the most important findings for these transition regimes. To date, the nonlinear regime, however, is still poorly understood.

1.1 Flat Plate

Many theoretical studies and numerical simulations during the last decades aimed at understanding the nonlinear regime of the supersonic transition process for a flat plate. The focus of these investigations (e.g Erlebacher & Hussaini, 1990; Masad & Nayfeh, 1990; Thumm et al., 1990) has been on nonlinear breakdown scenarios (secondary instability mechanisms) that are known as viable paths to turbulence in incompressible boundary layers. However, at present it is still not clear if these scenarios represent relevant breakdown mechanisms for flows at supersonic Mach numbers.
Already the linear regime is far more complex in supersonic boundary layers than in incompressible boundary layers; therefore, major differences between incompressible and compressible flows can also be expected for the nonlinear regime. For the linear stages at low supersonic Mach numbers, the exponential growth of oblique instability waves is more critical for the transition process than the exponential growth of two-dimensional waves (Mack, 1984). Following this lead, by using Direct Numerical Simulations (DNS), Thumm (1991) (see also Fasel et al., 1993) discovered a new breakdown mechanism—termed “oblique breakdown”—for a boundary layer at Mach 1.6. This breakdown is initiated by the nonlinear wave interaction of two oblique instability waves with equal but opposite wave angles. The viability of oblique breakdown was confirmed by Chang & Malik (1994) using nonlinear Parabolic Stability Equations (PSE). Chang & Malik confirmed that the two oblique instability waves form a wave–vortex triad with a steady, streamwise vortex mode. The oblique waves are linearly amplified until nonlinear saturation sets in, while the vortex mode grows mainly nonlinearly. Higher harmonics of the initially forced wave pair are generated by direct nonlinear interaction of the wave–vortex triad. These higher harmonics grow rapidly in streamwise direction until they reach high amplitudes in the order of the amplitudes of the original wave–vortex triad. Furthermore, oblique breakdown leads to a more rapid transition than the “classical” secondary instability mechanisms that have been investigated by Erlebacher & Hussaini (1990), Masad & Nayfeh (1990) and Thumm et al. (1990). It also requires much lower threshold disturbance amplitudes for the nonlinear development (see also Adams & Sandham, 1993; Sandham et al., 1995; Eissler & Bestek, 1996; Fezer & Kloker, 1999; Husmeier et al., 2005) when compared to the threshold amplitudes for secondary growth. To date, however, oblique breakdown has not been reported in any experimental study of transition in supersonic boundary layers.

Although oblique breakdown is more natural to occur in supersonic boundary layers since oblique waves experience strong exponential growth according to linear
stability theory (LST), this breakdown mechanism has also been studied for several incompressible flow configurations using Direct Numerical Simulations (Schmid & Henningson, 1992; Berlin et al., 1994). In these investigations, oblique breakdown is termed "oblique transition" by the authors. It is appropriate to distinguish between the incompressible and compressible oblique mechanisms and therefore, to use different terminology, since both mechanisms differ in a very important aspect. For supersonic flow, the steady, streamwise vortex mode undergoes nonlinear forcing from the oblique wave pair whereas, for incompressible flow, this vortex mode is amplified through transient growth leading to a streaky structure. The transient growth theory describes the non-modal growth of streamwise independent three-dimensional structures attributed to the non-normal linearized Navier–Stokes operator (Reddy et al., 1990; Trefethen et al., 1993). If these structures reach high amplitudes, they evolve into streamwise streaks, which are elongated regions of high and low streamwise velocity relative to the mean flow. The final breakdown of these streaks due to a secondary instability represents the final stage of oblique transition in incompressible flows.

Oblique transition has also been investigated experimentally for plane Poiseuille flow by Elofsson & Alfredsson (1998) and for incompressible boundary layers by Wiegel (1996) and Elofsson & Alfredsson (2000). Especially for plane Poiseuille flow, oblique transition provides a suitable explanation for the subcritical transition observed in laboratories. The experimental data from Wiegel (1996) for oblique transition in a Blasius boundary layer have been compared to numerical results obtained from the spatial Direct Numerical Simulations by Berlin et al. (1999). For improving the match between the simulation and the experimental data, Berlin et al. investigated different forcing techniques and introduced an adverse pressure gradient in streamwise direction. Their simulations, however, still predicted an earlier transition of the flow to turbulence than observed in the experiments. In their simulations the flow structures of the late stage of oblique transition showed many similarities to the structures observed in both numerical and experimental investigations of the classical
K- and H-type transition scenarios.

For supersonic flat-plate boundary layers, experiments are very challenging since it is very difficult to excite a pure wave. This is a consequence of the high background noise radiating from the turbulent boundary layers of the nozzle walls into the test section. Note that for the so-called “quiet” wind tunnels (Schneider, 2001), the background noise is significantly reduced. Several experimental studies were devoted to the linear stage of boundary-layer transition initiated by natural background disturbances in the wind tunnel at supersonic \( (M < 4) \) and hypersonic \( (M > 4) \) speeds (Demetriades, 1960; Laufer & Vrebalovich, 1960; Kendall, 1975; Lysenko & Maslov, 1984). Demetriades, Laufer & Vrebalovich, and Kendall also introduced controlled disturbances into the boundary layer and compared the data obtained from these forced transition experiments to the data from their natural transition experiments. In both types of experiments, they found similar trends. The early experimental investigations confirmed some basic features of linear stability theory for compressible boundary layers, as for example, the strong amplification of first-mode oblique instability waves for low supersonic Mach numbers and the stabilizing effect of wall-cooling on the stability behavior of first-mode disturbances and destabilizing effect on second-mode disturbances. These experiments, however, could not provide detailed information on the streamwise development of pure waves with a specific spanwise wavenumber. This issue was resolved by the controlled experiments by Kosinov & Maslov (1985) and Kosinov \textit{et al.} (1990) for a boundary layer layer at Mach 2. In both investigations, a wave train was produced in the boundary layer by a local harmonic point source (electrical discharge inside the flat plate). Measurements were taken for several streamwise and spanwise positions and the output of the hot-wire anemometer was Fourier-transformed with respect to the spanwise direction. Hence, Kosinov \textit{et al.} (1990) were able to study the linear development of the entire disturbance field.

The early nonlinear transition regime for a boundary layer at moderate supersonic Mach numbers has also been studied extensively by Kosinov and co-workers (Kosinov
et al., 1994a,b; Ermolaev et al., 1996; Kosinov et al., 1997). However, their focus has not been on finding evidence for the existence of oblique breakdown. Instead, they have mainly focused on a different breakdown mechanism, termed “asymmetric subharmonic resonance”. In this resonance mechanism, one oblique fundamental wave (20 kHz) interacts with two oblique (asymmetric) subharmonic waves forming a so-called resonance triad. This resonance mechanism was first detected in a supersonic boundary layer by Kosinov et al. (1994b), who carried out controlled transition experiments for a flow at Mach 2.

The existence of subharmonic resonance triads and a theoretical framework for these triads based on weakly-nonlinear theory was originally proposed by Craik (1971) for incompressible boundary layers and was further developed by Volodin & Zelman (1978) and Volodin & Zelman (1993). A different mathematical approach based on Floquet theory and the idea of secondary instability in a laminar, incompressible boundary layer, modulated by a high-amplitude, 2D Tollmien-Schlichting wave, was later introduced by Herbert (1988). Both theories converge in the limit of small disturbance amplitudes. Experimentally, subharmonic resonance (also called N-type breakdown) was first observed by Kachanov et al. (1977) and was later verified by several experimental studies (Saric & Thomas, 1984; Kachanov & Levchenko, 1984). In all these early investigations of incompressible boundary layers, the resonance triad is composed of one 2D Tollmien-Schlichting wave and two oblique, subharmonic instability waves with equal but opposite wave angle. During the transition process, the primary 2D Tollmien-Schlichting wave reaches higher amplitude values in downstream direction than the oblique waves because of its higher streamwise amplification in the linear regime. Due to resonance, the 2D primary wave transfers energy to the small-amplitude oblique instability waves resulting in a strong increase in their streamwise amplification rates (Kachanov, 1994).

For the transition process in a supersonic boundary layer at Mach 2, as investigated by Kosinov and co-workers (Kosinov et al., 1994a,b; Ermolaev et al., 1996;
Kosinov et al., 1997), the classical subharmonic resonance mechanism with a primary 2D instability wave does not play a crucial role. Instead, as already mentioned, three-wave resonance triads with a primary oblique instability wave are more natural to occur because, in contrast to incompressible boundary layers, oblique instability waves in supersonic boundary layers are usually more amplified in the linear regime than 2D instability waves (Mack, 1984). Further experimental studies by Kosinov and his co-workers (Ermolaev et al., 1996) revealed evidence for the existence of many different asymmetric subharmonic resonance triads. They concluded from their studies that depending on the amplitude ratio between the fundamental waves and the subharmonic waves a certain resonance triad is selected by the flow. The viability of Kosinov’s asymmetric, subharmonic resonance triad was confirmed theoretically by Tumin (Kosinov & Tumin, 1996; Tumin, 1996) using a perturbation method (Tumin, 1995) based on a generalization of the averaging method by Zelman & Maslennikova (1993).

1.2 Sharp Cone at Zero Angle of Attack

The flow field for a sharp cone is more complicated than for a flat plate. Therefore, a short summary of the main physical aspects for the mean flow past a sharp cone is given below. A graphical solution for the inviscid and irrotational flow field was first obtained by Busemann (1929). His paper on this subject, however, is only a short note and no details or results are discussed. Later, Taylor & Maccoll (1933) derived the so-called Taylor-Maccoll equation (Anderson, 2004), an ODE for the radial velocity \( U \) in figure 1.2 and solved it numerically (without a computer!). Taylor & Maccoll (1933) assumed that all flow properties are constant along all rays through the tip of the cone between the attached oblique shock and the cone surface (as also stated by Busemann, 1929). This assumption is reasonable since on the cone surface, which represents also a ray from the tip, the pressure has to be
constant. If the pressure did vary for a semi-infinite cone, its value would become unphysical at infinity. For rays close to the cone surface (small $\theta - \theta_c$ in figure 1.2), the velocity component perpendicular to the ray ($V$) has to decrease in its magnitude, approaching zero at the surface since the flow cannot penetrate the surface. Furthermore, continuity suggests that this velocity component points towards the surface entraining fluid into a control volume formed by rays from the tip of the cone. A streamline from the free-stream is deflected by the oblique shock away from the surface and becomes asymptotically parallel to the cone surface. All streamlines in the inviscid flow field of a cone behind the shock are therefore curved resulting in a higher pressure on a ray closer to the cone surface ($\theta - \theta_c$) than on a ray farther away. Moreover, Anderson (2004) states that the magnitude of the velocity vector decreases on rays closer to the cone surface. If a cone half angle is large enough, flow can even transition from supersonic to subsonic near the cone surface without a shock wave. All the above statements are only true for sharp cones with an attached shock wave. For a given Mach number, there is a maximum cone half angle where the oblique shock detaches from the cone tip and the Taylor-Maccoll equation loses its validity.

Accounting for viscous effects, the flow field of a cone changes close to the cone surface. As for a flat plate, a thin boundary layer develops resulting from the skin friction between the flow and the surface. The boundary layer equations for a cone exhibit azimuthal curvature terms and therefore, no similarity solution can be found (Mack, 1987; Malik & Spall, 1991). Mangler (1948) derived a similarity solution by neglecting any azimuthal curvature effects and introduced the so-called Mangler
transformation. This transformation is valid in regions where the cone radius is much larger than the boundary layer thickness. For these conditions, Mangler was able to show that the similarity equations for a flat plate are also the valid equations for a cone. The dimensional boundary-layer “edge” conditions for both types of flow are, however, different at the same streamwise position. For identical edge conditions, the cone mean-flow profiles at a particular streamwise position can be obtained from the flat-plate profiles at a three times smaller streamwise position (Mangler, 1948). At the same streamwise position the boundary layer of a flat plate is $\sqrt{3}$ times thicker than the boundary layer on a cone.

In the first boundary layer stability calculations for a cone boundary layer, Mangler’s transformation was applied i.e. flat-plate profiles were used as mean flow (Malik, 1984; Mack, 1987; Gasperas, 1987). Malik (1984) investigated a supersonic boundary layer on a sharp cone ($5^\circ$ half angle) at zero angle of attack at various Mach numbers. He neglected the curvature terms for the calculation of the mean-flow profiles and the eigenvalue analysis. The mean-flow conditions for his calculations matched the flight experiments from Fisher & Dougherty (1982) (Mach 1.2, 1.35, 1.6 and 1.92) and the quiet tunnel experiments from Beckwith et al. (1983) at Mach 3.5 (before shock). Malik (1984) showed that in these low free-stream disturbance experiments, transition could be predicted by the $e^N$ method with $N$ ranging from 9 to 11. The experimental and numerical studies by Chen et al. (1988, 1989) for a sharp cone and a flat plate at Mach 3.5 confirmed this finding. Furthermore, these studies resolved a controversy regarding transition on flat plates and cones. Earlier, Pate (1971) introduced a ratio of sharp-cone to flat-plate transition Reynolds number that was greater then unity, and particular at Mach 3.5 the ratio was about 2.2. The experiments that Pate (1971) used for his correlations were performed in conventional (noisy) supersonic wind tunnels. The radiated noise from the turbulent boundary layer on the nozzle walls in these wind tunnels significantly influenced the transition Reynolds number. A flat-plate boundary layer was particularly receptive to this type of free-
stream noise. In contrast to these earlier findings, Chen et al. (1989) showed that the ratio of a sharp-cone to flat-plate transition Reynolds number is smaller than unity and has a value of about 0.8 for Mach 3.5. The smaller transition Reynolds number on a cone, when compared to a flat plate, is due to the stronger integral amplification of disturbances in the linear regime (Fezer & Kloker, 2004).

The linear stability behavior of disturbances in hypersonic boundary layers on sharp cones at Mach 8 (6.8 after shock) was calculated by Mack (1987) and Gasperas (1987). Both authors used the Mangler transformation in order to generate the boundary layer profiles. Mack (1987), however, also investigated the stability behavior of nonsimilar cone boundary layer profiles by calculating the boundary layer directly including the transverse curvature. Curvature terms, however, were not included in the stability equations for his eigenvalue analysis. Gasperas (1987), on the other hand, implemented these terms in his analysis. Hence, when combined, the findings from both studies enable to distinguish between curvature effects resulting from the mean-flow profiles only or curvature effects due to the modified linear stability equations. The results from Mack (1987) obtained with the nonsimilar profiles did not show an important difference when compared to the results from the self-similar profiles whereas Gasperas (1987) found that curvature, when implemented in the stability equations, strongly stabilizes two-dimensional second-mode type disturbances. Note that a detailed discussion on first-mode type and second-mode type disturbances can be found in Mack (1969) or in chapter 5 in this dissertation.

Mack (1987) and Gasperas (1987) also compared their results to the experiments by Stetson et al. (1983) for a sharp cone at Mach 8. A complete summary of these and further experiments can be found in Stetson & Kimmel (1992). The results from Mack (1987) and Gasperas (1987) obtained from linear theory do not agree very well with the experimental data. This is most likely caused by nonlinear effects in the experiment since Stetson et al. (1983) used a conventional “noisy” wind tunnel. Nevertheless, the experimental data confirmed that two-dimensional instability waves
with a second-mode character are the most significant unstable disturbances (Mack, 1987) at Mach 8.

A thorough study of curvature and divergence effects for a Mach 5 (after shock) boundary layer on a hollow cylinder and a sharp cone was conducted by Malik & Spall (1991). Their major findings can be summarized as follows: Transverse curvature stabilizes axisymmetric (viscous first and second-mode type) disturbances, as also reported by Gasperas (1987) for Mach 8, while curvature destabilizes oblique, viscous first-mode type disturbances. An azimuthal wavenumber for the transition from a stabilizing to a destabilizing influence of curvature on viscous first-mode type instability waves had not been reported by Malik & Spall (1991). Body divergence generally stabilizes both symmetric and asymmetric disturbances.

Many numerical simulations using DNS and PSE have investigated the nonlinear transition regime for the flow conditions of Stetson’s hypersonic experiments since the data set from these experiments is extensive (e.g. Pruett et al., 1995; Pruett & Chang, 1995; Fezer & Kloker, 2004; Husmeier & Fasel, 2007). However, Stetson and co-workers did not try to monitor the development of a single instability wave nor did they identify a particular breakdown mechanism. In the earlier mentioned experiments by Beckwith et al. (1983) and Chen et al. (1989) for a sharp cone at Mach 3.5, the authors also did not study a particular transition route. They rather focused on the measurement of transition onset and the corresponding transition Reynolds number. The controlled experiments by Corke et al. (2002) and Matlis (2003) for a sharp cone at Mach 3.5 (half angle 7°) in the Mach 3.5 Quiet Wind Tunnel at NASA Langley are the only experiments that have studied a particular transition route for a cone. They observed asymmetric subharmonic resonance where one oblique instability wave with fundamental frequency resonates with oblique subharmonic instability waves. Note that the experimental efforts in the Mach 3.5 Quiet Wind Tunnel at NASA Langley are currently being resumed.
2. Present Research

The introduction of this dissertation emphasizes the significance of transition research for compressible boundary layers since the transition process has crucial design implications for a thermal protection system (TPS) of a high-speed vehicle. At supersonic speeds transition is most likely triggered by oblique instability waves and therefore, only nonlinear mechanisms initiated by primary, high-amplitude, oblique instability waves seem to play an important role in the transition process. The focus on oblique (viscous first-mode type) instability waves for supersonic boundary-layer transition is mainly due to the strong exponential growth of these modes in the linear regime. Note, however, that not only the exponential growth in the linear regime can generate high-amplitude disturbance waves but also strong receptivity to external disturbances, for example at the neutral branch points as discussed by Fedorov & Khokhlov (2002) for hypersonic boundary layers. At the Mach numbers range of interest for this thesis (Mach 2-3.5) leading edge receptivity is the most dominant process for the generation of instability waves inside the boundary layer and hence, high-amplitude disturbances are mainly generated through exponential growth in the linear regime.

The previous chapter summarized the known nonlinear transition mechanisms for supersonic boundary layers. Using DNS, Thumm (1991) and Fasel et al. (1993) discovered oblique breakdown for a flat-plate boundary layer at Mach 1.6. This breakdown mechanism is initiated by the nonlinear wave interaction of two oblique instability waves with equal but opposite wave angles. Instead of finding evidence for the existence of oblique breakdown, Kosinov et al. (1994b) and Ermolaev et al. (1996) observed three-wave resonance triads with a primary oblique instability wave and two oblique subharmonic waves with different spanwise wavenumbers in their experiments for a flat-plate boundary layer at Mach 2. Furthermore, Corke et al. (2002) and Matlis (2003) also reported a subharmonic resonance of oblique instability waves
in their experiments for a conical boundary layer at Mach 3.5. There is a clear discrepancy between the findings from numerical simulations (DNS) published in the literature and the experimental investigations by Kosinov and his co-workers and by Corke et al. (2002) and Matlis (2003). If oblique breakdown is as dominant as stated by several authors in the literature (as for example by Adams & Sandham, 1993 for the flow conditions of the experiments by Kosinov et al., 1994b) oblique breakdown should have naturally occurred in these experiments. The present research addresses this discrepancy and other unresolved questions concerning oblique breakdown and asymmetric subharmonic resonance. In the following sections, the objective of this thesis is formulated and discussed in detail. Furthermore, the structure of the remaining part of this thesis is outlined.

2.1 Objective

The purpose of this dissertation is to study possible transition scenarios for supersonic boundary layers. The emphasis is on identifying dominant breakdown scenarios and the simulation of the entire transition path to turbulence. In the literature, different transition scenarios for supersonic boundary layers have been studied as already mentioned before. Only in a few investigations, the numerical results were compared to experimental findings as for example in Mack (1987) for the linear transition regime on a sharp cone. For the nonlinear regime of transition, no detailed comparison between numerical simulations and experiments exists. One reason for the lack of detailed comparisons is due to the insufficient measurements of the experimentalists. In Corke et al. (2002) and Matlis (2003) for example, measurements are primarily made at one azimuthal position and the three-dimensional disturbance field introduced by the disturbance generator is not known. Hence, it is very difficult to setup a numerical simulation that can match these experiments.

The insufficient data from the experiments also influences the interpretation of
the data. It is, for example, easier to measure different frequencies at one particular position in the flow field than different spanwise or azimuthal wavenumbers for one particular frequency. The knowledge of the three-dimensional disturbance field for one disturbance frequency is however mandatory for the identification of an oblique breakdown mechanism. Thus, the lack of information may lead to a misinterpretation of the experimental findings. As a consequence, transition mechanisms, where a subharmonic frequency is involved, are easier to identify. The experiments by Kosinov and his co-workers are the only experiments for a supersonic boundary layer that provide the complete three-dimensional disturbance field and hence, will be the focus of the first part of this thesis. Linear stability investigations and direct numerical simulations have been performed for this case in order address the following question:

1. **Question:** Can oblique breakdown be identified in the experiments by Kosinov and his co-workers?

Both oblique breakdown and asymmetric subharmonic resonance seem to be a viable transition path for supersonic Mach numbers. To date, it is unclear what mechanism is most dominant. Thus, the second objective of this thesis is to study both mechanisms in order to answer the second question:

2. **Question:** What is the most dominant breakdown mechanism for a supersonic boundary layer?

The experiments by Kosinov and his co-workers provide already for this objective an extensive data set and will help to address this question. Since new experiments are planned in the Mach 3.5 Quiet Wind Tunnel at NASA Langley for a conical boundary layer, the second question will be primarily discussed for this flow configuration. In order to model a broad background disturbance environment as is present in a natural transition scenario, the transition process initiated by a wave packet will be investigated. The naturally strongest breakdown mechanism should then transition the wave packet to a turbulent spot and yield the answer to the second question.

The final part of this thesis focuses on oblique breakdown and will be a continua-
tion of the studies summarized in Mayer (2004) and Husmeier et al. (2005). In these references, the early nonlinear regime of oblique breakdown for a flat-plate boundary layer at Mach 3 was investigated. If oblique breakdown, however, leads indeed to a fully developed turbulent boundary layer has not been completely clarified yet since only one numerical study by Jiang et al. (2006) exists that simulated the transition path of oblique breakdown for a flat-plate boundary layer at Mach 4.5. Hence, the last question that this thesis is addressing is:

3. **Question:** Does oblique breakdown lead to a fully developed turbulent boundary layer?

### 2.2 Outline of the Thesis

In the following two chapters, the governing equations and the numerical methods are described. The governing equations are introduced for a cone geometry. Moreover, the limitations for these equations (with respect to free-stream temperature) are discussed in detail. The chapter for the numerical method (chapter 4) covers two Navier-Stokes codes used for the direct numerical simulations (DNS) in this thesis. Both codes were developed in our CFD (computational fluid dynamics) laboratory. The first code (see Harris, 1997) was employed for all flat-plate simulations (chapter 6 and chapter 8) while with the second code (see Laible et al., 2008, 2009), a wave packet in a cone boundary layer was computed (chapter 7). Chapter 5 gives a short introduction to linear stability theory (LST) and thus, to the linear stability solvers by Mack (Mack, 1965) and Tumin (Tumin, 2007). These solvers were utilized for all linear stability investigations in this dissertation. A separate chapter for validation of these different numerical methods is omitted since in each result chapter, DNS results are compared to either linear stability theory, experiments, or both. Thus, every chapter has its own validation included. There are three different result chapters (6, 7, and 8). Each chapter addresses one specific question as stated in the previous section so that each
chapter can be read independently of the other result chapters. The result chapters address the questions in the same order as listed in section 2.1. Hence, chapter 6 focuses on question 1 while chapter 7 and 8 discuss question 2 and 3, respectively. In chapter 9, the main ideas of this dissertation are summarized.
3. Governing Equations

The flow considered here is governed by the unsteady, three-dimensional, compressible Navier–Stokes equations, consisting of the equations for the conservation of mass, momentum and energy. Using conservative variables and conical coordinates these equations can be written as follows

$$\frac{\partial U}{\partial t} + \frac{1}{r} \frac{\partial \left( r(E_c + E_d) \right)}{\partial x} + \frac{1}{r} \frac{\partial \left( r(F_c + F_d) \right)}{\partial y} + \frac{1}{r} \frac{\partial \left( G_c + G_d \right)}{\partial \varphi} + H = 0. \quad (3.1)$$

Here, \( U \) represents the vector of the conservative variables and is given as 
\[
U = [ \rho, \rho u, \rho v, \rho w, E_t ] ,
\]
where the symbols \( \rho \), \( u \), \( v \), \( w \) and \( E_t \) denote the fluid density, streamwise velocity, wall-normal velocity, azimuthal/spanwise velocity and total energy, respectively. The convective flux vectors \( (E_c, F_c, G_c) \), the viscous and heat conduction terms \( (E_d, F_d, G_d) \) and the source term \( (H) \) can be calculated from the previously introduced flow quantities and pressure \( p \) using the following equations.

\[
E_c = \begin{bmatrix}
\rho u \\
\rho u^2 + p \\
\rho w \\
\rho w v \\
(E_t + p) u
\end{bmatrix}, \quad F_c = \begin{bmatrix}
\rho v \\
\rho v^2 + p \\
\rho w \\
\rho w u \\
(E_t + p) v
\end{bmatrix}, \quad G_c = \begin{bmatrix}
\rho w \\
\rho w u \\
\rho w v \\
\rho w^2 + p \\
(E_t + p) w
\end{bmatrix},
\]

\[
E_d = \begin{bmatrix}
0 \\
-\tau_{xx} \\
-\tau_{xy} \\
-\tau_{x\varphi} \\
-\rho u \tau_{xx} - \rho v \tau_{xy} - \rho w \tau_{x\varphi} + q_x
\end{bmatrix}, \quad F_d = \begin{bmatrix}
0 \\
-\tau_{xy} \\
-\tau_{yy} \\
-\tau_{y\varphi} \\
-\rho u \tau_{xy} - \rho v \tau_{yy} - \rho w \tau_{y\varphi} + q_y
\end{bmatrix},
\]

\[
G_d = \begin{bmatrix}
0 \\
-\tau_{x\varphi} \\
-\tau_{y\varphi} \\
-\tau_{\varphi\varphi} \\
-\rho u \tau_{x\varphi} - \rho v \tau_{y\varphi} - \rho w \tau_{\varphi\varphi} + q_\varphi
\end{bmatrix},
\]

\[
H = \begin{bmatrix}
0 \\
-\frac{1}{r} \sin \theta_c (\rho w^2 - \tau_{\varphi\varphi} + p) \\
-\frac{1}{r} \cos \theta_c (\rho u^2 - \tau_{\varphi\varphi} + p) \\
\frac{1}{r} \sin \theta_c (\rho w - \tau_{x\varphi}) + \frac{1}{r} \cos \theta_c (\rho u - \tau_{y\varphi}) \\
0
\end{bmatrix}.
\]
The total energy is computed from the internal energy $i_e$ and the velocities as

$$E_t = \rho \left( i_e + \frac{1}{2}(u^2 + v^2 + w^2) \right). \quad (3.2)$$

The stresses in the conical coordinate system are determined by

$$\tau_{xx} = \frac{\mu}{Re} \left( \frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \frac{\partial v}{\partial y} - \frac{21}{3} \frac{\partial w}{\partial \phi} - \frac{21}{3} \frac{r}{r} \left( u \sin \theta_c + v \cos \theta_c \right) \right), \quad (3.3a)$$

$$\tau_{yy} = \frac{\mu}{Re} \left( \frac{4}{3} \frac{\partial v}{\partial y} - \frac{2}{3} \frac{\partial u}{\partial x} - \frac{21}{3} \frac{\partial w}{\partial \phi} - \frac{21}{3} \frac{r}{r} \left( u \sin \theta_c + v \cos \theta_c \right) \right), \quad (3.3b)$$

$$\tau_{\phi\phi} = \frac{\mu}{Re} \left( \frac{4}{3} \frac{1}{r} \frac{\partial w}{\partial \phi} - \frac{2}{3} \frac{\partial u}{\partial x} - \frac{21}{3} \frac{\partial v}{\partial y} + \frac{41}{3} \frac{r}{r} \left( u \sin \theta_c + v \cos \theta_c \right) \right), \quad (3.3c)$$

$$\tau_{xy} = \frac{\mu}{Re} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right), \quad (3.3d)$$

$$\tau_{x\phi} = \frac{\mu}{Re} \left( \frac{1}{r} \frac{\partial u}{\partial \phi} + \frac{\partial w}{\partial x} - \frac{1}{r} w \sin \theta_c \right), \quad (3.3e)$$

$$\tau_{y\phi} = \frac{\mu}{Re} \left( \frac{1}{r} \frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial y} - \frac{1}{r} w \cos \theta_c \right), \quad (3.3f)$$

and the heat-flux vector is given as

$$q = -\frac{\mu}{(\gamma - 1)RePrM^2} \left( \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{1}{r} \frac{\partial T}{\partial \phi} \right), \quad (3.4)$$

with $\mu$ as dynamic viscosity.

The conical coordinate system used for the equations above is illustrated in figure 3.1. The streamwise direction, parallel to the cone surface, is denoted by $x$. The wall-normal direction, orthogonal to the cone surface, is specified by $y$ and $\phi$ represents the azimuthal direction. The local radius $r$ can be calculated from the streamwise and wall-normal direction according to $r = x \sin \theta_c + y \cos \theta_c$, with $\theta_c$ as cone half angle. The governing equations in a rectangular coordinate system, which are applied for all flat-plate simulations, can be obtained from the equations in the conical coordinate system for $\theta_c = 0$ and $r \to \infty$. The equations in rectangular coordinates are, for example, discussed in Harris (1997).
The flow quantities are nondimensionalized with their approach-flow values, indicated by the subscript \( \infty \), except for the pressure and the total energy, which are scaled by the dynamic pressure \( \rho_\infty^* U_\infty^* \). Furthermore, the independent variables \( x \), \( y \) and \( t \) are nondimensionalized using a reference length scale \( L^* \) (and the approach streamwise velocity \( U_\infty^* \) for \( t \)). In both Navier–Stokes codes used for the present work, this reference length scale is an arbitrary, constant value and will always be denoted by \( L^* \) throughout this thesis whereas all linear stability solvers applied to the different problems in this thesis are based on a so-called viscous length scale \( L_e^* \), which represents an estimate for the local boundary-layer thickness

\[
L_e^* = \sqrt{\frac{\nu_e^* x_e^*}{U_e^*}}.
\]  

Here, \( \nu_e^* \) is the dimensional kinematic viscosity at the boundary layer edge (subscript \( e \)).

The nondimensionalization of the governing equations introduces the Reynolds number \( Re \), Mach number \( M \), Prandtl number \( Pr \) and the ratio of specific heats \( \gamma \) as nondimensional parameters

\[
Re = \frac{\rho_\infty^* U_\infty^* L^*}{\mu_\infty^*}, \quad Pr = \frac{\mu^* c_{p_\infty}^*}{k^*},
\]

\[
M = \frac{U^*}{a_\infty^*} = \frac{U_\infty^*}{\sqrt{(\gamma - 1) c_{p_\infty}^* T_\infty^*}} , \quad \text{and} \quad \gamma = \frac{c_{p_\infty}^*}{c_{p_\infty}^* - R},
\]  

with \( c_{p_\infty}^* \), \( k^* \), \( a_\infty^* \) and \( R \) being the specific heat at constant pressure, the thermal conductivity, the speed of sound and the specific gas constant for air of the approach
flow, respectively. For all simulations, the Prandtl number is assumed to be constant and has a value of 0.71. Hence, the Prandtl number relates the thermal conductivity $k$ directly to the viscosity $\mu$. If this is justified is discussed in section 3.2.

The Reynolds number based on the viscous length scale $L^*_e$ is usually called local Reynolds number since it is only valid for one particular streamwise position. This Reynolds number is denoted by

$$R_x = \frac{U^*_e L^*_e}{\nu^*_e} = \sqrt{\frac{U^*_e x^*_e}{\nu^*_e}} \quad (3.7)$$

throughout this thesis.

Three additional constraints for the internal energy $i_e$, the pressure $p$ and the dynamic viscosity $\mu$ are necessary in order to close the set of governing equations. These constraints will be introduced in two separate sections in order to discuss their applicability and limitations in greater detail.

### 3.1 Perfect Gas Assumption

In many aerodynamic applications the moving fluid (air) is considered to be a perfect gas. This assumption is valid for 99 percent of all practical aerodynamic problems as stated by Anderson (2000). For a perfect gas, the intermolecular forces (attractive/repulsive force) between the different molecules are neglected since the average distance between the molecules is assumed to be large. This is, for example, the case for air at standard conditions. However, if the distance between the molecules decreases, the intermolecular forces have to be accounted for and hence, affect the gas properties. A gas where intermolecular forces are important is called a “real gas” (Anderson, 2000). Conditions that require the fluid to be treated as a real gas are very high pressures and/or low temperatures ($T^* \sim 30K$) according to Anderson (2000). For the free-stream temperature range of about 100K-160K (“cold” wind-tunnel conditions) discussed in this thesis, real-gas effects might therefore not play
an important role.

If the temperature of the fluid is strongly increased, as for example by the viscous dissipation in a supersonic/hypersonic boundary layer at a surface, high-temperature effects become dominant. Note that Anderson (2000) distinguishes between real-gas effects and high-temperature effects. After his definition, which is also used throughout this thesis, real-gas effects occur only at low temperatures and/or high pressures and are connected to intermolecular forces. High-temperature effects, on the other hand, characterize, for example, the influence of the excitation of vibrational energy, dissociation and ionization. The excitation of vibrational energy of the molecules becomes important at a temperature of approximately 800K (Anderson, 2000). At this condition, the fluid can still be treated as a perfect gas, but the specific heats \( c_p \) and \( c_v \) depend on temperature. This leads to the definitions of a calorically perfect gas where the specific heats are constant, independent of temperature, and thermally perfect gas with \( c_p \) and \( c_v \) as a function of temperature.

In order to determine whether the fluid can be considered to be a calorically or thermally perfect gas for this thesis, an estimation of the surface temperature for an adiabatic flat plate in a Mach 3 flow is provided. For this calculation, the fluid is assumed to be a calorically perfect gas since this leads to a higher surface temperature than for a thermally perfect gas and is therefore a “worst” case scenario. For a calorically perfect gas, the surface temperature can be estimated from the equation for the total temperature combined with a recovery factor \( r \) that accounts for the heat losses in the boundary layer due to conduction and convection (Anderson, 2000)

\[
T_{aw}^* = T_\infty^* \left( 1 + r \frac{\gamma - 1}{2} M_\infty^2 \right) \quad \text{and} \quad r \sim \sqrt{Pr}.
\]

This results in a wall temperature of about 252K for a free-stream temperature of 100K. Note that the similarity solution for a compressible flat-plate boundary layer using the calorically perfect gas assumption predicts a very similar value for Mach 3. Moreover, this temperature value is far below the limit (800K) for the excitation
of vibrational energy of the molecules as stated by Anderson (2000) and thus, the calorically perfect gas assumption is valid.

Figure 3.2 shows how the specific heat $c_p$ at a constant pressure changes with temperature for air according to the tables compiled in Vargaftik (1975). The horizontal dotted line represents the value of $c_p$ for a calorically perfect gas. Clearly, for a temperature below about 160$K$, $c_p$ of air starts to depend strongly on pressure and temperature. This dependence is due to real-gas effects. The temperature limit of $T^* \sim 30K$ as stated by Anderson (2000), where real-gas effects start to play a dominant role, seems to be too low or only valid for low pressure ($\leq 1bar$). With increasing temperature, $c_p$ loses its dependence on pressure and air behaves as a thermally perfect gas. The specific heat $c_p$ deviates from the value for calorically perfect gas at about 400$K$, which is below the limit from Anderson (2000). Nevertheless, figure 3.2 confirms that in the temperature range important for this thesis (100$K$-252$K$), air can be considered as a calorically perfect gas. Therefore, all simulations summarized in this document have been performed with this assumption.
For a calorically perfect gas, the internal energy is given explicitly by

\[ i_e^* = c_v^* T^* \Rightarrow i_e = \frac{T}{(\gamma - 1) \gamma M^2}, \tag{3.9} \]

with \( c_p = \text{constant} \) and the ratio of specific heats

\[ \gamma = \frac{c_p}{c_v - R} = 1.4 = \text{constant}. \tag{3.10} \]

The gas pressure is computed from the equation of state for a perfect gas

\[ p = \frac{\rho T}{\gamma M^2}. \tag{3.11} \]

### 3.2 Viscosity Law and Thermal Conductivity

The only relationship that is missing for the closure of the governing equations is an equation for the viscosity \( \mu \). Equations (3.3) employ already the so-called Stoke’s assumption, which means that the bulk viscosity is set to zero. The bulk viscosity is a measure of the difference between the mechanical pressure, which is an average of the normal stress, and the thermodynamic pressure, which is given by equation (3.11).
The remaining viscosity coefficient $\mu$ in the governing equations is commonly computed from Sutherland’s law, which has the following nondimensional form

$$\mu = T^\frac{3}{2} \frac{1 + \frac{C}{T^\infty}}{T + \frac{C}{T^\infty}}.$$  \hspace{1cm} (3.12)$$

Here, $C = 110.4K$ and $T^\infty$ represents the dimensional temperature of the approach flow. The viscosity is nondimensionalized by its approach-flow value.

Figure 3.3a shows a comparison between the temperature dependence of the viscosity for air at different pressures from Vargaftik (1975) and the value obtained from Sutherland’s law. This figure demonstrates that for air, there is only a small influence of pressure on the viscosity, and Sutherland’s law can accurately reproduce the values from Vargaftik (1975) for a temperature range of 100K-300K. Moreover, for a constant Prandtl number of 0.71, the thermal conductivity $k$ can be obtained from Sutherland’s law and equations (3.6) as illustrated in figure 3.3b. Also for the thermal conductivity, there is a good agreement between the values from Vargaftik (1975) and the computed values using equations (3.6) for the temperature range of 100K-300K. Thus, the assumption of a constant Prandtl number of 0.71 is justified for the temperature range of interest and will be used throughout this thesis.
4. Numerical Method

After introducing the governing equations in the last chapter, in this chapter, the computational methods used for all direct numerical simulations (DNS) in this thesis are presented. The results for a flat plate (chapter 6 and 8) have been obtained by employing the Navier–Stokes code NSCC (Navier–Stokes Compressible in C) while the cone results were computed with a new code. Both codes were developed in our CFD laboratory (Harris, 1997; Laible et al., 2008, 2009). Section 4.1 explains the numerical method and boundary conditions implemented in NSCC. Additional information regarding this code can also be found in von Terzi (2004) and Husmeier (2008). Note, however, that with the actual version of NSCC used in this thesis cone geometries cannot be calculated as in Husmeier (2008). Instead other features have been implemented, as for example full Fourier transformations in spanwise direction. The new Navier–Stokes code is discussed in section 4.2. Only minor modifications to this code were necessary in order to generate wave packets in a cone boundary layer as illustrated in chapter 7.

4.1 NSCC Code

The governing equations from chapter 3 are solved on a rectangular coordinate system. An explicit fourth-order Runge–Kutta method (Ferziger, 1998) is used as time integration. Harris (1997) optimized the time integration in NSCC in order to reduce storage requirements. Applied to the following model ordinary differential equation

\[ \frac{d\phi}{dt} = f(t, \phi) , \]  

(4.1)
the explicit fourth-order Runge–Kutta method can be written as

1st substep:
\[ \phi_1 := \phi_n + \frac{\Delta t}{2} f(t, \phi_n), \]

2nd substep:
\[ \phi_2 := \phi_n + \frac{\Delta t}{2} f(t + \frac{\Delta t}{2}, \phi_1), \]
\[ \phi_1 := \phi_1 + 2\phi_2, \]

3rd substep:
\[ \phi_2 := \phi_n + \Delta t f(t + \frac{\Delta t}{2}, \phi_2), \]
\[ \phi_1 := \frac{1}{3} (-\phi_n + \phi_1 + \phi_2), \]

4th substep:
\[ \phi_{n+1} := \phi_1 + \frac{\Delta t}{6} f(t + \Delta t, \phi_2). \]

The symbol “:=” in equations (4.2) indicates that these equations are assignments and thus are only valid in the context of a programming language. Variables with the subscript \( n \) are from the old timestep \( n \) while variables with subscript \( n + 1 \) are the result of the time integration (new timestep). \( \Delta t \) is the stepsize in time.

The spatial derivatives in streamwise and wall-normal direction \((x, y)\) of equations (3.1), (3.3), and (3.4) are discretized using second-order one-sided finite differences inside the integration domain. These finite differences have the following form for grid point \( i \) and timestep \( n \)

\[ \left( \frac{\partial \phi}{\partial x} \right)_{i}^{+ \ n} \approx \frac{-\phi_{i+2}^{n} + 8\phi_{i+1}^{n} - 7\phi_{i}^{n}}{6\Delta x}, \]  
\[ \left( \frac{\partial \phi}{\partial x} \right)_{i}^{- \ n} \approx \frac{\phi_{i-2}^{n} - 8\phi_{i-1}^{n} + 7\phi_{i}^{n}}{6\Delta x}, \]

where \((+)\) denotes forward differencing and \((-)\) indicates backward differencing. In the literature, these stencils are usually referred to as “4th-order split” finite differences since averaging equations (4.3) leads to a standard 4th-order central difference
stencil. For every Runge–Kutta substep in equations 4.2, the integration direction (forward/backward differencing) of these one-sided stencils is altered. Altering the integration direction increases the overall accuracy of the numerical scheme. Originally, the idea of using low-order one-sided finite differences and altering the integration direction for different substeps of the time integration scheme was introduced by MacCormack (Tannehill et al., 1997). He used first-order one-sided differences for the spatial derivatives and a second-order Runge–Kutta method (Heun’s method) for the time integration. Although first-order one-sided differences are applied in MacCormack’s method, the overall accuracy in time and space is second order. A similar behavior can also be observed in the present case. Harris (1997) showed that by applying his numerical scheme to different model equations (wave equation and diffusion equation) a third-order method can be recovered.

Equations (4.3) are only valid for first derivatives. However, in equation (3.1), also second derivatives with respect to all spatial directions and cross derivatives, second derivatives with respect to two spatial directions, are present (in the stress and heat-flux terms). In NSCC, these derivatives are computed by employing twice the finite differences for a first derivative (equations 4.3) with opposite integration direction instead of using a finite difference stencil for a second derivative directly. Harris (1997) denotes the derivatives in equation (3.1) as “outer” derivatives and the derivatives in equations (3.3) and (3.4) as “inner” derivatives.

4.1.1 Domain Boundaries

The finite difference stencils used inside the integration domain cannot be applied at the domain boundaries and at the first grid points adjacent to the boundary points since the values of all flow variables beyond the boundaries are not known. Hence, at these grid points new finite difference stencils have to be employed, which are generally referred to as “boundary closure” in the literature. For the method developed
Figure 4.1 Illustration of the computational grid and the stencils implemented in NSCC for the streamwise direction \( x \) at one timestep \( n \). (○) denotes grid point where the spatial derivative is computed. (●) represents grid points that are used for the calculation of the spatial derivative at grid point (○). Note that this notation follows Kloker (1993).

by Harris (1997), only the grid points adjacent to the boundary points need to be calculated by a different stencil while the boundary points are set by the boundary conditions (section 4.1.3) except for the free-stream boundary, where the boundary condition is applied to both grid points. Harris (1997) chose the following finite difference stencils for the calculation of grid points adjacent to the boundary points

\[
\frac{\left( \frac{\partial \phi}{\partial x} \right)_n^2}{6\Delta x} \approx \frac{\phi_n^5 - 5\phi_n^4 + 10\phi_n^3 - 3\phi_n^2 - 3\phi_n^1}{6\Delta x}, \tag{4.4a}
\]

\[
\left( \frac{\partial \phi}{\partial x} \right)_n^{n_x-1} \approx \frac{-\phi_n^{n_x-4} + 5\phi_n^{n_x-3} - 10\phi_n^{n_x-2} + 3\phi_n^{n_x-1} + 3\phi_n^{n_x}}{6\Delta x}. \tag{4.4b}
\]

Note that for these stencils, averaging with the corresponding stencil from equations (4.3) results again in a fourth-order finite difference stencil. An illustration of the computational grid and the stencils implemented in NSCC for the streamwise direction \( x \) is given in figure 4.1. In this figure, (○) denotes the grid point where the spatial derivative is computed while (●) indicates all grid points that are used for the calculation of the spatial derivative at grid point (○). Figure 4.1 also shows the coefficients used for the calculation of a particular finite difference stencil.
4.1.2 Spanwise Discretization

The spanwise domain is assumed to be periodic and therefore, transformed into spectral space using Fast Fourier transforms. This has an implication on the calculation of any spanwise derivative since in spectral space, a derivative with respect to \( z \) for Fourier mode \( k \) at timestep \( n \) is defined as

\[
\left( \frac{\partial \tilde{\phi}}{\partial z} \right)_n^k \simeq \beta_k \tilde{\phi}_n^k \quad \text{and} \quad \left( \frac{\partial \tilde{\phi}}{\partial z} \right)_n^k \simeq -\beta_k \tilde{\phi}_n^k
\]

(4.5)

for cosine and sine modes, respectively. Here, the spanwise wavenumber \( \beta_k \) is obtained from

\[
\beta_k = \frac{2\pi k}{\lambda_z}.
\]

(4.6)

The spanwise wave length \( \lambda_z \) determines the spanwise domain extent of the DNS.

NSCC has two options for the Fourier transformations: (i) All flow variables (i.e. \( u \)-velocity, \( v \)-velocity, etc.) are assumed to be symmetric to the centerline, except for the spanwise velocity \( w \), which is antisymmetric. Symmetric quantities are then transformed into Fourier space using a Fourier cosine transformation and antisymmetric variables (\( w \)) are transformed by a Fourier sine transformation. Thus, only one-half spanwise wave length \( \lambda_z \) has to be computed for this configuration. (ii) No symmetry is assumed and therefore, all variables are transformed using a full Fourier transformation. This option requires the computation of the entire wave length \( \lambda_z \) in spanwise direction.

The Fourier transformations in NSCC are based on the VFFTPK library, which can be downloaded from netlib (http://www.netlib.org/vfftpack/). According to this library and its implementation in NSCC, a Fourier cosine transformation into spectral
space and its back transformation into physical space are given by

**physical→spectral:**

\[
\tilde{\phi}_k^c = \mathcal{F} (\phi)_k^c \sim \frac{1}{2(n_z - 1)} \left[ \phi_0^c + 2 \sum_{l=1}^{n_z-1} \phi_l^c \cos \left( \frac{\pi k l}{n_z - 1} \right) \right]
\]  

(4.7a)

**spectral→physical:**

\[
\phi_l^c = \mathcal{F}^{-1} (\tilde{\phi})_l^c \sim \tilde{\phi}_0^c + 2 \sum_{k=1}^{K-1} \tilde{\phi}_k^c \cos \left( \frac{\pi k l}{n_z - 1} \right)
\]

(4.7b)

for \( k = 0, ..., K - 1 \) and \( l = 0, ..., n_z - 1 \), respectively. \( \tilde{\phi}_k^c \) represent the Fourier amplitudes for mode \( k \). Moreover, \( n_z \) indicates the number of grid points used for resolving the spanwise direction in physical space over the interval \([0, (n_z - 1)\Delta z]\) with

\[
\Delta z = \frac{\lambda_z}{2(n_z - 1)}
\]

(4.8)

and \( K \) represents the number of modes in Fourier space (for the simulations \( n_z = 2K - 1 \)).

The Fourier sine transformation to spectral space and its back transformation into physical space are as follows

**physical→spectral:**

\[
\tilde{\phi}_k^s = \mathcal{F} (\phi)_k^s \sim -\frac{1}{(n_z - 1)} \sum_{l=1}^{n_z-1} \phi_l^s \sin \left( \frac{\pi k l}{n_z - 1} \right)
\]

(4.9a)

**spectral→physical:**

\[
\phi_l^s = \mathcal{F}^{-1} (\tilde{\phi})_l^s \sim -2 \sum_{k=1}^{K-1} \tilde{\phi}_k^s \sin \left( \frac{\pi k l}{n_z - 1} \right)
\]

(4.9b)

for \( k = 0, ..., K - 1 \) and \( l = 0, ..., n_z - 1 \), respectively.

In contrast to a symmetric simulation where only one-half of the spanwise wave length has to be calculated, an asymmetric simulation requires the entire spanwise wave length as computational domain. Hence, for symmetric simulations \( n_z \) represents the number of grid points in one-half wave length, whereas for asymmetric simulations,
this number depicts the grid points in one full spanwise wave length. In this case, the
grid spacing in spanwise direction is therefore obtained from
\[
\Delta z = \frac{\lambda_z}{(n_z - 1)}.
\] (4.10)
The full Fourier transformation for an asymmetric simulation are implemented in
NSCC according to

\[
\text{physical} \rightarrow \text{spectral}:
\]
\[
\tilde{\phi}_0 \sim \frac{1}{2n_z} \sum_{l=0}^{n_z-1} \phi_l
\] (4.11a)
\[
\tilde{\phi}_k^c \sim \frac{1}{n_z} \sum_{l=0}^{n_z-1} \phi_l \cos \left( \frac{2\pi kl}{n_z} \right)
\] (4.11b)
\[
\tilde{\phi}_k^s \sim \frac{1}{n_z} \sum_{l=0}^{n_z-1} \phi_l \sin \left( \frac{2\pi kl}{n_z} \right)
\] (4.11c)

\[
\text{spectral} \rightarrow \text{physical}:
\]
\[
\phi_l \sim \tilde{\phi}_0 + \sum_{k=1}^{K-1} \left[ \tilde{\phi}_k^c \cos \left( \frac{2\pi kl}{n_z} \right) + \tilde{\phi}_k^s \sin \left( \frac{2\pi kl}{n_z} \right) \right]
\] (4.11d)

with \( k = 0, ..., K - 1 \) and \( l = 0, ..., n_z - 1 \). As for the symmetric case, \( K \) denotes
the number of Fourier modes. The entire storage space in NSCC for the Fourier
modes is however \( 2K - 1 \) since the cosine modes and the sine modes have to be stored
separately.

The representation of the spanwise direction in Fourier space has one disadvantage:
Nonlinear terms in the governing equations (chapter 3) cannot be easily calculated in
Fourier space. Thus, for the computation of these terms, the governing equations are
transformed into physical space and thereafter, transformed back into spectral space.
This approach is commonly referred to as “pseudo-spectral” (Canuto et al., 1988).

4.1.3 Boundary Conditions

At the inflow, the conservative quantities \( \rho, \rho u_i \) and \( E_t \), obtained from the similarity
solution of a compressible flat-plate boundary layer, are specified. The no-slip and
no-penetration conditions are used at the wall except for the disturbance hole/slot. In addition, the wall is assumed to be adiabatic for the base flow, whereas temperature fluctuations at the wall are assumed to vanish. At the outflow, a buffer domain technique (Meitz & Fasel, 2000) is implemented to avoid reflections of disturbance waves from the outflow boundary. At the free-stream boundary, all total flow quantities are separated into the mean and disturbance quantities. For the mean flow quantities, a von Neumann condition is applied, whereas for the disturbance quantities, an exponential decay condition is employed, which was derived for compressible flow using linear stability considerations (Thumm, 1991). Harris (1997) provides details on the implementation of these boundary conditions in NSCC.

4.2 New Code

Also in the new code (Laible et al., 2008, 2009), the Navier–Stokes equations are integrated in time using the explicit fourth-order Runge–Kutta method. The implementation of this method in the code follows exactly Harris (1997) as explained in the previous section (equation 4.2).

For the spatial differentiation inside the computational domain, several different options are available. In this section, only the options that were used for the simulation of the wave packet in chapter 7 are discussed (for more detail, see Laible et al., 2008, 2009). The interior spatial discretization is mainly based on high-order accurate finite differences. In particular, the inviscid fluxes (convective terms $E_c$ and $F_c$ in equation 3.1) are divided in an upwind flux and a downwind flux using van Leer’s splitting (van Leer, 1982). Then 9th-order grid centered upwind differences (Zhong, 1998) are applied to evaluate the derivatives for these fluxes in $x$ and $y$-direction. These grid centered upwind differences are derived using a factor $\sigma$, which prescribes
the degree of upwinding,

\[
\left( \frac{\partial \phi}{\partial x} \right)_n^i = \sum_{k=i-N}^{i+N} c_k \left( \sigma \right) \phi_n^k - \sigma \Delta x_n \left( \frac{\partial^{(2N)} \phi}{\partial x^{(2N)}} \right)_n^i + \ldots .
\]

(4.12)

For \( \sigma = 0 \), the upwind scheme reduces to a standard central difference scheme. The \( c_k \)'s in equation (4.12) are the stencil coefficients, which are dependent on the factor \( \sigma \). \( \sigma \) is obtained from an eigenvalue analysis of the discretized (in space) linear wave equation (Zhong, 1998). For the \( 9^{th} \)-order upwind scheme \( (N = 5) \), Laible et al. (2008) obtained a value of \( \sigma = -1500 \) in order to stabilize the numerical scheme. \( \Delta x \) is the averaged grid spacing over the stencil interval and has the order of \( 2N - 1 \). The parameter \( N \) determines the number of grid points used for the stencil. Moreover, equation (4.12) clearly shows that for \( \sigma \neq 0 \) an additional error term is introduced, which reduces the order of the centered upwind difference scheme by one when compared to the central difference scheme with the same number of grid points.

The derivatives of the viscous terms \( (E_d \text{ and } F_d \text{ in equation 3.1}) \) and the source term \( (H) \) are calculated using \( 8^{th} \)-order central differences in streamwise direction and wall-normal direction. It is important to note that the second derivatives in the viscous terms are calculated directly instead of applying twice a first derivative stencil. This is in contrast to the method by Harris introduced in section 4.1 and improves the stability of the numerical scheme (Zhong, 1998).

All finite difference stencils are derived on a non-uniform grid. The coefficients \( (c_k \text{'s in equation 4.12}) \) are obtained from a Lagrange polynomial interpolation (Zhong & Tatineni, 2003). For a stencil based on \( N \) grid points with coordinates \( x_i \) and node values \( \phi_i \), the \( N - 1 \) degree polynomial is given by

\[
P_N(x) = \sum_{i=1}^{N} l_i(x)\phi_i, \quad \text{with} \quad l_i(x) = \frac{\prod_{l=1,l\neq i}^{N}(x - x_l)}{\prod_{l=1,l\neq i}^{N}(x_i - x_l)}.
\]

(4.13)

For example, a second derivative at grid point \( x_i \) can be calculated by differentiating
the polynomial $P_N(x)$ twice according to

$$
\left( \frac{d^2 \phi(x)}{dx^2} \right)_{x=x_i} = \left( \frac{d^2 P_N(x)}{dx^2} \right)_{x=x_i} = \sum_{j=1}^{N} b_{i,j} \phi_j, \quad (4.14)
$$

where the coefficients $b_{i,j}$ have different values for different grid points with index $i$ (non-uniform grid), and are given by

$$
b_{i,j} = \frac{d^2}{dx^2} (l_j(x))_{x=x_i}. \quad (4.15)
$$

### 4.2.1 Domain Boundaries

Since boundary closures based on high-order finite difference schemes may develop oscillations and hence, are usually unstable, Laible used different methods in order to stabilize the numerical scheme at the boundaries. Figures 4.2 and 4.3 summarize Laible’s approach for the grid centered upwind difference stencils used for the discretization of the convective terms. In streamwise direction the numerical scheme is mainly stabilized by employing standard central difference stencils with reduced order up to the last three points where the discretization is switched to one-sided 5th-order finite differences. Although the order is reduced from 9th-order upwind finite differences to one-sided 5th-order finite differences near the outflow, only a limited upstream effect of the lower-order numerical scheme is expected since this numerical scheme is utilized for supersonic and hypersonic transition simulations. Near the inflow such boundary treatment is not required since the initial condition is known upstream of the inflow boundary of the high-order DNS (more details can be found in section 4.2.3). Therefore, grid centered upwind finite differences can be applied up to the first grid point of the computational domain.

For the discretization of the viscous terms (second derivatives) near the inflow, the same approach as for the convective terms was chosen. The flow field is known upstream of the inflow and hence, high-order central difference stencils can be used up to the inflow. Near the outflow however, one-sided difference stencils based on the
Figure 4.2 Illustration of the computational grid and the stencils implemented in the new high-order code by Laible (Laible et al., 2008, 2009) for the inviscid terms ($E_c$ and $F_c$ in equation 3.1) in streamwise direction $x$ at one timestep $n$. ($\circ$) denotes grid point where the spatial derivative is computed. (●) represents grid points that are used for the calculation of the spatial derivative at grid point ($\circ$).

same number of grid points as the central difference stencils are employed. Thus, the order of the numerical scheme utilized as boundary closure for the viscous terms near the outflow does not need to be as strongly reduced as for the convective terms to stabilize the overall numerical scheme.

In order to maintain high-order boundary closures in wall-normal direction, Laible follows an approach suggested by Zhong & Tatineni (2003). At the wall, the interior high-order finite difference schemes (convective and viscous) are coupled with high-order boundary closures using one-sided finite differences (figure 4.3). To ensure stable boundary schemes, the grid is clustered near the wall according to the following function:

$$y_j = \frac{\arcsin \left( -\eta \cos \left( \frac{\pi j}{2 n_y} \right) \right)}{\arcsin \left( \eta \right)}. \quad (4.16)$$

Here, $j$ denotes the grid point index in wall-normal direction, $n_y$ is the total number of grid points in this direction and the parameter $\eta$ determines the degree of grid
Figure 4.3 Illustration of the computational grid and the stencils implemented in the new high-order code by Laible (Laible et al., 2008, 2009) for the inviscid terms ($E_c$ and $F_c$ in equation 3.1) in wall-normal direction $y$ at one timestep $n$. (○) denotes grid point where the spatial derivative is computed. (●) represents grid points that are used for the calculation of the spatial derivative at grid point (○).

Note that stretching is only applied at the wall boundary and not (like in Zhong & Tatineni, 2003) at the wall and free-stream boundary. Applying grid stretching only at the wall is advantageous for boundary layer simulations, since typically a rather strong grid stretching away from the wall is employed. The free stream does not need to be as highly resolved as the region close to the wall. The resulting numerically unstable boundary closure at the free stream is circumvented by enforcing the Dirichlet boundary condition (section 4.2.4), not only on the free stream boundary point, but also at the points next to the boundary. Hence as at the inflow, high-order one-sided finite differences are avoided at the free stream.

### 4.2.2 Azimuthal Discretization

The azimuthal direction in the high-order code by Laible was assumed to be periodic and therefore transformed into Fourier space using the VFFTPK library as discussed
in section 4.1.2. Hence, all spatial derivatives with respect to the azimuthal direction were calculated according to equation (4.5). Moreover, all flow quantities are assumed to be symmetric with respect to the centerline except for the azimuthal velocity \( w \), which is antisymmetric. Since the same library for the Fast Fourier Transformation (FFT) as in NSCC was also implemented in the new high-order code, the definition of the Fourier transformation follows equations (4.7) and (4.9). However, Laible applies a different scaling to all Fourier modes with \( k \neq 0 \). In his case, these Fourier modes have a value that is twice the magnitude of the modes defined in equations (4.7) and (4.9). For example, the transformation into spectral space from equation (4.7) can therefore be recast to

\[
\tilde{c}_0^c = F \left( c_0 \right) \sim \frac{1}{2(n_z - 1)} \left[ \phi_0^c + 2 \sum_{l=1}^{n_z-1} \phi_l^c \right] \tag{4.17}
\]

for the \( "0^{th}" \) Fourier mode and

\[
\tilde{c}_k^c = F \left( c_k \right) \sim \frac{1}{(n_z - 1)} \left[ \phi_0^c + 2 \sum_{l=1}^{n_z-1} \phi_l^c \cos \left( \frac{\pi kl}{n_z - 1} \right) \right] \tag{4.18}
\]

for all higher modes with \( k \neq 0 \). Note that although Laible implemented a different scaling factor for the FFT’s in the Navier-Stokes code, all wave packet simulations in chapter 7 were post-processed using equations (4.7) and (4.9) in order to maintain consistency throughout this thesis.

### 4.2.3 Simulation Strategy and Initial Condition

In this section, the strategy is explained how to obtain appropriate initial conditions (IC) for cone simulations presented in this thesis. In contrast to flat-plate boundary layers, no similarity solution exists for a cone boundary layer if the transverse curvature terms are considered (Malik & Spall, 1991). Thus, a different strategy, when compared to flat-plate simulations, has to be found to obtain an accurate IC for a cone. This strategy is demonstrated in figure 4.4. In a precursor simulation, using a
finite volume code developed by Gross (Gross & Fasel, 2002, 2008), the steady base flow for the entire cone geometry including the nose tip is calculated (figure 4.4a). This finite volume code solves the Navier–Stokes equations on a generalized coordinate system. The convective fluxes are discretized using a second-order symmetric total variation diminishing (TVD) upwind scheme while the viscous terms are calculated with a second-order accurate control volume approach. The time integration is based on an implicit Euler. This code can accurately predict the entire steady flow field for a cone. However, for unsteady transition simulations, the numerical scheme is too diffusive for capturing the correct spatial development of instability waves with a reasonable number of grid points. Therefore, a smaller part of the computational domain from the precursor simulation is extracted for the actual transition simulations, for which the high-order code developed by Laible (figure 4.4b) is employed. The flow field of this smaller domain is then interpolated on a new computational grid suitable for the high-order computations. Note that in this smaller domain the shock is also included. If the high-order finite differences as described in section 4.2 were used for the calculation of the shock, strong oscillations would be introduced. To avoid such oscillations, in the code by Laible, the order of the interior numerical

Figure 4.4 Illustration of the simulation strategy used for all cone simulations in chapter 7: (a) Integration domain for the precursor simulations of the steady base flow using a coarse grid and a low-order numerical scheme. Note the nose tip of the cone is included in this simulation. (b) Integration domain for the high-order simulations using the code developed by Laible. Initial Condition (IC) from the precursor DNS is interpolated on a new computational grid.
integration scheme is strongly reduced in the near-shock region.

4.2.4 Boundary Conditions

The inflow is separated into two regions: a subsonic region \((M < 1)\) close to the wall and a supersonic region \((M > 1)\). In the supersonic region, Dirichlet conditions for \(u, v, w, T, p\) and \(\rho\) are specified (e.g. obtained from the precursor calculation) while for the subsonic region, a non-reflecting boundary condition is adopted as suggested by Poinset & Lele (1992). On the cone surface, the no-penetration \((v = 0)\) and the no-slip \((u = 0, w = 0)\) conditions are enforced. The wall is set to be adiabatic for the steady base flow and temperature fluctuations are assumed to vanish for the unsteady simulations. At the outflow, a buffer domain technique is applied, where finite amplitude disturbances are ramped to zero (see Meitz & Fasel, 2000). Since for the simulations presented in chapter 7, the free stream is located above the oblique shock, Dirichlet conditions (for \(u, v, w, T, p, \rho\)) can be enforced at this boundary.
5. Compressible Linear Stability Theory

The spatial and temporal evolution of infinitesimal small disturbances in a boundary layer is governed by linear stability theory (LST). In all simulations discussed in this dissertation, the boundary layer is forced through a hole or disturbance slot in the wall using small disturbance amplitudes. Hence, the initial disturbance development in the boundary layer for these simulations should follow linear stability behavior. In order to validate whether the Navier–Stokes solvers used for such simulations can capture the initial linear disturbance development accurately, results of the DNS are compared to theoretical predictions of LST in each result chapter. Furthermore, results from LST can also be utilized to identify possible Craik-type resonances (Craik, 1971). Some basic concepts of LST are introduced in the following sections. The main ideas and results of LST summarized in this chapter are obtained from Mack (1969).

5.1 Characterization of a Disturbance

In linear stability theory (LST), disturbances have the form

$$
\phi'(x, y, z, t) = \hat{\phi}(y) \exp \left( i (\alpha x + \beta z - \omega t) \right).
$$

This is a general equation for a plane wave travelling at a specific wave angle $\psi$ with respect to the streamwise direction $x$. The disturbance amplitude $\hat{\phi}$ is only dependent on the wall-normal direction $y$. The wave angle is given by

$$
\psi = \arctan \left( \frac{\beta}{\alpha_r} \right),
$$

with

$$
\alpha_r = \frac{2\pi}{\lambda_x} \quad \text{and} \quad \beta = \frac{2\pi}{\lambda_z}
$$

as streamwise and spanwise/azimuthal wavenumbers, respectively. The streamwise wavenumber $\alpha_r$ is the real part of the complex wavenumber $\alpha$ in equation (5.1). If
the streamwise wavenumber $\alpha$ is assumed to be complex

$$\alpha = \alpha_r + i\alpha_i,$$  \hspace{1cm} (5.4)

disturbances grow spatially with $\alpha_i$ as streamwise amplification rate. In the case of spatial LST, the disturbance frequency $\omega$ is real while for temporal LST, where disturbances grow temporally, $\omega$ is complex and $\alpha$ is real. Disturbances with a negative value of $\alpha_i$ (positive $\omega_i$) experience streamwise (temporal) amplification while for positive values of $\alpha_i$ (negative $\omega_i$), disturbances decay. Disturbances that neither grow nor decay are referred to as neutral. As already indicated in equation (5.3), the spanwise or azimuthal wavenumber $\beta$ is always real. For a cone, the spanwise wavenumber $\beta$ is dependent on the so-called azimuthal mode number $k_c$, which represents the ratio of the cone circumference and the azimuthal wave length $\lambda_z$ at the cone surface. Figure 5.1 illustrates that for $k_c = 1$ the azimuthal wave length $\lambda_z$

\[
\begin{array}{ccc}
\lambda_z : & 2\pi r & \pi r & \ldots & 2\pi r/k_c \\
\lambda_z : & 1 & 2 & \ldots & k_c
\end{array}
\]

Figure 5.1 Definition of azimuthal mode number and azimuthal wave length for a cone.

corresponds to the cone circumference. In general for a cone, the azimuthal wave length $\lambda_z$ can be calculated from

$$\lambda_z (x) = \frac{2\pi r (x)}{k_c}.$$  \hspace{1cm} (5.5)

The cone radius $r$ is a function of the streamwise direction $x$

$$r (x) = x\sin (\theta_c).$$  \hspace{1cm} (5.6)
Note that in the framework of LST, \( k_c \) can only have integer values and consequently, only a discrete set of azimuthal wavenumbers \( \beta \) can exist. This is in contrast to a flat plate (with infinite dimension in \( z \)-direction), where the spanwise wavenumber \( \beta \) can in principle take any value.

The streamwise and spanwise/azimuthal wavenumbers are nondimensionalized using the viscous length scale \( L_e^* \),

\[
\alpha = \alpha^* L_e^* \quad \text{and} \quad \beta = \beta^* L_e^* .
\] (5.7)

The nondimensionalization of the streamwise direction \( x \) with \( L_e^* \) results in the local Reynolds number as introduced by equation (3.7)

\[
R_x = \frac{x^*}{L_e^*} = \sqrt{\frac{U_e^* x^*}{\nu_e^*}} .
\]

For the frequency, there are two commonly used nondimensionalizations, which are as follows:

\[
\omega = 2\pi \frac{f^* L_e^*}{U_e^*} \quad \text{and} \quad F = \frac{\omega}{R_x} = 2\pi \frac{f^* \nu_e^*}{U_e^* x^*} .
\] (5.8)

In the literature, \( F \) is often called “reduced” frequency and in contrast to \( \omega \), is independent of the viscous length scale \( L_e^* \). If the boundary layer edge conditions are constant, a constant value of \( F \) represents a constant dimensional frequency \( f^* \). This is not the case for \( \omega \) since this frequency is proportional to \( \sqrt{x} \).

5.2 Linearization of the Governing Equations

For the derivation of the linear stability equations, all flow quantities are decomposed into a mean flow (denoted by capital letters) and a disturbance component (denoted by ‘\(^\prime\)’). The mean flow is assumed to be parallel (\( V = 0, U = U(y) \) and \( T = T(y) \))
and two-dimensional ($W = 0$):

\begin{align*}
    u &= U(y) + u', \\
    v &= v', \\
    w &= w', \\
    p &= 1 + p', \\
    T &= T(y) + T', \\
    \rho &= \frac{1}{T(y)} + \rho'.
\end{align*}

(5.9)

Note that in contrast to the Navier–Stokes equations for the DNS (equation 3.1), pressure is nondimensionalized by its boundary edge value $p_e^*$ instead of the dynamic pressure and the mean-flow pressure is assumed to be constant in wall-normal direction (boundary-layer assumption). The velocities are nondimensionalized by $U_e^*$, temperature by $T_e^*$, density by $\rho_e^*$ and viscosity by $\mu_e^*$. As length scale, $L_e^*$ (equation 3.5) is used. Furthermore, it is assumed that the dynamic viscosity and thermal conductivity are only functions of temperature. This is appropriate since in this dissertation a calorically perfect gas with constant Prandtl number $Pr = 0.71$ is investigated (section 3.1) and the viscosity is given by Sutherland’s law. A Taylor series approximation for the viscosity and conductivity, where all higher-order terms for the temperature disturbance are neglected, yields

\begin{align*}
    \mu &= \mu(T) + \frac{d\mu(T)}{dT}T' \quad \text{and} \quad k = k(T) + \frac{dk(T)}{dT}T'.
\end{align*}

(5.10)

The decompositions in equations (5.9) and (5.10) are then substituted into the compressible Navier–Stokes equations (for a flat plate since in this thesis, all curvature and divergence terms are neglected for the linear stability analysis). If the resulting equations are linearized and the mean flow is subtracted, the disturbance equations for the continuity, $x$-momentum, $y$-momentum, $z$-momentum and energy are obtained (Mack, 1969; Balakumar & Malik, 1992):
continuity

\[ \frac{\partial \rho'}{\partial t} + \frac{1}{T} \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right) + v' \frac{d(T^{-1})}{dy} + U \frac{\partial \rho'}{\partial x} = 0, \]  

(5.11)

\(x\)-momentum

\[ \left( \frac{\partial u'}{\partial t} + U \frac{\partial u'}{\partial x} + v' \frac{dU}{dy} \right) = - \frac{T}{\gamma M_e^2} \frac{\partial p'}{\partial x} + \frac{T}{R_x} \left[ \mu \left( \frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2} \right) \right. 
\[ + \frac{\mu}{3} \left( \frac{\partial^2 u'}{\partial x \partial y} + \frac{\partial^2 u'}{\partial x \partial z} + \frac{\partial^2 w'}{\partial y \partial z} \right) + \frac{d\mu}{dT} \frac{d(T)}{dy} \left( \frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} \right) 
\[ + \frac{d\mu}{dT} \left( \frac{dU}{dy^2} \frac{T'}{dy} + \frac{dU}{dT} \frac{dT'}{dy} \right) + \frac{d^2\mu}{dT^2} \frac{dT}{dy} \frac{T'}{dy} \right], \]  

(5.12)

\(y\)-momentum

\[ \left( \frac{\partial v'}{\partial t} + U \frac{\partial v'}{\partial x} \right) = - \frac{T}{\gamma M_e^2} \frac{\partial p'}{\partial y} + \frac{T}{R_x} \left[ \mu \left( \frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2} + \frac{\partial^2 v'}{\partial z^2} \right) \right. 
\[ + \frac{\mu}{3} \left( \frac{\partial^2 u'}{\partial x \partial y} + \frac{\partial^2 u'}{\partial x \partial z} + \frac{\partial^2 w'}{\partial y \partial z} \right) + \frac{d\mu}{dT} \frac{d(T)}{dy} \left( \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial x} \right) 
\[ + \frac{d\mu}{dT} \left( \frac{dU}{dy^2} \frac{T'}{dy} + \frac{dU}{dT} \frac{dT'}{dy} \right) + \frac{d^2\mu}{dT^2} \frac{dT}{dy} \frac{T'}{dy} \right], \]  

(5.13)

\(z\)-momentum

\[ \left( \frac{\partial w'}{\partial t} + U \frac{\partial w'}{\partial x} \right) = - \frac{T}{\gamma M_e^2} \frac{\partial p'}{\partial z} + \frac{T}{R_x} \left[ \mu \left( \frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2} \right) \right. 
\[ + \frac{\mu}{3} \left( \frac{\partial^2 u'}{\partial x \partial z} + \frac{\partial^2 u'}{\partial y \partial z} + \frac{\partial^2 w'}{\partial y \partial z} \right) + \frac{d\mu}{dT} \frac{d(T)}{dy} \left( \frac{\partial w'}{\partial y} + \frac{\partial v'}{\partial x} \right) 
\[ + \frac{d\mu}{dT} \left( \frac{dU}{dy^2} \frac{T'}{dy} + \frac{dU}{dT} \frac{dT'}{dy} \right) + \frac{d^2\mu}{dT^2} \frac{dT}{dy} \frac{T'}{dy} \right], \]  

(5.14)

energy

\[ \left( \frac{\partial T'}{\partial t} + U \frac{\partial T'}{\partial x} + v' \frac{dT}{dy} \right) = -(\gamma - 1) T \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right) 
\[ + \frac{\gamma T}{Pr R_x} \left[ \mu \left( \frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y^2} + \frac{\partial^2 T'}{\partial z^2} \right) + \frac{d\mu}{dT} \frac{d^2T}{dy^2} T' + 2 \frac{d\mu}{dT} \frac{dT}{dy} \frac{dT'}{dy} + \frac{d^2\mu}{dT^2} \left( \frac{dT}{dy} \right)^2 T' \right] 
\[ + \frac{\gamma(\gamma - 1) T M_e^2}{R_x} \left[ 2 \frac{dU}{dy} \left( \frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} \right) + \frac{d\mu}{dT} \left( \frac{dU}{dy} \right)^2 T' \right]. \]  

(5.15)
The equation of state is
\[ p' = \frac{T'}{T} + \rho' T. \] (5.16)

The boundary conditions are
\[ y = 0 : \quad u' = 0, v' = 0, w' = 0, \text{ and } T' = 0, \]
\[ y \to \infty : \quad \text{all disturbances are bounded}. \]

These equations further simplify if their solutions are limited to plane harmonic
waves in the form of equation 5.1. The disturbance quantities are then replaced by
\[
\begin{bmatrix}
  u' \\
  v' \\
  w' \\
  \rho' \\
  T'
\end{bmatrix}
= \begin{bmatrix}
  \hat{u} \\
  \alpha \hat{v} \\
  \hat{w} \\
  \hat{\rho} \\
  \hat{T}
\end{bmatrix}
\exp(i(\alpha x + \beta z - \omega t)) .
\] (5.18)

Hence, the linearized Navier–Stokes equations can be recast to the following eigen-
value problem:

**continuity**
\[
i (U\alpha - \omega) \hat{\rho} + \frac{1}{T} \left( i\alpha \hat{u} + \alpha \frac{d\hat{v}}{dy} + i\beta \hat{w} \right) + \frac{d(T^{-1})}{dy} \alpha \hat{v} = 0 ,
\] (5.19)

**x-momentum**
\[
i (U\alpha - \omega) \hat{u} + \frac{dU}{dy} \alpha \hat{v} = -iT\alpha \hat{\rho} + \frac{T}{R_x} \left[ \mu \left( \frac{d^2\hat{u}}{dy^2} - (\alpha^2 + \beta^2) \hat{u} \right) \\
+ \mu \left( \frac{i\alpha^2 \hat{v}}{dy} - \alpha^2 \hat{u} - \alpha \beta \hat{w} \right) + \frac{d\mu}{dT} \left( \frac{d\hat{u}}{dy} + i\alpha^2 \hat{v} \right) \\
+ \frac{d\mu}{dT} \left( \frac{d^2\hat{U}}{dy^2} + \frac{d\hat{U}}{dy} \frac{d\hat{T}}{dy} \right) + \frac{d^2\mu}{dT^2} \frac{dU}{dy} \frac{dT}{dy} \hat{T} \right] ,
\] (5.20)

**y-momentum**
\[
i (U\alpha - \omega) \hat{v} = -\frac{T}{\alpha \gamma M_e^2} \frac{d\hat{p}}{dy} + \frac{T}{R_x} \left[ \mu \left( \frac{d^2\hat{v}}{dy^2} - (\alpha^2 + \beta^2) \hat{v} \right) \\
+ \frac{i\beta \alpha^2 \hat{w}}{dy} - \alpha^2 \hat{w} - \alpha \beta \hat{u} \right) + \frac{d\mu}{dT} \left( \frac{d\hat{v}}{dy} + i\alpha^2 \hat{w} \right) \\
+ \frac{d\mu}{dT} \left( \frac{d^2\hat{U}}{dy^2} + \frac{d\hat{U}}{dy} \frac{d\hat{T}}{dy} \right) + \frac{d^2\mu}{dT^2} \frac{dU}{dy} \frac{dT}{dy} \hat{T} \right],
\]
\[
\begin{align*}
+ \frac{\mu}{3} & \left( i \frac{d\hat{u}}{dy} + \frac{d^2\hat{v}}{dy^2} + i \frac{\beta}{\alpha} \frac{d\hat{w}}{dy} \right) + \frac{d\mu}{dT} \left( 2 \frac{dT \frac{d\hat{v}}{dy}}{dy} + i \frac{dU}{dy} \right) \\
& - \frac{2}{3} \frac{d\mu}{dT} \frac{dT}{dy} \left( i \frac{d\hat{u}}{dy} + \frac{d\hat{v}}{dy} + i \frac{\beta}{\alpha} \frac{d\hat{w}}{dy} \right),
\end{align*}
\]

(5.21)

\[z\text{-momentum}\]

\[
\begin{align*}
& i (U\alpha - \omega) \hat{w} = -i T \frac{\beta \hat{p}}{\gamma M_e^2} + \frac{T}{R_x} \left[ \mu \left( \frac{d^2\hat{w}}{dy^2} - (\alpha^2 + \beta^2) \hat{w} \right) \\
& + \frac{\mu}{3} \left( i \alpha \beta \frac{d\hat{v}}{dy} - \alpha \beta \frac{d\hat{u}}{dy} - \beta^2 \hat{w} \right) + \frac{d\mu}{dT} \frac{dT}{dy} \left( \frac{d\hat{w}}{dy} + i \alpha^2 \hat{v} \right) \right],
\end{align*}
\]

(5.22)

\[\text{energy}\]

\[
\begin{align*}
& i (U\alpha - \omega) \hat{T} + \frac{dT}{dy} \alpha \hat{v} = -(\gamma - 1) T \left( i \alpha \frac{d\hat{u}}{dy} + \alpha \frac{d\hat{v}}{dy} + i \beta \frac{d\hat{w}}{dy} \right) \\
& + \frac{\gamma T}{Pr R_x} \left[ \mu \left( \frac{d^2\hat{T}}{dy^2} - (\alpha^2 + \beta^2) \hat{T} \right) + \frac{d\mu}{dT} \left( \frac{d^2\hat{T}}{dy^2} + 2 \frac{dT}{dy} \right) + \frac{d^2\mu}{dT^2} \left( \frac{dT}{dy} \right)^2 \hat{T} \right] \\
& + \frac{\gamma (\gamma - 1) T M_e^2}{R_x} \left[ 2 \mu \frac{dU}{dy} \left( \frac{d\hat{u}}{dy} + i \alpha^2 \hat{v} \right) + \frac{d\mu}{dT} \left( \frac{dU}{dy} \right)^2 \hat{T} \right],
\end{align*}
\]

(5.23)

\[\text{equation of state}\]

\[
\hat{p} = \frac{\hat{T}}{T} + \hat{\rho} T.
\]

(5.24)

The boundary conditions are

\[
y = 0 : \quad \hat{u} = 0, \hat{v} = 0, \hat{w} = 0, \quad \text{and} \quad \hat{T} = 0,
\]

\[
y \to \infty : \quad \text{all eigenfunctions are bounded}.
\]

5.3 Inviscid Theory

To simplify the numerical procedure for solving the eigenvalue problem introduced above, Mack (1969) applied a coordinate transformation to the linear stability equations. For the inviscid case \(R_x \to \infty\), this transformation is especially advantageous because then the three-dimensional eigenvalue problem can be reduced to a
Figure 5.2 Illustration of the tilde coordinate system introduced by Mack (1969) in order to reduce the three-dimensional inviscid eigenvalue problem to a two-dimensional form. Here, in contrast to the more general approach of Mack (1969), only a two-dimensional boundary layer is considered.

two-dimensional one. Figure 5.2 illustrates the rotated coordinate system \((\tilde{x}, y, \tilde{z})\) according to Mack. In this coordinate system, the \(\tilde{x}\)-axis is in the direction of the wave propagation (perpendicular to the wave front) while the \(x\)-axis is in the direction of the free stream. The angle between the \(x\)- and \(\tilde{x}\)-axes is the wave angle \(\psi\). Since the \(\tilde{z}\)-axis is along the wave front of the harmonic wave, equation (5.1) reduces to

\[
\phi' (\tilde{x}, y, t) = \hat{\phi}(y) \exp \left( i \left( |\vec{k}| \tilde{x} - \omega t \right) \right),
\]

with

\[
\tilde{x} = \cos(\psi) x + \sin(\psi) z,
\]

and

\[
\cos(\psi) = \frac{\alpha_r}{|k|} \text{ and } \sin(\psi) = \frac{\beta}{|k|},
\]

where \(\vec{k}\) represents the wavenumber vector

\[
\vec{k} = \begin{bmatrix} \alpha_r \\ \beta \end{bmatrix}.
\]

Hence, in the stability equations, all derivatives of the disturbance quantities are zero with respect to \(\tilde{z}\). Note, for convenience, in equation (5.26) and in the following discussion the temporal approach of LST is used. The disturbance frequency \(\omega\) is
therefore assumed to be complex and $\alpha$ is real ($\alpha = \alpha_r$)

$$\omega = \omega_r + i \omega_i = \alpha_r \left( c_{ph,x}^e + i c_{ph,x}^i \right),$$  \hspace{1cm} (5.30)$$

with $c_{ph,x}^e$ denoting the phase speed in streamwise direction ($x$) and $\alpha_r c_{ph,x}^i$ representing the temporal amplification rate.

The transformation relations for the eigenfunctions defined by equation (5.18) between both coordinate systems are as follows:

$$\ddot{u} = \dot{u} + \tan(\psi) \dot{w}, \hspace{1cm} (5.31a)$$
$$\ddot{v} = \dot{v}, \hspace{1cm} (5.31b)$$
$$\ddot{w} = -\tan(\psi) \dot{u} + \dot{w}, \hspace{1cm} (5.31c)$$
$$\ddot{p} = \dot{p}, \hspace{1cm} (5.31d)$$
$$\ddot{T} = \dot{T}, \hspace{1cm} (5.31e)$$
$$\ddot{\rho} = \dot{\rho}. \hspace{1cm} (5.31f)$$

5.3.1 Neutral Solutions

Dropping the viscous terms and transforming the stability equations into the new coordinate system leads to the two-dimensional inviscid stability equations. Mack (1969) states that there are several forms of the inviscid equations that are helpful to address different properties of the inviscid case. Lees & Lin (1946), for example, derived the compressible version of the Rayleigh equation

$$\frac{d}{dy} \left[ \frac{(U - c_{ph,x}) \frac{d\tilde{v}}{dy} - \frac{dU}{dy} \tilde{v}}{\left( 1 - \tilde{M}_{rel}^2 \right) T} \right] = \frac{|\tilde{K}|^2}{T} (U - c_{ph,x}) \tilde{v},$$  \hspace{1cm} (5.32)$$

where $\tilde{M}_{rel}$ denotes the relative Mach number

$$\tilde{M}_{rel} = \frac{U - c_{ph,x}}{\sqrt{T}} M_e,$$  \hspace{1cm} (5.33)$$
and $\tilde{M}_e$ is the Mach number in the tilde coordinate system

$$
\tilde{M}_e = M_e \cos (\psi ) .
$$

The physical meaning of $\tilde{M}_{rel}$ can be better understood if neutral disturbances are considered ($c^i_{ph,x} = 0$). In this case, $\tilde{M}_{rel}$ represents the local Mach number of the mean flow relative to the phase velocity in the direction of the wave propagation

$$
\tilde{M}_{rel} = \frac{U^*(y) - c^*_{ph,x}}{a^*(y)} \cos (\psi ) .
$$

Moreover, $\tilde{M}_{rel}$ is a function of the wall-normal direction $y$.

Equation (5.32) is a second-order differential equation for the wall-normal velocity eigenfunction $\tilde{v}$ with two boundary conditions:

$$
\tilde{v}(0) = 0 \text{ and } \tilde{v} \text{ bounded as } y \to \infty .
$$

Using

$$
\xi (y) = \frac{1}{(1 - \tilde{M}^2_{rel}) T},
$$

equation (5.32) can be rewritten in the following form (Henningson & Schmid, 2001)

$$
\frac{d}{dy} \left[ \xi (y) \frac{d\tilde{v}}{dy} \right] = \left[ \frac{d}{dy} \left( \frac{\xi (y) \frac{dU}{dy}}{U - c_{ph,x}} \right) + \frac{|k|^2}{T} \right] \tilde{v} .
$$

In this form, it is straightforward to recognize the singularity for $U(y_c) = c_{ph,x}$ ($\tilde{M}_{rel} = 0$) in equation (5.38) for a neutral disturbance. The wall-normal position $y_c$ of the singularity is called “critical layer”. A solution for equation (5.38) according to Frobenius’ method can only exist at $y_c$ if the singularity is regular. Therefore, the quantity

$$
\frac{d}{dy} \left( \xi (y) \frac{dU}{dy} \right) = \frac{d}{dy} \left( \frac{1}{T} \frac{dU}{dy} \right)
$$

has to vanish at the critical layer. The wall-normal position (critical layer), where this quantity has a value zero, is also called “generalized inflection point”. Furthermore, as
proven by Lees & Lin (1946), a generalized inflection point somewhere within the boundary layer is a necessary and sufficient condition for the existence of a neutral, inviscid subsonic disturbance. The terminology “subsonic” in the previous statement stems from a classification of neutral disturbances based on $\tilde{M}_{rel}$ at the boundary layer edge ($U = T = 1$ for $y \geq \delta$) introduced by Lees & Lin (1946):

$$|\tilde{M}_{rel}(y \geq \delta)| < 1 \quad \left( c_{ph,x} > 1 - \frac{1}{\tilde{M}_e} \right) : \text{subsonic,} \tag{5.40a}$$
$$|\tilde{M}_{rel}(y \geq \delta)| = 1 \quad \left( c_{ph,x} = 1 - \frac{1}{\tilde{M}_e} \right) : \text{sonic,} \tag{5.40b}$$
$$|\tilde{M}_{rel}(y \geq \delta)| > 1 \quad \left( c_{ph,x} < 1 - \frac{1}{\tilde{M}_e} \right) : \text{supersonic.} \tag{5.40c}$$

In addition, Lees & Lin (1946) could show that a generalized inflection point at a wall-normal position $y_c$ greater than $y_0$, where $y_0$ is the point at which $c_{ph,x} = 1 - \frac{1}{\tilde{M}_e}$, is a necessary condition for the existence of an amplified disturbance. The amplification rate is related to the difference in phase speed $c_{ph,x}(y_c) - c_{ph,x}(y_0)$; that is, as $c_{ph,x}(y_c) \to c_{ph,x}(y_0)$, $\alpha_r c_{ph,x} \to 0$. This is however only true for a particular type of disturbances, the so-called “first-mode” disturbances, which are an extension of the incompressible Tollmien-Schlichting instability waves. Further information on amplified disturbances can be found in section 5.3.2.

Mack (1969) first realized that different types of disturbances exist, which are solutions of the inviscid stability problem. If the compressible version of the Rayleigh equation (equation 5.32) is recast in terms of the disturbance pressure eigenfunction instead of wall-normal velocity

$$\frac{d^2 \tilde{p}}{dy^2} - \frac{d}{dy} \left[ \ln \left( \tilde{M}_{rel}^2 \right) \right] \frac{d \tilde{p}}{dy} - |\tilde{k}|^2 \left( 1 - \tilde{M}_{rel}^2 \right) \tilde{p} = 0, \tag{5.41}$$

one can see that this equation changes its behavior when $(1 - \tilde{M}_{rel})$ changes its sign. If the second term on the left-hand side is neglected (which is possible for large $|\tilde{k}|^2$ according to Mack, 1969), the remaining equation is elliptical in nature for $\tilde{M}_{rel} < 1$ and a wave equation for $\tilde{M}_{rel} > 1$. Hence, multiple solutions can exist when $\tilde{M}_{rel} > 1$
(“region of supersonic relative flow”) somewhere in the boundary layer since there is an infinite sequence of wave lengths which satisfy the boundary conditions. Moreover, Lees & Lin (1946) provided the theorem that there is only a unique wavenumber corresponding to one phase speed for the neutral subsonic disturbance if \( \tilde{M}_{rel} < 1 \) throughout the entire boundary layer.

Mack (1969) termed the multiple solutions of the inviscid stability problem modes. The different modes can be distinguished by the number of zeros in their pressure eigenfunction. If there is no zero in the pressure eigenfunction of a neutral subsonic disturbance, the disturbance is a so-called “first mode” while one zero in the pressure eigenfunction indicates a “second mode”. In general, an \( n^{th} \) mode of a neutral subsonic disturbance has \( n - 1 \) zeros in its pressure eigenfunction. For insulated wall boundary layers, the second mode and the higher modes only appear for a free-stream Mach number of \( 2.2 \) and higher.

A further consequence of a region of supersonic relative flow (\( \tilde{M}_{rel} > 1 \)) in the boundary layer is the existence of another class of disturbance waves that are independent of the generalized inflection point. These disturbances are characterized by having phase velocities in the range

\[
1 \leq c_{ph,x} \leq 1 + \frac{1}{M_c}
\]

and were first discovered by Mack (1969). For a phase velocity greater than unity, equation (5.38) does not have a singularity since the streamwise velocity \( U \) has a maximal value of one in the free stream \( (U(y \geq \delta) = 1) \). Because of the absence of a singularity and therefore a critical layer, Mack (1969) referred to these disturbances as “regular neutral solutions”.

A summary of all neutral, plane disturbances for the inviscid stability problem of a boundary layer is presented in figure 5.3. As already mentioned before (equations 5.40), disturbances can be classified according to their phase velocity. There are mainly two different groups of disturbances. Disturbances that result from the
boundary layer and are therefore commonly termed “boundary-layer modes” (discrete spectrum) and disturbances that are travelling in the free stream (continuous spectrum). The boundary-layer modes consist of the supersonic, subsonic and regular disturbances as discussed by Mack (1969) while the free-stream modes are of acoustical, entropic and vortical nature. “Slow” acoustic waves have the phase speed \( c_{ph,x} = 1 - 1/\tilde{M}_e \) and propagate upstream relative to the free stream. Entropy and vorticity waves propagate with the free stream and “fast” acoustic waves \( (c_{ph,x} = 1 + 1/\tilde{M}_e) \) travel downstream relative to the free stream.

5.3.2 Amplified and Damped Solutions

The previous section introduced the main classes of neutral disturbances for a high-speed boundary layer. The primary interest of linear stability theory, however, is on amplified disturbances since these disturbances will eventually transition the flow to turbulence. Figures 5.4a and 5.4b (all data presented in the figures of this chapter are digitized from Mack, 1969) show the real and imaginary part of the complex eigenvalue \( c_{ph,x}^r + ic_{ph,x}^i \) as a function of the streamwise wavenumber for two-dimensional disturbances. These diagrams are typical for low and moderate supersonic Mach numbers up to Mach 4.4 (Mack, 1969), which is also the Mach number range of interest for this dissertation. The streamwise wavenumbers of neutral subsonic disturbances and
neutral regular disturbances are also included in this figure. \( \alpha_{sn} \) denotes the streamwise wavenumber of the neutral subsonic mode \( n \). For example, a first-mode neutral subsonic disturbance has the streamwise wavenumber \( \alpha_{s1} \) and a second-mode neutral subsonic disturbance has \( \alpha_{s2} \) as its streamwise wavenumber. A similar nomenclature is used by Mack for the regular neutral disturbances. \( \alpha_{r1} \) represents a first-mode regular neutral solution while \( \alpha_{rn} \) would denote a \( n^{th} \)-mode regular neutral solution. The distinction between different regular neutral modes is according to the zeros (phase change) in the pressure eigenfunction as discussed for the subsonic neutral solutions in the previous section.

The eigenvalues for \( c_{ph,x}^i \neq 0 \) in figure 5.4a lie on two separate curves and therefore, form two distinct “families” of solutions. Mack (1969) states that it is easier to discuss the inviscid amplified or damped solutions in terms of these families than in terms of modes as for the neutral solutions from the previous section. He distinguishes between different families according to their origin in the complex phase speed diagram. For example in figure 5.4a, the solid curve originates from \( \alpha_r = 0 \) with a phase speed of \( c_{ph,x} = 1 - 1/M_e \) while the dashed curve starts at \( \alpha_{r1} \) with \( c_{ph,x} = 1 \). Since the mode
characteristics change along these eigenvalue curves and are often not well defined, it is not possible to assign one curve to a particular mode. For example, on the dashed curve between $\alpha_{r1}$ and $\alpha_{s2}$ in figure 5.4a, the pressure eigenfunctions for the different eigenvalues change from a “first-mode regular neutral”-type eigenfunction with zero phase change to a “second-mode subsonic neutral”-type eigenfunction with one phase jump. Because of this phase change, typical for second-mode neutral solutions (regular and subsonic), Mack (1969) terms the amplified disturbances between $\alpha_{r1}$ and $\alpha_{s2}$ “second-mode amplified solutions”. The solutions on the solid curve starting at $\alpha_r = 0$ are called “first-mode amplified solutions” in the region between $\alpha_r = 0$ and $\alpha_{s1}$ and “second-mode damped solutions” near the minimum between $\alpha_{r1}$ and $\alpha_{s2}$. Again, this distinction is based on the phase change in the pressure eigenfunction of the different solutions.

An important result of Mack’s numerical investigations is that first-mode amplified solutions have a phase speed $c_{ph,x}$ between the phase speed for a sonic disturbance $c_{ph,x}(y_0) = 1 - 1/\bar{M}_e$ and the phase speed at the critical layer (generalized inflection point) $c_{ph,x}(y_c)$, hence

$$c_{ph,x}(y_0) < c_{ph,x} < c_{ph,x}(y_c).$$

As a consequence of this restriction, the amplification rate of first-mode amplified solutions is directly related to the difference in phase speed $\Delta c_{ph,x} = c_{ph,x}(y_c) - c_{ph,x}(y_0)$. Figure 5.5a shows $c_{ph,x}(y_c)$ and $c_{ph,x}(y_0)$ as a function of edge Mach number $M_e$ for several different free-stream temperatures $T_e^*$. For all curves, the difference $\Delta c_{ph,x}$ has a minimum near $M_e = 1.6$ while for larger Mach numbers the difference increases and finally levels off. The maximal temporal amplification rate $(\alpha_r c_{ph,x}^i)_{max}$ for two-dimensional first-mode amplified disturbances exhibits a very similar behavior with a minimum near Mach 1.5. The influence of the phase difference $\Delta c_{ph,x}$ on the amplification rate of first-mode disturbances suggests that anything that changes $c_{ph,x}(y_c)$ or $c_{ph,x}(y_0)$ affects the amplification rate of first-mode disturbances. For
Figure 5.5 Effect of free-stream temperature on the stability behavior of a supersonic/hypersonic boundary layer with insulated wall: (a) Phase velocities as function of Mach number \( M_e \), (b) maximum amplification rates of two-dimensional first- and second-mode amplified disturbances for different Mach numbers. Reproduced from Mack (1969), figures 11.13 and 11.14.

example, by increasing the free-stream temperature and keeping the wall insulated, the curve for \( c_{ph,x}(y_c) \) moves closer to \( c_{ph,x}(y_0) \) (figure 5.5a). Note that \( c_{ph,x}(y_0) = 1 - 1/M_e \) is not dependent on the free-stream temperature and therefore, stays constant.

The maximal amplification rate as a function of the free-stream temperature is depicted in figure 5.5b for several Mach numbers. Clearly, as expected from figure 5.5a, an increase in free-stream temperature reduces the temporal amplification rate of first-mode amplified disturbances. Also shown in figure 5.5b are the maximal amplification rates for second-mode amplified disturbances. The effect of the free-stream temperature on a second-mode amplified disturbance can be either stabilizing or destabilizing depending on the Mach number. Since second-mode neutral disturbances (regular or subsonic) are primarily a result of the supersonic relative flow region (\( \tilde{M}_{rel} > 1 \)), the amplification rates of second-mode amplified disturbances are mainly influenced by the wall-normal extent of this region. Therefore, anything that changes the thickness of the supersonic relative flow region also affects second-mode (and higher) amplified disturbances. Figure 5.5b further indicates that for two-dimensional disturbances, second-mode amplified solutions have always the highest amplification rates when
compared to first-mode amplified solutions for Mach numbers $M_e \geq 2.2$.

So far, only the stability behavior of two-dimensional disturbances have been discussed in this section. Results for three-dimensional disturbances can be obtained from the two-dimensional stability equations in the tilde coordinate system from figure 5.2. In this coordinate system, the Mach number $\tilde{M}_e$ of the mean flow in direction of the wave propagation is defined by equation (5.34) with $\psi$ as the wave angle ($\tilde{M}_e = M_e \cos(\psi)$). The change of the Mach number $\tilde{M}_e$ with respect to the wave angle $\psi$ has direct implications on first-mode and second-mode amplified disturbances. The phase speed of a sonic disturbance $c_{ph,x}(y_0) = 1 - 1/\tilde{M}_e$ decreases as $\psi$ increases from 0° to 90° while the phase speed of the critical layer $c_{ph,x}(y_c)$ remains unchanged since the mean velocity and temperature profiles are fixed. Consequently, the difference $\Delta c_{ph,x} = c_{ph,x}(y_c) - c_{ph,x}(y_0)$ increases and this leads to a destabilization of first-mode amplified disturbances. At the same time, the wall-normal thickness of the supersonic relative flow region decreases with $\tilde{M}_e$ (equation 5.33) yielding a stabilization of second-mode and higher-mode amplified disturbances. Both trends are illustrated in figure 5.6 for a Mach 8.0 boundary layer with an insulated wall and a free-stream temperature of $T^* = 50 K$. This figure shows the temporal amplification rate as a function of the streamwise wavenumber. For two-dimensional waves, the amplified region of the first three modes are merged (solid line) and second-mode amplified disturbances are most unstable. With increasing wave angle $\psi$, first-mode amplified disturbances are destabilized until they reach their highest amplification rate at about $\psi = 56^\circ$. Since the streamwise wavenumber $\alpha_r$ approaches zero for $\psi \to 90^\circ$, the decrease in $\alpha_r$ eventually outweighs the increase in amplification rate due to the increase of $c_{ph,x}(y_c) - c_{ph,x}(y_0)$ and therefore, $\alpha_r c_{ph,x}^2$ starts to decrease with further increase of $\psi$ (graph for $\psi = 60^\circ$ in figure 5.6). As expected, the second-mode and third-mode amplified solutions are stabilized with increasing $\psi$. At about $\psi = 45^\circ$, only the first two unstable regions are still visible while at $\psi = 60^\circ$, the second-mode unstable region has completely vanished.
5.4 Viscous Theory

In order to investigate the influence of viscous effects on the stability behavior of a high-speed boundary layer the complete linearized Navier–Stokes equations (equation 5.19 to 5.24) have to be solved numerically. As for the inviscid case, Mack (1969) transformed equations (5.19) to (5.24) into the rotated (tilde) coordinate system illustrated in figure 5.2. In this coordinate system, the governing equations reduce to a “nearly two-dimensional form”. Despite the existence of the $\tilde{z}$-momentum equation, the three-dimensional form of the governing equations differs only from the two-dimensional form in a dissipation term in the energy equation involving the velocity component parallel to the wave front ($\tilde{w}'$). Mack (1969) showed that, by neglecting this dissipation term, only a small error in the calculation of the maximal temporal amplification rate for oblique disturbances is introduced, while the computational cost is considerable reduced since only two-dimensional equations have to be solved. Today, neglecting this dissipation term and therefore, reducing the three-dimensional problem to a two-dimensional form is not necessary since the computational resources...
increased significantly in the last 40 years.

The terminology introduced for the inviscid case is also used for the viscous theory. In particular, a first-mode amplified solution of the inviscid stability problem has a viscous counterpart, which is termed “viscous first-mode amplified disturbance” in this section. The same convention is also employed for second-mode amplified disturbances. The distinction between viscous first-mode and viscous second-mode disturbances is again based on the phase change in the pressure eigenfunction as explained in section 5.3.1. However, for the viscous problem, it is even harder to assign a disturbance a particular mode characteristic, especially for high Mach numbers.

In the following sections, the main results of viscous linear stability theory are summarized with an emphasis on lower Mach numbers.

5.4.1 Amplified and Damped Solutions

Mack (1969) reports that, depending on the Mach number, the wave angle and the mode type (first or second mode) of the disturbance, viscosity can have both, a stabilizing and destabilizing effect. In figure 5.7, neutral stability curves for two-dimensional disturbances are plotted against $1/R_x$ for different Mach numbers from 1.6 to 3.8. The streamwise wavenumbers $\alpha_{s1}$ for two-dimensional neutral subsonic disturbances at $1/R_x = 0$ denote the inviscid limit for the different Mach numbers.

At all Mach numbers, the neutral stability curves in figure 5.7 approach the inviscid limit for $R_x \to \infty$. For the inviscid case, the region between $\alpha_r = 0$ and $\alpha_{s1}$ includes first-mode amplified solutions as discussed for Mach 3.8 in figure 5.4a. Hence, it can be concluded that the neutral stability curves in figure 5.7 define the boundary of the viscous counterpart of a first-mode amplified region. At Mach 1.6, for low Reynolds numbers, the unstable region is at higher streamwise wavenumbers than the inviscid limit. Thus, for this Mach number and at low Reynolds numbers, viscosity destabilizes disturbances with streamwise wavenumber larger than $\alpha_{s1}$ and
Figure 5.7 Neutral stability curves for two-dimensional disturbances as a function of inverse of local Reynolds number $1/R_x$ and streamwise wavenumber $\alpha_r$ for several different Mach numbers (insulated wall and free-stream temperature $T_e = 80K$). Reproduced from Mack (1969), figure 12.2.

stabilizes disturbances for lower streamwise wavenumber. This destabilizing effect decreases with increasing Mach number. At $M_e = 3.8$, for example, viscosity only stabilizes viscous first-mode amplified disturbances at all finite Reynolds numbers.

Figure 5.8 provides a clearer picture of the influence of viscosity on the stability behavior of viscous first (figure 5.8a) and second-mode (figure 5.8b) amplified two-dimensional disturbances. In this figure, the maximal temporal amplification rate $(\alpha_r c_{ph,x})_{max}$ is plotted against local Reynolds number $R_x$ for several Mach numbers. As explained in figure 5.7, at low supersonic Mach numbers ($M_e = 1.6, 2.2,$ and $2.6$), viscosity has a destabilizing effect on viscous first-mode disturbances for low Reynolds numbers in figure 5.8a. At Mach 3.0, this effect disappears and viscosity stabilizes for all finite Reynolds numbers. Viscous second-mode amplified disturbance (figure 5.8b), on the other hand, are always stabilized by viscosity.

The destabilizing effect of viscosity on first-mode disturbances decreases with increasing wave angle. Therefore, three-dimensional first-mode disturbances are less destabilized than two-dimensional first-mode disturbances. Figure 5.9 shows the
maximal temporal amplification rate of inviscid and viscous first and second-mode amplified solutions as a function of edge Mach number. The thin lines represent the inviscid limit. While inviscid, two-dimensional second-mode amplified disturbances are always more amplified than two-dimensional, inviscid first-mode disturbances for Mach numbers larger than 2.2; this is not the case if three-dimensional, inviscid first-mode amplified disturbances are considered. The curve for two-dimensional, inviscid second-mode disturbances intersects the curve for three-dimensional, inviscid first-mode solutions at about $M_e = 2.66$. Viscosity strongly reduces the amplification rates of second-mode amplified disturbances. Below of about $M_e = 3.0$, first-mode disturbances become destabilized by viscosity. Since the destabilizing influence of viscosity decreases with increasing obliqueness, the wave angle decreases for viscous first-mode amplified disturbances for decreasing Mach numbers. The Mach number where the curve for two-dimensional, viscous second-mode solutions intersects with the curve for viscous first-mode solutions is approximately at $M_e = 4.0$. 
Figure 5.9 Effect of Mach number on the maximal temporal amplification rate of most unstable inviscid and viscous first and second-mode-type amplified disturbances at $R_x = 1500$. Wave angles of first-mode amplified solutions are specified for selected Mach numbers. Reproduced from Mack (1969), figures 11.11, 11.21, and 13.18.

5.4.2 Complex Eigenvalue Diagram

The final figure discussed in this chapter (figure 5.10) demonstrates the complex phase speed $c_{ph,x}^\text{r} + ic_{ph,x}^\text{j}$ of two-dimensional disturbances as a function of streamwise wavenumber $\alpha_r$ for the viscous linear stability problem at Mach number 3.8. The inviscid limit from figure 5.4 is also included for comparison. Recent studies of compressible linear stability theory (Fedorov, 2003; Tumin, 2007) introduced new concepts and terminology that can be explained using figure 5.10.

The dispersion relation in figure 5.10a is very similar to the inviscid case although the local Reynolds number is low ($R_x = 850$). There are two solutions with one solution originating from the slow acoustic wave spectrum $c_{ph,x}^\text{r} = 1 - 1/M_e$ and the other originating from the fast acoustic wave spectrum $c_{ph,x}^\text{r} = 1 + 1/M_e$. These are the viscous counterparts of the $\alpha_r = 0$ family and the $\alpha_r = 1$ family of solutions from figure 5.4. Fedorov (2003) and Tumin (2007) denote the family of solutions from the slow acoustic wave spectrum “slow mode” or “mode S” and consequently, the family
Figure 5.10 Complex phase speed of two-dimensional disturbances as a function of streamwise wavenumber $\alpha_r$ for $M_e = 3.8$, insulated wall and free-stream temperature $T_e = 80K$: (a) real part $c_{\text{ph},x}^r$, (b) imaginary part $c_{\text{ph},x}^i$. Reproduced from Mack (1969), figure 12.17.

of solutions from the fast acoustic wave spectrum “fast mode” or “mode F”. This notation will also be used throughout this dissertation.

The phase speed of mode S and mode F change with streamwise wavenumber $\alpha_r$ and coincide at about $\alpha_r = 0.3$, which is called point of “synchronism” (Fedorov & Khokhlov, 2002). Any point where two families of solutions coincide in their phase speed $c_{\text{ph},x}^r$ is termed “synchronism” by Fedorov (2003) and Tumin (2007). Hence, for example, mode S is synchronized with the slow acoustic wave spectrum at $\alpha_r = 0.0$, while mode F is synchronized with the fast acoustic wave spectrum for the same streamwise wavenumber. The “synchronism” mechanism between mode S and mode F at $\alpha_r = 0.3$ leads to the amplification of one of those modes and produces for the current Mach number ($M_e = 3.8$) the second mode unstable region. Figure 5.10b shows $c_{\text{ph},x}^i$ versus the streamwise wavenumber $\alpha_r$ for both modes at two different local Reynolds numbers and for the inviscid case. At $R_x = 850$, mode S contains the viscous first-mode amplified solutions and the viscous second-mode amplified solutions while mode F is damped throughout the entire wavenumber regime. At $R_x = 1500$, the picture changes and mode S contains only the viscous first-mode amplified solution while mode F, which is not shown, develops into a viscous second-
mode amplified disturbance, just as predicted by the inviscid theory. If the Reynolds number is further increased the curves for mode S and mode F approach the inviscid limit.

As a final conclusion for this chapter, it is important to note that there exists a misunderstanding in the research community about first-mode and second-mode unstable regions. As discussed for figure 5.10b, both regions are not always a result of two amplified independent families of disturbances (mode S and mode F). Even for the inviscid case at high Mach numbers, the second-mode unstable region (and even higher-mode unstable regions) can be contained within one family of solutions (the family originating from $c_{ph,x} = 1 - 1/M_e$). This fact, for example, is illustrated in figure 5.6 (for 0 deg) of the last section.
6. Transition Initiated by a Wave Train in a Flat-Plate Boundary Layer at Mach 2

This chapter is the first of three result chapters. Its main focus is on the first question raised in chapter 2: Can oblique breakdown be identified in the experiments by Kosinov and his co-workers? Hence, the experimental investigations by Kosinov et al. (1994b) and Ermolaev et al. (1996) serve as a reference and provide the physical conditions for the numerical setup, which is explained in sections 6.1 and 6.2. In these experiments, the weakly nonlinear regime of transition was studied. This led to the discovery of asymmetric subharmonic resonance triads, which appear to be relevant for transition in a Mach 2 boundary layer. These triads were composed of one primary oblique wave of frequency $20kH\text{z}$ and two oblique subharmonic waves of frequency $10kH\text{z}$. While the experimentalists have focused on this new breakdown mechanism, the experimental data also indicate the presence of another mechanism, which might be related to oblique breakdown. With the simulations presented in section 6.4.3, the possible presence of oblique breakdown mechanisms in the experiments is explored by deliberately suppressing subharmonic resonances in the DNS and by comparing the numerical results with the experimental data. In addition, the subharmonic transition route is also investigated in detail using LST (section 6.3) and DNS (section 6.4.4).

6.1 Physical Problem and Computational Setup

Supersonic flow at Mach 2 over a flat plate is investigated. The computational setup is designed to allow for a direct comparison of the DNS results with the experimental measurements by Ermolaev et al. (1996). The unit Reynolds number is $Re = 6.6 \times 10^6 m^{-1}$ and the free-stream temperature is $T_\infty = 160K$. The base flow of the DNS closely matches the compressible similarity solution for Mach 2. In earlier investigations by Kosinov et al. (1990), the mean flow profile in their experiments also
matches the similarity solution. Therefore, it can be assumed that the experiment (Ermolaev et al.) and the DNS presented in this chapter are performed under the same mean flow conditions.

\[ x^*, y^*, z^*, b^*, a^*, r^*, x, y, \ldots \]

\[ W, z, v, p, x, 1, 2, 1.0 - 1.0, 0.0 \]

Figure 6.1 Comparison of experimental flat-plate model (H×L×W = 10 × 450 × 200 mm) with the computational domain (a) and blowing and suction velocity profile in streamwise direction prescribed over the disturbance hole (b); M=2.0, T^\infty=160.0K, flat plate.

As illustrated in figure 6.1a, the flat plate used by Ermolaev et al. has a length of 0.45m and a width of 0.2m. Disturbances were generated by a glow discharge (harmonic point source) in an electrical discharge chamber placed inside the flat plate and penetrate the flow through a hole with a diameter of 0.42mm at x^* ≈ 0.038m. In order to minimize computational cost, the extent of the computational domain for the DNS discussed in this chapter covers only a portion of the experimental setup, as indicated by the box placed on top of the flat plate in figure 6.1a. The inflow of the domain is located 0.02m downstream of the leading edge of the plate (x_0^* = 0.02m), while the outflow is placed between x_L^* ≈ 0.11m and x_L^* ≈ 0.18m, depending on the specified goal of a particular simulation (parameter study or highly resolved simulation). The domain height of y_H^* ≈ 0.02m ≈ 24δ (24 laminar boundary layer thicknesses δ at the outflow) for simulations with a smaller streamwise domain extent or y_H^* ≈ 0.035m ≈ 34δ for simulations with a larger streamwise domain extent has been selected such that disturbance reflections from the free-stream boundary into
the domain of interest are prevented (figure 6.2a). A pseudo-spectral discretization using Fourier modes (Canuto et al., 1988) is employed in the spanwise direction of

![Figure 6.2](image)

**Figure 6.2** Computational setup: (a) Contours of disturbance pressure (20kHz) with the spanwise wavenumber \( \beta = 0.1 \text{mm}^{-1} \) indicating that reflections from the freestream boundary reach the near wall region inside the buffer domain downstream of the region of interest, (b) contours of disturbance pressure (20kHz) at the wall showing that the disturbances do not reach the spanwise boundaries; \( M=2.0, T_\infty=160.0K \), flat plate.

the computational domain (assuming spanwise periodicity). As a result, the local disturbances generated by the glow discharge in the experiments are replaced by a periodic series of localized disturbances spaced one fundamental wave length apart, \( \lambda_z = z_W^* = 0.063m \). This wave length corresponds to a fundamental spanwise wave-number of \( \beta^* = 2\pi/z_W^* = 0.1 \text{mm}^{-1} \). The domain width \( z_W \) has been chosen large enough that adjacent disturbances do not interact with each other inside the computational domain. In fact, computational cost is reduced even further by enforcing symmetry with respect to the \( z^* = 0.0m \) plane for the streamwise velocity \( u \), wall-
normal velocity $v$, density $\rho$ and temperature $T$ and antisymmetry for the spanwise velocity $w$; thus, in effect, the simulation is restricted to the interval $[0, z^*_W/2 = \lambda^*_z/2]$.

Time-harmonic disturbances with the fundamental frequency of $20kHz$ and the subharmonic frequency of $10kHz$ are introduced through a hole (section 6.2) located between $x^*_1 \simeq 0.029m \leq x^* \leq x^*_2 \simeq 0.038m$ ($x^*_2 - x^*_1 \simeq 0.7\lambda^*_x$ of the fundamental frequency) by disturbing the wall-normal velocity. The subharmonic frequency and therefore, the subharmonic resonance triad investigated by Kosinov et al. and Ermolaev et al. has been deliberately excluded in some of the simulations in order to focus only on the nonlinear wave interactions caused by forcing with $20kHz$. These simulations are listed in table 6.1 as CFUN 1-5. With cases CFUN 1-5, the question whether oblique breakdown was present in the experiments can be addressed. For simulations of the subharmonic resonance triad (CSUB 1-10 in table 6.1), the flow was also perturbed with the subharmonic frequency. In order to determine for the simulations a similar spanwise disturbance input (amplitude and phase) as generated by the glow discharge in the experiments, a calibration procedure is required such that the flow response downstream of the forcing location in the DNS closely matches that from the experiment. A detailed description of this procedure is provided in section 6.4.1.

The computational grid for all simulations has been stretched in wall-normal direction. For certain simulations, the grid resolution in streamwise direction was increased near the outflow (figure 6.3a). The number of grid points required in streamwise direction is determined by the domain length and the streamwise wave length of the relevant instability waves. The streamwise domain extends about 8 to about 13 wave lengths of a two-dimensional instability wave with frequency $20kHz$ (in the linear stage). One wave length of a two-dimensional fundamental wave is typically resolved with about 27 points inside the equidistant grid region before the resolution is increased towards the outflow (CFUN 5). The required wall-normal grid resolution has been determined using the shape of the wall-normal amplitude distribution for
Table 6.1 Main simulation parameters for all cases included in this chapter. All cases, where only the fundamental frequency is perturbed, are denoted as CFUN and all cases with both forcing frequencies are denoted as CSUB. Note that the differences between CSUB 1 to CSUB 9 are listed in table 6.2 and are discussed in more detail in section 6.4.4; M=2.0, $T_\infty=160.0\text{K}$, flat plate.

The $u$-velocity disturbance in the linear stage as a guideline. As an example, a small portion of this wall-normal amplitude distribution is illustrated in figure 6.3b for the case of frequency $20 kHz$, spanwise wavenumber $\beta = 0.5 mm^{-1}$ and the streamwise position $x^* = 0.06m$. For spanwise wavenumbers close to $\beta = 0.5 mm^{-1}$, a small second amplitude maximum appears near the wall, whereby, in this case, the distance to the wall is resolved by 6 points.
Figure 6.3 Computational setup: (a) Grid resolution in streamwise direction for a simulation with equidistant grid (−) and a simulation with increased resolution near the outflow (− −), grid resolution in wall-normal direction (...). For all simulations discussed in this chapter, (b) wall-normal amplitude and phase distribution close to the wall. Symbols indicate grid locations used to resolve the second maximum of the wall-normal amplitude distribution for the u-velocity disturbance with 20kHz (■■■) and its corresponding wall-normal phase distribution (●●●) at the spanwise wavenumber $\beta = 0.5mm^{-1}$ and the streamwise position $x^* = 0.06m$; $M=2.0$, $T^*_\infty=160.0K$, flat plate.

6.2 Disturbance Generation

Disturbances in the wall-normal velocity are introduced into the flow by time-harmonic blowing and suction over a disturbance hole at the wall near the inflow boundary (figure 6.1a). During the startup of the simulation, the forcing amplitude $A(\beta, t)$ is ramped up in time over one disturbance period. The streamwise velocity distribution $v_p$ over the hole (figure 6.1b) is represented by a fifth-order polynomial, which is smooth everywhere including at the end points and has the shape of a dipole (Harris, 1997):

$$v(x, y = 0, \beta, t) = A(\beta, t) v_p(x_p) \cos(-\omega t + \theta_p(\beta)),$$  

where $x_p$ is defined as

$$x_p = \frac{2x - (x_2 + x_1)}{x_2 - x_1}, \quad -1 \leq x_p \leq 1.$$  

(6.1)
For the fundamental frequency (20 kHz), the spanwise profile of the hole in spectral space was calculated using the calibration procedure explained in section 6.4.1. Disturbances with subharmonic frequency were excited by forcing each spanwise Fourier mode with the same amplitude $A(\beta) = A$ and phase $\theta_p(\beta) = \theta$ resulting in a finite approximation of the delta function.

### 6.3 Linear Stability Considerations

The linear behavior of instability waves with both frequencies (20kHz and 10kHz) obtained using linear stability theory (LST) is investigated to determine the spanwise wavenumbers of highly unstable waves at both frequencies. For these waves, possible resonance triads according to LST are determined and later compared to the DNS results in section 6.4.4.

The eigenvalue problem posed by LST is solved using the linear stability solver by Mack (1965) with self-similar compressible boundary-layer profiles as mean flow. For a given frequency $F$, local Reynolds number $R_x$ and spanwise wavenumber $\beta = \beta^*L^*_e$, this solver returns the complex streamwise wavenumber $\alpha = \alpha^*L^*_e = \alpha_r + i\alpha_i$ as an eigenvalue, where the real part represents the streamwise wavenumber $\alpha_r$ and the imaginary part the streamwise amplification rate $\alpha_i$ (see also equation 5.4). The length scale $L^*_e$ was previously defined in equation (3.5) and is commonly used for nondimensionalization in a linear stability analysis. The local Reynolds number $R_x$ based on this reference length is given by equation (3.7) and the frequency is nondimensionalized according to equation (5.8). Using the nondimensionalization in equation (5.8) for the fundamental (20kHz) and the subharmonic frequency (10kHz) leads to the corresponding nondimensionalized values $F = 3.8\cdot10^{-5}$ and $F = 1.9\cdot10^{-5}$, respectively.

The results obtained by LST are summarized as follows: Waves observed in the experiments with the fundamental frequency of 20kHz and in the spanwise wave-
Figure 6.4 LINES: Spanwise distribution of amplification rate $\alpha_i$ (a,b) and streamwise wavenumber $\alpha_r$ (c,d) at the positions $x^* = 0.06m$ (a,c) and $x^* = 0.13m$ (b,d). (—) 20kHz, ( - ) 10kHz. SYMBOLS: Spanwise distribution of normalized amplitude $N$ with $x_{ref} = 0.038$ (b). (■) 20kHz, (▼) 10kHz. Arrows illustrate a resonance triad (c,d); M=2.0, $T^*_\infty=160.0K$, flat plate.
number range of $\beta^* = 0.5 mm^{-1}$ to $\beta^* = 1.0 mm^{-1}$ are in the vicinity of the maximum amplification rate $\alpha_i$, at both locations $x = 0.06 m$ and $0.13 m$ as shown in figures 6.4a and 6.4b. For the subharmonic frequency $10 kHz$, figures 6.4a and 6.4b indicate that the range of maximum amplification rate is slightly shifted to lower spanwise wavenumbers at $x^* = 0.13 m$. A better criterion for the identification of instability waves reaching the highest amplitudes within the domain of interest is the normalized amplitude $N$. It is obtained by the integration of $\alpha_i$ between two streamwise locations or, equivalently, by taking the logarithm of the ratio of the disturbance amplitudes at these two locations:

$$N = \int_{x_{ref}}^{x} -\alpha_i(\hat{x}) \, d\hat{x} = \ln\left(\frac{A(x)}{A(x_{ref})}\right), \quad (6.3)$$

where $x_{ref}$ is the starting location for the integration, i.e., the reference location for the amplitude ratio. At location $x^* = 0.13$ in figure 6.4b the normalized amplitudes ($N$) are shown for both frequencies (symbols) at different spanwise wavenumbers. They were obtained by integrating the amplification rates $\alpha_i$ predicted by LST with $x_{ref}$ corresponding to the location where the disturbances were introduced in the experiments by Kosinov et al. (1994b) ($x_{ref}^* = 0.038 m$). For the fundamental frequency, waves with a spanwise wavenumber $\beta$ between $0.7 mm^{-1}$ and $0.8 mm^{-1}$ reach the highest normalized amplitudes, whereas for the subharmonic frequency, waves with a spanwise wavenumber $\beta^* = 0.6 mm^{-1}$ have the highest normalized amplitude.

Before the different three-wave resonance triads are determined using LST, a short introduction to the resonance conditions is given below. Generally, the resonance condition for a three-wave triad is a phase synchronization of all three instability waves (Kachanov & Levchenko, 1984),

$$\theta^1 = \theta^2 + \theta^3, \quad \theta^n(x, z, t) = \alpha^n x + \beta^n z - \omega^n t. \quad (6.4)$$

Here, $\theta$ denotes the phase, $\omega$ the angular frequency, $\alpha_r$ and $\beta$ the streamwise and spanwise wavenumber, respectively. By comparing the coefficients in front of the
independent variables \(x, z, t\) the following three conditions can be derived:

\[
\omega^1 = \omega^2 + \omega^3, \quad \alpha_r^1 = \alpha_r^2 + \alpha_r^3, \quad \beta^1 = \beta^2 + \beta^3. \tag{6.5}
\]

For the special case of a symmetric, subharmonic resonance triad with one primary two-dimensional wave \((\omega^1, \beta^1)\) and two symmetric, subharmonic oblique waves \((\omega^\frac{1}{2}, \beta^\frac{1}{2})\), equation (6.5) reduces to the well-known conditions for incompressible boundary layers (Kachanov & Levchenko, 1984)

\[
\omega^1 = 2\omega^\frac{1}{2}, \quad \alpha_r^1 = 2\alpha_r^\frac{1}{2}, \quad \beta^2 = -\beta^3. \tag{6.6}
\]

Equation (6.6) can be further utilized to derive an expression for the phase velocity \(c_{ph,x}^1\) of the two-dimensional instability wave

\[
c_{ph,x}^1 = v_x^\frac{1}{2}, \quad c_{ph,x}^1 = \frac{\omega^1}{\alpha_r^1}, \quad v_x^\frac{1}{2} = \frac{\omega^\frac{1}{2}}{\alpha_r^\frac{1}{2}}. \tag{6.7}
\]

where \(v_x^\frac{1}{2}\) is the velocity of the two oblique waves “with which it is necessary to move along the x-axis in order that the subharmonic phase should not depend on the time” (quote from Kachanov & Levchenko, 1984). It is important to note that \(v_x^\frac{1}{2}\) is not the phase speed of the oblique waves since their phase speed is defined as

\[
c_{ph,x}^\frac{1}{2} = \frac{\omega^\frac{1}{2}\alpha_r^\frac{1}{2}}{(\alpha_r^\frac{1}{2})^2 + (\beta^\frac{1}{2})^2}. \tag{6.8}
\]

Symmetric, subharmonic resonance triads do not play an important role in the transition process of a Mach 2 boundary layer. The resonance conditions for Kosinov’s asymmetric, subharmonic resonance triad are more complex. For this case, equations (6.5) become

\[
\omega^1 = 2\omega^2 = 2\omega^3, \quad \alpha_r^1 = \alpha_r^2 + \alpha_r^3, \quad \beta^1 = \beta^2 + \beta^3. \tag{6.9}
\]

Equation (6.9) is illustrated as an addition of three wave vectors in figures 6.4c and 6.4d for \(x^* = 0.06m\) and \(x^* = 0.13m\) forming a triad that was also observed by Kosinov
et al. (1994b) in their early experiment in 1994. The primary wave of this triad has the fundamental frequency of $20kHz$ and a spanwise wavenumber of $\beta^* = 0.8mm^{-1}$. The two subharmonic waves that close the triad have a spanwise wavenumber of $\beta^* = -0.6mm^{-1}$ and $\beta^* = 1.4mm^{-1}$.

In the following, the procedure used to locate a resonance triad using LST results (Zengl, 2006) is explained. Figure 6.5a shows the spanwise distribution of the streamwise wavenumber $\alpha_r$ for both frequencies at $x^* = 0.06m$. In order to satisfy the resonance condition for the spanwise wavenumber in equation (6.9) the difference in the spanwise wavenumbers of the two subharmonic waves of the triad must equal the spanwise wavenumber of the primary, fundamental wave

$$\beta^2 + \beta^3 = \Delta \beta = \beta^1.$$  \hspace{1cm} (6.10)

Therefore, the spanwise distribution of $\alpha_r$ for the subharmonic frequency in figure 6.5a is first mirrored with respect to $\beta^* = 0.0mm^{-1}$ and then shifted by $\Delta \beta$ as defined in equation (6.10). Note, for two-dimensional boundary layers the spanwise distribution of the complex wavenumber $\alpha$ is symmetric with respect to $\beta^* = 0.0mm^{-1}$. In this example, $\Delta \beta^*$ is equal to $0.5mm^{-1}$, which implies that we are looking for a triad with $\beta^* = 0.5mm^{-1}$ as the spanwise wavenumber for the primary wave. To satisfy the resonance condition for the streamwise wavenumber in equation (6.9), the values of both graphs of the streamwise wavenumber for the subharmonic frequency in figure 6.5a are added. The resulting graph exhibits the value of the streamwise wavenumber of the fundamental primary wave with $\beta^* = 0.5mm^{-1}$ at only one single point indicated by a black dot. The spanwise wavenumber at this position is about $\beta^* = 1.16mm^{-1}$. The black dot in figure 6.5a suggests the existence of a resonance triad, which is composed of one primary wave with fundamental frequency of $20kHz$ and spanwise wavenumber of $\beta^* = 0.5mm^{-1}$ and two subharmonic waves with $\beta^* = -0.66mm^{-1}$ and $\beta^* = 1.16mm^{-1}$. The same triad can be found at $x^* = 0.13mm^{-1}$ (figure 6.5b). Note that this triad was not reported by Kosinov and his co-workers.
In figures 6.5c and 6.5d, the black dot with the largest spanwise wavenumber ($\beta^* = 1.4 \, mm^{-1}$) represents the asymmetric subharmonic resonance triad discovered by Kosinov et al. (1994b). Furthermore, figure 6.5c shows an other black dot, which is related to a new triad (spanwise wavenumber of the primary wave: $\beta^* = 0.8 \, mm^{-1}$ and of the two subharmonic waves: $\beta^* \simeq 0.0 \, mm^{-1}$ and $\beta^* \simeq 0.8 \, mm^{-1}$). Note that for this case, the resonance conditions are not exactly fullfilled at $x^* = 0.06 \, m$. This triad cannot be identified farther downstream at $x^* = 0.13 \, m$. It might however still play an important role since this could explain the development of a peak near $\beta^* = 0.0 \, mm^{-1}$ in the subharmonic disturbance signal for the mass flux in the experiments (Kosinov et al., 1994b). More details on this topic can be found in section 6.4.4. Using this method for the identification of resonance triads, various asymmetric, subharmonic resonance triads can be found for a Mach 2 boundary layer. The spanwise wavenumber of each subharmonic wave is determined by the value of the spanwise wavenumber for the primary wave with the fundamental frequency. This dependency is illustrated in figure 6.6 for $x^* = 0.06 \, m$ and $x^* = 0.13 \, m$. At least one resonance triad exists for every spanwise wavenumber of the primary wave between $\beta^* = 0.5 \, mm^{-1}$ and $\beta^* = 1.0 \, mm^{-1}$. For the first triads, the subharmonic wave that would initially experience linear growth has a spanwise wavenumber close to the one connected to the maximum amplitude growth as shown in figure 6.4b ($\beta^* \simeq 0.6 \, mm^{-1}$). Triads with one subharmonic wave close to $\beta^* \simeq 0.0 \, mm^{-1}$ are called second triads throughout this thesis.

For the resonance triads discussed so far, the resonance conditions formulated in equation (6.9) have only been verified for two different streamwise locations ($x^* = 0.06 \, m$ and $x^* = 0.13 \, m$). One triad has been chosen for figure 6.7 to confirm whether these conditions are also satisfied for other streamwise locations. In figure 6.7, the downstream development of the streamwise wavenumber $\omega^*_\alpha$ for all three waves participating in the triad are plotted (lines). The symbols represent the sum of the streamwise wavenumber from both subharmonic waves. The values of this sum are
Figure 6.5 Illustration of the procedure used to determine a particular asymmetric, subharmonic resonance triad using LST results for different streamwise positions: (a,c) $x^* = 0.06m$, (b,d) $x^* = 0.13m$. (...) Spanwise distribution of streamwise wavenumber $\alpha_r$ for the fundamental frequency, (- -) spanwise distribution of $\alpha_r$ for the subharmonic frequency, (-.-) spanwise distribution of $\alpha_r$ for the subharmonic frequency shifted by $\Delta\beta = 0.5mm^{-1}$ (a,b) or $\Delta\beta = 0.8mm^{-1}$ (c,d), (--) $\alpha_r$ of the fundamental frequency at $\beta = 0.5mm^{-1}$ (a,b) or at $\beta = 0.8mm^{-1}$ (c,d), (o) summation of (- -) and (-.-), (●) resonance point; M=2.0, $T_\infty=160.0K$, flat plate.
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Figure 6.6 Dependency of the spanwise wavenumber of both subharmonic waves ($\beta_{10kHz}$) included in the triad on the spanwise wavenumber of the primary wave ($\beta_{20kHz}$). Subharmonic waves that would experience resonance growth: (o-o) 1. triad at $x^* = 0.06m$, (□-□) 1. triad at $x^* = 0.13m$, (▼-▼) 2. triad at $x^* = 0.06m$. Subharmonic waves that would experience linear growth: (●-●) 1. triad at $x^* = 0.06m$, (■-■) 1. triad at $x^* = 0.13m$, (▼-▼) 2. triad at $x^* = 0.06m$; $M=2.0$, $T_\infty=160.0K$, flat plate.

very close to the spanwise wavenumber of the primary wave as required by equation (6.9) for all streamwise positions.

It can be summarized that LST predicts various asymmetric, subharmonic resonance triads for the physical flow conditions of Kosinov’s experiments. Most likely, there is a mechanism that is responsible for the selection process of a specific triad since Kosinov reported only one triad. Furthermore, the second triads as indicated in figure 6.6 may have also been present in the experiments by Kosinov et al. (1994b) as this could explain the development of a peak near $\beta^* = 0.0mm^{-1}$ in the subharmonic disturbance signal of the mass flux.

6.4 DNS Results

A difficulty that arises when setting up simulations to match the experiments by Kosinov et al. (1994b) and Ermolaev et al. (1996) is the implementation of the disturbance generation into the Navier–Stokes solver. The diameter of the hole in the
plate model used in the experiments is far too small to be properly resolved in a DNS. Also modeling the electrical discharge from the experiments by a harmonic point source using an arbitrary monopole to disturb any flow quantity at the wall will not lead to the same flow response as observed in the experiments. Therefore, a different forcing method was applied for the present numerical simulations such that a similar disturbance development is produced in the boundary layer as in the experiments. This forcing method is based on blowing and suction at the wall as explained in section 6.2. The application of this different disturbance generation can be justified if the evolving instability waves near the forcing location are in the linear regime since their eigenbehavior does not depend on the details of the wave generation. Important are only the resulting amplitudes and phases of the different spanwise Fourier modes of the disturbance waves developing downstream of the disturbance hole. Therefore, a receptivity study has been performed in order to determine what spanwise amplitude and phase distribution of the $v$-velocity over the disturbance hole produces the same flow response as the localized forcing technique used in the experiments. Since the
experimental data are not known at the forcing location, a position slightly down-
stream \((x^* = 0.06m)\) serves as a reference location for the receptivity study. The
modeling of the harmonic point source in the experiments leads to a broad spanwise
forcing spectrum. This forcing method differs from previous investigations of oblique
and Jiang \textit{et al.} (2006), where only one discrete wave pair was forced and only the
higher-harmonic spanwise modes of these two waves were included in the simulations.

The results of the receptivity study are presented in the next section. Following
the receptivity study, DNS results for the linear regime (low disturbance amplitude)
are compared to data obtained from LST and experimental data by Kosinov (private
communication) at \(x^* = 0.06m\) in order to verify the validity of the numerical setup
and the applicability of the different disturbance generation. Results of the early
nonlinear regime are discussed in the succeeding sections. Here, the focus is on
answering the two previously raised questions: Can oblique breakdown be identified
in the experiments, and if oblique breakdown indeed occurs, what role does it play
when compared to asymmetric subharmonic resonance. To answer the first question,
DNS have been performed, where only the fundamental frequency was perturbed. In
order to address the second question, the “entire” experiment was simulated including
subharmonic forcing.

6.4.1 Receptivity Study

A receptivity study has been performed for the fundamental frequency \((20kHz)\). Dis-
turbances of this frequency still exhibit a linear behavior up to the streamwise position
\(x^* = 0.06m\) in both, the experiments and the DNS. An initial linear disturbance de-
velopment does not occur for subharmonic disturbances since the resonant interaction
of disturbances from both frequencies immediately alters the flow field downstream
of the disturbance hole.
Figure 6.8 Spanwise amplitude (a) and phase (b) distribution (20kHz) of normalized output signal \( (e^*)' / E^* \) (disturbance voltage over mean voltage) measured by a hot-wire anemometer in the experiments at streamwise location \( x^* = 0.06m \) and wall-normal location \( y^* / \delta^* = 0.53 \) (physical space, \( N = 161 \)): (□) experiments (Kosinov, 2006), (–) interpolated by a cubic spline; \( M=2.0, T^*_\infty=160.0K \), flat plate.

The experimental data employed for the receptivity study have been provided by Kosinov (2006). They were obtained using hot-wire measurements at the wall-normal location \( y^* / \delta^* = 0.53 \). The raw measurement data consist of the disturbance voltage over the mean voltage \( (e^*)' / E^* \) measured at various spanwise locations extending from \( z^* = -11.2mm \) to \( z^* = 7.3mm \). The experimental data exhibit a slight asymmetry with respect to \( z^* = 0.0 \). Since the setup of the computations is symmetric with respect to \( z^* = 0.0 \), only half of the experimental data points have been considered for the receptivity study (from \( z^* = 0.0 \) to \( z^* = 11.2mm \) in figure 6.8). The measurement data have been extended up to \( z^* = z_W^* / 2 \approx 0.0315 \) using additional data points with zero amplitude and zero phase, then interpolated onto an equidistant grid with \( n_z = 161 \) points, and finally transformed into Fourier space using equation (4.7) (figure 6.9).

The raw hot-wire output signal can be related to mass-flux and temperature fluctuations in the flow. The temperature fluctuations are assumed to be small (Kosinov, 2006) and therefore, the mass-flux disturbance can be directly calculated from
Figure 6.9 Spanwise amplitude (a) and phase (b) distribution (20kHz) of normalized output signal \((e^*)'/E^*\) from figure 6.8 transformed into Fourier space using equation (4.7) \((K = 161 \text{ and } \Delta \beta^* = 0.1\text{mm}^{-1})\): (−) output signal of hotwire, (−−) mass-flux disturbance nondimensionalized by mean values measured at \(y^*/\delta^* = 0.53\), (■) mass-flux disturbance nondimensionalized by mean values measured in the free stream; \(M=2.0\), \(T^* = 160.0\text{K}, \text{flat plate.}\)

\[
\frac{(\rho^* u^*)'}{(\rho^* U^*)} = K \frac{(e^*)'}{E^*}
\]  

(6.11)

(figure 6.9a). Here, the calibration factor \(K\) depends on many parameters, e.g. the temperature loading factor of the hot-wire. For the present experimental data, its value is approximately \(1/0.27\) (Kosinov, 2006). In the hot-wire measurements, the mass-flux disturbance (figure 6.9a) is measured relative to the local mean mass flux \((\rho^* U^*)\) at the measurement location \((x^*, y^*) = (0.06m, 0.53\delta^*)\). The DNS data, on the other hand, are normalized by the free-stream value at the inflow boundary (approach flow), \((\rho^* U^*)_\infty\). Therefore, as a final step, the measured mass-flux disturbance has been rescaled by \((\rho^* U^*)/(\rho^* U^*)_\infty \simeq 0.5\) so that the resulting amplitude distribution (figure 6.9a) can now be compared directly to the DNS data. As seen from figure 6.9a, the absolute disturbance amplitude at the reference location \(x^* = 0.06m\) is small enough so that the disturbance development from the forcing location up to \(x^* = 0.06m\) is assured to be linear. This is a necessary justification for the very different disturbance method (blowing and suction) employed in the simulations.
Figure 6.10 Spanwise amplitude (a) and phase (b) distributions (20kHz) of mass-flux disturbance \((\rho u)'\) at wall-normal location \(y^*/\delta^* = 0.53\) and streamwise location \(x^* = 0.06m\): (...) Forcing \((v\text{-velocity over the disturbance hole})\) applied in the DNS, (---) response of mass-flux disturbance due to the forcing in the DNS, (■) experimental data, (*) calculated using equation (6.12) and (6.13); \(M=2.0, T_\infty^*=160.0\), flat plate.

Having established the spectral composition of the disturbance at \(x^* = 0.06m\) measured in the experiments, the receptivity study for matching up the DNS with these data proceeds as follows. In a first simulation (CFUN 1), disturbances with the same fundamental frequency of 20kHz as in the experiments are introduced by time-harmonic blowing and suction with a very low (linear) amplitude (see equation 6.1). At this point, the proper spectral distribution of the forcing amplitude \(A(\beta)\) and phase \(\theta(\beta)\) for the \(v\)-velocity that would lead to a match with the experimental data is still unknown. Therefore, as a first guess, both \(A(\beta)\) and \(\theta(\beta)\) are kept constant over all spanwise Fourier modes (finite approximation of the delta function), as indicated by the dotted lines in the amplitude and phase distribution plots of figure 6.10. The amplitude and phase distribution at the reference location \(x^* = 0.06m\) of the disturbance that is generated by this type of forcing are also plotted in figure 6.10 as solid lines. Clearly, the DNS data do not match the corresponding experimental data points. However, considering that the disturbance development is linear, the appropriate forcing amplitude \(A(\beta)^{new}\) for the \(v\)-velocity can now be calculated from
Figure 6.11 Spanwise amplitude (a) and phase (b) distributions (20kHz) of mass-flux disturbance \((\rho u)'\) at wall-normal location \(y^*/\delta^* = 0.53\) and streamwise location \(x^* = 0.06m\) of the DNS (--) and the experiments (■ FFT by Kosinov, □ FFT using equation 4.7) for the new amplitude and phase distributions of the \(\nu\)-velocity over the disturbance hole (...); \(M=2.0, T_{\infty}^* = 160.0K\), flat plate.

The new forcing amplitude and phase distribution are plotted in figure 6.10 as the curves marked by stars and also in figure 6.11 as dotted lines. When using this adjusted forcing in a second DNS (CFUN 2, low forcing amplitude), the new simulation results match the experimental data almost exactly, as seen from figure 6.11. Note that the data in figure 6.11a are normalized by the amplitude value of the mass-flux disturbance at \(\beta^* = 0.0\). The experimental data in figure 6.11 (Kosinov, 2006) have been transformed into Fourier space by a full Fourier series expansion as specified in Ermolaev et al. (1996). Therefore, the experimental data are not entirely symmetric with respect to \(\beta^* = 0.0\). The
difference in the Fourier series expansion is also responsible for the small discrepancy between the phase distributions in figure 6.11b. The experimental data, when transformed using equation (4.7) in figure 6.9 match the numerical results in figure 6.11b exactly.

The calibration procedure, explained above, provides a tool to match the disturbance signal in the experiments at a location close to the disturbance generation. It is, however, not clear if the amplitude and phase distribution in figure 6.11 indeed uniquely represent the localized forcing of the experiments. Therefore, figure 6.12a shows this forcing signal in physical space at one streamwise position \( x^* = 0.032m \) over the hole. It demonstrates that the flow is mainly perturbed close to the center in spanwise direction, where the \( v \)-velocity has its maximum. Figure 6.12b shows also contours of the \( v \)-velocity at the wall and hence, illustrates that the calibration procedure leads to a localized forcing as in the experiments. For comparison, figures 6.12a and 6.12c confirm that for the subharmonic frequency a localized excitation is also achieved. As already mentioned in section 6.2, disturbances with the subharmonic

![Graph showing amplitude distribution](image)

![Graph showing contour of v-velocity](image)

Figure 6.12 Illustration of the localized forcing over the disturbance hole in physical space: (a) Spanwise amplitude distribution of the \( v \)-velocity at \( x^* = 0.032m \) in physical space for the fundamental frequency (□-□) and the subharmonic frequency (○-○). (b and c) Contours of the \( v \)-velocity at the wall \( (y^* = 0.0m) \) for the fundamental frequency with \( K = 41 \) (b) and the subharmonic frequency with \( K = 81 \) (c); \( M=2.0, T_{\infty}^* = 160.0K \), flat plate.
frequency were excited in all simulations presented in this chapter by forcing each spanwise Fourier mode with the same amplitude \( A(\beta) = A \) and phase \( \theta(\beta) = \theta \) resulting in a finite approximation of the delta function.

![Figure 6.13 Isosurface of constant \( v \)-velocity disturbance \((\pm 8.0 \cdot 10^{-7})\) for three different time instants: (a) \( t = T/4 \), (b) \( t = T/2 \), (c) \( t = 3T/4 \). The \( v \)-velocity at the wall is disturbed by the dotted amplitude and phase distribution in figure 6.11 \( \text{CFUN 2); M=2.0, } T^*_\infty=160.0\text{K, flat plate.} \]

This section is closed by providing a three-dimensional impression of the flow structures that develop over the disturbance hole in response to the blowing and suction at the wall with the amplitude and phase distribution shown in figure 6.11 \((20\text{kHz})\). In figure 6.13, the flow response to this disturbance method is illustrated by isosurfaces of the \( v \)-velocity disturbance for different time instants. It is clearly visible that the resulting disturbance is highly localized in the DNS, just as caused by the glow discharge in the experiments. The two different shades for the isosurfaces distinguish between the sign of the \( v \)-velocity. The larger structures travel along the Mach wave towards the free stream whereas smaller downstream travelling structures develop close to the surface. These smaller structures contain boundary layer modes that grow in downstream direction as indicated in figure 6.14.

### 6.4.2 Linear Disturbance Development

Two DNS with a very small forcing amplitude have been conducted in order to investigate the linear disturbance development. In one DNS, only the fundamental
Figure 6.14 Contours of constant \( v \)-velocity disturbance (maximum: \( 5.0 \cdot 10^{-6} \), minimum: \( -5.0 \cdot 10^{-6} \)) along the centerline \( (z^* = 0.0m) \) for one time instant illustrating the excitation of disturbance waves inside the boundary layer: (—) positive disturbance amplitude, (— -) negative disturbance amplitude; \( M=2.0, T_\infty^* = 160.0K \), flat plate.

Frequency is forced as already explained in section 6.4.1 (see CFUN 2 in table 6.1) and in the other DNS, only the subharmonic frequency is perturbed (figures 6.12a and 6.12c). Results from both simulations are shown in figure 6.15. The streamwise and spanwise amplitude and phase distributions of the mass-flux disturbance \( (\rho u)' \) at wall-normal location \( y^*/\delta^* = 0.53 \) are illustrated in figures 6.15a and 6.15b for the fundamental frequency and in figures 6.15c and 6.15d for the subharmonic frequency. Both cases show similar trends. The maximum in the amplitude distribution in both figures is caused by the exponential growth according to linear theory. For the fundamental frequency, the maximum is located near \( \beta^* = 0.8mm^{-1} \) (close to the outflow) and for the subharmonic frequency, the maximum is near \( \beta^* = 0.6mm^{-1} \) as predicted by LST in figure 6.4b. Two-dimensional disturbances and disturbances with a small wave angle do not experience any streamwise growth for the subharmonic frequency and only grow weakly for the fundamental frequency (figure 6.16).

Figure 6.16 demonstrates an important aspect for the setup of the asymmetric subharmonic resonance in the experiments. For the “classical” symmetric subharmonic resonance for incompressible flow, the primary wave of the resonance triad is a highly amplified two-dimensional wave. Similarly the oblique, primary waves of the
Figure 6.15 Contour levels of Fourier amplitude \((a,c)\) and phase \((b,d)\) for the mass-flux disturbance \((\rho u)\) at wall-normal location \(y^*/\delta^* = 0.53\) for different streamwise and spanwise locations (CFUN 2): \((a,b)\) fundamental frequency, \((c,d)\) subharmonic frequency. Note, the phase in \((b,d)\) is plotted downstream of \(x^* \simeq 0.04m\) marked by vertical dashed lines; \(M=2.0, T^*_\infty=160.0\)K, flat plate.
Figure 6.16 Spanwise amplitude (a) and phase (b) distribution for the mass-flux disturbance $(\rho u')$ at wall-normal location $y^*/\delta^* = 0.53$ for two different streamwise positions (CFUN 2): (no symbol) $x^* = 0.06m$, (□) $x^* = 0.13m$, (—) subharmonic frequency, (−−) fundamental frequency; $M=2.0$, $T_\infty=160.0K$, flat plate.

asymmetric, subharmonic resonance triads found in figure 6.6 also experience a high streamwise amplitude growth.

The receptivity study from the previous section provides a tool for matching the flow response to the localized forcing technique applied in the experiments for the fundamental frequency (20kHz). This matching procedure however is based on a single quantity, $(\rho^*u^*)'$ and location $(x^*, y^*) = (0.06m, 0.53\delta^*)$ within the entire flow field. In order to verify that near the forcing location the disturbance development is indeed linear, the simulation results have been compared to results from Linear Stability Theory. Shown in figure 6.17 are the wall-normal amplitude and phase distributions for selected spanwise wavenumbers of the streamwise velocity and the density disturbances from the DNS (CFUN 2) and the corresponding distributions from LST for the fundamental frequency. The amplitude distributions from both, linear theory and DNS, are normalized by their respective maximum value within the boundary layer. The excellent agreement between LST and DNS indicates that the linear eigenbehavior of the disturbances is correctly reproduced in the DNS. Furthermore, this agreement emphasizes that the DNS would match the mass-flux disturbance $(\rho^*u^*)'$
in the experiments throughout the linear regime if experimental data from different streamwise and wall-normal positions were available for comparison.

Figure 6.18 provides another indication that the disturbances in the DNS develop linearly according to their eigenbehavior for the fundamental frequency. The streamwise amplification rate \( \alpha_i \) and the wavenumber \( \alpha_r \) from LST, are compared to \( \alpha_i \) and \( \alpha_r \) from the DNS, which are computed from the wall-pressure disturbance as follows:

\[
\alpha_i = -\frac{d}{dx} \left[ \ln \left( A(x) | p'_{\text{wall}} \right) \right], \quad \alpha_r = \frac{d}{dx} \left[ \theta(x) | p'_{\text{wall}} \right]. \tag{6.14}
\]

The amplification rate from the DNS in figure 6.18a is most likely modulated by acoustic waves generated by the forcing. The amplification of disturbances in the DNS is slightly stronger than predicted by LST. This difference between LST and DNS has also been observed in previous investigations (Thumm et al., 1989; Thumm, 1991) and has been attributed to non-parallel effects resulting from the growth of the boundary layer. For the streamwise wavenumber \( \alpha_r \) (figure 6.18b), which is less sensitive to non-parallel effects, the agreement between DNS and LST is nearly perfect. A very similar observation can also be made for the subharmonic frequency implying that the resonance triads, identified using LST results in section 6.3, are only weakly affected by non-parallel effects.

In summary, the focus of this section has been on the linear behavior of disturbances for both frequencies (20 kHz and 10 kHz). It was shown that the DNS can reproduce the results predicted by linear theory for the wall-normal shape of the eigenfunctions and the downstream disturbance growth. In the experiments, the disturbance development up to \( x^* = 0.06m \) is assumed to be linear. Therefore, the receptivity study from section 6.4.1 can be employed to match the development of the mass-flux disturbance \((\rho u)'\) for the fundamental frequency (20 kHz) from the DNS and the experiments throughout the entire linear regime. The temporal evolution of the mass-flux disturbance \((\rho u)'\) at \( y^*/\delta^* = 0.53 \) and \( x^* = 0.06m \) in figure 6.19 illustrates this agreement between the experiment and the DNS. Furthermore, it was possible
Figure 6.17 Wall-normal amplitude and phase distribution of streamwise velocity disturbance ($a$, $b$, $c$) and density disturbance ($d$, $e$, $f$) for three different spanwise wave-numbers at $x^* = 0.06m$ for the fundamental frequency (20kHz): ($a,d$) $\beta^* = 0.5mm^{-1}$, ($b,e$) $\beta^* = 1.0mm^{-1}$, ($c,f$) $\beta^* = 1.5mm^{-1}$. Symbols represent results obtained from LST (($\star$): amplitude, ($\times$): phase) and lines represent DNS results (CFUN 2) (|---|): amplitude, (---): phase); $M=2.0$, $T_\infty^{*}=160.0K$, flat plate.
Figure 6.18 Downstream development of the complex streamwise wavenumber (\( a \) amplification rate, \( b \) streamwise wavenumber) predicted by LST and by DNS (CFUN 2) (from wall pressure disturbance) for three different spanwise wavenumbers and the fundamental frequency (20kHz). **Symbols** represent results obtained from LST: (■) \( \beta^* = 0.5mm^{-1} \), (●) \( \beta^* = 1.0mm^{-1} \), (▲) \( \beta^* = 1.5mm^{-1} \). **Lines** represent DNS results (CFUN 2): (--) \( \beta^* = 0.5mm^{-1} \), (- -) \( \beta^* = 1.0mm^{-1} \), (---) \( \beta^* = 1.5mm^{-1} \); \( M=2.0, T^*_\infty=160.0K \), flat plate.

to show that the resonance triads are only weakly affected by the streamwise growth of the boundary layer. Hence, the same resonance triads discovered using LST may also be present also in the following simulations with both disturbance frequencies and higher forcing amplitudes.

### 6.4.3 Identification of Oblique Breakdown in the Experimental Data

Before trying to identify the oblique breakdown mechanism in the simulations, a short summary of the most important characteristics of oblique breakdown is presented. In many previous numerical investigations (Thumm, 1991; Fasel *et al.*, 1993; Adams & Sandham, 1993; Fezer & Kloker, 1999), the authors concluded that oblique breakdown is a very dominant mechanism. For the computational setup of these simulations, oblique breakdown produced the highest growth rates for the nonlinearly generated modes. In all these previous simulations, oblique breakdown was triggered by forcing two oblique instability waves with equal but opposite wave angle. The wave angle
Figure 6.19 Temporal evolution of the mass-flux disturbance \((\rho u)'\) for the fundamental frequency \((20 k H z)\) at the wall-normal position \(y^* / \delta^* = 0.53\) and the streamwise position \(x^* = 0.06 m\) for the experiment \((a)\) (Kosinov, 2006) and the DNS \((b)\) (CFUN 2): \((-\) positive disturbance amplitude, \((...)\) negative disturbance amplitude; \(M=2.0, T_\infty = 160.0 K,\) flat plate.

of this wave pair was determined by forcing discretely at the most unstable spanwise wavenumber according to linear theory. This chosen spanwise wavenumber thus also defined the spanwise domain size \(z_w = 2\pi / \beta\) (Thumm, 1991; Fasel et al., 1993; Adams & Sandham, 1993; Fezer & Kloker, 1999). All waves with smaller spanwise wavenumbers were therefore excluded from the simulations.

One of the most important features of oblique breakdown, as observed in these earlier investigations, is the generation of steady longitudinal modes, which grow strongly in the streamwise direction. As oblique breakdown sets in, the steady modes (denoted by \([0, \pm 2]\)) start to play a dominant role, since they are directly generated by the forced wave pair \([1, \pm 1]\). The notation \([h, k]\) is used to identify a particular wave according to its frequency \(h\) and its spanwise wavenumber \(k\). \(h\) denotes multiples of the fundamental frequency and \(k\) multiples of the smallest spanwise wavenumber. The \([0, \pm 2]\) modes are responsible for the generation of various other modes, as for example the wave pair \([1, \pm 3]\) or the steady longitudinal modes \([0, \pm 4]\). A detailed description of the nonlinear wave interactions in the early nonlinear stages of oblique breakdown can be found in Thumm (1991). Thumm and Fasel et al. also stated a particular
characteristic of the nonlinear wave interactions in oblique breakdown: Modes with odd spanwise wavenumbers $k$ are only generated for odd harmonic frequencies $h$, and modes with even spanwise wavenumbers are generated only for even frequencies $h$.

Having discussed the main characteristics of oblique breakdown enables one to interpret the numerical results from nonlinear DNS and allows an analysis to see if this mechanism was present in the experiments. Four DNS of the early nonlinear stages of transition have been performed and are compared to the experimental findings in the following paragraphs. The flow was perturbed by the method outlined in section 6.4.1 with the single frequency of $20kH\Omega$. The four simulations differ in the absolute magnitude of the forcing amplitude ($v$-velocity over disturbance hole). The main simulation parameters are summarized in table 6.1. For the first simulation (CFUN 2, from the previous section), the forcing amplitude is small enough for the developing disturbances to be in the linear regime throughout the entire computational domain. The second simulation (CFUN 3) has a forcing amplitude that leads to an exact match of the experimental spanwise amplitude distribution of the mass-flux disturbance $(\rho u)'$ at the streamwise location $x^* = 0.06m$ in figure 6.9. For the next simulation (CFUN 4), the forcing amplitude is again increased and differs from CFUN 3 by a factor of $\sim 1.38$. Note that even for this higher forcing amplitude, the spanwise disturbance distribution at $x^* = 0.06m$ from the experiment is still matched in figure 6.11 and no departure from the linear wave development is noticeable. In the first three simulations, the computational domain ranges in downstream direction from $x_0^* = 0.02m$ to $x_L^* \simeq 0.17m$ so that the numerical results can be compared to the reported experimental data at locations $x^* = 0.11m$, $x^* = 0.12m$, and $x^* = 0.13m$. The final simulation (CFUN 5) has the highest forcing amplitude (increased by a factor of $\sim 1.73$ compared to CASE 3). For this simulation, the number of spanwise Fourier modes and the streamwise extent ($x_L^* \simeq 0.18m$) of the computational domain are increased. Therefore, this simulation covers a larger range of the transition process than the other simulations.
For CFUN 4, figure 6.20 illustrates the streamwise development of the disturbance spectrum generated by the localized forcing at \( x^* = 0.038m \). CFUN 3 exhibits a similar behavior and is therefore not shown. In figure 6.20a, the Fourier amplitudes for the mass-flux disturbance \((\rho u)^{'}\) at the wall-normal location \( y^*/\delta^* = 0.53 \) are plotted versus the streamwise direction \( x \) and the spanwise wavenumber \( \beta^* \). Only the positive half of the spanwise spectrum is shown since all simulations are symmetric with respect to \( \beta^* = 0.0mm^{-1} \). Initially both, the small-amplitude disturbances (CFUN 2, figure 6.15a) and the large-amplitude disturbances (CFUN 4, figure 6.20a), experience exponential amplitude growth resulting in a maximum near \( \beta^* = 0.8mm^{-1} \). For the larger amplitude disturbances in figure 6.20a, however, a second maximum forms near \( \beta^* = 2mm^{-1} \) downstream of \( x^* = 0.1m \) as a result of nonlinear wave interactions. These nonlinear wave interactions are also visible in the phase development for the mass-flux disturbance \((\rho u)^{'}\) at the same wall-normal position (figure 6.20b). In figure 6.20b, the near-field of the disturbance hole is excluded and the start position for the post-processing is indicated by the vertical dashed line.
In figure 6.21, the numerical results are compared to the experimental data at
\( x^* = 0.11m, \ x^* = 0.12m, \) and \( x^* = 0.13m \) downstream of the leading edge of the flat
plate. In this figure, the amplitude and phase distribution of the mass-flux disturbance
\((\rho u)^0\) are plotted for the first three DNS cases over a wider range of spanwise wave-
numbers. Noted as primary peaks and secondary peaks, the same maxima as in
figure 6.20 also appear in figure 6.21 for both DNS with a large forcing amplitude
(CFUN 3 and CFUN 4). The absolute values of the mass-flux disturbance of the DNS
in this figure are rescaled by a single, constant factor \( A_{ref} \) (CFUN 3: \( \simeq 3.008 \cdot 10^{-4} \),
CFUN 4: \( \simeq 4.794 \cdot 10^{-4} \)) to match the experimental data. The experimental results
were again provided by Kosinov (2006) and were transformed into spectral space using
a full (asymmetric) Fourier series expansion (Ermolaev et al., 1996). The streamwise
development of the primary maxima in CFUN 4 matches the behavior observed in
the experiment perfectly. This is not quite the case for the streamwise development
of the nonlinearly generated secondary maxima. In CFUN 3, these maxima are not
as pronounced as in CFUN 4, although the absolute disturbance amplitudes from
CFUN 3 at position \( x^* = 0.06m \) match the estimated values from the experiments
(figure 6.9).

The temporal evolution of the disturbances at the streamwise location \( x^* = 0.13m \)
from the experiment and the DNS (CFUN 4) is illustrated in the \((t, z)\) diagrams in figure 6.22. In the measurement data, shown in figure 6.22a, the subharmonic frequency
is clearly visible. Structures close to the centerline \( (z^* = 0.0m) \) repeat every second
period of the fundamental frequency. This is not the case for the DNS (figure 6.22b)
where the structures close to the centerline show no trace of a subharmonic. This
is not surprising, since the subharmonic frequency was not forced. Visible instead
are steady modes that modulate the disturbance signal between \( z^* = -0.004m \) and
\( z^* = 0.004m \) shifting the disturbance amplitude to purely positive or negative values.

The overall excellent agreement between the numerical results of the DNS and the
experimental data as observed in figure 6.21 indicates that even with the (deliberate)
Figure 6.21 Spanwise amplitude (a, c, e) and phase (b, d, f) distribution of the mass-flux disturbance ($\rho u'$) (20kHz) at wall-normal location $y^*/\delta^* = 0.53$ for the streamwise positions $x^* = 0.11m$ (a, b), $x^* = 0.12m$ (c, d) and $x^* = 0.13m$ (e, f): (—) low forcing amplitude (CFUN 2), (*) medium forcing amplitude (CFUN 3), (---) large forcing amplitude (CFUN 4), (□) experiment (Kosinov, 2006); $\text{M}=2.0$, $T^*_\infty=160.0K$, flat plate.
suppression of a subharmonic development (present in the experiment) the simulations
capture the development of the fundamental disturbances correctly. The results from

Figure 6.22 Temporal evolution of the mass-flux disturbance \((\rho u)\)' at the wall-normal
position \(y^*/\delta^* = 0.53\) and the streamwise position \(x^* = 0.13m\) for the experiment (a)
(Kosinov, 2006) and the DNS (b) (CFUN 4): (——) positive disturbance amplitude,
(...) negative disturbance amplitude; \(M=2.0, T^*_c = 160.0K,\) flat plate.

the DNS therefore suggest that in the experiments another mechanism coexisted with
the subharmonic resonance triad. The generation of steady modes in figure 6.22 and
the development of secondary peaks in figure 6.21 at a spanwise wavenumber that is a
higher harmonic of the spanwise wavenumber of the primary peaks are an indication
for the presence of oblique breakdown mechanisms.

The DNS with the highest forcing amplitude (CFUN 5) exhibits stronger non-
linear wave interactions and therefore also a more pronounced secondary maximum.
Shown in figure 6.23 is the nonlinear development of the mass-flux disturbance \((\rho u)\)' at
\(y^*/\delta^* = 0.53\) in response to the high-amplitude forcing. Clearly, numerous additional
maxima at equally spaced spanwise wavenumbers are visible for the fundamental
frequency in figure 6.23b. The time-average in figure 6.23a comprising the steady,
spanwise modes also exhibits several maxima that are equally spaced in spanwise
direction. The dashed lines indicate the spanwise wavenumber of the maxima for the
steady modes near the outflow. These dashed lines are also plotted in figure 6.23b to
Figure 6.23 Contour levels of the Fourier amplitude for the mass-flux disturbance $(\rho u)'$ at wall-normal location $y^*/\delta^* = 0.53$ for different streamwise and spanwise locations (CFUN 5): (a) steady modes, (b) fundamental frequency (20kHz); $M=2.0$, $T_\infty=160.0$K, flat plate.

Illustrate that the maxima of the steady modes are located in between the maxima for the fundamental frequency. This behavior is characteristic for oblique breakdown. The first maximum in figure 6.23b represents mode $[1, 1]$, the additional maxima at higher spanwise wavenumbers represent modes $[1, 3]$, $[1, 5]$ etc. Similarly, the maximum at $\beta \simeq 1.4mm^{-1}$ in figure 6.23a corresponds to mode $[0, 2]$ and the additional maxima to the modes $[0, 4]$, $[0, 6]$ etc. The Fourier amplitude $(\rho u)'$ at $y^*/\delta^* = 0.53$ for the spanwise wavenumbers of the maxima in figure 6.23 is plotted versus $x$ in figure 6.24 on a semi-logarithmic scale. For the steady modes, these wavenumbers are indicated by the dashed lines in figure 6.23a. As for oblique breakdown of a discrete wave pair, the steady modes are strongly amplified. Near the outflow, the peak corresponding to mode $[0, 2]$ reaches even higher amplitudes than mode $[1, 1]$ (compare figure 6.24a and figure 6.24b). Note, the localized drop in amplitude for several modes (e.g. mode $[0, 4]$ at $x^* \simeq 0.127$) is an artifact of plotting the amplitude at a constant $y^*/\delta^*$, where the streamwise change in shape of the wall-normal amplitude profiles does not scale with the boundary layer growth.
Figure 6.24 Streamwise amplitude distribution for the mass-flux disturbance \((\rho u)'\) at wall-normal location \(y^*/\delta^* = 0.53\). (a) \((-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\) \([0,0]\), \((-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\) \([0,2]\), \((-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\) \([0,4]\), \((-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\) \([0,6]\), (b) \((-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\) \([1,0]\), \((-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\) \([1,1]\), \((-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\) \([1,3]\), \((-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\) \([1,5]\); \(M=2.0\), \(T^*_\infty=160.0K\), flat plate.

To this end, it can be summarized that the nonlinear disturbance development in the simulations with high forcing amplitude is caused by oblique breakdown mechanisms for the fundamental frequency. Despite the absence of a subharmonic resonance, the disturbance development in the simulations agree very well with the experimental data for both, the linear and the weakly nonlinear stages. This fact suggests that oblique breakdown might have also been present in the experiments by Kosinov et al. (1994b) and Ermolaev et al. (1996). The next section discusses the subharmonic resonance triads. Therefore, in the following simulations, the flow was additionally disturbed with the subharmonic frequency as in the experiments.

6.4.4 Nonlinear Disturbance Development of the Resonance Triads

In order to investigate the dependency of the asymmetric subharmonic resonance on the relationship between fundamental and subharmonic disturbances, several numerical studies have been performed and are discussed in the following. First, the influence of the amplitude ratio between disturbances of both frequencies and then, the influence of the phase relation is studied. For all simulations presented in this
section, the absolute value of the forcing amplitude and the spanwise forcing pro-
file for the fundamental frequency over the disturbance hole inside the flat plate are
identical to case CFUN 4. Disturbances with subharmonic frequency are excited by
forcing each spanwise Fourier mode with the same amplitude $A(\beta) = A$ and phase
$\theta_p(\beta) = \theta$ (figure 6.12). To save computational cost, the DNS for both parametric
studies have smaller computational domains in both the streamwise and wall-normal
direction than the domain illustrated in figures 6.2a and 6.2b (see also table 6.1).

Figure 6.25 shows results from one DNS (CSUB 1) that will serve as a reference
for all other cases discussed in this section. The amplitude and phase of the forcing
for subharmonic disturbances is listed in column one of table 6.2. The flow response
to this type of forcing is illustrated in figure 6.25 for the subharmonic frequency. This
figure shows contour levels versus streamwise direction and spanwise wavenumber
of the Fourier amplitude and phase of the mass-flux disturbance $(\rho u)$ at wall-normal
location $y^*/\delta^* = 0.53$. It is clearly visible that at higher spanwise wavenumbers a
second maximum appears near $\beta^* = 1.7 \text{mm}^{-1}$ close to the outflow (figure 6.25a).

The nonlinear wave interactions caused by the resonance triads alter the flow field
immediately downstream of the disturbance generation. Note that this is the reason
why the calibration procedure (section 6.4.1) applied for the fundamental frequency
cannot be used for the subharmonic frequency. The phase is also significantly changed
by nonlinear effects leading to two phase jumps close to $\beta^* = 1.2 \text{mm}^{-1}$ and $\beta^* =
0.0 \text{mm}^{-1}$ in figure 6.25. The phase jump at $\beta^* = 0.0 \text{mm}^{-1}$ supports the previously
stated possibility (section 6.3) of additional triads from the second group in figure 6.6.

Cases CSUB 2 and CSUB 3 differ from CSUB 1 in the absolute value of the
forcing amplitude for the subharmonic frequency. In CSUB 2, the forcing amplitude
is two times larger than for CSUB 1. For CSUB 3, the forcing amplitude of the
subharmonic frequency is further increased by a factor of three when compared to
CSUB 2. A similar disturbance development as for CSUB 1 in figure 6.25 is also
found for CSUB 2 and CSUB 3. Figure 6.26 provides a more detailed comparison
Figure 6.25 Contour levels of Fourier amplitude (a) and phase (b) of the mass-flux disturbance \((\rho u)'\) for the subharmonic frequency \((10kHz)\) at wall-normal location \(y^*/\delta^* = 0.53\) for different streamwise and spanwise locations (CSUB 1); \(M=2.0, T_\infty=160.0K\), flat plate.

between all three cases (CSUB 1, 2 and 3). Here, the spanwise amplitude distribution of \((\rho u)'\) at \(y^*/\delta^* = 0.53\) for several streamwise positions are compared and do not show any noticeable differences between all three cases. This is a clear indication that the nonlinear resonant growth of subharmonic disturbances does not depend on the forcing amplitude for this frequency and that the amplitude ratio for disturbances of both frequencies does not determine a specific resonance triad in the simulations. This is in contrast to the experimental findings (Kosinov et al., 1994b; Ermolaev et al., 1996) where an influence of the forcing amplitude on the nonlinear wave development was reported.

In the simulations, however, the asymmetric subharmonic resonance can be strongly influenced by changing the phase relation between fundamental and subharmonic disturbances. The influence of the phase relation on the resonance mechanism of one particular triad was previously studied for incompressible boundary layers by Zelman & Maslennikova (1993) and for a Mach 3 boundary layer by Zengl (2006). In both investigations, it was possible to delay transition by changing the phase to a specific
Figure 6.26: Spanwise amplitude distribution for CSUB 1 (a), CSUB 2 (b) and CSUB 3 (c) of the mass-flux disturbance \((\rho u)'\) for the subharmonic frequency \((10\text{kHz})\) at wall-normal location \(y^*/\delta^* = 0.53\); (---) \(x^* = 0.06m\), (—-) \(x^* = 0.08m\), (---) \(x^* = 0.1m\), (×) linear behavior at \(x^* = 0.1m\), (■) spanwise amplitude distribution for CSUB 1 (a) at \(x^* = 0.1m\) (rescaled); \(M=2.0, T^*_\infty=160.0\text{K},\) flat plate.
Table 6.2 Amplitude and phase study: forcing amplitude and phase (of the \( v \)-velocity at the wall) for the subharmonic frequency; \( M=2.0, \ T^* =160.0K \), flat plate.

<table>
<thead>
<tr>
<th>( A (\beta) )</th>
<th>CSUB 1</th>
<th>CSUB 2</th>
<th>CSUB 3</th>
<th>CSUB 4</th>
<th>CSUB 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta \theta^{2D}_p / \pi )</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.15</td>
<td>-0.30</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( A (\beta) )</th>
<th>CSUB 6</th>
<th>CSUB 7</th>
<th>CSUB 8</th>
<th>CSUB 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta \theta^{2D}_p / \pi )</td>
<td>-0.45</td>
<td>-0.60</td>
<td>-0.75</td>
<td>-0.90</td>
</tr>
</tbody>
</table>

Several simulations have been performed in order to investigate the importance of the phase relation on the resonance mechanism. Some of these simulations are listed in table 6.2. For these DNS, the phase of subharmonic disturbances has been altered with respect to the phase of fundamental disturbances by introducing a phase shift \( \theta_p(\beta) \). This phase shift is quantified in table 6.2 as a phase difference \( (\Delta \theta^{2D}_p / \pi) \) between two-dimensional fundamental waves and two-dimensional subharmonic waves. Note that only for the subharmonic frequency, the phase of the forcing signal is constant over the spanwise wavenumbers. The spanwise phase distribution for the fundamental frequency is taken from CFUN 4 in section 6.4.1. Therefore, \( \Delta \theta_p / \pi \) varies with \( \beta \).

For the six cases summarized in table 6.2 (CSUB 4-9), the phase difference \( \Delta \theta^{2D}_p / \pi \) between two-dimensional fundamental waves and two-dimensional subharmonic waves was changed. The disturbance development for the subharmonic frequency (10 kHz) is illustrated in figure 6.27 for all these cases. Plotted versus streamwise location and spanwise wavenumber are contour levels of the Fourier amplitude of the mass-flux disturbance \( (\rho u)' \). The first two figures (CSUB 4 and 5) exhibit similar trends as already discussed for CSUB 1 in figure 6.25a. As before, the first maximum at small spanwise wavenumbers results from the exponential growth according to linear theory, whereas the second maximum at higher wavenumbers develops due to the resonance. However, for both DNS cases, this second maximum is shifted towards
Figure 6.27 Effect of phase difference $\Delta \theta_{p}^{2D}/\pi$ (phase difference between two-dimensional fundamental waves and two-dimensional subharmonic waves) on the contour levels of Fourier amplitude of the mass-flux disturbance ($\rho u$)' for the subharmonic frequency (10kHz) at wall-normal location $y^\alpha/\delta^* = 0.53$ for different streamwise and spanwise locations: (a) $\Delta \theta_{p}^{2D}/\pi \simeq -0.15$ (CSUB 4), (b) $\Delta \theta_{p}^{2D}/\pi \simeq -0.30$ (CSUB 5), (c) $\Delta \theta_{p}^{2D}/\pi \simeq -0.45$ (CSUB 6), (d) $\Delta \theta_{p}^{2D}/\pi \simeq -0.60$ (CSUB 7), (e) $\Delta \theta_{p}^{2D}/\pi \simeq -0.75$ (CSUB 8), (f) $\Delta \theta_{p}^{2D}/\pi \simeq -0.90$ (CSUB 9); $M=2.0$, $T_\infty^\alpha=160.0K$, flat plate.
smaller spanwise wavenumbers when compared to CSUB 1. Moreover, it is more pronounced for CSUB 5 (figure 6.27b) than for CSUB 1 and CSUB 4 (figure 6.25a) and close to the outflow, reaches higher amplitude levels than the first maximum. With increasing $\Delta \theta^2_{p}/\pi$, the second maximum moves even closer to the first, resulting in a merging of both maxima and in an increase of its amplitude (CSUB 7 and 8 in figures 6.27d and 6.27e).

Figure 6.28 provides further details of the simulation results from figure 6.27 and of results from additional simulations. A tool was developed that tracks the spanwise wavenumber and the amplitude value of both maxima in figure 6.27; these values are plotted versus $\Delta \theta^2_{p}/\pi$ in figure 6.28 for three different streamwise positions. The spanwise wavenumber of the first maximum (figure 6.28a), which is generated according to linear theory, does not match the spanwise wavenumber from the DNS with small forcing amplitude, indicating that nonlinear wave interactions have already altered the subharmonic disturbance field. At positions $x^* = 0.08m$ and $x^* = 0.1m$ and for a phase difference of $\Delta \theta^2_{p}/\pi \simeq -0.53$, the spanwise wavenumber experiences a sudden increase in its value to $\beta^* \simeq 1.0mm^{-1}$ and $\beta^* \simeq 1.2mm^{-1}$, respectively. This sudden increase is due to the merging of the amplitude maxima seen in figures 6.27d and 6.27e. The plots for the second maximum (figures 6.28c and 6.28d) show a general trend that can also be observed in the contour plots in figure 6.25 and figure 6.27 and in the amplitude plots in figure 6.26: The spanwise wavenumber of a maximum decreases with increasing streamwise position. The graph for $x^* = 0.1m$ in figure 6.28c also confirms that the second maximum can be shifted within a certain interval of the spanwise wavenumber ranging from $\beta^* \simeq 1.0mm^{-1}$ to $\beta^* \simeq 1.9mm^{-1}$. A similar range for the first triads in section 6.3 was also obtained using the results from LST in figure 6.6. Another important finding is indicated in figure 6.28 by the large symbols at $\Delta \theta^2_{p}/\pi = -0.60$. All graphs in figure 6.28 have a periodicity of $\Delta \theta^2_{p}/\pi = \pi$. The large symbols result from a simulation with $\Delta \theta^2_{p}/\pi = -1.60$. These results match exactly the DNS with $\Delta \theta^2_{p}/\pi = -0.60$. 
Figure 6.28 Spanwise wavenumber and amplitude value of first (a, b) and second maximum (c, d) in \((\rho u)\) at \(y^* / \delta^* = 0.53\) vs. phase difference \(\Delta \theta^{2D}_{\rho} / \pi\) between subharmonic and fundamental forcing. For (b) and (d) the amplitudes are normalized by the forcing amplitude \(A(\beta) = A\) (see also table 6.2): (—) linear behavior at \(x^* = 0.06m\), (—) linear behavior at \(x^* = 0.08m\), (—) linear behavior at \(x^* = 0.1m\), (●●●) nonlinear behavior at \(x^* = 0.06m\), (◼◼◼) nonlinear behavior at \(x^* = 0.08m\), (◆◆◆) nonlinear behavior at \(x^* = 0.1m\), large symbols (○, □, ◆) represent a phase shift of \(\Delta \theta^{2D}_{\rho} / \pi = -1.60\); \(M=2.0, T_{\infty}=160.0K\), flat plate.
The results shown in figure 6.28 can also be employed to determine the optimum $\Delta \theta_p^{2D}$ for which the DNS results exhibit similar characteristics in the spanwise mass-flux distribution for the subharmonic frequency to those obtained from the experiments by Kosinov et al. (1994b) and from theory by Tumin (1996). Figure 6.29 provides a quantitative comparison of these results (from a DNS with a larger computational domain) for different streamwise positions with $\Delta \theta_p^{2D}/\pi = -0.30$. All three results show a small first peak around $\beta^* \approx \pm 0.6mm^{-1}$ and a strong second peak at $\beta^* \approx \pm 1.4mm^{-1}$ generated by two resonance triads with two primary waves at $\beta^* = \pm 0.8$ and four subharmonic waves at $\beta^* = \mp 0.6$ and $\beta^* = \pm 1.4$. Note that figures 6.29a and 6.29c also have a very small peak at $\beta^* = 0.0mm^{-1}$, which might be generated by the second triads in section 6.3. Ermolaev et al. (1996), however, attributed these peaks to acoustic disturbances radiating from the supersonic boundary layer.

This section is closed by discussing figure 6.30, which illustrates contour levels of Fourier amplitude and phase of the mass-flux disturbance $(pu)'$ for both, the fundamental frequency and the subharmonic frequency, with $\Delta \theta_p^{2D}/\pi = -0.30$. The disturbance development for the fundamental frequency is of great interest. Section 6.4.3 makes the case that the nonlinear wave interactions for this frequency may be due to the oblique breakdown mechanism involving only the fundamental disturbances with frequency $20kHz$. The results in figure 6.30 are obtained from case CSUB 10. Despite the presence of subharmonic disturbances, the disturbance development for the fundamental frequency in figures 6.30a and 6.30b agrees with the results from case CFUN 4 (see figure 6.20 and also figure 6.31) confirming that two transition mechanisms coexist in the experiments by Kosinov et al. (1994b).
Figure 6.29 Spanwise amplitude distribution of the mass-flux disturbance \((\rho u)'\) for the subharmonic frequency \((10kHz)\): (a) experiment (Kosinov et al., 1994b), (b) theory (Tumin, 1996) and (c) DNS (CSUB 10) with \(\Delta\theta^{2D}/\pi = -0.30\): (—) or (1) \(x^* = 0.06m\), ([ ]) or (2) \(x^* = 0.12m\), (+—) or (3) \(x^* = 0.13m\). Note the amplitudes of all plots are arbitrarily scaled; M=2.0, \(T_\infty^* = 160.0K\), flat plate.
Figure 6.30 Contour levels of Fourier amplitude (a,c) and phase (b,d) of the mass-flux disturbance \((\rho u)’\) for the fundamental frequency \((20kHz)\) and the subharmonic frequency \((10kHz)\) at wall-normal location \(y^*/\delta^* = 0.53\) for different streamwise and spanwise locations (CSUB 10, with \(\Delta \theta^P_{2D}/\pi = -0.30\)); \(M=2.0, T^*_\infty =160.0K\), flat plate.
Figure 6.31 Spanwise amplitude (a) and phase (b) distribution (fundamental frequency, 20kHz) for the mass-flux disturbance ($\rho u'$) at wall-normal location $y^*/\delta^* = 0.53$ for two different streamwise positions ($x^* = 0.06m$: $(-, \times)$, $x^* = 0.13m$: $(-, +)$). Comparison between DNS of the “entire” experiment including the subharmonic resonance $(-, - -)$ and results from a simulation of the nonlinear disturbance development for the fundamental frequency only $(\times, +)$; M=2.0, $T_{\infty}^*$=160.0K, flat plate.

6.5 Summary

The early nonlinear transition regime of a wave train in a Mach 2 flat-plate boundary layer was investigated following the experimental studies by Kosinov and his co-workers (Kosinov et al., 1994a,b; Ermolaev et al., 1996; Kosinov et al., 1997). While the forcing method in the experiment was a glow discharge, in the numerical simulations, the flow was forced by perturbing the $v$-velocity using blowing and suction. A receptivity study was performed to adjust the forcing through the hole in order to allow matching of the flow response to the glow discharge with that of forcing through blowing and suction. In the experimental studies, the nonlinear disturbance development was attributed to asymmetric subharmonic resonance between one oblique instability wave with frequency $20kHz$ and two oblique subharmonic waves of different wave angles. Furthermore, the experimental findings indicated the existence of additional asymmetric subharmonic resonance triads and the experimentalists concluded that specific resonance triads dominate a particular transition process depending on
the amplitude ratio between fundamental and subharmonic waves.

However, scrutinizing the experimental data suggests the presence of an additional breakdown mechanism. Understanding this mechanism was one part of chapter 6 and therefore, only the fundamental frequency of $20\,kHz$ was perturbed for some simulations. Despite the absence of a subharmonic forcing, the disturbance development in these simulations agreed very well with the experimental findings for the fundamental frequency for both, the linear and the nonlinear stages. In the simulations, the nonlinear development was caused by oblique breakdown mechanisms. If confirmed, this would be the first experimental evidence of the oblique breakdown mechanism in a supersonic boundary layer.

Furthermore, using LST, various possible asymmetric, subharmonic resonance triads were identified for the physical flow conditions of Kosinov’s experiments. DNS with small forcing amplitudes were conducted in order to study the linear disturbance development including nonparallel effects resulting from the streamwise growth of the boundary layer. The primary wave from all subharmonic resonance triads identified by LST experienced strong exponential streamwise amplitude growth. The streamwise wavenumbers of all triad components were only weakly affected by nonparallel effects, leading to the conclusion that the same resonance triads discovered using LST are also present in the DNS with large forcing amplitudes. The results obtained from several DNS with large forcing amplitudes revealed that the amplitude ratio between disturbances with both frequencies does not affect the resonance triads, which is in contrast to the experimental findings. Furthermore, in the numerical simulations, the phase difference between disturbances of both frequencies plays an important role since it influences the absolute value of the maximum generated by the resonance and its spanwise wavenumber. By adjusting the phase difference to a certain value, it was possible to observe a similar resonance triad in the simulations as reported in the experiments (Kosinov et al., 1994a) and in the theoretical investigation by Tumin (1996).
7. Transition Initiated by a Wave Packet in a Conical Boundary Layer at Mach 3.5

The previous chapter outlined that additionally to asymmetric subharmonic resonance another breakdown mechanism was present in the experiments by Kosinov and his co-workers. Most likely this breakdown mechanism was related to oblique breakdown. If confirmed this would be the first experimental evidence of the presence of oblique breakdown in a supersonic boundary layer. Furthermore, chapter 6 also provided an idea regarding the relationship between asymmetric subharmonic resonance and oblique breakdown. However, it is still unclear what breakdown mechanism is most dominant and would transition a boundary layer in a natural disturbance environment with a broad background disturbance spectrum. Hence, this chapter discusses the second question stated in chapter 2: What is the most dominant breakdown mechanism for a supersonic boundary layer? To provide some understanding of a natural transition scenario, it is common to study the spatial development of a wave packet in the boundary layer since in a wave packet a wide spectrum of disturbance waves is present (Gaster & Grant, 1975; Gaster, 1975; Medeiros & Gaster, 1999a,b). Therefore, in this chapter, a cone boundary layer at Mach 3.5 is disturbed by a short duration pulse through a hole on the cone surface. As a consequence of this pulsed forcing, a wave packet is generated. The approach flow of the simulations match the flow conditions of the NASA Quiet Wind Tunnel at NASA Langley (section 7.1 and 7.2). The spatial development of the wave packet is investigated in detail for both, the linear (section 7.3) and the early nonlinear transition regime (section 7.5). Furthermore, using LST, subharmonic resonance triads and several new resonance triads are identified (section 7.4) that might explain early nonlinear disturbance development in the disturbance spectra of the wave packet.
7.1 Physical Problem and Computational Setup

The computational setup follows the experimental study by Corke et al. (2002) and Matlis (2003) in the NASA Langley Quiet Wind Tunnel and the numerical investigations by Laible et al. (2008). The cone has a semi vertex angle of $\theta_c = 7^\circ$, and a cone length of $L^* = 0.3556m$. The nose radius of the cone is given as $r_{nose}^* = 0.038mm$ and therefore, the cone can be considered as “sharp”. The approach flow of the ex-

Figure 7.1 Cone model used for the NASA experiments: (a) three-dimensional view, (b) cross section. The ‘sharp’ cone has a nose radius of 0.038mm, the semi vertex angle is 7° and the model length is 0.3556m; $M=3.5$, $T_\infty^* = 90.116K$, sharp cone.

periments in the NASA Quiet Wind Tunnel had a Mach number of 3.5 and a unit Reynolds number of $9.45E6m^{-1}$. The stagnation temperature and pressure were given as $310.9K$ and $172,368.93Pa$, respectively. Throughout this chapter all direct numerical simulations (DNS) are performed for this setup. A schematic of the cone model and the corresponding coordinate system for all simulations is shown in Figure 7.1.

As initial condition for the simulations of the wave packet, mean-flow results from earlier simulations (Laible et al., 2008) have been interpolated and converged onto a new computational grid. The entire procedure to obtain an accurate mean-flow solution for the conical geometry is explained in detail in section 4.2.3 and in Laible et al. (2008). The computational domain for all simulations in this chapter starts at $x_0^* = 0.105m$ and thus, does not include the nose tip. The outflow is positioned at
Figure 7.2 Estimation of the streamwise wave length $\lambda_x$ for disturbances included in the wave packet using linear stability theory (LST) (Mack, 1965): (a) streamwise wave length $\lambda_x$ as a function of frequency $F$ for azimuthal modenumber $k_c = 21$, (b) streamwise wave length $\lambda_x$ as a function of azimuthal modenumber for frequency $F=10.0E-5$, (c) linear stability diagram for azimuthal modenumber $k_c = 21$ indicating the forcing location (dashed line); $M=3.5$, $T_\infty=90.116K$, sharp cone.
$x^*_L = 0.327m$. The domain height is chosen as $y^*_H = 0.075m$, which corresponds to about 80 boundary layer thicknesses $\delta$ at the outflow. In azimuthal direction, only one third of the cone is simulated.

The required streamwise resolution was determined using linear stability considerations. Figures 7.2a and 7.2b show an estimate of the streamwise wave length $\lambda_x$ from LST as a function of frequency $F$ (equation 5.8), local Reynolds number $R_x$ (equation 3.7), and azimuthal modenumber $k_c$. Laible et al. (2009) reports that the most critical disturbances for transition (based on N-factor calculations) on a cone at Mach 3.5 have a frequency of about $2.0E-5$ and an azimuthal modenumber of $k_c = 32$. Hence, a disturbance band with $F = 10.0E-5$ as the highest disturbance frequency should include the most dominant frequencies for a wave packet on a cone at Mach 3.5. From figures 7.2a and 7.2b, it can be concluded that disturbances with frequency $F = 10.0E-5$ have the smallest streamwise wave length for $k_c = 21$ at $R_x = 2000$ ($\approx$ the end of the cone). This wave length has a dimensional value of about $0.0045m$ and is resolved by 10 points in all simulations discussed in this chapter. In wall-normal direction, the boundary layer is resolved by about 50 points at the outflow. Additionally, the grid is stretched in order to stabilize the numerical scheme as discussed in section 4.2. Pseudo-spectral discretization (Canuto et al., 1988) was employed in azimuthal direction with about 100 Fourier-modes and the flow was assumed to be symmetric to the centerline of the cone.

Disturbances have been introduced into the boundary layer by pulsing the wall-normal velocity through a hole on the cone surface. More details on the disturbance generation can be found in section 7.2. The simulation setup was validated by comparing mean-flow data to previous simulations summarized in Laible et al. (2008) and results from a simulation with a low forcing amplitude to linear stability theory (LST). The linear stability results will be discussed in section 7.3 and here, only the mean-flow data are presented. Figure 7.3a compares the streamwise development of the boundary layer thickness $\delta$ in dimensional form for two simulations with different
grids to data from Laible et al. (2008) and Mangler-transformed flat-plate results (Mangler, 1948). Clearly, there is an excellent agreement between all data sets. Even for the coarser grid (301 points in wall-normal direction) the boundary layer growth is correctly reproduced. Moreover, the wall-normal distribution of the mass flux in figure 7.3b also matches the result from the Mangler-transformed similarity solution at $x^* = 0.251m$. Hence, the chosen resolution for the simulation of a supersonic boundary layer over a cone for the previously mentioned flow condition is sufficient for the mean flow. Note that for all results discussed in this chapter the finer wall-normal resolution is used except for the figures where the coarser resolution is highlighted explicitly (see section 7.3).

Figure 7.3 Comparison of selected mean-flow properties to previous simulations (Laible et al., 2008) and similarity solution in order to validate the simulation setup for all simulations presented in this chapter: (a) streamwise development of boundary layer thickness $\delta$, (b) wall-normal distribution of mass flux; $M=3.5$, $T_\infty=90.116K$, sharp cone.

7.2 Disturbance Generation

The flow was forced through a hole on the cone surface by pulsing the wall-normal velocity with the streamwise and azimuthal distributions shown in figure 7.4a. The time signal of the pulse is plotted in figure 7.4b. The duration of the pulse is about
Figure 7.4 Wall-normal velocity at the wall as a function of streamwise and azimuthal direction (a) and as a function of time (b). In (a), the dashed line (---) represents the azimuthal distribution and the solid line (---) streamwise distribution of the wall-normal velocity; M=3.5, $T^*_{\infty}=90.116$K, sharp cone.

5% of the total simulation time. In dimensional form, this value corresponds to about $t_1 = 0.044$ms while the entire simulation time is about $0.854$ms.

The forcing signal was

$$v(x, y = 0, z, t) = \begin{cases} A \cos^3(\pi x_p) \cos^3(\pi z_p) \sin(-\omega t), & t < t_1 \\ 0, & t \geq t_1 \end{cases}$$

(7.1)

where $x_p$ and $z_p$ are defined as

$$x_p = \frac{x - 0.5(x_2 + x_1)}{x_2 - x_1} \quad \text{and} \quad z_p = \frac{z}{z_2 - z_1},$$

(7.2)

respectively. $A$ denotes the forcing amplitude and $\omega$ the forcing frequency. As demonstrated in figure 7.5, the monopole forcing starts at $x_1^* = 0.1414m$ in streamwise direction and ends at $x_2^* = 0.1449m$ yielding a diameter of about $3.5mm$ while in azimuthal direction the hole has a diameter of $1mm$.

### 7.3 Linear Regime

The linear regime is investigated by pulsing the flow through a hole (see figure 7.6a) with very small disturbance amplitudes ($A=0.001\%$ of the approach velocity $U_\infty$) so
that the wave packet remains within the linear regime throughout the entire computational domain. The flow response to such forcing is illustrated in figure 7.6 by showing contours of the instantaneous wall-pressure disturbances for different time instants on the unrolled cone surface. Note that throughout this chapter all figures show either wall pressure or quantities that are obtained from wall pressure since this simplifies strongly the post-processing of the large amount of data from the simulations.

Towards the end of the duration of the pulse ($t^* = 0.043ms$), an imprint of the developing wave packet on the wall pressure is clearly visible. The disturbance amplitudes of the wall pressure are in the order of $10^{-8}$ and for the streamwise velocity disturbance about 1-2 orders of magnitude higher. The flow structures that are visible in the wall pressure are strongly dependent on the contour levels chosen. In figure 7.6, the developing wave packet has a three-dimensional structure, which can be expected since at low and moderate supersonic Mach numbers oblique instability waves experience higher streamwise amplification rates than two-dimensional waves according to linear stability theory (LST). If the contour levels are decreased, different
Figure 7.6 Flow response in wall pressure on the unrolled cone surface to the forcing described in section 7.2 at different time instants. The physical time increases from (a)-(d); $M=3.5$, $T_\infty=90.116K$, sharp cone.
Figure 7.7 Excitation of 3 different wave packets illustrated by contours of wall pressure (on unrolled cone surface). The contour levels in this figure are about one order of magnitude smaller than in figure 7.6. The physical time increases from (a)-(c); M=3.5, $T_\infty^*=90.116$K, sharp cone.
flow structures become apparent. Additionally to the three-dimensional structures, which are composed of instability waves that grow in downstream direction, structures that have a curved (more "two-dimensional") wave front appear in figure 7.7. For figures 7.7a and 7.7b, these structures are still very close to the three-dimensional structures and become distinct farther downstream in figure 7.7c.

It seems that by pulsing the boundary layer three different wave packets are generated. The first two wave packets have a more two-dimensional wave front and are denoted by "1" and "2" in figure 7.7c. They travel at a higher group velocity than the third wave packet denoted by "3", which consists of the actual instability waves. Wave packet "1" is generated during the interval of the pulse when fluid is blown into the boundary layer while wave packet "2" evolves when fluid is sucked from the boundary layer (see also figure 7.4b). Both of these wave packets are damped in streamwise direction and vanish farther downstream. At this point it is unclear if wave packet "1" and "2" contain fast acoustic disturbances or damped discrete modes that are commonly referred to as fast modes (Fedorov, 2003). The nature of these wave packets, however, might not be of great importance since they most likely will not have an influence on the transition process, which is initiated by wave packet "3".

Figure 7.7c also emphasizes that the choice of the domain width (one third of the cone) is large enough for wave packet "3" while both faster traveling wave packets reach the azimuthal boundaries. Note that disturbances in both faster traveling wave packets are significantly lower in amplitude. In the following sections, the focus is on wave packet "3" since the amplitudes of its disturbance spectrum grow strongly in streamwise direction and therefore, this wave packet is mainly responsible for the transition process. The streamwise development of the disturbance spectrum obtained from a Fourier transformation of the wall-pressure disturbance using 100 modes in time and azimuthal direction is displayed in figure 7.8. The period of the smallest frequency, which has the value of $f^* = 1.171kHz$, corresponds to the simulation time $t_{sim} (0.85ms)$. Note, as discussed in section 7.1, the highest frequency that can
Figure 7.8 Streamwise development of the Fourier amplitude obtained from wall pressure in the frequency–azimuthal modenumber plane, with $f^*$ as dimensional frequency and $k_c$ as azimuthal modenumber. The smallest frequency has a period based on the simulation time $T = t_{sim}$ and has the value of about $f^* = 1.171kHz$; $M=3.5$, $T^*_{\infty}=90.116K$, sharp cone.

be accurately resolved by the streamwise resolution of the simulation has a value of $F = 10E-5$, which is above 110kHz. Hence, all instability waves that experience strong streamwise growth for a conical boundary layer at Mach 3.5 are included in the present simulations.

The disturbance spectra in figure 7.8 are typical for a supersonic boundary layer. The maximum has a finite azimuthal modenumber $k_c$ unequal to zero. Thus, as already known from linear stability theory (Mack, 1969), oblique disturbances have the strongest streamwise amplification rates. With increasing downstream location, the peak amplitude in figure 7.8 shifts to higher azimuthal modennumbers and lower frequencies. At the end of the computational domain, this peak is located close to frequency $f^* = 23.415kHz$ and to azimuthal modenumber $k_c = 21$.

In the following two figures, results from the DNS with the low forcing amplitude are compared to predictions from linear stability theory in order to further validate the computational setup. The first figure (figure 7.9) shows the complex wavenumber $\alpha = $
Figure 7.9 Comparison of complex wavenumber $\alpha$ versus streamwise direction (a-b), frequency (c-d) and azimuthal modenumber (e-f) obtained from wall pressure to theoretical predictions from LST and a simulation with a coarser grid; M=3.5, $T_{\infty}=90.116K$, sharp cone.
\( \alpha_r + i \alpha_i \), which is calculated from the DNS data using the wall-pressure disturbance (Eissler & Bestek, 1996; Ma & Zhong, 2003) and equation (6.14). Since a wave packet contains a wide range of frequencies and azimuthal modenumbers, the complex wave-number is plotted as a function of streamwise direction, frequency and azimuthal modenumber in figure 7.9. In figure 7.9, symbols represent results from Tumin’s linear stability solver while solid lines indicate results from Mack’s linear stability solver. Note that both solvers do not account for curvature or divergence effects resulting from the cone geometry. Since the computational domain is far downstream of the nose tip of the cone and the curvature parameter \( \chi_s \) as defined by Malik & Spall (1991) is below 5% at the inflow, curvature effects can be assumed to be very weak for the present work (see also Laible et al., 2008, figure 10). Hence, it is justified to compare the results of both LST solvers to the DNS data set.

In general, the streamwise wavenumber \( \alpha_r \) is less sensitive to the criterion used for its computation or non-parallel effects resulting from the growth of the boundary layer. Thus, the agreement between DNS and LST (Tumin’s solver) is nearly perfect in figures 7.9a, 7.9c and 7.9e. The difference between Mack’s solver and Tumin’s solver is only due to the different mean-flow profiles used for the stability analysis. For the linear stability analysis using Mack’s solver, self-similar compressible boundary-layer profiles are applied whereas for Tumin’s solver, the mean flow was obtained by numerically solving the Navier–Stokes equations with the DNS code.

For disturbances with weak streamwise amplification (see figures 7.9b, 7.9d and 7.9f), or close to the forcing hole, the streamwise wavenumber \( \alpha_r \) from the DNS is modulated by the fast traveling wave packets generated by the forcing. This modulation is even more pronounced for the streamwise amplification rate \( \alpha_i \) in figures 7.9b, 7.9d and 7.9f. The streamwise amplification rate \( \alpha_i \) is somewhat overpredicted by the DNS when compared to LST. This behavior has also been reported in earlier investigations for a wide range of Mach numbers (Thumm, 1991; Eissler & Bestek, 1996; Ma & Zhong, 2003; Laible et al., 2008). Note that Ma & Zhong (2003) and Sivasub-
Figure 7.10 Comparison of wall-normal amplitude (a-c) and phase (d-f) distributions to theoretical predictions from LST and a simulation with a coarser grid for frequency $f^* = 23.415 kHz$, azimuthal modenumber $k_c = 21$ and streamwise position $x^* = 0.23114m$. All LST results are computed using Tumin’s solver (Tumin, 2007, 2008) with mean-flow profiles from the DNS; $M=3.5$, $T_{∞}^* = 90.116K$, sharp cone.
ramanian et al. (2009) observe a better agreement close to the maximal streamwise amplification rate.

The wall-normal amplitude and phase distribution from the DNS for selected flow quantities at frequency $f^* = 23.415 \text{kHz}$, azimuthal modenumber $k_c = 21$ and streamwise position $x^* = 0.23114m$ are plotted in figure 7.10. Also included are the results from LST. The amplitude distributions from both, LST and DNS, are normalized by their respective maximum value within the boundary layer. The excellent agreement in figures 7.9 and 7.10 between LST and DNS indicates that the simulations capture the linear disturbance development accurately. Even the simulation with the coarser grid yields the same results.

The actual time signal of the wave packet, before it was Fourier transformed in time and azimuthal direction for the discussion above, is plotted for the wall pressure on the centerline of the wave packet in figure 7.11 for different streamwise positions. Plotting the wave packet in this way is very convenient for experimentalists since they only need to measure the time signal of one disturbance quantity, here the wall pressure, at a few streamwise positions on the centerline. Furthermore, this figure provides an impression of how the center region develops in downstream direction. Clearly, with increasing downstream location the wave packet spreads. Initially, at $x^* = 0.199m$, the wave packet consists of about two wave lengths while farther downstream about three wave lengths are visible. The envelopes of these time signals, computed using Fourier transformations according to Gaster & Grant (1975), exhibit one dominant peak. At $x^* = 0.199m$, this peak is close to the tail of the wave packet while farther downstream, it shifts towards the center. The propagation speeds of the front and the tail of the wave packet are indicated by the dotted lines in figure 7.11. The velocity of the wave packet front is about $550 m/s$ while the velocity of the tail is slower ($\approx 420 m/s$).

Contours of constant amplitude (envelopes) as a function of the azimuthal angle $\varphi$ are given in figure 7.12. This figure demonstrates the overall development and
therefore the three-dimensional shape of the wave packet. Initially, the wave packet has the form of a “butterfly” with the maximal amplitude values on the centerline and two smaller side peaks or “tongues” at about $\varphi \simeq \pm 0.2$ in the front part of the packet. Farther downstream, additional tongues at higher angles develop until in the last plot (figure 7.12f) a total of seven tongues appear. The maximal disturbance amplitude shifts from the centerline to the two side peaks at about $x^* = 0.256m$.

When transformed into the x-z plane, as shown in figures 7.13(right) for differ-

Figure 7.11 Temporal evolution of wall-pressure disturbance amplitude and its envelope for different streamwise positions along the centerline of the wave packet. Dotted lines (...) are an estimate for propagation speed of the front ($\sim 550m/s$) and the tail ($\sim 420m/s$) of the wave packet. Solid lines (—) represent the time signal and dashed lines (•−•) its envelope; M=3.5, $T^*_{\infty}$=90.116K, sharp cone.
Figure 7.12 Temporal evolution of contours of constant amplitudes (envelopes) from wall pressure for different azimuthal angles $\varphi$ at several streamwise positions. All plots use the same contour levels; $M=3.5$, $T_\infty=90.116K$, sharp cone.
Figure 7.13 Spatial evolution of wall pressure revealing the wave structure (left) and the envelopes (right) on the unrolled cone surface for different time instances; $M=3.5$, $T_\infty=90.116\text{K}$, sharp cone.
ent time instants on the unrolled cone surface, the wave packet has the shape of a hand. Again, initially the maximal disturbance amplitude is located on the centerline \( z^* = 0.0m \) and two smaller side peaks, which have the shape of fingers, are present. While the wave packet travels downstream, additional fingers arise and the maximal amplitude shifts from the centerline to the side peaks. The original three-dimensional signal that was used for the calculation of the envelope in figures 7.13(right) is illustrated in figures 7.13(left). One contour line from the envelope plots is also shown in order to roughly define the boundary of the wave packet. The packet structure is entirely three-dimensional. The three-dimensionality results from strongly amplified oblique instability waves. Moreover, the wave packet spreads rapidly in streamwise and azimuthal direction.

7.4 Identification of Possible Resonance Triads Using Linear Theory

Before discussing the nonlinear transition regime initiated by a high-amplitude wave packet, possible resonance triads according to Craik (1971) are identified for the flow conditions discussed in this chapter. As shown by Kosinov and co-workers (Kosinov et al., 1994b; Ermolaev et al., 1996), who investigated asymmetric subharmonic resonance in a wave train (forcing by harmonic point source instead of a pulse) at Mach 2, resonance triads may be important transition mechanisms in supersonic boundary layers. Craik (2001) states for incompressible boundary layers that for “all non-resonant cases, the forced quadratic disturbance remains relatively small” when compared to the resonant cases. A theoretical study of resonance triads for the present flow configuration, therefore, allows to predict possible weakly nonlinear wave interactions and might help to understand the weakly nonlinear development of a wave packet.

Since a wave packet consists of a wide range of instability waves, also a wide range of nonlinear wave interactions are possible. To simplify the following study, only resonance triads for instability waves with azimuthal modenumber \( k_c = 21 \) and
frequency \( f^* = 23.415kHz \) are determined. As discussed in section 7.3, instability waves with this azimuthal modenumber and frequency experience strong streamwise growth and thus, reach the highest amplitudes at the outflow for the present setup.

Utilizing the stability solver of Mack (1965), linear stability data for a large number of frequencies and azimuthal modenumbers were generated. These data were used as input for a software tool, which calculates the azimuthal modenumber \( k_c \) for possible resonance triads within the limits of the data set according to the resonance conditions as introduced by Craik (1971)

\[
\omega^1 = \omega^2 + \omega^3, \quad \alpha_r^1 = \alpha_r^2 + \alpha_r^3, \quad \beta^1 = \beta^2 + \beta^3, \tag{7.3}
\]

with \( \omega \) as angular frequency, \( \beta \) as azimuthal wavenumber and \( \alpha_r \) as streamwise wavenumber. For a cone, the resonance condition for the azimuthal wavenumber \( \beta \) can be recast to

\[
k_c^1 = k_c^2 + k_c^3. \tag{7.4}
\]

In order to locate any resonance triad in the data set, the frequency and the azimuthal modenumber \( k_c \) for the primary wave (here \( f^* = 23.415kHz, k_c = 21 \)) and the frequency of one secondary wave have to be specified.

Before determining possible resonance triads using the previously mentioned software tool, the procedure to locate a subharmonic resonance triad is explained (see also section 6.3). Note that the algorithm of the software tool follows closely this procedure. Figure 7.14 shows the streamwise wavenumber \( \alpha_r \) as a function of the azimuthal modenumber \( k_c \) for the fundamental and subharmonic frequency at \( R_x = 1600 (0.24m) \). In order to satisfy the resonance condition for the azimuthal modenumber \( (k_c^1 = k_c^2 + k_c^3) \), both subharmonic waves of the triad need to have a difference in their azimuthal modenumber that has the value of the azimuthal modenumber of the primary, fundamental wave, which is denoted by a vertical line in figure 7.14b. Therefore, the azimuthal distribution of \( \alpha_r \) for the subharmonic frequency in figure 7.14b can be shifted by \( \Delta k_c = 21 \) (- -), since we are looking for a triad with \( k_c^1 = 21 \) as
Figure 7.14 Illustration of the procedure used to determine a particular asymmetric, subharmonic resonance triad using LST results at $R_x = 1600$ (0.24m): (a) Azimuthal distribution of streamwise wavenumber for the fundamental and subharmonic frequency. (b) Value of azimuthal modenumber for the fundamental wave is set to $k_c = 21$. In order to fulfill the resonance condition for the azimuthal modenumber ($k_c^1 = k_c^2 + k_c^3$), the curve for the azimuthal distribution of the streamwise wavenumber for the subharmonic frequency is shifted by $\Delta k_c = 21$. (c) Both curves for the subharmonic frequency are added. (d) Determination of the resonance triads. Note that the arrows denote the different components (waves) of one triad; $M=3.5$, $T_\infty=90.116K$, sharp cone.
the azimuthal modenumber for the primary wave. To satisfy the resonance condition for the streamwise wavenumber \( (\alpha_r^1 = \alpha_r^2 + \alpha_r^3) \) both graphs of the streamwise wavenumber for the subharmonic frequency in figure 7.14c are added (-\.-). The resulting graph exhibits the value of the streamwise wavenumber of the fundamental, primary wave with \( k_c^1 = 21 \) at four points indicated by black dots in figure 7.14d (the points for negative \( k_c \) are not shown here). One subharmonic resonance triad is highlighted by the vertical line in figure 7.14d and the three arrows represent all three waves included in the triad. The azimuthal modenumber at this position is about 36. Hence, the black dot at 36 in figure 7.14d suggests the existence of a resonance triad, which is composed of one primary wave with fundamental frequency of \( f^* = 23.415 khz \) and azimuthal modenumber of \( k_c^1 = 21 \) and two subharmonic waves with \( k_c^2 \sim 15 \) and \( k_c^3 \sim 36 \).

Figure 7.15 summarizes the results of the search for subharmonic resonance triads using the previously mentioned tool and LST data for the fundamental frequency \( f^* = 23.415 khz \). The azimuthal modenumbers \( k_c^{2,3} \) of both subharmonic waves included in the triad as a function of the azimuthal modenumber of the fundamental wave \( k_c^1 \) is shown in figure 7.15a. Since in this work the focus is on \( k_c^1 = 21 \), figure 7.15b illustrates the streamwise development of the azimuthal modenumber for both subharmonic waves at exactly \( k_c^1 = 21 \). Both figures show three subharmonic resonance triads. For all three triads in figure 7.15b, the azimuthal modenumber of the subharmonic waves is changing in streamwise direction suggesting that at every streamwise position a different pair of subharmonic waves is necessary to close the triad. For a particular range of the local Reynolds number, this change is however very weak. Hence, the first subharmonic triad (no symbols) might be strong from \( R_x = 1200 \) to \( R_x = 1400 \) while the other two subharmonic resonance triads (\( \times \) and \( \Box \)) might be dominant from \( R_x = 1600 \) to \( R_x = 1800 \). Close to the end of the computational domain at about \( R_x = 1960 \), two subharmonic resonance triads become detuned (\( \times \) and \( \Box \)) since the resonance conditions are not satisfied anymore.
Figure 7.15 Possible subharmonic resonance triads for $k_1^c = 21$ obtained from LST data: (a) dependency of azimuthal modenumber of the waves with the subharmonic frequency on the azimuthal modenumber of the fundamental wave at $R_x = 1600$ ($0.24m$), (b) streamwise development of the azimuthal modenumber for both subharmonic waves; $M=3.5$, $T^*_\infty=90.116K$, sharp cone.

At this point it is not possible to determine the most dominant triad in figure 7.15. Several factors such as, the linear amplification of all three waves included in the triad, the phase relation between all three waves and the streamwise change in the azimuthal modenumber for the subharmonic waves included in the triads, influence the strength of the resonance interaction. The impact of these factors on the resonance interaction have to be studied in greater detail. Instead this chapter focuses on identifying resonance triads where the secondary waves do not have a subharmonic frequency. Such triads have never been reported for a supersonic boundary layer. Figure 7.16 summarizes all efforts in this regard for the streamwise position $R_x = 1600$ ($0.24m$) and $k_1^c = 21$. The dependency of the azimuthal modenumber of the secondary waves included in the triad on their frequency is demonstrated in figure 7.16a. The two line styles distinguish between the different secondary waves, where a solid line indicates the first secondary wave and a dashed line the second secondary wave. The different symbols represent different resonance triads. If the frequency of both secondary waves deviates from the subharmonic frequency (the frequency for one secondary wave has
Figure 7.16 Possible resonance triads where the secondary waves have a frequency unequal to a subharmonic frequency: (a) dependency of azimuthal modenumber of the secondary waves included in the resonance triads on their frequency for $R_x = 1600$ (0.24m) and $k_c^1 = 21$: different symbols indicate different resonance triads and the line style distinguishes between both secondary waves that close the triad, (b) illustration of the limit when one secondary wave reaches zero disturbance frequency; M=3.5, $T_{\infty} = 90.116$K, sharp cone.

to increase and for the other to decrease according to $\omega^1 = \omega^2 + \omega^3$) additional resonance triads appear. Every subharmonic resonance triad splits into two new triads (in figure 7.16a at $F_{sec} = 1.0E-5$). The limit for one secondary frequency converging to zero is depicted in figure 7.16b. The azimuthal distribution of the streamwise wavenumber tends to zero (—) for zero disturbance frequency (first secondary wave) whereas the azimuthal distribution of the streamwise wavenumber for the second secondary wave tends to the distribution of the fundamental frequency (- -) shifted by $k_c^1 = 21$.

The resulting resonance triads for one secondary frequency $F \to 0.0$ are summarized in figure 7.17. Three main cases can be observed, where the first case (figure 7.17a) represents the interaction of the primary instability wave with the mean flow, the second (figure 7.17b), a fundamental resonance of the oblique primary wave with an oblique secondary wave with the same frequency but different azimuthal modenumber $k_c$ and the third, oblique breakdown. All these “triads” have a coun-
Figure 7.17 Illustration of possible resonance triads with one secondary frequency reaching zero ($F \to 0.0$) using wave vectors ($R_x = 1600$ ($0.24m$) and $k^1_c = 21$): (a) interaction between the primary wave and the mean flow, (b) fundamental resonance between the oblique primary wave and an oblique secondary wave with the same frequency but different azimuthal modenumber $k_c$, (c) “oblique breakdown” (interaction between the oblique primary wave and an oblique secondary wave with the same frequency and the opposite azimuthal modenumber $k^2_c = -k^1_c$ than the primary wave); M=3.5, $T^*_\infty$=90.116K, sharp cone.
terpart, mirrored at $k_c = 0$, with instability waves traveling at opposite wave angles except for “oblique breakdown” (third case) where the counterpart is the same as the original mechanism. Note that the triad at $k_c \sim 33$ in figure 7.17 is of the same nature as the second case (figure 7.17b) but with a different azimuthal modenumber and therefore, is not listed separately as a fourth limiting case.

The outcome of this study can be used to address the question whether oblique breakdown or any other resonance triad is a dominant mechanism in the transition process of a boundary layer at supersonic Mach numbers. The three factors previously mentioned that influence the strength of a resonance interaction suggest that oblique breakdown is the strongest nonlinear mechanism because both oblique instability waves, initiating oblique breakdown, experience the strongest amplitude growth according to linear theory, and the wave–vortex triad is always synchronized in downstream direction. However, the effect of the phase relation between these modes still needs to be investigated in detail.

All triads found in this section are summarized in figure 7.18, which shows the disturbance spectrum at $R_x = 1600$ (0.24m) for the wave packet discussed in section 7.3. The white circle at $k_c = 21$ denotes the primary wave with $f^* = 23.415\text{kHz}$ while the other circles denote the subharmonic resonance triads. Filled circles indicate subharmonic waves that mainly experience linear growth whereas unfilled circles represent subharmonic waves that are expected to grow due to the resonance. This distinction between the subharmonic waves is based on their amplitude level in the frequency–azimuthal modenumber plane in figure 7.18. Lines demonstrate the frequency and azimuthal modenumber for the new resonance triads with secondary waves that do not have a subharmonic frequency. The dashed lines predict positions where strong resonance growth might appear.
Figure 7.18 Disturbance spectrum in the frequency–azimuthal modenumber plane for the wave packet of section 7.3 obtained from wall pressure at the streamwise position $R_x = 1600 (0.24m)$. Circles indicate the position of subharmonic resonance triads identified in figure 7.15 and lines demonstrate their limiting behavior for one frequency converging to zero (figure 7.16). The white circle at $k_c = 21$ denotes the primary wave while the subharmonic waves that are expected to experience resonance growth are denoted by the circles that are not filled. Each figure (a-c) illustrates different resonance triads. The dashed lines predict positions where strong resonance growth might appear; $M=3.5$, $T_\infty^*=90.116K$, sharp cone.
7.5 Early Nonlinear Regime

The study of resonance triads in the previous section provides an idea of the frequencies and azimuthal modenumbers where possible resonance growth might occur for the investigated wave packet (wave packet “3” in section 7.3). Since a wave packet is composed of a large number of waves, the focus on the resonance interactions of one primary wave, as applied in this study, is a strong simplification of the real weakly nonlinear development. Moreover, it is unclear if the new resonance triads, where the secondary waves have no subharmonic frequencies, indeed constitute possible transition mechanisms. The results presented in this section, however, suggest that the simplified study of resonance triads can explain the weakly nonlinear regime for a wave packet. A new wave packet simulation was performed with a higher forcing amplitude ($A=6\%$ of the approach velocity $U_\infty$). The initial spatial development of the wave packet is still linear for this simulation. Farther downstream, the flow field is altered by nonlinear wave interactions.

The streamwise development of the disturbance spectrum in the frequency–azimuthal modenumber plane for the simulation with the higher forcing amplitude is given in figure 7.19. In this figure, circles indicate the position of the subharmonic resonance triads from the previous section and vertical lines close to the abscissa denote the azimuthal modenumbers of the cases with $F \to 0.0$ for one of the secondary waves. Note that the triads found in section 7.4 usually have an azimuthal modenumber with a noninteger value. On a cone, however, only integer values of the azimuthal modenumber for instability waves in the linear regime can exist. Thus, it is very likely that only detuned Craik-type resonance triads play a role for transition on a cone.

For the first two spectra in figure 7.19, nonlinear wave interactions are very weak. Hence, these spectra are very similar to the spectrum in figure 7.8b. Farther downstream, the spectra are strongly altered by nonlinear wave interactions. Below the
Figure 7.19 Disturbance spectrum in the frequency–azimuthal modenumber plane for the simulation with the high forcing amplitude (6% of the approach velocity $U_\infty$) obtained from wall pressure at different streamwise positions. Circles indicate the position of the subharmonic resonance triads identified in section 7.4. The vertical dashed-dotted lines close to the abscissa represent the azimuthal modenumbers of the cases with $F \to 0.0$ for one of the secondary waves in section 7.4; $M=3.5$, $T_\infty^*=90.116\text{K}$, sharp cone.
two dominant regions of high amplitudes centered at about $k_c = 21$, “legs” develop, which connect disturbance waves from the two dominant regions with high frequency to disturbance waves with frequency approaching zero. These legs are amplified in streamwise direction through nonlinear wave interactions (compare figures 7.19c and 7.19d). It appears as if the cases with $F \to 0.0$ for one of the secondary waves (see previous section) predict the position of these legs for small frequencies (vertical dashed-dotted lines in figures 7.19c and 7.19d). The circles that reflect the positions of subharmonic resonance growth are located within those legs. Hence, the new resonance triads, identified in the previous section, could explain the development of the legs in figures 7.19c and 7.19d.

Additional to the legs, “ears” develop at $k_c \sim \pm 55$, which are higher-harmonics in azimuthal direction of the high-amplitude regions. These ears are also amplified in streamwise direction and the center region of those ears reach amplitude levels that are comparable in magnitude to the amplitude levels of the legs. Oblique breakdown could explain the appearance of such ears since they are at a position where one could expect mode $[1, 3]$ of oblique breakdown. Typically, in oblique breakdown simulations, the oblique wave pair $[1, \pm 1]$ is initially forced and the nonlinear development of this wave pair exhibits a characteristic feature: Modes with odd azimuthal numbers $k$ are only generated for odd harmonic frequencies $h$, and modes with even spanwise wavenumbers are generated only for even frequencies $h$. Hence, a signature feature for the initial nonlinear stage of oblique breakdown is the strong growth of the steady modes $[0, \pm 2]$ and wave modes $[1, \pm 3]$. As illustrated in figure 7.19d, the legs at about $k_c \sim \pm 40$ approach an azimuthal modenumber where modes $[0, \pm 2]$ would be if the flow were forced periodically, while the ears can be linked to modes $[1, \pm 3]$.

Figure 7.20 shows a very interesting result from a study performed in parallel to the present work (same flow parameters), which addresses the question whether an increasing number of primary disturbance modes can influence oblique breakdown. In this study, Laible et al. (2009) forced several pairs of oblique instability modes with
Figure 7.20 Streamwise development of disturbance spectra in the frequency–azimuthal modenumber plane from Laible et al. (2009) These spectra are obtained from a simulation (case B) that models natural transition for the same flow configuration as in the present work. Only oblique modes, as typical for oblique breakdown, are introduced into the computational domain by prescribing their linear eigenfunctions and a random phase at the inflow. Note that the computational domain is farther downstream than in the present work and only 1/32 of the cone is simulated. $h$ and $k$ denote multiples of the fundamental frequency and fundamental azimuthal modenumber ($k_c = 32$); $M = 3.5$, $T_{\infty}^* = 90.116$K, sharp cone.

Different frequencies and amplitude levels corresponding to linear N-factor calculations by prescribing their linear eigenfunctions and a random phase at the inflow. The disturbance spectrum for one of these simulations (case B in Laible et al., 2009) at three different streamwise positions is demonstrated in figure 7.20. Note that these spectra are farther downstream than the spectra discussed in this chapter. Furthermore, $k$ denotes multiples of the azimuthal modenumber $k_c = 32$, which is the azimuthal modenumber for the instability waves with the highest streamwise growth (according to N-factor calculations). At $x = 1.65$ ($x$ is nondimensionalized with the cone length of $0.3556m$), two dominant regions of high amplitudes are clearly visible for $k$ unequal to zero. These regions generate legs and peaks in a staggered pattern typical for oblique breakdown farther downstream (for $x = 1.75$ and $x = 1.85$). When compared to the spectra in this chapter (figure 7.19c and 7.19d), figure 7.20 exhibits two major similarities: (i) the generation of legs at $k = 2$ that end in the steady mode $[0, 2]$ and (ii) the generation of mode $[1, 3]$. This comparison confirms that oblique
breakdown may be also present in the high-amplitude wave packet from this chapter.

To close this section, different features of the wave packet in the weakly nonlinear regime are displayed using envelopes. As in figure 7.11 for the wave packet initiated by a small forcing amplitude, figure 7.21 shows the temporal evolution of wall-pressure envelope along the centerline of the wave packet generated by a high amplitude pulse. Differences between the weakly nonlinear stage and the linear transition regime can be immediately observed by comparing both figures. The dotted lines indicate the propagation speed of the front and the tail of the packet. Note that the actual speed of the wave packet, which is not equal to the propagation speed of the wave front or tail, is calculated from the group velocity. Initially, for the simulation with the higher forcing amplitude the propagation speeds of both, the front and the tail, are very close to the values for the linear regime. Note that the values in figure 7.21 and 7.11 are only estimated. Also the shape of both packets with different forcing amplitudes are comparable for the first four streamwise positions. Farther downstream, starting at $x^* = 0.270m$, the wave packet with higher disturbance amplitudes experiences however a strong change in its envelope. A second peak develops at the tail and the spreading of the packet significantly increases while the propagation speed of the tail decreases as highlighted by two ellipses for $x^* = 0.270m$ and $x^* = 0.284m$.

Figure 7.22 demonstrates the three-dimensional shape of the wave packet by displaying contours of constant amplitudes (envelopes) from wall pressure for different azimuthal angles $\varphi$. This figure gives an impression of how nonlinear wave interactions alter the structures analyzed in figure 7.12. Close to the disturbance hole at $x^* = 0.213m$, the same “butterfly” structure as before is visible. With increasing downstream position, this structure is slightly changing when compared to the linear case. The peak on the centerline starts to split in two peaks and longitudinal structures appear between $\varphi = 0.0$ and $\pm 0.1$. These longitudinal structures develop into a pronounced tail for figure 7.23 and cause a significant increase in the spreading of the wave packet. A different view of the wave packet is provided by showing con-
Figure 7.21 Temporal evolution of wall-pressure envelope for different streamwise positions along the centerline of the wave packet initiated by a high amplitude pulse. Dotted lines (...) are an estimate for propagation speed of the front ($\sim 570\text{m/s}$) and the tail ($\sim 410\text{m/s}$) of the wave packet. Ellipses highlight a significant change in the packet spreading; $M=3.5$, $T_\infty^*=90.116\text{K}$, sharp cone.

t ours of streamwise velocity disturbance at three wall-normal positions in figure 7.24. Close to the wall (figures 7.24a and 7.24d), two longitudinal structures develop near the centerline. For higher wall-normal positions the influence of these longitudinal structures vanishes. In typical oblique breakdown simulations, a similar behavior can be observed (compare figure 8.10 from the next chapter). The streamwise velocity disturbances exhibit features of the steady vortex-mode close to the wall while farther away structures that result from a superposition of the initially forced wave pair are
Figure 7.22 Temporal evolution of contours of constant amplitudes (envelopes) from wall pressure for different azimuthal angles $\varphi$ at several streamwise positions from the simulation with the high forcing amplitude. Figures (a)-(f) use the same contour levels; $M=3.5$, $T_\infty=90.116K$, sharp cone.
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Figure 7.23 Temporal evolution of contours of constant amplitudes (envelopes) from wall pressure for different azimuthal angles \( \varphi \) at a streamwise positions farther downstream than in figure 7.22 from the simulation with the high forcing amplitude; \( M=3.5, T^*_\infty=90.116K \), sharp cone.

more pronounced.

7.6 Summary

In the second result chapter, transition mechanisms initiated by a wave packet in a cone boundary layer at Mach 3.5 were studied. Disturbances were introduced into the boundary layer by pulsing the wall-normal velocity through a hole on the cone surface. The computational setup was very close to experiments by Corke et al. (2002) and Matlis (2003). The study was divided into three parts. In the first part, the linear development of a wave packet was documented in detail. Contour plots of the wall pressure close to the disturbance hole revealed that three different wave packets were generated by the applied forcing. Two wave packets traveled at a fast group velocity and consisted of disturbances that were damped in streamwise direction. The third wave packet was composed of unstable boundary layer modes that strongly grew in streamwise direction. The latter wave packet was the focus of the present study
Figure 7.24 Spatial evolution of streamwise velocity disturbance $u'$ at different wall-normal positions (a,d: $y^* = 0.125mm$, b,e: $y^* = 0.493mm$ and c,f: $y^* = 0.992mm$) and two different time instances (a,b,c: $t^* = 0.192ms$ and d,e,f: $t^* = 0.278ms$); $M=3.5$, $T^*_\infty=90.116K$, sharp cone.
since this wave packet was assumed to generate transition. The disturbance spectra for this wave packet in the frequency–azimuthal modenumber plane based on wall-pressure amplitudes exhibited features typical for the linear regime as predicted by linear stability theory. Oblique instability modes experienced the highest streamwise amplification. With increasing streamwise position of the disturbance spectra, the highest streamwise amplification shifted to lower disturbance frequencies and higher azimuthal modenumbers. On the unrolled cone surface in the x-z plane, contours of the envelope obtained from the wall pressure showed structures similar to a hand. The number of “fingers” increased in downstream direction.

The second part of the chapter 7 focused on the identification of possible, asymmetric resonance triads for the most dominant oblique instability wave of the wave packet. This instability wave has a dimensional frequency of about $f^\ast \simeq 23.415 kH z$ and an azimuthal modenumber of about $k_c = 21$ for the present computational setup. With this wave as primary wave of the resonance triads, three different asymmetric, subharmonic resonance triads have been found. Whether these subharmonic resonance triads constitute dominant breakdown mechanisms depends mainly on the following three factors: (i) the linear amplification of all three waves included in the triad, (ii) the phase relation between all three waves and (iii) the streamwise change in the azimuthal modenumber for the subharmonic waves included in the triads. Furthermore, new resonance triads have been documented where the secondary waves have no subharmonic frequencies. Such resonance triads have not been reported yet for a supersonic boundary layer and therefore, it is unclear if these new resonance triads indeed represent possible transition mechanisms. The study of these new triads however suggested that oblique breakdown might be a limiting case of these new resonance triads with one secondary wave having zero frequency. This result can be used to address the question whether oblique breakdown or any other resonance triad is a dominant mechanism in the transition process of a boundary layer at supersonic Mach numbers. The three factors previously mentioned that influence the
strength of a resonance interaction, suggest that oblique breakdown is the strongest nonlinear mechanism because both oblique instability waves, initiating oblique breakdown, experience the strongest amplitude growth according to linear theory, and the wave–vortex triad is always synchronized in downstream direction.

Although a study of possible resonance triads based on one dominant instability wave as primary wave is a strong simplification of the real weakly nonlinear development in a wave packet, it still might help to explain some major findings in the third and final part of chapter 7. This part focused on the weakly nonlinear development of a large amplitude wave packet. The initial disturbance development of this wave packet remained linear while farther downstream nonlinear wave interactions altered the shape and the disturbance spectrum of the wave packet. The overall development of the disturbance spectrum obtained from the wall pressure can be summarized as follows: (i) For the present setup, there is no pronounced asymmetric subharmonic resonance in the spectrum although asymmetric subharmonic resonance cannot be ruled out since there is a streamwise growth of disturbance waves that have a subharmonic frequency of the most dominant instability wave in the wave packet. (ii) “Legs” appear at high azimuthal modenumbers that connect instability waves with high frequency to waves with frequency approaching zero. These legs could be explained by the new resonance triads discussed in the second part of this study. (iii) Higher-harmonics of the highly amplified oblique instability waves generate “ears” centered at \( k_c \sim \pm 55 \). These ears might be a result of oblique breakdown. A study performed in parallel (Laible et al., 2009) further corroborates this finding.
8. Complete Transition to Turbulence Via Oblique Breakdown in a Flat-Plate Boundary Layer at Mach 3

The previous two result chapters clearly emphasized that oblique breakdown is a very important nonlinear transition scenario for supersonic boundary layers. Chapter 6 showed that oblique breakdown might have been present in the experiments by Kosinov and co-workers while chapter 7 suggested that oblique breakdown might be the most dominant nonlinear mechanism. However, a nonlinear mechanism is only relevant for the transition process if this mechanism can indeed completely transition a laminar boundary layer to turbulence. Hence, the following chapter discusses the final question of this thesis: Does oblique breakdown lead to a fully developed turbulent boundary layer? In this chapter, the entire transition path initiated by a pair of oblique time-harmonic waves at low amplitudes is studied in detail. The flow conditions for this study are explained in section 8.1. The forcing amplitudes are low enough so that the early disturbance development can be compared to stability theory (section 8.2). The known origin and characteristics of the disturbances introduced facilitate a more reliable interpretation of the results at the later nonlinear stages. In section 8.3, the characteristics of the oblique breakdown mechanism are studied for the early and late nonlinear stages. Finally, in section 8.4, it is assessed whether a fully turbulent flow can be reached and sustained.

8.1 Physical Problem and Computational Setup

The simulations in this chapter are a continuation of the earlier studies by Mayer (2004) and Husmeier et al. (2005) of flat-plate boundary layer at Mach 3. The physical conditions of the simulations match the Princeton wind-tunnel conditions (Graziosi & Brown, 2002): the unit Reynolds number formed with the free-stream velocity and free-stream viscosity at the inflow is $Re = 2.181 \times 10^6 m^{-1}$ and the free-stream
temperature is $T^*_\infty = 103.6 K$.

Figure 8.1a illustrates the computational setup for all simulations discussed in this chapter. The earlier simulation by Mayer (2004) and Husmeier et al. (2005) is denoted as CASE 1. CASE 3 and 4 are the main focus of this chapter and CASE 2, 5, 6 & 7 are used as grid, domain height and domain width studies. The overall resolution increases from CASE 1 to CASE 7. An example for the grid of CASE 3 & 4 is shown in figure 8.1b. The grid is clustered in the streamwise direction using a fifth-order polynomial (figure 8.1c) and in the wall-normal direction using a third-order polynomial (figure 8.1e). The computational grid in physical space consists of a total of roughly 212 million grid points for CASE 3 & 4 (see table 8.1 for the other cases). The inflow of the domain for all cases is located at $x^*_0 \simeq 0.258 m$ downstream of the leading edge of the plate, whereas the outflow ranges from approximately 11.3 to 14.5 streamwise wave lengths $\lambda_x$ of the oblique fundamental disturbance waves in the linear regime, i.e. $x^*_L \simeq 1.145 m$ for CASE 3 & 4. The domain height for CASE 3 & 4 is chosen as $y^*_H \simeq 0.030 m \approx 5$ boundary layer thicknesses $\delta$ (laminar) at the outflow, such that even with the high increase in boundary layer thickness caused by the transition process no turbulent flow structures reach the free-stream boundary. Pseudo-spectral discretization using Fourier modes (Canuto et al., 1988) is employed in the spanwise direction of the computations. For CASE 1-6, the flow is assumed to be symmetric to the centerline ($z^* = 0 m$) of the flat plate with respect to the streamwise velocity $u$, wall-normal velocity $v$, density $\rho$ and temperature $T$ and antisymmetric with respect to the spanwise velocity $w$, whereas for CASE 7, the symmetry condition is removed.

Time-harmonic disturbances with a fundamental frequency of about $f^* = 6.36 kHz$ ($F = 3 \times 10^{-5}$) are introduced through a blowing and suction slot (section 6.2) located between $x^*_1 \simeq 0.394 m$ and $x^*_2 \simeq 0.452 m$ ($x_2 - x_1 \approx \lambda_x$). A discrete wave pair of instability waves with equal but opposite wave angle is excited for all cases, except for CASE 4, where also a two-dimensional wave with the fundamental frequency and a
Figure 8.1 Computational setup for all cases: (a) computational domain, (b) grid used for CASE 3, for clarity only every 14th point in $x$ and every 6th point in $y$ are plotted, (c) grid clustering in the streamwise direction, (d) blowing and suction velocity profile $v_p$ over the disturbance slot, (e) grid stretching in the wall-normal direction; $M=3.0$, $T_\infty=103.6K$, flat plate.
very low disturbance amplitude is additionally forced to initialize waves that are not
directly generated through the nonlinear wave interactions of the forced wave pair.
The spanwise wavenumber of $\beta^* \simeq 211.52 m^{-1}$ for this wave pair is chosen to be such
that the generated instability waves experience strong amplification as predicted by
LST throughout the entire computational domain. This spanwise wavenumber deter-
mines also the domain width of all simulations, i.e. $z^*_w = \lambda^*_z = 2\pi/\beta^* \simeq 0.03 m$. The
major differences between the setup for all cases are summarized in table 8.1.

8.2 Linear Regime

The linear transition regime is studied using both numerical simulations and sev-
eral theoretical approaches. This section will discuss results from LST, PSE and
DNS. The linear stability solvers from Mack (Mack, 1965, 1987) and Tumin (Tumin,
2007, 2008) are employed. The latter utilizes a single-domain Chebyshev spectral
collocation method (Malik, 1990; Tumin, 2007) to solve for the entire spectrum for
a single disturbance frequency at a given streamwise position in the flow and for a
prescribed spanwise wavenumber. Tumin’s solver was adjusted to the Intel® Math
Kernel Library (based on LAPACK) for the solution of the generalized, nonsymmet-
ric eigenvalue problem. The spectra for the physical flow conditions discussed in this
chapter are shown in figure 8.2 for two frequencies and two spanwise wavenumbers at
the streamwise position $x^* = 0.5 m$ ($R_x = 1044$).

Figure 8.2a,b show the two-dimensional eigenvalue spectra (vanishing $\beta$) for the
frequencies $f^* = 6.36 kHz$ and $f^* = 84.78 kHz$. Two horizontal lines illustrated by
the circles in both figures represent the acoustic wave modes. Tumin (2007) denotes
the spectrum on the left side as the fast acoustic wave spectrum and on the right
side as the slow acoustic wave spectrum. These continuous spectra originate from
streamwise wavenumbers $\alpha_r$ corresponding to the phase velocities $c_{ph,x} = 1 \pm 1/M$.
The vertical line of circles in figure 8.2 indicates the continuous vorticity and entropy
Table 8.1 Main simulation parameters that differ between all cases. Note that the notation \([h, k]\) is used in order to identify a particular wave according to its frequency \(h\) and its spanwise wavenumber \(k\). \(h\) denotes multiples of the fundamental forcing frequency \(f^* = 6.36 kHz\) and \(k\) multiples of the smallest spanwise wavenumber \(\beta^* = 211.52 \text{m}^{-1}\); \(M=3.0\), \(T_{\infty}=103.6\text{K}\), flat plate.
Figure 8.2 Eigenvalue spectra at $x^* = 0.5m$ ($R_x = 1044$). Results denoted by ($\diamond$, $\circ$, $\Box$) are obtained from Tumin’s solver (DNS solution as base flow) whereas results denoted by ($\downarrow$, $\blacksquare$) are from Mack’s solver (similarity solution as base flow). (a) $f^* = 6.36kHz$ ($F = 3.0 \times 10^{-5}$, $\omega = 0.03132$), 2D ($\beta \sim 10^{-8}$), (b) $f^* = 84.78kHz$ ($F = 40.0 \times 10^{-5}$, $\omega = 0.4176$), 2D ($\beta \sim 10^{-8}$), (c) $f^* = 6.36kHz$ ($F = 3.0 \times 10^{-5}$, $\omega = 0.03132$), 3D ($\beta = 0.102$); M=3.0, $T^*_\infty=103.6K$, flat plate.
spectra \((c_{ph,x} = 1)\). Very close to the acoustic wave spectra are two discrete modes in figure 8.2a with mode F (fast mode) originating from the fast acoustic wave spectrum and mode S (slow mode) originating from the slow acoustic wave spectrum. These modes can be distinguished from the continuous spectrum by their eigenfunctions (see for example figure 8.4a).

![Figure 8.3](image.png)

Figure 8.3 Stability behavior of both discrete eigenmodes \((- : \text{ mode } S, \--\text{ : \text{ mode } F})\) from the spectrum in figure 8.2 as a function of nondimensionalized angular frequency \(\omega\) (equation 5.8) at \(R_x = 1044\) \((\beta \sim 10^{-8})\): (a) phase velocity \(c_{ph,x}\), (b) streamwise amplification rate \(\alpha_i\). Vertical solid line highlights the outflow boundary of the longest simulation (CASE 3) discussed in this chapter. The insert in (b) shows a close-up of the amplified region; \(M = 3.0, T_\infty^* = 103.6\text{K}, \text{flat plate.}\)

By changing the disturbance frequency or the streamwise position for the calculation of the spectrum, both discrete modes are moving through the complex \(\alpha\) plane as illustrated by figure 8.2b. Here, both modes are clearly separate from the continuous spectra and are strongly damped. A similar trend can be observed when the spanwise wavenumber is increased as shown in figure 8.2c. Tracking the eigenvalues of mode F and mode S with changing disturbance frequency at a constant streamwise position \(R_x\) leads to figure 8.3. This figure shows the phase velocities of both modes as a function of the nondimensionalized frequency \(\omega\). Note that \(\omega\) is dependent on the streamwise position and on the frequency (equation 5.8). The phase velocities, however, are only a function of \(\omega\) meaning that changing the frequency for a con-
stant streamwise position will result in the same phase-velocity plot as changing the streamwise position for a constant frequency (Ma & Zhong, 2003). This is not true for the amplification rate $\alpha_i$ in figure 8.3b.

The phase velocity and amplification rate development in figure 8.3 show some of the typical behavior as discussed for Mach 3.8 in section 5.4. For $\omega \to 0$, mode S synchronizes with the slow acoustic wave spectrum and mode F synchronizes with the fast acoustic wave spectrum represented by the horizontal lines at $c_{ph,x} = 1 - 1/M$ and $c_{ph,x} = 1 + 1/M$, respectively. For increasing frequency $\omega$, the phase velocity of mode S increases and for mode F decreases until they coincide at a specific frequency ($\omega \approx 0.44$ in figure 8.3a). This coincidence between the phase velocities of both discrete modes results from a “synchronism” mechanism (Fedorov & Khokhlov, 2002, and section 5.4), which can amplify one of the discrete modes. For higher Mach numbers, usually mode S is amplified leading to the so-called second-mode unstable region. In figure 8.3b, the “synchronism” amplifies mode F. The increase in amplification for this mode is however too small in order to generate a second-mode unstable region as can be expected when figure 5.9 is considered.

The “synchronism” mechanism between both discrete modes has also a strong effect on their eigenfunctions. Figure 8.4 demonstrates how the eigenfunctions of both modes change with increasing frequency $\omega$. When the phase velocities of both modes coincide, their eigenfunction profiles become very similar as apparent in the pressure disturbance in figure 8.4c.

For the setup of the simulations discussed in this chapter, it is important to know whether the synchronization point at $\omega \approx 0.44$ is located within the computational domain since this mechanism could have an influence on the nonlinear stages of transition and therefore, on oblique breakdown. In contrast to higher Mach numbers, the synchronization point at Mach 3 is at a large value for the frequency $\omega$. Figure 8.5, for example, illustrates how this value decreases with increasing Mach number. Note that this figure does not show the exact location where the phase velocities of both
Figure 8.4 Eigenfunctions of the u-velocity and the pressure disturbance for $R_x = 1044$ and $\beta \sim 10^{-8}$ from Tumin’s solver (Tumin, 2007, 2008): u-velocity: amplitude (—) and phase (—-·) of mode S, amplitude (-) and phase (---·) of mode F, pressure: real part (—) and imaginary part (---·) of mode S, real part (-.-) and imaginary part (-..-) of mode F. (a) $\omega = 0.03132$ ($f^* = 6.36kHz, F = 3.0 \times 10^{-5}$), results denoted by (◊, ■) are from Mack’s solver (Mack, 1965) for comparison, (b) $\omega = 0.2088$ ($f^* = 42.39kHz, F = 20.0 \times 10^{-5}$), (c) $\omega = 0.4176$ ($f^* = 84.78kHz, F = 40.0 \times 10^{-5}$), (d) $\omega = 0.4915$ ($f^* = 100.00kHz, F = 47.08 \times 10^{-5}$); $M=3.0$, $T^*_1=103.6K$, flat plate.
Figure 8.5 Angular frequency $\omega$ at the synchronization point of mode F with the entropy and vorticity continuous spectrum as a function of Mach number $M$ for two-dimensional disturbances (---) and as a function of spanwise wavenumber $\beta$ (——) for Mach 3 at $R_x = 1044$. According to Mack (1969), a second-mode unstable region exists for $M \gtrsim 3.8$ (see figure 5.9). $M=3.0$, $T_\infty=103.6K$, flat plate.

discrete modes coincides, but rather the location where mode F coalesces with the entropy and vorticity continuous spectrum ($c_{ph,x} = 1$ in figure 8.3a). The vertical lines in figure 8.3 indicate the position of the outflow for the simulation with the largest streamwise domain size (CASE 3). Clearly, the synchronization point is far downstream of the outflow and hence, will not play any role in the transition process for a flat-plate boundary layer at Mach 3. With increasing spanwise wavenumber this point even moves to higher values of $\omega$ (figure 8.5). Thus, it can be concluded that oblique breakdown, initiated for a spanwise wavenumber of $\beta^* = 211.52m^{-1}$ and a frequency of $f^* = 6.36kHz$ ($F = 3.0 \times 10^{-5}$), is only caused by mode S and thus, mode F is not of interest for the interpretation of the simulation data in the following sections.

For the rest of this section, the emphasis is on the stability behavior of mode S and whether the simulations can correctly reproduce its downstream behavior. The amplification rate of mode S is strongly dependent on the spanwise wavenumber. As demonstrated in figure 8.2c, mode S moves to higher streamwise amplification
rates and becomes separated from the slow acoustic wave spectrum for a spanwise wavenumber of \( \beta^* = 211.52 \text{m}^{-1} \) and a frequency \( f^* = 6.36 \text{kHz} \) when compared to the spectrum for two-dimensional disturbances with the same disturbance frequency in figure 8.2a. Note, also the origins of the slow and fast acoustic spectra move to different streamwise wavenumbers since their corresponding streamwise phase velocities change according to \( c_{ph,x} = 1 \pm 1/(M \cos(\psi)) \) with \( \psi \) being the wave angle (section 5.3).

Tracking the eigenvalues of mode S in figure 8.2 yields the stability diagrams in figure 8.6. In figure 8.6, contour levels of the amplification rate \( \alpha_i \) are plotted for different disturbance frequencies and at different streamwise positions for a given spanwise wavenumber. The solid lines constitute the stability diagram for the spanwise wavenumber \( \beta^* \approx 211.52 \text{m}^{-1} \) and the dashed line in figure 8.6a indicates the neutral curve for the corresponding two-dimensional instability waves. Figure 8.6b shows contour levels of \( \alpha_i \) at constant local Reynolds number \( R_x = 750 \) and figure 8.6c for a constant frequency \( f^* = 6.36 \text{kHz} \). The computational domains for CASE 1, 2, 3 & 4 are also included in these figures. Instability waves (modes S) with the frequency of interest for the DNS (6.36kHz) are amplified throughout the computational domain as can be seen in figure 8.6c. Therefore, linear stability suggests that, for the flow conditions and the domain of interest, transition may be triggered by these oblique instability waves.

In addition to the oblique breakdown simulations, two DNS with a considerably decreased forcing amplitude were performed such that the linear regime is maintained throughout the computational domain. These simulations facilitate a comparison of the DNS results to LST (with parallel flow assumption) and to PSE. The PSE calculations were conducted using NOLOT (Hein et al., 1996) during a stay at the Institut für Aerodynamik und Gasdynamik (IAG) in Stuttgart, Germany. Figure 8.7 compares the complex streamwise wavenumber \( \alpha \) from the DNS to LST and PSE for a constant frequency of \( 3.0 \times 10^{-5} \) (6.36kHz) at two different spanwise wavenumbers.
Figure 8.6 Contours of constant amplification rate $\alpha_i$ obtained from LST (Mack’s solver): (a) for constant spanwise wavenumber $\beta^* \simeq 211.52m^{-1}$, dashed line indicates neutral curve for two-dimensional waves, (b) for constant local Reynolds number $R_x = 750 \ (x^* = x_0^* \simeq 0.258m)$ (c) for constant frequency $f^* = 6.36kHz \ (F = 3.0 \times 10^{-5})$. Horizontal solid line in (a) and dotted lines in (b) and (c) indicate computational setup; $M=3.0$, $T_\infty=103.6K$, flat plate.
Figure 8.7 Streamwise development of the complex streamwise wavenumber $\alpha$ obtained by LST using Mack’s solver ($\times$: $\beta^* = 211.52m^{-1}$), PSE using NOLOT (- -: $\beta^* = 196.2m^{-1}$) and DNS (low forcing amplitude) for $F = 3.0 \times 10^{-5}$ (6.36kHz): (a) streamwise amplification rate $\alpha_i$, (b) streamwise wavenumber $\alpha_r$. For the DNS, $\alpha$ is calculated using two different criteria, the wall pressure (---: $\beta^* = 211.52m^{-1}$) and the maximum in the wall-normal amplitude distribution of the u-velocity disturbance ($\circ$: $\beta^* = 211.52m^{-1}$, $\square$: $\beta^* = 196.2m^{-1}$); $M$=3.0, $T_\infty$=103.6K, flat plate.

($\beta^* = 196.2m^{-1}$ and $\beta^* = 211.52m^{-1}$). From the DNS data, the complex streamwise wavenumber was calculated using equations (5.4) and (6.14). The streamwise amplification rate $\alpha_i$ in figure 8.7a is computed using either the wall-pressure disturbance amplitude (Eissler & Bestek, 1996; Ma & Zhong, 2003) or the maximum in the wall-normal amplitude distribution for the u-velocity. As observed in previous investigations for supersonic flat-plate boundary layers (Thumm et al., 1989; Eissler & Bestek, 1996; Husmeier et al., 2005), the amplification rate based on the latter predicts stronger growth rates than LST and this behavior was attributed to non-parallel effects by the authors. The excellent agreement between DNS and PSE results in figure 8.7a further corroborates this statement. Note that PSE is a nonlocal stability analysis, which accounts for non-parallel effects. When using the wall-pressure disturbance for the computation of the amplification rate $\alpha_i$, the agreement between LST and DNS data improves significantly. This improvement confirms that different criteria are differently affected by non-parallel effects as already observed for incom-
Figure 8.8 Comparison of wall-normal amplitude and phase distribution obtained from DNS (reduced forcing amplitude) with the eigenfunctions from LST for (a) u-velocity disturbance, (b) temperature disturbance and (c) pressure disturbance at $x^* = 0.5m$ and $\beta^* = 211.52m^{-1}$. DNS: (—) amplitude distribution, (- -) phase distribution; Tumin’s stability solver (DNS profiles as base flow): (×) amplitude distribution, (+) phase distribution; M=3.0, $T^*_{\infty}=103.6K$, flat plate.

The last figure for this section (figure 8.8) compares the wall-normal amplitude and phase distribution for the velocity, temperature and pressure from the DNS to results obtained by LST using Tumin’s stability solver at $x^* = 0.5m$, $\beta^* = 211.52m^{-1}$ and $f^* = 6.36kHz$. The amplitude distributions from both, linear theory and DNS, are normalized by their respective maximum values within the boundary layer. The excellent agreement between all results substantiates that the linear eigenbehavior of the unstable mode in figure 8.2c is correctly reproduced in the DNS. Furthermore, the agreement with theory confirms that the disturbances introduced via the blowing and suction slot indeed initiate a pair of oblique instability waves and that these waves are dominant for the present setup.
8.3 From the Weakly Nonlinear Regime to the Late Nonlinear Stages

In the last section, it was demonstrated that, for a laminar flat-plate boundary layer at Mach 3 with $T_\infty^* = 103.6K$ and an adiabatic wall, transition can only be initiated by a slow mode (mode S). For a frequency of $f^* = 6.36kHz$ and a spanwise wavenumber of $\beta^* = 211.52m^{-1}$, this mode experiences strong streamwise amplification and therefore, was forced in all simulations discussed below. The forcing amplitude was chosen as 0.3\% of the free-stream velocity $U_\infty$ for all cases. In addition, the wall temperature, even in the fully turbulent region, was fixed to the adiabatic wall temperature of the laminar boundary layer, i.e. $T_w = T_{ad}$.

To verify whether the transition process initiated in the simulations indeed passes through all transition regimes typical for a low-disturbance environment, figure 8.9b compares the streamwise amplification rate $\alpha_i$ obtained from the maximum in the wall-normal amplitude distribution of the streamwise velocity disturbance from figure 8.9a (—) with results from LST (Mack’s solver) and the DNS with a reduced

Figure 8.9 Initial disturbance development of the forced oblique instability wave with $f^* = 6.36kHz$ and $\beta^* = 211.52m^{-1}$ for cases with the high forcing amplitude (lines), a DNS with reduced forcing amplitude (o) and for LST (x). (a) streamwise development of maximum u-velocity disturbance: (—) [1, ±1] ( - - ) [0, ±2] (---) [1, ±3] (---) [0, ±4] (b) amplification rate $\alpha_i$ calculated according to equation (6.14) using the maximum of the u-velocity disturbance as criterion, (b) wall-normal amplitude and phase distribution of the streamwise velocity perturbation at $x^* = 0.65m$ (vertical dotted line in a and b); M=3.0, $T_\infty^*=103.6K$, flat plate.
forcing amplitude from the previous section. The streamwise amplification rate in figure 8.9b for the cases with the high forcing amplitude is either contaminated by acoustic disturbances or by the fast mode (see also the validation case in Eissler & Bestek, 1996) since its streamwise distribution exhibits an oscillation. For the simulation with the reduced forcing amplitude from the previous section, these modulations are weaker since the disturbance slot was positioned farther upstream when compared to the location of the disturbance slot for all simulations with the higher forcing amplitude. As a consequence, the amplitude levels of the damped acoustic disturbances or the damped fast mode were very small in the region of interest for the simulation with the small forcing amplitude (see figure 8.7a). Note a receptivity study as in Tumin et al. (2007) would provide more detailed information about the absolute amplitude values of the fast mode and the continuous part of the spectrum right downstream of the disturbance slot. Such a study, however, is beyond the scope of this chapter. Nevertheless, figure 8.9 confirms that in all cases with a large forcing amplitude the linear regime is correctly reproduced by all DNS results presented in this chapter.

The early nonlinear stages of oblique breakdown are dominated by the nonlinear interaction of a wave–vortex triad composed by the forced oblique discrete modes $[1, \pm 1]$ and the steady vortex modes $[0, \pm 2]$ (Fasel et al., 1993; Chang & Malik, 1994) as demonstrated by the streamwise development of the wall-normal maximum for the streamwise velocity in figure 8.9a. It is important to note that the $[0, \pm 2]$ modes are generated directly by the forced oblique modes $[1, \pm 1]$ and grow mainly through nonlinear interaction with these oblique modes (Chang & Malik, 1994). This is in contrast to the incompressible case where the growth of the $[0, \pm 2]$ modes is caused by transient growth (Berlin et al., 1999).

The wave–vortex triad also clearly dominates the flow structures in the early nonlinear stages of oblique breakdown. Figure 8.10 shows contours of streamwise velocity disturbance at two different wall-normal positions between $x^* = 0.546m$ and
Figure 8.10 Contours of streamwise velocity disturbance illustrating flow structures in the early stage of oblique breakdown between $x^* = 0.546m$ and $x^* = 0.670m$: (a) close to the wall at $y^* \simeq 0.9mm$, (b) farther away at $y^* \simeq 2.3mm$; $M=3.0$, $T_\infty = 103.6K$, flat plate.

$x^* = 0.670m$. Close to the wall at $y^* \simeq 0.9mm$ longitudinal regions of low and high streamwise velocity develop and are amplified in streamwise direction while for higher wall-normal positions (at $y^* \simeq 2.3mm$) the typical wave pattern of two superimposed oblique instability waves is visible. Hence, the influence of the vortex modes $[0, \pm 2]$ on the contours of streamwise velocity disturbance is more significant closer to the wall while for higher wall-normal positions the initially forced wave pair $[1, \pm 1]$ is dominant. The wall-normal dependency of the flow structures in the contours of streamwise velocity disturbance is caused by the absolute amplitude level of modes $[0, \pm 2]$ and $[1, \pm 1]$ and their wall-normal amplitude distribution. The maximum in the wall-normal amplitude distributions of modes $[0, \pm 2]$ is closer to the wall than for modes $[1, \pm 1]$ (see also Mayer, 2004).

Instantaneous flow structures identified by the Q-criterion (Hunt et al., 1988) in figure 8.11 confirm the structures shown in figure 8.10. Q is related to the second invariant of the velocity gradient tensor. Positive values of Q reveal flow regions where rotation dominates the flow field. Again, the flow structures in figure 8.11a
Figure 8.11 Flow structures identified by the Q-criterion: (a) \( Q = 10 \) between \( x^* = 0.546m \) and \( x^* = 0.670m \), (b) \( Q = 100 \) between \( x^* = 0.670m \) and \( x^* = 0.798m \). Also shown are the boundary layer thickness indicated by a solid black horizontal line and contours (between \(-0.1\) and \(-0.001\)) of streamwise velocity disturbance for a constant spanwise position (in (a) at \( z^* \approx 0.0149m = \lambda_z/2 \) and in (b) at \( z^* \approx 0.0114m \approx 0.38\lambda_z \)). When comparing the contours of streamwise velocity disturbance in figures 8.10 and 8.11, the Q-criterion predicts similar flow structures.

The characteristic disturbance amplitude curves for the early and late nonlinear stages of oblique breakdown is shown in figure 8.12. This figure demonstrates the streamwise development of the maximum u-velocity disturbance for various spanwise wavenumbers from CASE 1 and CASE 2 for modes that are directly created through nonlinear wave interactions of the wave-vortex triad. The nonlinear generation of higher-harmonic modes in time and spanwise direction follows a particular pattern as found for a Mach 1.6 boundary layer (Thumm, 1991; Fasel et al., 1993): Modes
Figure 8.12 Streamwise development of the maximum u-velocity disturbance for different spanwise wavenumbers from CASE 1 (symbols) and CASE 2 (lines): (a) stationary modes (○, —): [0, ±2], (□, - -): [0, ±4], (◊, --): [0, ±6], (△, .-): [0, ±8]), (b) modes with the fundamental frequency (○, —): [1, ±1], (□, - -): [1, ±3], (◊, --): [1, ±5], (△, .-): [1, ±7]), (c) modes with the first higher-harmonic frequency (○, —): [2, ±0], (□, - -): [2, ±2], (◊, --): [2, ±4], (△, .-): [2, ±6], (+, .-): [2, ±8]), (d) modes with the second higher-harmonic frequency (○, —): [3, ±1], (□, - -): [3, ±3], (◊, --): [3, ±5], (△, .-): [3, ±7], (+, .-): [3, ±9]); M=3.0, $T_\infty^* = 103.6K$, flat plate.
with odd spanwise wavenumbers $k$ are only generated for odd harmonic frequencies $h$, while modes with even spanwise wavenumbers are generated only for even frequencies $h$ resulting in an even value for the sum of $h$ and $k$.

The forced modes $[1, \pm 1]$ initially develop linearly up to about $x^* = 0.7m$ (see figure 8.9a and figure 8.12b). At roughly this position, modes $[0, \pm 2]$ and $[1, \pm 3]$ reach amplitude levels that are comparable to the levels of modes $[1, \pm 1]$. This event marks the end of the early nonlinear stage. A short distance downstream, at about $x^* = 0.8m$, higher harmonic modes reach amplitude levels of the same order of magnitude as the original wave–vortex triad and nonlinear saturation sets in. Note that the maximum of the $u$-velocity fluctuation does not directly display the energy transfer between separate modes. Nevertheless, it still reveals important events in the transition process of oblique breakdown, especially when modes that are not direct descendants of the wave–vortex triad are considered. These modes are generated by round-off errors due to the limited machine precision and their streamwise and spanwise amplitude development is displayed for the subharmonic frequency in figures 8.13a and 8.13b and for the fundamental frequency in figures 8.13c and 8.13d. Note that for other frequencies, which are not integer multiples of the fundamental frequency, similar amplitude distributions can be observed.

The streamwise amplitude development of all modes in figures 8.13a and 8.13c exhibit similar features as for the modes in figure 8.12. Up to about $x^* = 0.8m$, all modes experience streamwise amplitude growth while they start to saturate farther downstream. However, one major difference to figure 8.12 is a sudden increase in the streamwise growth rate for all modes at about $x^* = 0.9m$. The spanwise amplitude distributions in figures 8.13b and 8.13d broaden significantly in streamwise direction with the peak amplitude at small spanwise wavenumbers. An explanation for the initial growth up to $x^* = 0.8m$ of disturbances with frequencies that are no longer integer multiples of the fundamental frequency may be provided by the new resonance triads discussed in chapter 7 where all three instability waves possess a different
Figure 8.13 Streamwise (a,c) and spanwise (b,d) amplitude development for modes that are not directly generated by the wave–vortex triad ([1, ±1] and [0, ±2]) from CASE 3: (a,b) subharmonic frequency, (c,d) fundamental frequency. For (a): (—) [0.5, 0], ( - - ) [0.5, ±1], ( -.- ) [0.5, ±2], ( --- ) [0.5, ±3]. For (c): ( — ) [1, 0], ( - - ) [1, ±2], ( -.- ) [1, ±4], ( --- ) [1, ±6]. For (b,d): (○) $x^* = 0.6m$, (□) $x^* = 0.7m$, (◊) $x^* = 0.8m$; $M=3.0$, $T^*_\infty=103.6K$, flat plate.
Figure 8.14 Flow structures identified by the Q-criterion for $Q = 15000$ (CASE 3) between $x^* = 0.798m$ and $x^* = 0.924m$. Also shown are contours of spanwise vorticity at $z^* \approx -0.0087m$. (a) Entire three-dimensional view, (b) close-up of the breakdown region confirming that the Q-criterion predicts similar structures as illustrated by the spanwise vorticity; $M=3.0$, $T_{\infty}=103.6K$, flat plate.

disturbance frequency.

In the following figures, flow structures are shown for the region between $x^* = 0.798m$ and $x^* = 0.97m$ in order to highlight key features in the flow field that could be related to the sudden increase in the streamwise growth rate at about $x^* = 0.9m$ for all modes in figure 8.13 that are not directly generated by the wave-vortex triad. The instantaneous three-dimensional isosurfaces for $Q = 15000$ in figure 8.14a reveal that the longitudinal structures from figure 8.11 are lifted up from the wall and breakup into small-scale structures, which are similar to hairpin-like vortices. A close-up view of these flow structures is given in figure 8.14b. Contours of spanwise vorticity at $z^* \approx -0.0087m$ in figure 8.14 further corroborate that the Q-criterion clearly identifies the relevant flow structures.

The contours of spanwise vorticity for various spanwise positions in figure 8.15 provide a detailed view of the breakup region and the downstream development of the
Figure 8.15 Contours of spanwise vorticity at various spanwise positions for one time instant from CASE 3: (a) $z^* \approx -0.0087m$, (b) $z^* \approx -0.0076m$, (c) $z^* \approx -0.0064m$, (d) $z^* \approx -0.0052m$, (e) $z^* \approx -0.0041m$, (f) $z^* \approx -0.0029m$, (g) $z^* \approx -0.0017m$; $M=3.0$, $T_\infty=103.6K$, flat plate.
small-scale structures. Note that the streamwise extent in this figure reaches farther downstream than in figure 8.14. The contour levels are chosen such that the contrast in these plots is significantly increased. Figure 8.15 reveals rope-like structures (region of high shear) that coincide with the position of the longitudinal structures from the previous figure. The tip of these rope-like structures is lifted up from the wall and again breaks up into smaller scales. This breakup is further exemplified by the temporal evolution of contours of spanwise vorticity at $z^* \simeq -0.0087m$ in figure 8.16. At this spanwise position, the breakup region extends from about $x^* \simeq 0.84m$ to $x^* \simeq 0.9m$. The sudden increase in streamwise amplification of all modes downstream of $x^* \simeq 0.9m$ in figure 8.13 seems to be linked to the breakup into smaller scales. Downstream of this position the entire flow is rapidly contaminated by small-scale structures as demonstrated in figure 8.17 and the final breakdown to turbulence is initiated.

The flow structures in figures 8.14 to 8.17 are symmetric with respect to the centerline of the plate. This is to be expected since these figures are obtained from CASE 3, in which symmetry is enforced by the computational setup. If the symmetry condition is not enforced as for CASE 7, the picture does not change as demonstrated by figure 8.18. This figure illustrates contours of streamwise velocity $u$ of the first higher Fourier mode in spanwise direction from CASE 7 for the sine and cosine modes (equation 4.11), respectively. The minimum and maximum of the contour levels in figure 8.18a and b are different in order to emphasize the flow structures. The influence of asymmetric modes on oblique breakdown initiated by two oblique waves with exactly the same amplitude and phase is limited since these modes are only generated by the round-off error of the calculation. In CASE 3, the streamwise position of the final breakup into small-scale structures denotes the location where all modes with frequency unequal to integer multiples of the forcing frequency are strongly amplified. In CASE 7, a similar behavior can be observed. At exactly the same streamwise position (where the breakup into small-scale structures occurs)
Figure 8.16 Contours of instantaneous spanwise vorticity at one particular spanwise positions ($z^* \approx -0.0087m$) for various time instants highlighting the breakup into smaller structures (CASE 3): (a) $t = \frac{3}{20}T$, (b) $t = \frac{6}{20}T$, (c) $t = \frac{9}{20}T$, (d) $t = \frac{12}{20}T$, (e) $t = \frac{15}{20}T$, (f) $t = \frac{18}{20}T$; $M=3.0$, $T_\infty=103.6K$, flat plate.
Figure 8.17 Flow structures identified by the Q-criterion for $Q = 20000$ (CASE 3) between $x^* = 0.924m$ and $x^* = 1.071m$. (a) Entire three-dimensional view, (b) close-up of the early turbulent region; $M=3.0$, $T_{\infty}^*=103.6K$, flat plate.

The asymmetric modes also start to be amplified as illustrated by figure 8.18a. The amplitude values of the streamwise velocity for the sine mode in figure 8.18a provide a measure for the magnitude of asymmetry in CASE 7. For the $u$-velocity, this mode is set to zero in CASE 3. Since the contour levels for the sine mode in figure 8.18a are more than 10 orders of magnitude smaller than the contour levels for the cosine mode, CASE 7 remains symmetric even after the breakup into small-scale structures. This is true over the entire domain length of CASE 3. It is however visible that the asymmetric modes are strongly amplified in downstream direction and will eventually reach high amplitude values.

8.4 Final Breakdown to Turbulence

In the previous section a detailed description of the breakup into small-scale structures for oblique breakdown was given and therefore, the different transition regimes from
Figure 8.18 Contours of instantaneous streamwise velocity $u$ obtained from CASE 7 for the first higher Fourier mode in spanwise direction: (a) sine mode, contour levels from $-1.0E-12$ to $1.0E-12$, (b) cosine mode, contour levels from $-0.1$ to $0.1$; $M=3.0$, $T_{\infty}=103.6K$, flat plate.

the early nonlinear regime to the breakdown were discussed. Figure 8.17 clearly illustrates that the breakup is accompanied by a rapid spreading of small-scale structures over the entire flow field downstream of a particular streamwise position ($x^* \approx 0.9m$). Close to this position, the time signal however is still periodic although the flow field exhibits features of a turbulent boundary layer (small-scale structures). Thus, the final breakdown to turbulence did not occur yet. All cases listed in table 8.1, except of CASE 1, which has a very small streamwise domain extent, lead to the same results for the early and late nonlinear transition stages. The different grid resolutions and domain heights of the different cases mainly affect the final breakdown to turbulence. Therefore, this section will also assess what resolution is necessary in order to obtain a converged solution for the entire transition process of oblique breakdown at Mach 3.

In order to check when periodicity breaks down in the simulations, figure 8.19 compares the streamwise development of the wall-normal maximum for the $u$-velocity disturbance obtained from a Fourier transformation of the time-dependent flow data for CASE 2, CASE 3 and CASE 4. Note that CASE 4 only differs from CASE 3 in the forcing input. In CASE 4, additionally to the oblique instability waves with frequency $f^* = 6.36kHz$, a two-dimensional instability wave with one order of mag-
Figure 8.19 Streamwise development of the wall-normal maximum of the streamwise velocity $u$ for selected Fourier modes: (a) CASE 3, (b) CASE 4; the dotted line marks the end of the domain of CASE 2. The notation $[h,k]$ is used to identify a particular wave according to its frequency $h$ and its spanwise wavenumber $k$. $h$ denotes multiples of the fundamental frequency and $k$ multiples of the smallest spanwise wavenumber; $M=3.0$, $T^*_\infty=103.6K$, flat plate.

Figure 8.19 Streamwise development of the wall-normal maximum of the streamwise velocity $u$ for selected Fourier modes: (a) CASE 3, (b) CASE 4; the dotted line marks the end of the domain of CASE 2. The notation $[h,k]$ is used to identify a particular wave according to its frequency $h$ and its spanwise wavenumber $k$. $h$ denotes multiples of the fundamental frequency and $k$ multiples of the smallest spanwise wavenumber; $M=3.0$, $T^*_\infty=103.6K$, flat plate.

magnitude smaller amplitude but the same forcing frequency as the oblique waves was also introduced. Forcing a two-dimensional instability wave initializes waves that can otherwise not be directly generated through the nonlinear wave interactions of the forced wave pair and, therefore, are generated only at the level of round-off errors. As a consequence of the additional perturbation, the disturbance spectrum should be broader earlier for this simulation.

For figure 8.19, two different Fourier transforms have been performed using either a time signal with the length of one period of the forcing frequency or two periods. If the flow field remained periodic in time, the Fourier-modes from both time signals would develop identically in the streamwise direction. As can be seen in figure 8.19, this is not the case for CASE 3 & 4. Just downstream of the end of the domain of the simulation CASE 2 ($x^* = 1.05$), highlighted by a dotted vertical line in figure 8.19, the streamwise amplitude distribution of the initially forced mode $[1, \pm 1]$ differs for the two time signals. Hence, CASE 3 and CASE 4 clearly lose their periodicity close to the end of the computational domain. This is further emphasized by figure 8.20,
Figure 8.20 Temporal evolution of the streamwise velocity at \( y = 2.15 \text{mm} \ (y^+ \simeq 50) \) for CASE 3: (a) \( x^* = 0.942 \text{m} \), (b) \( x^* = 1.104 \); \( M = 3.0 \), \( T_\infty^* = 103.6 \text{K} \), flat plate.

which shows the original time-signal (12 forcing periods) for CASE 3 for two different streamwise positions at the wall-normal location \( y = 2.15 \text{mm} \ (y^+ \simeq 50) \). At \( x^* = 0.942 \text{m} \) (figure 8.20a), the signal is still strongly periodic whereas farther downstream, at \( x^* = 1.104 \) (figure 8.20b), a more random behavior becomes apparent. As figure 8.13 from the previous section, figures 8.19 and 8.20b illustrate the growth of disturbances with frequencies that are no longer integer multiples of the fundamental forcing frequency. In figure 8.19, for example, the subharmonic disturbances reach amplitude levels comparable to the initially forced waves \([1, \pm 1]\) in proximity to the end of the computational domain.

With the loss of the periodicity in CASE 3, one missing piece of evidence for the final breakdown to turbulence is found. Another piece of evidence is the streamwise decay of the skin-friction coefficient after the strong increase caused by transition. Figure 8.21a demonstrates that CASE 3 indeed experiences a decay in skin friction. All disturbances with a non-harmonic frequency (with respect to the original forcing frequency) start to saturate right after the skin-friction coefficient \( c_f \) drops in figure 8.21a. This is shown for the subharmonic frequency in figure 8.19. Hence, the
decay of the skin-friction coefficient is most likely linked to the loss of periodicity. Once, the periodicity in time is lost and the peak in skin friction is surpassed downstream of $x^* = 1.05$, the transition process ends and the flow has reached a turbulent state.

The skin-friction coefficient in figure 8.21a is calculated from

$$c_f = \frac{2\mu \frac{\partial u}{\partial y} |_{y=0}}{Re},$$

where the Reynolds number $Re$ is based on an arbitrary reference length $L^*$ and the flow quantities in the free stream. Note that symbols with an overline $\overline{\phi}$ represent the Reynolds-average, i.e. time and spanwise averaged flow quantities, throughout this chapter

$$\overline{\phi} = \frac{1}{L_x} \frac{1}{\Delta t} \int_{t_0}^{t_0 + \Delta t} \int_0^{L_z} \phi(t, z) \, dt \, dz.$$ (8.2)

Furthermore, fluctuations about the mean of a Reynolds-averaged quantity $\overline{\phi}$ are denoted by $\phi'$ and fluctuations about the mean of a Favre-averaged quantity $\overline{\rho \phi / \overline{\rho}}$ are denoted by $\phi''$. Note that in some figures the interval for the time average is indicated by the number of forcing periods $T_{forcing}$.

Also included in figure 8.21a are different values of the skin-friction coefficient from other numerical simulations published in the literature for turbulent supersonic flow (Guarini et al., 2000; Maeder et al., 2001) and a theoretical correlation for the fully-developed turbulent regime given by White (1991). This correlation is valid for non-adiabatic wall boundary conditions and has the following form

$$c_f \sim \frac{0.455}{S \ln \left( \frac{0.06}{S} Re_{ext} \frac{\mu_e}{\overline{\rho U}} \sqrt{\frac{T_e}{T_{wall}}} \right)^2},$$ (8.3)

where subscript $e$ refers to edge conditions and subscript $w$ to wall conditions. The factor $S$ can be obtained from

$$S = \frac{\left( \frac{T_{wall}}{T_e} - 1 \right)^{\frac{1}{2}}}{\arcsin(A) + \arcsin(B)},$$ (8.4)
Figure 8.21 Streamwise development of selected mean-flow properties from CASE 2 and CASE 3 in comparison to different values published in the literature for turbulent supersonic flow (Guarini et al., 2000; Maeder et al., 2001) and theoretical models (White, 1991): (a) skin-friction coefficient $c_f$, (b) Reynolds number based on momentum thickness $\Theta$, (c) shape factor $H_{12}$. Note that for CASE 2 and 3 the interval for the time average is indicated by the number of forcing periods $T_{forcing}$; $M=3.0$, $T_\infty=103.6\text{K, flat plate.}$
with $A$ and $B$ defined as

$$A = \frac{2a^2 - b}{\sqrt{b^2 + 4a^2}} \quad \text{and} \quad B = \frac{b}{\sqrt{b^2 + 4a^2}}, \quad (8.5)$$

where $a$ and $b$ are given by

$$a = \sqrt{\frac{\gamma - 1}{2}} M_c^2 T_e T_w \quad \text{and} \quad b = \left( \frac{T_{aw}}{T_w} - 1 \right), \quad (8.6)$$

respectively. Note that $T_{aw}$ denotes the adiabatic wall-temperature for a turbulent boundary layer at the same flow conditions. Since this value is not known, it has to be estimated using the turbulent recovery factor (White, 1991; Roy & Blottner, 2006)

$$r_{turb} \sim (Pr)^{\frac{1}{3}} \sim 0.9. \quad (8.7)$$

The streamwise development of the skin friction coefficient for CASE 3 in figure 8.21a approaches the theoretical estimate given by White’s correlation (White, 1991) towards the end of the computational domain. Moreover, the value computed by Guarini et al. (2000) using a Temporal Direct Numerical Simulation (TDNS) for a Mach 2.5 turbulent boundary layer is close to the DNS results. TDNS, with some assumptions, computes an approximation of a fully-developed turbulent boundary layer at a given streamwise location in the limit of a vanishing extent of the streamwise domain size. The data can then be used for comparison of skin-friction coefficients with data obtained via a full spatial DNS, if a location can be found where the Reynolds number based on the momentum thickness $Re_\theta$ and the shape factor $H_{12}$ for both simulations are in close enough agreement. Figures 8.21b and 8.21c confirm that the TDNS can be used as a rough estimate. These figures show the streamwise development of the Reynolds number based on the momentum thickness $Re_\theta$ and the shape-factor $H_{12}$ for CASE 2, CASE 3 and TDNS reference data from the literature. Both the Reynolds number and the shape-factor are close to values from Guarini et al. (2000) at the end of the computational domain, but do not exactly match. A better agreement cannot be expected since both simulations differ in the
Figure 8.22 Comparison of selected mean-flow properties from CASE 3 starting from $x^* = 1.047m$ to values published in literature for supersonic and hypersonic turbulent flat-plate boundary layers (Guarini et al., 2000; Maeder et al., 2001; Jiang et al., 2006; Maekawa et al., 2007; Martin, 2007; Coles, 1954): (a) shape factor $H_{12}$ vs. $Re_\theta$, (b) skin friction coefficient $c_f$ vs. $Re_\theta$; $M=3.0$, $T_{\infty}=103.6K$, flat plate.

flow speed and in the wall-temperature boundary condition. The TDNS of Maeder et al. (2001), although at the same Mach number, approximates the flow with a considerably higher momentum thickness, i.e. a location farther downstream than in simulations presented in this chapter. Consequently, their data should deliver a lower skin-friction coefficient, which is indeed the case.

Figure 8.22 shows the skin-friction coefficient and the shape factor as a function of the Reynolds number based on momentum thickness for the region downstream of the peak in skin friction of CASE 3 together with several additional temporal and full spatial DNS and experiments of supersonic and hypersonic turbulent boundary layers published in the literature (Guarini et al., 2000; Maeder et al., 2001; Jiang et al., 2006; Maekawa et al., 2007; Martin, 2007; Coles, 1954). When compared to these results, CASE 3 can be regarded as reaching realistic values for mean-flow properties of a wall-cooled supersonic turbulent boundary layer close to the end of the computational domain. In order to compare computed values of the skin-friction coefficient for different Mach numbers and wall-temperature boundary conditions, it is common
(Pirozzoli et al., 2004) to transform the compressible skin-friction coefficient $c_f$ at a given local Reynolds number $Re_x$ into an incompressible reference value $c_{f,i}$ at an incompressible reference Reynolds number $Re_{x,i}$ using the van Driest II transformation (White, 1991; Roy & Blottner, 2006). These transformations are as follows:

$$c_{f,i} = F_c c_f \quad \text{and} \quad Re_{x,i} = F_x Re_x,$$

where $F_c$ represents the skin-friction transformation function and $F_x$ denotes the Reynolds number transformation function. Both functions can be computed from equation (8.4) according to

$$F_c = S^2 \quad \text{and} \quad F_x = \frac{\mu}{\mu_w} F_c^{-1}.$$  

The transformed skin friction for CASE 3 is shown in figure 8.23a. For reference purposes, the friction coefficient for a Blasius boundary layer

$$c_{f,i}^B = \frac{0.664}{\sqrt{Re_{x,i}}},$$  

and a correlation by White (1991) for the estimate of the friction coefficient of an incompressible turbulent boundary layer

$$c_{f,i}^W = \frac{0.455}{\ln^2 (0.06 Re_{x,i})}$$

are also included in this figure. As seen in figure 8.23a, results for CASE 3 are approaching the theoretical curve of equation (8.11) for an incompressible turbulent boundary layer. This gives further confidence that CASE 3 transitioned to a turbulent state.

To assess whether CASE 3 indeed predicts a representative streamwise skin-friction distribution independent of grid and time-averaging influences, and typical for an oblique breakdown transition scenario at Mach 3, the skin-friction coefficients of the other simulation cases (CASE 1-2, 4-6) listed in table 8.1 are compared to CASE 3 in figure 8.23b. Clearly, all simulations, except for CASE 2, experience
Figure 8.23 Streamwise development of skin-friction coefficient for all simulations: (a) van Driest II transformed in order to compare with incompressible, turbulent skin-friction predictions. Simulation data is taken downstream of the streamwise position $x^* = 1.047m$ ($Re_{x,i} = 609479$) where the skin-friction coefficient starts to decay, (b) for the later stages of transition in order to assess the influence of different computational grid configurations and the length of the interval for time-averaging; $M=3.0$, $T_\infty=103.6K$, flat plate.

identical streamwise growth in skin friction up to the maximal value. Moreover, in the transitional regime the skin friction is also independent of the interval length used for the time-averaging (denoted by the number of forcing periods $T_{forcing}$) since here, the flow is still periodic. Hence, the plateaus at approximately $x^* \approx 0.86m$ and $x^* \approx 0.9m$ and the valley at $x^* \approx 1.0m$ seem to be a characteristic feature of oblique breakdown initiated by only two oblique instability waves. It is very important to note that these features correlate with the findings from the previous section. The first plateau at about $x^* \approx 0.86m$ is close to the streamwise position where the first breakup occurs in figure 8.16 while the second plateau marks the second breakup followed by the generation of hairpin-like vortices. The latter plateau also coincides with streamwise location where all disturbance modes that are not direct descendants of the wave–vortex triad start significantly to grow in figure 8.13. An imprint of the valley at $x^* \approx 1.0m$ for the skin-friction coefficient in figure 8.23b can also be found
Figure 8.24 Skin-friction coefficient for CASE 3 as a function of interval length for time-averaging indicated by the number of forcing periods $T_{\text{forcing}}$ at three different streamwise positions; $M=3.0$, $T_{\infty}=103.6$K, flat plate.

in the streamwise distribution of these modes in figure 8.13.

The skin-friction distributions in figure 8.23b deviate downstream of the maximum at about $x^* \simeq 1.04m$ for CASE 3 and CASE 4. Although in CASE 4, an additional two-dimensional mode ($[1,0]$) with the fundamental frequency is forced in order to broaden the disturbance spectrum, final breakdown to turbulence is not enhanced when compared to CASE 3. For CASE 4, the skin-friction coefficient in figure 8.23b is computed from time-averaged data with only two forcing periods as interval length for the averaging. Figure 8.24 demonstrates that a larger interval length is required for time-averaging in order to obtain convergence for CASE 3. For the first two streamwise positions reported in figure 8.24 convergence is achieved within 25 forcing cycles while for the last position ($x^* = 1.104m$) CASE 3 could be longer averaged.

An overview of the grid resolution in near-wall units utilized for CASE 3 and 4 and some main characteristic quantities as maximum in skin-friction coefficient are listed in table 8.2. When compared to other numerical simulations in the literature, CASE 3 and 4 have sufficient resolution close to the outflow boundary for a supersonic turbulent boundary layer. (Note that CASE 5 has an even higher domain height as CASE 3 and CASE 6 has a finer grid resolution in wall-normal and streamwise
Table 8.2 Grid resolution and domain size from CASE 3 (inner length scale taken at \( x^* = 1.087m \)) compared to other simulations in the literature; \( M=3.0, T^*_\infty = 103.6K \), flat plate.

direction than CASE 3.)

For CASE 3, wall-normal distributions of the Reynolds-averaged streamwise velocity (\( \overline{U} \)), Favre averaged streamwise velocity (\( \overline{pU}/\overline{p} \)) and Reynolds-averaged temperature (\( \overline{T} \)) are presented in figure 8.25 using outer scaling at different streamwise positions. For comparison, one profile from CASE 2 is also plotted in figures 8.25b and 8.25c at position \( x^* = 0.996m \). This profile matches the corresponding profile from CASE 3 perfectly in the outer part of the boundary layer. Close to the wall, however, differences become apparent, which is to be expected since the skin-friction coefficient does also not agree for both cases in figure 8.23b. The difference in the mean-flow profiles between CASE 2 and CASE 3 in figure 8.25 is most likely caused by the different wall-normal stretching in both simulations since the grid resolution at the wall is exactly the same for both simulations. Note that it is common in simulations of turbulent boundary layers to use strong grid stretching in wall-normal direction. For the simulations presented here, such a grid stretching is not applicable because it would be unlikely, if not impossible, to compute a correct disturbance eigenfunction in the linear regime of the transition process. Since the goal of the present investigation is to compute through the entire transition process, a more conservative approach to
Figure 8.25 Wall-normal distribution of several mean-flow quantities from CASE 2 and CASE 3 at different streamwise positions: (a) Reynolds-averaged streamwise velocity, (b) Favre-averaged streamwise velocity, (c) Reynolds-averaged temperature. $\delta_c$ denotes boundary layer thickness obtained from the Reynolds-averaged streamwise velocity $U$; $M=3.0$, $T_\infty=103.6K$, flat plate.

With increasing downstream position, all mean-flow profiles in figure 8.25 become fuller when compared to the laminar initial condition denoted by IC. For the last two downstream locations $x^* \simeq 1.051m$ and $x^* \simeq 1.104m$, the change in the profile shape is not as pronounced as for the upstream positions. These two positions are downstream of the maximum in skin friction in figure 8.23b (see also table 8.3) and therefore, the mean-flow profiles should be close to turbulent boundary layer profiles. For convenience, table 8.3 summarizes selected mean-flow properties at all downstream locations utilized for figure 8.25 and succeeding figures.

Using the van Driest transformation

$$U_c = \int_0^u \sqrt{\frac{T_w}{T}} du$$

(8.12)

for the mean-flow profiles in figure 8.25 allows for comparison with incompressible similarity profiles of turbulent boundary layers. The van Driest transformed stream-
Figure 8.26 Van Driest transformed streamwise velocity normalized by wall-shear velocity for different streamwise positions: (a) $x^* = 0.942m$, (b) $x^* = 0.996m$, (c) $x^* = 1.05m$, (d) $x^* = 1.104m$; for clarity only every fourth point is shown, except for (d) where every point is plotted to illustrate the near-wall resolution; $M=3.0$, $T_\infty=103.6K$, flat plate.
wise velocity in near-wall units is plotted in figure 8.26 for different streamwise positions. Also included is the theoretical similarity profile for incompressible turbulent boundary layers, which follows the form

\[ \frac{U_c}{\nu} = \frac{1}{\kappa} \ln \left( \frac{y^+}{\kappa} \right) + C \]  

(8.13)

For the von Kármán constant \( \kappa \), typically a value of \( \kappa = 0.41 \) is used while the constant \( C \) is about 5.2 (Roy & Blottner, 2006).

Downstream of the peak in skin friction \( c_f \), the van Driest transformed velocity approaches the theoretical curves in figures 8.26c and 8.26d. This behavior is also reported by Jiang et al. (2006), who investigated transition initiated by oblique breakdown in a flat-plate boundary layer at Mach 4.5. Their van Driest transformed mean velocity profile is compared to CASE 3 in figure 8.26d. Note that in figure 8.26 not every point in wall-normal direction is plotted except for figure 8.26d, which therefore illustrates the near wall resolution of CASE 3 (\( \Delta y^+ \approx 0.49 \)).

The first part of this section corroborated that oblique breakdown can lead to a turbulent flow by collecting two important pieces of evidence: (1) The loss of periodicity of the flow and (2) a clear decay of the skin friction in downstream direction after reaching a peak value at about \( x^* \approx 1.05 \text{m} \). So far, however, only mean-flow data for CASE 3 were discussed and compared to theoretical predictions for

<table>
<thead>
<tr>
<th>( x^* [\text{m}] )</th>
<th>upstream of ( c_f^{\text{max}} )</th>
<th>downstream of ( c_f^{\text{max}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Re_x )</td>
<td>2.041E6 2.165E6</td>
<td>2.292E6 2.378E6 2.416E6</td>
</tr>
<tr>
<td>( Re_{x,i} )</td>
<td>– –</td>
<td>6.125E5 6.383E6 6.481E6</td>
</tr>
<tr>
<td>( Re_y )</td>
<td>1458 1594</td>
<td>1714 1774 1827</td>
</tr>
<tr>
<td>( H_{12} )</td>
<td>4.704 4.606</td>
<td>4.721 4.961 4.935</td>
</tr>
<tr>
<td>( c_f )</td>
<td>2.317E−3 3.055E−3</td>
<td>2.993E−3 2.900E−3 2.898E−3</td>
</tr>
<tr>
<td>( c_f^{i} )</td>
<td>– –</td>
<td>4.878E−3 4.710E−3 4.730E−3</td>
</tr>
<tr>
<td>( \delta^* [\text{mm}] )</td>
<td>9.074 11.477</td>
<td>15.680 17.714 18.402</td>
</tr>
</tbody>
</table>

Table 8.3 Summary of mean-flow properties at five different streamwise locations for CASE 3; M=3.0, \( T_{\infty}^* \)=103.6K, flat plate.
supersonic turbulent flows and results from other numerical simulations. The focus of the following discussion is on turbulent statistics that are based on fluctuation quantities as for example the turbulent Mach number or r.m.s. values.

It is common in the literature to check the accuracy of simulations of turbulent boundary layers by analyzing one-dimensional power spectra of velocity components at given streamwise and wall-normal positions. Such results are displayed in figure 8.27 for $y^+ \approx 49$ at different streamwise locations. The spectra were computed using Fourier transforms and the following relation

$$E_{\alpha \alpha} = \overline{\mathcal{F}(\alpha) \mathcal{F}(\alpha)}, \quad (8.14)$$

where $\mathcal{F}(\alpha)$ symbolizes the Fourier transform of the velocity component $\alpha$. In figure 8.27, $\alpha$ represents either the streamwise, wall-normal or spanwise velocity, i.e. $u$, $v$ or $w$, respectively. Note that for the calculation of the power spectra in the spanwise direction (figures 8.27a, 8.27c and 8.27e) the overline in equation (8.14) indicates only a time average (Reynolds average) whereas for the computation of the power spectra in time (figures 8.27b, 8.27d and 8.27f) only a spanwise average is used.

In figure 8.27, all velocity components experience an energy decline as predicted by theory (Heisenberg, 1948) (dashed and dashed-dotted lines). For the calculation of the energy spectra in CASE 2 (figures 8.27a and 8.27b), only a limited amount of temporal data were available and, hence, the curves are not as smooth as for CASE 3 (figures 8.27c-f). Nevertheless, the energy decay in the spanwise direction close to the end of the computational domain (at $x^* = 1.051m$ in figure 8.27a) seems to be sufficient in CASE 2 whereas in figure 8.27b the lower resolution in the streamwise direction leads to a premature drop in energy for high frequencies. The higher resolution in the spanwise and streamwise directions for CASE 3 improves the power spectra (at $x^* = 1.051m$ in figures 8.27c, d and at $x^* = 1.087m$ in figures 8.27e, f). Figures 8.27c and 8.27e show a drop of the order of six decades in the spanwise direction while figures 8.27d and 8.27f show a drop of more than four decades.
Figure 8.27 One-dimensional lateral (left) and temporal (right) power spectra $E_{\alpha\alpha}$ for velocity components from CASE 2 and CASE 3 at $y^+ \simeq 49$: (a) $x^* = 1.051 m$ (CASE 2), (b) $x^* = 1.051 m$ (CASE 2), (c) $x^* = 1.051 m$ (CASE 3), (d) $x^* = 1.051 m$ (CASE 3), (e) $x^* = 1.087 m$ (CASE 3), (d) $x^* = 1.087 m$ (CASE 3); lines denote theoretical reference data from Heisenberg (1948); $M=3.0$, $T^*_{\infty}=103.6K$, flat plate.
Figure 8.28 Wall-normal distribution of turbulent Mach number $M_t$ (a) and fluctuation Mach number $M'$ (b) at different streamwise positions from CASE 3 and CASE 2; $M=3.0$, $T_\infty=103.6$K, flat plate.

One convenient measure of compressibility effects in a turbulent boundary layer is the fluctuation Mach number $M'$. A similar quantity widely used in the literature is the turbulent Mach number $M_t$. These are defined as

$$M' = \sqrt{\langle M - \overline{M} \rangle^2} \quad \text{and} \quad M_t = \frac{\sqrt{w''w'' + v''v'' + w''v''}}{\overline{u'}}.$$ (8.15)

Turbulence is only weakly affected by compressibility effects for a fluctuation Mach number smaller than about 0.3 (Morkovin’s hypothesis, see Fernholz & Finley, 1980). Note that different threshold values have been reported in the literature (Guarini et al., 2000). For the simulations discussed in this chapter, the fluctuation Mach number exceeds a value of 0.3 only slightly, as can be seen in figure 8.28b. As a consequence, Morkovin’s hypothesis is likely to hold and comparison of statistical values of the turbulence with incompressible data after proper transformation is justified.

For simulation CASE 3, the wall-normal distribution, for both the turbulent Mach number and the fluctuation Mach number, exhibits another peak close to the boundary layer edge. A similar peak is also present in the r.m.s. value of the streamwise
velocity fluctuation $u''$ in figure 8.29a. This peak might be caused by the strong coherent structures visible in the spanwise average of the wall-normal density gradient in figure 8.30. Although the turbulent Mach number for CASE 2 looks smoother in figure 8.29a, the r.m.s. value of the streamwise velocity fluctuation shows a similar peak as CASE 3 close to the boundary layer edge.

Figure 8.29 Wall-normal distribution of r.m.s. values for (a) streamwise, (b) wall-normal and (c) spanwise velocity at $x^* = 1.051m$ and $x^* = 1.087m$ for CASE 3; $M=3.0$, $T^*_\infty=103.6K$, flat plate.

For CASE 3, figure 8.29 compares the r.m.s. values of all velocity components (Favre average) to profiles published in the literature from DNS of an incompressible turbulent boundary layer (Spalart, 1988) and the previously mentioned DNS by
Guarini et al. (2000) of a compressible turbulent boundary layer at Mach 2.5. The profiles for CASE 3 and of Guarini et al. (2000) are rescaled employing Morkovin’s density scaling. Except for the peak at the boundary layer edge, the r.m.s. value for the streamwise velocity fluctuation matches the distribution from Guarini et al. (2000) and Spalart (1988) at $x^* = 1.087m$ while the r.m.s. values for the other velocity components have not yet reached the corresponding reference data. This suggests that the flow is turbulent, but still may not yet be fully developed.

Figure 8.30 Contours of spanwise averaged wall-normal density gradient $\partial \rho / \partial y$ at two different instances one fundamental forcing period apart (CASE 3): (a) $t_1$, (b) $t_2 = t_1 + T_{\text{forcing}}$; $M=3.0$, $T_{\infty}^* = 103.6K$, flat plate.

In the last figure of this section (figure 8.31), a topview ($x$-$z$ plane) of contours of instantaneous density is given for two different wall-normal positions. This figure illustrates the break-up region and the early turbulent region close to the wall (figure 8.31a) and farther away (figure 8.31b). Dark regions denote high density and brighter regions denote low density flow. Two-dimensional coherent structures seem to appear in figure 8.31b. These structures are also present in figure 8.30 and repeat with half the wave length of the initially forced oblique fundamental waves. The strong coherence might be an explanation for the overshoot of the skin friction in figure 8.21a and figure 8.23 when compared with the turbulent reference data.
8.5 Summary

In the final result chapter, the complete transition path of oblique breakdown in a supersonic flat-plate boundary layer at Mach 3 was simulated. The transition process was initiated by a discrete pair of oblique instability waves at low disturbance amplitudes with frequency $f^* = 6.36 kHz$. The downstream development of this wave pair and the concomitant process of laminar to turbulent transition was studied from the linear regime to the final breakdown to turbulence. Linear theory predicts and the DNS confirm that oblique instability waves with the frequency of interest and the spanwise wavenumber $\beta^* = 211.52 m^{-1}$ are strongly amplified throughout
the computational domain and can trigger oblique breakdown for the computational setup and realistic experimental conditions.

The early nonlinear transition regime exhibited the typical characteristics for oblique breakdown: The wavenumber spectrum filled up rapidly in the spanwise and streamwise direction and the well known pattern of nonlinear wave interactions initiated by the forced oblique wave pair was observed. Typical flow structures for the early stages of oblique breakdown were also identified. These structures were predominantly longitudinal structures with a rope-like shape in the sideview \( (x-y) \)-plane. In the later stages of transition, the tip of these structures was lifted up from the wall and broke down to small-scale structures. The breakup region extended from about \( x^* \approx 0.84m \) to \( x^* \approx 0.9m \) of the computational setup. Close to the end of this breakup region (downstream of \( x^* \approx 0.9m \)) a sudden increase in streamwise amplification of all modes that are not direct descendants of the original oblique wave pair occurred. This increase in streamwise amplification seemed to be linked to the breakup into small scales. Downstream of \( x^* \approx 0.9m \) the entire flow was rapidly contaminated by small-scale structures and the final breakdown to turbulence was initiated. In an ideal environment, where oblique breakdown is initiated by two oblique waves with exactly the same amplitude and phase, the influence of asymmetric modes on oblique breakdown is only limited. Hence, up to the early turbulent regime, oblique breakdown is mainly symmetric with respect to the spanwise direction. In the DNS using full Fourier transformation in spanwise direction, asymmetric modes were however amplified in downstream direction and may have eventually reached high amplitude values in the later turbulent region.

The DNS data provided strong evidence that a fully turbulent flow was reached. The most important results are: (1) The turbulent and fluctuating Mach numbers were sufficiently low such that Morkovin’s hypothesis holds and the comparison of properly transformed data with results obtained for incompressible turbulent boundary layers is justified. (2) The decay of the skin-friction coefficient in the streamwise
directions approached correlations and comparable data for turbulent boundary layers in the literature. (3) A loss of periodicity in the time signals for the investigated setup occurred downstream of the peak in skin friction. (4) A logarithmic region in the van Driest transformed mean streamwise velocity profile was formed. (5) The power spectra of velocity components exhibited well-known theoretical scaling laws. In conclusion, the DNS data clearly demonstrated that oblique breakdown is a viable path to sustained turbulence.
9. Conclusions

The nonlinear transition regime for supersonic boundary layers was investigated using linear stability theory (LST) and direct numerical simulations (DNS). To date, the most dominant nonlinear mechanism that eventually transitions a laminar, supersonic boundary layer to turbulence is still unknown. The knowledge of the relevant nonlinear mechanisms is however mandatory for the accurate determination of the transition onset. Previous investigations (Fasel et al., 1993; Kosinov et al., 1994b) of the nonlinear transition regime discovered two main nonlinear mechanisms, the so-called “oblique breakdown” mechanism and “asymmetric subharmonic resonance”. Several questions related to both mechanisms were still unresolved and hence, were the main focus of this thesis. Each result chapter of this dissertation focused on one single question as stated in chapter 2. Moreover, each result chapter was closed by providing a detailed summary of all important findings. Here, only the major findings are given.

Chapter 6 discussed whether oblique breakdown can be identified in the experiments by Kosinov and his co-workers who investigated experimentally asymmetric subharmonic resonance in a Mach 2 flat-plate boundary layer. This issue is of great importance since to date oblique breakdown has not been found in any experiment and has only been studied numerically. A calibration procedure was developed to obtain a similar disturbance generation in the direct numerical simulations (DNS) as was present in the experiments. By disturbing only the fundamental frequency from the experiments, it was possible to show that the nonlinear wave interactions for the fundamental frequency exhibited features of a new breakdown mechanism that could be linked to oblique breakdown. In numerical studies published in the literature, oblique breakdown has always been initiated by two oblique instability waves with equal disturbance amplitude and phase. Therefore, it is unclear how strong oblique
breakdown might be weakened when the flow is not completely symmetric. Since an experiment can never be perfectly symmetric, the results of the experiments by Kosi-nov et al. and his co-workers additionally suggest that in an asymmetric disturbance environment oblique breakdown does not significantly lose its relevance.

Chapter 7 addressed the question whether oblique breakdown or asymmetric sub-harmonic resonance is the most dominant breakdown mechanism in a supersonic boundary layer. A DNS of a broadband disturbance environment should provide answers to this question because for such a setup, the naturally strongest breakdown mechanism should prevail and transition the flow to turbulence. Thus, chapter 7 focused on the early nonlinear regime of a wave packet in a cone boundary layer at Mach 3.5. This particular Mach number and the cone geometry were chosen since new experiments of the early nonlinear regime for this setup are planned in the Mach 3.5 Quiet Wind Tunnel at NASA Langley. Therefore, the objective of this study was also to support the experimental efforts and to identify possible dominant breakdown mechanisms. The DNS in chapter 7 showed that also in a broadband disturbance environment, strong features of oblique breakdown are visible in the weakly nonlinear breakdown process. As explained using results from linear stability theory (LST) and theoretical considerations, oblique breakdown may be regarded as a limiting case of a resonance triad with one disturbance wave approaching zero frequency. The criteria used for the evaluation of the strength of a resonance triad summarized in chapter 7 are therefore also applicable to oblique breakdown. Since both oblique instability waves initiating oblique breakdown experience the strongest amplitude growth according to linear theory and are always synchronized in downstream direction, oblique breakdown is the strongest nonlinear transition mechanism.

The last result chapter (chapter 8) answered the question whether oblique breakdown can completely transition a laminar boundary layer to a turbulent state. Therefore, the entire transition path from the linear regime to the final breakdown to turbulence was simulated for a Mach 3 boundary layer using DNS. Oblique breakdown
was initiated by harmonically forcing two oblique instability waves with equal but opposite wave angle. The numerical simulations clearly demonstrated that oblique breakdown is capable of transitioning a laminar boundary layer to fully developed turbulence. Typical mean-flow properties of a turbulent, supersonic boundary layer were reached close to the end of the computational domain. In the transitional regime, the skin friction increased significantly in streamwise direction until a peak was reached. The following decay of the skin-friction coefficient approached correlations and comparable data for turbulent boundary layers in the literature. Downstream of the peak in skin friction, the flow lost its periodicity in time with respect to the initial forcing frequency. A logarithmic region in the van Driest transformed mean streamwise velocity profile was formed and the power spectra of velocity components exhibited well-known theoretical scaling laws.

As an overall summary of this dissertation it can be concluded that independent of the transition scenario (natural transition or controlled transition) oblique breakdown may be the most relevant nonlinear transition mechanism for the flow conditions and Mach number range investigated in this dissertation (2-3.5). Hence, for flat-plate or circular cone geometries at zero angle of attack, the transition onset can be determined by considering the nonlinear wave interactions of an oblique breakdown mechanism. As illustrated in chapter 8, the transition onset is located at the streamwise position where the skin friction coefficient deviates from its laminar distribution. This event occurs when the streamwise amplitude levels of the steady vortex modes \([0, \pm 2]\) surpasses the streamwise amplitude distribution of the initially forced wave modes \([1, \pm 1]\). This fact may serve as a simple criterion for the determination of the streamwise location of the transition onset.
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