DYNAMIC VOLUNTEER'S DILEMMAS, UNIQUE BID AUCTIONS, AND DISCRETE BOTTLENECK GAMES: THEORY AND EXPERIMENTS

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DEDICATION

To my lifelong mentor, Dr. Daisaku Ikeda.

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ABSTRACT

The main theme of my dissertation is the analysis of several interactive decision making situations with multiple decision makers whose interests do not fully coincide. *Non-cooperative game theory* is invoked to carry on this analysis.

The first chapter describes an experimental study of volunteer's dilemmas that evolve over time. Only a single volunteer is required for the public good to be provided. Because volunteering is costly, each prefers that some other players bear the full costs of volunteering. Reflecting on the observation that in many naturally occurring social dilemmas it is beneficial to volunteer earlier than later, I assume that the payoff to the volunteer and the (higher) payoff to each of the non-volunteers decrease monotonically over time. I derive symmetric and asymmetric subgame perfect equilibria. The experimental results provide little support to asymmetric equilibria in which only a single subject volunteers immediately. In comparison to the symmetric subgame perfect equilibrium, they show that subjects volunteer, on average, earlier than predicted.

The second chapter explores a new type of online auction, called the *unique bid auction*, that has recently emerged on the Internet and gained widespread popularity in many countries. In a sharp contrast to traditional auctions, the winner in this class of auctions is the bidder who submits the lowest (highest) *unique* bid; all ties are discarded. I propose an algorithm to numerically compute the symmetric mixed-strategy Nash equilibrium solution and then conduct a series of experiments to assess the predictive power of the equilibrium solution. The experimental results show that the solution

accounts quite well for the subjects' bidding behavior on the aggregate level, but not on the individual level.

The last chapter proposes a discrete version of William Vickrey's model of traffic congestion on a single road with a single bottleneck. In my model, both the strategy space and number of commuters are finite. An algorithm similar to the one used in the second chapter is proposed to numerically calculate the symmetric mixed-strategy Nash equilibrium. The discrete model is then compared with the original continuous model of Vickrey in terms of the equilibrium solution and its implications.

INTRODUCTION

> *"... although I believe that in the history of science it is always the theory and not the experiment, always the idea and not the observation, which opens up the way to new knowledge, I also believe that it is always the experiment which saves us from following a track that leads nowhere: which helps us out of the rut, and which challenges us to find a new way."*
>
> *Karl R. Popper, the Logic of Scientific Discovery, 1959*

In reflecting on a widespread use of game theory, an important question is how well the theory accounts for the actual behavior of decision makers in disparate interactive situations. Game theoretic models require very sophisticated and restrictive rationality assumptions: each agent forms beliefs (or expectations) of what the others might do (i.e., strategic thinking), and given these beliefs each agent chooses an optimal decision from her possible alternatives (i.e., optimization). It is now widely accepted that not every agent behaves rationally in complex contexts due to limited cognitive capabilities; either or both of these assumptions may be violated. The question is of great importance since the applications of game theory are clearly based on the implicit assumption that the theory has predictive value. There is little interest in a game theory, however elegant it may be, that does not have positive predictive power (Davis and Holt, 1993).

My attempt in this dissertation is to answer this question with the help of experimental methods. My motivation for this approach is by no means to refute the power of game theory. Rather, I wish to improve it by identifying systematic behavioral regularities (e.g., deviations from equilibrium), if any, from which we may glean for future development of an alternative theory that better accounts for the actual behavior of decision makers. The interplay between game theory and experiments does not only serve to throw light on the predictive power of equilibrium solutions, but also to establish behavioral regularities, which, in turn, will be distilled into new theory.

The theories that this dissertation subjects to experimental testing come from *non-cooperative* game theory (Nash, 1951). My dissertation consists of three independent topics, each of which falls within this framework: Dynamic Volunteer's Dilemmas (Chapter 1), Unique Bid Auctions (Chapter 2), and Discrete Bottleneck Game (Chapter 3). Common to all the three topics is the construction of equilibrium solutions. The first two chapters also include data collected in laboratory experiments to assess the predictive power of these equilibrium solutions.

Chapter 1 focuses on a model called the *dynamic volunteers' dilemma game*, which is formulated as a non-cooperative *n*-person game in extensive form with symmetric players, discrete time, finite horizon, and complete information. The model is motivated by many social dilemma situations where (i) people have to decide independently whether to volunteer for a costly action that requires only a single individual to accomplish it, (ii) the decision takes place in real time so potential volunteers can observe each other's behavior and wait for one of the others to act, and (iii) delay is costly both for volunteers and non-volunteers.

Players in the dynamic volunteer's dilemma game face the problem of equilibrium selection; the game possesses *asymmetric* subgame perfect equilibria in which a single player volunteers immediately, and a symmetric subgame perfect equilibrium (in behavioral strategies) in which each of *n* identical players fully randomizes over her two actions, volunteer or not, at each time period. In each of the asymmetric equilibria, the volunteer receives a lower payoff than each of the *n*-1 non-volunteers. Although these asymmetric equilibria are socially optimal, this payoff asymmetry may hamper a tacit coordination on any of them. In the absence of pre-play communication or sufficient opportunities of learning, players may fail to find a volunteer and thereby fail to coordinate their actions on a specific asymmetric equilibrium. In the symmetric environment, the symmetric equilibrium, in which each of the players receives the same expected payoff, is more appealing as a predictor.

Experimental results consist of several findings. First, as implied by the symmetric equilibrium, a lower cost of volunteering elicits earlier termination of the game. Second, subjects largely fail to achieve one of the asymmetric equilibria but on average they receive a better outcome than they would have ended up with under the symmetric equilibrium. Third, subjects revealed heterogeneous propensity to volunteer. For about 50 percent of the subjects, the hypothesis about the expected frequency of volunteering decisions cannot be rejected. The other 50 percent subjects are divided between "free riders," who volunteer significantly fewer times than predicted, and "hard core" cooperators, who volunteer significantly more often than predicted.

Chapter 2 explores a new type of on-line auction, called the *unique bid auction* (UBA), that recently has emerged on the Internet and gained widespread popularity in many countries. A new and major feature of this type of auction that sharply differentiates it from any other auctions is the *uniqueness* of the winning bid. In the *lowest* unique bid auction (LUBA), the winner of a prize is the one who submits the lowest unique bid. On the other hand, in the *highest* unique bid auction (HUBA), the winner is the one who submits the highest unique bid. Therefore, all ties are discarded, and with a positive probability the auction may terminate with no winner.

The introduction of this new feature renders the derivation of equilibrium solutions in closed-form extremely challenging. I propose an algorithmic procedure that uses non-stationary Markov chains to numerically compute the symmetric mixed-strategy Nash equilibrium solution. To keep the model tractable, the number of bidders is assumed commonly known and each bidder is allowed to only submit a single bid.

Results from laboratory experiments indicate that the equilibrium solutions do not account well for the bidding behavior on the individual level in both the unique lowest and unique highest bid auctions. This is partly due to the inclination of some subjects to choose the same bid repeatedly over iterations of the auction. On the other hand, the behavior on both the aggregate and group levels is accounted for quite well by the equilibrium solution in the lowest unique bid auction but not in the highest unique bid auction. However, with experience, the bidding behavior on the aggregate and group levels move in the direction of equilibrium in the last 30 rounds of the highest unique bid auction.

In Chapter 3, I propose a discrete version of William Vickrey's (1969) bottleneck model, which analyzes urban traffic congestion on a single road caused by a single bottleneck with a finite capacity. My motivation for discretizing Vickrey's continuous model is twofold. First, many researchers have followed a common practice in transportation science, economics, and other disciplines that uses continuous models (i.e., a continuum of players and continuous strategy space) to analyze phenomena that are essentially discrete. I argue that in some cases the predictions derived from the continuous model may not provide good approximations to phenomena that are discrete in nature. This is the case when congestion involves only a few decision makers such as ships seeking to use canal locks, and passengers queueing at airport security gates. I wish to compare the two formulations in order to determine how good the approximations are to the associated discrete case. Second, I wish to develop a model to account for the experimental implementation of the bottleneck game in the laboratory. Whether in an experiment or even naturally occurring settings, in order to implement a mechanism or test a theory one has to use a discrete strategy space and a finite number of players. Moreover, as the number of participants in experiments and in some traffic congestion applications is typically small, the approximations provided by the continuous model may fail.

A variant of the algorithmic procedure that is used in Chapter 2 is proposed to numerically compute the symmetric mixed-strategy Nash equilibrium solution to the discrete bottleneck game with a finite number of players and discrete strategy space. The discrete bottleneck game is formulated as a game of complete information in which the number of players, strategy space, and cost structure are commonly known.

Extensive numerical computations indicate the presence of systematic discrepancies when the continuous model is used to account for departure times in traffic networks in which the number of players is relatively small. The computations further show that as the population size grows the difference between the continuous and discrete models of traffic congestion caused by a bottleneck can safely be ignored.

To bridge the gap between theory and practice in traffic networks, two extensions of the discrete model are proposed. The first extension concerns the case where the number of commuters is a random variable, rather than a constant, whose distribution is commonly known. The second extension deals with the case in which an alternative transportation mode not subject to congestion is available.

Each chapter is self-contained. Therefore, no general conclusion is provided at the end of the dissertation. Instead, conclusions and future research associated with each topic are discussed at the end of each chapter.

CHAPTER 1: DYNAMIC VOLUNTEER'S DILEMMAS

1.1 Introduction

There is by now a voluminous body of research in public economics on volunteer's dilemmas in which one or more group members are required to make costly contributions in order to provide benefit to all the group members.[1] Closely associated with research on the war of attrition, which has been studied in economics and biology, this research has recently been extended from static to dynamic volunteer's dilemmas that evolve over time.[2] My interest here is in the class of *n*-person dynamic volunteer's dilemmas where only a single contributor, called *volunteer*, is required for the public good to be provided, and the value of the public good is commonly known to diminish over time. Participants in these social dilemmas have to decide whether to volunteer and if so, at what time to do so. Variants of the dynamic volunteers' dilemma game have been studied theoretically by several researchers (e.g., Bilodeau and Slivinski, 1996; Bliss and Nalebuff, 1984; Hendricks et al., 1988; Shapira and Eshel, 2000; Weesie, 1993, 1994). The models that have been proposed differ from one another depending on whether the game is formulated in a strategic or extensive form, time is continuous or discrete, the time horizon is finite or infinite, the information provided to the group members is

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[1] For the original volunteer's dilemma game and its extension, see Diekmann (1985, 1993).

[2] The problem of finding someone who incurs the full cost of providing public goods in the context of real-time decision making bears a resemblance to war of attrition. The war of attrition was intensively studied in biology, where Maynard Smith (1974) originally posed it as a conflict between two animals fighting over pray. Later on, it has widely been applied to economic analyses of, for example, firm exit from a duopoly market (Fudenberg and Tirole, 1986), an oligopoly market (Bulow and Klemperer, 1999), and many others.

complete or incomplete, and players are either symmetric or asymmetric. The equilibrium solutions to these models differ correspondingly.

Examples that have motivated these models include public services such as getting up at night to quiet a crying baby, cleaning public toilets (Bilodeau and Slivinski, 1996), and chairing a university department. Quoting Dawkins (1976), Shapira and Eshel give an example of masses of emperor penguins standing on the brink of the water and hesitating before jumping in because of the danger of falling prey to seals. At least one of them has to dive in for the rest to know whether there is a seal lurking in the water. Yet another example from biology can be found in the behavior of a group of foraging animals (e.g., marmots) in which one of them occasionally looks up, checks for a predator, and issues an alarm thereby increasing the risk of attracting the predator. The most infamous and disturbing example (see, e.g., Weesie, 1993, 1994) is the 1964 Kitty Genovese murder case in which the victim was sexually assaulted and stabbed to death in the courtyard of her apartment complex in the city of New York. Despite the fact that 38 people were watching the brutal attack from behind their windows, no spectator volunteered to help. In fact, the police was not called until the attack was over. Common to all of these examples is that the utility each group member derives from the provision of the collective good—stopping a baby from crying at night, issuing an alarm, or calling the police—diminishes over time.

In contrast to a large body of theoretical studies on dynamic volunteer's dilemmas and war of attrition games, little experimental work has been conducted to answer the question how people actually behave in such situations. One exception is Bilodeau et al.

(2004) in which they tested the predictive power of a unique subgame perfect equilibrium to their asymmetric dynamic volunteer's dilemma game.[3] I focus on a class of dynamic volunteer's dilemma games, which are formulated in an extensive form, with n symmetric players, discrete time, finite horizon, and complete information in which the payoffs to all the group members decrease over time. I first construct symmetric and asymmetric subgame perfect equilibria for the game and then test them experimentally in an attempt to observe and identify systematic behavioral regularities, if any, of financially motivated subjects in a controlled laboratory environment.

The experimental results show several major behavioral patterns. First, as the cost of volunteering is exogenously decreased, at least one volunteer emerges earlier. Second, the results show no coordination on one of the multiple socially optimal asymmetric equilibria in which a single subject volunteers at time period t=0. Only a minority of the games played in the experiment support the asymmetric equilibria. Third, the data also show systematic deviations from the symmetric subgame perfect equilibrium (SSPE) in which each player should fully randomize her actions at each point in time. Subjects volunteered, on average, significantly earlier than predicted. These deviations are attributed to the heterogeneity of the subjects in terms of their inclination to free ride. Substantial and consistent differences between subjects are observed with some subjects exhibiting a strong inclination to volunteer with little or no regard to the free riding of their cohort members, some subjects who *never* volunteer across all the 50 iterations of the base game, and yet other subjects who adhere to equilibrium play.

[3] See also Oprea et al. (2008) and Phillips and Mason (1997).

The rest of this chapter is organized as follows. In Section 1.2, I introduce notation, formally describe the game, and then construct the symmetric and asymmetric equilibria. Section 1.3 lists the research questions that I attempt to answer and describes the design of my experiment. Section 1.4 states the results, and Section 1.5 concludes this chapter with a brief discussion.

1.2 Dynamic Volunteer's Dilemma Game

1.2.1 Model

The dynamic volunteer's dilemma game examined in this chapter is a stylized model that depicts a situation in which each player attempts to free ride on the efforts of some other member of her group in a dynamic environment. There is a group of n symmetric, risk-neutral players. Time is discrete and finite: there are T+1 time periods, 0, 1, …, T. At each time period, the players are asked to decide independently and anonymously whether to contribute to the provision of a public benefit. The game starts at period t=0 and terminates either when at least one of the n players contributes at time period $t \leq T$ or when time period T ends with no contributor, whichever occurs first. The first player who decides to unilaterally contribute to the public good is called a *volunteer*. Each player may volunteer at most once; players may also opt not to volunteer at all.

Denote by H_t and L_t the respective payoffs of the non-volunteer and volunteer at time period $t \in \{0,1,...,T\}$. The two payoff functions satisfy the following assumptions:

1. $H_t > L_t$ for all $t \in \{0,1,...,T\}$.

2. Both H_t and L_t are strictly decreasing in t.

3. If time period T ends with no volunteer, then each of the n players receives a fixed payoff ε ($\varepsilon \geq 0$), which is strictly smaller than L_T.

The first assumption implies that for any stage of the game, volunteering is costly. Hence, each player prefers to have someone else volunteer at any time period. The second assumption reflects the observation that delays in volunteering are costly to *all* the n players, and that the delay costs increase in time. The third assumption is necessary because the game has a finite horizon. A finite horizon implies that the game may terminate with no volunteer. Therefore, when the game ends with no volunteer payoffs to all the n players must be specified.

The assumption of discrete time requires defining the volunteer's and non-volunteer's payoff functions for the case of multiple volunteers at the same time period. Under certain circumstances, it is reasonable to assume that the costs of volunteering are equally shared among the multiple volunteers (Weesie and Franzen, 1998). For example, if several penguins simultaneously dive in to check where there is a seal, then the chances that a given penguin is caught by seals decrease with the number volunteering penguins diving into the ocean. Therefore, emperor penguins could share the cost of volunteering. An alternative assumption was employed:

4. If multiple players volunteer simultaneously at period t, then each of them receives L_t and all the others receive H_t.

This assumption is reasonable in cases where no matter how many players volunteer, each of them has to incur the full cost of volunteering. For example, if several bystanders jump into a river simultaneously to rescue a drowning child, all of them risk their lives as much as when only one of them jumps in. If several witnesses simultaneously call the police in case of a crime, then all of them have to be called in to testify. Hence, in such situations, each of the multiple volunteers receives L_t.[4]

1.2.2 Equilibrium Analysis

This game has *n* asymmetric subgame perfect equilibria in which a single player volunteers at time period t=0.[5] Each of these equilibria is socially optimal because it maximizes the group's welfare. Such equilibria yield *asymmetric* payoffs; the volunteer receives a lower payoff of L_0 and each of the *n*-1 non-volunteers a higher payoff of H_0. There also exists a unique symmetric subgame perfect equilibrium (SSPE), in which each player fully randomizes over her actions (i.e., volunteer or do not volunteer) at each time period. In a sharp contrast to the first type of the equilibria, the SSPE yields the same equilibrium payoff to all the players.

To derive the SSPE, backward induction is invoked. Consider the subgame starting at time period t=T. Table 1.1 illustrates the payoff matrix of this subgame. This subgame has a unique symmetric Nash equilibrium in mixed strategies. Let σ_T denote the probability that each player volunteers at time period T. This probability is determined in such a way that this player is indifferent in terms of her expected payoff between volunteering and not volunteering. In other words,

[4] This assumption is not necessary to derive an equilibrium solution. However, it allows me to derive a closed-form solution.

[5] More precisely, at any time period in such equilibria, one of the *n* players volunteers whereas each of the other *n*-1 players does not.

$$L_T = \varepsilon(1-\sigma_T)^{n-1} + H_T[1-(1-\sigma_T)^{n-1}].$$[6]

Then, the equilibrium probability of volunteering at time period t=T is

$$\sigma_T = 1 - \left(\frac{H_T - L_T}{H_T - \varepsilon}\right)^{\frac{1}{n-1}}.$$

Each player's equilibrium payoff in this subgame is L_T.

The same procedure is now repeated backward until the initial time period t=0. Let σ_t denote the probability that each player volunteers at time period t. Then, the equilibrium probability of volunteering at time period t is

$$\sigma_t = 1 - \left(\frac{H_t - L_t}{H_t - L_{t+1}}\right)^{\frac{1}{n-1}}.$$

Each player's equilibrium payoff in this subgame is L_t. Then, the (behavioral) strategy profile $\sigma = \{\sigma_i\}_{i=1}^N$, where $\sigma_i = (\sigma_1,...,\sigma_T)$ for each player i, constitutes the unique symmetric subgame perfect equilibrium. Note that I have just derived a Nash equilibrium in behavioral strategies that specify the probability of volunteering at each time period, conditional on reaching it. It is not a strategic-form mixed strategy over the entire strategy space. It has long been established (Kuhn, 1953) that under perfect recall—a condition that I assume in the present study—they are equivalent (see, e.g., Fudenberg and Tirole 1991).

To compute an equilibrium payoff induced by the SSPE, the probability distribution of termination time of the game has to be computed. Let P_t denote the probability of termination at time period t (t=0, 1, ..., T). Given the equilibrium strategy

[6] This player can assure herself a payoff of L_T by volunteering.

profile σ, the probability of termination at time period *t*, provided that the game did not terminate before *t*, is computed from

$$1-(1-\sigma_t)^n = 1-\left(\frac{H_t - L_t}{H_t - L_{t+1}}\right)^{\frac{n}{n-1}}.$$

Notice that the probability of termination before *t* is $\prod_{\tau=0}^{t-1}(1-\sigma_\tau)^n$. Then, P_t is computed from

$$P_t = [1-(1-\sigma_t)^n]\prod_{\tau=0}^{t-1}(1-\sigma_\tau)^n \text{ for } t \in \{1,2...,T\}$$

with $P_0 = 1-(1-\sigma_0)^n$. Denote by P_{NV} the probability of termination with no volunteer. Then,

$$P_{NV} = \prod_{\tau=0}^{T}(1-\sigma_\tau)^n .$$

The equilibrium payoff induced by the SSPE is computed from

$$E = \sum_{t=0}^{T} P_t \cdot L_t + P_{NV} \cdot \varepsilon .$$

The SSPE σ yields the same equilibrium payoff of *E* to each of the *n* players. Note that the SSPE is a Pareto deficient equilibrium.

1.3 Research Questions and Experimental Design

1.3.1 Research Questions

The payoff functions used in the experiment were

$$H_t = 20(e^{-0.1t} - e^{-6}) + \varepsilon$$
$$L_t = 20\delta(e^{-0.1t} - e^{-6}) + \varepsilon$$

for t=0, 1, …, T, where 0<δ<1. Let n=3, T=30, and ε=1.00. Consider two experimental conditions where δ=0.3 and δ=0.6, respectively, and their associated payoffs are presented in Table 1.2. It is easily verified that these payoff functions satisfy all of the four assumptions mentioned before. The upper panel of Figure 1.1 exhibits the equilibrium probability distributions of termination time ($P_0, P_1, ..., P_{30}, P_{NV}$) for the two conditions, which are derived under the SSPE play ($\sigma = \{\sigma_i\}_{i=1}^{3}$, where $\sigma_i = (\sigma_0, \sigma_1, ..., \sigma_{30})$). The lower panel of Figure 1.1 displays their cumulative distributions for Conditions δ=0.3 and δ=0.6.

The following research questions are formulated:

Q.1 *Does δ=0.6 induce earlier volunteering than δ=0.3?*

Given the payoff functions above, δ=0.3 (δ=0.6) implies a relatively high (low) cost of volunteering at each point in time. Subjects may be inclined to wait longer and abstain from volunteering if δ=0.3 than if δ=0.6. Figure 1.1 shows that under the SSPE play, a volunteer is likely to emerge considerably earlier when δ=0.6 than when δ=0.3.

Q.2 *Can subjects coordinate their actions and achieve one of the socially optimal subgame perfect equilibria in which a single subject volunteers at time period t*=0*?*

One may anticipate failure to coordinate the group's actions on any of such asymmetric equilibria in which the volunteer receives a lower payoff than each of the n-1 non-volunteers. The reason is that this payoff asymmetry may render these equilibria difficult to realize because players have no clue of which equilibrium should be chosen. In the absence of pre-play communication or sufficient opportunities of learning, players may not succeed to coordinate their actions on one of such equilibria. Evidence supporting this prediction has already been documented by Bilodeau et al. (2004). They incorporated into their model heterogeneous characteristics of agents such as different costs, benefits, and lengths of life. These allow agents to single out a focal player who should volunteer at time period t=0. Their experimental results showed that even with heterogeneous players this unique socially optimal subgame perfect equilibrium was realized in only 133 out of 472 cases (28.2 percent). This suggests that in a symmetric environment, in which homogenous subjects face a coordination problem of determining *who* should volunteer at time period t=0, it would be even more difficult to achieve one of the asymmetric equilibria.

As shown in the previous section, the dynamic volunteer's dilemma game possesses a unique symmetric subgame perfect equilibrium in which each player fully randomizes over her actions at each time period. Therefore, I also want to answer the following question:

Q.3 *Is the behavior of subjects consistent with symmetric subgame perfect equilibrium play?*

1.3.2 Experimental Design

Subjects

The subjects were 90 University of Arizona undergraduate and graduate students, who volunteered to participate in a group decision making experiment for payoff contingent on performance. Subjects interacted with one another in cohorts of 9, five cohorts (sessions) in Condition δ=0.3 and five other cohorts in Condition δ=0.6. A between-group design was used in both conditions. Each session lasted about 90 minutes. Excluding a $5 show-up bonus, the mean individual payoff for Conditions δ=0.3 and δ=0.6 was $23.76 and $14.21, respectively.

Procedure

All the ten sessions were conducted in the same manner. The nine members of each cohort were randomly seated in a large computer laboratory and handed written instructions. The subjects were separated from one another by partitions that prevented any form of communication between them. In each session, the subjects participated in 50 identical iterations (called "*rounds*") of the dynamic volunteer's dilemma game described in Section 1.2. In each round, the 9 members of each cohort were randomly and anonymously assigned to 3 groups of 3 members each. They were instructed that the group composition would change randomly from round to round. Consequently, reputation building was not possible.

There were 31 *periods* in each round, including period 0. Each period lasted 1.5 seconds. Watching the game unfolding and the clock advancing, the subject's task on each period was to decide whether to stop the clock (= volunteering).[7] Stopping the clock was accomplished by simply moving the cursor outside of a red circle on the screen.[8] Once the first player in a group stopped the clock, the game for this group (but not necessarily for the other two groups in the same cohort) was over. If none of the group

[7] To prevent any social implications, the game was framed in neutral terms with "stopping the clock" substituted for "volunteering". The subject who was the first to stop the clock was designated as the Stopper, and the others will be designated as the Non-stoppers.

[8] This procedure was used to avoid the noise of clicking that might have conveyed information to members of the other two groups.

members stopped the clock, then the game lasted a total of 46.5 seconds (31×1.5). At the beginning of each round, the clocks of the three groups in each session were synchronized. All the 9 members of a cohort waited until the final period elapsed. The subjects were explicitly instructed that stopping the clock was not mandatory.

Payoffs to the stopper, if any, and to the non-stoppers were determined by a table in the instructions that the subjects could consult before and during the experiment. The payoffs could also be read directly from a screen that depicted the two separate payoff functions and, in addition, updated the payoffs on each period and listed them numerically on the screen. Payoffs were stated in experimental currency called "points". At the end of the session, points were converted into US dollars at the rate 20 points=$1.00 in Condition δ=0.3 and 50 points=$1.00 in Condition δ=0.6.

Once the round was completed, the outcome was presented on the individual screens. This screen informed the subject whether or not she stopped the clock on that round, recorded the period on which the game terminated, and presented each subject with her payoff for the round. Information about the decisions and payoffs of the two other groups in the same round was not disclosed.

The experiment started with practice rounds intended to familiarize the subjects with how to use the "mouse" to submit their decisions and get acquainted with the operation of the clock. Subjects were not paid for these practice rounds that they could repeat as many times as they wished. Once all the members of a cohort completed the practice rounds (typically 2-4 rounds) by each independently pressing the button "I'm ready to start playing" the session started.

1.4 Results

1.4.1 Four Major Findings

Finding 1: Effects of Costs of Volunteering

Figure 1.2 exhibits the observed cumulative relative frequency distributions of termination time for Conditions δ=0.3 and δ=0.6. Each of the two distributions is based on 750 observations (3×50×5 group by round by session). Comparison of these distributions (see statistical evidence below) shows that, as predicted by the SSPE (bottom panel of Figure 1.1), subjects in Condition δ=0.6 stopped the clock, on average, earlier than subjects in Condition δ=0.3.

The mean termination times across the five sessions in Conditions δ=0.3 and the five sessions in Condition δ=0.6 were 8.79 and 3.51, respectively.[9] The mean termination times in the five sessions in Condition δ=0.6 were 3.11, 5.11, 3.25, 2.51, and 3.57. All of these five means are smaller than 5.2. In contrast, the five mean stopping times in Condition δ=0.3 were 9.56, 9.67, 8.09, 7.94, and 8.69. All of them exceed 7.9. Taking the session as the unit of analysis, the null hypothesis that the two conditions are identical in terms of their mean termination times was rejected (p<0.01, using the Mann-Whitney U test). The results show that, on average, subjects stopped the clock in all sessions in Condition δ=0.6 earlier than in all the five sessions in Condition δ=0.3.

Finding 2: Coordination Failure on a Socially Optimal Asymmetric Equilibrium

[9] The case in which a round ends with no stopper was considered in the analysis by assigning the value of 31 to such a case.

Table 1.3 lists the observed frequency distributions of termination time for periods 0, 1, 2-4, … , 8-10, …, 17-30, and for the no stopping decision (i.e., no stopping or "NS"). The results are presented for each session separately and across sessions. Table 1.3 shows that the clock was stopped at time period t=0 113 times in Condition δ=0.3 and 233 times in Condition δ=0.6 (out of 750 observations in each condition).[10] These frequencies include the cases in which multiple subjects stopped the clock at t=0, which are not socially optimal. After removing these cases, the results show that subjects successfully achieved one of the socially optimal asymmetric equilibria 111 times in Condition δ=0.3 and 204 times in Condition δ=0.6. Compared with 14.8 percent of all the rounds on Condition δ=0.3 terminating with exactly one of the group members volunteering, the corresponding percent in Condition δ=0.6 almost doubled to a value of 27.2. This latter value is practically identical to the one reported by Bilodeau et al. (2004).

It could be argued that since the clock used in the experiment moved rather quickly, taking about 1.5 seconds per period, players wishing to stop the clock immediately might have been late in responding due to slow reaction time. Therefore, "stopping the clock immediately" is defined to mean stopping the clock at period t=0 or t=1. Under this definition, out of 150 decisions (50×3) in each session, there were 36, 20, 32, 38, and 51 games in Sessions 1 to 5, respectively, in Condition δ=0.3 in which only one subject stopped immediately. The corresponding frequencies in Condition δ=0.6 were 79, 45, 53, 75, and 61. Therefore, one of the socially optimal asymmetric subgame

[10] In each condition a minority of the subjects stopped the clock at t=0 far more frequently. The maximum number of stopping the clock at t=0 by a single subject was 29 times (out of 50 rounds) in Condition δ=0.3 and 40 times in Condition δ=0.6. Eighty percent of the subjects stopped the clock at t=0 no more than four times in Condition δ=0.3 and no more than nine times in Condition δ=0.6.

perfect equilibria was *almost* achieved 177 times (23.6 percent) in Condition δ=0.3 and 313 times (41.7 percent) in Condition δ=0.6.

In spite of realizing the benefit of volunteering early, subjects may start the session by stopping the clock too late. Then, as they gain more experience with the task and learn more about the behavior and frequency of free riders, they may progressively advance their stopping times and eventually achieve one of the socially optimal subgame perfect equilibria. Murphy, Rapoport and Parco (2006a) reported such patterns of behavior in trust dilemmas that also evolve over real time.[11] No evidence is found in support of this hypothesis. Figure 1.3 displays the mean termination time computed across the three groups, m_t, for rounds 1 through 50 (t=1, …, 50). The results are exhibited in ten panels by condition and by session within condition.

To investigate the (linear) association between the values of m_t with the round number t, I computed the Spearman rank-order correlation coefficient, r, for each session in each condition (see linear regression lines in Figure 1.3). The coefficient values ranged between -0.28 (Session 4) and 0.20 (Session 2) in Condition δ=0.3 and between -0.07 (Session 3) and 0.28 (Session 5) in Condition δ=0.6. In both conditions, only the correlation coefficients of two cohorts (Session 4 in Condition δ=0.3 and Session 5 in Condition δ=0.6) are statistically different from zero (p<0.05), positive in one case and negative in the other. There is no evidence for convergence to the socially optimal subgame perfect equilibria over the 50 rounds of play.

[11] In a sharp contrast to the present dynamic volunteer's dilemma game, however, their game possesses the unique subgame perfect equilibrium in which all players should stop the clock at t=0.

Finding 3: Earlier Volunteering than Predicted by the SSPE

The dynamic volunteer's dilemma game examined in the present experiment possesses (i) asymmetric subgame perfect equilibria in which a single player stops the clock immediately and (ii) a unique symmetric subgame perfect equilibrium (SSPE) in which each player randomizes over two actions, stop the clock or not, at each time period. The latter equilibrium prescribes a positive probability to stopping the clock immediately at *t*=0, i.e., $\sigma_0 > 0$, which implies that the game terminates at period *t*=0 with a positive probability, i.e., $P_0 > 0$ (see the upper panel of Figure 1.1). Therefore, all the observed games that ended immediately at period *t*=0 support *both* the asymmetric and symmetric equilibria.

To test the symmetric equilibrium, I employ the definition of "stopping the clock immediately" that was introduced earlier, which means stopping the clock at either *t*=0 or *t*=1. The probability distribution of termination time is normalized by subtracting the equilibrium probabilities of termination at periods *t*=0 and *t*=1 (i.e., P_0 and P_1). Then, the new equilibrium probability of termination at period *t* (*t*=2, 3, …, *T*) is $\tilde{P}_t = P_t /(1-P_0-P_1)$ and the new equilibrium probability of no volunteer is $\tilde{P}_{NV} = P_{NV} /(1-P_0-P_1)$. Figure 1.4 exhibits the new predicted and observed relative frequency distributions of termination time ($t \geq 2$) by session and across all sessions (upper left corner) for Condition δ=0.3. Figure 1.5 exhibits the corresponding distribution for Condition δ=0.6. Inspection of these figures shows similar group patterns for all five sessions in each condition. In every case, subjects stopped the clock, on average, earlier than predicted. The Kolmogorov-Smirnov (K-S) one-sample test—a test of goodness of fit—was invoked to test the null hypothesis that the observed termination time distributions at or after t=2 are drawn from a population having the (normalized) equilibrium distribution. The null hypothesis was rejected for each of the five sessions in Condition δ=0.3 and for Sessions 3 and 4 in Condition δ=0.6.

Stopping the clock earlier than predicted had significant monetary implications for the subjects. As illustrated in Table 1.2, group payoff is maximized if one of the three members of the group stops the clock immediately as the clock starts ticking. For each condition separately, I tested the null hypothesis that the mean payoff is equal to the expected payoff under equilibrium play. Using the single-sample t-test, the hypothesis was clearly rejected (t=15.36 and t=4.83 for Conditions δ=0.3 and δ=0.6, respectively, p<0.01 in each case). The expected payoff per session under equilibrium play is \$8.39 for Condition δ=0.3 and \$9.37 for Condition δ=0.6. On average, the subjects in Condition δ=0.3 earned 2.83 times *more* than expected, and the subjects in Condition δ=0.6 earned 1.52 times *more* than expected. Deviations from equilibrium behavior in the direction of early exit paid off handsomely.

Finding 4: Individual Differences in Volunteering

Recall that the experiment was designed in such a way that the computer software recorded only stopping decisions of the volunteers, at most one in each group and three in each round.[12] Therefore, non-volunteers were not given the opportunity to record their stopping decisions because each round terminated on the volunteer's move. This implies

[12] If none of group members stops the clock in a given round, then "No stopper" is recorded as the decision of the group in this round.

that even a few subjects who commit themselves to volunteering at early time periods might have caused deviations from the SSPE behavior. To determine whether deviations from the equilibrium solutions are due to a minority of "hard core" volunteers, I computed the expected number of stopping the clock in a session. Each subject either stops the clock or not on each of 50 adjacent rounds, independently of the previous outcome. Therefore, the total number of stopping decisions is binomial. The probability that a player becomes a volunteer in a round is computed from

$$\sigma_0 + \sigma_1(1-\sigma_0)^3 + \sigma_2\prod_{\tau<2}(1-\sigma_\tau)^3 ... + \sigma_{30}\prod_{\tau<30}(1-\sigma_\tau)^3 .$$

The probability for Condition δ=0.3 (δ=0.6) is 0.31 (0.36). Then, for each condition, I computed the probability distribution of the number of stopping decisions under the SSPE play. These distributions are approximately normal with expected number of 15.58 stopping decisions in Condition δ=0.3 and 17.80 in Condition δ=0.6. The corresponding standard deviations are 3.28 and 3.39. Then, I computed the central 99 percent intervals around the expected number of stopping decisions: [7, 25] and [9, 27] for Conditions δ=0.3 and δ=0.6, respectively. These are displayed in Figure 1.6.

Figure 1.6 illustrates a major finding concerning individual differences in the decision to be the first to volunteer. Approximately one half of the subjects (24 of 45 in each of the two conditions) behaved in agreement with the SSPE. Of the remaining 21 subjects in Condition δ=0.3, 12 stopped the clock fewer times than expected, and 9 stopped it more times than expected.[13] The results for Condition δ=0.6 are exactly the

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[13] In Condition δ=0.3, five subjects never volunteered, and one was the first to volunteer on 49 of the 50 rounds.

same: 24 of the 45 subjects behaved in agreement with the SSPE, 12 stopped the clock significantly less frequently than expected, and 9 stopped the clock significantly more often than expected. In each condition, the observed distribution of number of stopping decisions has a considerably larger variance than predicted, testifying to the considerable heterogeneity of the subjects with respect to the critical decision in the present study, namely, if and at what time to volunteer. The results also show that, in each condition, the subjects who volunteered fewer times than expected stopped the clock, on average, later than those subjects who volunteered more frequently than predicted.[14]

1.4.2 Other Findings

The decision whether to be the first to volunteer is most likely shaped by moral considerations and social conventions. If, in fact, this is the case, then under random assignment of subjects to groups in each of the experimental conditions the observed relative frequency distributions of individual number of stopping decisions should be the same for both conditions. In agreement with this prediction, the two-sample K-S test could not reject this null hypothesis (D=0.133, p>0.1).[15] Figure 1.7 exhibits the two observed cumulative relative frequency distributions of the individual number of stopping decisions, which are seen to track each other rather closely.

[14] Five of the 12 subjects in Condition δ=0.3 who volunteered fewer times than predicted never volunteered. The mean stopping time for the other 7 subjects in Condition δ=0.3 is 19.12, and the mean for the 9 subjects in this condition who volunteered more often than predicted is 5.82. The difference between these means is significant by the Mann-Whitney test (p<0.05). The corresponding means for Condition δ=0.6 are 9.34 and 2.14. The difference between these two means is also significant by the Mann-Whitney U test (p<0.05).

[15] D is the K-S test statistic.

Does it pay to be the first to volunteer? In both conditions of the experiment, the answer is positive if the volunteer's dilemma game is played only once. If it is played for multiple rounds, even when the group composition is determined randomly on each round, then the answer is not as clear due to the possibility of strategic play by some of the subjects. For example, a subject may decide to be the first to volunteer on early rounds of the session in order to establish a social norm of early stopping in her cohort. On later rounds, hiding behind the veil of anonymity, she may decide to free ride on the early stopping decisions of her cohort members. To answer this question, the individual payoff for the session was correlated with the individual number of exit decisions. The linear correlation values were negative and highly significant: -0.701 and -0.487 for Conditions δ=0.3 and δ=0.6, respectively (p<0.01 in each case). Figure 1.8 displays the scatter plot and the linear regression line for each condition. In Condition δ=0.3, any increase by a single decision to volunteer decreased the individual payoff by 3.9 points. The corresponding value for Condition δ=0.6 (2.6 points) is smaller, reflecting the difference in the cost of volunteering between these two conditions.

1.5 Conclusion

Chapter 1 examines a class of *n*-person dynamic volunteer's dilemma games which are formulated in an extensive form, with *n* symmetric players, discrete time, finite horizon, and complete information. In this class of dilemmas, only a single volunteer is required for the provision of a public good and the value of the good diminishes over time, i.e., the payoffs to all group members decrease over time. I have derived the symmetric and asymmetric subgame perfect equilibria for the game and tested their implications in the controlled environment of the laboratory.

The experimental results reveal several findings. First, a lower cost of volunteering (i.e., a higher value of δ) elicited earlier termination of the game. This suggests that the lower the cost of volunteering, the earlier one or more individuals emerge for the provision of a public benefit. Second, subjects struggled to achieve one of the socially optimal asymmetric equilibria but they received a better outcome than what they would have ended up with under SSPE play. Subjects largely failed to coordinate their actions on one of the socially optimal asymmetric subgame perfect equilibria in which only a single subject volunteers immediately at time period t=0. At the same time, they volunteered, on average, earlier than predicted by the SSPE. These results persisted in each of the two conditions, but more so in Condition δ=0.3. Third, subjects exhibited heterogeneous propensities to volunteer. About 50 percent of the subjects in each condition adhered to SSPE play in terms of the expected frequency of stopping decisions (see Figure 1.6). The other 50 percent were divided almost equally between “free riders” (left side of the lower bound of the central 99% interval), who stopped the clock significantly fewer times than predicted by the SSPE, and “hard core volunteers” (right side of the upper bound of the central 99% interval), who stopped the clock significantly more often than predicted. “Hard core volunteers” deserve this name because they persisted in stopping early despite a sharp decline in their earnings. “Free riders” greatly benefited, as they always do, from the presence of the “hard core volunteers.” The systematic deviation from the equilibrium distribution of stopping times is mostly due to a substantial fraction of the “hard core volunteers” who opted to stop the clock in the first 3-5 time periods of the game.

Experimental studies of iterated public good games almost invariably show that the propensity to free ride increases with more experience in playing these games. This holds for most experiments on public good provision that implement the Voluntary Contribution Mechanism (VCM, see, e.g., reviews of public good experiments by Camerer, 2003 and Ledyard, 1995) as well as for the continuous-time trust-based dilemmas experiment reported by Murphy et al. (2006a). In contrast, the results of the iterated dynamic volunteer’s dilemma present no discernible effects of learning. At least two explanations for this finding suggest themselves. The first has to do with the mechanism used to elicit responses. The *decision method*—the one used in the present study—records the stopping time of the volunteer; non-volunteers are not provided with the option of recording their intended stopping times because the game terminates on the volunteer’s action. Consequently, players never learn about the *intended* stopping times of non-volunteers. The *strategy* method provides players with information about the propensity to volunteer (or not volunteer) of all the group members. In a second study on continuous-time trust-based dilemmas, Murphy et al. (2006b) reported evidence that when credible signaling is possible the decline in cooperation over iterations of the stage game observed by Murphy et al (2006a) under the decision method did not occur. A comparison of the decision and strategy methods in the present dynamic volunteer’s dilemma game is called for to assess the impact of credible signaling, or lack of it, in the iterated dynamic volunteer’s dilemma game.

A second possible explanation is that, in contrast to other public good mechanisms (e.g., the VCM), a *single* player in the dynamic volunteer's dilemma can ensure the provision of the good. There is no need for a player intending to volunteer to depend on the actions of others. This hypothesis may be experimentally tested by extending the game in the present study to public good games where m ($m>1$) volunteers are required for the public good provision such that the value of the good for the m volunteers and the possibly different value for the $n-m$ non-volunteers are determined by the timing of the last player to act voluntarily.

CHAPTER 2: UNIQUE BID AUCTIONS

2.1 Introduction

This chapter considers a *unique bid auction*, a new type of online auction that has rapidly been gaining widespread popularity particularly in Great Britain, Sweden, Germany, Australia, and the US. The new feature of this type of auction, that sharply differentiates it from previous auctions, is the *uniqueness* of the winning bid. The common rule in classical auctions (e.g., first-price sealed-bid auction) is to break ties with some lottery mechanism. In contrast, in this new type of auction ties are not considered. Rather, a necessary condition for winning the auction is for the bid to be unique. This type of auction is called the unique bid auction (UBA).

In a typical UBA an auctioneer wishes to sell a particular good. Bids are integers in local currency such as US dollars. The auctioneer specifies the maximum amount of bid, which is usually set much lower than the value of the good, and the maximum number of bids required to close the auction. The auctioneer does not restrict the bidders to submit a single bid. For each bid, an entry fee is charged. In the lowest unique bid auction (LUBA), the winner is the bidder submitting the *lowest* unique bid. In the highest unique bid auction, the winner is the bidder who submits the *highest* unique bid. In the absence of a lowest (highest) unique bid, the auction terminates with no winner. If there is a winner, then she is awarded the right to purchase the good at her bid (i.e., at the winning bid). To ensure profit, the auctioneer closes the auction only after the minimum number of bids has been placed.[16] The attraction for the bidder is that she may acquire the good at well below its true value. For example, in a typical HUBA, a car valued at $20,000 might be offered to bidders at maximum bid price of $100. If the auctioneer charges $10 entry fee per bid, then he would need at least 2,000 bids to cover the cost of the car (excluding costs of conducting the auction and processing the bids). If the auction closes with a unique highest bid of $80, then the winner would purchase the car for 0.4% of its value and earn a profit of $20,000-(80+10). As reported by the *Boston Globe* (02/04/2006), one of the website specializing in LUBAs recently sold a laptop for $19, a living room suite for $43, and a Hummer SUV for less than $700. Another UBA auction site, Auction4acause.com, that specializes in the HUBA sold Apple iPhone 8GB (retail price $399) for $6.82, Home Depot $500 Gift Card for $8.37, and MacBook Air (retail price $1799.99) for $10.27.

In this chapter, generic UBAs are devised that differ from UBAs in the real world in several details but that still capture the essential feature of UBAs, namely, the uniqueness of the winning bid. In the generic UBAs, each bidder is asked to choose a single integer (i.e., bid) from a common, pre-specified set of integers (i.e., bids) $B = \{\underline{b}, \underline{b}+1, ..., \overline{b}\}$, where $\underline{b}$ and $\overline{b}$ are the minimum and maximum amounts of bid, respectively. The winner is the bidder who chooses the lowest (highest) unique integer in the generic LUBA (HUBA). If there is no unique bid, then the auction ends with no

[16] At some UBA sites, several conditions must be satisfied simultaneously to close an auction. For example, at Auction4acause.com, an auction will remain open until either (i) the maximum number of bids allocated for the auction is reached or (ii) the auction reaches a pre-specified maturation day and has received the minimum number of bids required to close. If the minimum number of bids has not been reached, the auction will be extended until the minimum number of bids has been reached.

winner. A major feature of the generic UBAs, that sharply differentiates from typical UBAs, is how to determine the payoff of the winner; if there is a winner, her payoff is the integer she picked. All the other bidders receive nothing. To achieve tractability, it is also assumed that the number of bidders *n* is fixed and commonly known before the auction commences. To simplify the procedure, no entry fee is charged. For more details of the generic UBAs, see Section 2.3.

The most similar studies to the present study are the first-price sealed-bid auctions conducted by Dufwenberg and Gneezy (2000, 2002) and later by Gneezy (2005). For example, in the experiment by Gneezy each of two bidders simultaneously selects an integer from the set $B=\{1, 2, \dots, \bar{b}\}$. The winner choosing the lowest bid is paid a dollar amount times the integer she bids whereas the other player gets 0. The main difference from these earlier studies is that in the UBAs the winning bid must be unique. Therefore, a UBA is not guaranteed to end with a winner. In contrast, if there is a tie in the auctions studied Dufwenberg and Gneezy and by Gneezy, then the earnings are equally split between the bidders.[17]

2.2 Previous Literature

In spite of a widespread popularity of the UBA around the world, both theoretical and experimental studies are scarce. To the best of my knowledge, there are three analyses of the equilibria of UBAs and their variant formats. The first is by Raviv and Virag (2007), who have offered an analysis under additional assumptions and, in addition,

[17] The uniqueness of the winning bid is not a new feature in the literature of auction. For example, Rapoport and Amaldoss (2000, 2004) analyze all-pay auctions in which the strategy space is discrete and ties are counted as losses.

have used empirical data provided to them by a UBA site to test their solution. Their analysis is based on several simplifying assumptions, namely, (i) the maximum amount of bid is set far below the value of an auctioned item; (ii) focus only on probability of winning (or a tie) rather than expected value, and (iii) repeat the auction (or return entry fee) in case there is no winner. Under these assumptions they provided the exact solution for a special case of the LUBA with constant net payoff of the value of the item minus the maximum amount of bid no matter what the winning bid. The constant payoff greatly simplifies their problem. In Section 2.3 I construct a solution to a different and larger class of auctions without making these assumptions.

Östling et al. (2007) reported a second analysis of the equilibrium solution for the LUBA.[18] They report an exact solution to a related case, one that I do not consider here, where the number of players n is uncertain. Using the theory of Poisson games developed by Myerson (1998, 2000), they assume that n is a random variable that has a commonly known Poisson distribution. Additionally, in an appendix to their paper they provide a solution for the more difficult case where n is fixed and known. However, their numerical results are restricted to the special case $n=\bar{b}\leq 8$. Using a very different approach to compute the mixed-strategy equilibrium solution, Section 2.2 describes an algorithm that is not subject to their restrictions.

Eichberger and Vinogradov (2007) suggested analytical solutions for the lowest unique bid auction in which the number of potential bidders is fixed and commonly

[18] More precisely, their game is a variant of the LUBA called the LUPI (lowest unique positive integer) game. In this game, unlike typical LUBAs, the winner does not need to pay her winning bid to the auctioneer.

known.[19] The major differences from the first two studies are that the outside option (i.e., not entering the auction) is available in the strategy space and, more importantly, each bidder is allowed to submit multiple bids. The former assumption implies that whether to enter the auction depends partly on a value of entry fee per bid. Therefore, the bidders' entry decisions are determined endogenously, and the number of active bidders (i.e., bidders who decided to enter the auction) is not necessarily the same as the number of potential bidders. Although the latter assumption makes their model closer to the lowest unique bid auctions in the real world, at the same time it makes it very difficult to derive the equilibrium solution. They have derived the equilibrium solution for a special case and left a characterization of the general solution of the game for future research.

Common to all of the three studies is an attempt to test their models with field data.[20] Critical to the models of Raviv and Virag and of Östling et al., as to the present model, is the assumption that each player submits a single bid. Therefore, the number of bids is the same as the number of bidders. In contrast, unique bid auctions conducted on the Internet do not restrict the number of bids per entrant. Clearly, if a bidder submits multiple bids, the bids will necessarily differ from one another. Consequently, multiple bids by the same player cannot be considered independent, and field studies using Internet data are therefore inappropriate for testing models postulating single bids. A

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[19] They call this type of auction as the least-unmatched price auction (LUPA).

[20] Östling et al. also subjected their model to laboratory testing. Their laboratory experiment has two important issues. First, theoretical speaking, the assumption that the number of players has a Poisson distribution cannot be replicated in the laboratory because its support is not bounded above. Therefore, no matter how much they scale down their game, there is always a positive probability that the number of players exceeds the number of available seats in their laboratory. Second, Östling et al. did not inform their subjects of the process by which the number of players in each round was determined. In contrast, their model assumes a commonly known Poisson distribution for the number of players.

second equally serious problem with using field data is the possibility of collusion between bidders. The model of Eichberger and Vinogradov explicitly allows dependency of bids within a bidder while it still requires independence between bidders. However, it is impossible to exclude cases of collusion between bidders from field data.[21] Therefore, the assumption of independence between the bidders' decisions may not be guaranteed in the field data.

The rest of the chapter is organized as follows. Section 2.3 presents and discusses the equilibrium solutions to LUBA and HUBA. It describes a computational procedure for constructing symmetric mixed-strategy equilibrium solutions to these two auctions. Section 2.4 describes alternative procedures for implementing the UBA. Section 2.5 describes two experiments designed to study the predictive power of the equilibrium solutions for the model proposed in Section 2.3 and identify deviations from equilibrium, if any. I focus on bid patterns with the same *n* for both the LUBA and HUBA. I have chosen the experimental method because it can implement the assumptions of the model with precision. My purpose in both experiments is not to mimic unique bid auctions played on the Internet. Rather, the experiments are designed to isolate the effect of uniqueness of the winning bid and study it experimentally within the framework of the model proposed in Section 2.3. Section 2.6 analyzes and discusses results of the two experiments. Section 2.7 concludes the chapter.

2.3 Equilibrium Solutions

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[21] This is not just a theoretical objection, as collusion between bidders has been reported in the field data examined by Östling et al. Eichberger and Vinogradov also reported that their field data might have included cases that a single bidder submits multiple sets of bids using different identities.

I now formally present the LUBA and HUBA. Note that they differ from LUBAs and HUBAs in the real world in several details.

2.3.1 LUBA and HUBA

There are *n* bidders, *n*>2. Each bidder chooses a single bid, which is an element in the common strategy set $B = \{\underline{b}, \underline{b}+1, ..., \overline{b}\}$. Bids are made simultaneously and anonymously. In the lowest unique bid auction (LUBA), the bidder making the lowest unique bid is the winner and she is paid the value of her bid, *b* ($b \in B$). In the highest unique bid auction (HUBA), the bidder making the highest unique bid is the winner. If there is no unique bid, then the auction ends with no winner. To simplify the analysis, no entry fee is charged.[22]

2.3.2 Asymmetric Pure-strategy Equilibria

Both the LUBA and HUBA have multiple asymmetric pure-strategy equilibria. With a single bid per bidder, denote by (b_1, b_2, … , b_n) a vector of *n* bids where the *n* bids are arranged in an ascending order (i.e., $b_1 \leq b_2 \leq ... \leq b_n$). Then, for a vector to be an equilibrium, it must have $b_1 = \underline{b}$ in the LUBA and $b_n = \overline{b}$ in the HUBA. There may be other conditions, which are not described here. Below I illustrate the equilibria with several examples.

Example Assume that *n*=7 and *B*={1, 2, … , 7}. Then the following vectors of bids are in equilibrium in LUBA (winning bids are represented in bold): (**1**, 2, 2, 3, 6, 6, 7), (1, 1, **2**, 3, 3, 4, 6), (1, 1, 1, **2**, 3, 4, 4), (1, 1, 1, 1, **2**, 3, 5), (1, 1, 1, 1, 1, **2**, 3), (1, 1, 2, 2, **3**, 4, 6), (1,

[22] If there is an entry fee then the auctioneer would need to refund the fee or allow bidders to repeat the auction in case of a tie. An auction under those assumptions will be solved as part of a future research agenda.

1, 2, 2, 2, **3**, 4), (1, 1, 1, 2, 2, **3**, 4). In HUBA, the following vectors of bids are in equilibrium: (1, 2, 2, 3, 6, 6, **7**), (1, 1, **5**, 6, 6, 7, 7), (1, 1, 1, **6**, 7, 7, 7), (**5**, 6, 6, 7, 7, 7, 7), (**4**, 5, 5, 6, 6, 7, 7), (**6**, 7, 7, 7, 7, 7, 7), (1, 2, 3, 4, 5, 6, **7**), (2, 2, 5, 5, **6**, 7, 7).

2.3.3 Symmetric Mixed-strategy Equilibrium

Because the bidders are assumed to be identical, it is the natural choice to focus on symmetric mixed-strategy equilibria (SMSE). This section describes a procedure that uses non-stationary Markov chains to numerically compute the SMSE only for the LUBA. The SMSE for the HUBA is computed in a similar way.

Denote by p a (symmetric) mixed strategy of a bidder. That is, $p = (p_{\underline{b}}, p_{\underline{b}+1}, ..., p_{\bar{b}})$, where p_b is the probability that a bidder who uses the mixed strategy p bids $b \in B$. Let one of the n bidders be a designated bidder. The expected payoff for this bidder for each bid b is computed and used to solve for the probabilities $p_{\underline{b}}, p_{\underline{b}+1}, ..., p_{\bar{b}}$. Note that each of the $n-1$ others, as well as the designated bidder, independently chooses the bids according to the probabilities $p_{\underline{b}}, p_{\underline{b}+1}, ..., p_{\bar{b}}$.

To construct the equilibrium probabilities, a non-stationary Markov chain is used. Suppose that the designated bidder bids an arbitrary bid $b \in B$. To determine whether the bid b is the winning bid, the only relevant bids of the n-1 others are the ones equal to or lower than b. The game can be viewed as an auction in which (i) an auctioneer starts at the lowest bid $\underline{b}$ and keep raising bid value until $\bar{b}$, (ii) the winner is the unique bidder who is the first to bid (e.g., raising hand), (iii) each bidder cannot observe the other bidders' timings of making a bid, and (iv) once the highest bid $\bar{b}$ is reached, the auction closes and the winner is announced. Therefore, the game occurs in time where time is equal to bid value. Making a bid anywhere lower than b is referred to as bidding before time b.

As time progresses (i.e., bid value increases), there are fewer other players who have not bid yet. Thus, the number of the other bidders remaining, from 0 to n-1, forms a stochastic process. Also, if one of the others bids at any bid before b, the designated bidder loses (no win, or NW). Therefore, the bidding process can be modeled over time (i.e., bid value) as a non-stationary Markov chain. The state space of the process is $S = \{0,1,...,n-1, NW\}$. Let $s_b \in S$ be a state at bid b. There are $|S| = n+1$ possible states for any time (i.e. any bid value).

Denote by $P(\underline{b}-1)$ a $1 \times (n+1)$ initial vector, whose elements are probabilities over possible states before the game starts. Before the game starts, the probability that $s_0 = n-1$ is 1, i.e., $P_{n-1}(\underline{b}-1) = 1$. Therefore,

$$P(\underline{b}-1) = [P_0(\underline{b}-1) \quad P_1(\underline{b}-1) \quad ... \quad P_{n-1}(\underline{b}-1) \quad P_{NW}(\underline{b}-1)] = [0 \quad 0 \quad ... \quad 1 \quad 0].$$

For $b \geq \underline{b}$, define a $(n+1)\times(n+1)$ transition matrix $P(b-1,b)$ with elements $P_{x,y}(b-1,b) = P(s_b = y \mid s_{b-1} = x)$ as follows:

$$P(b-1,b) = \left[\begin{array}{cccccc:c} 1 & 0 & 0 & ... & 0 & & 0 \\ 0 & 1-h_b & 0 & ... & 0 & & h_b \\ h_b^2 & 0 & (1-h_b)^2 & ... & & & 2h_b(1-h_b) \\ ... & ... & ... & ... & ... & & ... \\ h_b^{n-1} & 0 & \binom{n-1}{2} h_b^2 (1-h_b)^{n-3} & ... & (1-h_b)^{n-1} & & (n-1)h_b(1-h_b)^{n-2} \\ \hdashline 0 & 0 & 0 & ... & 0 & & 1 \end{array}\right],$$

where $h_b = \dfrac{p_b}{p_b + ... + p_{\bar{b}}} = \dfrac{p_b}{1 - \sum_{\beta < b} p_\beta}$.[23] h_b is the probability of bidding *b* if a bidder did not bid $\underline{b}, \underline{b}+1, ..., b-1$. This transition matrix gives the probabilities of transitions from a state at *b*-1 to a state at *b*. For example, the probability of a transition from $s_{b-1} = n-1$ to $s_b = 2$ is

$$P_{n-1,2}(b-1,b) = \binom{n-1}{2} h_b^2 (1-h_b)^{n-3}.$$

Then, the row vector that constitutes the probability distribution over state vectors at *b* is obtained by the following matrix multiplication:

$$P(b) = P(\underline{b}-1)P(\underline{b}-1,\underline{b})P(\underline{b},\underline{b}+1)...P(b-1,b).$$

Recall that the designated bidder who bids *b* will win provided that there was no unique bidder who bid before (i.e., lower than) *b*. This probability can be extracted from the row vector $P(b-1)$ by finding the probability of the state whose second component is 0. For example, $P_{[u\ \ 0]}(b-1)$ is the probability that *u* bidders (of the $n-1$ bidders) bid higher than $b-1$ and there was no unique bidder who bid less than or equal to $b-1$.

Suppose that the designated bidder bids *b* and $s_{b-1} = u$. Then, the designated bidder becomes the winner only if none of the *u* bidders bids *b*. The probability of winning by bidding *b* when $s_{b-1} = u$ is $(1-h_b)^u P_u(b-1)$. Hence, when each of the other bidders uses the mixed strategy *p*, the designated bidder's expected payoff of bidding *b* is

[23] To construct a transition matrix, all possible transitions from one state to another must be considered. However, it is impossible for some transitions to take place. The probability of such a transition is 0.

$$E(b, p) = b\sum_{u=0}^{n-1}(1-h_b)^u P_u(b-1).$$

To compute the SMSE, recall that the behavior of bidders who bid before *b* does not affect the payoffs of those who bid at or after *b*. Thus, the expected payoff of bidding *b* is a function of the equilibrium probabilities only through $p_{\underline{b}}, p_{\underline{b}+1},..., p_b$. To determine p_b, the values of $p_{\underline{b}}, p_{\underline{b}+1},..., p_{b-1}$ are fixed and p_b is varied. Since $p_{\underline{b}}, p_{\underline{b}+1},..., p_{b-1}$ are fixed, the designated bidder's expected payoff of bidding *b* is rewritten as

$$E(b, p_b) = b\sum_{u=0}^{n-1}(1-h_b)^u P_u(b-1).$$

Notice that $E(b, p_b)$ is continuous on $[0,1-\sum_{\beta<b} p_\beta]$ and strictly decreasing in p_b because the probability of a tie (i.e., losing) increases as p_b increases. This fact will be used to numerically search for p_b.

To find the equilibrium probabilities, the following general result is used. Suppose that *E* is an equilibrium expected payoff of the game. Then, (a) $E(b, p_b) \le E$ for any bid *b*, (b) $E(b, p_b) = E$ if $p_b > 0$, and (c) $p_b = 0$ if $E(b, p_b) < E$.[24] Since the true value of *E* is unknown, the algorithm must start with an estimate of *E*.[25] For a given value of *E*, the associated probabilities $p_{\underline{b}},..., p_{\bar{b}}$ are constructed sequentially through the following algorithm that starts at $\underline{b}$ and continues through $\bar{b}$.

Step 1 Set a value of *E*.

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[24] For proof of this general result, see sections 3.1.5, 3.4.2, and 3.4.3 in Vorob'ev (1977).

[25] If the value of *E* is set too high, the sum of the equilibrium probabilities may be much smaller than 1. On the other hand, if the value of *E* is set too low, no equilibrium solution may exist.

Step 2 Consider bid *b*. Given $p_{\underline{b}},\ldots,p_{b-1}$, compute $E(b,0)$.

a. If $E(b,0) \le E$, then keep $p_b = 0$. If $b < \bar{b}$, increase *b* by 1 unit and repeat Step 2. Otherwise, go to Step 3.

b. If $E(b,0) > E$, evaluate $E(b, 1-\sum_{\beta<b} p_\beta)$, where $1-\sum_{\beta<b} p_\beta$ is the maximum feasible value of p_b.

i. If $E(b, 1-\sum_{\beta<b} p_\beta) \le E$, then, there exists p_b ($0 < p_b \le 1-\sum_{\beta<b} p_\beta$) such that $E(b, p_b) = E$ since $E(b, p_b)$ is continuous on $[0, 1-\sum_{\beta<b} p_\beta]$ and strictly decreasing in p_b. If $b < \bar{b}$, then increase *b* by 1 unit and repeat Step 2. Otherwise, go to Step 3.

ii. If $E(b, 1-\sum_{\beta<b} p_\beta) > E$, then, the game has no solution for the given value of *E*. Terminate the algorithm. Go to Step 1, increase *E*, and repeat the algorithm.

Step 3 If $1-\sum_{b=\underline{b}}^{\bar{b}} p_b > \varepsilon$, where ε specifies how close is the sum of the probabilities to 1, then, go to Step 1, decrease *E*, and repeat the algorithm. Otherwise, $p_{\underline{b}},\ldots,p_{\bar{b}}$ are the equilibrium probabilities.

To illustrate the equilibrium solutions to the LUBA and HUBA, consider the following examples of the two auctions. For each auction, there is a group of *n*=50 bidders, and each of them selects a single integer from a common strategy space *B*={1, 2, … , 50}. Figure 2.1 exhibits the SMSE solutions to the two auctions, which are clearly not the mirror images of each other. First, in the equilibrium solution to the LUBA (upper panel), each of the bids in B is chosen with a positive probability. In contrast, in the equilibrium solution to the HUBA (lower panel) the bids 1 through 32 are never chosen at all. Second, in the equilibrium solution to the LUBA, the probabilities p_b first increase and then decrease as the bid b increases, whereas in the solution to the HUBA they increase monotonically in b. My computations show that the expected payoffs in the LUBA and HUBA are 0.107 and 0.835, respectively. These results suggest that in testing the equilibrium solutions experimentally, these two types of UBAs ought to be considered separately.

2.4 Alternative Implementations

It is well known that the same auction may be implemented ("framed") in alternative ways that, in theory, are strategically equivalent (Krishna, 2002). For example, the first-price auction may be implemented in several ways, as a sealed-bid auction in which bids are placed simultaneously, or as a Dutch auction. Bayesian Nash equilibrium theory suggests that these forms are isomorphic. In a similar way, each of the UBAs may be implemented in alternative ways. In what I call here "sequential implementation," the market operates like a Dutch auction with the auctioneer calling $\bar{b}$ and then lowering the price of the good in discrete steps (minimum bid increment is normalized to 1) until reaching $\underline{b}$. The major difference from the classical Dutch auction is that the n bids are not revealed until the clock reaches its minimum price $\underline{b}$. Under this implementation, the only difference between the LUBA and HUBA is whether the lowest unique bid or highest unique bid, respectively, wins the auction. Turocy et al. (2007) introduced the "silent" Dutch auction and studied it experimentally.[26] One of their major findings is that framing matters: market values in the "silent" Dutch implementation generally fell between those generated by the classical Dutch auction and the ones generated by the first-price sealed-bid auction. Another finding is that the two classical and "silent" Dutch auctions exhibited more heterogeneity across cohorts of subjects in the level of prices and in the way prices changed over time in comparison to the sealed-bid implementation.

For another implementation, I next show that an explicit prize *V* is not necessary in order to formulate and conduct UBAs. For each UBA with a prize *V* there exists a strategically equivalent auction with no exogenous prize in which the winner is awarded the value of her bid. To show this, consider a HUBA with a common strategy space $B = \{\underline{b}, \underline{b}+1, ..., \overline{b}\}$ and prize *V*. I refer to this class of auctions as auctions with *exogenous* prizes. It is easy to see that this auction is strategically equivalent to a LUBA with strategy space $\tilde{B} = \{V - \overline{b}, V - \overline{b} + 1, ..., V - \underline{b}\}$ and instead of an exogenous prize V, the player choosing the lowest unique bid is awarded the value of her bid. I refer to this class of auctions as LUBAs with *endogenous* prizes. Similarly, consider a LUBA with exogenous prize *V* and strategy space $B = \{\underline{b}, \underline{b}+1, ..., \overline{b}\}$. It is strategically equivalent to a HUBA with endogenous prize and strategy space $\tilde{B} = \{V - \overline{b}, V - \overline{b} + 1, ..., V - \underline{b}\}$ in which the player choosing the highest unique bid is awarded the value of her bid. I refer to this class of auctions as HUBAs with *endogenous* prizes. Although exogenous LUBAs (HUBAs) are strategically equivalent to endogenous HUBAs (LUBAs) (with different

[26] The "silent" Dutch implementation refers to the one in which a clock counts down as in the Dutch implementation, but in which the outcome of the auction is not revealed until the clock reaches the lowest price (Tulocy et al., 2007).

strategy space), it is an empirical question whether they yield the same pattern of bidding behavior. The focus in this chapter is on LUBAs and HUBAs with endogenous prizes. Hereafter, all references to LUBAs and HUBAs are these types.

2.5 Experimental Design

The purpose of this experiment was to test the predictive power of the SMSE in two UBAs that only differed from each other in the rule determining the winner: lowest unique bid in Condition LUBA and highest unique bid in Condition HUBA. In both conditions, a group of n=10 subjects participated in each auction and a common strategy space was B={1, 2, … , 25}.

Figure 2.2 exhibits the SMSE solutions for the two auctions, the LUBA (upper panel) and HUBA (lower panel). Similarly to Figure 2.1, Figure 2.2 shows that the equilibrium solutions to the LUBA and HUBA are not mirror images. A heuristic explanation for this is as follows. In placing her bid, a bidder in both auctions is driven by two motives, namely, to maximize the probability of choosing a winning bid and maximize her expected payoff. Both motives operate in the same direction in the HUBA: to win the auction, the bidder wishes to place a high bid. The higher the bid she places, the higher her payoff if she wins the auction. On the other hand, these two motives operate in opposite directions in the LUBA: to win the auction, the bidder wishes to place a low bid. However, the higher the bid she places, the higher her payoff if she wins the auction. The same two forces operate in the auction studied by Gneezy (2005), where in the case of tie the winning bidder is determined by lottery. In the auctions that he conducted, the equilibrium solution is always in pure strategies.

Subjects

One hundred University of Arizona undergraduate and graduate students participated in this experiment. They all volunteered to take part in a group decision making experiment for payoff contingent on performance. Male and female subjects participated in nearly equal proportions. Subjects were run in groups of 10, five groups in Condition LUBA and five other groups in Condition HUBA. None of the subjects was allowed to participate in another session. A session lasted about 75 minutes. Including a $5.00 show-up bonus, the mean payoff in Conditions LUBA and HUBA was $17.21 and $17.02, respectively.

Procedure

All the sessions were conducted in the same way. The five group members were randomly seated in a large computer laboratory and handed written instructions. No communication between the subjects was possible. The subjects were instructed that the purpose of the experiment was to study "a new type of auction that has become quite popular in the Internet." Implementing a between-subject design, each session included 60 identical rounds (auctions) that were structured as follows. On each round, the subject was asked to enter a bid by choosing one of the integers in the common strategy set *B*. Bids were entered anonymously. The subjects were instructed that the winner would be the one entering the lowest (highest) unique bid in Condition LUBA (HUBA). A winner would earn the value of her bid, whereas non-winners would earn nothing. No participation fee was charged.

Three screens were presented on each round. The Decision Screen listed the possible bids in *B* and asked each subject to choose and then enter one of them. The Results Screen presented all the five bids for the round, identified the winning bid (if at all), and recorded the subject's payoff for the round. Individual bidders were not identified. At any time, the subject could access a History Screen, which displayed the round number, all her previous bids from round 1 to the present round, and all the values of the previous winning bids. The experiment was self-paced.

At the end of the session, the subjects were paid in cash their cumulative earnings. In equilibrium, including the $5 show-up bonus, the expected earning in Conditions LUBA and HUBA were $16.09 and $16.04, respectively. To equalize mean earnings across the two conditions, the exchange rate was set at $1.00 per 1.5 points in Condition LUBA and 11 points in Condition HUBA.

2.6 Results

2.6.1 Aggregate/Group Level Results

<u>Bids</u>

Figures 2.2 exhibits side by side the observed aggregate (across sessions) relative frequency distributions of bids and the SMSE probability distributions of bids for rounds 1-30 (upper panel) and 31-60 (lower panel) of Conditions LUBA. The figure shows that the SMSE solution describes the aggregate results for Condition LUBA remarkably well. There are no systematic discrepancies between observed and predicted probabilities across the entire range of bids from 1 through 25 in both the first and last 30 rounds. The only possible exception is bid 25 that was chosen about four times as frequently as

expected (compare 1.6 in the first 30 rounds and 1.7 in the last 30 rounds to 0.4 percent). This discrepancy is mostly due to a few subjects who chose the maximum bid a disproportionally large number of times.

Figure 2.3 displays the same distributions for Condition HUBA. In a sharp contrast to Condition LUBA, systematic discrepancies between observed and predicted probabilities of bids were found for Condition HUBA. In equilibrium, bids equal to or smaller than 18 should never be placed and bid 19 should be chosen only 0.1 percent of the time. In contrast, bids 1-18 were actually chosen on 5.1 percent of the first 30 rounds and 1.6 percent of the last 30 rounds, and bid 19 was chosen on 4.1 percent of the first 30 rounds and 3.2 percent of the last 30 rounds. Figure 2.3 shows that, on the aggregate, bid values 21-25 were chosen less than predicted and bids 1-20 more than predicted.[27]

As a formal statistical analysis, the one-sample Kolmogorov-Smirnov (K-S) test was invoked to test the null hypothesis of SMSE play on the aggregate level for the first and last 30 rounds, separately (df=1500). Under this null hypothesis, both subjects within a group and rounds within a subject are independent. Given the excessively large number of degrees of freedom, it is not surprising that the null hypothesis was rejected on the aggregate level for both conditions (D=0.054 for rounds 1-30 of Condition LUBA, D=0.046 for rounds 31-60 of Condition LUBA, D=0.114 for rounds 1-30 of Condition HUBA, D=0.076 for rounds 31-60 of Condition HUBA, p<0.01 for each).[28]

[27] Bid 25 was chosen more frequently than predicted in the first 30 rounds.

[28] D is the K-S test statistic.

Tables 2.1 and 2.2 display the observed relative frequency distributions of bids and equilibrium probabilities on the group (session) and aggregate levels for rounds 1-30 and 31-60 of Conditions LUBA and HUBA, respectively. On the group level, the equilibrium solution accounted well for the relative frequency distributions of bids for both the first and last 30 rounds of Condition LUBA. In contrast, the same pattern of "stretching" the bids was observed in each of the five groups in Condition HUBA. The K-S test was used to test the null hypothesis of SMSE play on the group level for the first and last 30 rounds, separately (df=300). In the first 30 rounds, the null hypothesis was not rejected for three of the five groups (60%) in Condition LUBA and one of the five groups (20%) in Condition HUBA (p>0.05 for each condition). In the last 30 rounds, however, the null hypothesis was not rejected for three of the five groups (60%) in both Conditions LUBA and HUBA (p>0.05 for each condition). Table 2.3 shows that although the observed pattern of "stretching" the bids in the first 30 rounds did not completely disappear in the last 30 rounds of each of the five groups, the observed distributions of bids were more skewed to the left (i.e., higher bids) in the last 30 rounds than in the first 30 rounds. This trend is in the direction of the SMSE.

2.6.2 Individual Level Results

In Condition LUBA, the equilibrium solution accounted very well for the bidding behavior on the aggregate and group levels. Although the systematic deviations from the equilibrium solution on both the aggregate and group levels were observed in Condition HUBA, the bidding behavior moved closer to the equilibrium solution in the last 30 rounds than the first 30 rounds of the HUBA (see Figure 2.3 and Table 2.2). These results suggest two hypotheses. First, a majority of subjects may have independently randomized their bids according to the SMSE. The behavior on the aggregate and group levels could be an artifact of aggregation of the behavior of all, or, a majority of subjects who played the SMSE. Second, in both conditions, more subjects may have followed the SMSE in the last 30 rounds than in the first 30 rounds. In what follows, however, no evidence supporting these hypotheses will be discovered.

As before, a total of 60 rounds was divided into two blocks of 30 rounds and then the K-S test was invoked to test the null hypothesis of SMSE play on the individual level for each block (df=30). Under this hypothesis, subjects are independent of one another as well as are 30 iterations of the same auction for each subject. In other words, on each round subjects independently randomize their bids in the strategy set B according to the equilibrium probabilities. Tables 2.3 and 2.4 summarize the K-S test results for individual subjects of Conditions LUBA and HUBA, respectively ("R" stands for "Reject the null hypothesis of SMSE play" and "FR" for "Fail to reject the null hypothesis"). In the first 30 rounds, this null hypothesis could not be rejected for 29 of the 50 subjects (58%) in Condition LUBA and 23 of the 50 subjects (46%) in Condition HUBA (p>0.05 for each condition). In the last 30 rounds, in contrast, the null hypothesis could not be rejected for 22 of the 50 subjects (44%) in Condition LUBA and 20 of the 50 subjects (40%) in Condition HUBA (p>0.05 for each condition).

Figures 2.4 and 2.5 display the observed bid frequency distributions of the 50 subjects in the first and last 30 rounds of Condition LUBA, respectively. These figures show a wide variety of individual bidding patterns that defy a simple classification. Most of them are skewed to the right (i.e., in the direction of larger bids), as predicted by the equilibrium solution. Some are approximately uniform (e.g., subject 8 of Session 1 and subject 3 of Session 3 in the first 30 rounds, subject 10 of Session 3 in the last 30 rounds). Some are uni-modal (e.g., subject 7 of Session 1 over the 60 rounds, subject 9 of Session 4 in the last 30 rounds). Figure 2.6 and 2.7 organize the observed frequencies of bid for the 50 subjects in the first and last 30 rounds of Condition HUBA. Similarly, subjects in Condition HUBA demonstrate diverse bidding patterns. Common to both conditions is that although the bidding behavior of a minority of the 50 subjects is characterized well by the SMSE, when combined across subjects they yield the aggregate relative frequency distributions shown in Figures 2.2 and 2.3, which are accounted well by the SMSE.

Caution should be exercised to determine whether the subjects for whom the null hypothesis was not rejected by the K-S test have in fact followed SMSE play. Recall that SMSE play calls for subjects to independently randomize their bids on each round according to the equilibrium probabilities. This implies that if subjects repeatedly play the same auction, on each round each subject independently and stochastically decides whether to choose a different bid from her bid in previous round. Thus, subjects who actually follow the SMSE must not only

(A) generate their bid distributions that closely match with the predicted distribution,

which can be tested by the K-S test, but also

(B) switch their bids as frequently as predicted under SMSE play.

For example, the subject who has a bid distribution that resembles the predicted distribution very closely may not play the SMSE because she can still produce such a bid distribution by switching her bids considerably fewer times than predicted under SMSE play over iterations of the same auction. This subject may meet Requirement (A) but fail to satisfy Requirement (B).

To investigate subjects' switching behavior, and more importantly, to test whether the subjects for whom the null hypothesis of SMSE play was not rejected by the K-S test actually satisfied Requirement (B), I computed for each subject separately the number of switches in bids in Tables 2.5 (Condition LUBA) and 2.6 (Condition HUBA). Denote by w the number of switches ($0 \leq w \leq 29$). A switch occurs if a subject bids $b' \in B$ on round t and $b'' \in B$ on round t+1 (t=1, 2, … , 29 in the first 30 rounds and t=31, 32, ... , 59 in the last 30 rounds) and $b' \neq b''$. Then, I computed how many switches (out of 29 opportunities) would be observed in Conditions LUBA and HUBA under SMSE play. Each bidder either switches or not on each of 29 pairs of adjacent rounds, independently of the previous winning bid. Therefore, the total number of switches per bidder is binomial. The probability of not switching is given by conditioning on the result of a bid: $\sum_{i=1}^{25} p_i^2$. Then, the expected number of switches is $\mu_w = 29(1-\sum_{i=1}^{25} p_i^2)$ and the associated standard deviation is $\sigma_w = \sqrt{29(1-\sum_{i=1}^{25} p_i^2)}$. The mean number of switches in bids was computed to be 25.74 for Condition LUBA and 23.33 for Condition HUBA. The corresponding standard deviations were 1.70 and 2.14. Figure 2.8 shows the predicted probability distributions of the number of switches in bid for Conditions LUBA and HUBA, respectively. The distribution for Condition LUBA is slightly skewed to the left whereas the distribution for Condition HUBA is almost normally distributed. $\Pr(22 \le w \le 29)$ for Condition LUBA and $\Pr(18 \le w \le 28)$ for Conditions HUBA are approximately 0.99. In my analysis, I assume that a subject satisfies Requirement (B) if her number of switches in bid falls in the 99% central interval.

For each condition, each of the 50 subjects was classified into the following four categories based on which requirement(s) she satisfied: subjects who satisfied both Requirements (A) and (B), subjects who only satisfied Requirement (A), subjects who only satisfied Requirement (B), and subjects who satisfied neither of the two requirements. Tables 2.7 and 2.8 summarize the results of classification for Conditions LUBA and HUBA, respectively. Recall that the null hypothesis of SMSE play on the individual level was not rejected by the K-S test for 29 of the 50 subjects (58%) for Condition LUBA and 23 of the 50 subjects (46%) for Condition HUBA in the first 30 rounds. In the last 30 rounds, the corresponding number was 22 (44%) for Condition LUBA and 20 (40%) for Condition HUBA. These subjects fall in either the category of satisfying Requirement (A) or the category of satisfying both Requirements (A) and (B). In the first 30 rounds, the number of subjects who satisfied both requirements, i.e., played the SMSE, was 20 of the 50 subjects (40%) in each of the two conditions. In the last 30 rounds, this number declined to 15 (30%) in Condition LUBA and 13 (26%) in Condition HUBA. Therefore, in each of the two conditions, only a minority of the 50 subjects followed the SMSE both in the first and last 30 rounds. Also, the number of those who played the SMSE decreased over time. These results yield no support for the two hypotheses suggested at the beginning of this section.

Another observation is that in each of the two conditions the number of subjects who satisfied neither of the requirements increased by about 80% in the last 30 rounds (18 and 24 in Conditions LUBA and HUBA, respectively), compared to the corresponding number in the first 30 rounds (10 and 13 in Conditions LUBA and HUBA, respectively). One possible explanation is as follows. Subjects may have started the session by placing a different bid on almost every round. Then, as they gained more experience with the auction, they may have attempted to learn about other subjects' bidding patterns by staying on the same bid for short sequences of rounds and exploited the information for future rounds. Therefore, they may have had a stronger inclination to choose the same bid over short sequences of rounds in the last 30 rounds than in the first 30 rounds. By sticking to the same bid over many iterations of the auction, subjects do not only keep the number of switches in bid very small (i.e., violation of Requirement (B)) but also generate their bid distributions that deviate significantly from the predicted distribution (i.e., violation of Requirement (A)).

To illustrate the switching patterns of individual subjects, Figures 2.9 and 2.10 portray the 60 bids of each of the 50 subjects in Conditions LUBA and HUBA, respectively. Each individual graph plots the bids (y-axis) by round (x-axis). In Condition LUBA, the number of subjects who switched as many times as predicted under equilibrium play was 31 (62%) and 25 (50%) of the 50 subjects in the first and last 30 rounds, respectively. In Condition HUBA, the corresponding number is 34 (68%) in the first 30 rounds and 19 (38%) in the last 30 rounds. In both conditions, subjects showed a strong tendency to switch fewer times in the last 30 rounds than in the first 30 rounds. Some subjects switched remarkably fewer than predicted. For example, Subject 7 of Session 1 in Condition LUBA switched only 6 times in the first 30 rounds and never switched in the last 30 rounds (see Table 2.5). This subject continued choosing bid 3 from round 7 until the end of the session. Her switches in bid occurred in the first 6 pairs of adjacent rounds. Similarly, subject 2 of Session 4 in Condition HUBA switched only 5 times in the first 30 rounds and never switched in the last 30 rounds.

What affected the bidding behavior of individual subjects during the experiment? As seen before, some subjects stuck to the same bid for a long period of rounds (e.g., Subject 7 in Session 1 of Condition LUBA). At the same time, the data show that some subjects tended to choose a higher (lower) bid if the winning bid of the previous round was high (low), which is also documented by Östling et al. (2007). To analyze to what extent the subjects' bidding behavior at a current round relied on the previous winning bids, I conducted a fixed effects regression separately for each of the five sessions of each condition. As the independent variables I included lagged values of the winning bid.[29]

Table 2.9 reports the regression results. No general trend in the bidding behavior of subjects was observed in both conditions. In Condition LUBA, the bidding behavior of subjects in Sessions 2, 3, and 4 shows a significant dependency on the winning bid of the previous round; they tend to submit higher bids when the winning bid was high in the previous round. The twice lagged winning bid has a significant effect on the bidding

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[29] The current round number to control for time trend was not included because neither systematic nor replicable individual bidding pattern was discernable over the 60 rounds (see Figures 2.9 and 2.10).

behavior of subjects in Session 1. No influence of the past winning bids on subjects' current bidding behavior was observed in Session 5.

In Condition HUBA, similarly to Condition LUBA, in some sessions subjects tended to choose a higher (lower) bid when the winning bid was high (low) in the previous round (e.g., Sessions 2, 4, and 5). Subjects' bidding behavior was also influenced by the twice lagged winning bid in Sessions 4 and 5. There was no significant effect of the past winning bids on the bidding behavior of subjects in Sessions 1 and 3.

2.6.3 Discussion

By analyzing the bid patterns of subjects in the first and last 30 rounds separately for LUBA and HUBA, it is observed that a majority of the subjects deviated significantly from SMSE play. Rather, only a minority of the subjects followed SMSE play, and the number of such subjects became even smaller in the last 30 rounds. The subjects who deviated from equilibrium play did so by switching their bids between rounds less frequently than predicted, and this tendency was on average strengthened in the last 30 rounds. Not even a single subject placed the same bid in all 60 rounds; the lowest number of switches per subject that was recorded is 6 in Condition LUBA (subject 7 of Session 1) and 5 in Condition HUBA (subject 2 of Session 4). Both stayed on the same bid on the last 30 rounds. Rather than switching their bid on almost every round, most subjects often placed the same bid for short sequences of rounds, perhaps in an attempt to discover the ever changing patterns of bids and then exploit this information by best responding with a different bid.

On the group level, and even more so across all the groups, the equilibrium solution accounted for the distribution of bids in the LUBA very well. I observe heterogeneous patterns of bidding behavior on the individual level coupled with systematic and replicable patterns of bidding on the group and aggregate levels that seem to differ very little from equilibrium play. This is no longer the case when the rule for winning is changed by choosing the highest, rather than lowest, unique bid. When participating in the HUBA, the subjects deviated from the equilibrium solution by occasionally bidding below the predicted minimum bid. This tendency of stretching the bids was somewhat weakened over time, which resulted in a smaller discrepancy between the bidding behavior on both the aggregate and group levels and the equilibrium solution in the last 30 rounds.

2.7 Conclusion

Chapter 2 explored a recently emerged auction called the unique bid auction. In a sharp contrast to the traditional auctions, the winning bid must be unique; the winning bid is the lowest (highest) unique bid in the lowest (highest) unique bid auction. This new feature makes apparently no connection between the winning bid and value of the prize, which may lead people to consider the unique bid auctions as lotteries. The unique bid auction has been becoming popular around the world.

I have constructed the equilibrium solutions for the LUBA and HUBA, which, as presented in Section 2.3, have several details not shared by real unique bid auctions. Rather, the games were framed as unique bid auctions with no participation fee in which the winner is paid her bid. The solutions assume that the number of bidders is commonly known. To achieve tractability, the restriction was imposed that each player can only place a single bid. I have presented a procedure for numerically computing the probability distribution of bids that can be used with both the LUBA and HUBA. Theoretically, it is not limited by the number of players or the number of strategies. However, it is restricted in practice mostly by the number of players, as computation time increases exponentially in *n*.

Two experiments were conducted, namely Conditions LUBA and HUBA, that differed from one another with respect to how to determine the winning bid. Taken together, these experiments resulted in three major findings. First, only a minority of the subjects generated sequences of bids across iterations of the auction that did not deviate significantly from mixed-strategy equilibrium play. The major reason for deviating from equilibrium play was the inclination of some subjects to repeat the same bid too frequently. Second, in most cases the bidding behavior on the group and aggregate level for the LUBAs did not deviate significantly from equilibrium play. Similar results of heterogeneous patterns of bidding behavior on the individual level coupled with systematic and replicable behavior on the aggregate level that adheres to the symmetric mixed-strategy equilibrium have been reported in previous studies of market entry behavior (e.g., Rapoport, Seale, & Winter, 2002; Seale & Rapoport, 2000) and arrival times in single-server queues (Rapoport et al., 2004). Thirdly, subjects' bidding behavior on the group or aggregate level for the HUBAs did deviate significantly from equilibrium play due to a minority of bids that were placed below the values predicted to be chosen by the equilibrium solution.

These findings suggest two directions in which additional experimental research on unique bid auctions might proceed. The first direction is to test the difference between the LUBA and HUBA more extensively by using different group sizes and different strategy spaces. A second direction is to frame the experimental games as auctions with exogenous prizes or Dutch auctions with no observability of bids. The results on private value auctions reported by Turocy et al. (2007), who tested and consequently rejected the null hypothesis that alternative framings of strategically equivalent games as first-price sealed bid auctions and Dutch auctions result in the same bidding behavior, suggest that the particular framing of auctions matters. A third direction is to endogenize the number of bidders by charging a participation fee.

CHAPTER 3: DISCRETE BOTTLENECK GAMES

3.1 Introduction

The seminal paper on urban traffic congestion by Vickrey (1969) assumes that congestion on a single road takes on the form of multiple cars queueing behind a bottleneck. Vickrey's major contribution has been to *endogenize* the departure time decisions and to let the evolution of congestion over the rush hour be determined within the model (Arnott et al., 1998). Vickrey considered a situation, quite typical of morning rush hour, where a fixed and very large number of identical commuters have to travel from a single origin (e.g., home) to a single destination (e.g., work) along a single road. This road has a single bottleneck with a fixed and commonly known capacity. If the arrival rate at the bottleneck exceeds its capacity, a queue forms. Although all the commuters wish to arrive at the common destination at the same time, this is not physically possible because the bottleneck capacity is finite. Consequently, some commuters must arrive early and incur the costs of waiting whereas others may arrive late and pay the penalty for doing so. As noted by Arnott et al. (1990, 1998), in determining her departure time each commuter faces a tradeoff between *journey time* and *schedule delay* (early or late arrival at her destination). She can choose to depart in the tails of the rush hour when journey time is relatively low and incur the cost of arriving at work early or late. Alternatively, she can choose to depart at the peak hour when travel time is relatively high but schedule delay costs are low.

Vickrey's bottleneck model was independently formulated by Hendrickson and Kocur (1981), and subsequently extended by Smith (1983), Daganzo (1985), and influential papers by Arnott et al. (1990, 1993). In all of these formulations, the commuters are treated as a continuum. In making this assumption, these researchers have followed a common practice in transportation science and economics to use continuous models for analyzing phenomena that are essentially discrete. Quite often, but not always, the predictions derived from the continuous model provide good approximations to the phenomena that are discrete in nature. But in some cases (e.g., Swarthout and Walker, 2007), differences between the continuous and discrete versions of the same model may matter. Swarthout and Walker give as an example the simple Cournot model in which the continuous version of the model yields a unique equilibrium whereas the discrete version may have several pure-strategy equilibria. A second, more dramatic example is of a single economy with a public good in which the well-known mechanism proposed by Groves and Ledyard (1977) is used to determine how much each participant will pay to finance the public good and, consequently, how much of the public good will be provided. Swarthout and Walker show that in this case the correspondence between the continuous and discrete versions of the same model fails. In the case of continuous strategy spaces, the mechanism has a unique Pareto optimal equilibrium. But when the strategy spaces are discrete, in general the mechanism has multiple pure-strategy equilibria, only a small fraction of them are Pareto optimal. They conclude that one could easily go astray using continuous models to predict outcomes in discrete implementations.

But the issue is not only the goodness of the approximation. Whereas roadway congestion has normally been examined in contexts including thousands of commuters, where the effect of each commuter is negligible, congestion may involve a relatively small number of commuters who cause negative externalities. Examples of congestion that only involve a relatively small number of commuters are common in transportation markets and queueing (e.g., Glazer and Hassin, 1987). In many small company-owned or mining communities, many of the inhabitants who work at the same factory or same mine on the same shift, wish to arrive to their destination at the same time. Bridges, tunnels, and security checks will result in bottlenecks. Another example is of "lead time" in supply chain management. Here, consumers order a new product from a manufacturer for some special occasion (e.g., Christmas). If it takes time to produce a unit of the new product because of short supply, costly delays ("congestion") may arise. Check-in for departure on international flights by passengers wishing to check-in their luggage is yet another example of congestion at the bottleneck, common arrival time (same plane departure), and schedule costs. Other examples that have motivated this study include economics and transportation laboratory experiments designed to assess the descriptive power of equilibrium models of endogenous departure time in traffic networks (e.g., Daniel et al., 2007; Schneider and Weimann, 2004; Stein et al., 2007; Ziegelmeyer et al., 2008). These laboratory experiments only study a relatively small number of participants. In the present formulation, that continues previous research on the micro-foundations of congestion in traffic networks with endogenous arrivals (Levinson, 2005; Zou and Levinson, 2006), traffic congestion is modeled as a non-cooperative n-person game with identical commuters and a finite strategy space. Equilibrium obtains when no commuter has an incentive to alter her departure time, given that all the other commuters adhere to equilibrium play. Because schedule delay cannot be the same for all commuters, they must adjust their travel time over the rush hour to satisfy the equilibrium condition.

The motivation for this work is twofold. First, as mentioned earlier, the continuous and discrete versions of the same model yield different results. I wish to compare the two formulations in order to determine how good the continuous approximations are to the associated discrete case. Second, I wish to develop a model to account for the experimental implementation of the bottleneck game in the laboratory. Whether in an experiment or even in naturally occurring settings (see, e.g., Levinson, 2005), in order to implement a mechanism or test a theory one has to use a discrete strategy space and a finite number of commuters. Moreover, as the number of participants in experiments and in some traffic congestion applications (see above) is typically small, the approximations provided by the continuous model may not be satisfying.

The rest of this chapter is organized as follows. Section 3.2 introduces notation and then describes Vickrey's continuous model and the deterministic equilibrium solution constructed by Arnott et al. (1990). I do not review previous research that formulated the bottleneck situation as a continuous model. Rather, in Section 3.3 I present and briefly discuss three recent papers by Levinson (2005), Zou and Levinson (2006), and Ziegelmeyer et al. (2008) that focus on the discrete version of the bottleneck model. In Section 3.4 that constitutes the main section of the chapter, a numerical procedure is presented for computing a symmetric mixed-strategy equilibrium solution for the discrete version of the bottleneck model. Using a non-stationary Markov chain approach, I conclude this section with the presentation of an algorithm for computing the equilibrium probabilities. Section 3.5 compares the continuous and discrete versions of the bottleneck model in terms of travel time and travel cost. In Section 3.6, the discrete model is extended to the case where the number of commuters is a random variable whose distribution is commonly known, and a second case where an alternative transportation mode that is not subject to congestion is available. The equilibrium solutions to these two extensions are computed and illustrated. Section 3.7 concludes with a brief discussion.

3.2 Vickrey's Continuous Bottleneck Model

In the model of Arnott et al. (1990), a fixed number, *n*, of identical commuters travel every morning from home (*O*—origin) to work (*D*—destination). They do so along a single road with a bottleneck. In this model, commuters are treated as a continuum of measure *n*. All of them wish to arrive at the same destination at time t^*. Travel is not congested except at the bottleneck at which at most *s* commuters can pass per unit time. If the rate of arrival at the bottleneck exceeds *s*, then a queue develops behind the bottleneck.

Travel time from *O* to *D* is denoted by $T^f+T(t)$, where T^f is the fixed component of travel from *O* to *D*. With no restriction, assume that $T^f=0$ implying that a commuter arrives at the bottleneck as soon as she leaves home, and arrives at work as soon as she leaves the bottleneck. $T(t)$ is the waiting time at the bottleneck, and *t* is the departure time from home. Let $D(t)$ denote the length of the queue at time *t*. Then,

$$D(t) = \int_{\hat{t}}^{t} r(u)du - s(t - \hat{t}) ,$$

where $r(t)$ is a departure function, and $\hat{t}$ is the most recent time at which there was no queue. Travel time by departing at time t is computed from

$$T(t) = \frac{D(t)}{s}.$$

In words, the commuter's travel time equals queue length at the time she joins the queue divided by the service rate of the bottleneck. This equation also implies that she passes through the bottleneck instantaneously once her turn comes.

Following Vickrey (1969), Arnott et al. assume that the cost of the trip, denoted by C, is linear in journey time and schedule delay:

$$\begin{aligned} C(t) &= \text{travel time costs} + \text{time early costs} + \text{time late costs} \\ &= \alpha T(t) + \beta \max\{0, t^* - (t + T(t))\} + \gamma \max\{0, (t + T(t)) - t^*\}, \end{aligned}$$

where, as defined earlier, t^* is the desired arrival time.[30] Each commuter independently chooses a departure time, t, to minimize her travel cost.

Equilibrium obtains when no commuter has an incentive to unilaterally alter her departure time. Arnott et al. show that in equilibrium the commuters depart at a piecewise constant rate given by

$$r(t) = \begin{cases} \dfrac{\alpha s}{\alpha - \beta} & \text{for } t \in [t_F, t_O) \\ \dfrac{\alpha s}{\alpha + \gamma} & \text{for } t \in (t_O, t_L] \end{cases}$$

where t_F and t_L are the times at which the first and the last commuters depart, respectively. t_O is such a time that $t_O + T(t_O) = t^*$. Solving the following equations simultaneously yields t_F, t_L, and t_O:

[30] In accordance with empirical results by Small (1982), Arnott et al. assume that $\gamma>\alpha>\beta$. The assumption that $\gamma>\alpha$ is not required to assure existence of a pure-strategy equilibrium.

$$t_L - t_F = \frac{n}{s},$$

$$\beta(t^* - t_F) = \gamma(t_L - t^*),$$

$$t_O + \frac{\beta}{\alpha - \beta}(t_O - t_F) = t^*.$$

The first equation specifies that the length of congestion is *n*/*s*, the second equation states that the travel costs of the first and last commuters are equal, and the last equation follows from the definition of t_O. Then,

$$t_F = t^* - \left(\frac{\gamma}{\beta + \gamma}\right)\left(\frac{n}{s}\right),$$

$$t_L = t^* + \left(\frac{\beta}{\beta + \gamma}\right)\left(\frac{n}{s}\right),$$

$$t_O = t^* - \left(\frac{\beta\gamma}{\alpha(\beta + \gamma)}\right)\left(\frac{n}{s}\right).$$

The travel cost (*C*) of departing at t_F is $\beta(t^* - t_F)$ because there is no congestion at the bottleneck. Therefore, the travel cost is

$$C = \left(\frac{\beta\gamma}{\beta + \gamma}\right)\left(\frac{n}{s}\right).$$

Since all the commuters have the same travel cost of departing at the times between t_F and t_L, the total travel cost (*TC*) is given by

$$TC = nC = \left(\frac{\beta\gamma}{\beta + \gamma}\right)\left(\frac{n^2}{s}\right).$$

The total travel time (*TTT*) is computed as

$$TTT = \int_{t_F}^{t_L} D(t)dt = \left(\frac{\beta\gamma}{2\alpha(\beta+\gamma)} \right) \left(\frac{n^2}{s} \right) = \frac{TC}{2\alpha}.$$

Therefore, the total travel time cost (*TTC*) is given by

$$TTC = \alpha \times TTT = \frac{TC}{2}.$$

Brief comments on the results obtained by Arnott et al. are in order. First, notice that t_F, t_L, *C*, *TC*, and *TTC* are all independent of *α*. Recall that the first and the last commuters never encounter congestion at the bottleneck. Thus, any change in *α* does not influence their departure times, namely t_F and t_L, and their associated travel costs. Since all the commuters must have the same travel cost as the first and the last commuters, *C* and *TC* must be independent of *α*. The total travel time is proportional to 1/*α* so that the total travel time cost remains independent of *α*. Second, in the model of Arnott et al., the total travel time cost is half of the total travel cost. Thirdly, the equilibrium is deterministic; because the commuters are treated as a continuum and time is continuous, they can choose their departure times in a way that the travel cost of each commuter is constant over the rush hour.

3.3 Review of Previous Literature

Only a few recent studies have attempted an equilibrium analysis of the discrete version of the bottleneck model. Levinson (2005) studied two non-cooperative games: one with two commuters who only have three choices of departure time (early, on-time, late), and the other with three commuters who only have six choices of departure time (very early, early, on-time, late, really late, super late). In a subsequent study, Zou and

Levinson (2006) extended the previous model into a non-cooperative n-person game based on several simplifying assumptions. Their analysis is based on a restrictive assumption, due to the computational difficulties that they encountered, that the number of commuters, n, is smaller than or equal to seven. Zou and Levinson also limited the number of strategies (departure times) to $n+1$. Common to both studies are the following assumptions: (i) the service capacity per unit of time is a single commuter, (ii) ties are broken randomly among commuters arriving at the same time, and (iii) the analysis is restricted to pure-strategy Nash equilibria.[31]

The study of Ziegelmeyer et al. (2008) is the most closely related to the present study. To the best of my knowledge, Ziegelmeyer et al. are the first to construct a symmetric mixed-strategy Nash equilibrium of the discrete version of the bottleneck model. They first characterized pure-strategy Nash equilibria, and then constructed symmetric mixed-strategy equilibrium solutions for their laboratory experiments. Just as in the first two studies by Levinson and by Zou and Levinson, Ziegelmeyer et al. assume in their main study that the service capacity per unit of time is a single commuter. This is a restrictive assumption, which is relaxed in the next section, that considerably reduces computational complexity. A special feature of their model is the handling of ties. They handle ties as follows. Suppose that a commuter arrives at the bottleneck with j other commuters at time t. Then, in order to pass through the bottleneck, each of the $(j+1)$ commuters is assumed to spend $(j+1)$ units of time if the bottleneck is not congested at all

[31] Nash (1951) proved that every n-person game with a finite number of players and finite strategy space possesses at least one equilibrium in pure or mixed strategies. A pure-strategy Nash equilibrium may not exist in some finite games (e.g., the Matching Pennies game). See, e.g., Osborne and Rubinstein (1994).

at time t, and (j+1+v) units of time if v other commuters are waiting in a queue behind the bottleneck at time t, respectively. The present model does not impose any restriction on the service capacity. Also, in contrast to Ziegelmeyer et al., it invokes the more natural assumption that ties are broken randomly with equal probability among the commuters who arrive at the bottleneck simultaneously.

3.4 Discrete Bottleneck Game

3.4.1 Model

Vickrey's model of departure time (as elaborated by Arnott et al., 1990, 1993) is formulated as follows. There are n identical commuters who travel along a single road connecting a common origin O and a common destination D. Each commuter *independently* and *simultaneously* chooses a departure time, $t \in \{1,...,t^*,...,t_{\max}\}$. As before, travel is assumed to be uncongested anywhere except at a single segment of the road called a *bottleneck*. A first-come first-served (FCFS) queue discipline is applied at the bottleneck. Denote by s (>0) the service capacity per unit of time. If $s \geq 1$, then at most s commuters are served per unit time. If $s<1$, service capacity is constrained to take values of the form $s=1/d$, where d is an integer larger than 1. Then, only a single commuter is served at a time, and it takes each commuter d units of time to pass through the bottleneck.[32] It is assumed that for any s, if multiple commuters arrive at the bottleneck simultaneously, then they are served in a random order with equal probability.

[32] The parameter d can be interpreted as the service time per commuter, i.e., the number of units of time for one commuter to pass through the bottleneck.

The travel cost of a commuter departing at time t consists of three types of cost: travel time cost, early arrival cost, and late arrival cost. The linear cost structure of Arnott et al. is maintained: a commuter's travel cost is linear in travel time, early arrival time, and late arrival time.

Table 3.1 presents two examples, one for s=1/3 (i.e., d=3) and the other for s=3, that illustrate the discrete bottleneck game and computation of the travel costs. Let n=10, α=1, β=0.6, γ=2.4, $t \in \{1, 2, \ldots, 60\}$, and t^*=50. For both cases, the departure times, travel time, arrival time, and travel cost are listed in columns 2, 3, 4, and 5, respectively. In the example for s=1/3 (top panel), there is a tie among commuters 2, 3, and 4, who departed home at time 34. This tie was broken randomly (with probability 1/3), so that commuter 2 was the first to be served, Commuter 3 was the second, and commuter 4 was the third. Consequently, commuters 3 and 4 had to spend 6 and 9 units of time at the bottleneck, respectively, while commuter 2 only spent 3 units of time. Commuters 1 and 2 never encountered congestion at the bottleneck. On the other hand, commuter 10 had to join a queue formed by commuters 7, 8, and 9 when she arrived.

In the example for s=3 (bottom panel), four commuters departed at time 48, and six others at time 49. The first and second ties were randomly broken with probabilities 1/4 and 1/6, respectively. Among the first four commuters, commuter 4 was randomly selected to wait for an additional time unit and pass though the bottleneck with commuters 5 and 6, who departed at time 49. Commuters 7, 8, and 9 had to wait in the queue for an additional unit of time and then travel the bottleneck together. Although commuter 10 departed at time period 48, she had to wait in the line for 2 units of time before being served.

3.4.2 Computational Procedure

Depending on its parameter values, the discrete bottleneck game may or may not possess pure-strategy Nash equilibria. If pure-strategy Nash equilibria exist, then they are asymmetric with some commuters departing early and others departing late.[33] Because the *n* commuters are assumed to be identical, the model focuses on symmetric equilibria. Dasgupta and Maskin (1986) prove in their Lemma 6 that a finite symmetric game possesses a symmetric mixed-strategy Nash equilibrium (SMSE). Therefore, there exists at least one SMSE in the discrete bottleneck game.

Denote by p a (symmetric) mixed strategy of a commuter. That is, $p = (p_1, p_2, ..., p_{t_{max}})$, where p_t is the probability that a commuter who uses the mixed strategy p departs from the origin (i.e., arriving at the bottleneck) at time t. Let one of the n commuters be a designated commuter. The expected travel cost for this commuter for each departure time t is computed and used to solve for the equilibrium probabilities $p_1, p_2, ..., p_{t_{max}}$. Note that each of the n-1 other commuters is assumed to independently use the mixed strategy p.

To construct the equilibrium probabilities, once again I use a non-stationary Markov chain. Using an indirect approach, it is possible to compute the mixed-strategy equilibrium for a considerably larger number of commuters than in the previous studies

[33] In general, it is too complicated to fully characterize the set of pure-strategy Nash equilibria of the discrete bottleneck game. It may require certain restrictions on the set of strategies (i.e., departure times) and parameter values. Ziegelmeyer et al. (2008) identify the set of pure-strategy Nash equilibria under suitable restrictions on parameter values.

by Levinson (2005), Zhu and Levinson (2006), and Zieglemeyer et al. (2008). To this end, I need to define a proper state space that describes the stochastic nature of the traffic network. Two cases are considered: $s<1$ and $s\geq1$.

Case 1: $s<1$ As defined earlier, service capacity takes values of the form $s=1/d$, where d is an integer larger than 1. This means that the service time per commuter is d units of time. Note that for any time t, commuters can be in one of four locations in the system, namely, either at the origin, in a queue behind the bottleneck (i.e., waiting for service), within the bottleneck (i.e., receiving service), or at the destination. Denote by Ω a set of possible states and by $\omega_t \in \Omega$ a state at time t (more precisely, a state of the system immediately after all movements of the commuters occurred at time t). Each state is a vector of three elements. The first and the second elements specify the numbers of commuters (out of the n-1 commuters) at the origin and in a queue behind the bottleneck, respectively. They take on integer values from 0 to n-1. The third element is used to keep track of the time periods elapsed since the last commuter was served in the bottleneck; it takes on one of the integer values 0, 1, …, d-1. The value 0 indicates that no commuter is currently being served, whereas d-1 indicates that d-1 units of time elapsed since the last commuter was served. For example, suppose that d=5, and $\omega_{t-1}=[4 \quad 2 \quad 4]$. If none of the four commuters at the origin departs at time t, then the state at t is $\omega_t=[4 \quad 2 \quad 0]$. Note that the number of possible states is computed from

$$|\Omega| = \frac{n(n+1)}{2} + (d-1)\frac{(n-1)n}{2}.$$ [34]

[34] If the third element is zero, it implies that all of the n-1 commuters are not being served in the bottleneck. Then, the number of possible states is $n(n+1)/2$. On the other hand, if the third element takes a larger

Suppose that $\omega_{t-1} = [u \quad v \quad e]$, i.e., at time *t*-1, *u* commuters were at the origin, *v* commuters were waiting in a queue behind the bottleneck, and *e* units of time elapsed since a commuter was served. Suppose that the designated commuter and *j* commuters (out of *u*) depart from the origin (i.e., arrive at the bottleneck) at time *t*. Let *k* denote the number of commuters (out of *j*) being served before the designated commuter. Then, the designated commuter's travel time of departing at time *t*, given ω_{t-1} and *k*, $T(t \mid \omega_{t-1}, k)$, is computed from one of the following eight cases:

$$T(t \mid \omega_{t-1}, k) = \begin{cases} d & \text{if } u = 0, v = 0, e = 0 \\ dv + (d-1) & \text{if } u = 0, v > 0, e = 0 \\ d(v+1) + (d-e-1) & \text{if } u = 0, \text{any } v, 0 < e < d-1 \\ d(v+1) & \text{if } u = 0, \text{any } v, e = d-1 \\ d(k+1) + (d-1) & \text{if } u > 0, v = 0, e = 0 \\ d(v+k) + (d-1) & \text{if } u > 0, v > 0, e = 0 \\ d(v+k+1) + (d-e-1) & \text{if } u > 0, \text{any } v, 0 < e < d-1 \\ d(v+k+1) & \text{if } u > 0, \text{any } v, e = d-1 \end{cases}$$

<u>Case 2: *s*≥1</u> In this case, at most *s* commuters can pass through the bottleneck per unit time. A state of the system can be represented by a vector that only has two elements: the first and second elements state the numbers of commuters (out of the *n*-1 commuters) at the origin and in a queue behind the bottleneck. For example, suppose that *s*=2, and $\omega_{t-1} = [3 \quad 3]$. If none of the three commuters at the origin departs at time *t*, then the state of the system at time *t* is $\omega_t = [3 \quad 1]$. Since each element takes an integer value 0 to *n*-1, and the sum of the two elements cannot exceed *n*-1, the number of possible states is

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integer than zero (there are *d*-1 cases), then one of the *n*-1 commuters is being served. In this case, the number of possible states is (*d*-1)(*n*-1)*n*/2. Hence, the total number of possible states is *n*(*n*+1)/2+(*d*-1)(*n*-1)*n*/2.

$$|\Omega| = \frac{n(n+1)}{2}.$$

Suppose that $\omega_{t-1} = [u \quad v]$, and that the designated commuter and j commuters (out of u) depart from the origin (i.e., arrive at the bottleneck) at time t. Let k denote the number of commuters (out of j) waiting before the designated commuter. Define by $g(a)$ a function that rounds a number a to the nearest integer greater or equal to a. Then, the designated commuter's travel time of departure at t, given ω_{t-1} and k, $T(t \mid \omega_{t-1}, k)$, is given by one of the following four cases:

$$T(t \mid \omega_{t-1}, k) = \begin{cases} 1 & \text{if } u = 0, v \le s \\ g\left(\dfrac{v-s+1}{s}\right) & \text{if } u = 0, v > s \\ g\left(\dfrac{k+1}{s}\right) & \text{if } u > 0, v \le s \\ g\left(\dfrac{v-s+k+1}{s}\right) & \text{if } u > 0, v > s \end{cases}$$

For example, suppose that s=2, $\omega_{t-1} = [4 \quad 5]$, and k=3. If the designated commuter departs at t, then his travel time is $g((5-2+3+1)/2) = g(3.5) = 4$ units of time.

The number of departures at t follows the binomial distribution with parameters u and h_t, where $h_t = \dfrac{p_t}{p_t + ... + p_T} = \dfrac{p_t}{1 - \sum_{\tau<t} p_\tau}$. h_t is the probability of departure at t, given that departure before t did not occur. Then, the probability of j other commuters departing at t is computed from $f(u, j, h_t) = \binom{u}{j}(h_t)^j (1-h_t)^{u-j}$. When j other commuters depart at t, the designated commuter becomes the $(k+1)^{\text{th}}$ commuter among (j+1)

commuters with probability 1/(j+1). Then, the designated commuter's expected travel cost, given ω_{t-1} and k, is

$$C(t \mid \omega_{t-1}, j) = \frac{1}{j+1}\sum_{k=0}^{j} C(t \mid \omega_{t-1}, k),$$

where, as before,

$$\begin{aligned} C(t \mid \omega_{t-1}, k) &= \alpha T(t \mid \omega_{t-1}, k) \\ &\quad + \beta \max\{0, t^* - (t + T(t \mid \omega_{t-1}, k))\} + \gamma \max\{0, (t + T(t \mid \omega_{t-1}, k)) - t^*\} \end{aligned}.$$

Denote by $C(t, p \mid \omega_{t-1})$ the designated commuter's expected travel cost of departure at time t when each of the other n-1 commuters uses the mixed strategy p and the state at t-1 is ω_{t-1}, which is computed from

$$C(t, p \mid \omega_{t-1}) = \sum_{j=0}^{u} f(u, j, h_t) C(t \mid \omega_{t-1}, j).$$

To determine the designated commuter's expected travel cost of departure at t, the probability distribution over possible states at time t-1 must be derived. Denote by $P(0)$ a $1 \times |\Omega|$ initial vector, whose elements are probabilities over possible states at time 0. Note that all the n-1 commuters are at the origin at $t = 0$. Therefore, the probability that all the n-1 commuters are at the origin is 1, i.e., $P_{[n-1 \;\; 0 \;\; 0]}(0) = 1$ if s<1, and $P_{[n-1 \;\; 0]}(0) = 1$ if s≥1. The other elements in $P(0)$ take the value of 0.

For $t \geq 1$, define a $|\Omega| \times |\Omega|$ transition matrix $P(t-1, t)$ with elements $P_{x,y}(t-1, t) = P(\omega_t = y \mid \omega_{t-1} = x)$. To construct a transition matrix, all possible transitions from one state to another must be considered.[35] Then, the $1\times|\Omega|$ row vector that constitutes the probability distribution over states at time t-1 is obtained by the following matrix multiplication:

$$P(t-1)=P(0)P(0,1)P(1,2)...P(t-2,t-1)\,.$$

Then, the designated commuter's expected travel cost of departure at t when each of the other n-1 commuters uses the mixed strategy p is computed from

$$C(t,p)=\sum_{\omega_{t-1}\in\Omega}P_{\omega_{t-1}}(t-1)C(t,p\,|\,\omega_{t-1})\,.$$

To compute the SMSE, note that the FCFS queue discipline is used and thereby future arrivals cannot affect the costs of those commuters who have already arrived at the bottleneck. Thus, the expected travel cost of departure at time t is a function of the mixed strategy only through the probabilities $p_1,p_2,...,p_t$. To determine p_t, $p_1,p_2,...,p_{t-1}$ are fixed and p_t is varied. Since $p_1,p_2,...,p_{t-1}$ are fixed, the designated commuter's expected travel cost of departing at time t when each of the other n-1 commuters chooses p is rewritten as

$$C(t,p_t)=\sum_{\omega_{t-1}\in\Omega}P_{\omega_{t-1}}(t-1)C(t,p_t\,|\,\omega_{t-1})\,.$$

Notice that for all t, $C(t,p_t)$ is continuous on $[0,1-\sum_{\tau<t}p_\tau]$. Then, the following theorems are derived.

Theorem 3.1 *Given* $p_1,...,p_{t-1}$, *if* $p_t>\tilde{p}_t$, *then* $C(t,p_t)>C(t,\tilde{p}_t)$.

[35] It is impossible for some transitions to take place. For example, consider the case when s<1. Then, state [3 0 0] cannot be reached from state [2 1 0]. Thus, the probability of such a transition is 0.

<u>Proof of Theorem 3.1</u> Since $p_1,\ldots,p_{t-1}$ are fixed, all the components of $P(t-1)$, i.e., probabilities over possible states at t-1, are determined. Thus, to prove that $C(t,p_t) > C(t,\tilde{p}_t)$, it suffices to show that, for any ω_{t-1},

$$C(t,p_t\,|\,\omega_{t-1}) = \sum_{j=0}^{u} f(u,j,h_t)C(t\,|\,\omega_{t-1},j) \geq \sum_{j=0}^{u} f(u,j,\tilde{h}_t)C(t\,|\,\omega_{t-1},j) = C(t,\tilde{p}_t\,|\,\omega_{t-1}),$$

with strict inequality for some ω_{t-1}. Recall that $\omega_{t-1} = [u \quad v \quad e]$ if s<1 and $\omega_{t-1} = [u \quad v]$ if s≥1. All possible states at t-1 are divided into two exclusive cases with respect to the value of u: u=0 and u>0.

<u>Case 1: u=0</u> Since $C(t,p_t\,|\,\omega_{t-1}) = C(t\,|\,\omega_{t-1},0)$ and $C(t,\tilde{p}_t\,|\,\omega_{t-1}) = C(t\,|\,\omega_{t-1},0)$,

$$C(t,p_t\,|\,\omega_{t-1}) = C(t,\tilde{p}_t\,|\,\omega_{t-1}).$$

<u>Case 2: u>0</u> Rearrange $C(t,p_t\,|\,\omega_{t-1})$:

$$C(t,p_t\,|\,\omega_{t-1}) = \sum_{j=0}^{u} f(u,j,h_t)C(t\,|\,\omega_{t-1},j)$$

$$= f(u,0,h_t)\{C(t\,|\,\omega_{t-1},0) + C(t\,|\,\omega_{t-1},1) - C(t\,|\,\omega_{t-1},1) + \ldots + C(t\,|\,\omega_{t-1},u) - C(t\,|\,\omega_{t-1},u)\}$$

$$+ f(u,1,h_t)\{C(t\,|\,\omega_{t-1},1) + C(t\,|\,\omega_{t-1},2) - C(t\,|\,\omega_{t-1},2) + \ldots + C(t\,|\,\omega_{t-1},u) - C(t\,|\,\omega_{t-1},u)\}$$

$$+ \ldots..$$

$$+ f(u,u-1,h_t)\{C(t\,|\,\omega_{t-1},u-1) + C(t\,|\,\omega_{t-1},u) - C(t\,|\,\omega_{t-1},u)\}$$

$$+ f(u,u,h_t)C(t\,|\,\omega_{t-1},u)$$

$$= C(t\,|\,\omega_{t-1},u) - \sum_{j=0}^{u-1}\left(\{C(t\,|\,\omega_{t-1},j+1) - C(t\,|\,\omega_{t-1},j)\}\sum_{k=0}^{j} f(u,k,h_t)\right)$$

For $0 \le j \le u-1$, (a) $C(t \mid \omega_{t-1}, j+1) > C(t \mid \omega_{t-1}, j)$ and (b) $\sum_{k=0}^{j} f(u,k,h_t)$ is increasing in h_t, in other words, in p_t. Therefore, if $p_t > \tilde{p}_t$, $C(t, p_t \mid \omega_{t-1}) > C(t, \tilde{p}_t \mid \omega_{t-1})$.

By combining the results of the two exclusive cases, $C(t, p_t) > C(t, \tilde{p}_t)$. □

The intuition behind this theorem is that the bottleneck will stochastically become more congested as p_t increases, and thereby $C(t, p_t)$ is strictly increasing in p_t. This fact will be used to search for values of p_t.

Theorem 3.2 *Suppose that C is the equilibrium expected travel cost of the discrete bottleneck game. Then, there exists a unique symmetric mixed-strategy Nash equilibrium for the game.*

Proof of Theorem 3.2 Suppose that there are two symmetric mixed-strategy Nash equilibria, $p = (p_1, ..., p_{t_{\max}})$ and $\tilde{p} = (\tilde{p}_1, ..., \tilde{p}_{t_{\max}})$, each of which yields the same equilibrium expected travel cost, C.

Consider t=1. Recall that all the n-1 commuters are at the origin at t=0, i.e., $P_{[n-1\ \ 0\ \ 0]}(0) = 1$ if s<1, and $P_{[n-1\ \ 0]}(0) = 1$ if s≥1. Since all the components of $P(0)$, i.e., probabilities over possible states at t=0, are the same for the two symmetric mixed-strategy Nash equilibria, $C(1, p_1) = C(1, \tilde{p}_1) = C$ implies that for any ω_0,

$$\sum_{j=0}^{n-1} f(n-1, j, h_1) C(1 \mid \omega_0, j) = \sum_{j=0}^{n-1} f(n-1, j, \tilde{h}_1) C(1 \mid \omega_0, j).$$

Therefore, $h_1 = \tilde{h}_1$. Since $h_1 = p_1$ and $\tilde{h}_1 = \tilde{p}_1$, $p_1 = \tilde{p}_1$.

Consider t=2. Given that $p_1 = \tilde{p}_1$, all components of $P(1)$ are the same for the two symmetric mixed-strategy Nash equilibria. Then, $C(2, p_2) = C(2, \tilde{p}_2) = C$ implies that for any ω_1,

$$\sum_{j=0}^{n-1} f(n-1, j, h_2) C(2 \mid \omega_1, j) = \sum_{j=0}^{n-1} f(n-1, j, \tilde{h}_2) C(2 \mid \omega_1, j).$$

Therefore, $h_2 = \tilde{h}_2$. Since $h_2 = p_2/(1-p_1)$ and $\tilde{h}_2 = \tilde{p}_2/(1-\tilde{p}_1)$, $p_2 = \tilde{p}_2$.

Suppose that $p_\tau = \tilde{p}_\tau$ for $\tau < t$ $(2 < t \le t_{\max})$. Then, all components of $P(t-1)$ are the same for the two symmetric mixed-strategy Nash equilibria. Then, $C(t, p_t) = C(t, \tilde{p}_t) = C$ implies that for any ω_{t-1},

$$\sum_{j=0}^{n-1} f(n-1, j, h_t) C(t \mid \omega_{t-1}, j) = \sum_{j=0}^{n-1} f(n-1, j, \tilde{h}_t) C(t \mid \omega_{t-1}, j).$$

Therefore, $h_t = \tilde{h}_t$. Since $h_t = p_t/(1-\sum_{\tau<t} p_\tau)$ and $\tilde{h}_t = \tilde{p}_t/(1-\sum_{\tau<t} \tilde{p}_\tau)$, $p_t = \tilde{p}_t$.

Thus, $p_t = \tilde{p}_t$ whenever $p_\tau = \tilde{p}_\tau$ for $\tau < t$. By mathematical induction, $p_t = \tilde{p}_t$ for all $t \in \{1, ..., t_{\max}\}$. □

Although Theorem 3.2 guarantees the uniqueness of the SMSE for a specific value of *C*, it does not rule out the possibility of multiple symmetric mixed-strategy Nash equilibria for different values of *C*.[36] Based on extensive numerical results, which are not

[36] Finite symmetric games may possess multiple symmetric mixed-strategy Nash equilibria. See example (iv) in Baye et al. (1994).

reported here, showing that the sum of probabilities $\sum_{t=1}^{t_{\max}} p_t$ is strictly increasing in the value of *C*, I conjecture that the SMSE is unique.[37]

To find the equilibrium probabilities, the following general result is used. Suppose that *C* is an equilibrium expected travel cost of the game. Then, (a) $C(t, p_t) \geq C$ for any time *t*, (b) $C(t, p_t) = C$ if $p_t > 0$, and (c) $p_t = 0$ if $C(t, p_t) > C$.[38] Since the value of *C* is unknown, the algorithm must start with an estimate of *C*.[39] For a given value of *C*, the associated probabilities $p_1, ..., p_{t_{\max}}$ are constructed sequentially through the following algorithm that starts at *t*=1 and continues through $t = t_{\max}$.

Step 1: Set a value of *C*.

Step 2: Consider time period t. Given $p_1, ..., p_{t-1}$, compute $C(t,0)$.

a. If $C \leq C(t,0)$, then keep $p_t = 0$. If $t < t_{\max}$, increase *t* by 1 unit and repeat Step 2. Otherwise, go to Step 3.

b. If $C > C(t,0)$, evaluate $C(t, 1 - \sum_{\tau<t} p_\tau)$, where $1 - \sum_{\tau<t} p_\tau$ is the maximum feasible value of p_t.

   iii. If $C \leq C(t, 1 - \sum_{\tau<t} p_\tau)$, then, there exists p_t ($0 < p_t \leq 1 - \sum_{\tau<t} p_\tau$) such that $C(t, p_t) = C$ since $C(t, p_t)$ is continuous on $[0, 1 - \sum_{\tau<t} p_\tau]$

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[37] The numerical results show that (i) if the value of *C* is set too low, the sum of the associated probabilities is much smaller than 1, (ii) if the value of *C* is set too high, then no equilibrium solution exists, and (iii) as the value of *C* increases the sum of the associated probabilities strictly increases. This implies that there exists an equilibrium expected travel cost such that the associated probabilities sum up to 1. These probabilities are the SMSE.

[38] For proof of this result, see sections 3.1.5, 3.4.2, and 3.4.3 in Vorob'ev (1977).

[39] The smallest possible value of *C* is αd if $s<1$ and α if $s \geq 1$.

and strictly increasing in p_t by Theorem 3.1. If $t < t_{\max}$, then increase t by 1 unit and repeat Step 2. Otherwise, go to Step 3.

iv. If $C > C(t, 1 - \sum_{\tau<t} p_\tau)$, then, the game has no solution for the given value of *C*. Terminate the algorithm. Go to Step 1, decrease *C*, and repeat the algorithm.

Step 3: If $1 - \sum_{t=1}^{t_{\max}} p_t > \varepsilon$, where ε specifies how close is the sum of the probabilities to 1, then go to Step 1, increase *C*, and repeat the algorithm. Otherwise, $p_1, ..., p_{t_{\max}}$ are the equilibrium probabilities.

3.4.3 Comparison with Ziegelmeyer et al.

To verify that the proposed algorithm works properly, consider two examples studied by Ziegelmeyer et al. (2008). As mentioned earlier, a special feature that differentiates their model from the model in the present study is how to handle ties. I now show that the procedure successfully constructs the same equilibrium solutions to the two examples in Ziegelmeyer et al. upon a slight modification of the algorithm that computes the travel time. Service capacity per unit time, *s*, is assumed to be one.

Suppose that $\omega_{t-1} = [u \quad v]$, and that the designated commuter and *j* commuters (out of *u*) depart from the origin (i.e., arrive at the bottleneck) at time *t*. Then, the designated commuter's travel time of departure at *t*, given ω_{t-1} and *j*, $T(t \mid \omega_{t-1}, j)$, is given by one of the following four cases:

$$T(t \mid \omega_{t-1}, j) = \begin{cases} 1 & \text{if } u = 0, v = 0 \\ v & \text{if } u = 0, v > 0 \\ j+1 & \text{if } u > 0, v = 0 \\ v + j & \text{if } u > 0, v > 0 \end{cases}$$

The designated commuter's expected travel cost of departure at time *t*, given ω_{t-1}, is given by

$$C(t, p \mid \omega_{t-1}) = \sum_{j=0}^{u} f(u, j, h_t) C(t \mid \omega_{t-1}, j),$$

where

$$C(t \mid \omega_{t-1}, j) = \alpha T(t \mid \omega_{t-1}, j) \\ + \beta \max\{0, t^* - (t + T(t \mid \omega_{t-1}, j))\} + \gamma \max\{0, (t + T(t \mid \omega_{t-1}, j)) - t^*\}.$$

With the exception of this change, the algorithm is exactly the same as described in Section 3.4.2.

Table 3.2 presents the SMSE solutions for the cases β=0.25 (top panel) and β=0.5 (bottom panel), where n=4, s=1, α=1, γ=2, $t \in \{t^*-8, \ldots, t^*, \ldots, t^*+8\}$, and t^* is the desired arrival time. For both cases β=0.25 and β=0.5, the side-by-side comparison of the solutions generated by my method (O&R* in Table 2) and by Ziegelmeyer et al. (ZKMD in Table 2) shows no discrepancy between the SMSE and the expected travel costs (*C*). This comparison supports the accuracy of the algorithmic procedure. Based on these computation, when ties are not broken, the expected travel time (*T*) is 1.774 for β=0.25 and 2.129 for β=0.5, respectively.

How does the presence of a tie-breaking rule change the departure pattern, the expected travel cost, and the expected travel time? To answer these questions, for each β I constructed the SMSE (O&R in Table 2), associated with the expected travel cost and expected travel time by the algorithm described in Section 3.4.2. There are two major differences between the solutions due to Zielgemeyer et al. and O&R. First, for both β=0.25 and β=0.5, the expected travel cost computed by Ziegelmeyer et al. exceed the values computed by the O&R's algorithm. This is due to the fact that no tie-breaking rule is implemented in their study. Second, the distribution of the equilibrium probabilities reported in their study is considerably flatter than that of my solution. For each β, the two solutions have almost the same supports, and peak departures (stochastically) occur at the same time, namely, at t^*-3. However, for both β=0.25 and β=0.5 the equilibrium probability assigned to t^*-3 is considerably higher in my solution than in Ziegelmeyer et al.[40]

3.5 Comparison with Vickrey's Continuous Model

This section compares the continuous (Vickrey) and discrete (O&R) bottleneck models in an environment with a finite number of departure times with respect to several indices of travel cost and travel time. For this comparison, assume that α=1, β=0.6, γ=2.4, $t\in$ {1, 2, …, 60}, and t^*=50. In Section 3.5.1, I only change the service capacity, keeping the number of commuters fixed at n=10. In Section 3.5.2, I increase the number of commuters, keeping the service capacity constant at s=1. I do not study the effect of partitioning the strategy space into a small number of time intervals, as do Levinson (2005) and Zou and Levinson (2006). Rather, in both extensions the number of strategies (i.e., departure times) is relatively large.

[40] Zielgemeyer et al. also reported a second, large-scale experiment with n=16 (rather than n=4 as before), s=4, α=1, β=0.5, γ=2, and $t\in$ {t^*-8, …, t^*, …, t^*+8}, where, as before, t^* is the desired arrival time. However, they were unable to compute the symmetric mixed-strategy equilibrium "due to computational problems" that have not been specified.

3.5.1 Changing the Service Capacity

Table 3.3 reports the results of the comparison between the two models for seven bottleneck capacities $s \in \{1/4, 1/3, 1/2, 1, 2, 3, 4\}$. The two models are compared to each other in terms of the travel cost (*C*), total travel cost across the *n* commuters (*TC*), total travel time (*TTT*), total travel time cost (*TTC*), time at which the first commuter departs home (t_F), and time at which the last commuter departs home (t_L). For each capacity value *s*, Vickrey's model is shown to underestimate travel cost, total travel cost, total travel time, and total travel time cost in comparison with the O&R's model. The ratio of the value of the continuous model divided by the value of the discrete model is displayed in the third row of each panel (as a percentage ratio). The percentage ratios are relatively low for large *s* because travel time cost in Vickrey's model becomes negligible when the service capacity *s* becomes large, whereas it remains a major part of travel cost in the discrete model. The percentage ratios increase as the service capacity becomes smaller. But even at s=1/4 they are still considerably smaller than 1.

Vickrey's continuous model also underestimates the ratio of total travel time cost to total travel cost. As shown earlier in Section 3.2, the total travel time cost (*TTC*) in Vickrey's model is 50% of the total travel cost (*TC*). This is not the case in the O&R's model. In my specific example for n=10, the total travel time cost is 72%, 72%, 65%, 59%, 57%, 57%, and 57% of the total travel cost for s= 4, 3, 2, 1, 1/2, 1/3, and 1/4, respectively.

Figures 3.1a, 3.1b, and 3.1c depict the cumulative relative frequency distributions of departure time of the two models for the cases s=4, s=1, and s=1/4, respectively. For all three cases (and for other cases not reported here), the cumulative relative frequency distribution of departure times under the Vickrey's model is to the *right* of the O&R's model. This indicates that Vickrey's continuous model predicts commuters to depart *later* than they should depart under the O&R's discrete model (see also columns 6 and 7 in Table 3.3).

3.5.2 Changing the Number of Commuters

Next, fix s=1 and systematically increase the number of commuters. The three cost parameters α, β, and γ, and the desired arrival time t^* are the same as in Section 3.5.1. Table 3.4 summarizes the numerical results for $n \in \{5, 10, 15, 20, 30, 40, 50\}$. Once again, Vickrey's continuous model underestimates the values of the four indices *C*, *TC*, *TTT*, and *TTC*. It also predicts commuters to depart later than they should under the O&R's model and arrive later to their common destination. Comparison of the percentage ratios shows that the approximation provided by the deterministic equilibrium solution slowly increases in *n*, with these ratios exceeding 95 percent for the indices *C* and *TC* and 91 percent for the indices *TTT* and *TTC* when n=50 (bottom line of Table 3.4).

It is noteworthy that by constructing a function through the percentage ratios in Table 3.4, it is possible to extrapolate for the value of *n* for which the percentage ratio for each of the indices *C* and *TTT* reaches 99%. A cubic spline extrapolation method yields n=119.73 and n=142.44 for the indices *C* and *TTT*, respectively.

The effect of change in the value of *n* is exhibited most clearly in Figure 3.2. This figure displays the cumulative relative frequency distributions of departure times for the Vickrey's and O&R's models for n=30 (Figure 3.2a) and n=50 (Figure 3.2b). Except for the bias to depart later (that may exceed 7 percent of the population at t_F), the Vickrey's continuous model approximates the O&R's discrete model rather accurately.

3.6 Extensions

This section describes two extensions of the discrete bottleneck model. Both are proposed in order to narrow the gap between theory and practice in traffic networks subject to congestion. Importantly, both are based on variants of the same algorithm described in Section 3.4.2. The first extension maintains the assumption that the number of commuters choosing the congestible road is *exogenously* determined but replaces the assumption of a fixed *n* by a random *n* with commonly known distribution.[41] By allowing the choice of an alternative transportation mode that is not subject to congestion (e.g., train), the second extension allows for the number of commuters choosing the congestible road to be *endogenously* determined.

3.6.1 Random Number of Commuters

Throughout the current study, the bottleneck model of O&R was studied under the assumption that the number of commuters, *n*, is fixed and commonly known. However, under most general circumstances the exact value of *n* may not be known with precision. I propose to capture this uncertainty with the assumption that *n* is a random variable whose distribution is commonly known. As in the original model of O&R, the number of commuters who choose departure times on the congestible road is *exogenously* determined.

[41] The literature on transportation has recognized the practical importance of uncertainty. For example, Arnott et al. (1991, 1999) study stochastic environments with respect to road capacity and demand.

When considering games with an uncertain number of commuters, caution should be exercised in distinguishing between the probability distribution of the number of commuters perceived by an outside observer and the probability distribution perceived by a commuter who participates in the game.[42] To illustrate this distinction, consider a discrete bottleneck game in which the number of commuters is either 8 or 12 with equal probability. An observer looking at the game from the outside would conclude that the expected number of commuters is 10. A commuter who actually participates in the game would conclude that she is 1.5 times as likely to interact with 11 than with 7 other commuters. She then updates the conditional probability of interacting with 7 other commuters to 2/5 and the conditional probability of interacting with 11 other commuters to 3/5. From her perspective, the expected number of other commuters she plays with is 7 × 2/5 + 11 × 3/5 = 9.4. Therefore, including herself, she expects 10.4 (rather than 10) commuters to be in the game.

A major advantage of the present computational approach is that the modifications of the algorithm that take care of this distinction are minimal; one only needs to modify the initial vector and transition matrix. Recall that the elements of the initial vector specify probabilities over possible states at time 0. When the number of commuters, *n*, is fixed and known, the value of 1 is assigned to the state in which all the *n*-1 commuters are at the origin. All the other elements in the initial vector take the value of 0. Suppose now that *n* is either n_L with probability $\Pr(n = n_L)$ or n_H with probability $\Pr(n = n_H)$, where $n_L < n_H$, $\Pr(n = n_L) + \Pr(n = n_H) = 1$, and that this distribution is

[42] For the importance of this distinction, see Cooper (1981) and Myerson (1998).

common knowledge. Denote by $\hat{\Omega}$ a set of possible states and by $\hat{\omega}_t \in \hat{\Omega}$ a state at time *t*. Then, the number of possible states $|\hat{\Omega}|$ is computed from

$$|\hat{\Omega}| = \begin{cases} \dfrac{n_H(n_H+1)}{2} + (d-1)\dfrac{(n_H-1)n_H}{2} & \text{if } s<1 \\ \dfrac{n_H(n_H+1)}{2} & \text{if } s \geq 1 \end{cases}.$$

Denote by $\hat{P}(0)$ a $1 \times |\hat{\Omega}|$ initial vector whose elements are probabilities over possible states at time 0. To construct this vector, one must compute probabilities of the following two possible states: (i) all the $n_L - 1$ other commuters are at the origin, and (ii) all the $n_H - 1$ other commuters are at the origin. Denote by $\Pr(n = n_L \mid In)$ the conditional probability of $n = n_L$, given that a commuter is one of the participants in the game. Then,

$$\begin{aligned} \Pr(n = n_L \mid In) &= \frac{\Pr(n = n_L \cap In)}{\Pr(In)} \\ &= \frac{\Pr(n = n_L \cap In)}{\Pr(n = n_L \cap In) + \Pr(n = n_H \cap In)} \\ &= \frac{\Pr(n = n_L)\Pr(In \mid n = n_L)}{\Pr(n = n_L)\Pr(In \mid n = n_L) + \Pr(n = n_H)\Pr(In \mid n = n_H)}. \end{aligned}$$

Assuming that each commuter is equally likely to participate in the game, a commuter is (n_H / n_L) times as likely to be in the game if $n = n_H$ than if $n = n_L$. In other words, $\Pr(In \mid n = n_H) = (n_H / n_L)\Pr(In \mid n = n_L)$. Then,

$$\begin{aligned} \Pr(n = n_L \mid In) &= \frac{\Pr(n = n_L)\Pr(In \mid n = n_L)}{\Pr(n = n_L)\Pr(In \mid n = n_L) + (n_H / n_L)\Pr(n = n_H)\Pr(In \mid n = n_L)} \\ &= \frac{\Pr(n = n_L)}{\Pr(n = n_L) + (n_H / n_L)\Pr(n = n_H)}. \end{aligned}$$

Similarly, the conditional probability of $n = n_H$, given that a commuter is one of the participants in the game, $\Pr(n = n_H \mid In)$, is computed from

$$\Pr(n = n_H \mid In) = \frac{\Pr(n = n_H)}{\Pr(n = n_H) + (n_L / n_H)\Pr(n = n_L)}.$$

In the initial vector $\hat{P}(0)$, $\Pr(n = n_L \mid In)$ is assigned to the state in which all the $n_L - 1$ commuters are at the origin, while $\Pr(n = n_H \mid In)$ is assigned to the state in which all the $n_H - 1$ commuters are at the origin. A $|\hat{\Omega}| \times |\hat{\Omega}|$ transition matrix whose elements are probabilities of transition from one state to another must be defined. Then, the same algorithm described in Section 3.4.2 applies with these new initial vector and transition matrix.

Table 3.5 exhibits the SMSE for the case *n*=10 and for the case where *n* is either 8 with probability 0.6 or 12 with probability 0.4. Each commuter in both cases expects the same number of other commuters; any commuter in the former case knows that there are 9 other commuters whereas for any commuter who participates in the latter case the expected number of other commuters is $7 \times \Pr(n = 8 \mid In) + 11 \times \Pr(n = 12 \mid In) = 9$. For both cases, s=1, α=1, β=0.6, γ=2.4, $t \in \{1, \ldots, 60\}$, and t^*=50. Comparison of the two cumulative probability distributions of departure time in Table 3.5 shows that although each commuter in these two cases shares the same expectation about the number of other commuters, the uncertainty about the number of commuters induces earlier departure times and results in a higher travel cost.

3.6.2 Augmented Strategy Space

The previous sections have considered the case where all the n commuters choose their time of departure on the same congestible road. Consequently, in the original model of O&R and in its extension in Section 3.6.1, commuters are not given the opportunity to choose an alternative mode of transportation (such as a commuter train) that is not subject to congestion. The present section relaxes this assumption by incorporating the option of an alternative transportation mode into the strategy space. To do so, suppose that each commuter has to choose a departure time from the augmented strategy space, $\{1,...,t_{\max}\} \cup \{alt\}$, in which alt denotes the decision to use an alternative transportation mode. Let p_{alt} and C_{alt} denote the associated probability and cost, respectively.[43] Commuters deciding to travel on the congestible road choose their departure times without knowledge of the group size. Therefore, in contrast to the model in Section 3.4, the number of commuters who choose traveling on the congestible road is *endogenously* determined.

The algorithm that computes the SMSE with the augmented strategy space is similar to the one developed in Section 3.4.2. Denote by $C^{(i)}$ the expected travel cost when i commuters ($1 \le i \le n$) use the congestible road.[44] First, assuming that all the n commuters use the road, i.e., assuming that $p_{alt} = 0$, derive the SMSE $p_1,..., p_{t_{\max}}$ and compute the associated expected travel cost, $C^{(n)}$. Then, one of the following three cases takes place.

[43] The discussion here assumes that C_{alt} is a fixed value. C_{alt} can also be assumed to be a function of the number of commuters who choose an alternative transportation mode.

[44] If only one commuter uses the road, this commuter can achieve the smallest total travel cost. Thus, $C^{(1)} = \alpha d$ if $s<1$ and $C^{(1)} = \alpha$ if $s \ge 1$.

a. If $C^{(n)} < C_{alt}$ then none of the n commuters is willing to use the alternative transportation mode. Keep $p_{alt} = 0$. Then, the probabilities $p_{alt}, p_1 ..., p_{t_{\max}}$ are the SMSE and the equilibrium expected travel cost is $C^{(n)}$.

b. If $C^{(1)} \le C_{alt} \le C^{(n)}$, then using the congestible road is no longer a dominant strategy. Thus, each of the n commuters stochastically chooses the alternative transportation mode, i.e., $p_{alt} > 0$. Given C_{alt}, derive the associated probabilities $p_1, ..., p_{t_{\max}}$ and then calculate $p_{alt} = 1 - \sum_{t=1}^{t_{\max}} p_t$. The probabilities $p_{alt}, p_1 ..., p_{t_{\max}}$ constitute the unique SMSE with the equilibrium cost C_{alt}.

c. If $C_{alt} < C^{(1)}$, then it is a dominant strategy to use the alternative transportation mode. Therefore, the probabilities $p_{alt} = 1$ and $p_1 = p_2 = ... = p_{t_{\max}} = 0$ constitute the unique (degenerate) SMSE with the equilibrium cost C_{alt}.

Table 3.6 presents the SMSE for three cases of different costs of using the alternative transportation mode. As before, s=1, α=1, β=0.6, γ=2.4, $t \in \{1, …, 60\}$, and t^*=50. When $C_{alt} = 7$, it is too expensive for any commuter to use the alternative transportation mode. Thus, all the commuters use the congestible road (i.e., $p_{alt} = 0$), which results in the travel cost of 5.858 (see case s=1 in Table 3.3). As the value of C_{alt} decreases, each commuter increases her probability of choosing the alternative transportation mode (i.e., p_{alt}= 0.235 and 0.681 for C_{alt}= 5 and 3, respectively). Table 3.6 further shows that as C_{alt} decreases, commuters deciding to travel on the congestible road choose later departure times.

A close inspection of Tables 3.5 and 3.6, as well as Figures 3.1b, 3.1c, 3.2a, and 3.2b, shows oscillatory patterns in the equilibrium probabilities. These are not unique to the present model (see, e.g., Rapoport et al., 2004 and Seale et al., 2005 for similar oscillatory patterns in single-server queues with finite populations and endogenous arrival times).

3.7 Conclusion

It is common practice in economics, transportation science, and related disciplines to use continuous models in analyzing behavior that is essentially discrete. The strategy space, number of agents, or both, often are assumed to be continuous in order to gain analytical tractability, where in fact they are discrete. When a continuum of agents is assumed, one may derive closed-form solutions that, in turn, allow for the study of comparative statics. When this assumption is dropped, and the congestion model assumes any finite number of commuters, departure times can be computed exactly and approximations by the continuous model are no longer required. However, there is a trade-off: numerical and sometimes brute-force computations are substituted for the elegance and simplicity of the closed-form solutions. The numerical computations in this chapter indicate the presence of systematic errors, which for a small *n* are substantial, when models assuming a continuum of commuters are used to account for departure times in traffic networks in which the number of participants is relatively small.[45] They further suggest that as the population size grows the difference between the continuous and discrete models of traffic congestion diminishes and can safely be ignored.

[45] See, e.g, the road pricing experiments reported by Schneider and Weimann (2004) that were designed to test the continuous bottleneck congestion model of Arnott et al. (1990, 1993).

No explanation has been offered for why the equilibrium solution presented in this chapter yields appreciably higher expected travel costs (*C*) than the deterministic equilibrium of Arnott et al. My model differs from theirs by having discrete time periods and, more importantly, a finite number of commuters. It yields a mixed-strategy equilibrium where a commuter may not wish to deviate if she knows the strategies of the other n-1 commuters but may benefit from a change in departure time if she knows their actual choices. This is not the case when the equilibrium is deterministic. Under the stochastic equilibrium, two or more commuters may fail to coordinate their departure times whereas under the deterministic equilibrium they may not. This suggests that lack of coordination in departure time decisions is one reason for the higher travel costs. The present results are consistent with the results of Rapoport et al. (in press) who reported higher travel costs under mixed-strategy than pure-strategy equilibria in a study of route choice (rather than departure time) in traffic networks. A second reason for the higher travel costs is that under the assumption of a finite number of commuters, each commuter must spend a finite length of time passing through the bottleneck. This would increase her travel time and thereby her expected travel cost. In contrast, under Vickrey's formulation she passes through the bottleneck instantaneously once all the commuters who preceded her in a queue clear.

The SMSE solution is the natural choice under the assumption of identical commuters. But, clearly, commuters are in general not identical. They differ from one another, among other dimensions, in official work hours (Wilson, 1988), work flexibility (Emmerink and van Beek, 1995), unit schedule delay costs (Small, 1982), and cost of waiting in the queue. One would expect heterogeneity of the commuters to be conducive towards the existence of asymmetric pure-strategy equilibria due to self-selection of departure times by different segments of the population (e.g., Daniel, 2001; Xin and Levinson, 2007). Therefore, it is important to extend the theoretical analysis of endogenous departure times to encompass the case of heterogeneous commuters.

There are several directions that could be pursued in the future. The first direction is to experimentally test the discrete bottleneck model with a large number of commuters. In this chapter, a numerical algorithm has been constructed that overcomes computational problems which restricted the previous research to models with a small number of commuters. This direction will complement the contributions of Ziegelmeyer et al. (2008) by experimentally investigating the impact of a larger size of the population on the commuters' ability of a tacit coordination in a decentralized environment. The second direction is to extend the discrete model by incorporating uncertainty about the service capacity of a bottleneck. In the current study, I have assumed that the service capacity is fixed and commonly known. However, this assumption is restrictive because service facilities that result in a bottleneck are prone to congestion due to external random factors such as bad weather and accident. This direction is as important as the first extension of Section 3.6. The third direction is another extension of the present model by introducing "business hours" during which a bottleneck opens. The model in this chapter assumes that service facilities causing a bottleneck always open. Commuters may alter their departure patterns by responding to changes in business hours.

APPENDIX A: FIGURES

LIST OF FIGURES

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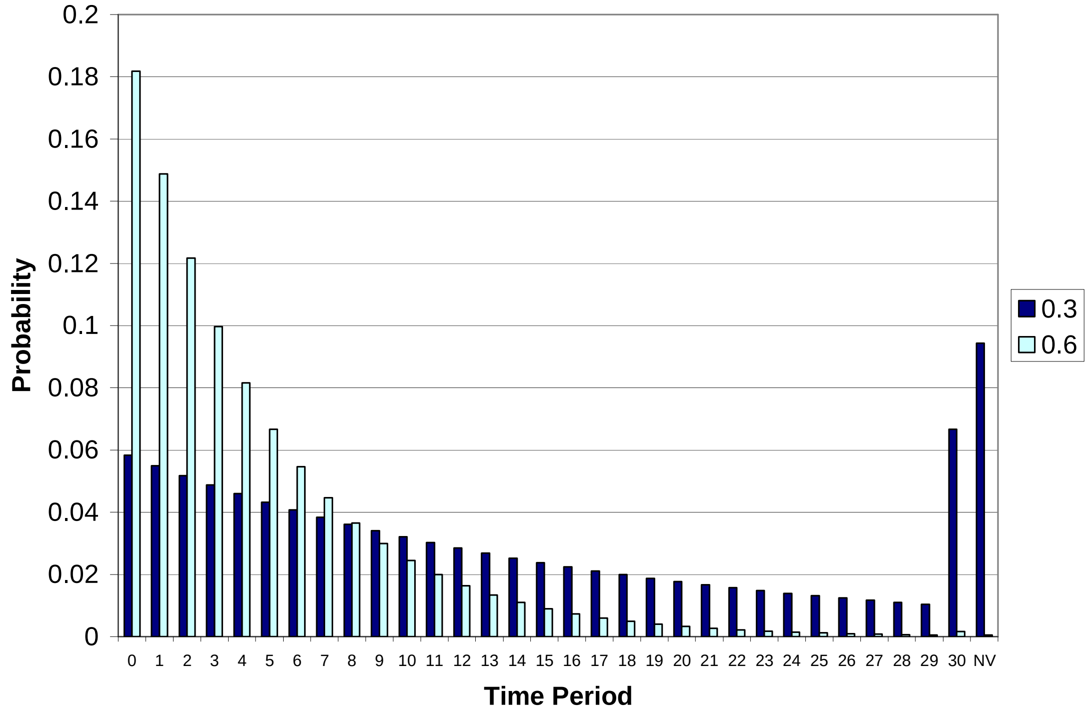
FIGURE 1.1: Equilibrium probability distributions of termination time (upper panel) and their cumulative probability distributions (lower panel) when n=3 and T=30.



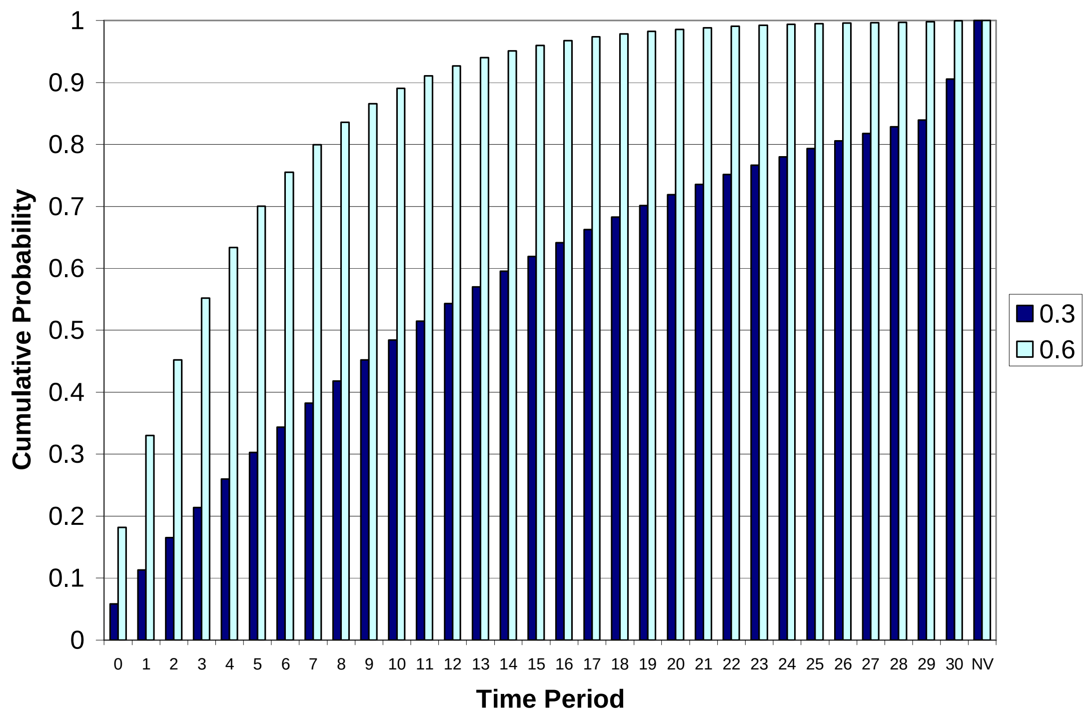


FIGURE 1.2: Observed cumulative relative frequency distributions of termination time for Conditions δ=0.3 and δ=0.6.

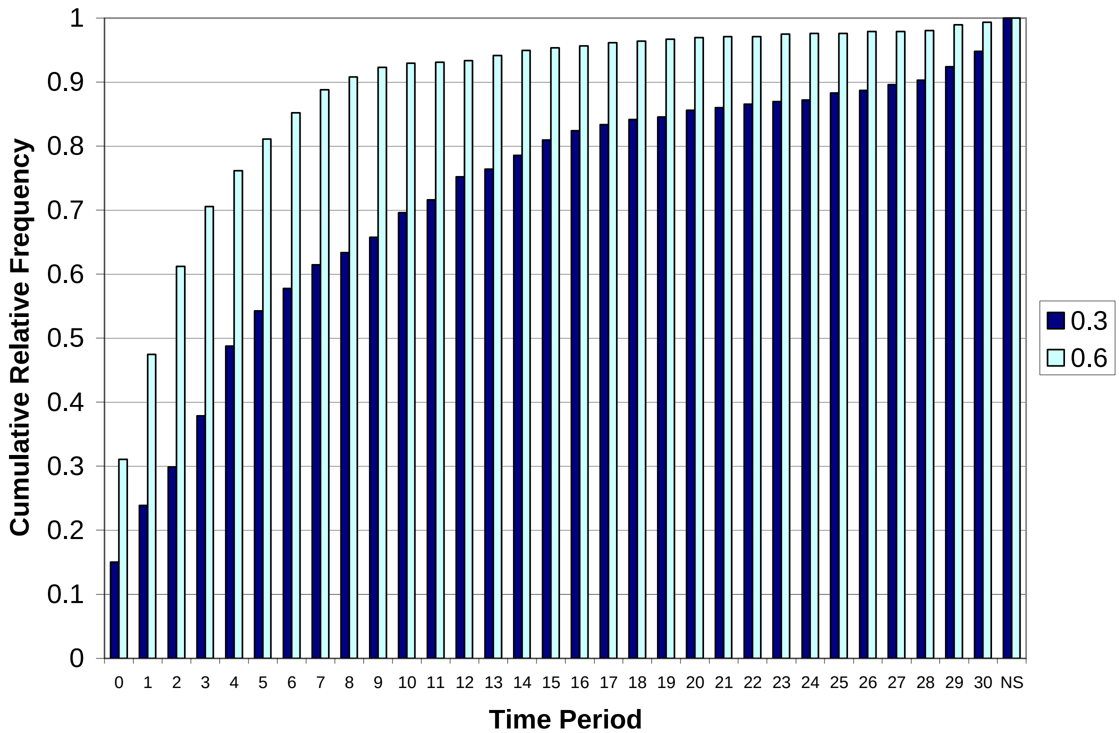


FIGURE 1.3: Mean termination time across the three groups over 50 rounds (left column: Condition δ=0.3, right column: Condition δ=0.6)

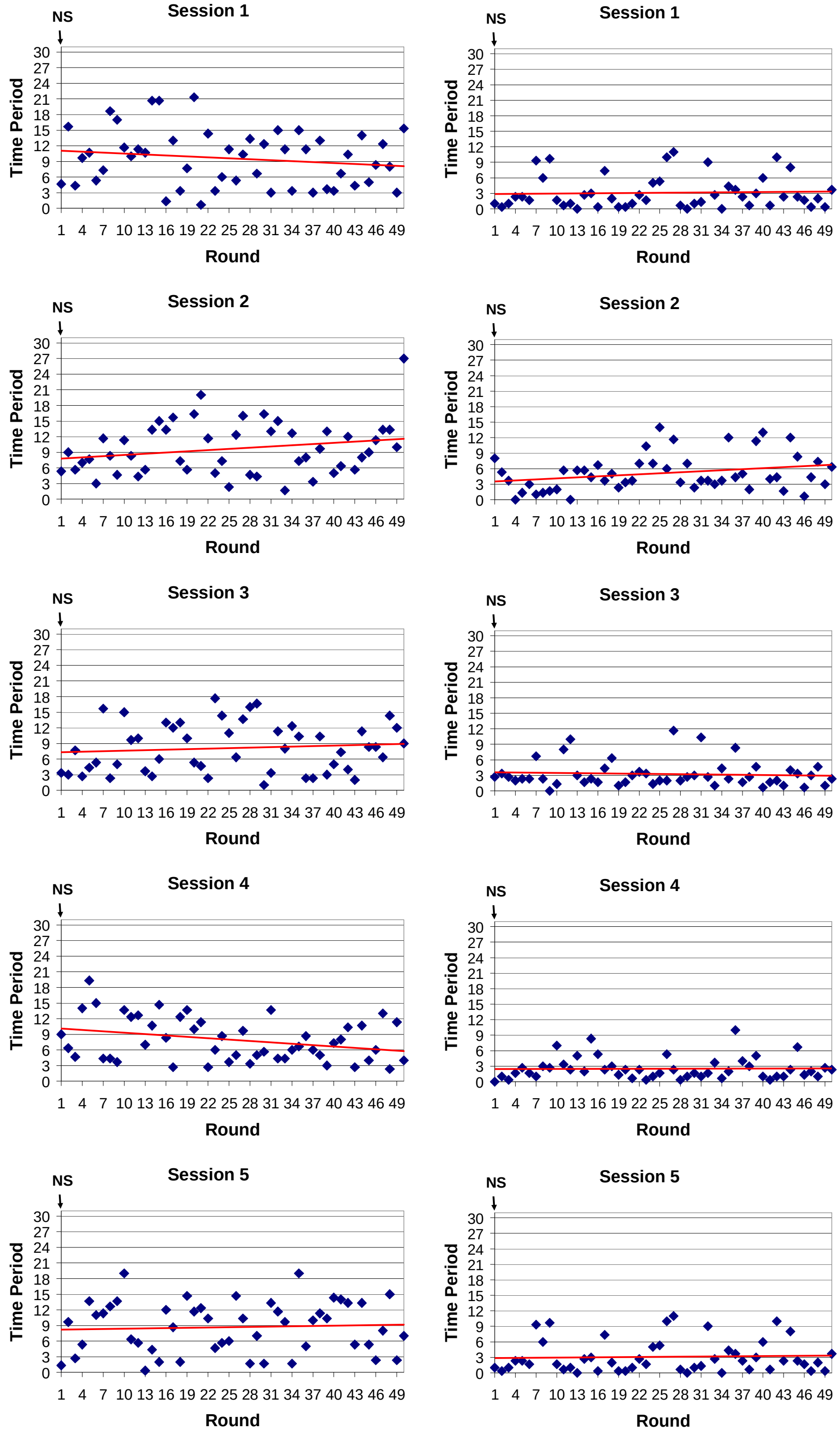


FIGURE 1.4: Predicted and observed relative frequency distributions of termination time (at t=2 or later) in Condition δ=0.3

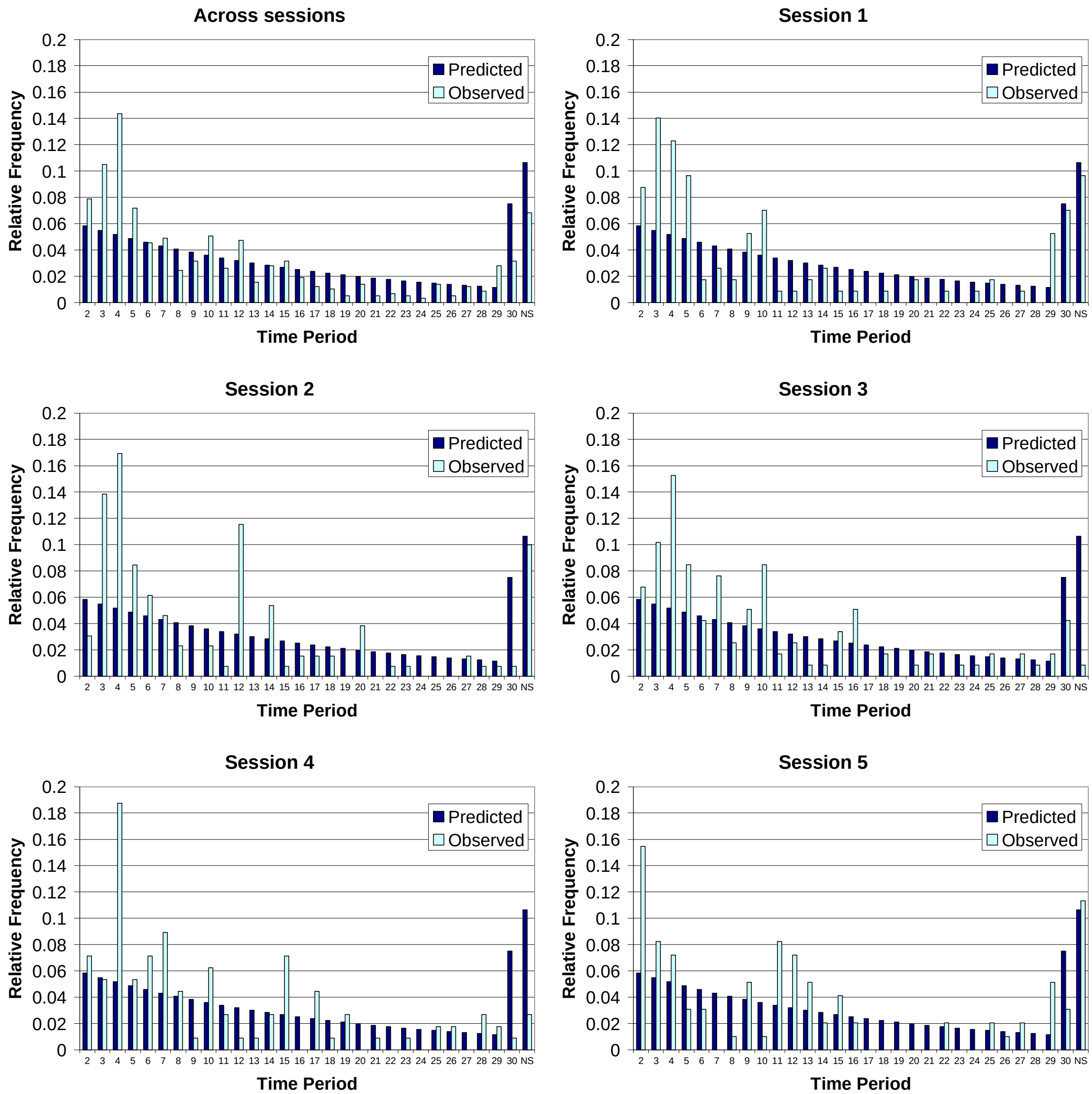


FIGURE 1.5: Predicted and observed relative frequency distributions of termination time (at t=2 or later) in Condition δ=0.6

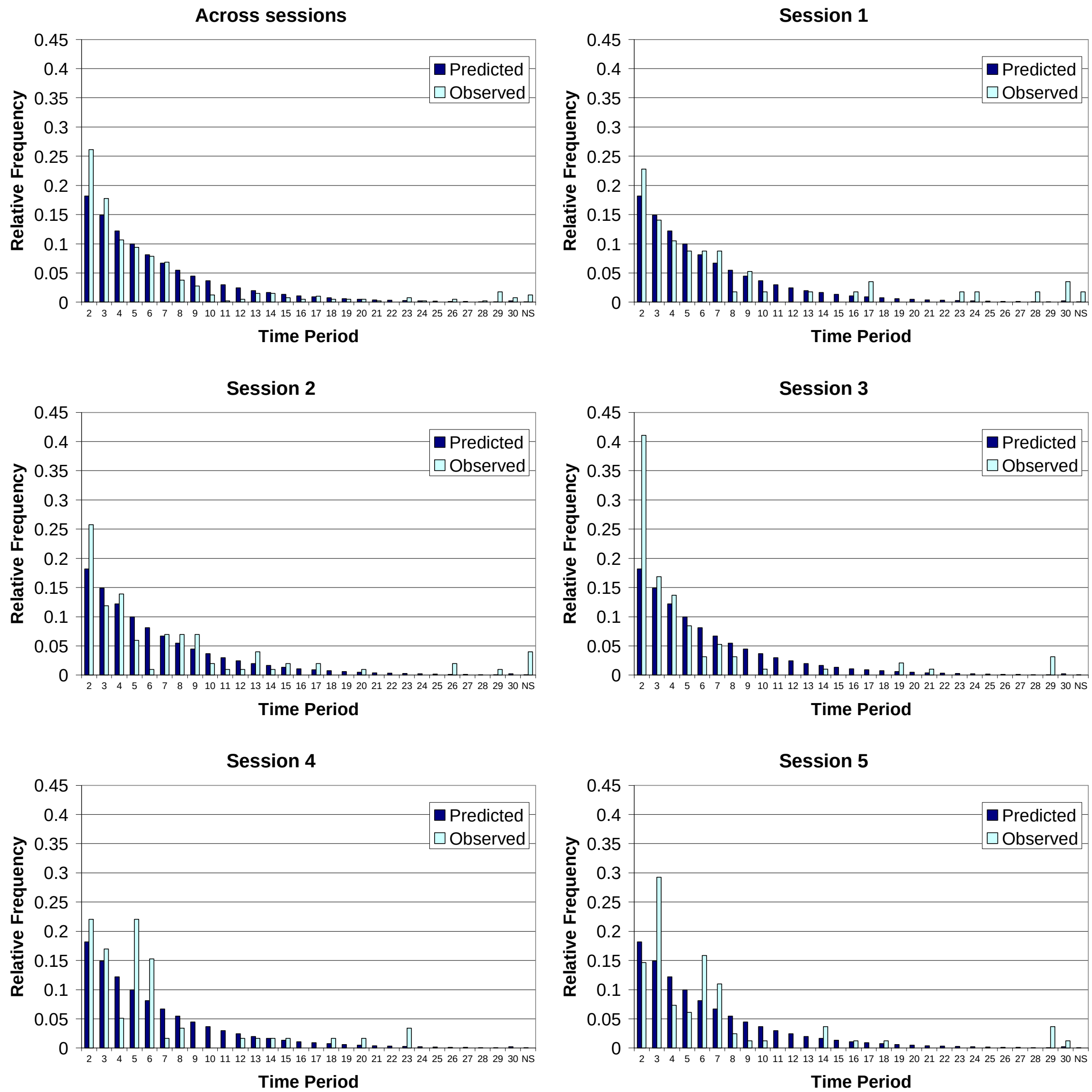


FIGURE 1.6: Frequency distributions of individual number of stopping decisions (upper panel: Condition δ=0.3, lower panel: Condition δ=0.6)

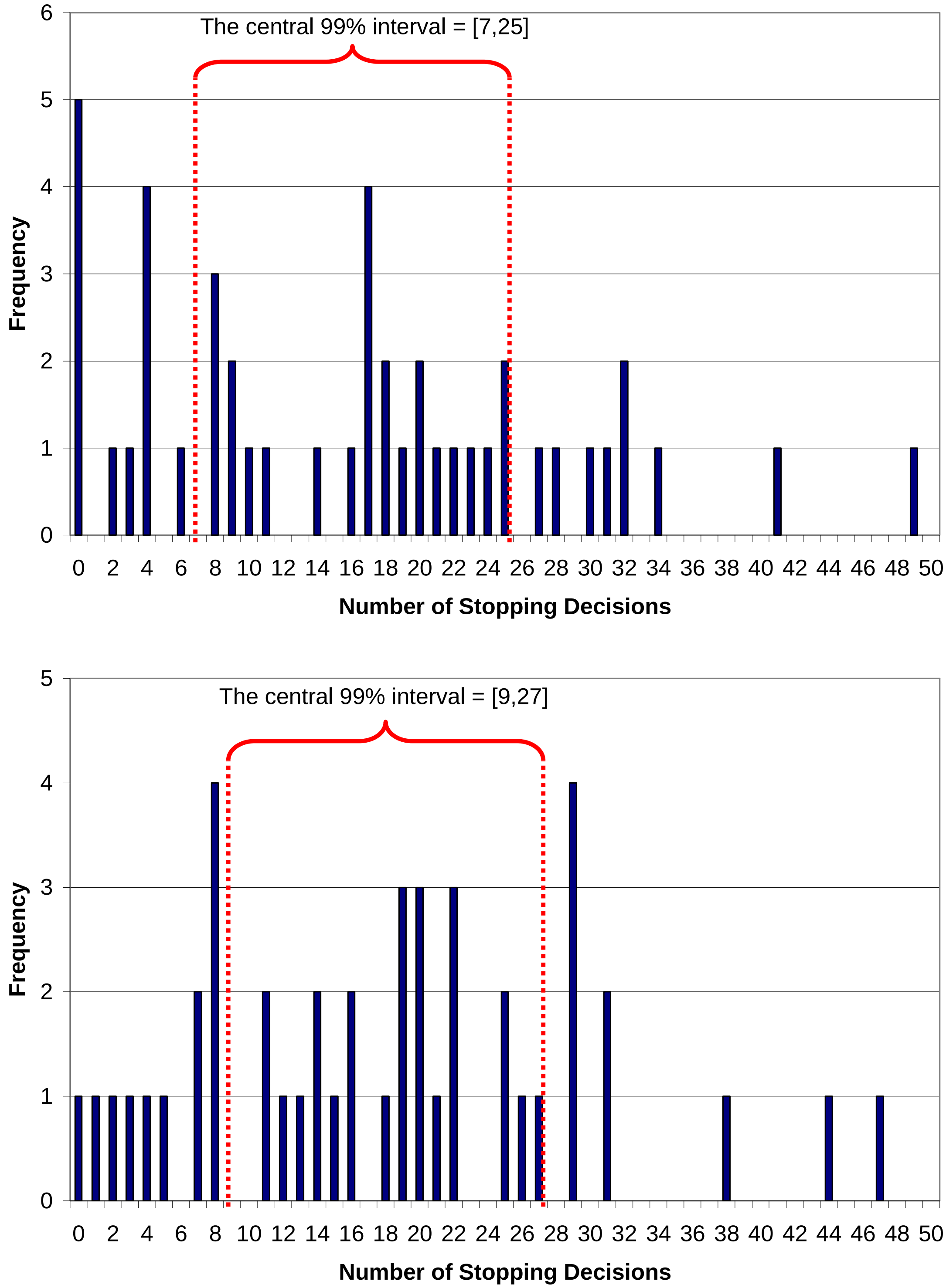


FIGURE 1.7: Cumulative relative frequency distributions of individual number of stopping decisions

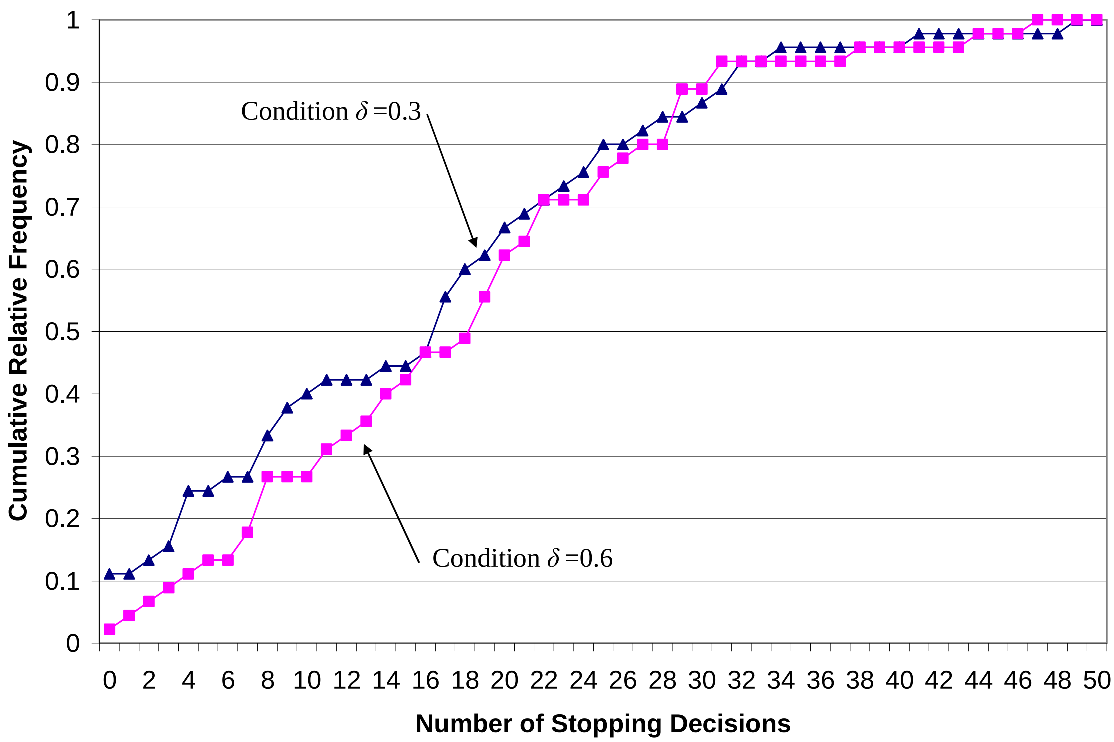
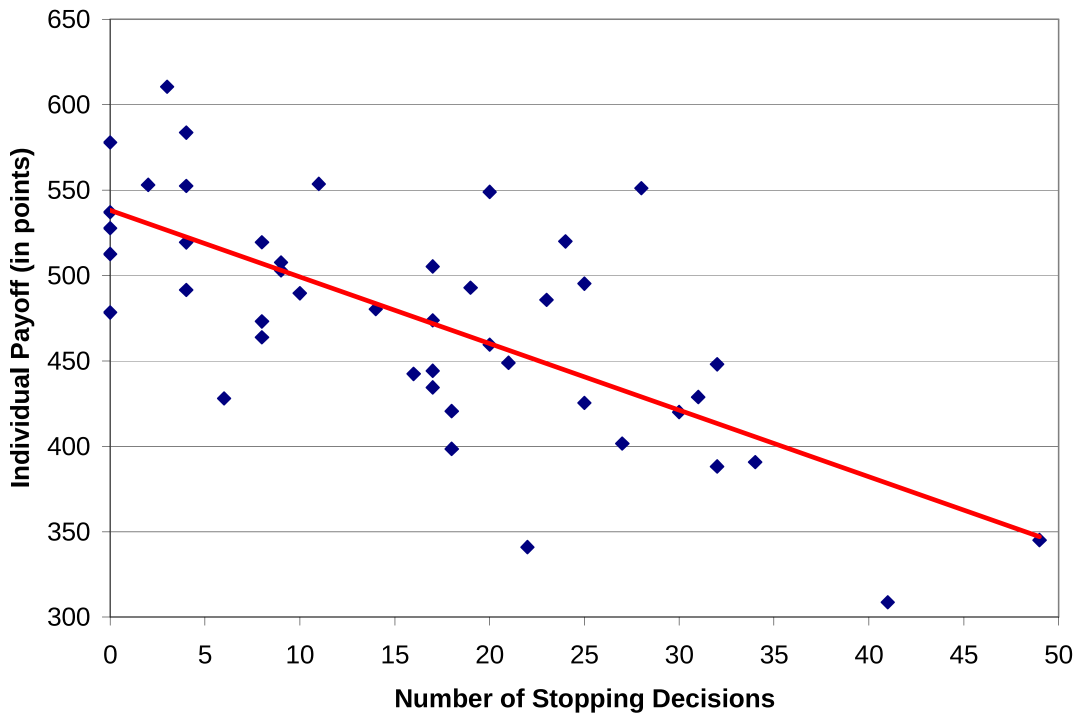


FIGURE 1.8: Individual payoff as a function of individual number of stopping decisions (upper panel: Condition δ=0.3, lower panel: Condition δ=0.6)



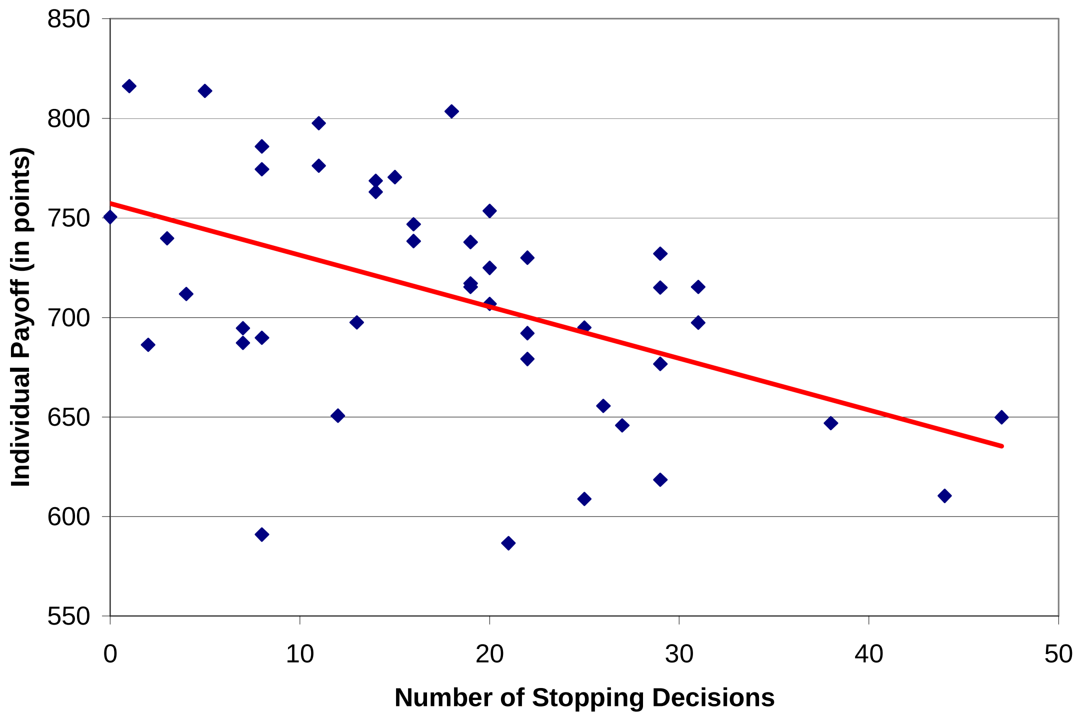


FIGURE 2.1: Symmetric mixed-strategy Nash equilibrium solutions for the distributions of bids with n=50 and B={1,...,50} (upper panel: LUBA; lower panel: HUBA)

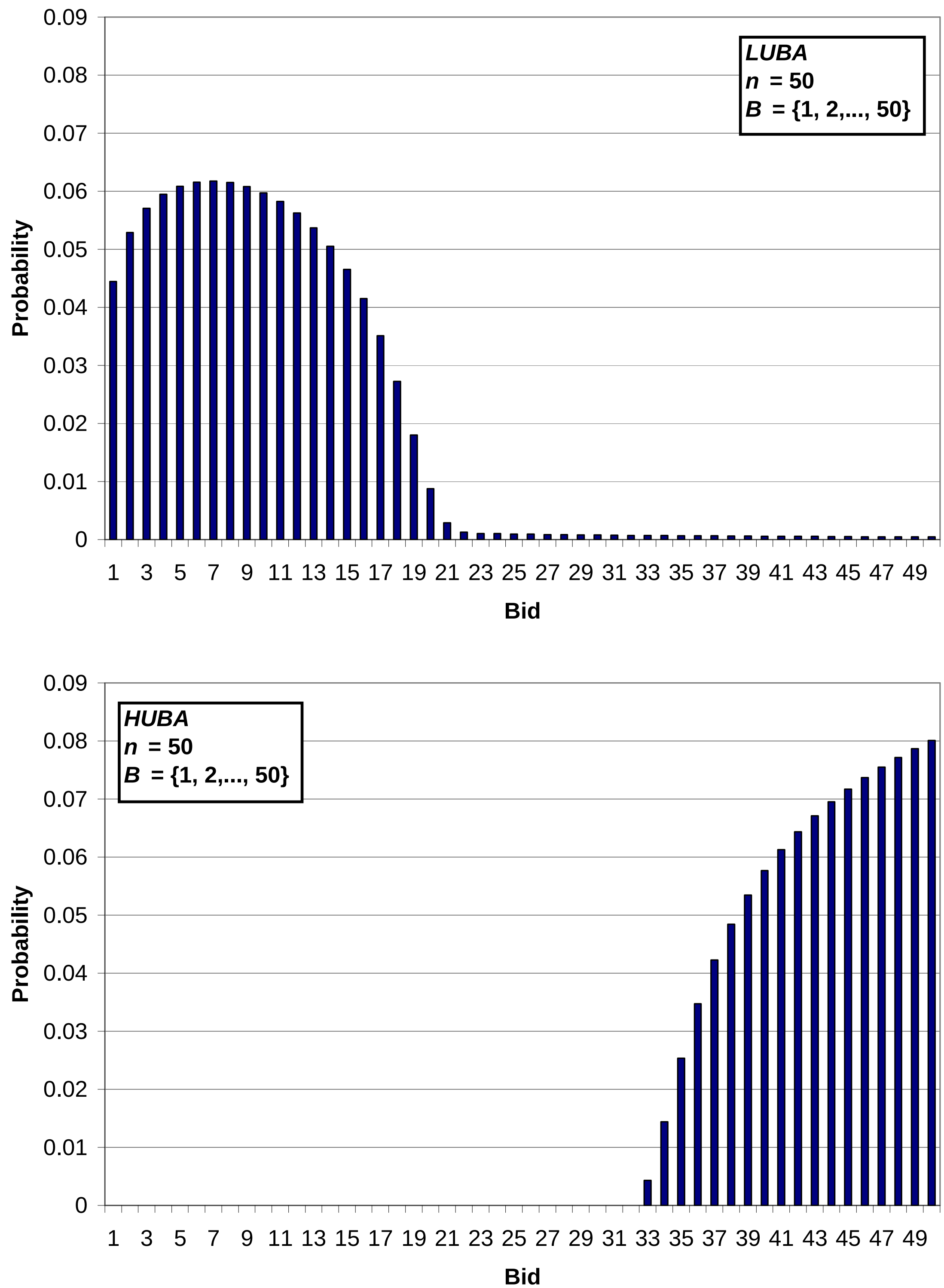
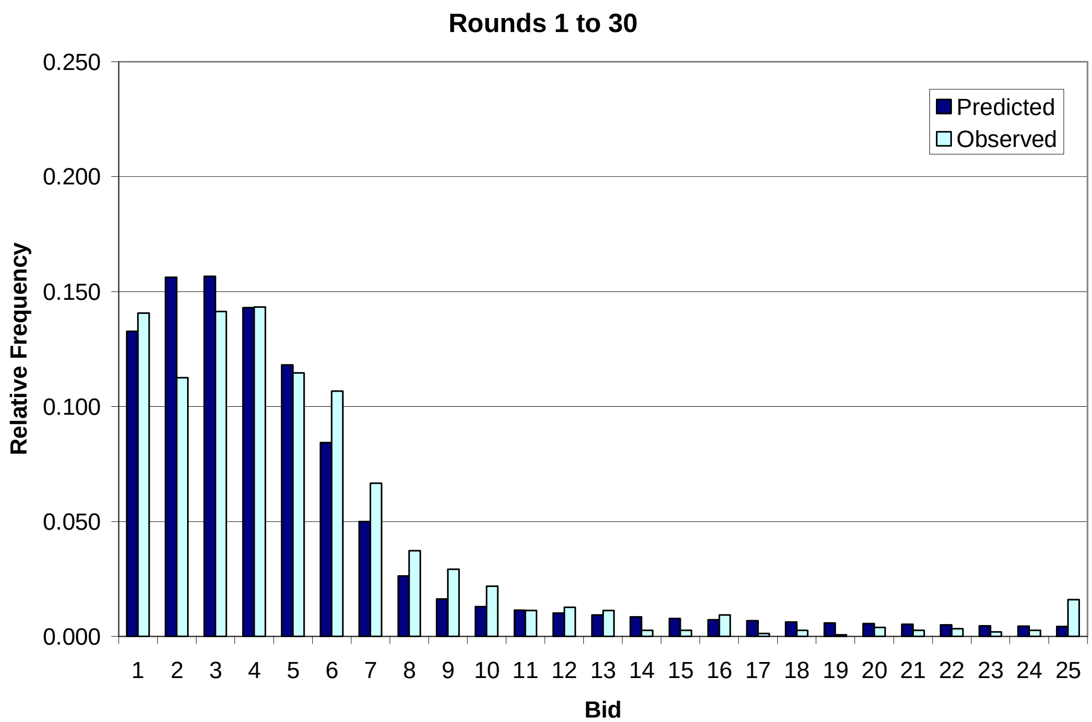


FIGURE 2.2: Predicted probabilities and observed relative frequency distributions of bids on the aggregate level for Condition LUBA



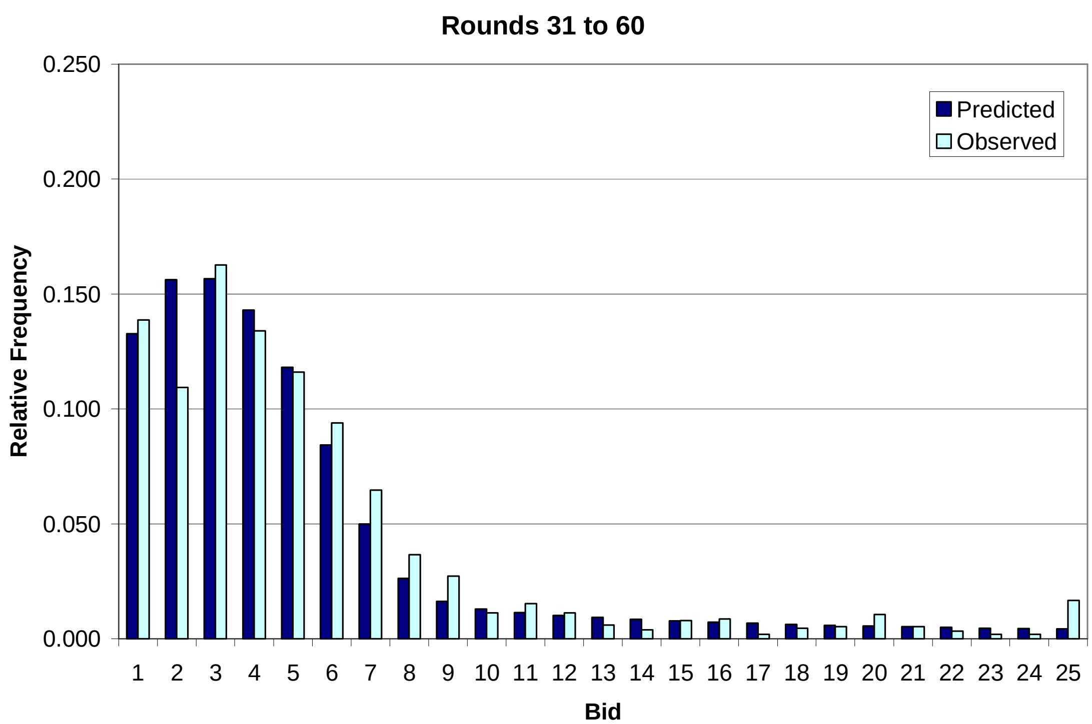
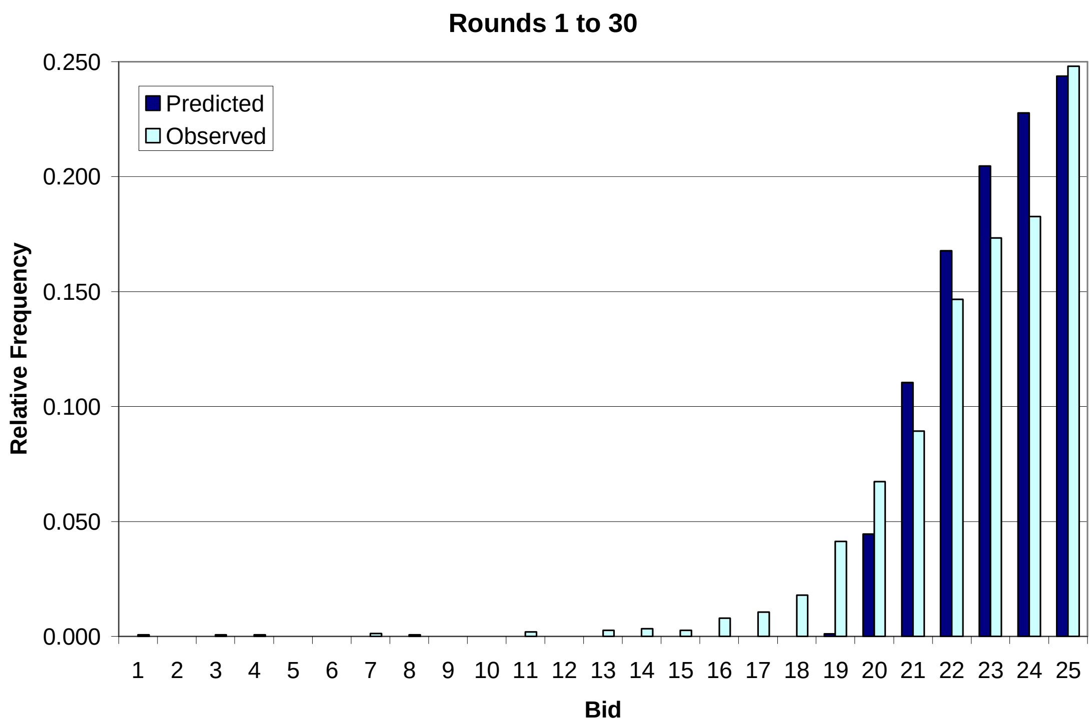


FIGURE 2.3: Predicted probabilities and observed relative frequency distributions of bids on the aggregate level for Condition HUBA



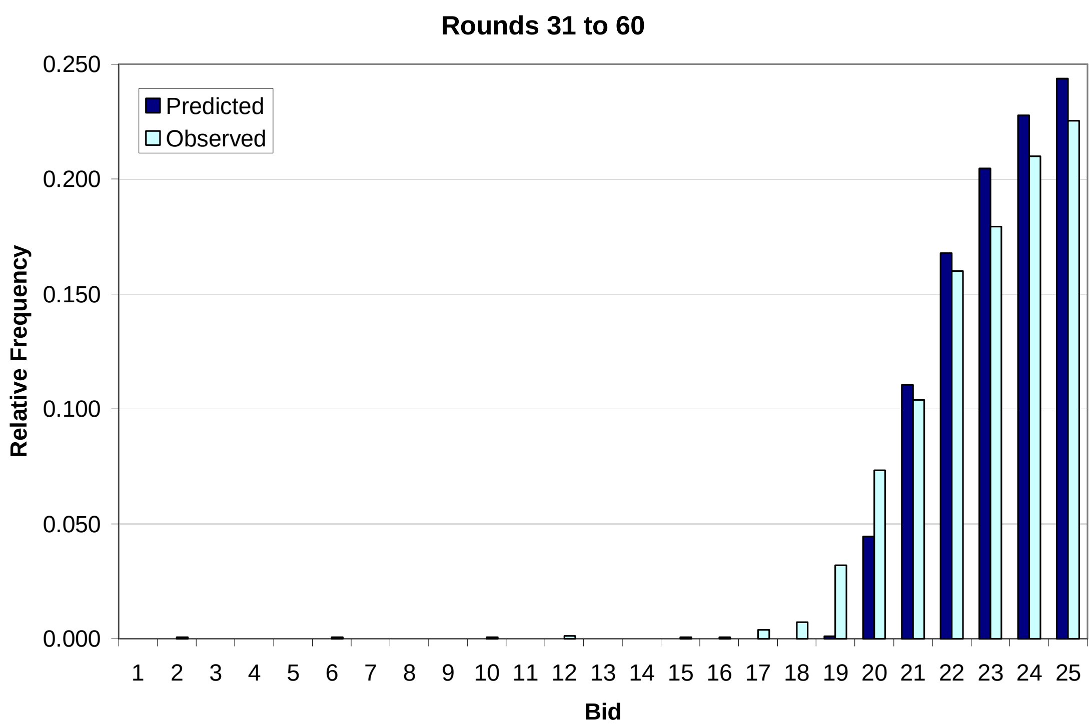


FIGURE 2.4: Observed relative frequency (y-axis) of bids (x-axis) of each of the fifty subjects in rounds 1 to 30 of Condition LUBA

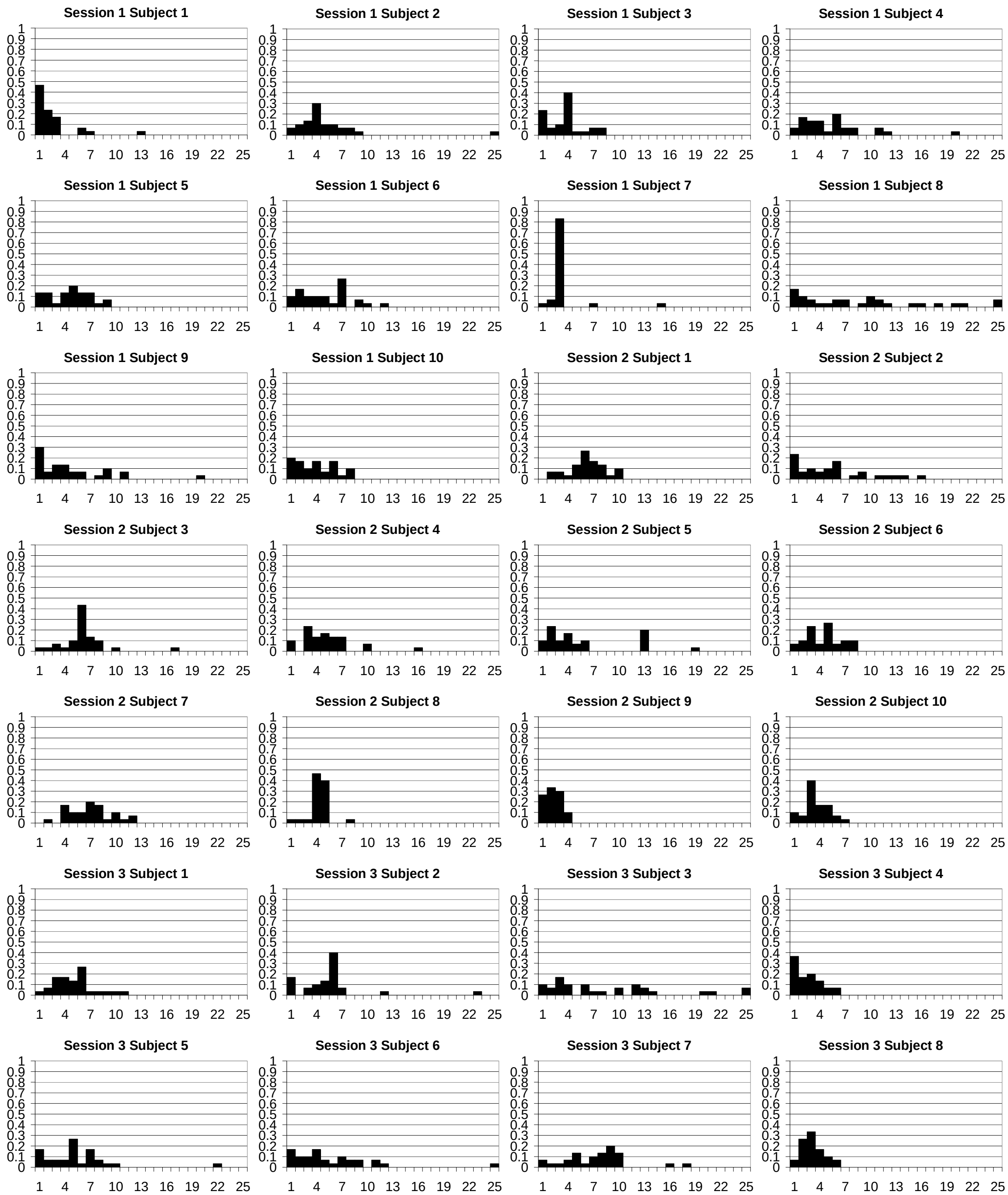


FIGURE 2.4 (Continued): Observed relative frequency (y-axis) of bids (x-axis) of each of the fifty subjects in rounds 1 to 30 of Condition LUBA

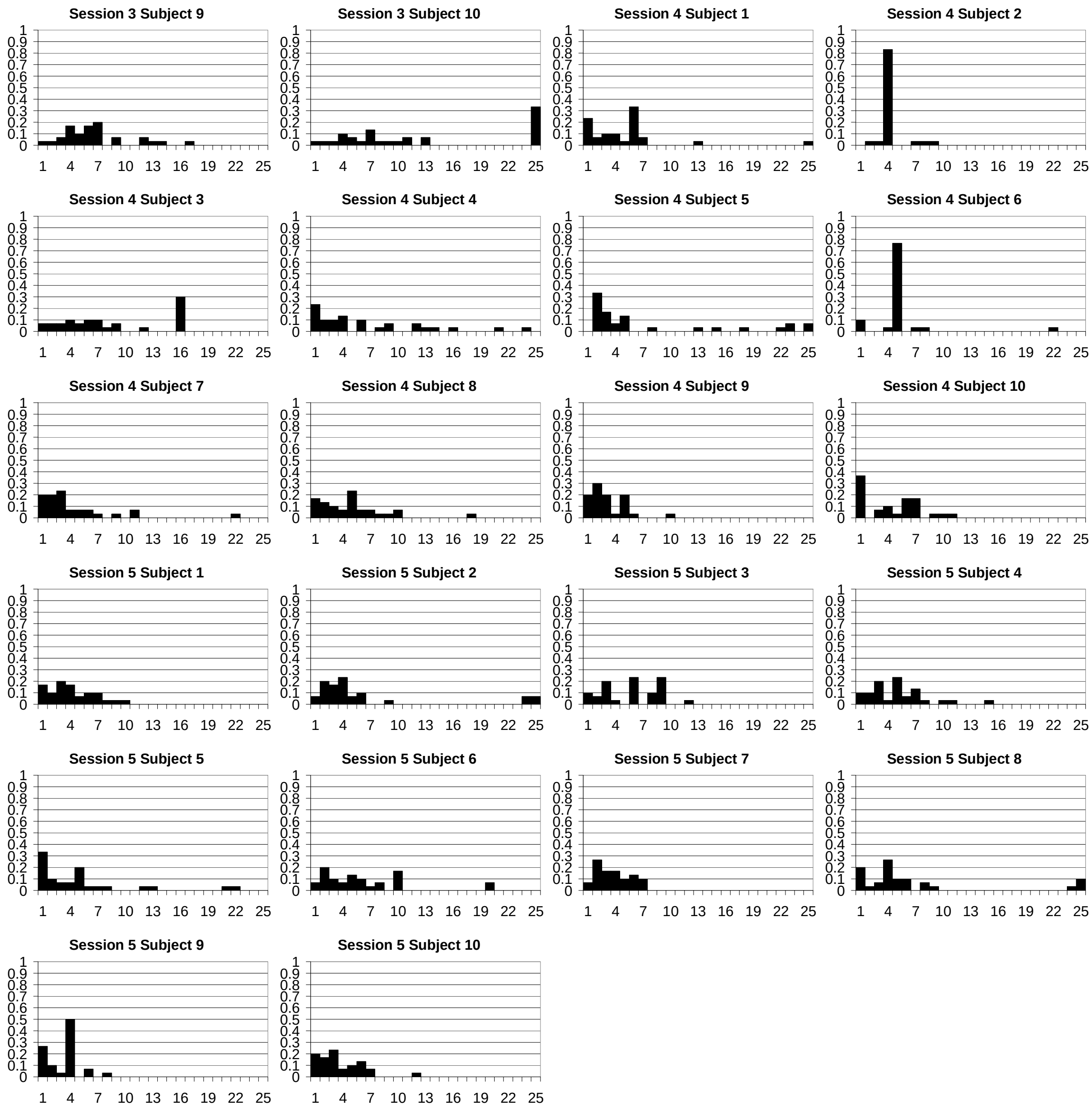


FIGURE 2.5: Observed relative frequency (y-axis) of bids (x-axis) of each of the fifty subjects in rounds 31 to 60 of Condition LUBA

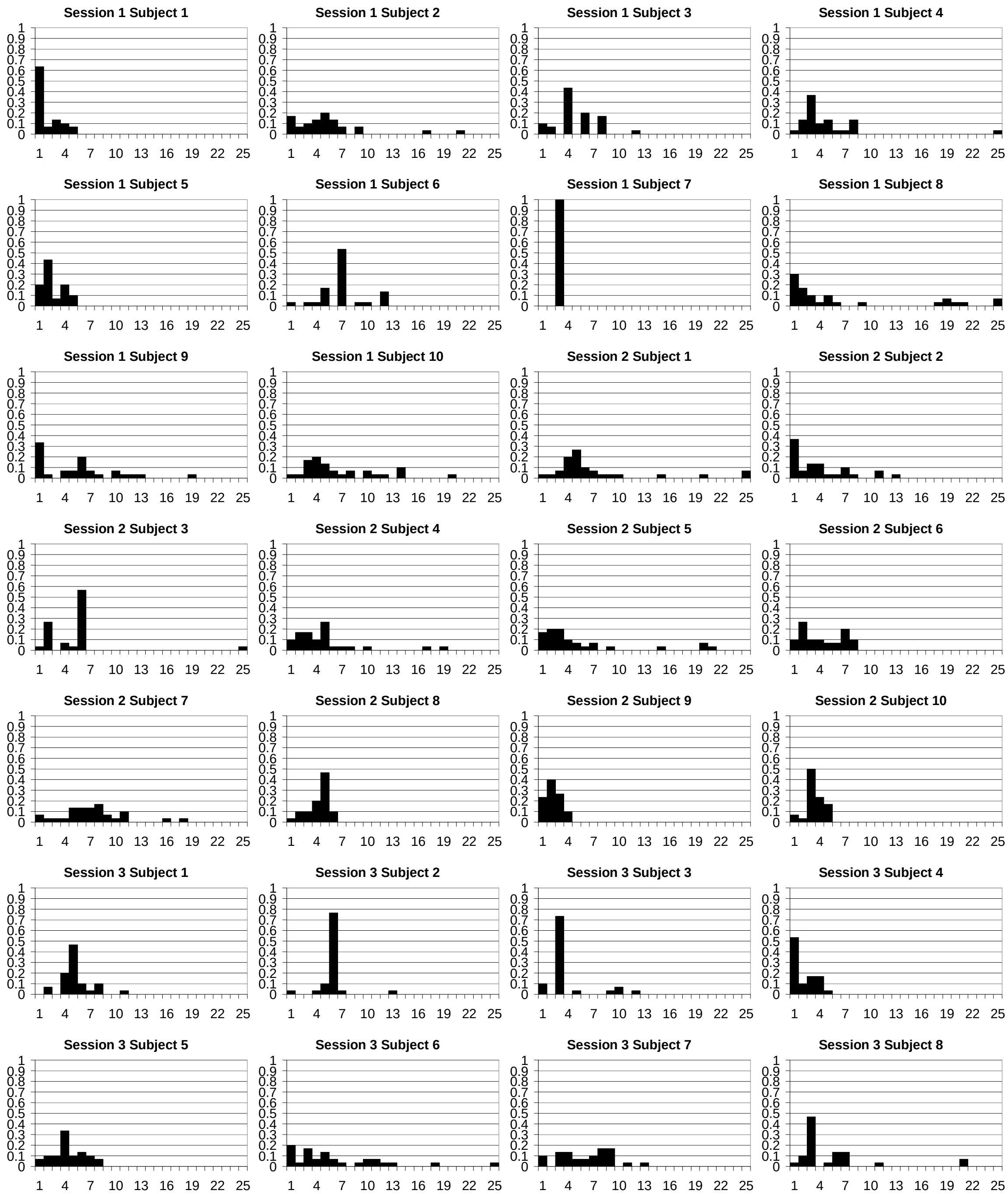


FIGURE 2.5 (Continued): Observed relative frequency (y-axis) of bids (x-axis) of each of the fifty subjects in rounds 31 to 60 of Condition LUBA

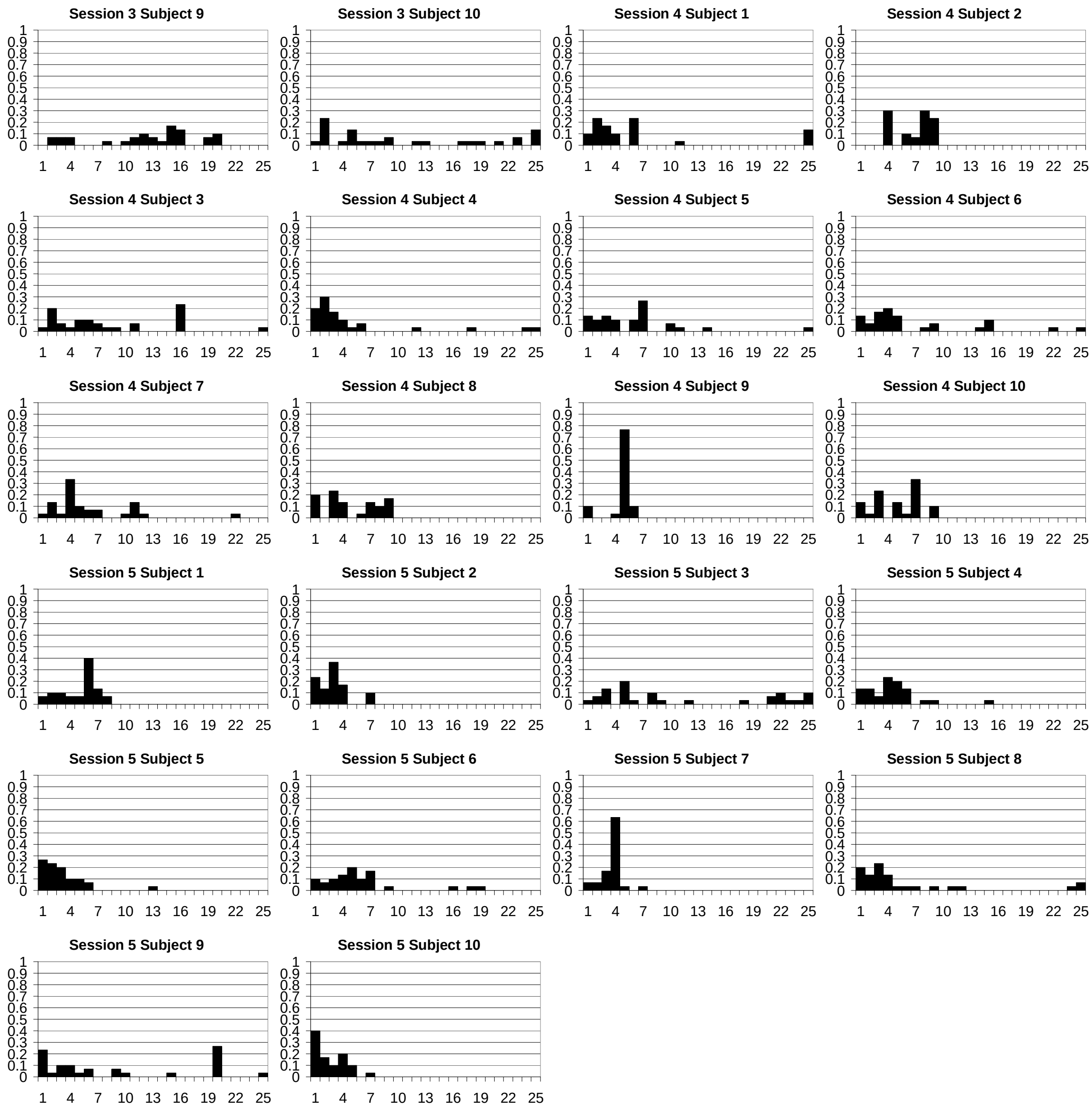


FIGURE 2.6: Observed relative frequency (y-axis) of bids (x-axis) of each of the fifty subjects in rounds 1 to 30 of Condition HUBA

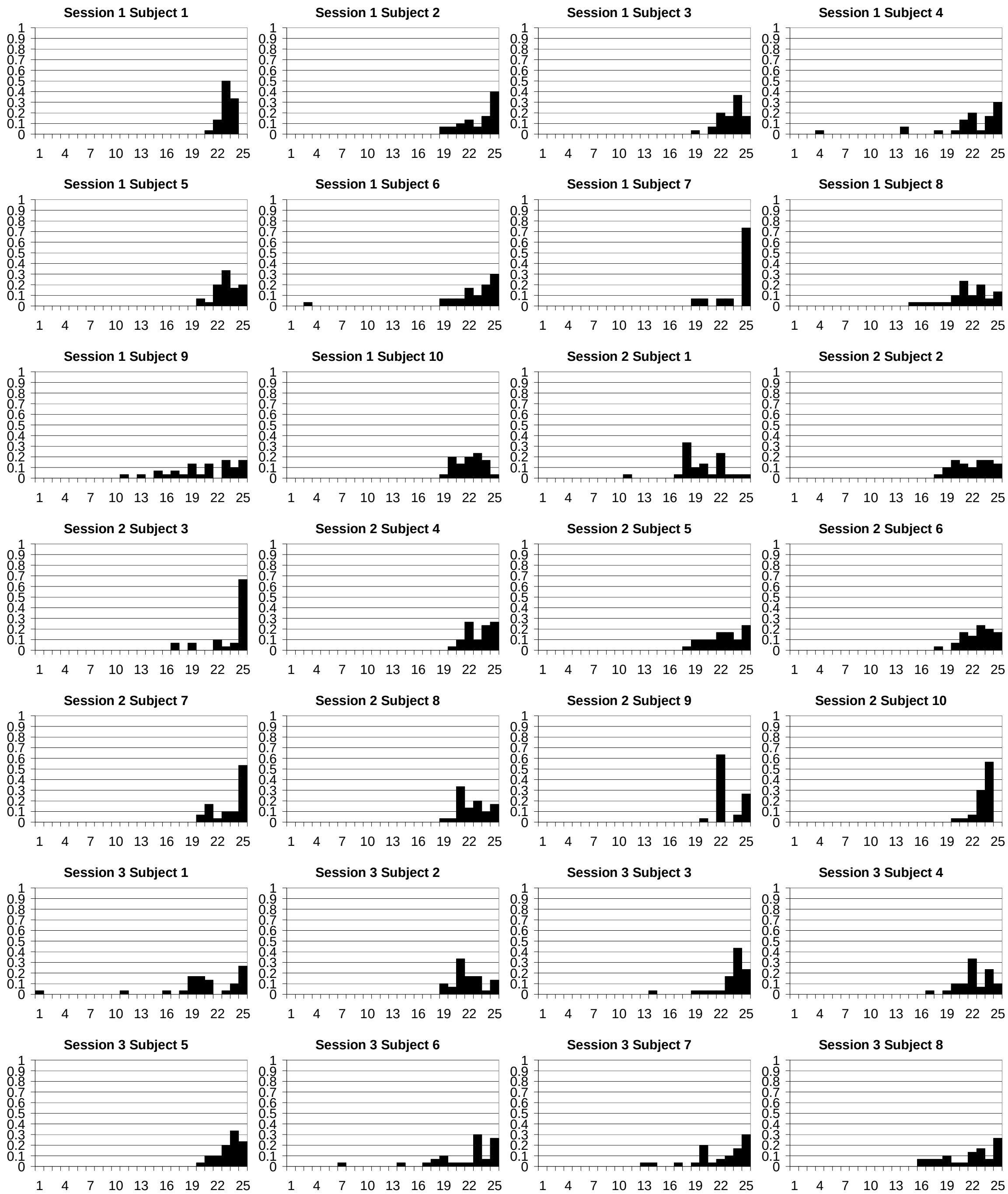


FIGURE 2.6 (Continued): Observed elative frequency (y-axis) of bids (x-axis) of each of the fifty subjects in rounds 1 to 30 of Condition HUBA

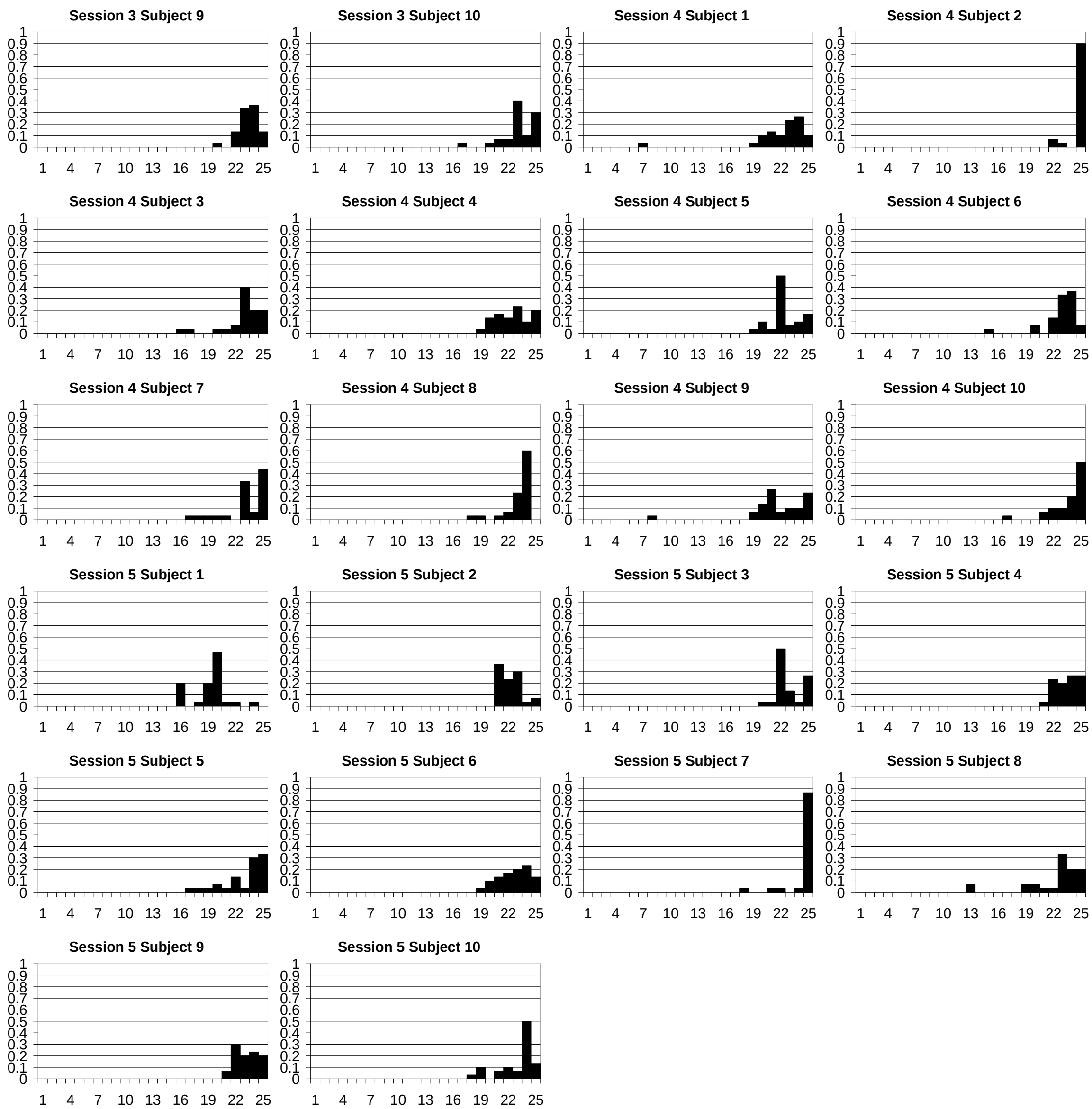


FIGURE 2.7: Observed relative frequency (y-axis) of bids (x-axis) of each of the fifty subjects in rounds 31 to 60 of Condition HUBA

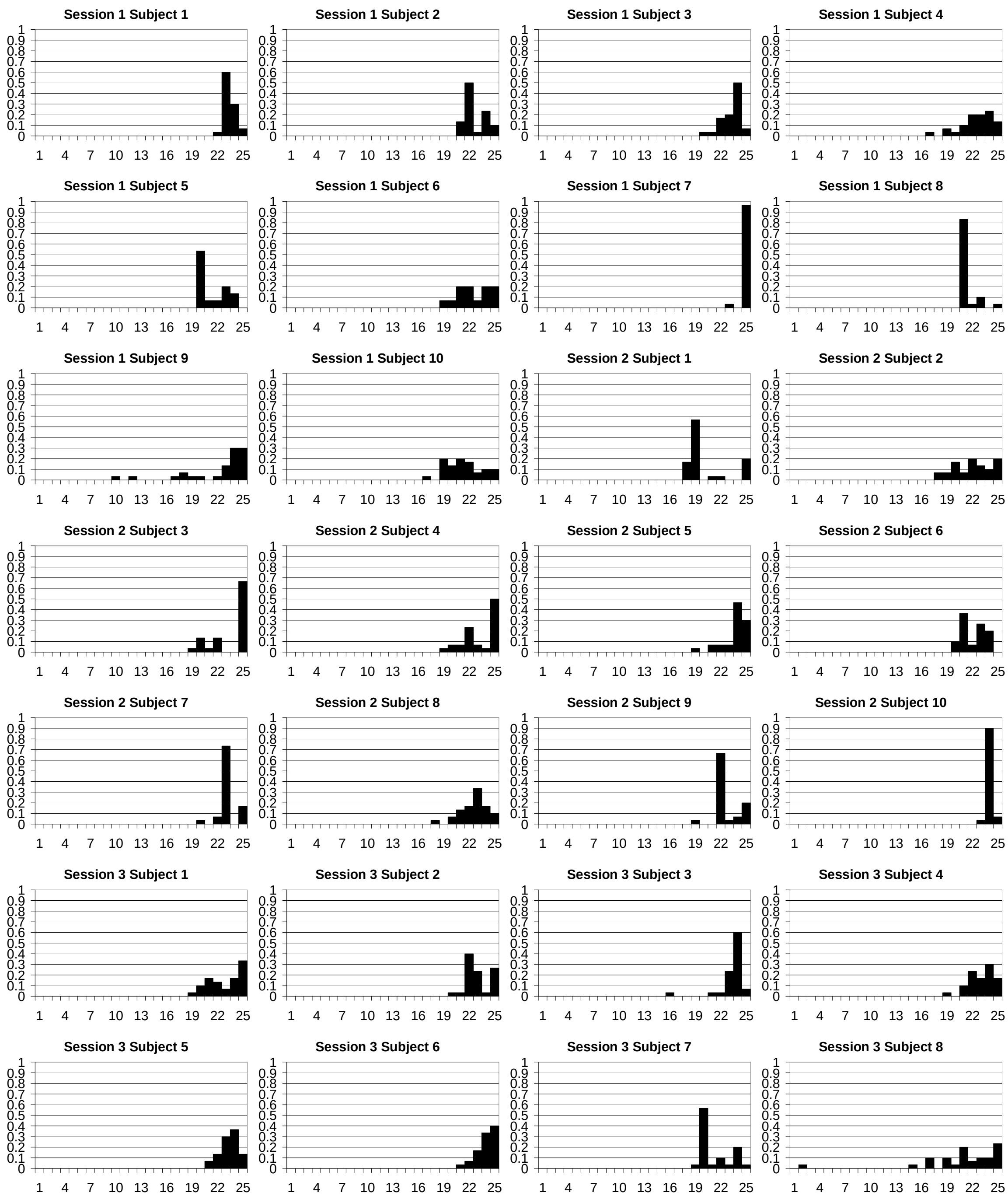


FIGURE 2.7 (Continued): Observed relative frequency (y-axis) of bids (x-axis) of each of the fifty subjects in rounds 31 to 60 of Condition HUBA

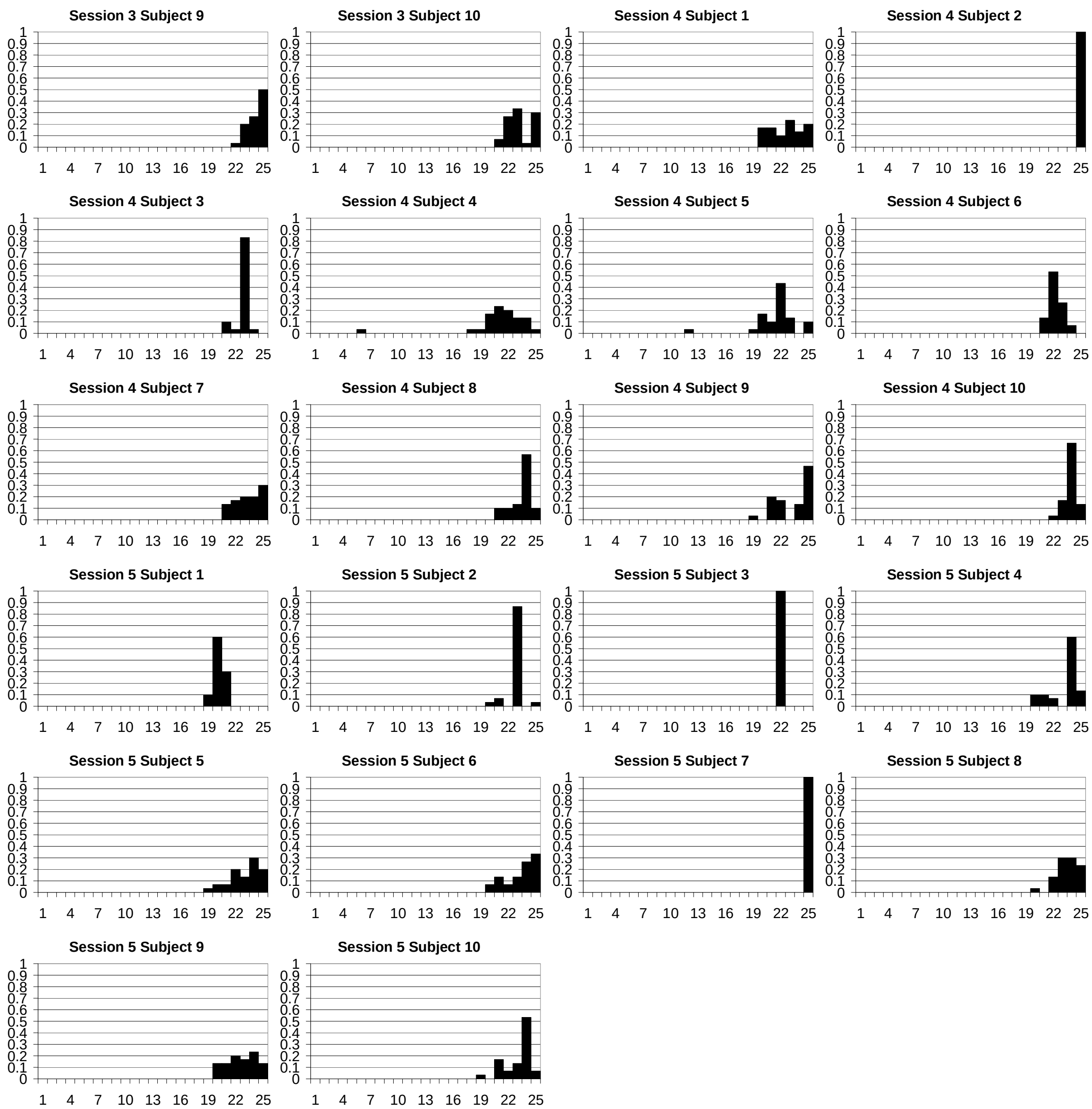


FIGURE 2.8: Predicted probability distributions of the number of switching bids for Conditions LUBA (upper panel) and HUBA (lower panel)

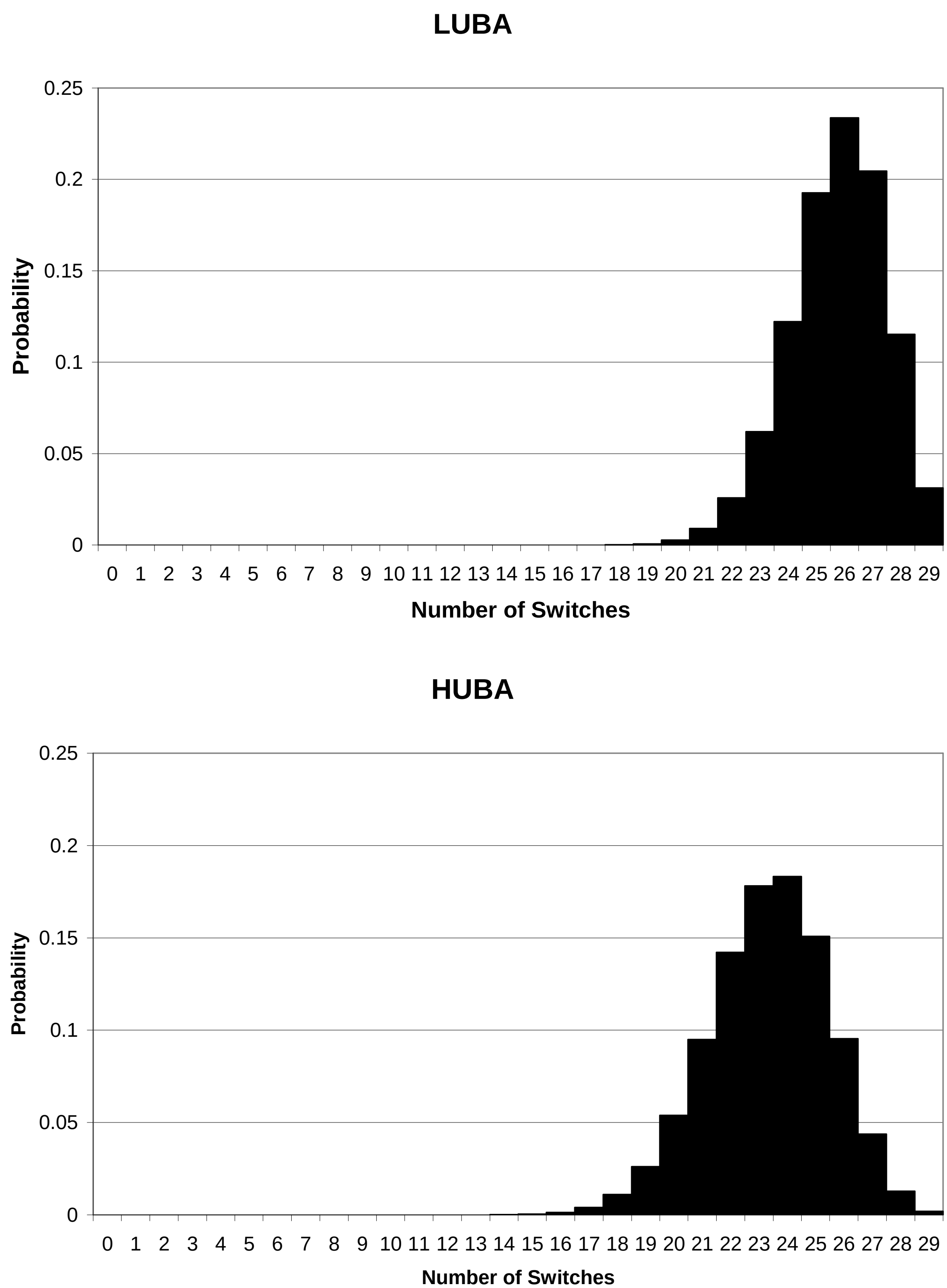


FIGURE 2.9: Bids (y-axis) by round (x-axis) of each of the fifty subjects of Condition LUBA

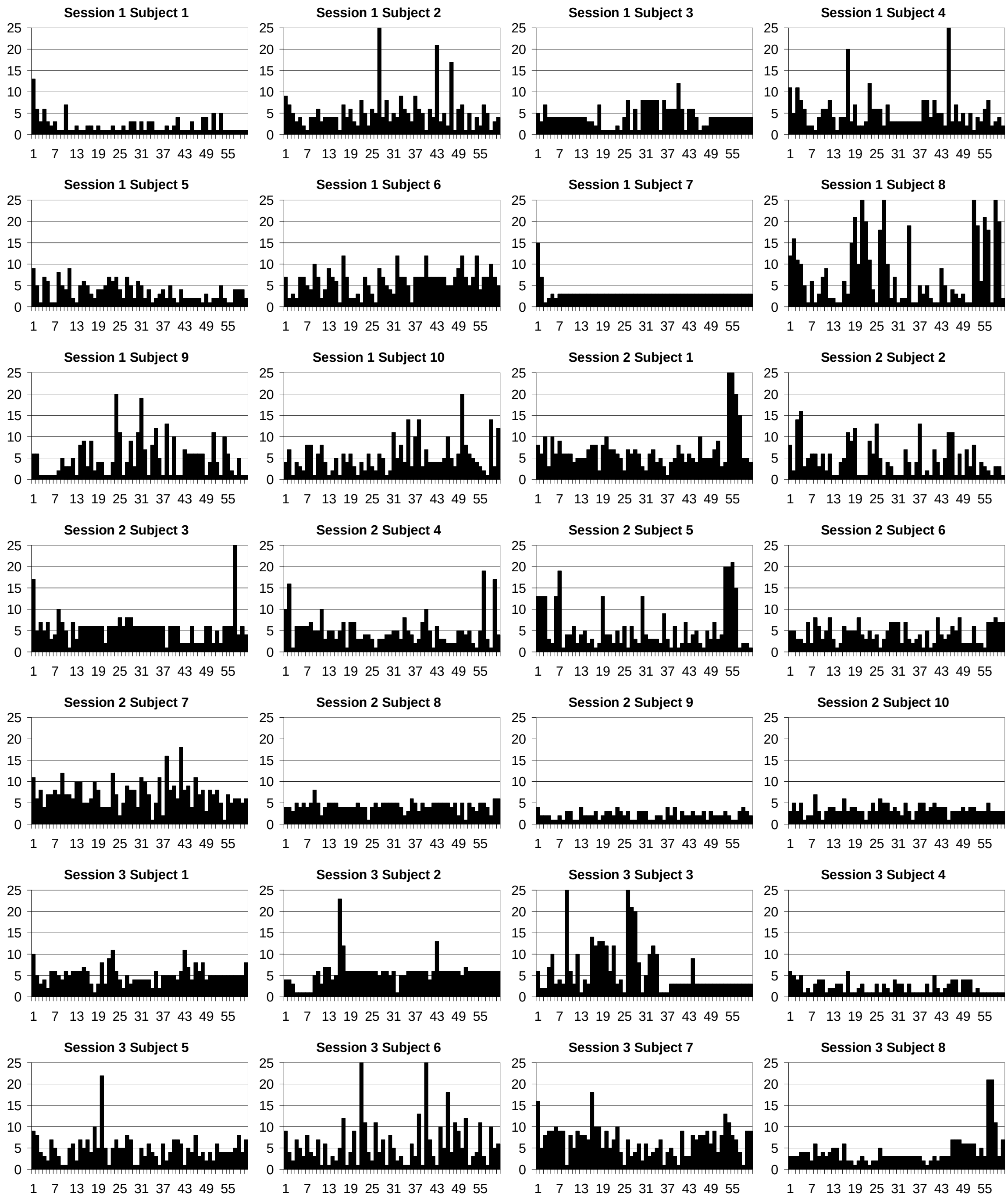


FIGURE 2.9 (Continued): Bids (y-axis) by round (x-axis) of each of the fifty subjects of Condition LUBA

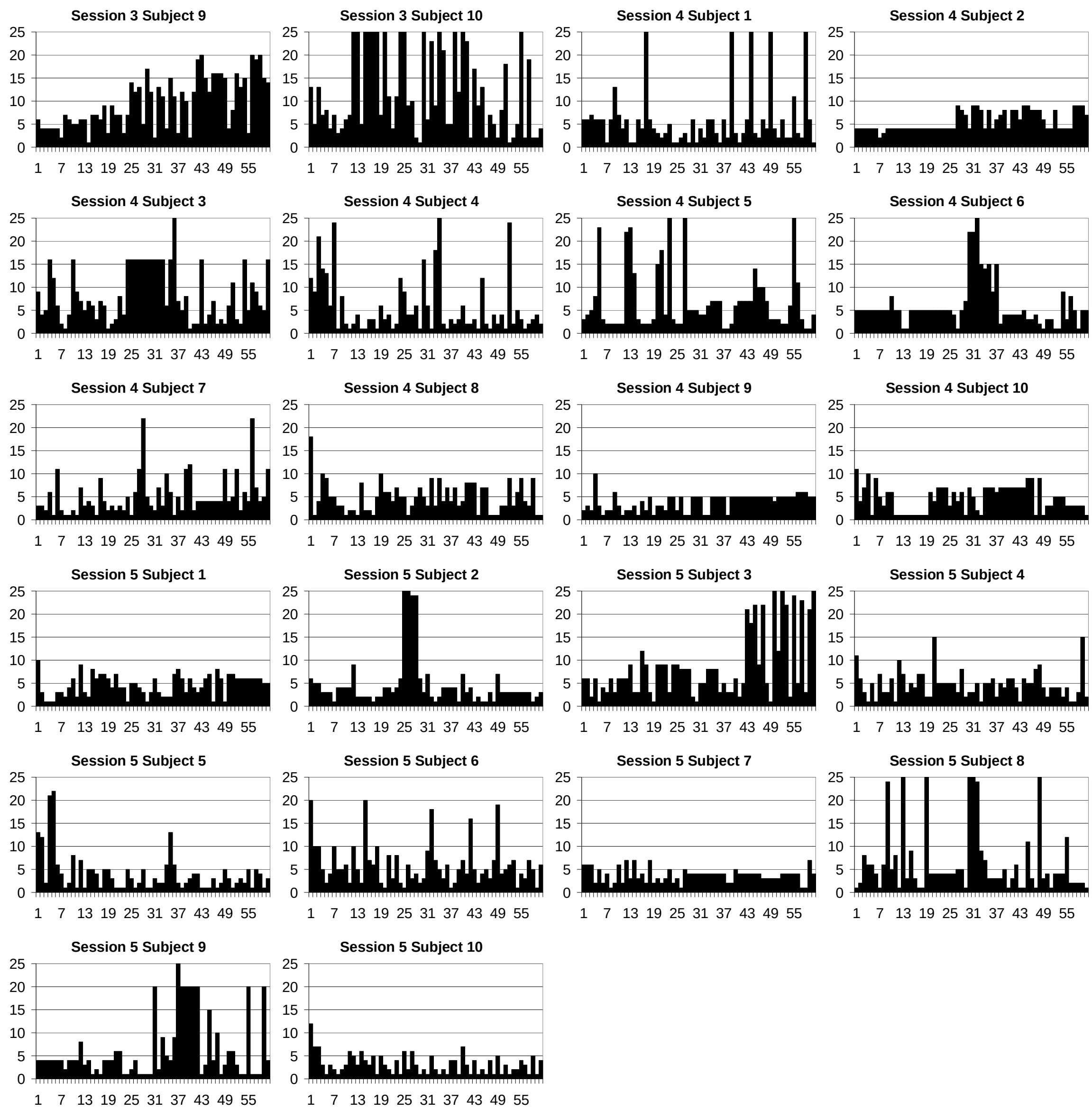


FIGURE 2.10: Bids (y-axis) by round (x-axis) of each of the fifty subjects of Condition HUBA

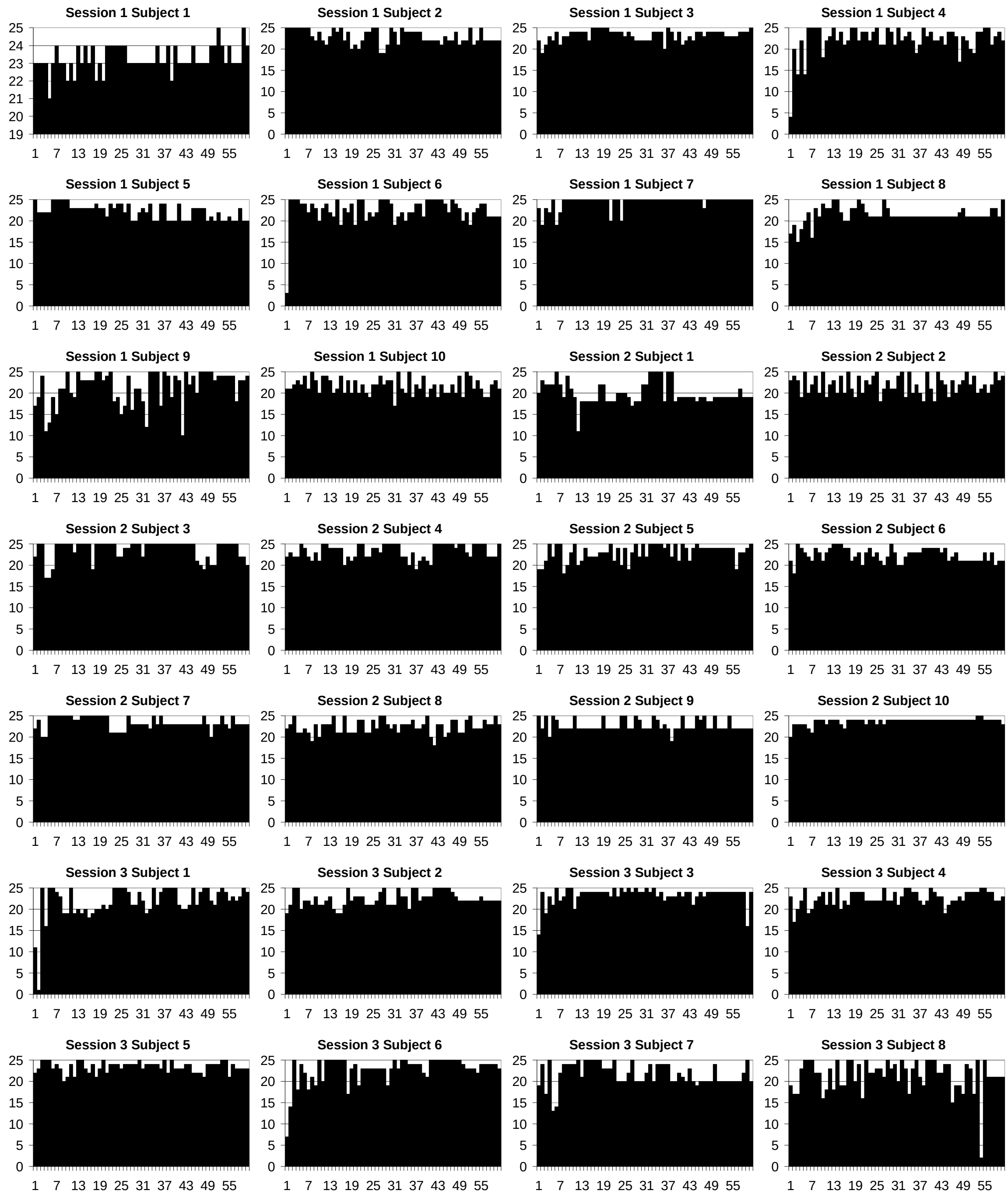


FIGURE 2.10 (Continued): Bids (y-axis) by round (x-axis) of each of the fifty subjects of Condition HUBA

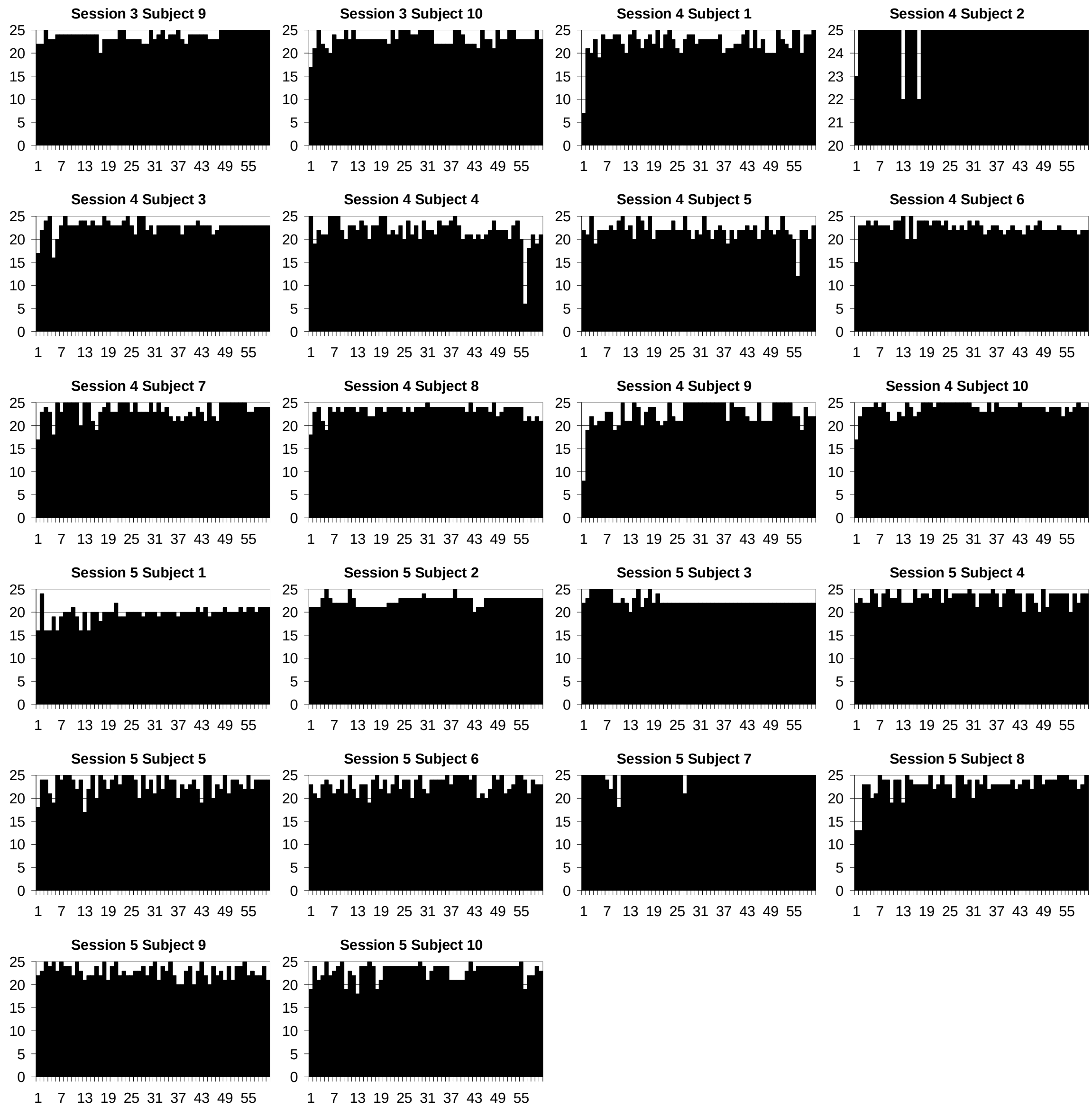
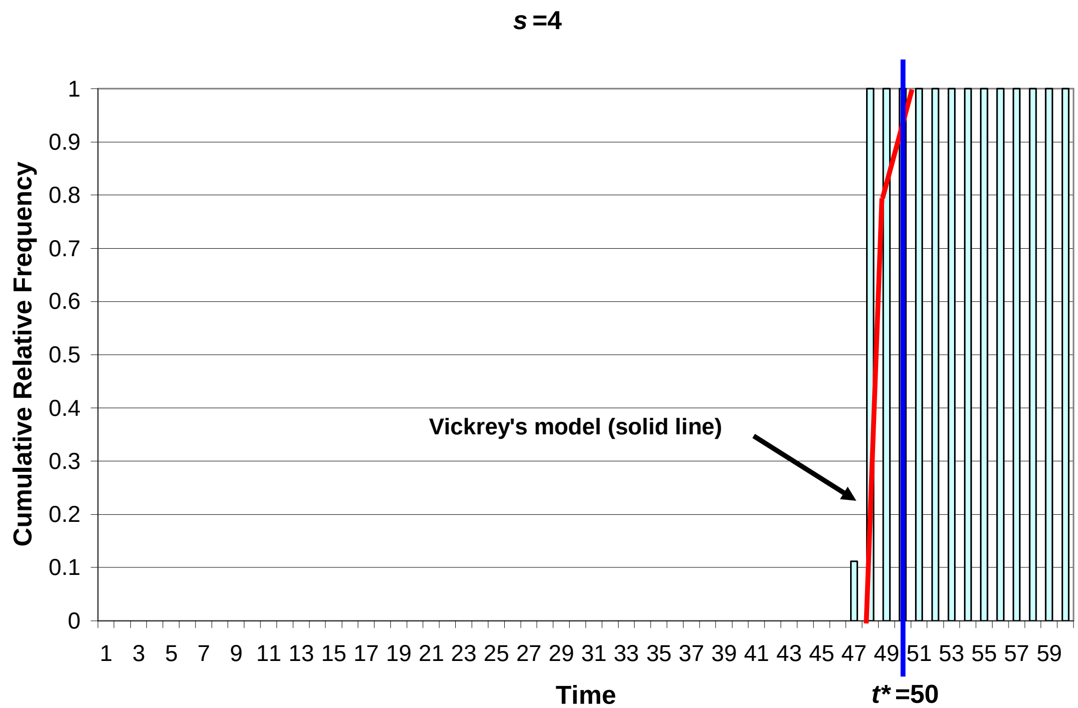


FIGURE 3.1: Cumulative Relative Frequency Distributions of Departure Times for the Vickrey's and O&R's bottleneck models for (3.1a) *s*=4, (3.1b) *s*=1, and (3.1c) *s*=1/4.



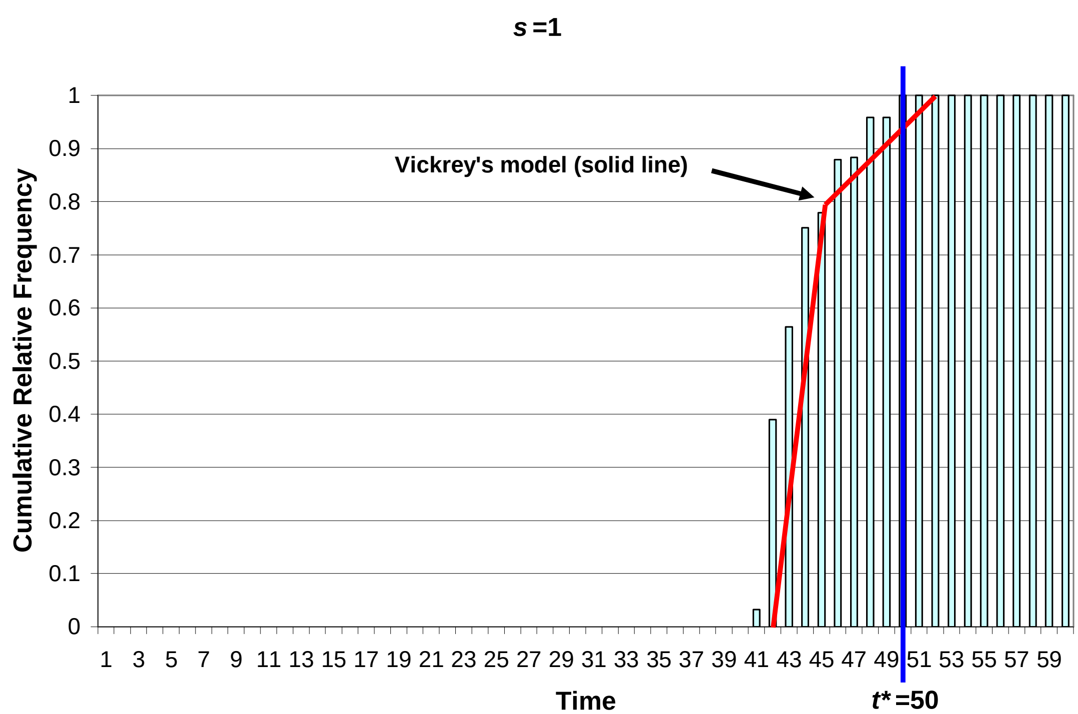


FIGURE 3.1(Continued): Cumulative Relative Frequency Distributions of Departure Times for the Vickrey's and O&R's bottleneck models for (3.1a) *s*=4, (3.1b) *s*=1, and (3.1c) *s*=1/4.

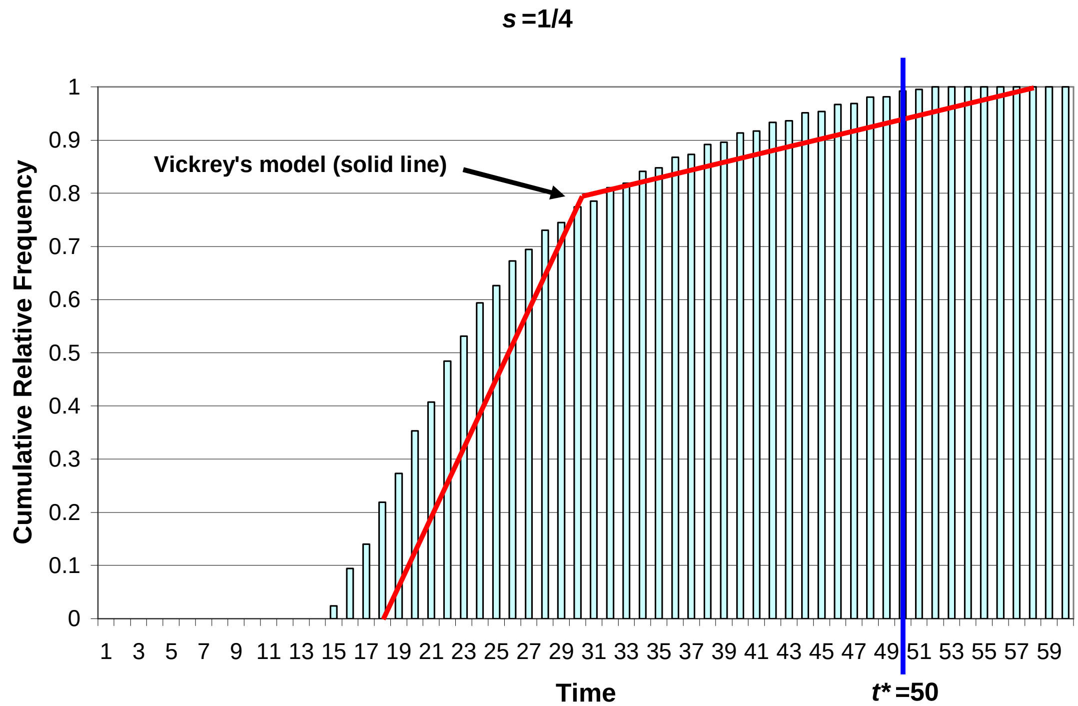
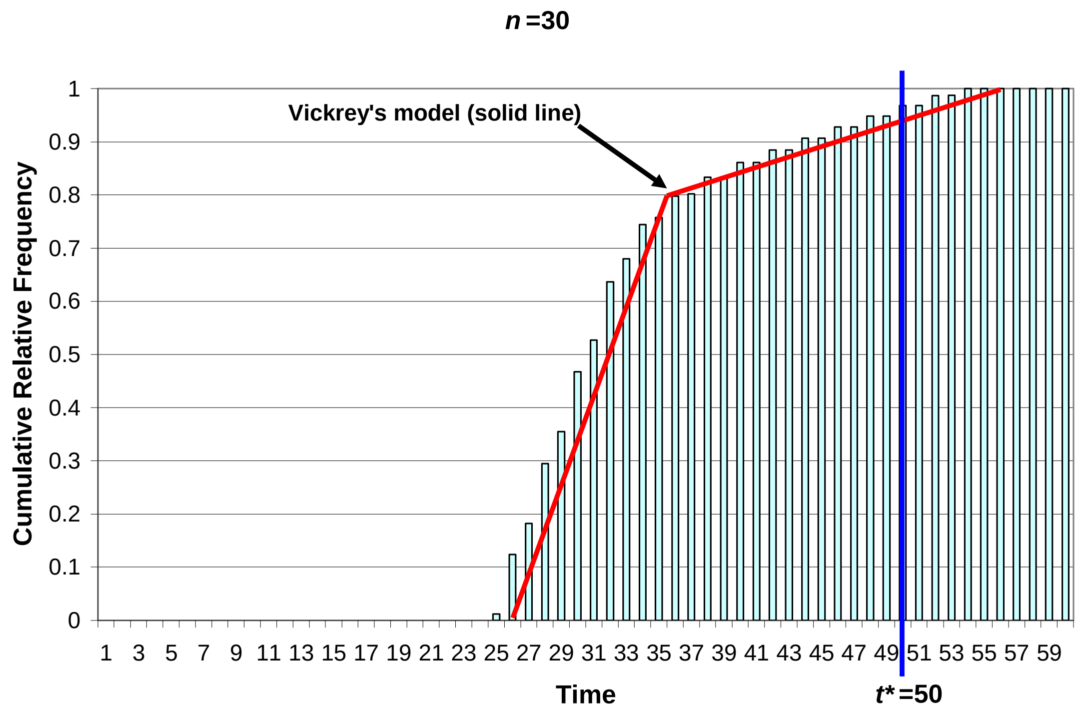
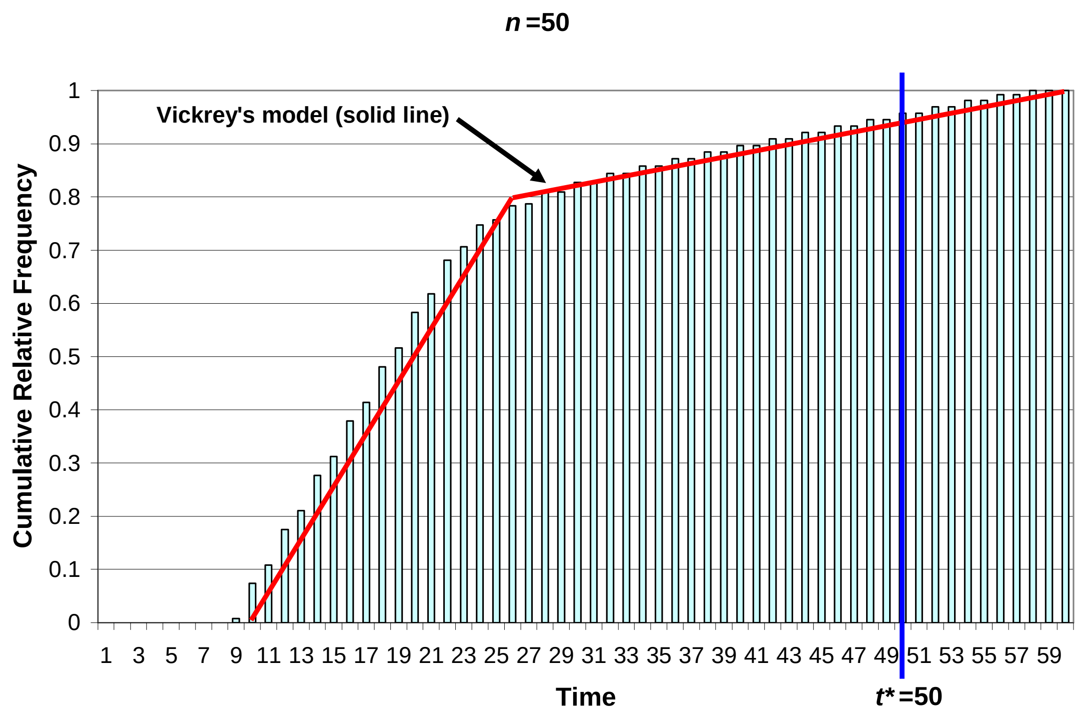


FIGURE 3.2: Cumulative Relative Frequency Distributions of Departure Times for the Vickrey's and O&R's models for (3.2a) n=30 and (3.2b) n=50





APPENDIX B: TABLES

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TABLE 1.1: Payoff matrix of the subgame starting at time period $t = T$

	Number of the other players who choose to volunteer			
	0	1	…	*n*-1
Volunteer	L_T	L_T	…	L_T
Don’t volunteer	ε	H_T	…	H_T

TABLE 1.2: Payoff tables for the experiment by condition

Time Period	Payoffs			Time Period	Payoffs		
	Volunteer (δ=0.3)	Volunteer (δ=0.6)	Non-volunteer		Volunteer (δ=0.3)	Volunteer (δ=0.6)	Non-volunteer
0	6.99	12.97	20.95	16	2.20	3.39	4.99
1	6.41	11.83	19.05	17	2.08	3.16	4.60
2	5.90	10.80	17.33	18	1.98	2.95	4.26
3	5.43	9.86	15.77	19	1.88	2.77	3.94
4	5.01	9.01	14.36	20	1.80	2.59	3.66
5	4.62	8.25	13.08	21	1.72	2.44	3.40
6	4.28	7.56	11.93	22	1.65	2.30	3.17
7	3.96	6.93	10.88	23	1.59	2.17	2.96
8	3.68	6.36	9.94	24	1.53	2.06	2.76
9	3.42	5.85	9.08	25	1.48	1.96	2.59
10	3.19	5.38	8.31	26	1.43	1.86	2.44
11	2.98	4.96	7.61	27	1.39	1.78	2.29
12	2.79	4.58	6.97	28	1.35	1.70	2.17
13	2.62	4.24	6.40	29	1.32	1.63	2.05
14	2.46	3.93	5.88	30	1.28	1.57	1.95
15	2.32	3.65	5.41	NV*	1	1	1

* “NV” means that the game ends with no volunteer.

TABLE 1.3: Observed frequency distributions of termination time by session for Conditions δ=0.3 and δ=0.6

Condition δ=0.3	Termination Time								NS*	Total
	0	1	2 to 4	5 to 7	8 to 10	11 to 13	14 to 16	17 to 30		
Session 1	22	14	40	16	16	4	5	22	11	150
Session 2	14	6	44	25	6	16	10	16	13	150
Session 3	20	12	38	24	19	6	11	19	1	150
Session 4	23	15	35	24	13	5	11	21	3	150
Session 5	34	19	30	6	7	20	8	15	11	150
Across sessions	113	66	187	95	61	51	45	93	39	750

Condition δ=0.6	Termination Time								NS*	Total
	0	1	2 to 4	5 to 7	8 to 10	11 to 13	14 to 16	17 to 30		
Session 1	64	29	27	15	5	1	1	7	1	150
Session 2	28	21	52	14	16	6	3	6	4	150
Session 3	32	23	68	16	4	0	1	6	0	150
Session 4	62	29	26	23	2	2	2	4	0	150
Session 5	47	21	42	27	4	0	4	5	0	150
Across sessions	233	123	215	95	31	9	11	28	5	750

* “NS” means that the game ends with no stopper.

TABLE 2.1: Predicted probabilities and observed relative frequencies of bids on the group and aggregate levels for Condition LUBA

Bid	Session 1		Session 2		Session 3		Session 4		Session 5		Aggregate		Predicted
	1 to 30	31 to 60	1 to 30	31 to 60	1 to 30	31 to 60	1 to 30	31 to 60	1 to 30	31 to 60	1 to 30	31 to 60	
1	0.177	0.183	0.093	0.120	0.120	0.110	0.157	0.107	0.157	0.173	0.141	0.139	0.133
2	0.127	0.100	0.097	0.157	0.083	0.070	0.123	0.107	0.133	0.113	0.113	0.109	0.156
3	0.180	0.197	0.153	0.157	0.123	0.183	0.107	0.120	0.143	0.157	0.141	0.163	0.157
4	0.140	0.130	0.140	0.127	0.123	0.103	0.153	0.133	0.160	0.177	0.143	0.134	0.143
5	0.063	0.097	0.150	0.150	0.107	0.110	0.153	0.127	0.100	0.097	0.115	0.116	0.118
6	0.087	0.067	0.133	0.107	0.120	0.130	0.087	0.083	0.107	0.083	0.107	0.094	0.084
7	0.077	0.073	0.077	0.060	0.083	0.047	0.050	0.093	0.047	0.050	0.067	0.065	0.050
8	0.037	0.040	0.057	0.037	0.037	0.040	0.020	0.047	0.037	0.020	0.037	0.037	0.026
9	0.030	0.013	0.013	0.013	0.043	0.030	0.027	0.060	0.033	0.020	0.029	0.027	0.016
10	0.013	0.017	0.030	0.010	0.030	0.017	0.013	0.010	0.023	0.003	0.022	0.011	0.013
11 to 25	0.070	0.083	0.057	0.063	0.130	0.160	0.110	0.113	0.060	0.107	0.085	0.105	0.103

TABLE 2.2: Predicted probabilities and observed relative frequencies of bids on the group and aggregate levels for Condition HUBA

Bid	Session 1		Session 2		Session 3		Session 4		Session 5		Aggregate		Predicted
	1 to 30	31 to 60	1 to 30	31 to 60	1 to 30	31 to 60	1 to 30	31 to 60	1 to 30	31 to 60	1 to 30	31 to 60	
1 to 15	0.030	0.007	0.003	0.000	0.023	0.007	0.010	0.007	0.007	0.000	0.015	0.004	0.000
16	0.007	0.000	0.000	0.000	0.010	0.003	0.003	0.000	0.020	0.000	0.008	0.001	0.000
17	0.010	0.010	0.010	0.000	0.020	0.010	0.010	0.000	0.003	0.000	0.011	0.004	0.000
18	0.010	0.007	0.043	0.027	0.017	0.000	0.007	0.003	0.013	0.000	0.018	0.007	0.000
19	0.043	0.037	0.040	0.077	0.057	0.020	0.023	0.010	0.043	0.017	0.041	0.032	0.001
20	0.063	0.083	0.067	0.057	0.073	0.073	0.060	0.050	0.073	0.103	0.067	0.073	0.045
21	0.093	0.157	0.107	0.077	0.087	0.073	0.077	0.117	0.083	0.097	0.089	0.104	0.111
22	0.140	0.140	0.187	0.163	0.107	0.147	0.123	0.177	0.177	0.173	0.147	0.160	0.168
23	0.187	0.163	0.133	0.167	0.193	0.183	0.207	0.210	0.147	0.173	0.173	0.179	0.205
24	0.173	0.200	0.163	0.193	0.190	0.240	0.200	0.193	0.187	0.223	0.183	0.210	0.228
25	0.243	0.197	0.247	0.240	0.223	0.243	0.280	0.233	0.247	0.213	0.248	0.225	0.244

TABLE 2.3: Results of the Kolmogorov-Smirnov test for Condition LUBA ("R": Reject the null hypothesis of SMSE play; "FR": Fail to reject the null hypothesis)

Subject	Session1		Session 2		Session 3		Session 4		Session 5	
	1 to 30	31 to 60	1 to 30	31 to 60	1 to 30	31 to 60	1 to 30	31 to 60	1 to 30	31 to 60
1	R	R	R	R	FR	R	FR	FR	FR	R
2	FR	FR	FR	FR	R	R	R	R	FR	R
3	FR	R	R	R	R	R	R	R	R	R
4	FR	FR	FR	FR	R	R	FR	FR	FR	FR
5	FR	R	FR	FR	FR	FR	FR	FR	FR	R
6	FR	R	FR	FR	FR	FR	R	FR	FR	FR
7	R	R	R	R	R	R	FR	R	FR	R
8	R	FR	R	FR	R	FR	FR	FR	FR	FR
9	FR	FR	R	R	R	R	R	R	R	R
10	FR	FR	FR	R	R	R	FR	FR	FR	R

TABLE 2.4: Results of the Kolmogorov-Smirnov test for Condition HUBA (“R”: Reject the null hypothesis of SMSE play; “FR”: Fail to reject the null hypothesis)

Subject	Session1		Session 2		Session 3		Session 4		Session 5	
	1 to 30	31 to 60	1 to 30	31 to 60	1 to 30	31 to 60	1 to 30	31 to 60	1 to 30	31 to 60
1	R	R	R	R	R	FR	FR	FR	R	R
2	FR	R	R	R	R	FR	R	R	R	R
3	FR	FR	R	R	FR	FR	FR	R	R	R
4	FR	FR	FR	R	R	FR	FR	R	FR	R
5	FR	R	FR	R	FR	FR	R	R	FR	FR
6	FR	FR	FR	R	R	R	FR	R	FR	FR
7	R	R	R	R	R	R	FR	FR	R	R
8	R	R	R	FR	R	R	R	FR	FR	FR
9	R	FR	R	R	FR	R	R	FR	FR	FR
10	R	R	R	R	FR	FR	R	R	FR	FR

TABLE 2.5: Number of switching bids for Condition LUBA (at most 29 opportunities of switching for each of the first and last 30 rounds)

Subject	Session1		Session 2		Session 3		Session 4		Session 5	
	1 to 30	31 to 60	1 to 30	31 to 60	1 to 30	31 to 60	1 to 30	31 to 60	1 to 30	31 to 60
1	21	16	23	25	25	15	24	27	24	27
2	25	29	25	26	14	10	7	18	7	18
3	15	11	19	14	27	7	23	26	23	26
4	22	21	19	24	23	16	26	28	26	28
5	27	21	25	25	26	25	20	17	20	17
6	27	20	24	24	29	28	9	21	9	21
7	6	0	22	28	25	26	27	23	27	23
8	27	25	19	21	19	15	23	22	23	22
9	21	23	19	22	21	27	23	8	23	8
10	28	26	23	18	23	18	18	13	18	13

TABLE 2.6: Number of switching bids for Condition HUBA (at most 29 opportunities of switching for each of the first and last 30 rounds)

Subject	Session1		Session 2		Session 3		Session 4		Session 5	
	1 to 30	31 to 60	1 to 30	31 to 60	1 to 30	31 to 60	1 to 30	31 to 60	1 to 30	31 to 60
1	17	16	17	11	22	24	26	19	19	16
2	20	15	28	29	20	12	5	0	10	5
3	15	16	11	9	20	16	22	8	14	0
4	23	25	19	16	21	19	23	24	20	19
5	13	17	23	16	21	14	23	27	25	23
6	23	19	26	14	18	11	22	20	27	21
7	11	2	8	14	16	16	20	19	7	0
8	21	6	20	20	21	20	19	15	20	18
9	23	19	15	17	9	10	21	12	25	26
10	25	27	15	3	23	18	16	15	20	12

TABLE 2.7: Four categories of subjects for Condition LUBA

Rounds 1 to 30	Session 1	Session 2	Session 3	Session 4	Session 5	Total
Neither	2	3	3	2	0	10
(A) only	2	1	0	2	4	9
(B) only	1	2	4	2	2	11
(A) & (B)	5	4	3	4	4	20

Rounds 31 to 60	Session 1	Session 2	Session 3	Session 4	Session 5	Total
Neither	5	2	5	2	4	18
(A) only	1	1	1	3	1	7
(B) only	0	3	2	2	3	10
(A) & (B)	4	4	2	3	2	15

TABLE 2.8: Four categories of subjects for Condition HUBA

Rounds 1 to 30	Session 1	Session 2	Session 3	Session 4	Session 5	Total
Neither	2	5	1	2	3	13
(A) only	2	0	1	0	0	3
(B) only	3	2	5	3	1	14
(A) & (B)	3	3	3	5	6	20

Rounds 31 to 60	Session 1	Session 2	Session 3	Session 4	Session 5	Total
Neither	5	9	3	3	4	24
(A) only	1	0	3	2	1	7
(B) only	1	0	1	3	1	6
(A) & (B)	3	1	3	2	4	13

TABLE 2.9: Results from a fixed effects regression for each of the five sessions by condition

Independent variables	Dependent variable: Individual bid at *t*				
	Session 1	Session 2	Session 3	Session 4	Session 5
Condition LUBA					
Winning bid at *t*-1	0.0069 (0.0673)	0.2083** (0.0787)	0.1252** (0.0480)	0.1480** (0.0476)	0.1662 (0.1145)
Winning bid at *t*-2	0.2029** (0.0689)	0.1373 (0.0780)	0.0455 (0.0480)	0.0591 (0.0478)	0.1320 (0.1164)
Winning bid at *t*-3	-0.0072 (0.0693)	-0.1285 (0.0786)	0.0470 (0.0482)	0.0156 (0.0474)	-0.0132 (0.1155)
R^2	0.015428	0.023460	0.015638	0.018310	0.005571
Condition HUBA					
Winning bid at *t*-1	0.0159 (0.0187)	0.0381* (0.0164)	0.0108 (0.0162)	0.0546** (0.0128)	0.0532** (0.0170)
Winning bid at *t*-2	0.0117 (0.0189)	-0.0090 (0.0164)	0.0194 (0.0162)	0.0273* (0.0128)	0.0384* (0.0170)
Winning bid at *t*-3	0.0088 (0.0187)	-0.0156 (0.0164)	0.0171 (0.0162)	0.0215 (0.0131)	0.0021 (0.0170)
R^2	0.002047	0.011780	0.005170	0.047791	0.025474

Number of observations = 570 (*df* = 557)
Standard errors appear in parenthesis.
*   : At 5% significance level
** : At 1% significance level

TABLE 3.1: Examples of a discrete bottleneck game with parameters n=10, α=1, β=0.6, γ=2.4, $t\in$ {1, 2, …, 60} and t^*=50: s=1/3 (top panel) and s=3 (bottom panel)

Player number	Departure time	Travel time	Arrival time	Travel cost
1	30	3	33	13.2
2	34	3	37	10.8
3	34	6	40	12
4	34	9	43	13.2
5	40	6	46	8.4
6	42	7	49	7.6
7	44	8	52	12.8
8	45	10	55	22
9	47	11	58	30.2
10	48	13	61	39.4

Player number	Departure time	Travel time	Arrival time	Travel cost
1	48	1	49	1.6
2	48	1	49	1.6
3	48	1	49	1.6
4	48	2	50	2
5	49	1	50	1
6	49	1	50	1
7	49	2	51	4.4
8	49	2	51	4.4
9	49	2	51	4.4
10	49	3	52	7.8

TABLE 3.2: Symmetric mixed-strategy equilibrium solutions for the cases n=4, s=1, α=1, $\beta\in$ {0.25, 0.5}, γ=2, and $t\in$ {t*-8, …, t*, …, t*+8} (NP = Not Provided)

	t*-8	t*-7	t*-6	t*-5	t*-4	t*-3	t*-2	t*-1	t*	C	T
β=0.25											
O&R*	0	0	0.038	0.148	0.239	0.288	0.200	0.086	0	2.336	1.774
ZKMD	0	0	0.038	0.148	0.239	0.288	0.200	0.086	0	2.34	NP
O&R	0	0	0	0.076	0.287	0.347	0.212	0.079	0	2.085	1.583
β=0.5											
O&R*	0	0	0	0	0.262	0.414	0.219	0.105	0	2.893	2.129
ZKMD	0	0	0	0	0.262	0.414	0.219	0.105	0	2.89	NP
O&R	0	0	0	0	0.172	0.602	0.054	0.172	0	2.629	1.786

TABLE 3.3: Comparison of the Vickrey's and O&R's bottleneck models with parameters n=10, α=1, β=0.6, γ=2.4, $t\in\{1, 2, \ldots, 60\}$, t^*=50, and $s\in\{1/4, 1/3, 1/2, 1, 2, 3, 4\}$

Model	*C*	*TC*	*TTT*	*TTC*	t_F	t_L
s=4						
Vickrey	1.2	12	6	6	48	50.5
O&R	2.201	22.011	15.884	15.884	47	48
Vickrey/O&R (%)	54.5	54.5	37.8	37.8		
s=3						
Vickrey	1.6	16	8	8	47.333	50.667
O&R	2.802	28.018	20.230	20.230	46	47
Vickrey/O&R (%)	57.1	57.1	39.5	39.5		
s=2						
Vickrey	2.4	24	12	12	46	51
O&R	3.431	34.313	22.151	22.151	45	50
Vickrey/O&R (%)	69.9	69.9	54.2	54.2		
s=1						
Vickrey	4.8	48	24	24	42	52
O&R	5.858	58.579	34.402	34.402	41	50
Vickrey/O&R (%)	81.9	81.9	69.8	69.8		
s=1/2 (*d*=2)						
Vickrey	9.6	96	48	48	34	54
O&R	11.403	114.028	65.566	65.566	33	51
Vickrey/O&R (%)	84.2	84.2	73.2	73.2		
s=1/3 (*d*=3)						
Vickrey	14.4	144	72	72	26	56
O&R	17.081	170.808	98.160	98.160	24	51
Vickrey/O&R (%)	84.3	84.3	73.3	73.3		
s=1/4 (*d*=4)						
Vickrey	19.2	192	96	96	18	58
O&R	22.773	227.727	130.894	130.894	15	52
Vickrey/O&R (%)	84.3	84.3	73.3	73.3		

TABLE 3.4: Comparison of the Vickrey's and O&R's bottleneck models with parameters s=1, α=1, β=0.6, γ=2.4, $t\in$ {1, 2, …, 60}, t^*=50, and $n\in$ {5, 10, 15, 20, 30, 40, 50}

Model	C	TC	TTT	TTC	t_F	t_L
n=5						
Vickrey	2.4	12	6	6	46	51
O&R	3.470	17.352	11.272	11.272	45	48
Vickrey/O&R (%)	69.2	69.2	53.2	53.2		
n=10						
Vickrey	4.8	48	24	24	42	52
O&R	5.858	58.579	34.402	34.402	41	50
Vickrey/O&R (%)	81.9	81.9	69.8	69.8		
n=15						
Vickrey	7.2	108	54	54	38	53
O&R	8.282	124.233	70.092	70.092	37	51
Vickrey/O&R (%)	86.9	86.9	77.0	77.0		
n=20						
Vickrey	9.6	192	96	96	34	54
O&R	10.666	213.320	117.053	117.053	33	52
Vickrey/O&R (%)	90.0	90.0	82.0	82.0		
n=30						
Vickrey	14.4	432	216	216	26	56
O&R	15.470	464.106	247.788	247.788	25	54
Vickrey/O&R (%)	93.1	93.1	87.2	87.2		
n=40						
Vickrey	19.2	768	384	384	18	58
O&R	20.273	810.901	426.554	426.554	17	56
Vickrey/O&R (%)	94.7	94.7	90.0	90.0		
n=50						
Vickrey	24	1200	600	600	10	60
O&R	25.074	1253.704	653.337	653.337	9	58
Vickrey/O&R (%)	95.7	95.7	91.8	91.8		

TABLE 3.5: Symmetric mixed-strategy equilibrium solutions for the case where n=10 (columns 2 and 3) and the case where n is either 8 with probability 0.6 or 12 with probability 0.4 (columns 4 and 5) with parameters s=1, α=1, β=0.6, γ=2.4, $t \in \{1, \ldots, 60\}$, and t^*=50.

Time	$n = 10$		$\Pr(n=8) = 0.6$ and $\Pr(n=12) = 0.4$	
	Probability	Cumulative Probability	Probability	Cumulative Probability
1 to 39	0	0	0	0
40	0	0	0	0
41	0.032	0.032	0.239	0.239
42	0.358	0.390	0.285	0.524
43	0.175	0.564	0.142	0.666
44	0.187	0.751	0.097	0.763
45	0.028	0.779	0.061	0.824
46	0.100	0.879	0.061	0.885
47	0.004	0.884	0.044	0.929
48	0.075	0.959	0.048	0.977
49	0	0.959	0.023	1
50	0.041	1	0	1
51 to 60	0	1	0	1
C	5.858		6.231	

TABLE 3.6: Symmetric mixed-strategy equilibrium solutions for three different costs of choosing an alternative transportation mode not subject to congestion with n=10, s=1, α=1, β=0.6, γ=2.4, $t \in \{1, \ldots, 60\}$, and t^*=50.

Time	C_{alt}		
	7	5	3
1 to 40	0	0	0
41	0.032	0	0
42	0.357	0	0
43	0.174	0.222	0
44	0.187	0.267	0
45	0.028	0.068	0
46	0.100	0.107	0.107
47	0.004	0.022	0.159
48	0.075	0.070	0
49	0	0.008	0.053
50	0.041	0	0
51 to 60	0	0	0
p_{alt}	0	0.235	0.681
C	5.858	5	3

APPENDIX C: INSTURCTIONS

Instructions for Chapter 1, Condition δ=0.3

Group Decisions in Real Time: Subject Instructions

Introduction

Welcome to the "Group Decisions in Real Time" experiment. During this experiment you will be asked to make a large number of decisions and so will the other participants. Your decisions, as well as the decisions of the other participants, will determine your monetary payoff according to the rules that will be explained shortly. The money that you earn during the experiment has been provided by a grant agency. It will be paid to you in cash at the end of the session.

Please read the instructions carefully. If you have any questions, please feel free to raise your hand. One of the experimenters will come to assist you.

From now on communication between the participants is strictly prohibited. If the participants communicate with one another by any shape or form, the experiment will be terminated.

Description of the Task

A total of **9** persons participate in this experiment (i.e., you and 8 other participants). During the experiment, all of these persons will participate in a series of **50** identical rounds. At the beginning of each round, the computer will randomly divide the 9 players into **3** groups of **3** members each. The composition of your group will change randomly from round to round (i.e., the people you play with in one round will not necessarily be the same people you played with in the previous round).

Throughout all 50 rounds, your identity will not be disclosed to the other group members and their identities will not be disclosed to you.

Decisions. Each round is played over time, which is measured by a clock. Rather than dealing with fractions of a second, the time in each round is divided into **30** steps, each lasting about 1.5 seconds. In each round, you, as well as the other two members of your group, will be asked to decide whether and what time to stop the clock. This is the only decision you'll be asked to make. The mechanics of doing so will be explained later. The player who is the **first** to stop the clock will be designated as the **Stopper**, and the others will be designated as **Non-stoppers**. In case multiple group members stop the clock at the same period, they will all be designated as Stoppers. No player is compelled to stop the clock; it is entirely possible that 30 steps elapse without any player stopping the clock.

Payoffs. Payoffs are associated with the time the clock is stopped. As the clock progresses through the round, the payoffs of the Stopper and of the Non-stoppers will **decrease** but not at the same rate. The basic idea here is to capture the fact that stopping the clock is costly, and that this cost increases with time. Therefore, Stoppers will always earn less than Non-stoppers. If no player stops the clock, or if the clock is stopped relatively late in the round, all group members will earn less than if the clock is stopped early in the round. **The payoffs will stop decaying once the clock is stopped**.

Examples

To illustrate how payoffs depend on the time of stopping the clock, please consult Table 1 at the end of the instructions. This table shows the relationship between the time at which the clock is stopped and the payoffs (measured in points) of the Stopper and Non-stoppers. See also Figure 1 that exhibits this relationship graphically.

Please consider the following numerical examples:

Example 1: Thirty steps have elapsed with no player stopping the clock. In this case, each of the group members earns 1 point.

Example 2: One of the group members was the first to stop the clock at step 5. Then, the Stopper earns 4.62 points whereas each of the Non-stoppers earns 13.08 points.

Example 3: Two members in your group stopped the clock at step 10. Then, each of these two stoppers earns 3.19 points whereas the other group member earns 8.31 points.

Example 4: All group members simultaneously stopped the clock at step 20. Then, each of them earns 1.80 points.

Description of the Decision Screen

Figure 1 displays a copy of the computer screen on your PC that will be presented to you on each round. A box right above a red circle is a clock, which runs from 0 to 30 steps. A diagram at the middle of the screen plots the payoffs of the Stopper (black curve) and each of the Non-stoppers (blue curve) against the elapsed steps. As the clock progresses, you will observe red dots moving along the two payoff curves indicating the current payoffs of the Stopper and Non-stoppers. Along the right side of the payoff diagram, two boxes display the payoffs of the Stopper and Non-stoppers. These payoffs will be updated on each step as the clock progresses to a new step. A purple horizontal bar perpendicular to the x-axis (elapsed steps) at step 30 indicates that step 30 is the last step that you can choose to stop the clock. Finally, a box on the top-left corner displays the number of the current round, your total score (= total points), and the payment you will receive at the end of the session.

Stopping the Clock.

We'll now explain how to stop the clock. Once all the participants have indicated that they are ready to start the current round, the computer screen showed in Figure 1 will appear on your PC. Until the moment the clock starts, the computer will keep the mouse pointer inside the red circle. Once the clock starts, you can stop the clock at any step you want. **All you need to do to is simply move the mouse pointer outside of the red circle**. We use this procedure to eliminate any noise due to clicking. If someone else in your group is the first to stop the clock, all the other group members will immediately be notified. No other player will be in a position to stop the clock as the computer will be automatically immobilized. In that case, please wait until all 30 steps elapse.

Interpreting the Results

At the end of each round, after 30 steps have elapsed, a Results screen will appear. Figure 2 at the end of the instructions shows an example of this screen. The Results screen shows whether you stopped the clock, at which step you stopped the clock, if you did so, at which step the first participant stopped the clock, and your payoff for the current round. In the example shown in Figure 2,

- *You did not stop the clock.*
- *One of the two other group members was the first to stop the clock at step 15.*
- *Your payoff for this round was 5.41 points.*

<u>End of Experiment</u>

After completing all **50** rounds, a summary screen will display the total points you have accumulated and the corresponding earnings in dollars (Points will be converted to money at the rate 20 points = $1.00). During the experiment, the cumulative number of points and dollar payoff that you have earned will be displayed in the box on the top-left corner of your computer display.

Please remain at your desk until asked to come forward and receive payment for the experiment.

Please place the instructions on the table in front of you to indicate that you have completed reading them. The experiment will begin shortly. Initially, you will play **unpaid** practice rounds that familiarize you with how to stop the clock by moving the mouse. You may repeat the practice rounds as many times as you wish until you feel comfortable with the program. Once you are finished with the practice rounds, click on the “I’m ready to start playing” button. Once everyone has clicked on this button, the **50 paid** rounds will follow.

Please remember that no communication is allowed during the experiment. If you encounter any difficulties please raise your hand and someone will assist you.

Table 1: Payoff Table

Step	Points		Step	Points	
	Stopper	Non-stopper		Stopper	Non-stopper
0	6.99	20.95	16	2.20	4.99
1	6.41	19.05	17	2.08	4.60
2	5.90	17.33	18	1.98	4.26
3	5.43	15.77	19	1.88	3.94
4	5.01	14.36	20	1.80	3.66
5	4.62	13.08	21	1.72	3.40
6	4.28	11.93	22	1.65	3.17
7	3.96	10.88	23	1.59	2.96
8	3.68	9.94	24	1.53	2.76
9	3.42	9.08	25	1.48	2.59
10	3.19	8.31	26	1.43	2.44
11	2.98	7.61	27	1.39	2.29
12	2.79	6.97	28	1.35	2.17
13	2.62	6.40	29	1.32	2.05
14	2.46	5.88	30	1.28	1.95
15	2.32	5.41	NS**	1	1

** “NS” implies that a round ends with no stopper.

Figure 1: Decision Screen

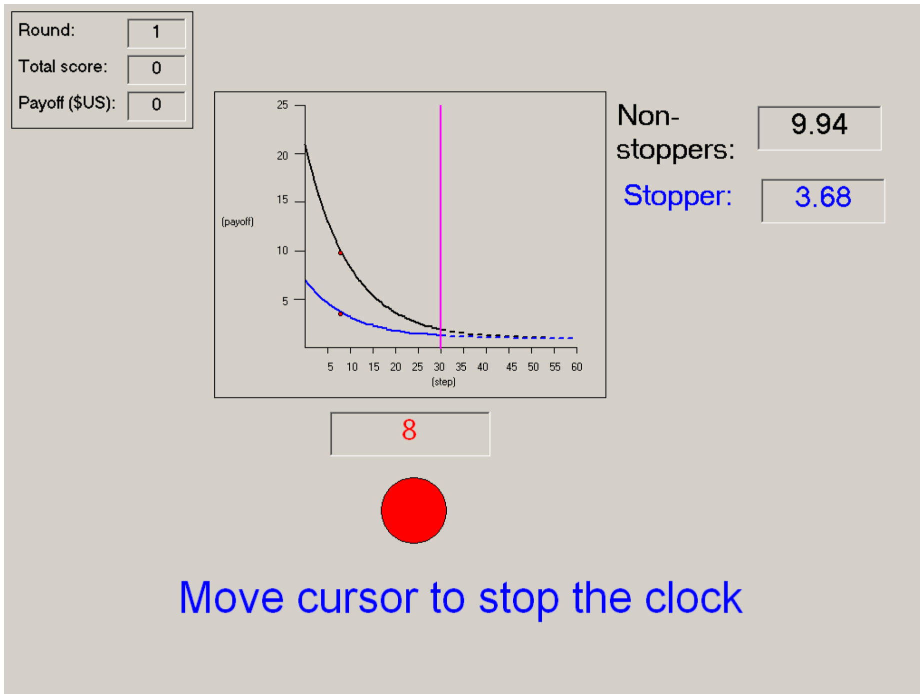
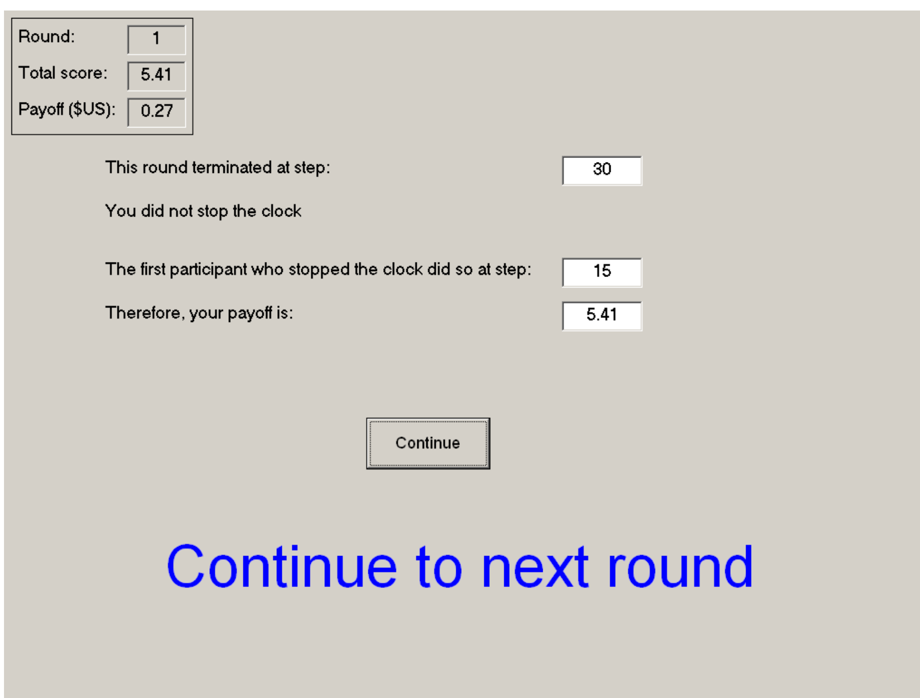


Figure 2: Results Screen



Instructions for Chapter 2, Condition LUBA

Lowest Unique Bid Auction: Instructions

Introduction

Welcome to the "*Lowest Unique Bid Auction*" experiment. The purpose of this experiment is to study a variant of a new type of auction that has become quite popular on the Internet. During the experiment, you'll be asked to make a large number of decisions and so will the other participants. Your decisions, as well as the decisions of the other participants, will determine your earnings according to the rules of the auction that will be explained below. The money that you'll earn will be paid to you in cash at the end of the session.

Please read the instructions carefully. If you have any questions, please raise your hand and one of the experimenters will come to assist you.

From now on communication between the participants is forbidden. If the participants communicate in any shape or form, the experiment will terminate.

Description of the Experiment

A total of **10** participants (yourself included) will take part in this experiment. During the experiment, all the ten players will participate in a series of **60** identical rounds. Although you'll be informed at the end of each round of the decisions made by the other participants, their identities will not be disclosed to you. Nor will you be able to associate any decision with any participant. Similarly, your identity will not be revealed to the other participants.

Rules of the Auction

On each round, you will participate in an auction. To do so, you will be asked to enter a bid, which is an integer between **1** and **25**. You may enter any bid within this range. Your bid, as well as the other participants' bids, will be entered independently and anonymously.

The winner of the auction will be the player who enters the **lowest** bid, provided that this bid is **unique** (no other player enters the same bid). If two or more players enter the same bid, then their bids will be discarded. If there is no lowest unique bid, then there will be no winner in the round. The winner, if there is one, will earn the **value of his/her bid** (in points). Every other group player will earn nothing.

Here is a brief summary of the rules of the auction. Each player enters a bid. The winner is the player who enters the lowest bid, provided that it is unique. If there is a winner, then he/she earns the value of his/her bid, whereas every other player wins nothing.

Examples

To illustrate the auction mechanism, please refer to the examples below. In the examples, 10 bids are presented in an ascending order (the same way they will presented to you during the experiment), one bid per player.

Example 1

Bids	2	2	6	6	6	**8**	16	21	24	24

Outcome: The winning bid is 8, which yields 8 points to its bidder.

Example 2

Bids	**10**	11	11	11	17	17	18	20	20	25

Outcome: The winning bid is 10, which yields 10 points to its bidder.

Example 3

Bids	5	5	5	9	9	12	12	12	12	**18**

Outcome: The winning bid is 18, which yields 18 points to its bidder.

Example 4

Bids	1	1	4	4	4	9	9	21	21	21

Outcome: As there is no unique bid, there is no winner in this auction.

Description of the Screens

Decision screen. Please type in your bid for the round by using the mouse cursor to press the numbered keys. At any point you can change your bid by pressing the “C” clear button. The range of the bids appears on this screen. The computer will not accept any bid smaller than 1 or higher than 25.

Once you are satisfied with your bid, please press the Confirm button.

On the right-hand side of the screen, there is a button labeled History. Please press this button whenever you wish to receive information about the auctions that took place in the previous rounds (see below).

At the upper right-hand part of the screen there is a box showing the current round number, the total number of points you have earned, and your cumulative payoff (in dollars) for the session.

History screen. You obtain access to this screen by pressing the “History” button on the Decision screen. The History screen presents a table of all the auctions that were held in the previous rounds. Each auction is presented in a separate column as follows:

Row 1: The round number.

Row 2: Your previous bids.

Row 3: The values of the winning bid~~s~~.

Other rows: All the bids entered in the auction.

By pushing the arrows left or right, you may view auctions that were held at earlier or later rounds, respectively. Pressing the “Back” button will take you back to the Decision screen.

The History screen is for your convenience only. It keeps track of the entire history of the session in case you wish to consult it.

Results screen. This screen lists your bid for the round, all the bids on that round, the winning bid, whether you won the auction on that round, and your payoff for the round. After examining the results of the round, please press the Next button to continue to the next round.

Payoff

At the end of the experiment, you’ll be paid in cash **$1.00 for every 1.5 points** you earn. You’ll be asked to sign a consent form and a receipt for your payment.

Please remember once again that no communication is allowed during the experiment. If you encounter any difficulties during the experiment, please raise your hand and one of the experimenters will come to assist you.

The experiment will begin shortly once all the participants complete reading the instructions.

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