PROBING GRAVITY: FROM THE ALTERNATIVE TO THE EFFECTIVE

by

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Dedication

A ma mère et à mon frère, qui ont fait preuve d'une patience infinie.

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Table of Contents

TABLE OF CONTENTS — *Continued*

List of Figures

List of Tables

Abstract

While general relativity is a very successful theory of gravity, having thus far passed all observational tests with flying colors, it is thought to be incomplete. Indeed, we lack an ultimate high energy theory in which general relativity and quantum mechanics are both valid. We consider extensions to the action of general relativity, and seek to place constraints on these alternative theories using astrophysical tests. General relativity has been extensively tested in the solar system, but not with precision in strong gravity systems. We discuss constraints that could be placed on alternative theories using neutron stars. We find that we may not be able to distinguish between general relativity and some alternative theories in the spacetimes around black holes. We also discuss constraints from cosmological tests, and show that instabilities can appear.

Adding higher-order terms to the action of general relativity can introduce new dynamical degrees of freedom and instabilities. From the standpoint of effective field theory however, these alternative theories are inconsistent because they are not unitary. In an effective field theory, no new degree of freedom is introduced. This also means that extra polarizations of gravitational waves, which are predicted by some alternative theories, would not be present in an effective field theory.

We then consider an effective field theoretic formulation for gravitational radiation called Non-Relativistic General Relativity (NRGR). We study the gravitational wave emission in non-relativistic coalescing compact binaries, which are thought to be powerful emitters of gravitational waves. While NRGR is based on the post-newtonian (PN) approximation to general relativity, and should therefore be in complete agreement with other post-newtonian methods, the effective

field theory approach provides two major advantages: it provides a consistent framework for the dynamics using a lagrangian formulation; also, one can in principle compute observables to all orders in the orbital velocity in a systematic way. We provide a brief overview of NRGR methods, and present the NRGR calculation of the subleading spin-orbit correction to the newtonian potential.

CHAPTER 1

INTRODUCTION

Modern physics is based on two great theories: general relativity and quantum mechanics. The former describes gravity at large distance scales, while the latter, with the formulation of field theory, describes the other three forces (electromagnetic, weak and strong) at small scales. General relativity assumes a smooth space-time everywhere, while quantum fluctuations make space-time chaotic at small scales. It is however misleading to say that general relativity and quantum mechanics are incompatible: they both do just fine at ordinary, low energies far from the Planck scale. Our ignorance lies in the nature of the ultimate high energy theory where both general relativity and quantum mechanics need to be valid [1]. The end goal is then to find a *quantum theory of gravity* or a *theory of everything* which encompasses both theories. String theory and other theories involving extra dimensions are candidates. They are still work in progress however, and experiments at the Large Hadron Collider [2] at CERN may confirm or exclude many theories including supersymmetry.

While a very successful theory thus far, general relativity has been proven to be insufficient or incomplete in several instances. First, general relativity predicts singularities (where the density becomes infinite), such as at the center of black holes, which does not seem physical. This suggests a breakdown of the theory at those points. Second, recent observations of supernovae [3] and the cosmological microwave background [4] make the presence of *dark energy* in the universe very probable. The presence of this new form of *missing* energy also indicates that general relativity is not the end of the story. Therefore, as elegant as general relativity may be, it makes sense to be looking for extensions to it.

General relativity has been tested in environments of weak gravity such as the solar system, with measurements of the precession of Mercury and the deflection of light around the Sun. Environments of strong gravity such as the spacetimes around neutron stars and black holes have also been studied, and agree qualitatively with the predictions of general relativity. However because general relativity has not been confirmed with precision, this leaves room for the possibility that gravity might deviate from general relativity in environments of strong gravity. It therefore makes sense to try and constrain deviations from general relativity by using strong-gravity astrophysical systems such as neutron stars and black holes.

We propose to extend general relativity by adding higher-order terms to the Einstein-Hilbert action, from which the Einstein equations are derived. The addition of these higher-order terms is actually predicted by string theory in the low-energy limit. The main part of this work consists in exploring the type of astrophysical tests that can constrain these new terms. We consider observations from neutron stars, black holes, as well as cosmology. We also briefly discuss observations from gravitational waves.

Neutron stars are ideal strong-gravity environments to test higher-order theories of gravity. One test that can be done using neutron stars is related to the mass and size of neutron stars. Recent advances in X-ray astronomy have led to the very first measurement of the mass-to-radius ratio of a neutron star via observations of gravitationally redshifted lines from its surface [5]. The predicted mass-to-radius ratio depends very sensitively on the properties of our theory of gravity and can, therefore, provide a strong constraint on deviations from general relativity [6]. We will show that neutron stars can help us constrain the importance of the higher-order terms by 14 orders of magnitude more than solar system tests could.

We also consider the study of black holes to help place constraints. While stellar black holes and neutron stars have roughly the same mass and size, they have different characteristics: there is no matter at the event horizon. One can study the spacetime around a neutron star and match it with the boundary at the surface of the star. However there is no matter at the event horizon of a black hole that one could match the spacetime metric to. The solutions that we study in black hole spacetimes are vacuum solutions.

Cosmological tests can also be used to constrain alternative theories of gravity. These tests are becoming more stringent because of the high-precision cosmological measurements that have recently been made. For example, the combination of supernovae data [3] and cosmic microwave background observations with WMAP [4] gives a precise estimate on the current acceleration rate of the universe. Since the rate of expansion of the universe is dependent upon the theory of gravity, we can use it to place constraints on any deviation from general relativity. Specifically, we look at the evolution of the scale factor as a function of time for different theories of gravity, and see how the evolution differs from general relativity.

This work is timely as much research is currently being done to try and bridge general relativity and quantum mechanics, in the quest to find a *quantum theory of gravity*. String theory and other theories involving extra dimensions, as well as deviations from general relativity using scalar tensor fields or higher-order terms in the curvature, are being investigated. There is no plenty of data coming from astrophysical systems, including neutron stars, black holes and cosmology to help us constrain these theories.

Chapter 2

Astrophysical constraints on alternative theories of gravity

2.1 Introduction

While general relativity is entirely self-consistent at the classical level, the action of the gravitational field may be more general than the Einstein-Hilbert action. The Einstein field equation was indeed constructed to obey only four simple requirements [7]. Of all the possibilities that meet these requirements, the field equations that are derived from the Einstein-Hilbert action are the only ones that are also linear in the Riemann tensor.

Albeit simple and elegant, there is no deep theoretical or experimental reason to choose this over a more general classical action. At the quantum level, one can interpret the metric tensor as a quantum field and the Einstein-Hilbert action as a quantum-field theoretic action. If we proceed to perform quantum-mechanical calculations, radiative corrections will induce a series of counterterms which cannot be reabsorbed into the original Lagrangian. Such terms would also appear as new, higher-derivative corrections terms in the Einstein-Hilbert action. Moreover, string theory incorporates general relativity while predicting an infinite set of correction terms to the Einstein-Hilbert action [8]. Such terms are considered to be string-theoretic corrections to general relativity. Therefore, there are good reasons to believe that general relativity is probably only an effective theory of gravity which is accurate only at relatively small curvatures.

In the limit of weak gravitational fields (or small curvatures), the parametric post-newtonian framework provides the means with which tests of metric gravitational theories can be performed [9]. However this method cannot be used when performing tests of gravity in the strong-field regime, where, by construction, any post-newtonian expansion fails. Other attempts to experimentally constrain deviations from general relativity have been numerous; they were made in the cases where terms added to the action were: scalar fields [10, 11], non-linear functions $f(R)$ of the Ricci scalar [12, 13, 14, 15] or higher-order expansions in the Ricci curvature [16].

Use of astrophysical data is of course crucial for constraining (or even disproving in some cases) these theories. Solar system tests, e.g. the precession of Mercury or the deflection of light around the Sun [9], have been used earlier and, so far, confirm the validity of general relativity. As these involve low curvatures –and the Ricci scalar is a measure of curvature– solar system tests cannot be used to detect with precision corrections to an action with higher-order terms like R^2. Indeed, while general relativity has been extensively tested in astrophysical environments of weak gravity, it still needs to be accurately tested in environments of strong gravity [17].

We look specifically at actions where the added terms are quadratic in the curvature:

$$S \,=\, M_{Pl}^2 \int \sqrt{-g}\,(\mathcal{R} + \alpha\,\mathcal{R}^2 + \beta\,\mathcal{R}^{\mu\nu}\,\mathcal{R}_{\mu\nu} + \gamma\,\mathcal{R}^{\alpha\beta\gamma\delta}\,\mathcal{R}_{\alpha\beta\gamma\delta})\,d^4x. \tag{2.1}$$

Under specific conditions, adding higher-order terms to the Einstein-Hilbert action is equivalent to the addition of scalar fields or additional tensor fields [18] [19]. Here, we will not follow that direction, but will rather work directly with the curvature terms.

There are also two ways of deriving Einstein's equations: through the metric formalism, where the action is minimized with respect to the metric tensor, and the Palatini formalism, where the affine connection is assumed to be independent from the metric tensor and thus the action is minimized with respect to both the metric tensor and the affine connection. In general relativity, these two formulations lead to the same field equations (Einstein's equations), but with extra terms in the action, this is no longer the case. Each formulation leads to different field equations. Also, the metric formalism is by construction torsion-free, since the connection described by Christoffel symbols is symmetric. In the Palatini formalism, the independent connection can be made to be symmetric or not symmetric, i.e. the theory can have torsion or no torsion. Capozziello et al. discuss torsion in f(R) theories [20]. Here we will focus on the torsion-free metric formalism.

Starting from the action (2.1), we derive the field equations and in particular the equations that describe the evolution of the cosmological scale factor and the structure of neutron stars. As mentioned above, strong gravity will better constrain these higher-order terms. Tests of gravity using neutron stars and black holes therefore seem best suited for the task [21]. Recent advances in X-ray astronomy have led to the first measurement of the mass-to-radius ratio of a neutron star via observations of gravitationally redshifted lines from its surface [5]; additionally, one can use the X-ray properties of accreting neutron stars and black holes [21]. While these are environments of high curvature, so is early-universe cosmology. Big bang nucleosynthesis [22] can therefore be studied to constrain $R+R^2$ gravity. Comparing the predictions of this equation with actual cosmological data is essential. This is timely, as much cosmological data has recently been released: supernovae data [3] as well as WMAP data [4], which together predict the current acceleration of the universe.

2.2 Motivation

Astrophysical and cosmological phenomena provide us with the unique ability to constrain the magnitudes of additional terms in the Einstein-Hilbert action in distinct and complementary ways. We focus on the contributions of second-order terms in the Einstein-Hilbert action. The magnitude of the constraints imposed on the parameters of the generalized theory will depend on the curvature of the gravitational field probed by each test. Curvature –viewed as the source of strong gravity– is represented by the Riemann curvature tensor or its contractions, the Ricci tensor and the Ricci scalar. Indeed, with an Einstein-Hilbert action of the form

$$S = M_{Pl}^2 \int \sqrt{-g}\,(\mathcal{R} + \alpha\,\mathcal{R}^2)\,d^4x, \tag{2.2}$$

the higher-order terms will introduce negligible deviations from what is predicted by General Relativity when

$$\alpha\mathcal{R}^2 \ll \mathcal{R}, \tag{2.3}$$

or equivalently when

$$\alpha \ll \frac{1}{\mathcal{R}} \equiv r_{curv}^2, \tag{2.4}$$

where we have defined the "radius of curvature" of the gravitational field as $r_{curv} \equiv \mathcal{R}^{-1/2}$. Clearly, the smaller the radius of curvature of the gravitational field, the better the constraint on the contribution of the higher-order terms to the Lagrangian action.

For tests involving stellar and terrestrial objects, the characteristic scale of the Riemann curvature tensor at a distance r away from the center of an object of mass M is $\mathcal{R}_{\alpha\beta\gamma\delta} \sim GM/r^3c^2$ (we use this here as an order of magnitude estimate and do not consider the fact that, if this distance is larger than the radius of the object, the Ricci tensor and Ricci scalar vanish). Figure 2.1 shows the radii of curvature of the gravitational fields probed by different tests of general relativity. In this case, deviations from general relativity are negligible when

$$\alpha \ll r_{curv}^2 \equiv \left(\frac{r^3c^2}{GM}\right). \tag{2.5}$$

The gravitational fields of neutron stars and stellar black holes have a radius of curvature that is smaller by about 5 orders of magnitude compared to all other tests. Observations of neutron stars and black holes can therefore place more stringent constraints on the higher-order terms by about 10 orders of magnitude. For a neutron star, $r_{curv} \approx 10^7$ cm and hence we expect to be able to impose constraints on the parameters of order $\alpha \leq (10^7$ cm$)^2$ by comparing our model calculations to the observed neutron-star properties (such as their radii, maximum masses or maximum spins). We note that the possible detection of gravitational waves by LIGO or LISA can also be used to test general relativity.

For tests involving the evolution of the universe, and assuming small deviations from general relativity, the curvature radius is of the order of the Hubble distance, $r_{curv} \sim D_H \equiv c/H$ and hence the contribution of higher-order terms will be negligible when:

$$\alpha \ll \left(\frac{c}{H}\right)^2 = (c\, t_H)^2\,, \tag{2.6}$$

where H is the Hubble parameter and $t_H \equiv 1/H$ is the Hubble time. Figure 2.2 shows the characteristic Hubble time at which the evolution of the universe would be affected by the higher-order terms in the action. The early universe is an ideal laboratory for constraining higher-order terms. The best constraints on strong-field gravity from cosmological observations can be achieved by comparing the elemental abundances predicted by Big Bang nucleosynthesis calculations with observations. Such phenomena probe the gravitational field in the Universe when the Hubble time was of order ~ 1 s, and hence place constraints on the coefficients of the higher-order terms that are of order $(3 \times 10^{10}$ cm$)^2$. Inflation would provide an even better constraint at (10^{-34} cm$)^2$. R^2 terms were in fact first suggested by Starobinsky [13] to generate inflation. Current experiments however are ill-suited to probe such early time periods.

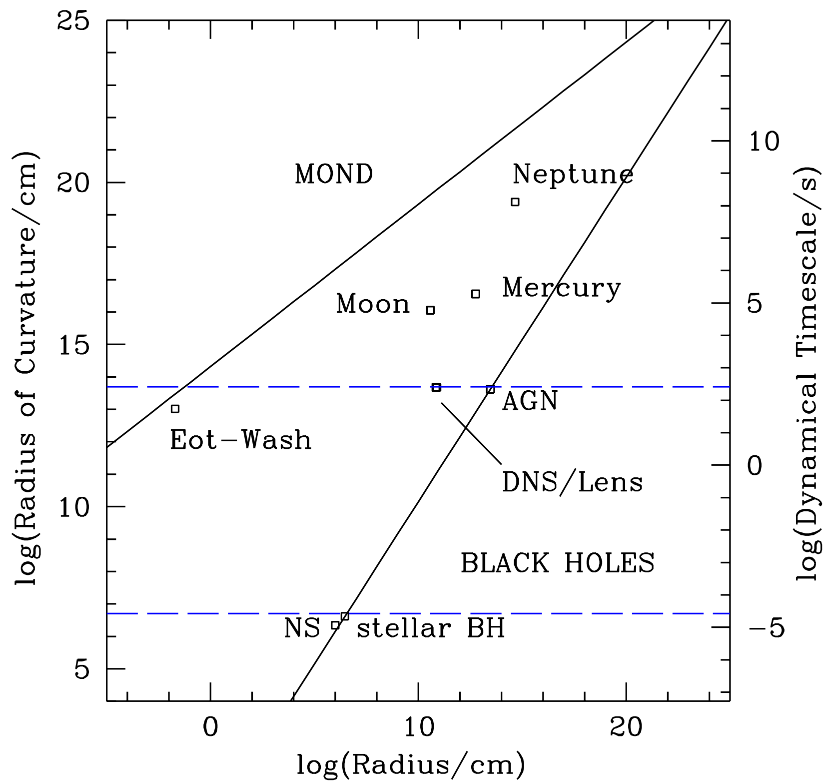


Figure 2.1: Tests of general relativity placed on an appropriate parameter space [23]. The y-axis represents the scale of the "radius of curvature", defined as $r_{curv} \equiv \sqrt{r^3c^2/GM}$, of the gravitational field probed by a test performed at a distance r away from a central object of mass M. The dynamical time is defined as the characteristic orbital time $r_{curv}/2\pi c$ of test particles at the same place in the gravitational field, and is relevant for tests of gravity with timing studies of compact objects. The lower right diagonal line represents the event horizons of Schwarzschild black holes, whereas the upper left diagonal line represents the MOND acceleration scale ($a_0 = 1.2 \times 10^{-10}\ m/s^2$). The lower dashed horizontal line corresponds to frequencies of gravitational waves probed by LIGO, while the upper dashed horizontal line represents frequencies probed by LISA. Not plotted on the graph is the possibility of a LHC black hole, which would give us even better constraints, but we are not yet sure these exist. [24]

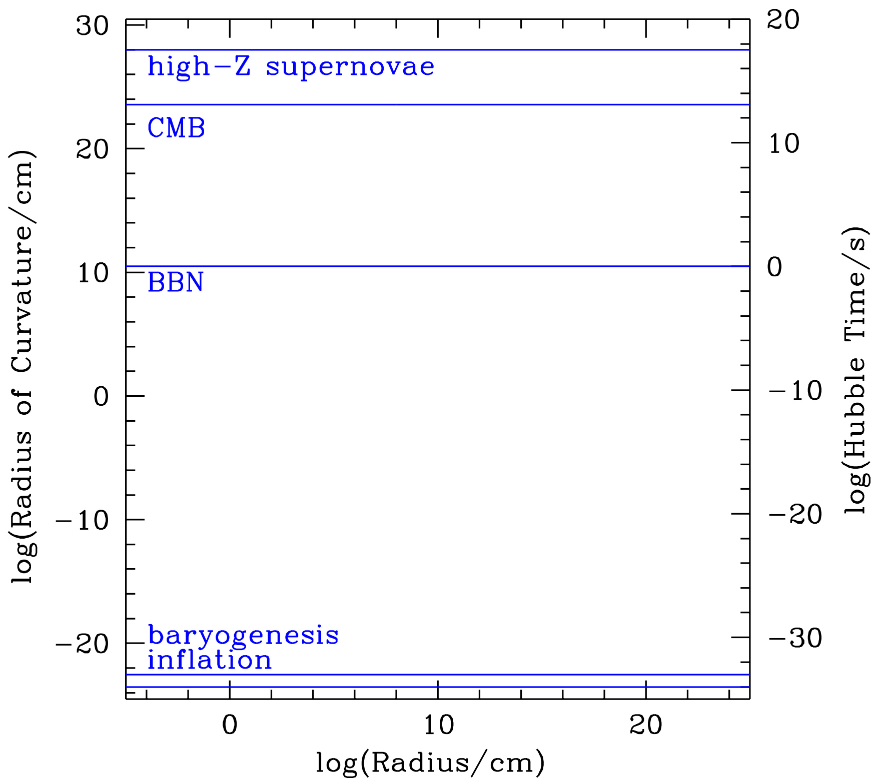


Figure 2.2: Cosmological tests of general relativity. The y-axis represents the radius of curvature (or Hubble distance) and on the right, the Hubble time (equal to r_{curv}/c) of the universe down to which general relativity should accurately describe gravitational interactions. [4, 22]

2.3 Field equations for alternative theories of gravity

2.3.1 Sign conventions

Here we use the Misner-Thorne-Wheeler $(+++)$ convention [7] for the metric signature:

$$+g = -(\omega^0)^2 + (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2, \tag{2.7}$$

the Riemann tensor

$$+R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}{}_{\nu\beta} - \partial_\beta \Gamma^{\mu}{}_{\nu\alpha} + \Gamma^{\mu}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\nu\beta} - \Gamma^{\mu}{}_{\sigma\beta}\Gamma^{\sigma}{}_{\nu\alpha}, \tag{2.8}$$

the Einstein equation

$$G_{\mu\nu} = +8\pi G\, T_{\mu\nu}, \tag{2.9}$$

where we define the Christoffel symbols to be

$$\Gamma^{\mu}{}_{\nu\beta} = \frac{1}{2} g^{\alpha\delta} \left(\frac{\partial g_{\alpha\beta}}{\partial x^\nu} + \frac{\partial g_{\alpha\nu}}{\partial x^\beta} - \frac{\delta g_{\beta\nu}}{\partial x^\alpha} \right), \tag{2.10}$$

the Ricci tensor

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}, \tag{2.11}$$

the Ricci scalar

$$R = R^{\mu}{}_{\mu} = R_{\mu\nu} g^{\mu\nu}, \tag{2.12}$$

and finally the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \tag{2.13}$$

The choice of sign convention leads to different signs in the Einstein equations, although the physical significance of the terms does not change. Different notations are used in the literature. Barraco and Hamity [16] also use the MTW $(+++)$ convention. Buchdahl [25] however uses the $(---)$ notation. Davies [26] shows how to go from one notation to the other. DeWitt [27] uses a different notation altogether $(+-+?)$ but finds the same field equation as for $(---)$.

2.3.2 Generalized Einstein-Hilbert action

We are interested in an Einstein-Hilbert action that is second-order in the curvature:

$$S = M_{Pl}^2 \int \sqrt{-g}\left(R + \alpha R^2 + \beta R_{\sigma\tau}R^{\sigma\tau} + \gamma R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\right) d^4x, \tag{2.14}$$

where $M_{Pl}^2 = 1/(16\pi G)$.

Classically, because of the Gauss-Bonnet identity, variations with respect to the metric of the term proportional to $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ can be expressed as variations of the terms proportional to R^2 and $R_{\sigma\tau}R^{\sigma\tau}$ [12]:

$$\frac{\delta}{\delta g_{\mu\nu}} \int d^4x\sqrt{-g}\left(R^2 - 4R_{\sigma\tau}R^{\sigma\tau} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\right) = 0. \tag{2.15}$$

Therefore in classical cases, the predictions of the theory described by the action (2.14) are identical to those of the action

$$S = M_{Pl}^2 \int \sqrt{-g}\left(R + \alpha' R^2 + \beta' R_{\sigma\tau}R^{\sigma\tau}\right), \tag{2.16}$$

where

$$\alpha' = \alpha - \gamma, \tag{2.17}$$

$$\beta' = \beta + 4\gamma. \tag{2.18}$$

When the spacetime is isotropic and homogeneous, as is the case in cosmology, an additional identity is satisfied [12], i.e.,

$$\frac{\delta}{\delta g_{\mu\nu}} \int d^4x\sqrt{-g}\left(R^2 - 3R_{\sigma\tau}R^{\sigma\tau}\right) = 0. \tag{2.19}$$

This implies that in cosmology, the predictions of the theory described by the action (2.14) are identical to those of the action

$$S = M_{Pl}^2 \int \sqrt{-g}\left(R + \alpha'' R^2\right), \tag{2.20}$$

where

$$\alpha'' = \alpha + \frac{1}{3}\beta + \frac{1}{3}\gamma. \tag{2.21}$$

2.3.3 Generalized Einstein Equations in $R + R^2$-Gravity

As seen above, in the context of R^2–gravity, the most general classical action is:

$$S = M_{Pl}^2 \int \sqrt{-g}\left(R + \alpha R^2 + \beta R_{\sigma\tau} R^{\sigma\tau}\right) d^4x. \tag{2.22}$$

In order to find the generalized Einstein equations, we minimize the action with respect to the metric tensor $g^{\mu\nu}$. There are three contributions: the GR term

$$S_0 = M_{Pl}^2 \int \sqrt{-g}\, R\, d^4x = M_{Pl}^2 \int \sqrt{-g}\, R_{\mu\nu}\, g^{\mu\nu}\, d^4x, \tag{2.23}$$

for which

$$\frac{\delta S_0}{\delta g^{\mu\nu}} = M_{Pl}^2 \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R\right). \tag{2.24}$$

The term proportional to the square of the Ricci curvature:

$$S_1 = M_{Pl}^2 \int \sqrt{-g}\, R^2\, d^4x, \tag{2.25}$$

for which

$$\frac{\delta S_1}{\delta g^{\mu\nu}} = M_{Pl}^2 \sqrt{-g}\left(-2\, R_{;\mu\nu} + 2\, g_{\mu\nu}\, \Box R + 2\, R\, R_{\mu\nu} - \frac{1}{2}\, R^2\, g_{\mu\nu}\right), \tag{2.26}$$

and where

$$R_{;\mu\nu} = \frac{\partial^2}{\partial x^\mu \partial x^\nu} R, \tag{2.27}$$

$$\Box R = \frac{\partial^2}{\partial x^\mu \partial x_\mu} R, \tag{2.28}$$

and the term proportional to the square of the Ricci tensor:

$$S_2 = M_{Pl}^2 \int \sqrt{-g}\, R_{\sigma\tau}\, R^{\sigma\tau}\, d^4x, \tag{2.29}$$

for which

$$\frac{\delta S_2}{\delta g^{\mu\nu}} = M_{Pl}^2 \sqrt{-g}\big(-2\, R_\mu{}^\sigma{}_{;\sigma\nu} + \Box R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R_{\sigma\tau} R^{\sigma\tau} + \frac{1}{2}\, g_{\mu\nu}\, \Box R + 2\, R_\mu{}^\alpha\, R_{\alpha\nu} \big). \tag{2.30}$$

The generalized field equations are therefore

$$R_{\mu\nu} - \frac{1}{2}\, R\, g_{\mu\nu} + \alpha\, K_{\mu\nu} + \beta\, L_{\mu\nu} = 8\pi G\, T_{\mu\nu}, \tag{2.31}$$

where

$$K_{\mu\nu} \equiv -2\, R_{;\mu\nu} + 2\, g_{\mu\nu}\, \Box R - \frac{1}{2}\, R^2\, g_{\mu\nu} + 2\, R\, R_{\mu\nu}, \tag{2.32}$$

and

$$L_{\mu\nu} \equiv -2\, R_\mu{}^\sigma{}_{;\sigma\nu} + \Box R_{\mu\nu} + \frac{1}{2}\, g_{\mu\nu}\, \Box R - \frac{1}{2}\, g_{\mu\nu}\, R_{\sigma\tau}\, R^{\sigma\tau} + 2\, R_\mu{}^\alpha\, R_{\alpha\nu}. \tag{2.33}$$

CHAPTER 3

CONSTRAINTS WITH NEUTRON STARS

3.1 Motivation

As seen in Chapter 2, using the observed properties of neutron stars can lead to more stringent constraints on the contributions from higher-order terms. For neutron stars, the action is described by Equation (2.22):

$$S = M_{Pl}^2 \int \sqrt{-g} \left(R + \alpha R^2 + \beta R_{\sigma\tau} R^{\sigma\tau} \right) d^4x. \tag{3.1}$$

and the generalized field equations by (2.31), (2.32) and (2.33).

One idea that has been explored in theories like scalar tensor theories is that of observing the orbits of particles in the external spacetime of a neutron star [21]. This will not be useful however when studying $R + R^2$ gravity. Indeed, any solution to general relativity's vacuum field equations (for which $R_{\mu\nu} = 0$) is also a solution to the generalized field equations of $R + R^2$, and this is true even in the presence of a cosmological constant. Therefore one will not be able to distinguish between the two theories when solely working in the external spacetime of a neutron star (we explore this in detail in Chapter 4).

Solutions in the interior of a neutron star will however depend on the theory of gravity. One can derive the expressions $M(r)$ (mass of neutron star as a function of its radius) and dp/dr (pressure differential along the radius of the neutron star), which together are called the Oppenheimer-Volkoff equations, the relativistic equivalent of hydrostatic equilibrium. They are simply the reduction of the generalized Einstein equations for the interior of static and spherically symmetric stars. Given an equation of state for the neutron star, these two equations can be solved simultaneously [28]. From this we can obtain expressions that determine the structure of the neutron star: mass, pressure and density as a function of radius.

One astrophysical test involving neutron stars is the determination of their mass-to-radius ratio. Heavy elements falling into the surfaces of neutron stars can lead to the emission of surface atomic lines which, due to the strong gravitational field surrounding neutron stars, become gravitationally redshifted. A relation between mass and radius can be determined from such a measurement of the gravitational redshift in surface atomic lines. [5, 6, 29]. Atomic lines are rarely observed in neutron stars however. The most promising candidates are young cooling stars and accreting X-ray bursters, in which heavy elements are common. A gravitational redshift was observed in the slowly rotating neutron star EXO 0748-676 during thermonuclear flashes. The mass-to-radius relation was found to be: $M \simeq 1.4\,(R/10\ \text{km})\,M_\odot$ [30]. A theory-dependent expression $M(r)$ (as explained above) can then be compared to the mass-to-radius ratio determined from atomic line observations, which are theory-independent. Comparisons of theory-dependent calculations and observations enable us to constrain the field of possible theories.

In this chapter, we work toward calculating the Oppenheimer-Volkoff equations for neutron stars in $R+R^2$ gravity. We note that calculations of the Oppenheimer-Volkoff equations for relativistic stars in $R+R^2$ gravity were done by Stelle [31] as well as Barraco and Hamity [16]. They linearized the metric, meaning that the calculations are valid for relativistic stars in the weak-field limit only. Because we are studying the strong-field gravity of neutron stars, an approximation will not be valid here. We attempt instead to compute the equations exactly. I show in this chapter how we get started, where the equations get complicated, and how we may be able to simplify them. Overall, the full calculation of the Oppenheimer-Volkoff equations in $R+R^2$ gravity turns out to be a complex numerical problem.

3.2 Deriving the Oppenheimer-Volkoff equations

To derive these equations, we first assume the interior of a neutron star to be static and spherically symmetric, and thus consider the metric:

$$ds^2 = -B(r)\,dt^2 + A(r)\,dr^2 + r^2\,(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{3.2}$$

where $A(r)$ and $B(r)$ are functions of r only. For the matter content, we use the energy-momentum tensor of a perfect fluid and assume spherical symmetry:

$$T_{\mu\nu} = p\,g_{\mu\nu} + (p+\rho)\,U_\mu\,U_\nu, \tag{3.3}$$

where p is the pressure, ρ the total energy density (functions of r only) and U^μ the velocity four-vector. For a fluid at rest, $U_r = U_\theta = U_\phi = 0,\ \ U_t = -\sqrt{B(r)}$. We are left with

$$T_{\mu\nu} = \begin{pmatrix} \rho\,B(r) & 0 & 0 & 0 \\ 0 & p\,A(r) & 0 & 0 \\ 0 & 0 & p\,r^2 & 0 \\ 0 & 0 & 0 & p\,r^2\,\sin^2\theta \end{pmatrix}, \tag{3.4}$$

or in mixed form

$$T_\mu{}^\nu = g^{\alpha\nu}T_{\mu\alpha}, \tag{3.5}$$

we have

$$T_\mu{}^\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{3.6}$$

Using this metric and energy-momentun tensor, we write the generalized field equations (2.31) in mixed tensor form. The t-t equation becomes

$$-\frac{1}{r^2 A}\left[-1 + A + rA'/A + (...)\right] = -8\pi\, G\, \rho, \tag{3.7}$$

where $(...)$ stands for the 35 terms listed in Table 3.1. The r-r component leads to

$$\frac{1}{r^2 A}\left[1 - A + rB'/B + (...)\right] = 8\,\pi\, G\, p, \tag{3.8}$$

where $(...)$ stands for the 26 terms listed in Table 3.2.

3.3 Looking for terms of leading order

Many new terms appear in the t-t and r-r components of the generalized Einstein tensor, making the equations quite complicated. We sought to simplify the equations by finding which terms were dominant, if any. We estimated the magnitude of these terms by using the general relativistic values of the metric components $A(r)$ and $B(r)$. In general relativity, we have [28]

$$A(r) = g_{rr} = \frac{1}{1 - 2M(r)/r}, \tag{3.9}$$

We also know that:

- $M'(r) = 4\pi r^2 \rho(r)$,
- $\frac{d \ln \rho}{d \ln r} = 1$, leading to $r\rho' = \rho$.

From this we find:

$$A' = 2A^2 \left(4\pi r \rho - \frac{M}{r^2}\right), \tag{3.10}$$

as well as higher derivatives in $A(r)$. On the other hand,

$$B(r) = -g_{00} = e^{2\,\nu(r)}, \tag{3.11}$$

1	$(10\alpha/r^2 + 3\beta/r^2)1/A$
2	$-4(3\alpha/r^2 + \beta/r^2)$
3	$(2\alpha/r^2 + \beta/r^2)A$
4	$2(4\alpha/r^2 + \beta/r^2)r\frac{A'}{A^2}$
5	$(-2\beta/r^2)r\frac{B'}{B}$
6	$(2\beta/r^2)r\frac{B'}{AB}$
7	$1/2(4\alpha/r^2 + \beta/r^2)r^2\frac{A'}{A^2}\frac{B'}{B}$
8	$-1/2(32\alpha/r^2 + 17\beta/r^2)r^3\frac{A'}{A^2}\frac{B''}{B}$
9	$-3(2\alpha/r^2 + \beta/r^2)r^4\frac{A'}{A^2}\frac{B^{(3)}}{B}$
10	$1/2(30\alpha/r^2 + 15\beta/r^2)r^3\frac{A'}{A^2}(\frac{B'}{B})^2$
11	$-29/8(2\alpha/r^2 + \beta/r^2)r^4\frac{A'}{A^2}(\frac{B'}{B})^3$
12	$27/4(2\alpha/r^2 + \beta/r^2)r^4\frac{A'}{A^2}\frac{B'}{B}\frac{B''}{B}$
13	$-(4\alpha/r^2 + \beta/r^2)r^2\frac{A''}{A^2}$
14	$-1/2(12\alpha/r^2 + 5\beta/r^2)r^3\frac{A''}{A^2}\frac{B'}{B}$
15	$-2(2\alpha/r^2 + \beta/r^2)r^4\frac{A''}{A^2}\frac{B''}{B}$
16	$3/2(2\alpha/r^2 + \beta/r^2)r^4\frac{A''}{A^2}(\frac{B'}{B})^2$
17	$-(4\alpha/r^2 + \beta/r^2)r^3\frac{A^{(3)}}{A^2}$
18	$-1/2(2\alpha/r^2 + \beta/r^2)r^4\frac{A^{(3)}}{A^2}\frac{B'}{B}$
19	$1/4(40\alpha/r^2 + 11\beta/r^2)r^2\frac{A'^2}{A^3}$
20	$1/4(56\alpha/r^2 + 25\beta/r^2)r^3\frac{A'^2}{A^3}\frac{B'}{B}$
21	$19/4(2\alpha/r^2 + \beta/r^2)r^4\frac{A'^2}{A^3}\frac{B''}{B}$
22	$-57/16(2\alpha/r^2 + \beta/r^2)r^4\frac{A'^2}{A^3}(\frac{B'}{B})^2$
23	$-7(4\alpha/r^2 + \beta/r^2)r^3\frac{A'^3}{A^4}$
24	$-7/2(2\alpha/r^2 + \beta/r^2)r^4\frac{A'^3}{A^4}\frac{B'}{B}$
25	$13/2(4\alpha/r^2 + \beta/r^2)r^3\frac{A'A''}{A^3}$
26	$13/4(2\alpha/r^2 + \beta/r^2)r^4\frac{A'A''}{A^3}\frac{B'}{B}$
27	$-1/4(8\alpha/r^2 + 5\beta/r^2)r^2\frac{1}{A}(\frac{B'}{B})^2$
28	$1/4(52\alpha/r^2 + 25\beta/r^2)r^3\frac{1}{A}(\frac{B'}{B})^3$
29	$-49/16(2\alpha/r^2 + \beta/r^2)r^4\frac{1}{A}(\frac{B'}{B})^4$
30	$-9/4(2\alpha/r^2 + \beta/r^2)r^4\frac{1}{A}(\frac{B''}{B})^2$
31	$-1/2(44\alpha/r^2 + 23\beta/r^2)r^3\frac{1}{A}\frac{B'}{B}\frac{B''}{B}$
32	$29/4(2\alpha/r^2 + \beta/r^2)r^4\frac{1}{A}(\frac{B'}{B})^2\frac{B''}{B}$
33	$4(2\alpha/r^2 + \beta/r^2)r^3\frac{1}{A}\frac{B^{(3)}}{B}$
34	$-3(2\alpha/r^2 + \beta/r^2)r^4\frac{1}{A}\frac{B'}{B}\frac{B^{(3)}}{B}$
35	$(2\alpha/r^2 + \beta/r^2)r^4\frac{1}{A}\frac{B^{(4)}}{B}$

Table 3.1: Terms in the t-t equation

1	$(14\alpha/r^2 + 5\beta/r^2)1/A$		14	$-1/4(28\alpha/r^2 + 3\beta/r^2)r^3\frac{A'^2}{A^3}\frac{B'}{B}$
2	$-4(3\alpha/r^2 + \beta/r^2)$		15	$-7/16(2\alpha/r^2 + \beta/r^2)r^4\frac{A'^2}{A^3}(\frac{B'}{B})^2$
3	$-(2\alpha/r^2 + \beta/r^2)A$		16	$1/4(32\alpha/r^2 + 11\beta/r^2)r^2\frac{1}{A}(\frac{B'}{B})^2$
4	$(-2\beta/r^2)r\frac{A'}{A}$		17	$-1/4(8\alpha/r^2 - \beta/r^2)r^3\frac{1}{A}(\frac{B'}{B})^3$
5	$(2\beta/r^2)r\frac{A'}{A^2}$		18	$-7/16(2\alpha/r^2 + \beta/r^2)r^4\frac{1}{A}(\frac{B'}{B})^4$
6	$1/2(16\alpha/r^2 + 7\beta/r^2)r^2\frac{A'}{A^2}\frac{B'}{B}$		19	$1/4(2\alpha/r^2 + \beta/r^2)r^4\frac{1}{A}(\frac{B''}{B})^2$
7	$-1/2(2\alpha/r^2 - 3\beta/r^2)r^3\frac{A'}{A^2}(\frac{B'}{B})^2$		20	$(6\alpha/r^2)r^3\frac{1}{A}\frac{B'}{B}\frac{B''}{B}$
8	$-3/8(2\alpha/r^2 + \beta/r^2)r^4\frac{A'}{A^2}(\frac{B'}{B})^3$		21	$3/4(2\alpha/r^2 + \beta/r^2)r^4\frac{1}{A}(\frac{B'}{B})^2\frac{B''}{B}$
9	$1/2(2\alpha/r^2 + \beta/r^2)r^4\frac{A'}{A^2}\frac{B'}{B}\frac{B''}{B}$		22	$-(4\alpha/r^2 + \beta/r^2)r^3\frac{1}{A}\frac{B^{(3)}}{B}$
10	$(8\alpha/r^2 + 3\beta/r^2)r^2\frac{A''}{A^2}$		23	$-1/2(2\alpha/r^2 + \beta/r^2)r^4\frac{1}{A}\frac{B'}{B}\frac{B^{(3)}}{B}$
11	$(4\alpha/r^2 + \beta/r^2)r^3\frac{A''}{A^2}\frac{B'}{B}$		24	$2(4\alpha/r^2 + \beta/r^2)r\frac{1}{A}\frac{B'}{B}$
12	$1/4(2\alpha/r^2 + \beta/r^2)r^4\frac{A''}{A^2}(\frac{B'}{B})^2$		25	$-2(4\alpha/r^2 + \beta/r^2)r^2\frac{1}{A}\frac{B''}{B}$
13	$-1/4(56\alpha/r^2 + 17\beta/r^2)r^2\frac{A'^2}{A^3}$		26	$(4\alpha/r^2 - \beta/r^2)r^3\frac{A'}{A^2}\frac{B''}{B}$

Table 3.2: Terms in the r-r equation

so that [28]:

$$\frac{B'}{B} = (lnB)' = 2\,\frac{d\nu}{dr} = 2\,\frac{M + 4\pi r^3 p}{r^2(1 - 2M/r)}. \tag{3.12}$$

Assuming that the pressure p and its derivatives are negligible, we get:

$$B' = 2\,A\,B\,\frac{M}{r^2}, \tag{3.13}$$

from which higher derivatives in $B(r)$ can be calculated.

Once all the derivatives were calculated, we used the following approximation: $\rho = \frac{M}{4/3\pi r^3}$. Finally, we were able to estimate the magnitude of each of the tt and rr terms as functions in M/r. We plotted in Fig 3.1 and Fig 3.2 the relative amplitudes of each of these terms.

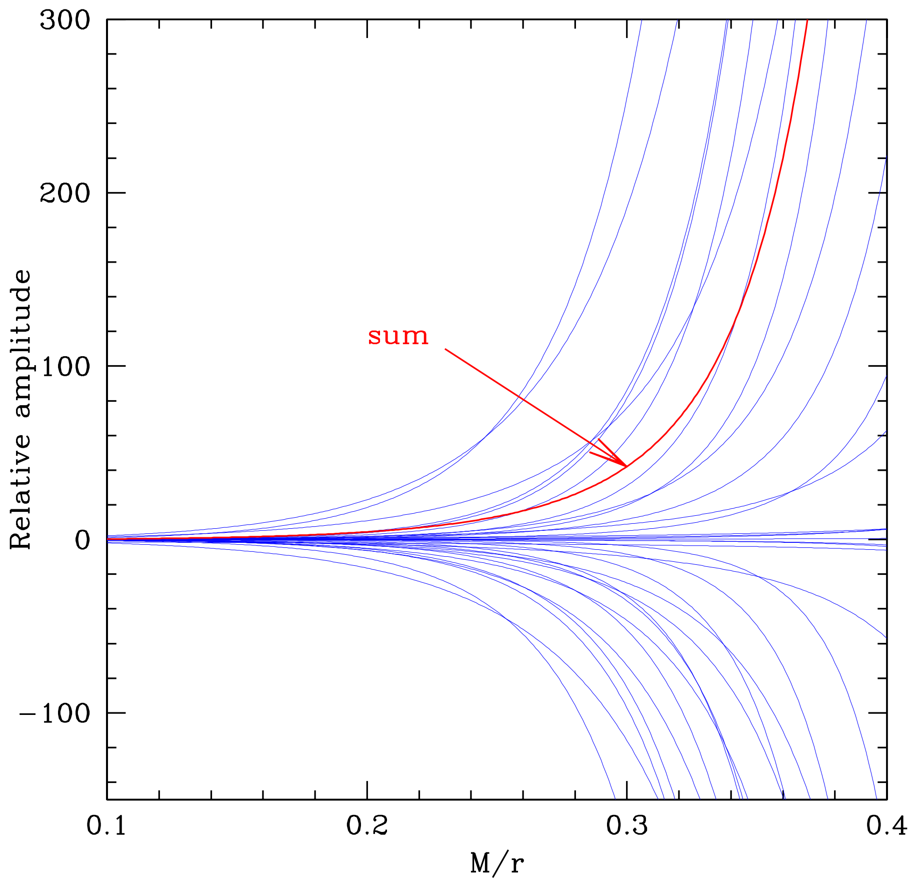


Figure 3.1: We plotted the relative amplitude of each of the 35 terms in the t-t component of the generalized Einstein tensor as a function of M/r. The relative amplitude is the estimated amplitude of each term using general relativity, divided by the amplitude of the first term in the series, with r^2/α factored out. β is taken to be equal to α, but even for smaller (or null) values of β, the curves have the same overall behavior. The relative sum of all 35 terms is also plotted. We find that none of the terms closely matches the overall sum, meaning that none of the terms can be safely ignored. We therefore won't be able to simplify the t-t equation.

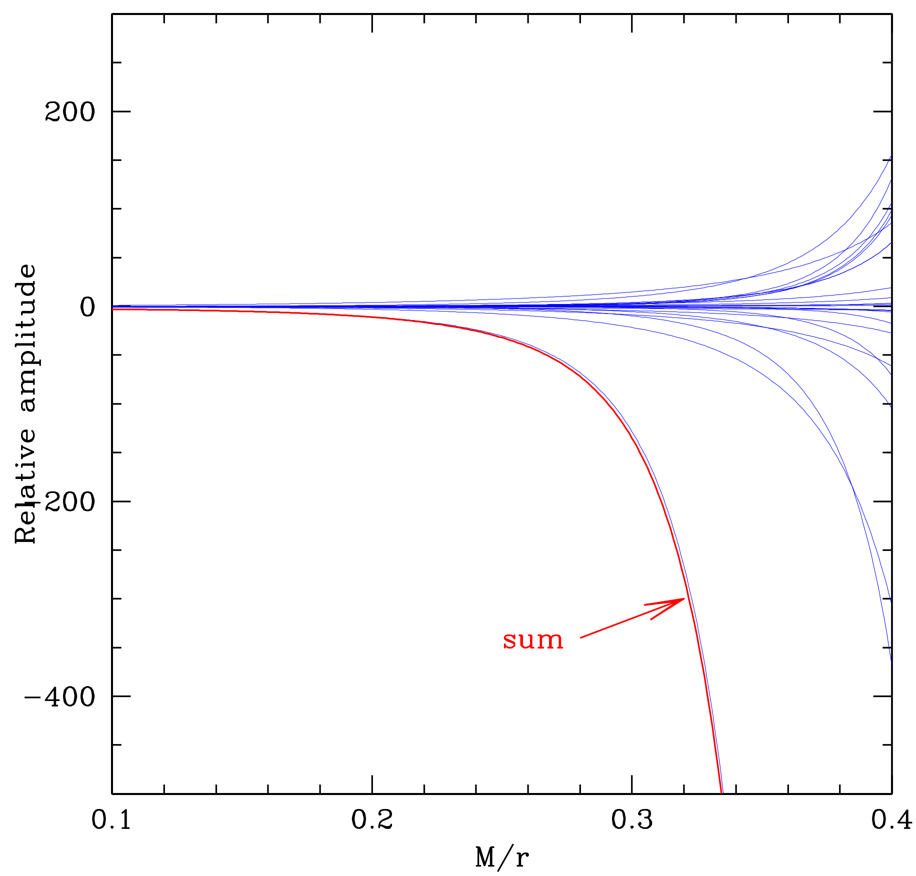


Figure 3.2: In a similar fashion, we plotted the relative amplitude of the 26 terms in the r-r component of the generalized Einstein tensor as a function of M/r. When the relative sum is plotted, we find that it is very closely matched by one of the 26 terms. Term 8 in Table 3.2 of the form $-\frac{A'}{A^2}(\frac{B'}{B})^3$ is overwhelmingly dominant. While the magnitudes were estimated using general relativity (and not higher-order gravity), it is a safe approximation in further calculations to keep the dominant term and neglect the remainder of the terms.

Replacing the (...) in Equation (3.8), we get:

$$\frac{1}{r^2 A}\left[1 - A + rB'/B - 3/8(2\alpha+\beta)r^2\frac{A'}{A^2}\left(\frac{B'}{B}\right)^3\right] = 8\,\pi\,G\,p, \tag{3.14}$$

This is a coupled differential equation with only first order derivatives in $A(r)$ and $B(r)$. Higher order derivatives in the metric, which naturally occur when adding higher order terms to the action, are absent in the r-r equation. One could use this simplified r-r equation to derive the generalized Oppenheimer-Volkoff equations, following the methods of Glendenning [28]. However for the t-t equation which gives the density $\rho(r)$, we need to keep all of the terms.

3.4 Discussion

A perturbative approach – as was done in the stellar case [16, 31] – is not appropriate in the strong-gravity environment of a neutron star. For the time-time component, as seen in Figure 3.1, none of the terms can be neglected. However for the r-r component, as seen in Figure 3.2, the equation simplifies drastically as one of the terms closely matches the sum. These two equations can be used together to derive the Oppenheimer-Volkov equations –the equivalent of hydrostatic equilibrium– which express the changes in pressure and density along the radius of the neutron star. After selecting an appropriate equation of state, we can numerically solve for the Oppenheimer-Volkov equations. Results can then be matched against observed values of mass-to-radius ratio for neutron stars, thus constraining the α and β coefficients of the higher order theory.

CHAPTER 4

CONSTRAINTS WITH BLACK HOLES

4.1 Introduction

We now explore the role that black holes can play in constraining alternative theories of gravity [32].

Black holes are among the most extreme astrophysical objects predicted by general relativity. They are vacuum solutions of the Einstein field equations realized astrophysically at the end stages of the collapse of massive stars. According to a variety of no-hair theorems, a general relativistic black hole is characterized only by three parameters identified with its gravitational mass, spin, and charge. Any additional "hair" on the black hole, associated with the properties of the progenitor star or the collapse itself, are radiated away in the form of gravitational waves over a finite amount of time.

Black holes might look different if general relativity is only an effective theory of gravity, valid at the curvature scales probed by current terrestrial and astrophysical experiments. If the more fundamental gravity theory has additional degrees of freedom, they might appear as additional "hair" to the black hole. This would be important for a number of reasons. First, additional degrees of freedom appear naturally in all attempts to quantize gravity, either in a perturbative approach [33] or within the context of string theory [8]. Detecting observational signatures of these additional degrees of freedom in black-hole spacetimes would serve as a confirmation of quantum gravity effects. Second, black-hole solutions not described by the Kerr-Newman metric may follow a set of thermodynamic relations different than those calculated by Bekenstein [34] and Hawking [35] with important implications for string theory [36]. Finally, the external spacetimes of astrophysical black hole will soon be mapped with gravitational-wave [37] and high-energy observations [30] and the means for searching for black holes with additional degrees of freedom will become readily available.

Introducing additional degrees of freedom to the Einstein-Hilbert action of the gravitational field does not necessarily alter the resulting field equations and hence the black-hole solutions. For example, the addition of a Gauss-Bonnet term to the action leaves the field equation completely unchanged [33]. Moreover, a large class of gravity theories in the Palatini formalism for which the action is a general function $f(R)$ of the Ricci scalar curvature R, lead to field equations that are indistinguishable from the general relativistic ones [38]. In all these situations, no astrophysical observation of a classical phenomenon, such as test particle orbits or gravitational lensing, can distinguish between these theories. Nevertheless, this leaves a large number of Lagrangian gravity theories that incorporate general relativity as a limiting case but are described by more general field equations.

The most widely studied such extension of general relativity is the Brans-Dicke gravity, which incorporates a dynamical scalar field in addition to the metric tensor. Black hole solutions in this theory were studied by Thorne & Dykla [39]. Following a conjecture by Penrose, these authors showed that the Kerr solution of general relativity is also an exact solution of the field equations in Brans-Dicke gravity and offered a number of arguments to support the claim that the collapse of a star in this gravity theory will produce uniquely a Kerr black hole. Additional analytic [35, 40, 41] and numerical [42] arguments were offered by other authors providing further evidence for the uniqueness of the Kerr solution in Brans-Dicke gravity.

We show that black-hole solutions of the general relativistic field equations are indistinguishable from solutions of a wide variety of gravity theories that arise by adding dynamical vector and tensor degrees of freedom to the Einstein-Hilbert action. Although we do not prove that the general relativistic vacuum solution is the *unique* solution of the extended Lagrangian theories, we use our results to argue that an observational verification of the Kerr solution for an astrophysical object cannot be used in distinguishing between general relativity and other Lagrangian theories such as those considered here. Note that we are only considering four-dimensional theories that obey the equivalence principle, and hence we are not studying theories with prior geometry [43], that are Lorentz violating [44], or braneworld gravity theories [45]. Although several of these extensions lead to predictions of an unstable quantum vacuum and of ghosts, we are focusing here on their classical black-hole solutions.

In general relativity, the external spacetimes of black holes that are astrophysically relevant, i.e., with zero charge, are completely specified by the relation

$$R_{\mu\nu} = \frac{R}{4} g_{\mu\nu} \ , \tag{4.1}$$

with $R_{,\mu} = 0$. Here $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar curvature. When the cosmological constant Λ is considered to be non-zero, then $R = 4\Lambda$.

It is our aim to show that the external spacetimes of general relativistic black holes, which satisfy equation (4.1), are practically indistinguishable from solutions in a number of gravity theories that arise by adding vector or tensor degrees of freedom to the Einstein-Hilbert action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(-2\Lambda + R \right) \ . \tag{4.2}$$

It is important to make here a distinction between the external spacetimes of black holes and those of stellar objects, which also satisfy equation (4.1) in general relativity. The field equation of a gravity theory is a high-order partial differential equation and its solutions depend on the boundary conditions imposed. In particular, when solving for the external spacetime of a stellar object, a number of regularity conditions need to be satisfied at the stellar surface, inside which the field equations are altered by the presence of matter. As a result, proving that the external spacetime of a general relativistic star satisfies the vacuum field equation of a different gravity theory is not a guarantee that it will be a valid solution for that theory, as well. It also needs to meet the altered regularity conditions at the stellar surface. This issue was recently explored for $1/R$ gravity in the metric [46] and in the Palatini formalism [47] with important implications for the viability of this theory. This concern, however, is not relevant for black-hole solutions, in which there is no matter anywhere outside the horizon and hence no regularity conditions need to be met. Indeed the vacuum field equation is valid throughout the entire spacetime accessible to a distant observer and only the boundary conditions at radial infinity need to be checked.

4.2 $f(R)$ Gravity in the Metric Formalism

A self-consistent theory of gravity can be constructed for any Lagrangian action that obeys a small number (four) of simple requirements [7]. Of all the possibilities, the field equations that are derived from the Einstein-Hilbert action (4.2) are the only ones that are also linear in the Riemann tensor and result in field equations that are of second order. However, any other action $f(R)$ that depends only on the Ricci curvature scalar will also satisfy the above four requirements [48], while being free of the Ostrogradski instability [49].

The field equation that results from extremizing an action that is a general function of the Ricci scalar, $f(R)$, is

$$\begin{aligned}(-R_{;k}R_{;l} + g_{kl}R_{;m}R^{;m})\, f'''(R) \\ + (-R_{;kl} + g_{kl}\Box R)\, f''(R) \\ + R_{kl}f'(R) - \frac{1}{2}g_{kl}f(R) = 0\ ,\end{aligned} \tag{4.3}$$

where primes denote differentiation with respect to R and we have used the sign convention of Ref. [7].

A general relativistic black-hole solution, i.e., one that satisfies equation (4.1) with $R_{,\mu} = 0$, will also be a solution of the field equation (4.3) if

$$\frac{1}{2}Rf'(R) - f(R) = 0\ . \tag{4.4}$$

We will now consider non-pathological functional forms of $f(R)$ that can be expanded in a Taylor series of the form

$$f(R) = a_0 + R + a_2R^2 + a_3R^3 + ...a_nR^n + ...\ , \tag{4.5}$$

where we have normalized all coefficients with respect to the coefficient of the linear term. The Einstein-Hilbert action is the specific case of equation (4.5) for $a_0 = -2\Lambda$, and $a_{n\geq 2} = 0$. We can then write the condition (4.4) for the existence of a constant curvature solution as

$$-a_0 - \frac{1}{2}R + \frac{1}{2}a_3R^3 + ... + \frac{n-2}{2}a_nR^n + ... = 0\ . \tag{4.6}$$

There are three cases to consider: *(i)* If $a_0 = 0$, then the Kerr solution, which corresponds to $R = 0$, will always be a solution of the field equations of a general $f(R)$ theory. Thus, in the absence of a cosmological constant, we conclude that the Kerr solution of general relativity remains an exact solution to all $f(R)$ theories as long as $f(R)$ has a Taylor expansion of the form in Eq. (4.5).

(ii) Moreover, independent of the value of a_0, all of the constant-curvature solutions of General Relativity in vacuum – including the Kerr solution – remain exact solutions of the $f(R)$ theory, if the Taylor series for $f(R)$ terminates after the quadratic term (i.e., if $a_{n\geq 3} = 0$). Indeed, this statement remains true independently of the value of a_0, and thus holds for both vanishing and non-vanishing cosmological constants.

(iii) Finally, if $a_0 \neq 0$ and the Taylor expansion extends beyond the quadratic term, then Kerr-like black-hole solution will always be possible. The only change is that the value of its constant curvature will be shifted relative to the value predicted in General Relativity. Since terrestrial and solar-system tests require any extra non-linear terms in the gravity action to be perturbative, this shift in the curvature will also be correspondingly small. However, even in this case, it is straightforward to show that the corrections to the curvature are actually suppressed by additional powers of the cosmological constant relative to what might naively have been expected on the basis of dimensional analysis. For example, given the expansion for $f(R)$ in Eq.(4.5), we would have expected the curvature term to have a leading correction term which scales as $R = -2a_0[1+\mathcal{O}(a_0 a_2)+...]$. However, explicitly solving Eq. (4.6), we find that the true leading correction is actually given by

$$R = -2a_0 \left(1 + 4a_0^2 a_3 + ...\right) \ . \tag{4.7}$$

Thus the deviations of the vacuum curvature solutions of $f(R)$ gravity from those of General Relativity are particularly suppressed.

4.3 $f(R)$ Gravity in the Palatini Formalism

In deriving the field equation (4.3), we extremized the action of the gravitational field with respect only to variations in the metric. In the so-called Palatini formalism, field equations of lower order can be derived from the same action of the gravitational field, by extremizing it over both the metric and the connection [50]. A large class of $f(R)$ theories in the Palatini formalism are known to result in the same field equations as general relativity [38].

Applying this procedure for a gravitational action that is a general function $f(R)$ of the Ricci scalar curvature, we obtain the well-known set of equations [50]

$$R_{\rm kl} f'(R) - \frac{1}{2} g_{kl} f(R) = 0 \tag{4.8}$$

$$\nabla_\sigma \left[\sqrt{-g} f'(R) g^{\mu\nu}\right] = 0 \ . \tag{4.9}$$

In order to look for constant curvature solutions in vacuum for this theory, we first take the trace of equation (4.8). The result is simply the algebraic equation (4.4), which we can solve for the value of the constant curvature (4.7) as before. For a solution with constant curvature, the factor $f'(R)$ in equation (4.9) is a constant, and the solutions to this equation are simply the Christoffel symbols of general relativity. As a result, any general relativistic solution of constant curvature, such as the black-hole solutions with cosmological constant, will also be solutions (with the same or slightly different value of the cosmological constant) to the field equations of an $f(R)$ gravity in the Palatini formalism.

4.4 General Quadratic Gravity

We shall now consider a gravitational action that incorporates all combinations of the Ricci curvature, Ricci tensor, and Riemann tensor, up to second order, i.e.,

$$\begin{aligned} S &= \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(-2\Lambda + R + \alpha R^2 + \beta R_{\sigma\tau} R^{\sigma\tau} \right. \\ & \qquad \left. + \gamma R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}\right) \ . \end{aligned} \tag{4.10}$$

with α, β, and γ the parameters of the theory. Such terms appear naturally as radiative corrections to the Einstein-Hilbert action in perturbative approaches to quantum gravity [33] or in string theory [8]. Note, however, that in general such theories are not free of the Ostrogradski instability [49].

Because of the Gauss-Bonnet identity, the predictions of the theory described by the action (4.10) in calculating classical properties of astrophysical black holes are identical to those of the action [12]

$$S = \frac{1}{16\pi G}\int \sqrt{-g}\left(-2\Lambda + R + \alpha' R^2 + \beta' R_{\sigma\tau}R^{\sigma\tau}\right), \tag{4.11}$$

where $\alpha' = \alpha - \gamma$ and $\beta' = \beta + 4\gamma$.

The field equation for this action in the metric formalism is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \alpha' K_{\mu\nu} + \beta' L_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{4.12}$$

where

$$K_{\mu\nu} \equiv -2R_{;\mu\nu} + 2g_{\mu\nu}\Box R - \frac{1}{2}R^2 g_{\mu\nu} + 2RR_{\mu\nu}, \tag{4.13}$$

$$L_{\mu\nu} \equiv -2R_{\mu\ \ ;\sigma\nu}^{\ \sigma} + \Box R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\Box R - \frac{1}{2}g_{\mu\nu}\, R_{\sigma\tau}R^{\sigma\tau} + 2R_{\mu}^{\ \alpha}R_{\alpha\nu}\ . \tag{4.14}$$

It is trivial to show that, for any black-hole solution satisfying equation (4.1), $K_{\mu\nu} = L_{\mu\nu} = 0$ and the field equation of quadratic gravity reduces to that of general relativity. As a result, the Kerr solution is also a solution of the general quadratic theory considered here.

4.5 Vector-Tensor Gravity

We finally consider a gravitational theory that incorporates a dynamical vector field in addition to the metric tensor. A priori, such an addition to the Einstein-Hilbert action appears to have the highest probability of requiring black-hole solutions that are not described by the Kerr metric. This is because the vector field has the same spin as photons, the geodesics of which are used to define the event horizon of a black hole. We restrict our attention to Lagrangian theories that are linear and at most of second-order in the vector field. The most general action for such a theory is [51]

$$\begin{aligned} S &= \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(-2\Lambda + R + \omega R K_\mu K^\mu \\ &\quad + \eta K^\mu K^\nu R_{\mu\nu} - \epsilon F_{\mu\nu}F^{\mu\nu} + \tau K_{\nu;\mu}K^{\mu;\nu})\ , \end{aligned} \tag{4.15}$$

with

$$F_{\mu\nu} = K_{\nu;\mu} - K_{\mu;\nu}\ . \tag{4.16}$$

The vector field K_μ at large distances from an object is meant to asymptote smoothly to a background value determined by a cosmological solution. Note that the values of the model parameters ω, η, ϵ, and τ are not independent [51].

As in the case of previous investigations of scalar-tensor gravity [39], we will be seeking vacuum solutions that are characterized by constant curvature, as well as by a constant vector K_μ. In this case, the field equations that are derived from the action (4.15) are [51]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \omega\Theta^{(\omega)}_{\mu\nu} + \eta\Theta^{(\eta)}_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{4.17}$$

$$\omega K_\mu R + \eta K^\alpha R_{\mu\alpha} = 0\ , \tag{4.18}$$

where $K^2 \equiv K_\mu K^\mu$,

$$\Theta^{(\omega)}_{\mu\nu} = K_\mu K_\nu R + K^2 R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}K^2 R\ , \tag{4.19}$$

$$\begin{aligned} \Theta^{(\eta)}_{\mu\nu} &= 2K^\alpha K_\mu R_{\nu\alpha} - 2K^\alpha K_\nu R_{\mu\alpha} \\ &\quad - \frac{1}{2}g_{\mu\nu}K^\alpha K^\beta R_{\alpha\beta}\ . \end{aligned} \tag{4.20}$$

We now multiply equation (4.18) by K_ν, combine it with equation (4.17), and look for the constant curvature solution (4.1) to obtain

$$\left[\Lambda - \frac{R}{4}\left(1+\omega K^2\right)\right] g_{\mu\nu} \\ -\eta \frac{R}{4}\left(K_\mu K_\nu + \frac{1}{2}K^2 g_{\mu\nu}\right) = 0 \ . \tag{4.21}$$

Contracting equation (4.21) with $g^{\mu\nu}$, we obtain for the constant curvature

$$R = \frac{16\Lambda}{4+(4\omega+3\eta)K^2} \simeq 4\Lambda\left[1-\left(\omega+\frac{3\eta}{4}\right)K^2\right] \ . \tag{4.22}$$

As in the previous cases, a black-hole solution that differs only in the value of the constant curvature from the general relativistic one is possible for the vector-tensor gravity theory that we have considered.

4.6 Discussion

Our results have important implications for current attempts to test general relativity in the strong-field regime using astrophysical black holes. On the one hand, we appear to be lacking a parametric theoretical framework with which to interpret observational data and quantify possible deviations from the general relativistic predictions for astrophysical black holes. On the other hand, the detection of deviations from the Kerr metric in the spacetime of an astrophysical black hole will be a very strong indication for the need of a major change in our understanding of gravitation.

4.7 Conclusion

We showed that the black-hole solutions of theories obtained by extending general relativity with additional scalar, vector or tensor degrees of freedom are essentially indistinguishable from those of general relativity. Thus, we conclude that a potential observational verification of the Kerr metric around an astrophysical black hole cannot, in and of itself, be used to distinguish between these theories. On the other hand, it remains true that detection of deviations from the Kerr metric will signify the need for a major change in our understanding of gravitational physics.

CHAPTER 5

CONSTRAINTS WITH COSMOLOGY

After exploring tests using neutron stars and black holes, let us see how cosmological data can constrain higher-order theories of gravity. We look at the evolution of the scale factor $a(t)$, which represents the size of the universe, and see how it is affected by higher-order terms in the action. We plug the Friedmann-Robertson-Walker metric into the generalized Einstein equations, and find the modified time evolution of the scale factor. This leads us to a differential equation of the third order in the scale factor.

5.1 Evolution of scale factor

In cosmology, the spacetime is assumed to be isotropic and homogeneous, therefore for an $R + R^2$ type of gravity theory, the action is that of Equation (2.20):

$$S = M_{Pl}^2 \int \sqrt{-g}\,(\mathcal{R} + \alpha\,\mathcal{R}^2)\, d^4x, \tag{5.1}$$

and the generalized field equations follow from Equation (2.31) where $\beta = 0$. The universe's isotropic and homogeneous spacetime is described by the Friedmann-Robertson-Walker metric:

$$ds^2 \;=\; -dt^2 + a^2(t)\left(\frac{1}{1-\kappa\,r^2}\,dr^2 + r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right). \tag{5.2}$$

The energy content of the universe, like for the interior of a neutron star, can be modeled by the energy-momentum tensor of an isotropic perfect fluid

$$T_\mu{}^\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{5.3}$$

By plugging the Friedmann-Robertson-Walker metric into the generalized field equations, we get two differential equations in the scale factor $a(t)$. In the case $\kappa = 0$ for a flat spacetime, the time-time equation – the generalized Friedmann equation– is a third-order differential equation in $a(t)$:

$$- \left(\frac{\dot{a}}{a}\right)^2 + 6\alpha \left[3\left(\frac{\dot{a}}{a}\right)^4 - 2\frac{\dot{a}^2\ddot{a}}{a^3} + \left(\frac{\ddot{a}}{a}\right)^2 - 2\frac{\dot{a}a^{(3)}}{a^2}\right] = -8\pi G\rho/3. \tag{5.4}$$

Define the Hubble parameter

$$H = \frac{\dot{a}}{a}, \tag{5.5}$$

and the deceleration parameter

$$q = -\frac{\ddot{a}a}{\dot{a}^2}, \tag{5.6}$$

then the generalized Friedmann equation becomes

$$-H^2 + 6\alpha\left[-3(q^2-1)H^4 + 2\dot{q}H^3\right] = -8\pi G\rho/3. \tag{5.7}$$

The energy density of the universe ρ can however be expressed as a function of the scale factor a. To see this, we use the conservation of the energy-momentum tensor

$$T^{\mu\nu}_{\ ;\nu} = 0, \tag{5.8}$$

which leads to the continuity equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \tag{5.9}$$

We assume the universe to be made of radiation (ρ_r), matter (ρ_m) and a cosmological constant component (ρ_Λ). As is the case for general relativity, the continuity equation leads to

- $\rho_r = \rho_{r0}\,(\frac{a}{a_0})^{-4}$ for radiation, since $p = \rho/3$,
- $\rho_m = \rho_{m0}\,(\frac{a}{a_0})^{-3}$ for matter, since $p = 0$,
- $\rho_\Lambda = \rho_{\Lambda 0}$ for vacuum, since $p = -\rho$,

and since $a_0 = 1$, the energy density is

$$\rho = \rho_{r0}\,a^{-4} + \rho_{m0}\,a^{-3} + \rho_{\Lambda 0} \tag{5.10}$$

In general relativity, the critical density is defined by

$$\rho_c = \frac{3H^2}{8\pi G}, \tag{5.11}$$

and the ratio of energy density to critical density is defined by

$$\Omega = \frac{\rho}{\rho_c}, \tag{5.12}$$

then

$$8\pi G\rho/3 = H^2\frac{\rho}{\rho_c} = H^2\,\Omega, \tag{5.13}$$

and

$$8\pi G\rho_0/3 = H_0^2\,\Omega_0, \tag{5.14}$$

leading to

$$8\pi G\rho/3 = H_0^2\left(\frac{\Omega_{r0}}{a^4} + \frac{\Omega_{m0}}{a^3} + \Omega_{\Lambda 0}\right). \tag{5.15}$$

While this is true in general relativity but not necessarily in $R+R^2$ gravity, we use this expression for density as an approximation. The Friedmann equation then becomes:

$$-\left(\frac{\dot{a}}{a}\right)^2 + H_0^2\left(\frac{\Omega_{r0}}{a^4} + \frac{\Omega_{m0}}{a^3} + \Omega_{\Lambda 0}\right) + 6\alpha\left[-3(q^2-1)H^4 + 2\dot{q}H^3\right] = 0. \tag{5.16}$$

This third-order differential equation in $a(t)$ can also be rewritten as 3 first-order equations in $a(t)$, $H(t)$ and $q(t)$:

$$\dot{a} = a\,H, \tag{5.17}$$

$$\dot{H} = -(1+q)\,H^2, \tag{5.18}$$

and for $\alpha \neq 0$:

$$\dot{q} = \left[\frac{1}{H} - \frac{H_0^2}{H^3}\left(\frac{\Omega_{r0}}{a^4} + \frac{\Omega_{m0}}{a^3} + \Omega_{\Lambda 0}\right)\right]/(12\alpha) + \frac{3}{2}H(q^2-1). \tag{5.19}$$

We use observed current values for density: $\Omega_{m0} = 0.25$, $\Omega_{r0} = 4.6 \times 10^{-5}$ and $\Omega_{\Lambda 0} = 0.75$. The scale factor a and deceleration parameter q are already defined as unitless quantities, and their initial values were taken to be, as general relativity would predict: $a_0 = 1$ and $q = -0.625$. We only consider scenarios with a flat universe ($\kappa = 0$). The Hubble parameter H however is dimensionful. We rewrite H as:

$$H = H_0\,H' \tag{5.20}$$

and time as:

$$t = t_H\,t' = t'/H_0 \tag{5.21}$$

so that the quantities H' and $t' = t/t_H$ are dimensionless. Our 3 differential equations are rewritten as:

$$\dot{a}' = a\,H', \tag{5.22}$$

$$\dot{H}' = -(1+q)\,H'^2, \tag{5.23}$$

and for $\alpha \neq 0$:

$$\dot{q}' = \left[\frac{1}{H'} - \frac{1}{H'^3}\left(\frac{\Omega_{r0}}{a^4} + \frac{\Omega_{m0}}{a^3} + \Omega_{\Lambda 0}\right)\right]/(12\alpha') + \frac{3}{2}H'(q^2-1), \tag{5.24}$$

where we define:

$$\alpha' = \alpha\,H_0^2, \tag{5.25}$$

and $H_0 \cong 2.3 \times 10^{-18} s^{-1}$ or in equivalent natural units $\sim 10^{-28}$ cm^{-1}. Therefore $\alpha = \alpha' \times 10^{56}$ cm^2.

We integrate these 3 equations together using a Runge-Kutta routine. In Figure 5.1, we plot the scale factor $a(t)$ vs t for different values of α'. We observe that adding a nonzero α term has a direct effect on the evolution of the scale factor. Whether α is positive or negative, the age of the universe will be lower than the current prediction of 13.7 Gyr. For positive α's, the age of the universe varies between 90% and 100% of the Hubble time. As α goes to zero, we recover general relativity as expected. However, for negative α's, the behavior is the opposite. Low absolute values of α lead to nonphysical behavior where the age of the universe could be arbitrarily small.

5.2 Age of the universe

In Figure 5.2, we plot the age of the universe as a function of $|1/\alpha'|$. We verify that negative values of α are non-physical, as small values of α give arbitrarily small values for the age of the universe. While this may be of interest to creationists, here we are interested in real data: observed lower limits of the age of the universe. These are constraints due to observations from radioactive dating, globular clusters and white dwarfs. The age estimate of 13.7 billion years from WMAP data was not used as a constraint, because it is model dependent: the estimate was derived using general relativity. White dwarfs give us our best model-independent lower limit on the age of the universe at approximately 12 billion years. Observations of white dwarfs however are not sufficient to provide an upper limit on positive, physical α's.

5.3 Dolgov-Kawasaki instability

We find that positive α's are well-behaved, and that as expected, we recover the general relativistic evolution when $\alpha \to 0$. Surprisingly, we find that negative α's

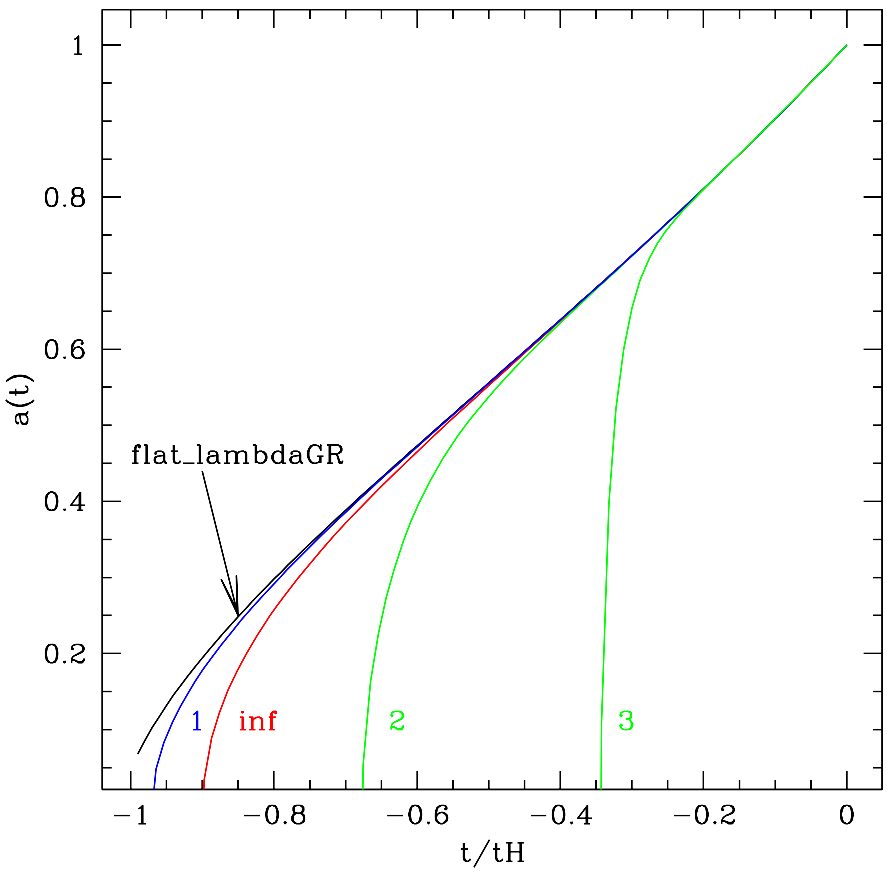


Figure 5.1: Scale factor a(t) vs. time for a wide range of α' values. $t/t_H = 0$ corresponds to the current time, and $t/t_H = -1$ corresponds to the time of the Big Bang, as predicted by general relativity in a flat universe with a cosmological constant Λ. Note that we did not restrict ourselves to the range where the perturbation is valid (relatively small α' values), rather we investigated all α' values from 0 to infinity. Graph 1 shows the evolution of the scale factor for $\alpha' = 10^{-3}$; Graph 2 corresponds to $\alpha' = -10^{-3}$ and Graph 3 corresponds to $\alpha' = -10^{-5}$. We also plotted the evolution for α' at infinity (positive or negative infinity values of α' yield the same evolution).

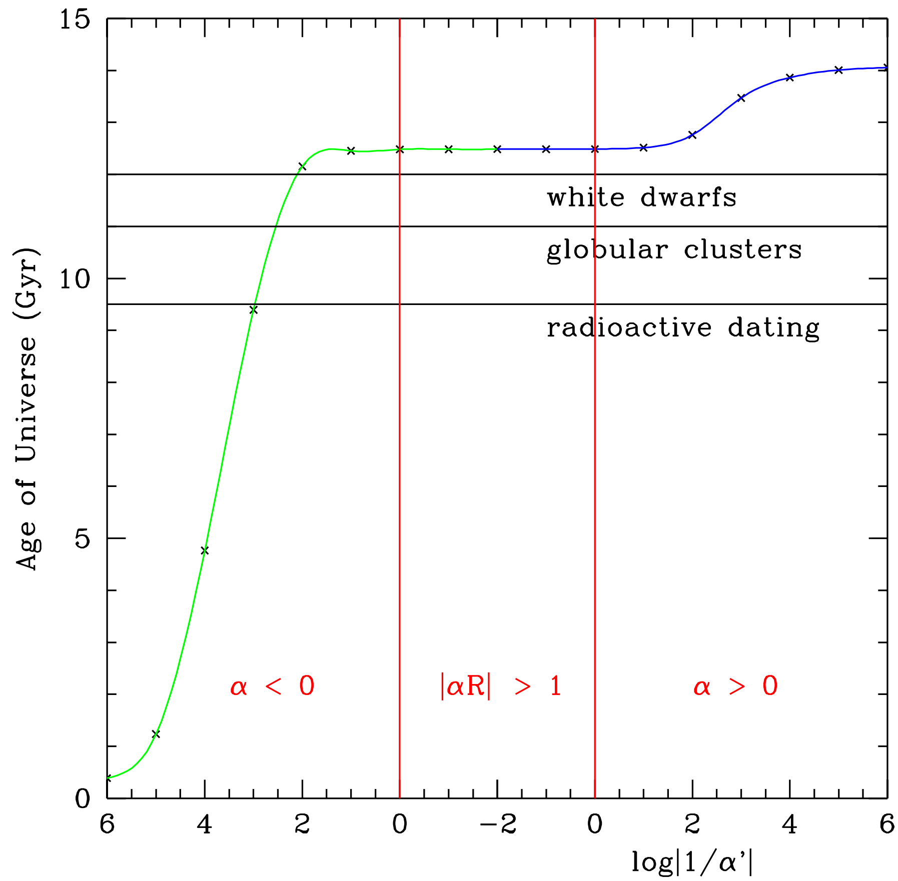


Figure 5.2: Age of the universe as a function of $log|1/\alpha'|$. The left section corresponds to $\alpha < 0$, the right section corresponds to $\alpha > 0$, and the central section outlines where the perturbation breaks down: $|\alpha\mathcal{R}| > 1$, which is equivalent to $|\alpha'| > 1$. The only physical part is the right section with positive α's. Horizontal lines represent observed lower limits for the age of the universe from radioactive dating, globular clusters and white dwarfs. These lower limits are not sufficient to constrain positive α's.

are *not well-behaved* as $\alpha \to 0$. The physics is not smooth in α but rather in $1/\alpha$. This nonperturbative effect, where general relativity is not recovered as we take its limit (for a negative sign of α), can be shown to have its root in singular perturbation theory. It is a property of classical differential equations. For example, in the equation:

$$\epsilon \ddot{x} + \dot{x} + x = 0, \tag{5.26}$$

one does not necessarily recover the lower-order differential equation $\dot{x} + x = 0$ when applying $\epsilon \to 0$.

In the literature of higher-order theories of gravity (which induce differential equations of higher order than general relativity), this instability has been called the Dolgov-Kawasaki instability. Hu [52] shows that a $f(R)$ theory is stable only when $d^2 f/dR^2 > 0$. In the case of the action $\sim R + \alpha R^2$ that we have studied in this chapter, this corresponds to $\alpha > 0$ as expected.

We note the difference between solving exact differential equations, and considering effective field theories. In the above example, ϵ is the parameter that is adjusted and for which a limit is taken. In an effective field theory on the other hand, it is not the parameter or coefficient ϵ which is adjusted, rather we say that the higher-order operator itself is suppressed. This would correspond here to $\ddot{x}$. This distinction is important when we consider the effect of higher order terms, and will be discussed in more detail in Chapter 6.

5.4 Discussion and Problems

We encountered problems and made approximations in this Chapter which need careful consideration.

- As mentioned above, some of the solutions ($\alpha < 0$) are not stable. This may suggest a breakdown of our perturbation.

- We used general relativity to find the initial (current) value for the deceleration parameter q. An alternative theory could have different boundary conditions.

- In the initial values for the density parameters, we assumed that the density Ω was 1. However in $R+R^2$ gravity, $\kappa = 0$ in the metric (flat space) does not necessarily lead to $\Omega = 1$ anymore. We have too many parameters. We really would need to explore the parameter space $(a\,, H\,, q\,, \Omega_r\,, \Omega_m\,, \Omega_\Lambda\,, \alpha, \kappa)$.

CHAPTER 6

ALTERNATIVE VS. EFFECTIVE: THE BATTLE

So far we have been exploring alternative theories of gravity, which consist in adding extensions to the action of general relativity. There are two schools of thought however, usually depending on whether one belongs to the general relativity or the high energy theory/effective field theory communities. One the one hand, alternative theorists explore extensions to general relativity, which could be considered to be *ad hoc* and sometimes lead to nonperturbative effects. On the other hand, effective field theorists view perturbations of the action very differently. If a perturbative term in the action appears to have nonperturbative effects, then any such effect will be canceled out by yet a higher order in the expansion. This is examplified by a field redefinition as shown in the following section.

6.1 Vacuum solutions of higher-order gravity

As seen in Chapter 4, the vacuum solution $R = 0$ to general relativity is also solution to a number of higher-order or alternative theories of gravity. Observation of the Kerr metric around a spinning black hole, for example, would therefore not constitute a verification of general relativity. In that case one would not be able to distinguish between theories of gravity. This thinking can be further demonstrated with the use of a field redefinition in the context of *effective field theory*. Following Burgess [33], we take the Einstein-Hilbert action:

$$S_{EH} = \frac{M_P^2}{2} \int d^4x \sqrt{-g}\, R \tag{6.1}$$

Minimizing the Einstein-Hilbert action leads to:

$$\frac{M_P^2}{2}\, \delta \left(\int d^4x \sqrt{-g}\, R \right) = -\frac{M_P^2}{2} \int d^4x \sqrt{-g}\, \left[R^{\mu\nu} - 1/2\, R\, g^{\mu\nu} \right] \delta g_{\mu\nu} \tag{6.2}$$

We make a field redefinition:

$$\delta g_{\mu\nu} = Y_{\mu\nu} = \frac{1}{M_P^2}\left[2aR_{\mu\nu} - (a+2b)Rg_{\mu\nu}\right] \tag{6.3}$$

When plugging this into equation (6.2), the minimization of the Einstein-Hilbert action leads to terms in R^2 and $R^{\mu\nu}R_{\mu\nu}$. Therefore in an extended action of the form:

$$S_{eff} = \int d^4x\, \sqrt{-g}\, \left[\frac{M_P^2}{2}R + aR_{\mu\nu}R^{\mu\nu} + bR^2\right] \tag{6.4}$$

the terms in R^2 and $R^{\mu\nu}R_{\mu\nu}$ are canceled exactly by the field redefinition of equation (6.3), regardless of the values of a and b. Therefore, as seen before, the R^2 terms should not affect the dynamics.

This is only true in vacuum, and there is discussion in the *alternative* school of thought as to whether general relativity's vacuum solutions ($R = 0$) would be viable solutions in some alternative theories [53]. This certainly does not apply in environments filled with a fluid such as in cosmology. An R^2 term could have an effect and possibly generate inflation [13].

An important question is how this applies to the emission of gravitational waves in inspiralling compact binaries. Far enough from each of the two compact objects, the vacuum solution $R = 0$ should apply and one could argue that R^2 terms in an extended action would not affect the dynamics of the binary or the emission of gravitational waves. Therefore gravitational waves coming from these systems would not constrain alternative theories of the $R + R^2$ type or the theories mentioned in Chapter 4. This is however at odds with the *alternative* school of thought.

6.2 Alternative vs. Effective: The Battle

Here is the showdown.

6.2.1 The Alternative

The first approach studies alternative theories of gravity where specific finite terms are added to the Einstein-Hilbert action. $f(R)$ theories are one example. In previous chapters we studied $R + R^2$ theories where no other term (such as R^3 or R^4) was considered to follow, even if negligible. The R^2 extension is considered to be *ad hoc*. This is also the case of $1/R$ theories proposed to explain the acceleration of the universe. They were not proposed as an infinite series of terms such as a polynomial expansion in the curvature.

In the case of $R + R^2$ theories, the Einstein equations become fourth order in the metric instead of second order. The higher orders in the differential equation can be interpreted as additional degrees of freedom. This is well illustrated as one makes a conformal transformation from the Jordan frame with the original R^2 term, to the Einstein frame which introduces scalar fields instead [19]. These extra degrees of freedom can be dynamical or not. Dynamical degrees of freedom can introduce instabilities, such as the Dolgov-Kawasaki instability observed in Chapter 5.

In the case of gravitational waves, the extra dynamical degrees of freedom are said to imply extra polarizations in the waves. This is remarkable. It implies that gravitational waves would be great observatories for detecting an alternative theory of gravity. Detecting one of these extra polarizations in a gravitational wave signal would effectively constitute a disagreement with general relativity.

6.2.2 The Effective

Stelle suggested that adding R^2 terms to the Einstein-Hilbert action could make the theory *renormalizable* [54]. While a remarkable fact, the theory also predicts the presence of negative energy modes, which make a $R+R^2$ theory non-unitary. By anyone in the high energy theory or effective field theory community, a theory that is not unitary is not consistent and needs not be considered.

From the effective field theory standpoint, an R^2 term would merely be a term in an infinite perturbative expansion about the leading-order solution to the equation of motion, in this case $R = 0$. Terms in the perturbative expansion are suppressed by increasing powers of $1/M_P^2$. Having an R^2 term would not add any additional degrees of freedom and would not have a nonperturbative effect on the dynamics such as creating an instability or an additional gravitational wave polarization. Any nonperturbative effect such as these would be viewed as an artifact of the expansion that could be corrected at higher order.

Therefore from the effective field theory perspective, there should be no measurement effect in vacuum from the presence of an R^2 term (assuming there are higher order terms completing the expansion). This is illustrated by the field redefinition above. Of course this is not true in cosmology or where matter is involved.

6.3 Constraints with gravitational waves

The question remains: can we use gravitational wave polarization to constrain $R + R^2$ gravity?

- If we consider a perturbative expansion such as in effective field theory, then there will no observable effect in gravitational waves, therefore gravitational radiation cannot be used to discriminate among theories of gravity.
- If we consider a finite, ad hoc theory such as $R+R^2$, then extra polarizations of the gravitational wave should be present and could be detected. The question is whether it makes sense to consider a theory that is not unitary.

We note that Cooney, DeDeo and Psaltis [55] have sought to eliminate these degrees of freedom with the method of *perturbative constraints*, first introduced by

Simon [56].

In the next chapter, we consider newtonian gravity as an effective field theory of general relativity. We use this effective approach in studying gravitational dynamics and gravitational radiation of inspiralling compact binaries.

CHAPTER 7

EFFECTIVE FIELD THEORY OF GRAVITATIONAL RADIATION

We now turn our attention to the gravitational dynamics and gravitational radiation from inspiralling compact binaries. We consider newtonian gravity to be an effective field theory of a higher order theory, in this case, general relativity. Such a perturbation around newtonian gravity is called the Post-Newtonian (PN) approximation to general relativity. While the PN approximation has been very useful thus far in computing gravitational observables, it lacks a coherent framework such as a lagrangian formalism. We explore in this Chapter the formalism of Non Relativistic General Relativity (NRGR) which, while based on the Post-Newtonian expansion, provides a consistent framework, and where observables can in principle be computed at any order in the velocity expansion.

7.1 Introduction

In light of possible detection of gravitational waves in the near future by LIGO or LISA, research in gravitational wave physics is very timely and relevant.

Coalescing compact binaries are thought to be powerful emitters of gravitational radiation. During the inspiral phase of coalescing compact binaries, when the compact objects can be considered to be non-relativistic ($v/c < 0.5$), one can treat the dynamics as a perturbative expansion in the orbital velocity. This is called the Post-Newtonian (PN) approximation to general relativity. However, current PN methods run into divergences because they approximate the compact objects to be point masses. They also lack a coherent framework such as a lagrangian formalism.

First introduced by Goldberger and Rothstein [57], Non-Relativistic General

Relativity (NRGR) offers an alternative. This effective field theoretic formalism takes into account the physical extent of the compact objects, therefore avoiding divergence issues, and provides a consistent framework using a lagrangian formalism where observables can be calculated at any order in the velocity. Effective field theories are used to describe the low-energy limits of more fundamental, possibly unknown, theories. They are widely used in the field of high energy physics and nuclear physics, as with non-relativistic quantum chromodynamics (NRQCD) and chiral perturbation theory. General relativity, as a theory of gravity, is considered to be the classical low-energy limit of a more fundamental theory. Newtonian gravity itself can be considered to be an effective field theory of Einstein's general relativity. Effective field theories can be applied to any physical systems, including classical ones, which contain distinct length scales (or equivalent energy scales). Such a framework allows us to describe the physics of a system at each particular scale by parametrizing our ignorance of the high energy effects.

During the inspiral phase of coalescing compact binaries, there are 3 widely separated length scales: 1) the size of each compact object 2) the size of the orbit 3) the wavelength of gravitational radiation that escapes the binary and reaches our detectors on Earth. By integrating out the physics at small length scales (size of compact object and size of orbit), one can extract gravitational wave observables such as energy, power and phase.

The methods of effective field theories enable us to provide a consistent framework with which to study gravitational dynamics and gravitational wave emission, using a lagrangian formalism. By describing the physics at different length scales separately, any divergences are resolved and renormalized into the coefficients of the parametric expansion. One can then calculate gravitational observables to any order in the velocity. NRGR enables us to 1) calculate the leading and subleading corrections (in powers of v/c) of the newtonian potential in order to better model the gravitational dynamics 2) calculate corrections to the predicted gravitational wave emission and provide more accurate templates. These calculations can be compared to other PN results (and should be in complete agreement), and can also be completed at higher orders in a systematic way, surpassing other PN calculations.

Note: there has been puzzlement in the gravitational wave community as to the choice of the name "Non-Relativistic General Relativity (NRGR)", or as to how relativity could possibly be non-relativistic. The *non-relativistic* part actually refers to the low orbital speeds of the compact objects, while general relativity is of course a theory of gravity. This is in analogy with the effective field theories of NRQCD and NRQED which nuclear physicists are familiar with. NRGR is in essence the same thing as the post-newtonian approximation to general relativity. The difference comes with the methods used (lagrangian formalism and path integral formulation).

7.2 What has been done in NRGR

A number of corrections to the newtonian potential have been successfully calculated using NRGR. Goldberger & Rothstein first calculated the spinless 1PN correction to the newtonian potential [57] (the so-called Einstein-Infeld-Hoffman potential). Gilmore & Ross calculated the 2PN correction [58] using the so-called Kol variables [59]. The spinless 1PN and 2PN NRGR calculations are in perfect agreement with other post-newtonian calculations. The 3PN spinless correction remains to be calculated and compared with PN calculations.

Porto and Rothstein have developed a formalism in which to take into account the spins of each of the compact objects [60]. Spin(1)-spin(2) coupling – the coupling of the spin of one compact object with the spin of the other compact object – at 2PN and 3PN was calculated by Porto & Rothstein [61]. After initial disagreements which were resolved, these NRGR results agreed with the post-newtonian results from Schaefer et al. [62]. Spin(1)-Spin(1) or Spin(1) squared – the coupling of the spin of one compact object with itself – was also calculated in NRGR by Porto & Rothstein [63]. This was also calculated by Schaefer et al. using post-newtonian methods [64].

The coupling of compact object spin with the angular momentum of the binary (*spin-orbit* coupling) was calculated by Porto & Rothstein at 1.5PN [60], which is the leading order term for spin-orbit. The next-to-leading order spin-orbit term at 2.5PN had not yet been calculated using NRGR. We show in the next Chapter the calculation of this term. Calculations of this term have been done by the post-newtonian teams of Buonanno et al. [65], Tagoshi at al. [66] and Damour et al. [67]. We note that Kol & Smolkin have also explored spin-orbit at 2.5PN using NRGR and the so-called Kol variables. They have so far only presented diagrams [68].

7.3 NRGR Tutorial

In calculating the next-to-leading order spin-orbit term at 2.5PN, we use the same effective field theory methods that were used by Goldberger & Rothstein to calculate the spinless 1PN correction [57]. We also use the same spin formalism and methods used by Porto & Rothstein to calculate the 3PN spin(1)-spin(2) term [61].

As outlined above, the major advantage of NRGR is that, unlike other post-newtonian methods, it is based on a lagrangian formalism. As in general relativity, the NRGR action is the sum of the Einstein-Hilbert (gravitational) action and the matter action, which is modeled here as 2 point particles + corrections:

$$S = S_{EH} + S_{pp} \tag{7.1}$$

where:

$$S_{EH} = -2M_P^2 \int d^4x \sqrt{-g}\, R \tag{7.2}$$

and to lowest order:

$$S_{pp} = -m \int d\tau = -m \int \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \tag{7.3}$$

The NRGR effective action is found from the path integral:

$$\exp[iS_{eff}[x_a]] = \int Dh_{\mu\nu} \exp[iS] \tag{7.4}$$

From the Einstein-Hilbert gravitational action, we can write the expression for a graviton propagator, as well as 3-graviton or in general n-graviton vertices. From the point particle (matter) action, we can find the *mass vertices* showing the coupling of compact object mass with gravitons. Combining mass vertices with graviton propagators enable use to calculate the gravitational potential corrections at different orders.

Note: in our discussions of alternative theories of gravity, we had been using a $(- + ++)$ sign convention for the metric. In this Chapter, we switch to a sign convention which is normally used in field theory: $(+ - --)$. The Planck mass is also defined differently: $M_P^2 = 1/(32\,\pi G)$.

While we are using methods of quantum mechanics (path integrals), the system here of two coalescing compact sources remains purely classical. Quantum corrections are actually quite negligible. From the real (conservative) part of the effective action, one extracts the gravitational dynamics: potential and equations of motion. The imaginary (nonconservative) part gives us the rate of gravitational radiation or energy loss.

One cannot however compute a path integral directly. This is where a perturbative expansion is needed. Such a perturbative expansion can be represented as a sum of Feynman diagrams, i.e. we can represent each order in the velocity as a set of Feynman diagrams. First, we expand the metric around Minkowski spacetime. This is a valid assumption since the compact objects are assumed to be far apart ($r >> R$), therefore the curvature is small:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{Pl}} \tag{7.5}$$

While we keep track of the order in the velocity, we also need to keep track of the order in the gravitational perturbation h (which can be linear, quadratic etc.). Expanding the point particle action in linear terms of h:

$$S_{pp} = -m \int dt \left[-\frac{1}{2}v^2 + \frac{1}{M_{Pl}}(\frac{1}{2}h_{00} - h_{0i}v_i + ...) \right] \tag{7.6}$$

gives us the coupling of the compact object worldlines (mass m) with the gravitational field or graviton h. At zeroth order in v, one recovers the Newtonian potential. Calculating diagrams at second order in v (1 PN) enabled Goldberger & Rothstein [57] to successfully recover the Einstein-Infeld-Hoffman potential. Gilmore & Ross have recently calculated the 2PN corrections [58] using Kol variables [59].

7.4 Spin degrees of freedom

In many cases, compact objects spin about their axes in addition to their orbit. The original formalism [57] for NRGR however did not include spin degrees of freedom. In [60], Porto showed that one can allow the local frame basis to rotate by adding spin degrees of freedom to the point particle action. The lowest order lagrangian term for spin is:

$$L = \frac{1}{2M_{Pl}} h_{i0,k} S^{ik} \tag{7.7}$$

Spin(1)-spin(2) (the coupling of the spin of one compact object with the spin of the other object) and spin-orbit (the coupling of the spin of one compact object with the orbital angular momentum) can in principle be calculated at any order. Porto & Rothstein recently computed the spin(1)-spin(2) potential at 3PN [61], in agreement with other PN calculations [62].

7.5 Spin-orbit corrections

In order to calculate spin-orbit corrections to the potential, one needs to combine a spin term with a simple mass insertion, keeping track of the order in the velocity. The leading order spin-orbit interaction at 1.5 PN is the sum of 2 diagrams:

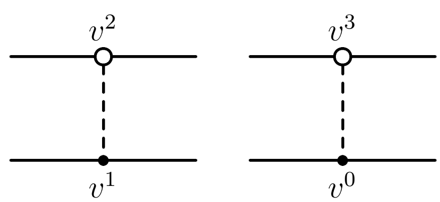


Figure 7.1: Leading order spin-orbit interaction (1.5 PN or v^3). The blob represents a spin insertion, while the other vertex consists of a simple mass insertion at various orders in the velocity (hence a spin-orbit coupling).

Each diagram contributes to the gravitational potential: the sum of the diagram amplitudes corresponds to $-iV$, where V is the potential. The potential correction that is calculated from these 2 diagrams is the following [60]:

$$V^{so}_{1.5PN} = \frac{G\,m_2}{r^2} n^j \left(S_1^{j0} + S_1^{jk}(v_1^k - 2v_2^k) \right) + 1 \leftrightarrow 2 \tag{7.8}$$

At next-to-leading order, we write down all the diagrams that sum up to 2.5PN, or equivalently v^5. This is the subject of the next Chapter.

CHAPTER 8

SUBLEADING SPIN-ORBIT CORRECTION IN DYNAMICS OF COMPACT BINARIES

8.1 The Toolbox

We are interested in the next-to-leading order (2.5PN) spin-orbit correction to the newtonian potential. Taking into account the spins of the compact objects adds many subtleties to the calculation. In NRGR, following the works of [57, 61]:

- The gauge we work in is not fully harmonic. Buonanno et al. [65] use a fully harmonic gauge.
- The spin is defined in the local frame of each compact object, as opposed to the Post-Newtonian/orbital frame used by Buonanno et al. [65].
- A spin supplementary condition (SSC) is needed. We use the covariant SSC $S^{j0} = S^{jk} v^k$.

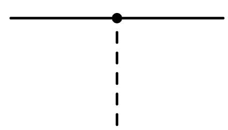
In order to calculate diagrams for the potential, we first need to write down the Feynman rules. We need all of the vertices: linear (spinless) mass vertices; quadratic (spinless) mass vertices; linear spin vertices; quadratic spin vertices. In spin-orbit, unlike spin-spin, there is only one spin term. The rule of the game is to mix and match one spin vertex with one spinless mass vertex. At 2.5PN, the allowed combinations of vertices and propagators are those that add up to 2.5PN or equivalently v^5.

8.1.1 Mass vertices

We obtain the mass vertices (or mass-graviton coupling) from expanding the point particle action:

$$S_{pp} = -m \int d\tau = -m \int dt \, [\, 1 - \vec{v}^2 + \frac{1}{M_P} \, (H_{00} + 2\, H_{0i}\, v_i + H_{ij}\, v_i\, v_j \,)]^{1/2} \quad (8.1)$$

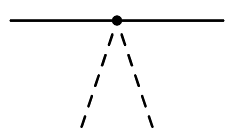
Linear mass vertices are shown in Figure (8.1) and quadratic mass vertices in Figure (8.2).



$$\begin{aligned} L_{0PN} &= -\frac{1}{2}\frac{m}{M_P} H_{00} \\ L_{0.5PN} &= -\frac{m}{M_P} H_{0i}\, v_i \\ L_{1PN} &= -\frac{1}{2}\frac{m}{M_P} [\, H_{ij}\, v_i\, v_j + \frac{1}{2} H_{00}\, \vec{v}^2\,] \\ L_{1.5PN} &= -\frac{1}{2}\frac{m}{M_P} H_{0i}\, v_i\, \vec{v}^2 \end{aligned}$$

Figure 8.1: Linear mass vertices

Quadratic terms:



$$\begin{aligned} L_{1PN} &= \frac{1}{8}\frac{m}{M_P^2} H_{00}\, H_{00} \\ L_{1.5PN} &= \frac{1}{2}\frac{m}{M_P^2} H_{00}\, H_{0i}\, v_i \end{aligned}$$

Figure 8.2: Quadratic mass vertices

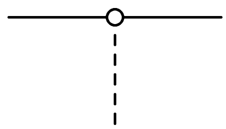
8.1.2 Spin vertices

We extract the linear spin term in the point particle lagrangian:

$$L = \frac{1}{2M_p} H_{\alpha\gamma,\beta}\, S^{\alpha\beta}\, u^{\gamma} \tag{8.2}$$

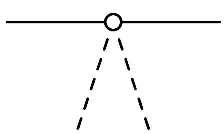
Using power counting, we write the linear spin vertices in Figure (8.3). The quadratic spin term in the point particle lagrangian is:

$$L = \frac{1}{4M_p^2} S^{\beta\gamma}\, u^{\mu}\, H_{\gamma}{}^{\lambda} (\frac{1}{2} H_{\beta\lambda,\mu} + H_{\mu\lambda,\beta} - H_{\mu\beta,\lambda}) \tag{8.3}$$



$$
\begin{aligned}
L_{1PN} &= \frac{1}{2M_p}\, H_{io,k}\, S^{ik} \\
L_{1.5PN} &= \frac{1}{2M_p}\, (H_{ij,k}\, S^{ik}\, u^j \,+\, H_{00,k}\, S^{0k}) \\
L_{2PN} &= \frac{1}{2M_p}\, (H_{0j,k}\, S^{0k}\, u^j \,+\, H_{i0,0}\, S^{i0}) \\
L_{2.5PN} &= \frac{1}{2M_p}\, H_{ij,0}\, S^{i0}\, u^j
\end{aligned}
$$

Figure 8.3: Linear spin vertices



$$
\begin{aligned}
L_{2PN} &= \frac{1}{4\,M_p^2}\, S^{ij}\, (H_j^{\lambda}\, H_{0\lambda,i} \,-\, H_j^k\, H_{0i,k}) \\
L_{2.5PN} &= \frac{1}{4\,M_p^2}\, \{S^{ij}(\frac{1}{2}H_j^{\lambda}(H_{i\lambda,0} + H_{i\lambda,k}u^k + 2H_{k\lambda,i}u^k) - H_j^0 H_{0i,0} \\
&\quad - H_j^l H_{ki,l}u^k) + S^{0j}(H_0^k H_{0j,k} - H_j^k H_{00,k} - H_0^{\lambda} H_{0\lambda,j})\}
\end{aligned}
$$

Figure 8.4: Quadratic spin vertices

from which we read quadratic spin vertices as listed in Figure (8.4).

8.2 Potential

We find the diagrams for the 2.5PN spin-orbit potential by mixing and matching mass and spin vertices to the right order. The diagrams are shown in Figures (8.5), (8.6), (8.7) and (8.8).

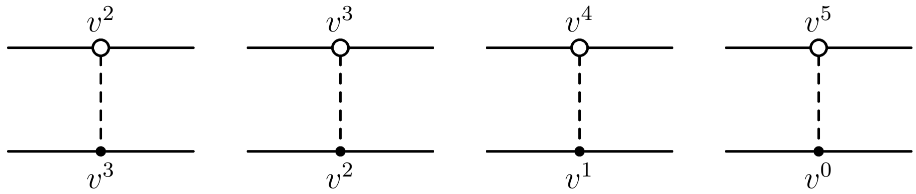


Figure 8.5: Feynman diagrams with one-graviton propagator

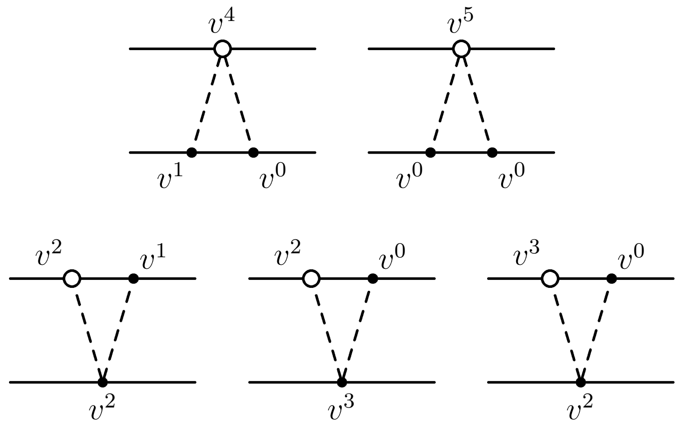


Figure 8.6: Quadratic terms

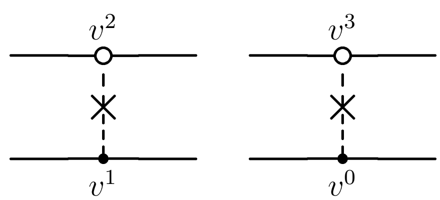


Figure 8.7: Propagator correction

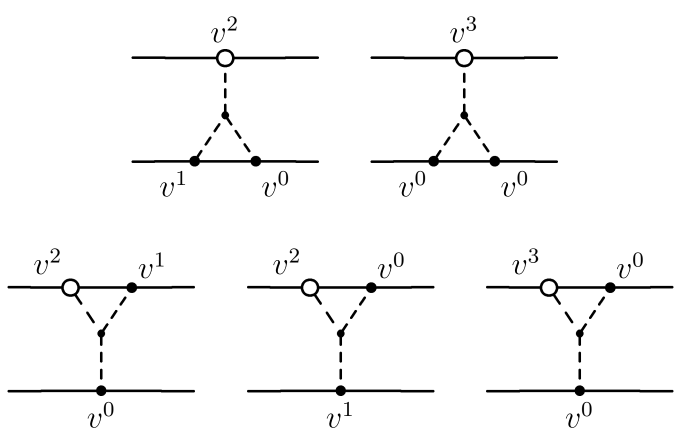


Figure 8.8: 3-graviton terms

8.2.1 Amplitudes

The amplitude of each Feynman diagram is shown in Tables (8.1), (8.2), (8.3) and (8.4).

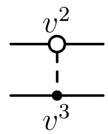
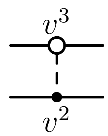
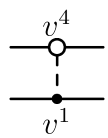
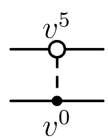
	$i\,G\,m_2\,S_1^{ik}\,\vec{v_2}^2\,v_{2i}\,n_k/r^2$
	$i\,G\,m_2\,\{(2\,\vec{v_1}.\vec{v_2}\,v_{2i} - 3/2\,\vec{v_2}^2\,v_{1i})\,S_1^{ik} - 1/2\,\vec{v_2}^2\,S_1^{0k}\}\,n_k/r^2$
	$2\,i\,G\,m_2\,S_1^{0k}\,(r\,a_{2k} + \vec{v_1}.\vec{v_2}\,n_k + v_{2k}\,\vec{v_2}.\vec{n})/r^2$
	$-i\,G\,m_2\,S_1^{0k}\,v_{1k}\,\vec{v_2}.\vec{n}/r^2$

Table 8.1: Feynman diagrams with single-graviton exchange

Total amplitude for linear terms:

$$-i\,V \;=\; i\,\frac{G\,m_2}{r^2}\,\{S_1^{ik}\,n_k\,((v_{2i}-\frac{3}{2}v_{1i})\,\vec{v_2}^2+2\,v_{2i}\,\vec{v_1}.\vec{v_2})+S_1^{0k}((2\,\vec{v_1}.\vec{v_2}-\frac{1}{2}\vec{v_2}^2)n_k+(2\,v_{2k}-v_{1k})\vec{v_2}.\vec{n}+2ra_{2k})\} \tag{8.4}$$

Total amplitude for quadratic terms:

$$-i\,V \;=\; i\,\frac{G^2 m_2}{r^3} n_k\{m_1(S_1^{ik}(2\,v_{2i} - v_{1i}) + S_1^{0k}) + 2\,m_2\,S_1^{ik}\,(v_{1i} - v_{2i})\} \tag{8.5}$$

Total amplitude for 3-graviton terms:

$$-i\,V \;=\; -i\,\frac{G^2 m_2}{r^3} n_k\{m_1(S_1^{ik}(\frac{1}{2}\,v_{1i} + 2\,v_{2i}) + 2\,S_1^{0k}) + m_2(S_1^{ik}(\frac{3}{2}\,v_{1i} + v_{2i}) + 2\,S_1^{0k})\} \tag{8.6}$$

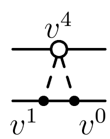
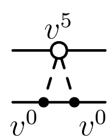
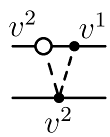
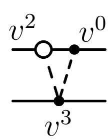
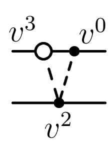
	$-2\, i\, G^2\, m_2^2\, S_1^{ik}\, v_{2i}\, n_k / r^3$
	$2\, i\, G^2\, m_2^2\, S_1^{ik}\, v_{1i}\, n_k / r^3$
	0
	$2\, i\, G^2\, m_1\, m_2\, S_1^{ik}\, v_{2i}\, n_k / r^3$
	$i\, G^2\, m_1\, m_2\, (\, S_1^{0k} - S_1^{ik}\, v_{1i}\,)\, n_k / r^3$

Table 8.2: Feynman diagrams with quadratic terms

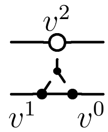
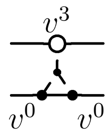
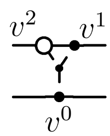
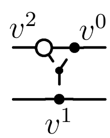
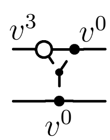
	$-i\,G^2\,m_2^2\,S_1^{ik}\,v_{2i}\,n_k/r^3$
	$-i\,G^2\,m_2^2\,(3/2\,S_1^{ik}\,v_{1i} + 2\,S_1^{0k}\,)\,n_k/r^3$
	$-3\,i\,G^2\,m_1\,m_2\,S_1^{ik}\,v_{1i}\,n_k/r^3$
	$-2\,i\,G^2\,m_1\,m_2\,S_1^{ik}\,v_{2i}\,n_k/r^3$
	$2\,i\,G^2\,m_1\,m_2\,(5/4\,S_1^{ik}\,v_{1i} - S_1^{0k}\,)\,n_k/r^3$

Table 8.3: Feynman diagrams with 3-graviton vertex

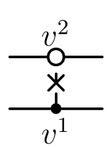
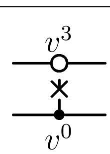
	$i\,G\,m_2\,S_1^{ik}/r^2\,\{\,r\,a_{2i}\,(\,v_{1k} - n_k\,\vec{v_1}.\vec{n}\,)$ $+\,v_{2i}\,(\,-3\,n_k\,\vec{v_1}.\vec{n}\,\vec{v_2}.\vec{n} + n_k\,\vec{v_1}.\vec{v_2} + \vec{v_1}.\vec{n}\,v_{2k} + v_{1k}\,\vec{v_2}.\vec{n})\,\}$
	$i\,G\,m_2\,S_1^{0k}\,\{n_k\,\vec{v_2}^{\,2} + r\,a_{2k} - r\,\vec{a_2}.\vec{n}\,n_k + 2\,v_{2k}\,\vec{v_2}.\vec{n} - 3\,n_k\,(\vec{v_2}.\vec{n})^2\,\}/(2r^2)$

Table 8.4: Feynman diagrams with propagator correction

Total amplitude for terms with a propagator correction:

$$
\begin{aligned}
-i\,V &= i\,\frac{Gm_2}{2\,r^2}\{S_1^{0k}(ra_{2k}+2\,v_{2k}\,\vec{v_2}.\vec{n}+n_k\,(\vec{v_2}^{\,2}-3\,(\vec{v_2}.\vec{n})^2-r\vec{a_2}.\vec{n}))+2\,S_1^{ik}(ra_{2i}\,(v_{1k}-n_k\,\vec{v_1}.\vec{n}) \\
&\quad +v_{2i}\,(n_k(\vec{v_1}.\vec{v_2}-3\,\vec{v_1}.\vec{n}\,\vec{v_2}.\vec{n})+v_{2k}\,\vec{v_1}.\vec{n}+v_{1k}\,\vec{v_2}.\vec{n}))\} \qquad (8.7)
\end{aligned}
$$

Adding all of the amplitudes, and taking amp = -iV gives us the potential V.

$$
\begin{aligned}
V_{S2.5PN} &= \frac{G^2\,m_2}{2\,r^3}\,n_k\,\{S_1^{0k}(2\,m_1+4\,m_2)+S_1^{ik}((3\,m_1-m_2)\,v_{1i}+6\,m_2\,v_{2i})\} \\
&\quad +\frac{Gm_2}{2\,r^2}\{S_1^{0k}\,(-5\,ra_{2k}+r\,n_k\,\vec{a_2}.\vec{n}+2\,(v_{1k}-3\,v_{2k})\,\vec{v_2}.\vec{n}+n_k\,(3(\vec{v_2}.\vec{n})^2-4\,\vec{v_1}.\vec{v_2})) \\
&\quad +S_1^{ik}(-2\,ra_{2i}(v_{1k}-n_k\,\vec{v_1}.\vec{n})-2\,v_{2i}\,(v_{2k}\,\vec{v_1}.\vec{n}+v_{1k}\,\vec{v_2}.\vec{n}) \\
&\quad +n_k\,(3\,v_{1i}\,\vec{v_2}^{\,2}-2\,(\vec{v_2}^{\,2}-3\,\vec{v_1}.\vec{n}\,\vec{v_2}.\vec{n}+3\,\vec{v_1}.\vec{v_2})\,v_{2i}))\}+(1\leftrightarrow 2) \qquad (8.8)
\end{aligned}
$$

or equivalently:

$$
\begin{aligned}
V_{S2.5PN} &= \frac{G^2\,m_2}{2\,r^3}\,n_k\,\{S_1^{0k}(2\,m_1+4\,m_2)+S_1^{ik}((3\,m_1-m_2)\,v_{1i}+6\,m_2\,v_{2i})\} \\
&\quad -\frac{G^2\,m_1}{2\,r^3}\,n_k\,\{S_2^{0k}(2\,m_2+4\,m_1)+S_2^{ik}((3\,m_2-m_1)\,v_{2i}+6\,m_1\,v_{1i})\} \\
&\quad +\frac{Gm_2}{2\,r^2}\{S_1^{0k}\,(-5\,ra_{2k}+r\,n_k\,\vec{a_2}.\vec{n}+2\,(v_{1k}-3\,v_{2k})\,\vec{v_2}.\vec{n}+n_k\,(3(\vec{v_2}.\vec{n})^2-4\,\vec{v_1}.\vec{v_2})) \\
&\quad +S_1^{ik}(-2\,ra_{2i}(v_{1k}-n_k\,\vec{v_1}.\vec{n})-2\,v_{2i}\,(v_{2k}\,\vec{v_1}.\vec{n}+v_{1k}\,\vec{v_2}.\vec{n}) \\
&\quad +n_k\,(3\,v_{1i}\,\vec{v_2}^{\,2}-2\,(\vec{v_2}^{\,2}-3\,\vec{v_1}.\vec{n}\,\vec{v_2}.\vec{n}+3\,\vec{v_1}.\vec{v_2})\,v_{2i}))\} \\
&\quad +\frac{Gm_1}{2\,r^2}\{S_2^{0k}\,(-5\,ra_{1k}+r\,n_k\,\vec{a_1}.\vec{n}-2\,(v_{2k}-3\,v_{1k})\,\vec{v_1}.\vec{n}-n_k\,(3(\vec{v_1}.\vec{n})^2-4\,\vec{v_1}.\vec{v_2})) \\
&\quad +S_2^{ik}(-2\,ra_{1i}(v_{2k}-n_k\,\vec{v_2}.\vec{n})+2\,v_{1i}\,(v_{1k}\,\vec{v_2}.\vec{n}+v_{2k}\,\vec{v_1}.\vec{n}) \\
&\quad -n_k\,(3\,v_{2i}\,\vec{v_1}^{\,2}-2\,(\vec{v_1}^{\,2}-3\,\vec{v_1}.\vec{n}\,\vec{v_2}.\vec{n}+3\,\vec{v_1}.\vec{v_2})\,v_{1i}))\}. \qquad (8.9)
\end{aligned}
$$

8.3 Conclusion

The methods of effective field theory enable us to provide a consistent framework with which to study gravitational dynamics and gravitational wave emission. By describing the physics at each length scale separately, divergences are resolved and renormalized into the coefficients of the parametric expansion. One can then calculate gravitational observables to any order in the velocity. NRGR enables us to calculate the corrections to 1) the Newtonian potential in order to better model the gravitational dynamics 2) the gravitational wave emission and will help provide more accurate templates for LIGO detection. Using the methods of NRGR, we have calculated the next-to-leading order (2.5PN) spin-orbit correction to the gravitational potential. Because we are not working in the same gauge, a direct comparison of this potential with that of Buonanno et al. is not possible. This will require calculating a gauge-invariant quantity such as the energy of a circular orbit as a function of frequency. Other subtleties such as different definitions for spin also need to be considered when comparing results.

References

[1] J. F. Donoghue, gr-qc/9712070

[2] http://lhc.web.cern.ch/lhc/

[3] S. Perlmutter et al., Nature **391**, 51 (1998) (astro-ph/9712212); S. Perlmutter et al., Astrophys. J. **517**, 565 (1999); A. G. Riess et al., Astron. J. **116**, 1009 (1998) (astro-ph/9805201)

[4] D. N. Spergel et al., Astrophys. J. Suppl. **148** (2003) (astro-ph/0302209); E. L Wright, astro-ph/0306132

[5] J. Cottam, F. Paerels, and M. Mendez, Nature **420**, 51 (2002)

[6] S. DeDeo, and D. Psaltis, Phys. Rev. Lett. **90**, 141101 (2003) (astro-ph/0302095)

[7] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (San Francisco: W.H. Freeman and Co., 1973)

[8] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring theory, Vols. I and II* (Cambridge University Press, 1987); J. Polchinski, *String Theory, Vols. I and II* (Cambridge University Press, 1988).

[9] C. M. Will, Living Rev. Relativity **9**, 3 (2006)

[10] C. H. Brans, and R. H. Dicke, Phys. Rev. **124**, 925 (1961)

[11] T. Damour, and G. Esposito-Farese, Class. Quantum Grav. **9**, 2093 (1992)

[12] J. D. Barrow, and A. C. Ottewill, J. Phys. A: Math. Gen. **16**, 2757 (1983)

[13] A. A. Starobinsky, Phys. Lett. **91B**, 1 (1980)

[14] S. M. Carroll, V. Duvvuri, M. Trodden, and M. S. Turner, Phys. Rev. D **70**, 043528 (2004) (astro-ph/0306438)

[15] T. Clifton, and J. D. Barrow, Phys. Rev. D **72**, 103005 (2005) (gr-qc/0509059)

[16] D. E. Barraco, and V. H. Hamity, Phys. Rev. D **57**, 954 (1998)

[17] D. Psaltis, astro-ph/0402213 (2004)

[18] A. A. Starobinsky, Class. Quantum Grav. **7**, 893 (1990)

[19] G. Magnano, and L. M. Sokolowski, Phys. Rev. D **50**, 5039 (1994) (gr-qc/9312008)

[20] S. Capozziello, R. Cianci, C. Stornaiolo, and S. Vignolo, Phys. Scripta **78**, 065010 (2008) (arXiv: 0810.2549 gr-qc)

[21] S. DeDeo, and D. Psaltis, astro-ph/0405067 (2004)

[22] S. Burles, K. M. Nollet, and M. S. Turner, Phys. Rev. D **63**, 063512 (2001) (astro-ph/0008495)

[23] I. I. Shapiro, *General Relativity and Gravitation: 12th International Conference on General Relativity and Gravitation* (Cambridge University Press, 1990), p.313.; J. O. Dickey, et al., Science **265**, 482 (1994); R. A. Hulse, Rev. Mod. Phys. **66**, 699 (1994); C. D. Hoyle, et al., Phys. Rev. Lett. **86**, 1418 (2001) (hep-ph/0011014); E G Adelberger, E.G, http://www.wlap.org/wl-repository/aps/meetings/general/2004/april/20040501-umwlcd0001-01/realaudio/sld024.htm; M. Milgrom, ApJ **270**, 365 (1983)

[24] S. Dimopoulos, and G. Landsberg, Phys. Rev. Lett.**87**, 161602 (2001) (hep-ph/0106295)

[25] H. A. Buchdahl, Mon. Not. R. astr. Soc. **150**, 1 (1970)

[26] P. C. W. Davies, S. A. Fulling, S. M. Christensen, and T. S. Bunch, Annals of Physics **109**, 108 (1977)

[27] B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, 1965)

[28] N. K. Glendenning, *Compact Stars* (Springer, 2000)

[29] D. Psaltis, Phys. Rev. D **77**, 064006 (2008) (arXiv: 0704.2426 astro-ph)

[30] D. Psaltis, Living Rev. Relativity **11**, 9 (2008)

[31] K. S. Stelle, Gen. Rel. Grav. **9**, 4, 353 (1978)

[32] D. Psaltis, D. Perrodin, K. R. Dienes, and I. Mocioiu, Phys. Rev. Lett. **100**, 091101, 1101 (2008)

[33] C. P. Burgess, Living Rev. Relativity **7**, 5 (2004)

[34] J. Bekenstein, Phys. Rev. D **7**, 2333 (1973); **9**, 3292 (1974)

[35] S. W. Hawking, Comm. Math. Phys. **43**, 199, (1975); Phys. Rev. D **13**, 191 (1976)

[36] J. Maldacena, A. Strominger, and E. Witten, JHEP **9712**, 002 (1997)

[37] F. D. Ryan, Phys. Rev. D **52**, 5707 (1995); N. A. Collins, and S. A. Hughes, Phys. Rev. D **69** 124022 (2004); K. Glampedakis, and S. Babak, Class. Quan. Gr. **23**, 4167 (2006)

[38] M. Ferraris, M. Francaviglia, and I. Volovich, CQG **11**, 1505 (1994); G. Allemandi, M. Francaviglia, M. L. Ruggiero, and A. Tartaglia, A. GRG, **37**, 1891 (2005).

[39] K. S. Thorne, and J. J. Dykla, ApJ **166**, L35 (1971)

[40] J. D. Bekenstein, Phys. Rev. D **5**, 2403 (1972)

[41] J. D. Bekenstein, and A. Meisels, Phys. Rev. D **18**, 4378 (1978)

[42] M. A. Scheel, S. L. Shapiro, and S. A. Teukolsky, Phys. Rev. D **51**, 4208 (1995); **51**, 4236 (1995)

[43] For example, Rosen's bimetric theory allows for naked singularities as discussed in Ref. [51].

[44] C. Eling, and T. Jacobson, CQG **23**, 5643; J. W. Moffat, gr-qc/0702070

[45] R. Gregory, R. Whisker, K. Beckwith, and C. Done, JCAP **0410**, 013 (2004).

[46] A. L. Erickcek, T. L. Smith, and M. Kamionkowski, Phys. Rev. D **74**, 121501 (2006)

[47] E. Barausse, T. P. Sotiriou, and J. C. Miller, Class. Quant. Grav. **25**, 062001 (2008) (gr-qc/0703132)

[48] R. Wald, *General Relativity* (University of Chicago Press, 1984)

[49] R. P. Woodard, Lect. Notes Phys. **720**, 403 (2007) (astro-ph/0601672)

[50] T. P. Sotiriou, and S. Liberati, Annals Phys. **332**, 935 (2007)

[51] C. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, 1993)

[52] Y. Song, W. Hu, and I. Sawicki, Phys. Rev. D **75**, 04404 (2007) (astro-ph/0610532); I. Sawicki, and W. Hu, Phys. Rev. D **75**, 127502 (2007)

[53] T. P. Sotiriou, and V. Faraoni, arXiv: 0805.1726 (gr-qc)

[54] K. S. Stelle, Phys. Rev. D**16**, 4 (1977)

[55] A. Cooney, S. DeDeo, and D Psaltis, Phys. Rev. D **79**, 044033 (2009) (arXiv: 0811.3635)

[56] J. Z. Simon, Phys. Rev. D **41**, 3720 (1990)

[57] W. D. Goldberger, and I. Z. Rothstein, Phys. Rev. D **73**, 104029 (2006) (hep-th/0409156)

[58] J. B. Gilmore, and A. Ross, Phys. Rev. D **78**, 124021 (2008) (arXiv:0810.1328)

[59] B. Kol, and M. Smolkin, Phys. Rev. D **77**, 064033 (2008) (arXiv: 0712.2822 hep-th)

[60] R. A. Porto, Phys. Rev. D **73**, 104031 (2006) (gr-qc/0511061)

[61] R. A. Porto, and I. Z. Rothstein, Phys. Rev. D **78**, 044012 (2008) (arXiv: 0802.0720)

[62] J. Steinhoff, S. Hergt, and G. Schaefer, Phys. Rev. D **77**, 081501 (2008) (arXiv: 0712.1716 gr-qc)

[63] R A Porto, and I. Z Rothstein, Phys. Rev. D **78**, 044013 (2008) (arXiv: 0804.0260 gr-qc)

[64] J. Steinhoff, S. Hergt, and G. Schaefer, Phys. Rev. D **78**, 101503 (2008) (arXiv: 0809.2200 gr-qc)

[65] G. Faye, L. Blanchet, and A. Buonanno, Phys. Rev. D **74**, 104033 (2006) (gr-qc/0605139)

[66] H. Tagoshi, A. Ohashi, and B. Owen, Phys. Rev. D **63**, 044006 (2001) (gr-qc/0010014)

[67] T. Damour, P. Jaranowski, and G. Schaefer, Phys. Rev. D **77**, 064032 (2008) (arXiv: 0711.1048); Phys. Rev. D **78**, 024009 (2008) (arXiv: 0803.0915 gr-qc)

[68] B. Kol, and M. Smolkin, Class. Quant. Grav. **25**, 145011 (2008) (arXiv: 0712.4116 hep-th)