Regularization of Simultaneous Binary Collisions in Some Gravitational Systems

by

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As members of the Dissertation Committee, we certify that we have read the dissertation prepared by Predrag Punoševac entitled *Regularization of Simultaneous Binary Collisions in Some Gravitational Systems* and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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Predrag Punoševac: ______________________________

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Dedication

To the memory of my mother Ljubica Punoševac. Your vision, encouragement, and ultimate sacrifice have enabled me to pursue my boyhood dreams in far away lands. May your soul rest in peace among the heavenly bodies, the study of which have taken me to this epic journey.

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Abstract

The aim of this study is to construct coordinate transforms that regularize the singularities of simultaneous binary collisions in the Newtonian gravitational systems. Explicit regularization transforms are introduced for a pair of decoupled Kepler problems, a restricted collinear four-body problem, and the collinear four-body problem. This is the first time such transforms are produced for collisions involving more than one colliding pairs in the study of the Newtonian gravitational systems.

1. Introduction

1.1. The N-body Problem

Consider n particles moving in a 3-dimensional Euclidean space $\mathbb{R}^3$. Let m_i, $\mathbf{q}_i$ denote the mass and the position of the i-th particle, respectively, and let t denote time. Assuming that the interactions among the particles are governed by the Newton Law of Gravitation, the motion of the particles is described by the following set of differential equations:

$$m_k \frac{d^2\mathbf{q}_k}{dt^2} = \frac{\partial U}{\partial \mathbf{q}_k}, \ k = 1, ..., n; \tag{1.1}$$

where U is the potential function

$$U = \sum_{1 \le j < i \le n} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|}.$$

Restricting the number of particles to two, three, or four, we have the so called two-body, three-body, or four-body problem, respectively. The two-body problem is also commonly referred to as the Kepler's problem.

Kepler's problem is *completely integrable.* The orbit of the particles in $\mathbb{R}^3$ are conic sections, and the positions of the particles for prescribed value t of time can be explicitly calculated.

The three-body problem is not integrable and it is notoriously complicated to study. People have also introduced and studied various systems which are more complex than the two-body problem but less complicated than the full three-body problem. We mention a few: the planar three-body problem (by assuming a two dimensional physical space), the collinear three-body problem (by assuming a one dimensional physical space), and the restricted three-body problem (by allowing particles of zero masses).

1.2. Collisions

We call the space of $\mathbf{q} := (\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_n) \in \mathbb{R}^{3n}$ the space of positions. Let

$$\Omega = \{\Omega_1, ..., \Omega_k\}, \ |\Omega_1| + ... + |\Omega_k| = n$$

be a partition of the set $N = \{1, 2, ..., n\}$ into k-subsets, $k < n$. We define the *singularity set* of a partition Ω as

$$\Delta_\Omega = \{\mathbf{q} \in \mathbb{R}^{3n} | \ \mathbf{q}_i = \mathbf{q}_j \text{ iff } i, j \in \Omega_l \text{ for some } \Omega_l \in \Omega\}. \tag{1.2}$$

The potential energy U is a real-analytic function on $\mathbb{R}^{3n} \setminus \Delta$ where $\Delta := \bigcup_\Omega \Delta_\Omega$. For a given initial condition $(\mathbf{q}_0, \mathbf{v}_0) \in (\mathbb{R}^{3n} \setminus \Delta) \times \mathbb{R}^{3n}$, there exists a solution of (1.1) of maximum extensions passing through $(\mathbf{q}_0, \mathbf{v}_0)$, the corresponding time interval of existence which we denoted as (t_1, t_2). We say that this solution has a singularity at $t = t_2$ if $t_2 < \infty$. In this case, $(\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_n) \to \Delta$ as $t \to t_2^-$. We say that t_2 is a *singularity of collision* if we further assume that there exists a point $\mathbf{L} \in \Delta$ such that $(\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_n) \to \mathbf{L}$ as $t \to t_2^-$. In this dissertation, we will only consider singularities of collisions. Note that, it is possible for a solution to end with a singularity that is not a collision as established in [X].

Let $\mathbf{q}(t) = (\mathbf{q}_1(t), \mathbf{q}_2(t), ..., \mathbf{q}_n(t))$ be a solution of equation (1.1) and

$$\lim_{t \to t_2^-} \mathbf{q}(t) = \mathbf{L} \in \Delta.$$

Then according to the location of $\mathbf{L}$ in Δ, we have different types of collision singularities, such as the singularity of *binary collision*, *triple collision*, and so on. We also have singularities of *simultaneous collisions* if we have more than two groups of particles colliding at t_2. In particular, we have a singularity of *simultaneous binary collisions*, in brief, a *SBC*, if each colliding group is with only two particles.

1.3. Regularizations of Binary Collisions

Let us now consider Kepler's problem on one dimensional physical space. Assume that the particles move along x-axis. Denote the distance between the two particles as x and let $v := \frac{dx}{dt}$. We can write the equation of motion with masses normalized as

$$\frac{dx}{dt} = v, \qquad \frac{dv}{dt} = -\frac{1}{x^2}. \tag{1.3}$$

We have the integral of energy as

$$h = \frac{v^2}{2} - \frac{1}{x}, \tag{1.4}$$

where h is the energy constant. Let us assume that $x(t)$, $t \in (t_1, t_2)$ is a solution of (1.3), and as $t \to t_2^-$, $x \to 0$. It is easily derived from (1.4) that

$$x(t) \sim (t - t_2)^{\frac{2}{3}} \tag{1.5}$$

as $t \to t_2^-$.

We introduce the new phase variables (ξ, η, h) by letting

$$x = \frac{\xi^2}{2}, \quad v = \frac{\eta}{\xi}, \quad h = \frac{v^2}{2} - \frac{1}{x}. \tag{1.6}$$

From (1.4), we have

$$2\xi^2 h = \eta^2 - 4. \tag{1.7}$$

We also introduce a new time τ by letting

$$d\tau = \frac{dt}{x}. \tag{1.8}$$

From (1.5) we have

$$\int_{t_2}^{t} \frac{1}{x(t)} dt \sim \int_{t_2}^{t} \frac{1}{(t - t_2)^{\frac{2}{3}}} dt = 3(t - t_2)^{\frac{1}{3}} \to 0 \text{ as } t \to t_2^-$$

So $\tau \to \tau_0 < \infty$ as $t \to t_2^-$. It follows that a solution $(x(t), v(t))$ of equation (1.3) is a solution of the new equations

$$\frac{d\xi}{d\tau} = \frac{\eta}{2}, \quad \frac{d\eta}{d\tau} = h\xi, \quad \frac{dh}{d\tau} = 0 \tag{1.9}$$

restricted to the algebraic variety (1.7). Notice that the singularity of collision $x = 0$ is now removed from equation (1.9). Coordinate transformations (1.6) and (1.8) are commonly referred to as a *regularization transform* for the Kepler's problem.

In general cases, coordinate transforms similar to (1.6) and (1.8) have been introduced to regularize the singularities of binary collisions of the Newtonian n-body problem. Consequently, if a solution $\mathbf{q}(t)$ of the n-body problem approaches to a singularity of binary collision at $t = t_2 < \infty$, then $\mathbf{q}(t)$ can be extended analytically beyond the time of collision $t = t_2$. In fact, $\mathbf{q}(t)$ can be written as a convergent power series in $\tau := (t - t_2)^{\frac{1}{3}}$ around $t = t_2$.

1.4. Triple Collisions

It has turned out that, unlike binary collisions, the singularities of triple collision (and all other collisions involving more than three particles) are not regularizable. This was first proved by Siegel [SM]. The main idea of Siegel's proof is to re-scale the coordinates of colliding particles by using the time factor $(t - t_2)^{-\frac{2}{3}}$. Together with an appropriate re-scaling of velocities, Siegel proved that all solutions of triple collisions in the three-body problem form the stable manifolds of a collection of normally hyperbolic fixed points in the phase space of the re-scaled coordinates. Since the eigenvalues of these normally hyperbolic fixed points are non-trivial functions of masses of the particles involved in collision, the corresponding singularities are essential for generic combinations of masses, therefore can not be removed by coordinate transforms.

The geometric significance of Siegel's proof is perhaps best demonstrated by McGehee's study of the total collisions in the collinear three-body problem. Let $\mathbf{x} = (x_1, x_2, x_3)^T$, (m_1, m_2, m_3) be respectively positions and masses of three particles moving on an Euclidean line $\mathbb{R}^1$. Let $M := \text{diag}(m_1, m_2, m_3)$, $\mathbf{p} := (p_1, p_2, p_3)^T$ where $p_i := m_i \frac{dx_i}{dt}$. The equation of motion for the collinear three-body problem is written as

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= M^{-1}\mathbf{p}, \\ \frac{d\mathbf{p}}{dt} &= \nabla U. \end{aligned} \tag{1.10}$$

where

$$U(\mathbf{x}) = \sum_{1\le j<i\le 3} \frac{m_i m_j}{|x_i - x_j|}. \tag{1.11}$$

The integral of energy is given as

$$T - U = h \tag{1.12}$$

where

$$T = \frac{1}{2}\mathbf{p}^T M^{-1}\mathbf{p}. \tag{1.13}$$

Let

$$I := \sum_{i=1}^{3} m_i x_i^2. \tag{1.14}$$

Following [McG1], we introduce McGehee's coordinates as follows. Let $(r, \mathbf{s}, p_r, \mathbf{p}_s)$ be the new phase coordinates defined by

$$\begin{aligned} r &= I^{\frac{1}{2}}, & \mathbf{s} &= \frac{\mathbf{x}}{r}, \\ p_r &= r^{\frac{1}{2}}\mathbf{p}^T\mathbf{s}, & \mathbf{p}_s &= r^{\frac{1}{2}}\mathbf{p} - p_r M\mathbf{s}, \end{aligned} \tag{1.15}$$

and τ be the new time defined by

$$\frac{d\tau}{dt} = r^{\frac{3}{2}}.$$

Roughly speaking, $(r, \mathbf{s})$ are polar coordinates for positions and $(p_r, \mathbf{p}_s)$ are the corresponding radian and angular velocities. The equations of motions in these new coordinates are easily derived as

$$\begin{aligned} \frac{dr}{d\tau} &= r p_r, \\ \frac{dv}{d\tau} &= \frac{1}{2}p_r^2 + \mathbf{p}_s^T M^{-1}\mathbf{p}_s - U(\mathbf{s}), \end{aligned}$$

$$\begin{aligned}
\frac{d\mathbf{s}}{d\tau} &= M^{-1}\mathbf{p}_s, \\
\frac{d\mathbf{p}_s}{d\tau} &= -\frac{1}{2}p_r\mathbf{p}_s - \left(\mathbf{p}_s^T M^{-1}\mathbf{p}_s\right) M\mathbf{r}_s + U(s)M\mathbf{s}_s + \nabla U(\mathbf{s}),
\end{aligned} \tag{1.16}$$

and the integral of energy is written as

$$\frac{1}{2}(\mathbf{p}_s^T M\mathbf{p}_s + p_r^2) - U(\mathbf{s}) = rh. \tag{1.17}$$

Substituting $r = 0$ into (1.16) and (1.17), we have introduced, in the places of triple collision in the phase space of old variables, a co-dimension one manifold defined by

$$\frac{1}{2}(\mathbf{p}_s^T M\mathbf{p}_s + p_r^2) - U(\mathbf{s}) = 0. \tag{1.18}$$

which we call the *triple collision manifold*. Further, we have the following set of equations

$$\begin{aligned}
\frac{dv}{d\tau} &= \frac{1}{2}p_r^2 + \mathbf{p}_s^T M^{-1}\mathbf{p}_s - U(\mathbf{s}), \\
\frac{d\mathbf{s}}{d\tau} &= M^{-1}\mathbf{p}_s, \\
\frac{d\mathbf{p}_s}{d\tau} &= -\frac{1}{2}p_r\mathbf{p}_s - \left(\mathbf{p}_s^T M^{-1}\mathbf{p}_s\right) M\mathbf{r}_s + U(s)M\mathbf{s}_s + \nabla U(\mathbf{s}),
\end{aligned} \tag{1.19}$$

obtained by setting $r = 0$ in (1.16). These equations are well-defined on the collision manifold. Upon regularization of binary collisions, the triple collision manifold looks like a "pants" (Figure 1.1) being topologically equivalent to a sphere minus four points.

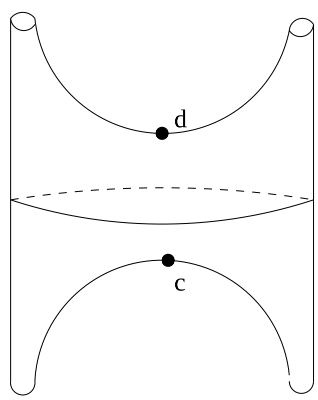

FIGURE 1.1. The triple collision manifold

Further studies revealed that, the flow (1.19) defined on the triple collision manifold (1.18) is a gradient-like flow with two hyperbolic fixed points (c, d in Fig.2). All solutions of triple collisions form the stable manifold of c and all solutions of triple ejection (coming out of triple collision) form the unstable manifold of d. Let us now take two points on the different sides of the stable manifold of c, arbitrarily close. These two solutions would end up following different arms of the collision manifold. This implies that, for the solutions of the collinear three-body problem around the orbits of triple collisions, there is no *continuity* with respect to initial conditions *after* collision, making regularization an impossibility. See (Figure 1.2).

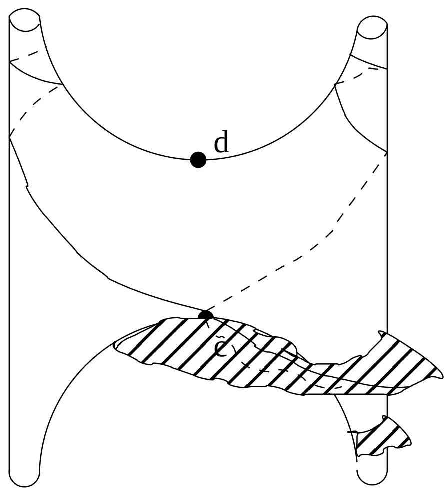

FIGURE 1.2. Flow on the triple collision manifold

With Siegel's proof and the studies of McGehee's collision manifold, it is clear that any collisions involving more than two particles can not be regularized.

1.5. Simultaneous Binary Collisions

Now the only remaining case for consideration, with respect to the issue of regularization, is the singularities of simultaneous binary collisions. As the first step, it is natural to consider the collinear four-body problem, the simplest version of the n-body problem in which this type of singularities can occur. Let x_1, x_2, x_3, and x_4 be the respective positions of four gravitational masses m_1, m_2, m_3, and m_4. The equations of motion is described by the following set of ordinary differential equations:

$$m_k \frac{d^2 x_k}{dt^2} = \frac{\partial U}{\partial x_k}, \ \ k = 1, 2, 3, 4;$$

or equivalently,

$$\frac{dx_k}{dt} = m_k^{-1} p_k, \ \ \frac{dp_k}{dt} = \frac{\partial U}{\partial x_k}, \tag{1.20}$$

where U is the potential function,

$$U = \sum_{1 \le j < i \le 4} \frac{m_i m_j}{|x_i - x_j|}.$$

As before, we call the space of $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ the space of positions. Recall that, for a partition Ω of the index set $\{1, 2, 3, 4\}$, Δ_Ω is the singular set for Ω and $\Delta = \bigcup_\Omega \Delta_\Omega$.

Let $\vec{\varphi}(t) = (x_1(t), x_2(t), x_3(t), x_4(t), v_1(t), v_2(t), v_3(t), v_4(t))$ be a solution of equation (1.20) defined on (t_1, t_2) experiencing collision singularity as $t \to t_2^-$. Assume that $\lim_{t \to t_2^-} \mathbf{x}(t) = \mathbf{L} = (L_1, L_2, L_3, L_4) \in \Delta$. According to the locations of $\mathbf{L}$ in Δ, collision singularities in the collinear four-body problem are put into the categories of (a) binary collisions, (b) simultaneous binary collisions, (c) triple collisions and (d) four-body (total) collision. (a)-(d) are in fact the only singularities allowed by equation (1.20) for $x(t)$.[1] In particular, we have an SBC (simultaneous binary collision) if $\mathbf{L}$ is such that $L_1 = L_2, L_3 = L_4$ but $L_1 \neq L_3$. Let us denote the set of $\mathbf{L}$ satisfying these restrictions as $\Delta_{12,34}$.

This work will address the following question:

Main Question: *If a solution $\vec{\varphi}(t)$ of a system of equations (1.20) for a given initial condition $\vec{\varphi}(t_0)$ experience SBC at t_2 i.e. $\lim_{t \to t_2^-} \mathbf{x}(t) = \mathbf{L} \in \Delta_{12,34}$, do there exist a*

[1] We note that the non-collision singularity in the collinear four-body problem, the existence of which is proved in [MM], only occurs after the singularity of binary collisions are regularized.

new time τ, an analytic invertible function $t = T(\tau)$ satisfying $\tau_2 := T^{-1}(t_2) < \infty$, and a phase space change of variables $\vec{\psi} = S(\vec{\varphi})$, such that the equations

$$\frac{d\vec{\psi}}{d\tau} = \mathbf{F}(\vec{\psi}) \tag{1.21}$$

obtained from (1.20) by the coordinate transformation S and the change of time T satisfy the following properties:

(a) a solution $\vec{\psi}(\tau)$ of (1.21) satisfying the initial condition $\vec{\psi}(T^{-1}(t_0))$ is well-defined at τ_2 i.e. $\vec{\psi}(\tau_2) := \lim_{\tau \to \tau_2^-} \vec{\psi}(\tau)$ exists;

(b) the equations (1.21) are real analytic at $\vec{\psi}(\tau_2)$?

1.6. Some Historical Remarks

The regularization of binary collisions played a pivotal role in Sundman's construction of global power series solutions for the three-body problem. Partly through the influences of Sundman's work ([Su], [SM]), regularization became an important theme. As we mentioned earlier, it turned out that the dynamical nature of collisions of more than three bodies are entirely different from that of two bodies. They are in general not regularizable. This was originally proved by Siegel ([S]). The underlining implications of Siegel's analysis on the phase space geometry have been thoroughly investigated through the introduction of McGehee's transformation ([McG1]), made possible many progresses, including the proofs on the existence of non-collision singularities ([MM], [X]), and the construction of global power series solutions ([Wa]).

For the issue of regularizations, the singularity of SBC is the only case left to be investigated. The results obtained in the past have been rather puzzling. On one hand, investigations based on Siegel's analysis and McGehee's transformation ([Sa1], [El1], [SL], [B]) have indicated that, geometrically, SBC are not that different from two independent binary collision in nature. On the other hand, however, no regularization variables have been found, not even for the system of decoupled Kepler's problems.

The aim of this study is to construct coordinate transforms that regularize the singularities of simultaneous binary collisions in the Newtonian gravitational systems. Explicit regularization transforms are introduced for a pair of decoupled Kepler problems, a restricted collinear four-body problem and the collinear four-body problem. This is the first time such transforms are produced for collisions involving more than one colliding pairs in the study of the Newtonian gravitational systems.

2. On a Pair of Decoupled Kepler's Problems

We obtain a pair of the decoupled Kepler's problems by dropping the interactions between mass groups $\{m_1, m_2\}$ and $\{m_3, m_4\}$ in the collinear four-body problem. Let $x_1, x_2 \in \mathbb{R}^+$ denote the distances between $m_1,\ m_2$ and $m_3,\ m_4$ respectively. In this chapter, we study the following set of differential equations

$$\frac{d^2x_1}{dt^2} = -\frac{1}{x_1^2}, \qquad \frac{d^2x_2}{dt^2} = -\frac{1}{x_2^2} \tag{2.1}$$

that describe the motion of particles. Let $v_1 = \frac{dx_1}{dt}$, $v_2 = \frac{dx_2}{dt}$. We take (x_1, x_2, v_1, v_2) as phase variables to rewrite equation (2.1) as

$$\frac{dx_1}{dt} = v_1, \quad \frac{dv_1}{dt} = -\frac{1}{x_1^2}; \qquad \frac{dx_2}{dt} = v_2, \quad \frac{dv_2}{dt} = -\frac{1}{x_2^2}. \tag{2.2}$$

For $i = 1, 2$, let $\Delta_i = \{(x_1, x_2) \in \overline{(\mathbb{R}^2)^+}, x_i = 0\}$, $\Delta = \Delta_1 \cup \Delta_2$ and $\Delta_{1,2} = \Delta_1 \cap \Delta_2$. Positions in $\Delta \setminus \Delta_{1,2}$ are positions of *binary collision* and those in $\Delta_{1,2}$ are positions of *simultaneous binary collisions*.

2.1. Preliminaries

Let us denote $\vec{\varphi} := (x_1, x_2, v_1, v_2)$ for short. Equations (2.2) have two first integrals of energy

$$h_1 = \frac{v_1^2}{2} - \frac{1}{x_1}, \qquad h_2 = \frac{v_2^2}{2} - \frac{1}{x_2}. \tag{2.3}$$

Let

$$\mathcal{U}_\rho := \{\vec{\varphi} = (x_1, x_2, v_1, v_2) \in (\mathbb{R}^2)^+ \times \mathbb{R}^2 :\ |h_1 x_1|,\ |h_2 x_2| < \rho\}$$

where $\rho < 1$ is positive. Throughout this section we fix ρ and consider only solutions of system of equations (2.2) in $\mathcal{U}_\rho$. We also let

$$F(h, u) = \int_0^u \frac{du}{\sqrt{2(h + \frac{1}{u})}} = \frac{\sqrt{2}}{3} u^{\frac{3}{2}} [1 + X(h, u)] \tag{2.4}$$

where

$$X(h,u)=\sum_{n=1}^{\infty}\frac{3c_n}{2n+3}h^n u^n \tag{2.5}$$

and c_n are such that

$$(1+x)^{-\frac{1}{2}}=1+\sum_{n=1}^{\infty}c_n x^n. \tag{2.6}$$

Lemma 2.1.1. *Let $\vec{\varphi}(t)=(x_1(t),x_2(t),v_1(t),v_2(t))$ be a solution of system of equations (2.2) in $\mathcal{U}_\rho$. Then*

$$t=\pm(F(h_1,x_1(t))-F(h_1,x_1(0)))=\pm(F(h_2,x_2(t))-F(h_2,x_2(0))) \tag{2.7}$$

where $\pm$ indicates that there is a sign that could go either way.

Proof: We have from (2.3) that

$$\int_0^t dt=\pm\int_{x_1(0)}^{x_1(t)}\frac{dx_1}{\sqrt{2(h_1+\frac{1}{x_1})}}=\pm\int_{x_2(0)}^{x_2(t)}\frac{dx_2}{\sqrt{2(h_2+\frac{1}{x_2})}},$$

from which (2.7) follows. □

For $t_1,t_2>0$ let

$$W_1(t_1)=\{\vec{\varphi}=(x_1,x_2,v_1,v_2)\in\mathcal{U}_\rho:\ F(h_1,\ x_1)=t_1\},$$
$$W_2(t_2)=\{\vec{\varphi}=(x_1,x_2,v_1,v_2)\in\mathcal{U}_\rho:\ F(h_2,\ x_2)=t_2\}.$$

Corollary 2.1.1. *Let $\vec{\varphi}(t)=(x_1(t),x_2(t),v_1(t),v_2(t))$ be a solution of system of equations (2.2) in $\mathcal{U}_\rho$. Then $x_1(t)\to 0$ as $t\to t_1^-$ if and only if $\vec{\varphi}(0)\in W_1(t_1)$. Similarly, $x_2(t)\to 0$ as $t\to t_2^-$ if and only if $\vec{\varphi}(0)\in W_2(t_2)$.*

Proof: Observe that $W_1(t_1)$ is defined by $F(h_1,x_1)=t_1$, an equation obtained by letting $t=t_1$, $x_1(t)=0$ in (2.7). The $\pm$ sign in (2.7) is forced to be negative since $\frac{dx_1}{dt}<0$ as $t\to t_1^-$. The situation for x_2 is similar. □

Let

$$\begin{aligned}Y&=F(h_2,\ x_2)-F(h_1,\ x_1)\\&=\frac{\sqrt{2}}{3}x_2^{\frac{3}{2}}\left[1+X(h_2,x_2)\right]-\frac{\sqrt{2}}{3}x_1^{\frac{3}{2}}\left[1+X(h_1,x_1)\right]\end{aligned} \tag{2.8}$$

where h_1, h_2 are as in (2.3) and $X(h, x)$ is as in (2.5). Y is a crucial new variable. Let us now make the following observations.

(a) Let $\vec{\varphi}(t)$ be a given solution of equation(2.2). Then

$$\frac{dY}{dt}(\vec{\varphi}(t)) = 0.$$

(b) The algebraic variety defined by $Y(\vec{\varphi}(t)) = 0$ and its backward images in time form a co-dimensional one immersed sub-manifold in phase space containing all solutions heading toward SBC.

(c) We can solve for $\frac{x_1}{x_2}$ and $\frac{x_2}{x_2}$ from (2.8) to obtain

$$\frac{x_1}{x_2} = \sqrt[\frac{2}{3}]{\frac{1 + X(h_2, x_2) - \frac{3\sqrt{2}Y}{2x_2^{3/2}}}{1 + X(h_1, x_1)}}; \qquad \frac{x_2}{x_1} = \sqrt[\frac{2}{3}]{\frac{1 + X(h_1, x_1) + \frac{3\sqrt{2}Y}{2x_1^{3/2}}}{1 + X(h_2, x_2)}} \tag{2.9}$$

where $X(h, x)$ is as in (2.5).

2.2. A change of variables and the regularized equations

We are now ready to introduce regularization variables. Let us denote the new phase variables as $\vec{\psi} := (\xi_1, \xi_2, \eta_1, \eta_2, h_1, h_2, Y)$, and the new time as τ.

First, (ξ_1, η_1) and (ξ_2, η_2) are determined by (x_1, v_1) and (x_2, v_2) through

$$x_1 = \frac{\xi_1^2}{2}, \quad v_1 = \frac{\eta_1}{\xi_1}; \qquad x_2 = \frac{\xi_2^2}{2}, \quad v_2 = \frac{\eta_2}{\xi_2}. \tag{2.10}$$

These are the well known Levi-Civita changes of coordinates. Second, (h_1, h_2) are defined by using (2.3) and Y by using (2.8). Third, τ is defined through

$$d\tau = \frac{1}{2}\left(\frac{1}{x_1} + \frac{1}{x_2}\right) dt, \tag{2.11}$$

and in reverse we have

$$dt = \left(\frac{1}{\xi_1^2} + \frac{1}{\xi_2^2}\right)^{-1} d\tau. \tag{2.12}$$

The new equations in phase space variables $\vec{\psi} = (\xi_1, \xi_2, \eta_1, \eta_2, h_1, h_2, Y)$ derived from equations (2.2) are as follows:

$$\frac{d\xi_1}{d\tau} = \frac{1}{1+f_1}\eta_1, \qquad \frac{d\eta_1}{d\tau} = \frac{2}{1+f_1}h_1\xi_1; \tag{2.13}$$

$$\frac{d\xi_2}{d\tau} = \frac{1}{1+f_2}\eta_2, \qquad \frac{d\eta_2}{d\tau} = \frac{2}{1+f_2}h_2\xi_2; \tag{2.14}$$

$$\frac{dh_1}{d\tau} = 0, \qquad \frac{dh_2}{d\tau} = 0; \tag{2.15}$$

$$\frac{dY}{d\tau} = 0. \tag{2.16}$$

where

$$\begin{aligned} f_1 = \frac{\xi_1^2}{\xi_2^2} &= \sqrt[\frac{2}{3}]{\frac{1+\sum_{n=1}^{\infty}\frac{3c_n}{2^n(2n+3)}{h_2}^n\xi_2^{2n} - \frac{6Y}{\xi_2^3}}{1+\sum_{n=1}^{\infty}\frac{3c_n}{2^n(2n+3)}h_1^n\xi_1^{2n}}}, \\ f_2 = \frac{\xi_2^2}{\xi_1^2} &= \sqrt[\frac{2}{3}]{\frac{1+\sum_{n=1}^{\infty}\frac{3c_n}{2^n(2n+3)}h_1^n\xi_1^{2n} + \frac{6Y}{\xi_1^3}}{1+\sum_{n=1}^{\infty}\frac{3c_n}{2^n(2n+3)}h_2^n\xi_2^{2n}}}. \end{aligned} \tag{2.17}$$

For the solutions of the equations (2.13)-(2.16) to represent the solutions of equation (2.2), we also have constraints

$$2\xi_1^2h_1 = \eta_1^2 - 4, \qquad 2\xi_2^2h_2 = \eta_2^2 - 4 \tag{2.18}$$

and

$$Y = \left[\frac{1}{6} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{c_n}{2^n(n+\frac{3}{2})}h_2^n\xi_2^{2n}\right]\xi_2^3 - \left[\frac{1}{6} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{c_n}{2^n(n+\frac{3}{2})}h_1^n\xi_1^{2n}\right]\xi_1^3. \tag{2.19}$$

(2.18) is derived from (2.3) and (2.19) from (2.8). This also shows that we didn't alter the dimension of the phase space since there are three relations relating seven new phase space variables which implies that the phase space is still four dimensional.

Derivations of equations (2.13)-(2.16): (2.15) and (2.16) follows from the fact that h_1, h_2 and Y are first integrals of equation (2.2). For the first item of the equations (2.13) we differentiate $\xi_1^2 = 2x_1$ to obtain

$$\frac{d\xi_1}{dt} = \frac{v_1}{\xi_1} = \frac{\eta_1}{\xi_1^2}.$$

We have by using (2.12)

$$\frac{d\xi_1}{d\tau} = \frac{\eta_1}{1 + \frac{\xi_1^2}{\xi_2^2}}.$$

We then substitute f_1 for $\frac{\xi_1^2}{\xi_2^2}$ using (2.9).

For the second item of the equations (2.13) we differentiate $\eta_1 = v_1\xi_1$ to obtain

$$\frac{d\eta_1}{dt} = v_1\frac{d\xi_1}{dt} + \xi_1\frac{dv_1}{dt} = \frac{v_1^2}{\xi_1} - \frac{\xi_1}{x_1^2}$$

where equation (2.2) is used for the second equality. We then use (2.3) and (2.12) to conclude

$$\frac{d\eta}{d\tau} = \frac{2h_1\xi_1}{1 + \frac{\xi_1^2}{\xi_2^2}}.$$

We again substitute f_1 for $\frac{\xi_1^2}{\xi_2^2}$ by using (2.9). The derivations for equations in (2.14) are similar. □

Let

$$\mathcal{V}_\rho = \{\vec{\psi} = (\xi_1, \xi_2, \eta_1, \eta_2, h_1, h_2, Y) : \ |h_1\xi_1^2|, |h_2\xi_2^2| < 2\rho\}$$

be the correspondence of $\mathcal{U}_\rho$ in the new phase space variables, and $\mathcal{A}_\rho$ be the algebraic variety defined by (2.18) and (2.19) in $\mathcal{V}_\rho$. Our next lemma assures that (2.18) and (2.19) are natural constraints for equations (2.13)-(2.16).

Lemma 2.2.1. *Let $\vec{\psi}(\tau), \tau \in (\tau_1, \tau_2)$ be a solution of equations (2.13)-(2.16) in $\mathcal{V}_\rho$. If $\{\vec{\psi}(\tau), \tau \in (\tau_1, \tau_2)\} \cap \mathcal{A}_\rho \neq \emptyset$, then it is included in $\mathcal{A}_\rho$.*

Proof: Recall that $\vec{\psi} = (\xi_1, \xi_2, \eta_1, \eta_2, h_1, h_2, Y)$ is used to denote the new phase variables and $\vec{\varphi} = (x_1, x_2, v_1, v_2)$ the old phase variables. Let $\vec{\psi}(\tau)$ be a solution of equation (2.13)-(2.16) and assume that $\vec{\psi}(\tau_0)$ satisfies (2.18) and (2.19). Useing (2.10) we obtain a corresponding value $\vec{\varphi}(t_0)$. Let $\vec{\varphi}(t)$ be the solution of equation (2.2) satisfying $\vec{\varphi}(0) = \vec{\varphi}(t_0)$. We then use (2.3), (2.8), (2.10) and

$$\tau = \tau_0 + \frac{1}{2}\int_0^t \left(\frac{1}{x_1(t)} + \frac{1}{x_2(t)}\right) dt \tag{2.20}$$

to convert $\vec{\varphi}(t)$ to a function of $\vec{\psi}$ in τ, which we denote as $\vec{\Psi}(\tau)$. We caution that there are more than one way to make the last conversion but we can always chose to make $\vec{\Psi}(\tau_0) = \vec{\psi}(\tau_0)$. We then observe that

(i) By the previous derivation of the new equations for $\vec{\psi}$, $\vec{\Psi}(\tau)$ satisfies (2.13)-(2.16). Then by the uniqueness of solutions of the system of ODEs $\vec{\Psi}(\tau) = \vec{\psi}(\tau)$ for all τ;

(ii) On the other hand, $\vec{\Psi}(\tau)$ satisfies (2.18) and (2.19) by default. □

Equations (2.13)-(2.16) defined on $\mathcal{A}_\rho$ are our *regularized equations* for (2.2). Other solutions of equations (2.13)-(2.16) are not relevant.

Remark: It is sometimes helpful to derive from (2.13)-(2.16) equations of different forms by using constraints (2.18) and (2.19). Since the phase variables and the time are kept the same, the corresponding solutions for equations of different forms derived by using (2.18) and (2.19) are identical. For instance, replacing f_1 in equation (2.13) by $\frac{\xi_1^2}{\xi_2^2}$ while keeping all other equations the same would give us a set of equations of new look, but the set of functions $\vec{\psi}(\tau)$ for solutions remains the same.

2.3. Regularization result

We are now ready to prove

Theorem 1. *Let $\vec{\varphi}(t) = (x_1(t), x_2(t), v_1(t), v_2(t))$, $t \in (t_1, t_2)$ be a solution of equation (2.2) in $\mathcal{U}_\rho$. Assume that $\lim_{t\to t_2^-}(x_1(t), x_2(t)) = \mathbf{L} \in \Delta$. Let $\tau(t)$ be defined by (2.20) and $\vec{\psi}(\tau)$, $\tau \in (\tau_1, \tau_2)$ be the functions obtained from $\vec{\varphi}(t)$ through (2.10), (2.3) and (2.8). Then*

(a) $\vec{\psi}(\tau)$ is a solution of equations (2.13)-(2.16) on $\mathcal{A}_\rho$;

(b) $\tau_2 := \tau(t_2) < \infty$, and $\vec{\psi}(\tau_2) := \lim_{\tau\to\tau_2^-} \vec{\psi}(\tau)$ is well-defined; and

(c) equations (2.13)-(2.16) defined on $\mathcal{A}_\rho$ are real analytic at $\vec{\psi}(\tau_2)$.

Proof: (a) This follows from the derivations of equations (2.13)-(2.16) in Section 2.2. We caution that (2.10) allows different ways to convert $\vec{\varphi}(t)$ to $\vec{\psi}(\tau)$ because ξ_i ($= \pm\sqrt{2x_i}$ by (2.10)) can assume different signs. This is a well known characteristic of Levi-Civita variables. For definiteness, let us chose the positive sign so that $\xi_i = \sqrt{2x_i}$. We also note that τ_0 in (2.20) is arbitrary.

(b) It is well known that when a collision singularity occur at t_2,

$$U(t) := \frac{1}{x_1(t)} + \frac{1}{x_2(t)} \sim (t - t_2)^{-\frac{2}{3}}.$$

From this it follows that

$$\tau_2 = \tau_0 + \frac{1}{2}\int_0^{t_2} U(t)dt < \infty.$$

Now for $\vec{\psi}(\tau_2)$: $h_1(\tau_2), h_2(\tau_2)$ and $Y(\tau_2)$ are integral constants determined by initial conditions. Observe that $x_i(t),\ i = 1, 2$ tend to definite limits as $t \to t_2^-$, which we denote as $x_i(t_2)$. We let $\xi_i(\tau_2) = \sqrt{2x_i(t_2)}$. Finally for $\eta_i(\tau_2)$ we use $\eta_i(\tau_2) = \frac{v_i(t_2)}{\xi_i(\tau_2)}$ if $\xi_i(\tau_2) \neq 0$ (In this case $v_i(t) \to v_i(t_2)$, is a definite limit as it follows from (2.3)). If $\xi_i(\tau_2) = 0$, then $\eta_i^2(\tau_2) = 4$ according to (2.18), from which it follows that $\eta_i(\tau_2) = -2$. $\eta_i(\tau_2)$ is negative because we have used positive sign for $\xi_i(\tau_2)$.

(c) We have three cases to consider depend on what happens at t_2: (1) $x_1(t_2) = 0$ and $x_2(t_2) = 0$; (2) $x_1(t_2) = 0$ but $x_2(t_2) \neq 0$; and (3) $x_2(t_2) = 0$ but $x_1(t_2) \neq 0$. They correspond to the cases of $Y = 0$, $Y > 0$ and $Y < 0$ respectively.

Case $Y = 0$: This is the case of simultaneous binary collisions. Set $Y = 0$ in equations (2.13) and (2.14). It is clear that the functions on the right hand are all analytic at the values of $\vec{\psi}(\tau_2)$ given in the above. We conclude that the singularity of simultaneous binary collisions is regularized.

Case $Y < 0$: This is a case of binary collision at which $x_2(t_2) = 0$. In this case $\xi_1(\tau_2) \neq 0$, $\xi_2(\tau_2) = 0$. To see that this singularity is removed in the first item of equation (2.13), we rewrite it as

$$\frac{d\xi_1}{d\tau} = \frac{\eta_1 \xi_2^2}{\xi_2^2 + \sqrt[\frac{2}{3}]{\frac{\frac{2}{3}\xi_2^3 + \sum_{n=1}^{\infty} \frac{c_n}{2^n(n+\frac{3}{2})} {h_2}^n \xi_2^{2n+3} - 4Y}{\frac{2}{3} + \sum_{n=1}^{\infty} \frac{c_n}{2^n(n+\frac{3}{2})} h_1^n \xi_1^{2n}}}}. \tag{2.21}$$

It is clear that $\xi_2 = 0$ is not a singularity of the function on the right hand because $-4Y > 0$ by assumption. The second item is handled similarly.

For the first item of equation (2.14) we replace f_2 by $\frac{\xi_2^2}{\xi_1^2}$ to rewrite this equation as

$$\frac{d\xi_2}{d\tau} = \frac{\xi_1^2 \eta_2}{\xi_1^2 + \xi_2^2}.$$

See the remark we made at the end of Section 2.2. The function on the right hand is obviously real analytic at $\vec{\psi}(\tau_2)$ since $\xi_1(\tau_2) \neq 0$. The argument for the second item follows the same line.

Case $Y > 0$**:** Similar to the case $Y < 0$. □

Theorem 1 is a precise way to state that all singularities of collision in equation (2.2) are removed by transferring to equations (2.13)-(2.16) on $\mathcal{A}_\rho$.

3. On a Restricted Four-body Problem

In this chapter we introduce regularization variables for the singularity of SBC in a restricted four-body problem. This restricted collinear four-body problem is obtained by letting $m_1 = m_4 = 0$. New issues arise as we move from the decoupled Kepler's problems studied in Chapter 2 to this new gravitational system that is not integrable.

3.1. Equations of motion

We consider gravitational particles m_1, m_2, m_3 and m_4 positioned at $x_1 \leq x_2 < x_3 \leq x_4$ respectively in $\mathbb{R}$. In this section, we assume $m_1 = m_4 = 0$. To simplify the writing we also assume $m_2 = m_3 = 1$. Our assumption on m_2 and m_3 is not necessary and the construction presented in this section applies in principle to arbitrary combinations of positive m_2 and m_3.

Let

$$u_1 = x_2 - x_1, \quad u_2 = x_4 - x_3, \quad \hat{u} = x_3 - x_2$$

and $v_i = \frac{du_i}{dt}$ for $i = 1, 2$, $\hat{v} = \frac{d\hat{u}}{dt}$. $(u_1, u_2, \hat{u}, v_1, v_2, \hat{v})$ are the phase variables. Let

$$\mathcal{K}(u, \hat{u}) = \frac{1}{u + \hat{u}}.$$

We denote

$$\mathcal{K}_1 = \mathcal{K}(u_1, \hat{u}), \quad \mathcal{K}_2 = \mathcal{K}(u_2, \hat{u}), \quad \hat{\mathcal{K}} = \mathcal{K}(0, \hat{u})$$

and write the equations of motion as

$$\begin{aligned}
\frac{du_1}{dt} &= v_1, \quad & \frac{dv_1}{dt} &= -\frac{1}{u_1^2} + \frac{\partial \mathcal{K}_1}{\partial u_1} - \frac{\partial \hat{\mathcal{K}}}{\partial \hat{u}}; \\
\frac{du_2}{dt} &= v_2, \quad & \frac{dv_2}{dt} &= -\frac{1}{u_2^2} + \frac{\partial \mathcal{K}_2}{\partial u_2} - \frac{\partial \hat{\mathcal{K}}}{\partial \hat{u}}; \\
\frac{d\hat{u}}{dt} &= \hat{v}, \quad & \frac{d\hat{v}}{dt} &= 2\frac{\partial \hat{\mathcal{K}}}{\partial \hat{u}}.
\end{aligned} \tag{3.1}$$

Let

$$h_1 := \frac{v_1^2}{2} - \frac{1}{u_1}, \qquad h_2 := \frac{v_2^2}{2} - \frac{1}{u_2}. \tag{3.2}$$

It follows from (3.1) that

$$\frac{dh_1}{dt} = v_1 \left(\frac{\partial \mathcal{K}_1}{\partial u_1} - \frac{\partial \hat{\mathcal{K}}}{\partial \hat{u}} \right), \qquad \frac{dh_2}{dt} = v_2 \left(\frac{\partial \mathcal{K}_2}{\partial u_2} - \frac{\partial \hat{\mathcal{K}}}{\partial \hat{u}} \right). \tag{3.3}$$

Remarks: (1) $(u_1, u_2, \hat{u}) \in (\mathbb{R}^2)^+ \times \mathbb{R}^+$ is now the space of positions and

$$\Delta_{1,2} = \{(u_1, u_2, \hat{u}) \in (\mathbb{R}^2)^+ \times \mathbb{R}^+ : u_1 = u_2 = 0,\ \hat{u} \in \mathbb{R}^+\}$$

is the singular set for the SBC.

(2) Observe that we would get back to the decoupled Kepler's problems considered in Chapter 2 by letting $\mathcal{K}(u, \hat{u}) =$ constant in equation (3.1).

(3) We intend to follow the ideas developed in Chapter 2. However, because $\mathcal{K}_i$, $\hat{\mathcal{K}}$ are non-trivial, h_i are no longer first integrals. Consequently, the correspondence of the new variable Y is much less straight forward to define.

(4) Let us also note that, for the restricted four-body problem introduced above, $\frac{\partial^2 \mathcal{K}_i}{\partial u_1 \partial u_2} = 0$ by design. The fact that the correspondences of these mixed derivatives are not zero in the full collinear four-body problem will present itself a major hurdle in similar constructions of regularization variables, as we will see in Sect. 3.4.

3.2. Variable Y: formal definition and convergence

3.2.1. Outline of strategy

Let $K > 1$ be fixed and $\rho < (100K^8)^{-2}$ be positive. In this section the phase variables are $\vec{\varphi} := (u_1, u_2, \hat{u}, v_1, v_2, \hat{v})$ and

$$\mathcal{U}_{K,\rho} = \{\vec{\varphi} \in (\mathbb{R}^3)^+ \times \mathbb{R}^3 : \quad u_1, u_2 < \rho;\ K^{-1} < \hat{u} < K;\ |h_1|, |h_2| < K\}.$$

We only consider solutions of equation (3.1) in $\mathcal{U}_{K,\rho}$.

Lemma 3.2.1. *Let $\vec{\varphi}(t)$, $t \in (t_1, t_2)$ be a solution of equation (3.1) in $\mathcal{U}_{K,\rho}$. Then the limits of $u_i, \hat{u}, \hat{v}$ are well defined as $t \to t_2^-$. Furthermore, if $u_i(t) \to u_i(t_2) \neq 0$, then $v_i(t)$ has a well defined limit as $t \to t_2^-$.*

The proof of Lemma 3.2.1 is well documented in literature. See, for instance, [SM]. We also give a proof in the Appendix B.

We have from (3.2) that

$$dt = \pm \frac{du_1}{\sqrt{2\left(h_1 + \frac{1}{u_1}\right)}}.$$

Let $\vec{\varphi}(t) \subset \mathcal{U}_{K,\rho}$ be a solution of equation (3.1). Integrating on both sides we obtain

$$t - t_1 = \pm \int_0^{u_1(t)} \frac{du_1}{\sqrt{2\left(h_1 + \frac{1}{u_1}\right)}} \tag{3.4}$$

where t_1 is such that $u_1(t_1) = 0$. Let us denote

$$F := \int_0^{u_1} \frac{du_1}{\sqrt{2\left(h_1 + \frac{1}{u_1}\right)}}.$$

Since h_1 is no longer a first integral of equation (3.1), F as written above is not precisely a well defined definite integral. Let us, however, put this subtlety aside for now and treat F formally as if it is well defined. We then expand the integrand to obtain

$$F = \frac{1}{\sqrt{2}} \left[\frac{2}{3} u_1^{\frac{3}{2}} + \sum_{n=1}^{\infty} c_n \int_0^{u_1} h_1^n u_1^{n+\frac{1}{2}} du_1 \right] \tag{3.5}$$

where $c_n, n > 0$ are as in (2.6) in Chapter 2.

To each of the integrals in (3.5) (as well as the new ones we will soon encounter), a **degree** is assigned according to the power of u_1 in the integrand. For instance, the integral

$$I_n = \int_0^{u_1} h_1^n u_1^{n+\frac{1}{2}} du_1$$

is an integral of degree $n+\frac{1}{2}$. Our strategy is to use integration by part together with equation (3.1) to replaces all integrals in (3.5) with integrals of degrees higher and higher to eventually write F explicitly in phase variables. Let us take I_n as an example. We have

$$\begin{aligned} I_n &= \frac{1}{n+\frac{3}{2}}\int_0^{u_1} h_1^n du_1^{n+\frac{3}{2}} \\ &= \frac{1}{n+\frac{3}{2}}h_1^n u_1^{n+\frac{3}{2}} - \frac{n}{n+\frac{3}{2}}\int_0^{u_1} h_1^{n-1}\frac{dh_1}{du_1}u_1^{n+\frac{3}{2}}du_1 \\ &= \frac{1}{n+\frac{3}{2}}h_1^n u_1^{n+\frac{3}{2}} - \frac{n}{n+\frac{3}{2}}\int_0^{u_1} h_1^{n-1}\frac{\partial \mathcal{K}_1}{\partial u_1}u_1^{n+\frac{3}{2}}du_1 + \frac{n}{n+\frac{3}{2}}\int_0^{u_1} h_1^{n-1}\frac{\partial \hat{\mathcal{K}}}{\partial \hat{u}}u_1^{n+\frac{3}{2}}du_1 \end{aligned}$$

where for the last equality we replaced $\frac{dh_1}{du_1}$ by using (3.3). I_n is then the summation of a term that is explicit in u_1 and h_1 and two integrals of one degree higher.

We now go one step further to transfer the new integrals obtained in the above to integrals of degree even higher. We have for instance

$$\begin{aligned} I &:= \int_0^{u_1} h_1^{n-1}\frac{\partial \mathcal{K}_1}{\partial u_1}u_1^{n+\frac{3}{2}}du_1 \\ &= \frac{1}{n+\frac{5}{2}}\int_0^{u_1} h_1^{n-1}\frac{\partial \mathcal{K}_1}{\partial u_1}du_1^{n+\frac{5}{2}} \\ &= \frac{1}{n+\frac{5}{2}}h_1^{n-1}\frac{\partial \mathcal{K}_1}{\partial u_1}u_1^{n+\frac{5}{2}} - \frac{n-1}{n+\frac{5}{2}}\int_0^{u_1} h_1^{n-2}\frac{\partial \mathcal{K}_1}{\partial u_1}u_1^{n+\frac{5}{2}}\frac{dh_1}{du_1}du_1 \\ &\quad -\frac{1}{n+\frac{5}{2}}\int_0^{u_1} h_1^{n-1}\frac{\partial^2 \mathcal{K}_1}{\partial u_1^2}u_1^{n+\frac{5}{2}}du_1 - \frac{1}{n+\frac{5}{2}}\int_0^{u_1} h_1^{n-1}\frac{\partial^2 \mathcal{K}_1}{\partial u_1 \partial \hat{u}}u_1^{n+\frac{5}{2}}\frac{\hat{v}}{v_1}du_1. \end{aligned}$$

The first two integrals can be further converted to integrals of one degree higher the same way. The last one, however, is with a new factor $\hat{v}v_1^{-1}$. We will keep $\hat{v}$, which is bounded therefore harmless, but rewrite v_1^{-1} through (3.2) as

$$v_1^{-1} = \frac{1}{\sqrt{2}}(u_1^{\frac{1}{2}} + \sum_{n=1}^{\infty} c_n h_1^n u_1^{n+\frac{1}{2}}) \tag{3.6}$$

where c_n is as in (2.6). The third integral is then replaced by a sequence of integrals of ascending degrees by using (3.6).

Based on the computations of similar nature, we now proceed as follows. Let us start with (3.5). First we replace the integral of degree $\frac{3}{2}$ in (3.5) by a function written explicitly in phase variables and a number of integrals of higher degree. We then move up to replace all integrals of degree $\frac{5}{2}$ the same way and so on.[1] This process goes forever and, at the end, we hope to be able to write F explicitly as a function of $u_1, h_1, \hat{u}$ and $\hat{v}$. In another word we hope to have a well defined, convergent replacement process.

3.2.2. A formal inductive process

We now formally introduce the desired replacement process following the strategy outlined in Sect. 3.2.1 for

$$F = \frac{1}{\sqrt{2}} \int_0^{u_1} \frac{du_1}{\sqrt{h_1 + \frac{1}{u_1}}}.$$

Initially we let

$$F = \mathcal{F}^{(3)} := \frac{1}{\sqrt{2}} \left[\frac{2}{3} u_1^{\frac{3}{2}} + \sum_{n=1}^{\infty} \int_0^{u_1} c_n h_1^n u_1^{n+\frac{1}{2}} du_1 \right]. \tag{3.7}$$

Proposition 3.2.1. *Let $m \geq 3$. We have*

$$\begin{aligned} F = \mathcal{F}^{(m)} := &\sum_{n=3}^{m+1} \left(\sum_{j \leq \hat{S}(m,n)} \hat{f}_j^{(n)}(u_1, h_1, \hat{u}, \hat{v}) \right) u_1^{\frac{n}{2}} \\ &+ \sum_{n=m}^{\infty} \left(\sum_{j \leq S(m,n)} \int_0^{u_1} f_j^{(n)}(u_1, h_1, \hat{u}, \hat{v}) u_1^{\frac{n}{2}} du_1 \right) \end{aligned} \tag{3.8}$$

where

(a) **on non-integral terms***:*

[1] Initially the degree of integral are moved up by one but very soon the increment get down to $\frac{1}{2}$ because of replacements that invokes (3.6).

(i) for every $j \le \hat{S}(m,n)$*, there exists coefficients* $\hat{C}_{n,j}$ *satisfying* $|\hat{C}_{n,j}| < 10^n$ *and integers* i_k *for* $k = 1$ *to* 4 *satisfying* $i_k \le 2n$ *such that*

$$\hat{f}_j^{(n)}(u_1, h_1, \hat{u}, \hat{v}) = \hat{C}_{n,j} h_1^{i_1} \hat{v}^{i_2} (u_1 + \hat{u})^{-i_3} \hat{u}^{-i_4};$$

(ii) $\hat{S}(m, m+1) < 5^{m-1}$*;*

(b) **on integrals***:*

(i) for every $j \le S(m,n)$*, there exists coefficients* $C_{n,j}$ *satisfying* $|C_{n,j}| < 10^n$ *and integers* i_k *for* $k = 1$ *to* 4 *satisfying* $i_k \le 2n$*, such that*

$$f_j^{(n)}(u_1, h_1, \hat{u}, \hat{v}) = C_{n,j} h_1^{i_1} \hat{v}^{i_2} (u_1 + \hat{u})^{-i_3} \hat{u}^{-i_4};$$

(ii) $S(m, m) < 5^m$*.*

Remark: $\mathcal{F}^{(m)}$ in (3.8) represents the integral F obtained at the end of stage m of a replacement process we will introduce momentarily in the proof of Proposition 3.2.1. According to (3.8) and (a)(i), the non-integral part is a finite sum of terms of ascending degrees in u_1, the highest of which is $\frac{m+1}{2}$. The term of degree $\frac{n}{2}, n \le m+1$ in this summation is in turn a summation of $\hat{S}(m,n)$ terms, each of which is in the form assumed in Proposition (a)(i). Similarly, according to (3.8) and (b)(i), the integral part is a series of integrals of ascending degrees, the lowest of which is $\frac{m}{2}$. We have in total $S(m,n)$ integrals of degree $\frac{n}{2}$ for $n \ge m$, each of which is in the form assumed in b(i). (a)(ii) and b(ii) claim that the growth of the number of terms created by replacement is slower than exponential, a crucial fact for convergence. Let us also note that the increment of power in u_1 is half instead of one in $\mathcal{F}^{(m)}$ because the use of (3.6). However, through integration by part the non-integral terms obtained from an integral of degree $\frac{m}{2}$ is of degree $\frac{m}{2} + 1$. This is why the non-integral part is a summation up to $n = m + 1$ instead of $n = m$.

Proof of Proposition 3.2.1: First we prove (a)(i), (b)(i) and (3.8) inductively. For $m = 3$, $\mathcal{F}^{(3)}$ is as in (3.7). It is obviously in the form assumed by (3.8) satisfying Proposition 3.2.1a(i) and b(i).

Let us now inductively assume that $\mathcal{F}^{(m)}$, $m = 3, \cdots, M$ are well-defined in the form assumed by (3.8), and Proposition 3.2.1a(i) and b(i) hold up to $m = M$. $\mathcal{F}^{(M+1)}$ is derived from replacing all integrals of degree $\frac{M}{2}$ in $\mathcal{F}^{(M)}$ as follows. Let

$$I = \int_0^{u_1} f_j^{(M)}(u_1, h_1, \hat{u}, \hat{v}) u_1^{\frac{M}{2}} du_1$$

be an integral of degree $\frac{M}{2}$ in $\mathcal{F}^{(M)}$,

$$\begin{aligned} I &= C_{M,j} \int_0^{u_1} h_1^{i_1} \hat{v}^{i_2} (u_1 + \hat{u})^{-i_3} \hat{u}^{-i_4} u_1^{\frac{M}{2}} du_1 \\ &= \frac{C_{M,j}}{\frac{M}{2}+1} \int_0^{u_1} h_1^{i_1} \hat{v}^{i_2} (u_1 + \hat{u})^{-i_3} \hat{u}^{-i_4} du_1^{\frac{M}{2}+1} \\ &= \frac{C_{M,j}}{\frac{M}{2}+1} \left(h_1^{i_1} \hat{v}^{i_2} (u_1 + \hat{u})^{-i_3} \hat{u}^{-i_4} u_1^{\frac{M}{2}+1} - \int_0^{u_1} u_1^{\frac{M}{2}+1} d\left(h_1^{i_1} \hat{v}^{i_2} (u_1 + \hat{u})^{-i_3} \hat{u}^{-i_4} \right) \right). \end{aligned}$$

Hence

$$I = \frac{C_{M,j}}{\frac{M}{2}+1} h_1^{i_1} \hat{v}^{i_2} (u_1 + \hat{u})^{-i_3} \hat{u}^{-i_4} u_1^{\frac{M}{2}+1} - \sum_{i=1}^{4} \mathcal{I}_i. \tag{3.9}$$

The first term on the right hand is the contribution of I to the non-integral part of $\mathcal{F}^{(M+1)}$. To be more precise we get one $\hat{f}_j^{(M+1)}$ from I such that

$$\hat{f}_j^{(M+1)} = \frac{C_{M,j}}{\frac{M}{2}+1} h_1^{i_1} \hat{v}^{i_2} (u_1 + \hat{u})^{-i_3} \hat{u}^{-i_4}.$$

For this $\hat{f}_j^{(M+1)}$ term Proposition 3.2.1a(i) is satisfied with

$$\hat{C}_{M+1,j} = \frac{C_{M,j}}{\frac{M}{2}+1}.$$

We also have in (3.9)

$$\begin{aligned}
\mathcal{I}_1 &= \frac{i_1 C_{M,j}}{\frac{M}{2}+1}\int_0^{u_1} h_1^{i_1-1}\hat{v}^{i_2}(u_1+\hat{u})^{-i_3}\hat{u}^{-i_4}\frac{dh_1}{du_1}u_1^{\frac{M}{2}+1}du_1,\\
\mathcal{I}_2 &= \frac{i_2 C_{M,j}}{\frac{M}{2}+1}\int_0^{u_1} h_1^{i_1}\hat{v}^{i_2-1}(u_1+\hat{u})^{-i_3}\hat{u}^{-i_4}\frac{d\hat{v}}{dt}\frac{1}{v_1}u_1^{\frac{M}{2}+1}du_1,\\
\mathcal{I}_3 &= -\frac{i_3 C_{M,j}}{\frac{M}{2}+1}\int_0^{u_1} h_1^{i_1}\hat{v}^{i_2}(u_1+\hat{u})^{-i_3-1}\hat{u}^{-i_4}\left(1+\frac{\hat{v}}{v_1}\right)u_1^{\frac{M}{2}+1}du_1,\\
\mathcal{I}_4 &= -\frac{i_4 C_{M,j}}{\frac{M}{2}+1}\int_0^{u_1} h_1^{i_1}\hat{v}^{i_2}(u_1+\hat{u})^{-i_3}\hat{u}^{-i_4-1}\frac{\hat{v}}{v_1}u_1^{\frac{M}{2}+1}du_1.
\end{aligned}$$

All these integrals are to be further transformed as follows.

(a) On $\mathcal{I}_1$: From (3.3) we have

$$\begin{aligned}
\mathcal{I}_1 &= \frac{i_1 C_{M,j}}{\frac{M}{2}+1}\int_0^{u_1} h_1^{i_1-1}\hat{v}^{i_2}(u_1+\hat{u})^{-i_3}\hat{u}^{-i_4}(-(u_1+\hat{u})^{-2}+(\hat{u})^{-2})u_1^{\frac{M}{2}+1}du_1\\
&= -\frac{i_1 C_{M,j}}{\frac{M}{2}+1}\int_0^{u_1} h_1^{i_1-1}\hat{v}^{i_2}(u_1+\hat{u})^{-(i_3+2)}\hat{u}^{-i_4}u_1^{\frac{M}{2}+1}du_1\\
&\quad+\frac{i_1 C_{M,j}}{\frac{M}{2}+1}\int_0^{u_1} h_1^{i_1-1}\hat{v}^{i_2}(u_1+\hat{u})^{-i_3}\hat{u}^{-(i_4+2)}u_1^{\frac{M}{2}+1}du_1.
\end{aligned}\tag{3.10}$$

Hence $\mathcal{I}_1$ is the sum of two integrals of degree $\frac{M}{2}+1$, each of which satisfies Proposition 3.2.1b(i): note that the degree of these new integrals is $\frac{M}{2}+1$ and recall that we have assumed inductively that

$$C_{M,j}\le 10^M,\qquad i_1,\ i_2,\ i_3,\ i_4\le 2M,$$

from which it follows that

$$\frac{i_1 C_{M,j}}{\frac{M}{2}+1}\le 10^{M+2},\qquad i_1-1,\ i_2,\ i_3+2,\ i_4+2\le 2(M+2).$$

(b) On $\mathcal{I}_2$: Note that $\frac{d\hat{v}}{dt}=-2\hat{u}^{-2}$ and for v_1^{-1} we use (3.6). We have

$$\begin{aligned}
\mathcal{I}_2 &= \frac{2i_2 C_{M,j}}{\frac{M}{2}+1}\int_0^{u_1} h_1^{i_1}\hat{v}^{i_2-1}(u_1+\hat{u})^{-i_3}\hat{u}^{-(i_4+2)}v_1^{-1}u_1^{\frac{M}{2}+1}du_1\\
&= \frac{\sqrt{2}i_2 C_{M,j}}{\frac{M}{2}+1}\sum_{k=0}^{\infty}\int_0^{u_1} c_k h_1^{i_1+k}\hat{v}^{i_2-1}(u_1+\hat{u})^{-i_3}\hat{u}^{-(i_4+2)}u_1^{\frac{M}{2}+k+\frac{3}{2}}du_1.
\end{aligned}\tag{3.11}$$

Hence $\mathcal{I}_2$ is a summation of infinitely many integrals of ascending degrees, each of which again satisfies Proposition 3.2.1b(i). Observe that to any given $n > M$, $\mathcal{I}_2$ contains only one integral of degree $\frac{n}{2}$.

(c) On $\mathcal{I}_3$*:* Similarly we have

$$\begin{aligned}\mathcal{I}_3 = &-\frac{i_3 C_{M,j}}{\frac{M}{2}+1}\int_0^{u_1} h_1^{i_1}\hat{v}^{i_2}(u_1+\hat{u})^{-i_3-1}\hat{u}^{-i_4}u_1^{\frac{M}{2}+1}du_1 \\ &-\frac{\sqrt{2}i_3 C_{M,j}}{M+2}\sum_{k=0}^{\infty}\int_0^{u_1} c_k h_1^{i_1+k}\hat{v}^{i_2+1}(u_1+\hat{u})^{-i_3-1}\hat{u}^{-i_4}u_1^{\frac{M}{2}+k+\frac{3}{2}}du_1\end{aligned} \tag{3.12}$$

$\mathcal{I}_3$ is again an infinite summation of integrals of ascending degrees, each of of which satisfies Proposition 3.2.1b(i). Again for any given $n > M$, $\mathcal{I}_3$ contains at most one integral of degree $\frac{n}{2}$.

(d) On $\mathcal{I}_4$*:* Similarly we have

$$\mathcal{I}_4 = -\frac{\sqrt{2}i_4 C_{M,j}}{M+2}\sum_{k=0}^{\infty}\int_0^{u_1} c_k h_1^{i_1+k}\hat{v}^{i_2+1}(u_1+\hat{u})^{-i_3}\hat{u}^{-i_4-1}u_1^{\frac{M}{2}+k+\frac{3}{2}}du_1 \tag{3.13}$$

This is similar to $\mathcal{I}_2$ and $\mathcal{I}_3$.

We are now ready to define $\mathcal{F}^{(M+1)}$. For every integral I of degree $\frac{M}{2}$ in $\mathcal{F}^{(M)}$, we replace I by using (3.9)-(3.13). This proves Proposition 3.2.1a(i), b(i) and (3.8).

For Proposition 3.2.1a(ii) and b(ii) we observe that from (a)-(d) above

Lemma 3.2.2. *(1)* $\hat{S}(M,n) = \hat{S}(M+1,n)$ *for* $n \leq M+1$*,*

(2) $\hat{S}(M+1, M+2) = S(M,M)$*,*

(3) $S(M+1,n) \leq S(M,n) + 4S(M,M)$ *for* $n > M$*.*

Proof: From $\mathcal{F}^{(M)}$ to $\mathcal{F}^{(M+1)}$, non-integral terms of degree $< \frac{M+2}{2}$ in u_1 are not effected so (1) holds. (2) follows from the observation that every integral I of degree $\frac{M}{2}$ in $\mathcal{F}^{(M)}$ contributes exactly one non-integral term (See (3.9)) of power $\frac{M}{2}+1$ in u_1 to $\mathcal{F}^{(M+1)}$. (3) follows from the fact that, for any $n > M$ fixed, replacing an integral I

of degree $\frac{M}{2}$ in $\mathcal{F}^{(M)}$ by using (3.9) and (a)-(d) above adds at most four more integrals of degree $\frac{n}{2}$ to the collection of integrals of the same degree in $\mathcal{F}^{(M)}$. $\square$

We now use Lemma 3.2.2(3) inductively to prove Proposition 3.2.1(ii). Note that Lemma 3.2.2(3) holds for all $M \geq 3$. We have

$$\begin{aligned} S(M+1, M+1) &\leq S(M, M+1) + 4S(M, M) \\ &\leq S(M-1, M+2) + 4S(M-1, M-1) + 4S(M, M) \\ &\leq S(3, M+1) + 4\sum_{n=3}^{M} S(n, n). \end{aligned}$$

Note that $S(3, M+1) = 1$. We have

$$S(M+1, M+1) \leq 4S(M, M) + 4S(M-1, M-1) + \cdots + 4S(3,3) + 1 \tag{3.14}$$

for all $M \geq 3$. Use (3.14) inductively we obtain

$$S(M, M) \leq 5^M,$$

from which it also follows that

$$\hat{S}(M+1, M+2) \leq S(M, M) < 5^M.$$

Here Lemma 3.2.2(2) is used for the first inequality. This finishes our proof of Proposition 3.2.1(a)(ii) and b(ii). $\square$

Finally we let

$$F(u_1, h_1, \hat{u}, \hat{v}) = \frac{\sqrt{2}}{3} u_1^{\frac{3}{2}} + \sum_{n=4}^{\infty} \left(\sum_{\hat{S}(n,n+1)} \hat{f}_j^{(n+1)}(u_1, h_1, \hat{u}, \hat{v}) \right) u_1^{\frac{n+1}{2}}. \tag{3.15}$$

Proposition 3.2.2. *Under the assumption that* $\rho < (100K^8)^{-2}$*,* $F(u_1, h_1, \hat{u}, \hat{v})$ *in (3.15) is convergent.*

Proof: From Proposition 3.2.1a(i) we have for every j,

$$\hat{f}_j^{(n)} = \hat{C}_{n,j} h_1^{i_1} \hat{v}^{i_2} (u_1 + \hat{u})^{-i_3} \hat{u}^{-i_4} u_1^{\frac{n}{2}},$$

from which it follows that

$$|\hat{f}_j^{(n)}| < 10^n K^{8n} \rho^{\frac{n}{2}} \tag{3.16}$$

on $\mathcal{U}_{K,\rho}$. Combining Proposition 3.2.1a(ii), b(ii) and (3.16) we have

$$|F(u_1, h_1, \hat{u}, \hat{v})| < \sum_{n=3}^{\infty} (100K^8)^n \rho^{\frac{n}{2}}.$$

Hence F converges provided that $\rho < (100K^8)^{-2}$. This proves Proposition 3.2.2. □

Remark: Let us caution that $F_1 = F(u_1, h_1, \hat{u}, \hat{v})$ is not analytic in u_1 at $u_1 = 0$ because the power of u_1 ascends by half instead of one. To get analyticity we need to replace u_1 by a new variable ξ_1 through $\xi_1^2 = u_1$. F_1 is then analytic in ξ_1 at $\xi_1 = 0$.

3.2.3. The new variable Y

Let $F(u_1, h_1, \hat{u}, \hat{v})$ be as in (3.15) and

$$Y := F(u_1, h_1, \hat{u}, \hat{v}) - F(u_2, h_2, \hat{u}, \hat{v}). \tag{3.17}$$

We claim that Y is a first integral of equation (3.1). This claim is proved as follows. For a given solution $\vec{\varphi}(t)$ of equation (3.1) in $\mathcal{U}_{K,\rho}$, let t_1 be such that $u_1(t_1) = 0$ and t_2 be such that $u_2(t_2) = 0$. We have from the way $F(u_1, h_1, \hat{u}, \hat{v})$ is defined that

$$t - t_1 = F(u_1(t), h_1(t), \hat{u}(t), \hat{v}(t)), \qquad t - t_2 = F(u_2(t), h_2(t), \hat{u}(t), \hat{v}(t)), \tag{3.18}$$

from which we obtain

$$Y(t) = t_2 - t_1.$$

It then follows that

$$\frac{dY}{dt} = 0. \tag{3.19}$$

.

Let

$$f(u_1, h_1, \hat{u}, \hat{v}) = \frac{3\sqrt{2}}{2} \sum_{n=4}^{\infty} \left(\sum_{\hat{S}(n,n+1)} \hat{f}_j^{(n+1)}(u_1, h_1, \hat{u}, \hat{v}) \right) u_1^{\frac{n-2}{2}}. \tag{3.20}$$

We have

$$F(u_1, h_1, \hat{u}, \hat{v}) = \frac{\sqrt{2}}{3} u_1^{\frac{3}{2}} (1 + f(u_1, h_1, \hat{u}, \hat{v})). \tag{3.21}$$

Note that $f(u_1, h_1, \hat{u}, \hat{v})$ is again in a form of power series in $\sqrt{u_1}$ and $f(0, h_1, \hat{u}, \hat{v}) = 0$. From (3.15) and (3.17) we obtain

$$\frac{u_1}{u_2} = \sqrt[\frac{2}{3}]{\frac{1 + f(u_2, h_2, \hat{u}, \hat{v}) + \frac{3\sqrt{2}Y}{2u_2^{\frac{3}{2}}}}{1 + f(u_1, h_1, \hat{u}, \hat{v})}} \tag{3.22}$$

and

$$\frac{u_2}{u_1} = \sqrt[\frac{2}{3}]{\frac{1 + f(u_1, h_1, \hat{u}, \hat{v}) - \frac{3\sqrt{2}Y}{2u_1^{\frac{3}{2}}}}{1 + f(u_2, h_2, \hat{u}, \hat{v})}}. \tag{3.23}$$

(3.22) and (3.23) are the correspondence of (2.8) in Chapter 2.

3.3. Variables of regularization

The rest of this section is in parallel to Section 2.3. Let us denote the regularization variables as $\vec{\psi} = (\xi_1, \xi_2, \eta_1, \eta_2, \hat{u}, \hat{v}, h_1, h_2, Y)$ and the new time as τ. (ξ_1, η_1) and (ξ_2, η_2) are again determined by (u_1, v_1) and (u_2, v_2) through Levi-Civita change of coordinates

$$u_1 = \frac{\xi_1^2}{2}, \quad v_1 = \frac{\eta_1}{\xi_1}; \qquad u_2 = \frac{\xi_2^2}{2}, \quad v_2 = \frac{\eta_2}{\xi_2}; \tag{3.24}$$

$\hat{u}, \hat{v}$ remains the same; (h_1, h_2) are defined by using (3.2); and Y is defined by using (3.17). The new time τ is defined through

$$\tau = \tau_0 + \frac{1}{2} \int_0^t \left(\frac{1}{u_1} + \frac{1}{u_2} \right) dt, \tag{3.25}$$

and in reverse we have

$$dt = \left(\frac{1}{\xi_1^2} + \frac{1}{\xi_2^2} \right)^{-1} d\tau. \tag{3.26}$$

The new equations for $\vec{\psi} = (\xi_1, \xi_2, \eta_1, \eta_2, \hat{u}, \hat{v}, h_1, h_2, Y)$ derived from equation (3.1), in the same fasion as in the previous section, are as follows.

$$\begin{aligned}
\frac{d\xi_1}{d\tau} &= \frac{1}{1+X_1}\eta_1, \qquad \frac{d\eta_1}{d\tau} = \frac{1}{1+X_1}\left(2h_1\xi_1 - \frac{\xi_1^3}{(\frac{1}{2}\xi_1^2+\hat{u})^2} + \frac{\xi_1^3}{\hat{u}^2}\right);\\
\frac{d\xi_2}{d\tau} &= \frac{1}{1+X_2}\eta_2, \qquad \frac{d\eta_2}{d\tau} = \frac{1}{1+X_2}\left(2h_2\xi_2 - \frac{\xi_2^3}{(\frac{1}{2}\xi_2^2+\hat{u})^2} + \frac{\xi_2^3}{\hat{u}^2}\right);\\
\frac{d\hat{u}}{d\tau} &= \frac{1}{1+X_1}\xi_1^2\hat{v}, \qquad \frac{d\hat{v}}{d\tau} = \frac{-1}{1+X_1}\frac{\xi_1^2}{\hat{u}^2};\\
\frac{dh_1}{d\tau} &= \frac{1}{1+X_1}\xi_1\eta_1\left(-\frac{1}{(\frac{1}{2}\xi_1^2+\hat{u})^2} + \frac{1}{\hat{u}^2}\right),\\
\frac{dh_2}{d\tau} &= \frac{1}{1+X_2}\xi_2\eta_2\left(-\frac{1}{(\frac{1}{2}\xi_2^2+\hat{u})^2} + \frac{1}{\hat{u}^2}\right);\\
\frac{dY}{d\tau} &= 0
\end{aligned} \tag{3.27}$$

where

$$X_1 = \sqrt[\frac{2}{3}]{\frac{1+f(\frac{1}{2}\xi_2^2, h_2, \hat{u}, \hat{v}) + \frac{6Y}{\xi_2^3}}{1+f(\frac{1}{2}\xi_1^2, h_1, \hat{u}, \hat{v})}}, \qquad X_2 = \sqrt[\frac{2}{3}]{\frac{1+f(\frac{1}{2}\xi_1^2, h_1, \hat{u}, \hat{v}) - \frac{6Y}{\xi_1^3}}{1+f(\frac{1}{2}\xi_2^2, h_2, \hat{u}, \hat{v})}}$$

and f is as in (3.20). For $i = 1$ and 2, $f(\frac{1}{2}\xi_i^2, h_i, \hat{u}, \hat{v}, Y)$ is analytic in ξ_i at $\xi_i = 0$. We also have $f(0, h_i, \hat{u}, \hat{v}) = 0$.

The following constraints are further imposed on equation (3.27)

$$2\xi_1^2 h_1 = \eta_1^2 - 4, \qquad 2\xi_2^2 h_2 = \eta_2^2 - 4; \tag{3.28}$$

$$Y = \frac{1}{6}\xi_1^3(1 + f(\frac{1}{2}\xi_1^2, h_1, \hat{u}, \hat{v})) - \frac{1}{6}\xi_2^3(1 + f(\frac{1}{2}\xi_2^2, h_2, \hat{u}, \hat{v})). \tag{3.29}$$

The derivations of equation (3.27) and the constraints (3.28) and (3.29) are straight forward: For the first item of the system (3.27), we differentiate $\xi_1^2 = 2x_1$ to obtain

$$\frac{d\xi_1}{dt} = \frac{v_1}{\xi_1} = \frac{\eta_1}{\xi_1^2}.$$

We have by using (3.26)

$$\frac{d\xi_1}{d\tau} = \frac{\eta_1}{1 + \frac{\xi_1^2}{\xi_2^2}}.$$

We then substitute X_1 for $\frac{\xi_1^2}{\xi_2^2}$ using (3.22).

For the second item of the equations (3.27) we differentiate $\eta_1 = v_1\xi_1$ to obtain

$$\frac{d\eta_1}{dt} = v_1\frac{d\xi_1}{dt} + \xi_1\frac{dv_1}{dt} = \frac{v_1^2}{\xi_1} - \frac{\xi_1}{u_1^2} + \xi_1\frac{\partial\mathcal{K}_1}{\partial u_1} - \xi_1\frac{\partial\hat{\mathcal{K}}}{\partial\hat{u}}$$

where equation (3.1) is used for the second equality. We then use (3.2), (3.26), the definition of $\mathcal{K}$, and the definition of $\hat{\mathcal{K}}$ to conclude

$$\frac{d\eta}{d\tau} = \frac{1}{1+\frac{\xi_1^2}{\xi_2^2}}\left(2h_1\xi_1 - \frac{\xi_1^3}{(\frac{1}{2}\xi_1^2+\hat{u})^2} + \frac{\xi_1^3}{\hat{u}^2}\right).$$

We again substitute X_1 for $\frac{\xi_1^2}{\xi_2^2}$ by using (3.22).

The derivations of other equations in (3.27), except the last three, are similar. Note that, unlike in the Chapter 2, variables h_1 and h_2 are not the first integrals of the system (3.1) but rather satisfy the equations (3.3) which must be used in the derivation of the second the last and third the last item of (3.27). The last item is merely restatement of (3.19).

The constraint (3.28) is derived from (3.2). Similarly we obtain (3.29) from (3.17). □

Let $\mathcal{V}_{K,\rho}$ be the correspondence of $\mathcal{U}_{K,\rho}$ in new phase variables and $\mathcal{A}_{K,\rho}$ be the algebraic variety defined by (3.28) and (3.29) in $\mathcal{V}_{K,\rho}$. Let $\vec{\psi}(\tau), \tau \in (\tau_1, \tau_2)$ be a solution of equation (3.27) in $\mathcal{V}_{K,\rho}$. We claim once more that, if $\{\vec{\psi}(\tau), \tau \in (\tau_1, \tau_2)\} \cap \mathcal{A}_{K,\rho} \neq \emptyset$, then it is included in $\mathcal{A}_{K,\rho}$. Our proof for this last claim is identical to that of Lemma 2.2 in Section 2. Equations (3.27) defined on $\mathcal{A}_{K,\rho}$ are our *regularized equations* for (3.1). In parallel to Theorem 1 in Chapter 2, we have

Theorem 2. *Let $\vec{\varphi}(t) = (u_1(t), u_2(t), \hat{u}(t), v_1(t), v_2(t), \hat{v}(t)), t \in (t_1, t_2)$ be a solution of equation (3.1) in $\mathcal{U}_{K,\rho}$. Assume that $\lim_{t\to t_2^-}(u_1(t), u_2(t), \hat{u}(t)) = \mathbf{L} \in \Delta$. Let $\tau(t)$ be defined by (3.25) and $\vec{\psi}(\tau), \tau \in (\tau_1, \tau_2)$ be the functions obtained from $\vec{\varphi}(t)$ through (3.24), (3.2) and (3.17). Then*

(a) $\vec{\psi}(\tau)$ is a solution of equation (3.27) on $\mathcal{A}_{K,\rho}$;

(b) $\tau_2 := \tau(t_2) < \infty$*, and* $\vec{\psi}(\tau_2) := \lim_{\tau\to\tau_2^-} \vec{\psi}(\tau)$ *is well-defined; and*

(c) equation (3.27) defined on $\mathcal{A}_{K,\rho}$ *is real analytic at* $\vec{\psi}(\tau_2)$*.*

Proof: The proof of this theorem is completely parallel to that of Theorem 1 in Chapter 2.

(a) This follows from the derivations of equations (3.27). We caution that (3.24) allows different ways to convert $\vec{\varphi}(t)$ to $\vec{\psi}(\tau)$ because ξ_i ($= \pm\sqrt{2u_i}$ by (3.24)) can assume different signs. This is a well known characteristic of Levi-Civita variables. For definiteness, let us chose the positive sign so that $\xi_i = \sqrt{2u_i}$. We also note that τ_0 in (3.23) is arbitrary.

(b) It is well known that when a collision singularity occur at t_2,

$$U(t) := \frac{1}{u_1(t)} + \frac{1}{u_2(t)} \sim (t - t_2)^{-\frac{2}{3}}.$$

From this it follows that

$$\tau_2 = \tau_0 + \frac{1}{2}\int_0^{t_2} U(t)dt < \infty.$$

Now for $\vec{\psi}(\tau_2)$: $Y(\tau_2)$ is integral constants determined by initial conditions. Observe that $(u_1(t), u_2(t), \hat{u}(t), \hat{v}(t), h_1(t), h_2(t))$ tend to definite limits as $t \to t_2^-$ by Lemma 3.2.1. In particular, we denote as $u_i(t_2) = \lim_{t\to t_2} u_i(t)$. We let $\xi_i(\tau_2) = \sqrt{2u_i(t_2)}$. Finally for $\eta_i(\tau_2)$ we use $\eta_i(\tau_2) = \frac{v_i(t_2)}{\xi_i(\tau_2)}$ if $\xi_i(\tau_2) \neq 0$ (In this case $v_i(t) \to v_i(t_2)$, is a definite limit as it follows from Lemma 3.2.1. If $\xi_i(\tau_2) = 0$, then $\eta_i^2(\tau_2) = 4$ according to (3.28), from which it follows that $\eta_i(\tau_2) = -2$. $\eta_i(\tau_2)$ is negative because we have used positive sign for $\xi_i(\tau_2)$.

(c) We have three cases to consider depend on what happens at t_2: (1) $u_1(t_2) = 0$ and $u_2(t_2) = 0$; (2) $u_1(t_2) = 0$ but $u_2(t_2) \neq 0$; and (3) $u_2(t_2) = 0$ but $u_1(t_2) \neq 0$. They correspond to the cases of $Y = 0$, $Y > 0$ and $Y < 0$ respectively.

Case $Y = 0$**:** This is the case of simultaneous binary collisions. Set $Y = 0$ in the system (3.27). It is clear that the functions on the right hand are all analytic at the values of $\vec{\psi}(\tau_2)$ given in the above. We conclude that the singularity of simultaneous binary collisions is regularized.

Case $Y < 0$**:** This is a case of binary collision at which $u_2(t_2) = 0$. In this case $\xi_1(\tau_2) \neq 0$, $\xi_2(\tau_2) = 0$. To see that this singularity is removed in the first item of equation (3.27), we rewrite it as

$$\frac{d\xi_1}{d\tau} = \frac{\eta_1 \xi_2^2}{\xi_2^2 + \sqrt[\frac{2}{3}]{\frac{\frac{2}{3}\xi_2^3 + \sum_{n=1}^{\infty} \frac{c_n}{2^n(n+\frac{3}{2})} {h_2}^n \xi_2^{2n+3} - 4Y}{\frac{2}{3} + \sum_{n=1}^{\infty} \frac{c_n}{2^n(n+\frac{3}{2})} h_1^n \xi_1^{2n}}}}.$$

It is clear that $\xi_2 = 0$ is not a singularity of the function on the right hand because $-4Y > 0$ by assumption. The second, fifth, sixth, and seventh items are handled similarly.

For the third item of equation (3.27) we replace X_2 by $\frac{\xi_2^2}{\xi_1^2}$ to rewrite this equation as

$$\frac{d\xi_2}{d\tau} = \frac{\xi_1^2 \eta_2}{\xi_1^2 + \xi_2^2}.$$

This follows from the fact that the system (3.27) could be derived in different form by using constraints (3.28) and (3.29). Since the phase variables and the time are kept the same, the corresponding solutions for equations of different forms derived by using (3.28) and (3.29) are identical. For instance, replacing X_1 in equation (3.27) by $\frac{\xi_1^2}{\xi_2^2}$ while keeping all other equations the same would give us a set of equations of new look, but the set of functions $\vec{\psi}(\tau)$ for solutions remains the same.. The function on the right hand is obviously real analytic at $\vec{\psi}(\tau_2)$ since $\xi_1(\tau_2) \neq 0$. The argument for the fourth and eight items follow the same line.

Case $Y > 0$**:** Similar to the case $Y < 0$. □

4. On The Collinear Four-Body Problem

4.1. Preliminaries

In this chapter we consider equations (1.20) for the collinear four-body problem assuming $x_1 \leq x_2 < x_3 \leq x_4$. To avoid messy formulas we set $m_1 = m_2 = m_3 = m_4 = 1$ in (1.20). We also assume that the center of masses of the four bodies are fixed at the origin, i.e.,

$$\sum_{i=1}^{4} x_i = 0, \qquad \sum_{i=1}^{4} \frac{dx_i}{dt} = 0. \tag{4.1}$$

This helps in cutting the dimension of phase space down by two.

Let

$$u_1 = x_2 - x_1, \quad u_2 = x_4 - x_3, \quad \hat{u}_1 = x_1 + x_2, \; \hat{u}_2 = x_3 + x_4, \tag{4.2}$$

and $v_i = \frac{du_i}{dt}, \hat{v}_i = \frac{d\hat{u}_i}{dt}$ for $i = 1, 2$. Note that by (4.1),(4.2) $\hat{u}_1 = -\hat{u}_2$, $\hat{v}_1 = -\hat{v}_2$ so there are only six independent variables out of $(u_i, \hat{u}_i, v_i, \hat{v}_i)$, $i = 1, 2$. We denote $\hat{u} = \hat{u}_1, \hat{v} = \hat{v}_1$ and use $\vec{\varphi} := (u_1, u_2, \hat{u}, v_1, v_2, \hat{v})$ as new phase space variables to rewrite equation (1.20) as

$$\begin{aligned} \frac{du_1}{dt} &= v_1, \quad & \frac{dv_1}{dt} &= -\frac{2}{u_1^2} + 2\frac{\partial \mathcal{K}}{\partial u_1}; \\ \frac{du_2}{dt} &= v_2, \quad & \frac{dv_2}{dt} &= -\frac{2}{u_2^2} + 2\frac{\partial \mathcal{K}}{\partial u_2}; \\ \frac{d\hat{u}}{dt} &= \hat{v}, \quad & \frac{d\hat{v}}{dt} &= \frac{\partial \mathcal{K}}{\partial \hat{u}} \end{aligned} \tag{4.3}$$

where

$$\mathcal{K} = \frac{2}{-2\hat{u} + u_2 - u_1} + \frac{2}{-2\hat{u} + u_2 + u_1} + \frac{2}{-2\hat{u} - u_2 - u_1} + \frac{2}{-2\hat{u} - u_2 + u_1}.$$

$(u_1, u_2, \hat{u}) \in (\mathbb{R}^2)^+ \times \mathbb{R}^-$ is now the space of positions and

$$\Delta_{12,34} = \{(u_1, u_2, \hat{u}) \in (\mathbb{R}^2)^+ \times \mathbb{R}^- : u_1 = u_2 = 0, \hat{u} \in \mathbb{R}^-\}$$

is the singular set for **SBC**. The potential function in new phase space variables is given as $U = \frac{1}{u_1} + \frac{1}{u_2} + \mathcal{K}$ where $\mathcal{K}$ is analytic on $\Delta_{12,34}$. We also have the integral of energy

$$\frac{\hat{v}^2}{2} + \frac{v_1^2}{4} + \frac{v_2^2}{4} - \frac{1}{u_1} - \frac{1}{u_2} - \mathcal{K}(u_1, u_2, \hat{u}) = h.$$

Let

$$h_1 := \frac{v_1^2}{4} - \frac{1}{u_1}, \quad h_2 := \frac{v_2^2}{4} - \frac{1}{u_2}. \tag{4.4}$$

From (4.3) it follows

$$\frac{dh_1}{dt} = v_1 \frac{\partial \mathcal{K}}{\partial u_1}, \qquad \frac{dh_2}{dt} = v_2 \frac{\partial \mathcal{K}}{\partial u_2}. \tag{4.5}$$

Remark: If we apply the replacement strategy of Section 3.2 to the collinear four-body problem, a technical difficulty caused by interactions between colliding pairs occur. Let us explain in details.

We follow the same strategy aiming at writing the integral F explicitly in phase variables through inductive use of integration by part and equations (4.3) and (4.5). Let us start again from (3.5) in Section 3.2.1, and repeat the computation appeared in Section 3.2.1 for I_n. We have

$$\begin{aligned} I_n &= \frac{1}{n+\frac{3}{2}} \int_0^{u_1} h_1^n du_1^{n+\frac{3}{2}} \\ &= \frac{1}{n+\frac{3}{2}} h_1^n u_1^{n+\frac{3}{2}} - \frac{n}{n+\frac{3}{2}} \int_0^{u_1} h_1^{n-1} \frac{dh_1}{du_1} u_1^{n+\frac{3}{2}} du_1 \\ &= \frac{1}{n+\frac{3}{2}} h_1^n u_1^{n+\frac{3}{2}} - \frac{n}{n+\frac{3}{2}} \int_0^{u_1} h_1^{n-1} \frac{\partial \mathcal{K}}{\partial u_1} u_1^{n+\frac{3}{2}} du_1 \end{aligned}$$

where for the last equality we replaced $\frac{dh_1}{du_1}$ by using (4.5). I_n is then the summation of a term that is explicit in u_1 and h_1 and one integral of one degree higher.

We now go one step further to transfer the new integral obtained in the above to integrals of degree even higher. We have for instance

$$\begin{aligned}
I &:= \int_0^{u_1} h_1^{n-1}\frac{\partial\mathcal{K}}{\partial u_1}u_1^{n+\frac{3}{2}}du_1 \\
&= \frac{1}{n+\frac{5}{2}}\int_0^{u_1} h_1^{n-1}\frac{\partial\mathcal{K}}{\partial u_1}du_1^{n+\frac{5}{2}} \\
&= \frac{1}{n+\frac{5}{2}}h_1^{n-1}\frac{\partial\mathcal{K}}{\partial u_1}u_1^{n+\frac{5}{2}} - \frac{n-1}{n+\frac{5}{2}}\int_0^{u_1} h_1^{n-2}\frac{\partial\mathcal{K}}{\partial u_1}u_1^{n+\frac{5}{2}}\frac{dh_1}{du_1}du_1 \\
&\quad -\frac{1}{n+\frac{5}{2}}\int_0^{u_1} h_1^{n-1}\frac{\partial^2\mathcal{K}}{\partial u_1^2}u_1^{n+\frac{5}{2}}du_1 - \frac{1}{n+\frac{5}{2}}\int_0^{u_1} h_1^{n-1}\frac{\partial^2\mathcal{K}}{\partial u_1\partial\hat{u}}u_1^{n+\frac{5}{2}}\frac{\hat{v}}{v_1}du_1 \\
&\quad -\frac{1}{n+\frac{5}{2}}\int_0^{u_1} h_1^{n-1}\frac{\partial^2\mathcal{K}}{\partial u_1\partial u_2}u_1^{n+\frac{5}{2}}\frac{v_2}{v_1}du_1.
\end{aligned}$$

The first two integrals can be further converted to integrals of one degree higher the same way. The third one, however, is with a factor $\hat{v}v_1^{-1}$. We keep $\hat{v}$, which is bounded therefore harmless, but rewrite v_1^{-1} through (4.4) as

$$v_1^{-1} = \frac{1}{2}(u_1^{\frac{1}{2}} + \sum_{n=1}^{\infty} c_n h_1^n u_1^{n+\frac{1}{2}}) \tag{4.6}$$

where c_n is as in (2.6). The third integral is then replaced by a sequence of integrals of ascending degrees by using (4.6).

Finally we try to converte the last integral

$$\hat{I} = \int_0^{u_1} h_1^{n-1}\frac{\partial^2\mathcal{K}}{\partial u_1\partial u_2}\frac{v_2}{v_1}u_1^{n+\frac{5}{2}}du_1.$$

into the sequence of integrals of the higher degree. We try to replace $v_2v_1^{-1}$ by rewriteing v_2 and v_1^{-1} through (4.4) as

$$\begin{aligned}
v_1^{-1} &= \frac{1}{2}(u_1^{\frac{1}{2}} + \sum_{n=1}^{\infty} c_n h_1^n u_1^{n+\frac{1}{2}}), \\
v_2 &= 2(u_2^{-\frac{1}{2}} + \sum_{n=1}^{\infty} \hat{c}_n h_2^n u_2^{n-\frac{1}{2}})
\end{aligned}$$

where c_n is as in (2.6) and $\hat{c}_n$ as in

$$(1+x)^{\frac{1}{2}} = 1 + \sum_{n=1}^{\infty} \hat{c}_n x^n.$$

But the very first integral we obtain is of the form

$$\int_0^{u_1} h_1^{n-1} \frac{\partial^2 \mathcal{K}}{\partial u_1 \partial u_2} \sqrt{\frac{u_1}{u_2}} u_1^{n+\frac{5}{2}} du_1 \tag{4.7}$$

which is of the same degree as the original integral $\hat{I}$. Consequently, the replacement scheme of Chapter 3 fails to work for the collinear four-body problem.

4.1.1. The Distance Ratio Problem

Let

$$\mathcal{U}_{K,\rho} := \{\vec{\varphi}(t) \in (\mathbb{R}^2)^+ \times \mathbb{R}^- \times \mathbb{R}^3 : u_1, u_2 < \rho \ll 1;\ K^{-1} < |\hat{u}| < K;\ |h_1|, |h_2| < K\}$$

The following lemma summarizes some well-known technical results about colliding pairs in the collinear 4-body problem experiencing **SBC**.

Lemma 4.1.1. *Let $\vec{\varphi}(t), t \in (t_1, t_2)$ be a solution of equation (4.3) in $\mathcal{U}_{K,\rho}$. Furthermore assume that $\lim_{t\to t_2^-}(u_1(t), u_2(t), \hat{u}(t)) = L \in \Delta_{12,34}$. Then the following limits exist and are finite:*

(i) $\lim_{t\to t_2^-} \hat{v}(t)$;

(ii) $\lim_{t\to t_2^-} h_1(t)$, $\lim_{t\to t_2^-} h_2(t)$;

(iii) $\lim_{t\to t_2^-} \frac{u_1(t)}{u_2(t)}$. □

A proof of this lemma in included in Appendix B.

Define $u^{-1} := 2U$, $\Lambda := \frac{u}{\hat{u}}$ and $\tilde{v} := u^{\frac{1}{2}}\hat{v}$. We now state the main technical result of this chapter.

Proposition 4.1.1 (Main Proposition). *Let $\vec{\varphi}(t), t \in (t_1, t_2)$ be a solution of equations (4.3) in $\mathcal{U}_{K,\rho}$. Furthermore assume that $\lim_{t\to t_2^-}(u_1(t), u_2(t), \hat{u}(t)) = L \in \Delta_{12,34}$. Then*

(a) *The ratio* $\frac{u_1}{u_2}$ *is given by a formal power series*

$$\frac{u_1}{u_2} = 1 + \sum_{|i|=1}^{\infty} H^{|i|}(u, \Lambda, \tilde{v}); \tag{4.8}$$

where $H^{|i|}(u, \Lambda, \tilde{v})$ *is a homogeneous polynomial of degree* $|i|$ *in listed variables.*

(b) *For* K *and* ρ *small enough such that* $|uh_1| \ll 1$ *and* $|uh_1| \ll 1$ *for all* $\vec{\varphi}(t) \in \mathcal{U}_{K,\rho}$ *heading towards SBC the series (4.8) is convergent.*

We postpone the proof of the Main proposition for the subsection 4.2.3. Finally, let

$$Q(\vec{\varphi}) := 1 + \sum_{|i|=1}^{\infty} H^{|i|}(u, \Lambda, \tilde{v}). \tag{4.9}$$

4.1.2. Regularization result

In this subsection we define regularization variables and prove the regularization result under assumption that the main proposition is valid. The proof of the main proposition will be provided in the next section. We define the new phase space variables, in notation $\vec{\psi} := (\xi_1, \xi_2, \eta_1, \eta_2, \hat{u}, \hat{v}, h_1, h_2)$, and the new time τ as follows:

(i) First, (ξ_1, η_1) and (ξ_2, η_2) are determined by (u_1, v_1) and (u_2, v_2) through

$$u_1 = \frac{\xi_1^2}{2}, \quad v_1 = \frac{\eta_1}{\xi_1}; \qquad u_2 = \frac{\xi_2^2}{2}, \quad v_2 = \frac{\eta_2}{\xi_2}; \tag{4.10}$$

the well known Levi-Civita changes of coordinates;

(ii) We keep $\hat{u}, \hat{v}$ unchanged;

(iii) (h_1, h_2) are defined by using (4.4) and (4.10);

(iv) τ is defined through

$$d\tau = \frac{1}{2}\left(\frac{1}{u_1} + \frac{1}{u_2}\right) dt, \tag{4.11}$$

or equivalently

$$dt = \left(\frac{1}{\xi_1^2} + \frac{1}{\xi_2^2}\right)^{-1} d\tau. \tag{4.12}$$

The equations (4.3) and (4.5) induce the equations

$$\begin{aligned}
\frac{d\xi_1}{d\tau} &= \frac{1}{1+f_1}\eta_1, & \frac{d\eta_1}{d\tau} &= \frac{1}{1+f_1}\left(4h_1\xi_1 + 2\xi_1^2\frac{\partial\mathcal{K}}{\partial\xi_1}\right);\\
\frac{d\xi_2}{d\tau} &= \frac{1}{1+f_2}\eta_2, & \frac{d\eta_2}{d\tau} &= \frac{1}{1+f_2}\left(4h_2\xi_2 + 2\xi_2^2\frac{\partial\mathcal{K}}{\partial\xi_2}\right);\\
\frac{d\hat{u}}{d\tau} &= \frac{1}{1+f_1}\xi_1^2\hat{v}, & \frac{d\hat{v}}{d\tau} &= \frac{1}{1+f_1}\xi_1^2\frac{\partial\mathcal{K}}{\partial\hat{u}};\\
\frac{dh_1}{d\tau} &= \frac{1}{1+f_1}\eta_1\frac{\partial\mathcal{K}}{\partial\xi_1}, & \frac{dh_2}{d\tau} &= \frac{1}{1+f_2}\eta_2\frac{\partial\mathcal{K}}{\partial\xi_2};
\end{aligned} \tag{4.13}$$

in the new phase space variables, where the analytic part of potential function $\mathcal{K}$ is now given as

$$\mathcal{K} = \frac{2}{-2\hat{u}+\frac{1}{2}(\xi_2^2-\xi_1^2)} + \frac{2}{-2\hat{u}+\frac{1}{2}(\xi_2^2+\xi_1^2)} + \frac{2}{-2\hat{u}-\frac{1}{2}(\xi_2^2-\xi_1^2)} + \frac{2}{-2\hat{u}-\frac{1}{2}(\xi_2^2+\xi_1^2)}$$

the functions f_1, f_2 are defined via

$$f_1 := Q\left(\vec{\psi}\right), \quad f_2 := \frac{1}{f_1},$$

and $Q\left(\vec{\psi}\right)$ is obtained from (4.9) by the change of phase variables $\vec{\varphi}$ into $\vec{\psi}$.

Remark: Observe that for the solutions of the equations (4.13) to represent the solutions of equation (4.3), they must satisfy the constraints

$$4\xi_1^2h_1 = \eta_1^2 - 8, \quad 4\xi_2^2h_2 = \eta_2^2 - 8. \tag{4.14}$$

The derivations of equations (4.13) are identical to those of (3.27).

Let $\mathcal{V}_{K,\rho}$ be the set corresponding to $\mathcal{U}_{K,\rho}$ in the new phase space $\vec{\psi}$ and let $\mathcal{A}_{K,\rho}$ be the algebraic variety defined by (4.14) in $\mathcal{V}_{K,\rho}$.

Theorem 3. *(i) Let $\vec{\varphi}(t)$, $t\in(t_1,t_2)$ be a solution of equation (4.3) in $\mathcal{U}_{K,\rho}$. Assume that* $\lim_{t\to t_2^-}(u_1(t),u_2(t),\hat{u}(t)) = \mathbf{L}\in\Delta_{12,34}$. *Let $\tau(t)$ be defined by (4.11) and $\vec{\psi}(\tau)$ be the functions obtained from $\vec{\varphi}(t)$ through (4.10) and (4.4). Then*

(a) $\vec{\psi}(\tau)$ is a solution of equations (4.13) on $\mathcal{A}_{K,\rho}$;

(b) $\tau_2 := \tau(t_2) < \infty$, and $\vec{\psi}(\tau_2) = \lim_{\tau\to\tau_2^-}\vec{\psi}(\tau)$ is well defined;

(c) (4.13) defined on $\mathcal{A}_{K,\rho}$ are real analytic at $\vec{\psi}(\tau_2)$.

(ii) Let $\vec{\psi}(\tau)$, $\tau \in (\tau_1, \tau_2)$ be a solution of equation (4.13) on $\mathcal{A}_{K,\rho} \subset \mathcal{V}_{K,\rho}$ and let $\vec{\varphi}(t)$ be the function obtained from $\vec{\psi}(\tau)$ by (4.12) and (4.10). Then $\vec{\varphi}(t)$ is a solution of (4.3) in $\mathcal{U}_{K,\rho}$.

Proof: (i)(a) This follows from the derivations of equations (4.13). We caution that (4.10) allows different ways to convert $\vec{\varphi}(t)$ to $\vec{\psi}(\tau)$ because ξ_i ($= \pm\sqrt{2u_i}$ by (4.10)) can assume different signs. This is a well known characteristic of Levi-Civita variables. For definiteness, let us chose the positive sign so that $\xi_i = \sqrt{2u_i}$. We also note that from (4.11) it follows

$$\tau = \tau_0 + \frac{1}{2}\int_0^t \left(\frac{1}{u_1(t)} + \frac{1}{u_2(t)}\right) dt \tag{4.15}$$

where τ_0 is arbitrary.

(b) It is well known that when a collision singularity occur at t_2,

$$U(t) := \frac{1}{u_1(t)} + \frac{1}{u_2(t)} \sim (t - t_2)^{-\frac{2}{3}}.$$

From this it follows that

$$\tau_2 = \tau_0 + \frac{1}{2}\int_0^{t_2} U(t)dt < \infty.$$

and $u = 0$ at τ_2. Now for $\vec{\psi}(\tau_2)$: $h_1(\tau), h_2(\tau), \hat{u}(\tau)$ and $\hat{v}(\tau)$ have definite limits as $\tau \to \tau_2^-$ by lemma 4.1.1. Observe that $u_i(t) \to 0$ as $t \to t_2^-$, which we denote as $u_i(t_2)$. We let $\xi_i(\tau_2) = \sqrt{2u_i(t_2)}$. Finally for $\eta_i(\tau_2)$ we use $\eta_i(\tau_2) = \frac{v_i(t_2)}{\xi_i(\tau_2)}$ if $\xi_i(\tau_2) \neq 0$ ($v_i(t) \to v_i(t_2)$, a definite limit in this case). If $\xi_i(\tau_2) = 0$, then $\eta_i^2(\tau_2) = 4$ according to (4.14), from which it follows that $\eta_i(\tau_2) = -2$. $\eta_i(\tau_2)$ is negative because we have used positive sign for $\xi_i(\tau_2)$.

(c) It is sufficient to show f_1 and f_2 are real analytic at $\vec{\psi}(\tau_2)$. We substitute $\frac{\xi_1^2}{2}$ and $\frac{\xi_2^2}{2}$ for u_1 and u_2 into (4.8). It is clear that the power series on the right still represents a real analytic function at the values $q(\tau_2)$ given above.

(ii) Let $\vec{\psi}(\tau)$ be a solution of (4.13) and assume that initial condition $\vec{\psi}(\tau_0)$ satisfies (4.14). Let $\vec{\varphi}(t)$ be a solution of (4.3) satisfying the initial condition $\vec{\varphi}(t_0)$ obtained from $\vec{\psi}(\tau_0)$ by (4.12) and (4.10). We then use (4.10) to convert $\vec{\varphi}(t)$ into the function of $\vec{\psi}$, which we denote $\vec{\Psi}(\tau)$. Then it follows $\vec{\Psi}(\tau)$ is a solution of (4.13) by the derivation of equations (4.13). On the another hand $\vec{\psi}(\tau) = \vec{\Psi}(\tau)$ for all τ by the uniqueness of solutions for ODE. □

4.2. The Distance Ratio Problem Revisited

In this section, we complete the proof of regularization result for the collinear four-body problem given in Theorem 3 by providing the proof for the Main Proposition. The statement of the Main Proposition was a key step in the proof of the Theorem 3 but so far we have not proved its validity. The proof of the Main Proposition will be given in three steps. In step one, we use modification of McGehee's transformation to construct collision manifold for SBC for the collinear four-body problem along the general theory presented in Appendix A. Following the same theory, we see that SBC correspond to the rest points of the vector field given in the new variables. In step two, we introduce yet another change of variables in which the manifold of rest points points corresponding to SBC is now collapsed into a single hyperbolic rest point of the vector-field. We then explicitly compute the stable manifold of the rest point corresponding to SBC and prove its analyticity in the sufficient small neighborhood of the SBC singularity. In step three, we use power series expansion of the stable manifold computed in the step two to construct the power series expansion (4.8) and show that the expansion is convergent under the hypothesis of the Main Proposition.

4.2.1. Collision Manifold

Recall that equations of motion of the collinear four body problem in phase space variables $\vec{\varphi}$ are given as (4.3). Following [Wa], we introduce modified McGehee's variables as follows. Let $\hat{u}$ be as before and let $(u, F_1, F_2, G_1, G_2, \tilde{v})$ and new time $\bar{\tau}$ be given as:

$$u^{-1} = 2U; \quad F_i = u^{-1}u_i; \quad G_i = u^{\frac{1}{2}}v_i; \quad \tilde{v} = u^{\frac{1}{2}}\hat{v}; \quad d\bar{\tau} = u^{-\frac{3}{2}}dt \tag{4.16}$$

Using (4.3) the differential equations for modified McGehee's variables are:

$$\begin{aligned} &\frac{du}{d\bar{\tau}} = -2Pu \\ \frac{dF_1}{d\bar{\tau}} &= G_1 + 2PF_1, \quad \frac{dG_1}{d\bar{\tau}} = 2\left(-\frac{1}{F_1^2} + \frac{\partial \mathcal{K}}{\partial F_1}\right) - PG_1, \\ \frac{dF_2}{d\bar{\tau}} &= G_2 + 2PF_2, \quad \frac{dG_2}{d\bar{\tau}} = 2\left(-\frac{1}{F_2^2} + \frac{\partial \mathcal{K}}{\partial F_2}\right) - PG_2, \\ \frac{d\hat{u}}{d\bar{\tau}} &= u\tilde{v}, \quad \frac{d\tilde{v}}{d\bar{\tau}} = -\tilde{v}P + u\frac{\partial \mathcal{K}}{\partial \hat{u}}, \end{aligned} \tag{4.17}$$

where

$$P := \left(-\frac{1}{F_1^2}G_1 - \frac{1}{F_2^2}G_2\right) + \left(\frac{\partial \mathcal{K}}{\partial F_1}G_1 + \frac{\partial \mathcal{K}}{\partial F_2}G_2 + u\frac{\partial \mathcal{K}}{\partial \hat{u}}\tilde{v}\right).$$

The integral h in new coordinates becomes

$$2 + 4uh = G_1^2 + G_2^2 + 2\tilde{v}^2. \tag{4.18}$$

Using the definition of the potential function and equations (4.16), we also get the identity

$$\frac{1}{2} = \frac{1}{F_1} + \frac{1}{F_2} + \mathcal{K}(F_1, F_2, u, \hat{u}) \tag{4.19}$$

where

$$\mathcal{K}(F_1, F_2, u, \hat{u}) = \frac{2u}{-2\hat{u} + uQ_1} + \frac{2u}{-2\hat{u} + uQ_2} + \frac{2u}{-2\hat{u} - uQ_1} + \frac{2u}{-2\hat{u} - uQ_2}.$$

$$Q_1 := F_2 - F_1, \qquad Q_2 := F_2 + F_1.$$

We eliminate constrains (4.18) and (4.19) by introducing new variables (α, β) instead of (F_i, G_i). Let

$$\begin{aligned} F_1 &= r\sin\alpha, \qquad G_1 = R\sin\beta; \\ F_2 &= r\cos\alpha, \qquad G_2 = R\cos\beta; \end{aligned} \tag{4.20}$$

where $R > 0, r > 0$ are auxiliary variables. The differential equations for $\mathbf{s} := (u, \alpha, \beta, \hat{u}, \tilde{v})$ are

$$\begin{aligned} \frac{du}{d\bar{\tau}} &= -\frac{2Ru}{r^2}E_1 - 2RuL_1 - 2u^2\frac{\partial\mathcal{K}}{\partial\hat{u}}\tilde{v}, \\ \frac{d\alpha}{d\bar{\tau}} &= \frac{R}{r}\sin(\beta-\alpha), \\ \frac{d\beta}{d\bar{\tau}} &= \frac{2}{Rr^2}E_2 + \frac{2}{R}L_2, \\ \frac{d\hat{u}}{d\bar{\tau}} &= u\tilde{v}, \\ \frac{d\tilde{v}}{d\bar{\tau}} &= -\tilde{v}\left(\frac{R}{r^2}E_1 + RL_2 + u\frac{\partial\mathcal{K}}{\partial\hat{u}}\tilde{v}\right) + u\frac{\partial\mathcal{K}}{\partial\hat{u}}. \end{aligned} \tag{4.21}$$

where

$$E_1 := -\frac{\sin\beta}{\sin^2\alpha} - \frac{\cos\beta}{\cos^2\alpha}, \quad E_2 := -\frac{\cos\beta}{\sin^2\alpha} + \frac{\sin\beta}{\cos^2\alpha};$$
$$L_1 := \frac{\partial\mathcal{K}}{\partial r}\cos(\beta-\alpha) + \frac{1}{r}\frac{\partial\mathcal{K}}{\partial\alpha}\sin(\beta-\alpha),$$
$$L_2 := -\frac{\partial\mathcal{K}}{\partial r}\sin(\beta-\alpha) + \frac{1}{r}\frac{\partial\mathcal{K}}{\partial\alpha}\cos(\beta-\alpha).$$

The identities (4.18) and (4.19), written in the new variables, determine auxiliary variables R, r implicitly as the functions of (α, β) as follows

$$2 + 4uh = R^2 + 2\tilde{v}^2 \tag{4.22}$$

$$\frac{1}{2} = \frac{1}{r\sin\alpha} + \frac{1}{r\cos\alpha} + \mathcal{K}(r, \alpha, u, \hat{u}), \tag{4.23}$$

where

$$\mathcal{K}(r, \alpha, u, \hat{u}) = \frac{2u}{-2\hat{u} + urQ_1} + \frac{2u}{-2\hat{u} + urQ_2} + \frac{2u}{-2\hat{u} - urQ_1} + \frac{2u}{-2\hat{u} - urQ_2},$$

and

$$Q_1 := \cos\alpha - \sin\alpha, \qquad Q_2 := \cos\alpha + \sin\alpha.$$

Theoretically we could eliminate R, r from (4.21) using relations (4.22) and (4.23) and work with such vector field. Due to the implicit nature of relations this is not convenient.

The rest points of the vector field (4.21) corresponding to collision orbits are given as: $\mathbf{s}_c = (u_c, \alpha_c, \beta_c, \hat{u}_c, \tilde{v}_c) = (0, \frac{\pi}{4}, \frac{\pi}{4} + \pi, \hat{u}, 0)$; while the values of the auxiliary variables (R, r) at the rest points are $R = \sqrt{2}$ and $r = 4\sqrt{2}$. Note that we actually have whole manifold of rest points, since $\hat{u} \in \mathbb{R}^-$.

The matrix of the linearization for the vector field (4.21) at the rest point corresponding to collision is given as $\mathrm{diag}(-\frac{1}{4}, \begin{bmatrix} +\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{8} \end{bmatrix}, 0, -\frac{1}{8})$. Diagonalization of this matrix is given as $\mathrm{diag}(-\frac{1}{4}, -\frac{1}{4}, \frac{3}{8}, 0, -\frac{1}{8})$. This completes the step one of the proof of the Main proposition.

4.2.2. Stable Manifold

In this subsection we present the step two of the proof of the main proposition. We follow the strategy outlined in the introduction of the section. The goal of the following three coordinate changes is to transform vector field (4.21) into the one that has the origin as the rest point and diagonal operator as its linear part.

(i) Let

$$\hat{\alpha} = \frac{1}{5}\alpha + \frac{2}{5}\beta, \quad \hat{\beta} = -\tfrac{2}{5}\alpha + \tfrac{1}{5}\beta \tag{4.24}$$

while $(u, \hat{u}, \tilde{v})$ remain as before. This transformation gives coordinates in which linearization matrix of equation of motion takes form $\mathrm{diag}(-\frac{1}{4}, -\frac{1}{4}, \frac{3}{8}, 0, -\frac{1}{8})$.

(ii) Let

$$\tilde{\alpha} = \frac{1}{5}\tan(5\hat{\alpha} - \frac{3}{4}\pi - 2\pi), \quad \tilde{\beta} = \tfrac{1}{5}\tan(5\hat{\beta} + \tfrac{1}{4}\pi - \pi), \tag{4.25}$$

and $(u, \hat{u}, \tilde{v})$ remain as before which will move the rest points of the vector field corresponding to collision to $(0, 0, 0, \hat{u}, 0)$.

(iii) Finally we introduce a singular transformation that collapses 1−dimensional manifold of the rest points of the vector field into a single rest point. Let $\Lambda = \frac{u}{\hat{u}}$ while $(u, \tilde{\alpha}, \tilde{\beta}, \tilde{v})$ are as before.

The vector field in new coordinates $\tilde{\mathbf{s}} := (u, \tilde{\alpha}, \tilde{\beta}, \Lambda, \tilde{v})$ becomes

$$\begin{aligned}
\frac{du}{d\bar{\tau}} &= u\left(-\frac{2R}{r^2}E_1 + 2RL_1\right) - \Lambda^3\frac{\partial\mathcal{K}}{\partial\Lambda}\tilde{v},\\
\frac{d\tilde{\alpha}}{d\bar{\tau}} &= (1+25\tilde{\alpha}^2)\left(-\frac{R}{5r}S_1 + \frac{4}{5Rr^2}E_2 + \frac{4}{5R}L_2\right),\\
\frac{d\tilde{\beta}}{d\bar{\tau}} &= (1+25\tilde{\beta}^2)\left(\frac{2R}{5r}S_1 + \frac{2}{5Rr^2}E_2 + \frac{2}{5R}L_2\right),\\
\frac{d\Lambda}{d\bar{\tau}} &= \Lambda\left(-\frac{2R}{r^2}E_1 + 2RL_1 - \Lambda^2\frac{\partial\mathcal{K}}{\partial\Lambda}\tilde{v} - \Lambda\tilde{v}\right),\\
\frac{d\tilde{v}}{d\bar{\tau}} &= -\tilde{v}\left(\frac{R}{r^2}E_1 - RL_1 + \Lambda^2\frac{\partial\mathcal{K}}{\partial\Lambda}\tilde{v}\right) + \Lambda^2\frac{\partial\mathcal{K}}{\partial\Lambda},
\end{aligned} \tag{4.26}$$

where

$$\begin{aligned}
S_1 &:= \sin\left(\frac{1}{5}\tan^{-1}(5\tilde{\alpha}) + \frac{3}{5}\tan^{-1}(5\tilde{\beta})\right), & C_1 &:= \cos\left(\frac{1}{5}\tan^{-1}(5\tilde{\alpha}) + \frac{3}{5}\tan^{-1}(5\tilde{\beta})\right);\\
S_2 &:= \sin\left(\frac{2}{5}\tan^{-1}(5\tilde{\alpha}) + \frac{1}{5}\tan^{-1}(5\tilde{\beta}) + \frac{\pi}{4}\right), & C_2 &:= \cos\left(\frac{2}{5}\tan^{-1}(5\tilde{\alpha}) + \frac{1}{5}\tan^{-1}(5\tilde{\beta}) + \frac{\pi}{4}\right);\\
S_3 &:= \sin\left(\frac{1}{5}\tan^{-1}(5\tilde{\alpha}) - \frac{2}{5}\tan^{-1}(5\tilde{\beta}) + \frac{\pi}{4}\right), & C_3 &:= \cos\left(\frac{1}{5}\tan^{-1}(5\tilde{\alpha}) - \frac{2}{5}\tan^{-1}(5\tilde{\beta}) + \frac{\pi}{4}\right);
\end{aligned}$$

$$E_1 := \frac{S_2}{S_3^2} + \frac{C_2}{C_3^2},\ E_2 := \frac{C_2}{S_3^2} - \frac{S_2}{C_3^2};$$

$$\begin{aligned}
L_1 &:= \frac{\partial\mathcal{K}}{\partial r}C_1 + \frac{1}{r}\left(\frac{1}{5}\frac{\partial\mathcal{K}}{\partial\tilde{\alpha}}(1+25\tilde{\alpha}^2) - \frac{2}{5}\frac{\partial\mathcal{K}}{\partial\tilde{\beta}}(1+25\tilde{\beta}^2)\right)S_1\\
L_2 &:= \frac{\partial\mathcal{K}}{\partial r}S_1 - \frac{1}{r}\left(\frac{1}{5}\frac{\partial\mathcal{K}}{\partial\tilde{\alpha}}(1+25\tilde{\alpha}^2) - \frac{2}{5}\frac{\partial\mathcal{K}}{\partial\tilde{\beta}}(1+25\tilde{\beta}^2)\right)C_1
\end{aligned}$$

$$2 + 4uh = R^2 + 2\tilde{v}^2, \tag{4.27}$$

$$\frac{1}{2} = \frac{1}{rS_3} + \frac{1}{rC_3} + \mathcal{K}(r, \tilde{\alpha}, \tilde{\beta}, \Lambda), \tag{4.28}$$

$$\mathcal{K}(r, \tilde{\alpha}, \tilde{\beta}, \Lambda) = \frac{2\Lambda}{-2 + \Lambda r Q_1} + \frac{2\Lambda}{-2 + \Lambda r Q_2} + \frac{2\Lambda}{-2 - \Lambda r Q_1} + \frac{2\Lambda}{-2 - \Lambda r Q_2}, \tag{4.29}$$

and

$$Q_1 := C_3 - S_3,\ \ Q_2 := C_3 + S_3$$

The rest point of the vector field (4.26) is $\tilde{\mathbf{s}}_{\mathbf{c}} = (u_c, \tilde{\alpha}_c, \tilde{\beta}_c, \Lambda_c, \tilde{v}_c) = (0, 0, 0, 0, 0)$.

Lemma 4.2.1. *The rest point $\tilde{\mathbf{s}}_{\mathbf{c}}$ of the vector field (4.26) is hyperbolic.*

Proof: Since the vector field (4.26) is real analytic in all its arguments at the rest point, it has the convergent Taylor power series expansion. By direct computation, using relations (4.27) and (4.28) to get expansions for R and r, we get

$$\frac{d\tilde{\mathbf{s}}}{d\bar{\tau}} = \lambda\tilde{\mathbf{s}} + \sum_{|i|>1} \mathbf{H}^{|i|}(\tilde{s}),\ \ i \in \mathbb{N}_0^5 \tag{4.30}$$

where $\lambda := \mathrm{diag}(-\frac{1}{4}, -\frac{1}{4}, \frac{3}{8}, -\frac{1}{4}, -\frac{1}{8})$ and $\mathbf{H}^{|i|} := (H_u^{|i|}, H_{\tilde{\alpha}}^{|i|}, H_{\tilde{\beta}}^{|i|}, H_{\Lambda}^{|i|}, H_{\tilde{v}}^{|i|})$ is a vector consisting of homogeneous polynomials of degree $|i|$ in variables $\tilde{s} := (u, \tilde{\alpha}, \tilde{\beta}, \Lambda, \tilde{v})$ as defined earlier. Thus it follows that the rest point is hyperbolic. . □

In the sequel, we identify (4.26) with its power series (4.30) for the local analysis around the fixed point $\tilde{\mathbf{s}}_{\mathbf{c}}$. We also use the method of majorants.

Recall, if

$$f(\mathbf{x}) = \sum_{|k|=1}^{\infty} c_k \mathbf{x}^k;\ \ \ F(\mathbf{x}) = \sum_{|k|=1}^{\infty} C_k \mathbf{x}^k$$

are two formal power series, F is said to be *majorant* of f, in notation $f \prec F$, if

$$|c_k| \leq C_k,\ \ \ \forall k \in \mathbb{N}_0^n.$$

Note that the coefficients of F must be non-negative real numbers which implies that f must have at least as large a radius of convergence as F.

Proposition 4.2.1. *(i) The local stable manifold of the vector field (4.26) at the rest point* $\tilde{\mathbf{s}}_{\mathbf{c}}$ *is given as*

$$\tilde{\beta} = -\frac{8h}{35}u\tilde{\alpha} + \frac{32}{35}\tilde{\alpha}\Lambda + \sum_{|k|=3}^{\infty} h^{|k|}(u, \tilde{\alpha}, \Lambda, \tilde{v}), \;\; k \in \mathbb{N}_0^4, \tag{4.31}$$

where $h^{|k|}$ *are inductively defined homogeneous polynomials in listed variables.*

(ii) The local stable manifold is real analytic in sufficiently small neighborhood of the rest point $\tilde{\mathbf{s}}_{\mathbf{c}}$*.*

Proof: First, we prove part 4.2.1(i). Lemma 4.2.1 implies existence of the invariant splitting $T_cS = T_c^uS \bigoplus T_c^sS$ (where S stands for the phase space) at the the fixed point. According to stable manifold theorem there exists a unique manifold of co-dimension one. It is locally given as the graph of a function $g : T_c^sS \to T_c^uS$. Equivalently we set

$$\tilde{\beta} = \sum_{|k|=1}^{\infty} h^{|k|} \tag{4.32}$$

where $h^{|k|} = \sum_{|k|} c_k u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4}$. We determine coefficients c_k inductively from (4.30).

We formally differentiate equation (4.32) with respect to $\bar{\tau}$ and get

$$\frac{d\tilde{\beta}}{d\bar{\tau}} = \sum_{|k|=1}^{\infty} \left(\partial_u h^{|k|}\frac{du}{d\bar{\tau}} + \partial_{\tilde{\alpha}} h^{|k|}\frac{d\tilde{\alpha}}{d\bar{\tau}} + \partial_{\Lambda} h^{|k|}\frac{d\Lambda}{d\bar{\tau}} + \partial_{\tilde{v}} h^{|k|}\frac{d\tilde{v}}{d\bar{\tau}} \right) \tag{4.33}$$

We insert the vector field (4.30) into (4.33) to get

$$\begin{aligned} \frac{3}{8}\tilde{\beta} + \sum_{|i|>1} H_{\tilde{\beta}}^{|i|} = \sum_{|k|=1}^{\infty} \Bigg(& \partial_u h^{|k|}(-\frac{1}{4}u + \sum_{|i|>1} H_u^{|i|}) + \partial_{\tilde{\alpha}} h^{|k|}(-\frac{1}{4}\tilde{\alpha} + \sum_{|i|>1} H_{\tilde{\alpha}}^{|i|}) \\ & + \partial_{\Lambda} h^{|k|}(-\frac{1}{4}\Lambda + \sum_{|i|>1} H_{\Lambda}^{|i|}) + \partial_{\tilde{v}} h^{|k|}(-\frac{1}{8}\tilde{v} + \sum_{|i|>1} H_{\tilde{v}}^{|i|}) \Bigg) \end{aligned} \tag{4.34}$$

equivalently,

$$\frac{3}{8}\tilde{\beta} + \sum_{|i|>1} H^{|i|}_{\tilde{\beta}} = \sum_{|k|=1}^{\infty} \Bigg((-\frac{1}{4}u\partial_u h^{|k|} - \frac{1}{4}\tilde{\alpha}\partial_{\tilde{\alpha}} h^{|k|} - \frac{1}{4}\Lambda\partial_\Lambda h^{|k|} - \frac{1}{8}\tilde{v}\partial_{\tilde{v}} h^{|k|}) + \partial_u h^{|k|} \sum_{|i|>1} H^{|i|}_u + \partial_{\tilde{\alpha}} h^{|k|} \sum_{|i|>1} H^{|i|}_{\tilde{\alpha}} + \partial_\Lambda h^{|k|} \sum_{|i|>1} H^{|i|}_\Lambda + \partial_{\tilde{v}} h^{|k|} \sum_{|i|>1} H^{|i|}_{\tilde{v}} \Bigg). \tag{4.35}$$

We insert (4.32) for $\tilde{\beta}$ in (4.35)

$$\frac{3}{8}\sum_{|k|=1}^{\infty} h^{|k|} + \sum_{|i|>1} H^{|i|}_{\tilde{\beta}}(u, \tilde{\alpha}, \sum_{|m|=1}^{\infty} h^{|m|}, \Lambda, \tilde{v}) = \sum_{|k|=1}^{\infty} \Bigg((-\frac{1}{4}u\partial_u h^{|k|} - \frac{1}{4}\tilde{\alpha}\partial_{\tilde{\alpha}} h^{|k|} - \frac{1}{4}\Lambda\partial_\Lambda h^{|k|} - \frac{1}{8}\tilde{v}\partial_{\tilde{v}} h^{|k|}) + \partial_u h^{|k|} \sum_{|i|>1} H^{|i|}_u(u, \tilde{\alpha}, \sum_{|m|=1}^{\infty} h^{|m|}, \Lambda, \tilde{v}) + \partial_{\tilde{\alpha}} h^{|k|} \sum_{|i|>1} H^{|i|}_{\tilde{\alpha}}(u, \tilde{\alpha}, \sum_{|m|=1}^{\infty} h^{|m|}, \Lambda, \tilde{v}) + \partial_\Lambda h^{|k|} \sum_{|i|>1} H^{|i|}_\Lambda(u, \tilde{\alpha}, \sum_{|m|=1}^{\infty} h^{|m|}, \Lambda, \tilde{v}) + \partial_{\tilde{v}} h^{|k|} \sum_{|i|>1} H^{|i|}_{\tilde{v}}(u, \tilde{\alpha}, \sum_{|m|=1}^{\infty} h^{|m|}, \Lambda, \tilde{v}) \Bigg) \tag{4.36}$$

where $m \in \mathbb{N}_0^4$. We further simplify (4.36) to get

$$\sum_{|k|=1}^{\infty} \left(\frac{3}{8}h^{|k|} + \frac{1}{4}u\partial_u h^{|k|} + \frac{1}{4}\tilde{\alpha}\partial_{\tilde{\alpha}} h^{|k|} + \frac{1}{4}\Lambda\partial_\Lambda h^{|k|} + \frac{1}{8}\tilde{v}\partial_{\tilde{v}} h^{|k|} \right) = - \sum_{|i|>1} H^{|i|}_{\tilde{\beta}}(u, \tilde{\alpha}, \sum_{|m|=1}^{\infty} h^{|m|}, \Lambda, \tilde{v}) + \sum_{|k|=1}^{\infty} \Bigg(\partial_u h^{|k|} \sum_{|i|>1} H^{|i|}_u(u, \tilde{\alpha}, \sum_{|m|=1}^{\infty} h^{|m|}, \Lambda, \tilde{v}) + \partial_{\tilde{\alpha}} h^{|k|} \sum_{|i|>1} H^{|i|}_{\tilde{\alpha}}(u, \tilde{\alpha}, \sum_{|m|=1}^{\infty} h^{|m|}, \Lambda, \tilde{v}) + \partial_\Lambda h^{|k|} \sum_{|i|>1} H^{|i|}_\Lambda(u, \tilde{\alpha}, \sum_{|m|=1}^{\infty} h^{|m|}, \Lambda, \tilde{v}) + \partial_{\tilde{v}} h^{|k|} \sum_{|i|>1} H^{|i|}_{\tilde{v}}(u, \tilde{\alpha}, \sum_{|m|=1}^{\infty} h^{|m|}, \Lambda, \tilde{v}) \Bigg)$$

We now rewrite previous expression by truncating the polynomials of degree greater then N.

$$\sum_{|k|=1}^{N}\left(\frac{3}{8}h^{|k|}+\frac{1}{4}u\partial_u h^{|k|}+\frac{1}{4}\tilde{\alpha}\partial_{\tilde{\alpha}}h^{|k|}+\frac{1}{4}\Lambda\partial_\Lambda h^{|k|}+\frac{1}{8}\tilde{v}\partial_{\tilde{v}}h^{|k|}\right)=$$
$$-\sum_{|i|>1}^{N}H_{\tilde{\beta}}^{|i|}(u,\tilde{\alpha},\sum_{|m|=1}^{N-1}h^{|m|},\Lambda,\tilde{v})$$
$$+\sum_{|k|=1}^{N-1}\left(\partial_u h^{|k|}\sum_{|i|>1}^{N}H_u^{|i|}(u,\tilde{\alpha},\sum_{|m|=1}^{N-1}h^{|m|},\Lambda,\tilde{v})+\partial_{\tilde{\alpha}}h^{|k|}\sum_{|i|>1}^{N}H_{\tilde{\alpha}}^{|i|}(u,\tilde{\alpha},\sum_{|m|=1}^{N-1}h^{|m|},\Lambda,\tilde{v})\right.$$
$$\left.+\partial_\Lambda h^{|k|}\sum_{|i|>1}^{N}H_\Lambda^{|i|}(u,\tilde{\alpha},\sum_{|m|=1}^{N-1}h^{|m|},\Lambda,\tilde{v})+\partial_{\tilde{v}}h^{|k|}\sum_{|i|>1}^{N}H_{\tilde{v}}^{|i|}(u,\tilde{\alpha},\sum_{|m|=1}^{N-1}h^{|m|},\Lambda,\tilde{v})\right) \tag{4.37}$$

We point out that as it is written the left hand side of (4.37) might contain some additional terms of the order higher then N. We disregard such terms to get equality in (4.37). We also note that all sums on the right hand side involving the coefficients of the stable manifold terminate at $N-1$ because sums

$$\sum_{|i|>1}H_*^{|i|}(u,\tilde{\alpha},\sum_{|m|=1}^{\infty}h^{|m|},\Lambda,\tilde{v})$$

begin with quadratic terms. In particular for the homogeneous polynomials of the specific degree $|k|$ we obtain from (4.37),

$$\frac{3}{8}h^{|k|}+\frac{1}{4}u\partial_u h^{|k|}+\frac{1}{4}\tilde{\alpha}\partial_{\tilde{\alpha}}h^{|k|}+\frac{1}{4}\Lambda\partial_\Lambda h^{|k|}+\frac{1}{8}\tilde{v}\partial_{\tilde{v}}h^{|k|}=p^{|k|} \tag{4.38}$$

where $p^{|k|}$ is a homogeneous polynomial in variables $(u,\tilde{\alpha},\Lambda,\tilde{v})$ with coefficients involving only the coefficients of $h^{|l|}$, $|l|<|k|$ and the coefficients of polynomials $H_u^{|i|}, H_{\tilde{\alpha}}^{|i|}, H_{\tilde{\beta}}^{|i|}, H_\Lambda^{|i|}, H_{\tilde{v}}^{|i|}$, $|i|\le|k|$. From (4.38) we get that particular coefficient c_k is given as

$$c_k=\left(\frac{3}{8}+\frac{1}{4}k_1+\frac{1}{4}k_2+\frac{1}{4}k_3+\frac{1}{8}k_4\right)^{-1}\cdot P \tag{4.39}$$

where P is a polynomial in coefficients c_l, $|l|<|k|$ and coefficients of polynomials $H_u^{|i|}, H_{\tilde{\alpha}}^{|i|}, H_{\tilde{\beta}}^{|i|}, H_\Lambda^{|i|}, H_{\tilde{v}}^{|l|}$, $|i|\le|k|$. This determines coefficients c_k uniquely by recursion starting at $c_{0,0,0,0}=0$.

Finally we list the nonzero coefficients of $h^{|1|}$ and $h^{|2|}$

$$c_{1,1,0,0} = -\frac{8h}{35}, \quad c_{0,1,1,0} = \frac{32}{35}. \tag{4.40}$$

We see that the stable manifold starts with the terms of order two. This is the well-known property of the stable manifold.

We now prove part (ii) of the proposition 4.2.1. In order to show that the stable manifold is real analytic in sufficiently small neighborhood of the rest point $\tilde{\mathbf{s}}_{\mathbf{c}}$ we must show the convergence of the power series expansion (4.31). We do this in two steps. In the first step, we show that stable manifold of the majorant vector field (to be defined in the proof) is a majorant for the stable manifold of the original vector field. In the step two, we find a suitable majorant vector filed of the vector field 4.30 which has analytical stable manifold which in turns proves convergence (analyticity) of the power series expansion found in the proof of the part 4.2.1(i).

In this first step along the system (4.30), we consider a vector field

$$\frac{d\tilde{\mathbf{s}}}{d\tilde{\tau}} = \lambda\tilde{\mathbf{s}} + \sum_{|i|>1} \mathbf{M}^{|i|}(\tilde{\mathbf{s}}) \tag{4.41}$$

where $\mathbf{M}^{|i|} := (M_u^{|i|}, M_{\tilde{\alpha}}^{|i|}, M_{\tilde{\beta}}^{|i|}, M_{\Lambda}^{|i|}, M_{\tilde{v}}^{|i|})$ is such that $H_u^{|i|} \prec M_u^{|i|}$, $H_{\tilde{\alpha}}^{|i|} \prec M_{\tilde{\alpha}}^{|i|}$, $H_{\tilde{\beta}}^{|i|} \prec -M_{\tilde{\beta}}^{|i|}$, $H_{\Lambda}^{|i|} \prec M_{\Lambda}^{|i|}$ and $H_{\tilde{v}}^{|i|} \prec M_{\tilde{v}}^{|i|}$. In words, the new vector field has the same liner part as the original field. The difference is that the coefficients of nonlinear terms of the components of the vector field (4.41) with the negative linear terms are all positive and greater or equal than the absolute values of the coefficients of the original vector field (4.30). On the another hand the coefficients of nonlinear terms of the component $\frac{d\tilde{\beta}}{d\tilde{\tau}}$ of the vector field (which has a positive linear coefficient) are all negative and their absolute values are greater or equal than the absolute values of the coefficients of the original vector field (4.30). We refer to new vector field (4.41) as a majorant vector field.

The existence of at least one such majorant vector field for our vector field (4.30) follows from its analyticity.

The new majorant vector field is also hyperbolic at the rest point $\tilde{s}_c$ by its definition and has the stable manifold of the same dimension as the original vector field. Let

$$\tilde{\beta}_M = \sum_{|k|=1}^{\infty} h_M^{|k|} \tag{4.42}$$

be the stable manifold of the filed (4.41). We claim

$$\sum_{|k|=1}^{\infty} h^{|k|} \prec \sum_{|k|=1}^{\infty} h_M^{|k|},$$

i.e. that the stable manifold of the majorant vector field is a majorant (in the standard sense of that word) for the stable manifold of the original field.

In order to prove the claim, observe that the same procedure for evaluating the coefficients of the stable manifold c_k can be applied to the coefficients C_k of the stable manifold of majorant vector field (4.41) provided that we replace coefficients $\mathbf{H}^{|i|}$ by the coefficients of $\mathbf{M}^{|i|}$. In particular we obtain

$$\begin{aligned}\sum_{|k|=1}^{N}\left(\frac{3}{8}h^{|k|}+\frac{1}{4}u\partial_u h^{|k|}+\frac{1}{4}\tilde{\alpha}\partial_{\tilde{\alpha}}h^{|k|}+\frac{1}{4}\Lambda\partial_\Lambda h^{|k|}+\frac{1}{8}\tilde{v}\partial_{\tilde{v}}h^{|k|}\right) = \\ -\sum_{|i|>1}^{N} M_{\tilde{\beta}}^{|i|}(u,\tilde{\alpha},\sum_{|m|=1}^{N-1}h^{|m|},\Lambda,\tilde{v}) \\ +\sum_{|k|=1}^{N-1}\Bigg(\partial_u h^{|k|}\sum_{|i|>1}^{N} M_u^{|i|}(u,\tilde{\alpha},\sum_{|m|=1}^{N-1}h^{|m|},\Lambda,\tilde{v})+\partial_{\tilde{\alpha}}h^{|k|}\sum_{|i|>1}^{N} M_{\tilde{\alpha}}^{|i|}(u,\tilde{\alpha},\sum_{|m|=1}^{N-1}h^{|m|},\Lambda,\tilde{v}) \\ +\partial_\Lambda h^{|k|}\sum_{|i|>1}^{N} M_\Lambda^{|i|}(u,\tilde{\alpha},\sum_{|m|=1}^{N-1}h^{|m|},\Lambda,\tilde{v})+\partial_{\tilde{v}}h^{|k|}\sum_{|i|>1}^{N} M_{\tilde{v}}^{|i|}(u,\tilde{\alpha},\sum_{|m|=1}^{N-1}h^{|m|},\Lambda,\tilde{v})\Bigg)\end{aligned} \tag{4.43}$$

Now it becomes apparent why we chose $M_{\tilde{\beta}}^{|i|}$ to be negative unlike other components of the majorant vector field. This makes the right hand side of (4.43) the polynomial in positive coefficients $M_u^{|i|}$, $M_{\tilde{\alpha}}^{|i|}$, $-M_{\tilde{\beta}}^{|i|}$, $M_\Lambda^{|i|}$, $M_{\tilde{v}}^{|i|}$ and in turn by induction all coefficients of the polynomials $h_M^{|k|}$ are positive. Again, we get for the homogeneous polynomials of degree $|k|$,

$$\frac{3}{8}h_M^{|k|} + \frac{1}{4}u\partial_u h_M^{|k|} + \frac{1}{4}\tilde{\alpha}\partial_{\tilde{\alpha}} h_M^{|k|} + \frac{1}{4}\Lambda\partial_\Lambda h_M^{|k|} + \frac{1}{8}\tilde{v}\partial_{\tilde{v}} h_M^{|k|} = p_M^{|k|}$$

where $p_M^{|k|}$ is a polynomial in variables $(u, \tilde{\alpha}, \Lambda, \tilde{v})$ with the positive coefficients involving only the positive coefficients of $h_M^{|l|}$, $|l| < |k|$ and the positive coefficients of polynomials $M_*^{|i|}$, $|i| \leq |k|$. Moreover, by induction,

$$|c_k| = (\frac{3}{8} + \frac{1}{4}k_1 + \frac{1}{4}k_2 + \frac{1}{4}k_3 + \frac{1}{8}k_4)^{-1} \cdot |P| \leq C_k = (\frac{3}{8} + \frac{1}{4}k_1 + \frac{1}{4}k_2 + \frac{1}{4}k_3 + \frac{1}{8}k_4)^{-1} \cdot P_M$$

where P_M is a polynomial in positive coefficient C_l, $|l| < |k|$ and positive coefficients of polynomials $M_u^{|i|}$, $M_{\tilde{\alpha}}^{|i|}$, $-M_{\tilde{\beta}}^{|i|}$, $M_\Lambda^{|i|}$, $M_{\tilde{v}}^{|i|}$, $|i| \leq |k|$.

This is the step two of the proof of the convergence. We find a suitable majorant vector field, of the vector field (4.30), having the analytic stable manifold. Since the vector field (4.30) is analytic there exist constants B such that

$$H_*^{|i|} \prec \sum_{i_1+i_2+i_3+i_4+i_5=|i|} B^{|i|} u^{i_1} \tilde{\alpha}^{i_2} \tilde{\beta}^{i_3} \Lambda^{i_4} \tilde{v}^{i_5},$$

where subscript $*$ stands for different components of the vector field. We chose for the majorant vector field

$$\begin{aligned}
\frac{du}{d\bar{\tau}} &= -\tfrac{1}{4}u + \textstyle\sum_{|i|>1} B^{|i|} u^{i_1} \tilde{\alpha}^{i_2} \tilde{\beta}^{i_3} \Lambda^{i_4} \tilde{v}^{i_5}, \\
\frac{d\tilde{\alpha}}{d\bar{\tau}} &= -\tfrac{1}{4}\tilde{\alpha} + \textstyle\sum_{|i|>1} B^{|i|} u^{i_1} \tilde{\alpha}^{i_2} \tilde{\beta}^{i_3} \Lambda^{i_4} \tilde{v}^{i_5}, \\
\frac{d\tilde{\beta}}{d\bar{\tau}} &= \tfrac{3}{8}\tilde{\beta} - \textstyle\sum_{|i|>1} B^{|i|} u^{i_1} \tilde{\alpha}^{i_2} \tilde{\beta}^{i_3} \Lambda^{i_4} \tilde{v}^{i_5}, \\
\frac{d\Lambda}{d\bar{\tau}} &= -\tfrac{1}{4}\Lambda + \textstyle\sum_{|i|>1} B^{|i|} u^{i_1} \tilde{\alpha}^{i_2} \tilde{\beta}^{i_3} \Lambda^{i_4} \tilde{v}^{i_5}, \\
\frac{d\tilde{v}}{d\bar{\tau}} &= -\tfrac{1}{8}\tilde{v} + \textstyle\sum_{|i|>1} B^{|i|} u^{i_1} \tilde{\alpha}^{i_2} \tilde{\beta}^{i_3} \Lambda^{i_4} \tilde{v}^{i_5}.
\end{aligned} \tag{4.44}$$

We use induction on the order $|k|$ of a coefficients C_k of the formal power series expansion

$$\tilde{\beta}_M = \sum_{|k|=1}^{\infty} C_k u^{k_1} \tilde{\alpha}^{k_2} \Lambda^{k_3} \tilde{v}^{k_4} \tag{4.45}$$

for the stable manifold of the vector field (4.44) to establish its analyticity. In what follows we establish the following:

- For $|k| = 1$ by direct computation we have $C_k = 0 \leq B$.
- Suppose $|C_k| \leq B^{|k|}$, in small enough neighborhood of SBC, for all $|k| < N$ then $|C_k| \leq B^{|k|}$ for $|k| = N$ in the same neighborhood.

We formally differentiate (4.45) to get

$$\begin{aligned}\frac{d\tilde{\beta}_M}{d\tilde{\tau}} = \; & \textstyle\sum_{|k|=1}^{\infty} C_k \left(k_1 u^{k_1-1}\frac{du}{d\tilde{\tau}}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4} + k_2 u^{k_1}\tilde{\alpha}^{k_2-1}\frac{d\tilde{\alpha}}{d\tilde{\tau}}\Lambda^{k_3}\tilde{v}^{k_4} + \right. \\ & \left. k_3 u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3-1}\frac{d\Lambda}{d\tilde{\tau}}\tilde{v}^{k_4} + k_4 u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4-1}\frac{d\tilde{v}}{d\tilde{\tau}}\right). \end{aligned} \tag{4.46}$$

We now insert the vector field (4.44) into (4.46) to get

$$\begin{aligned} \frac{3}{8}\tilde{\beta} - \sum_{|i|>1} B^{|i|} u^{i_1}\tilde{\alpha}^{i_2}\tilde{\beta}^{i_3}\Lambda^{i_4}\tilde{v}^{i_5} = \sum_{|k|=1}^{\infty} C_k \\ \Bigg(k_1 u^{k_1-1}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4}\Big(-\frac{1}{4}u + \sum_{|i|>1} B^{|i|} u^{i_1}\tilde{\alpha}^{i_2}\tilde{\beta}^{i_3}\Lambda^{i_4}\tilde{v}^{i_5}\Big) \\ + k_2 u^{k_1}\tilde{\alpha}^{k_2-1}\Lambda^{k_3}\tilde{v}^{k_4}\Big(-\frac{1}{4}\tilde{\alpha} + \sum_{|i|>1} B^{|i|} u^{i_1}\tilde{\alpha}^{i_2}\tilde{\beta}^{i_3}\Lambda^{i_4}\tilde{v}^{i_5}\Big) \\ + k_3 u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3-1}\tilde{v}^{k_4}\Big(-\frac{1}{4}\Lambda + \sum_{|i|>1} B^{|i|} u^{i_1}\tilde{\alpha}^{i_2}\tilde{\beta}^{i_3}\Lambda^{i_4}\tilde{v}^{i_5}\Big) \\ + k_4 u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4-1}\Big(-\frac{1}{8}\tilde{v} + \sum_{|i|>1} B^{|i|} u^{i_1}\tilde{\alpha}^{i_2}\tilde{\beta}^{i_3}\Lambda^{i_4}\tilde{v}^{i_5}\Big)\Bigg) \end{aligned} \tag{4.47}$$

which is formally equivalent to

$$\begin{aligned} \frac{3}{8}\tilde{\beta} - \sum_{|i|>1} B^{|i|} u^{i_1}\tilde{\alpha}^{i_2}\tilde{\beta}^{i_3}\Lambda^{i_4}\tilde{v}^{i_5} = \sum_{|k|=1}^{\infty} C_k\Big(-\frac{1}{4}k_1 - \frac{1}{4}k_2 - \frac{1}{4}k_3 - \frac{1}{8}k_4\Big) u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4} \\ + \Big(\sum_{|i|>1} B^{|i|} u^{i_1}\tilde{\alpha}^{i_2}\tilde{\beta}^{i_3}\Lambda^{i_4}\tilde{v}^{i_5}\Big)\cdot \sum_{|k|=1}^{\infty} C_k \left(k_1 u^{k_1-1}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4} + k_2 u^{k_1}\tilde{\alpha}^{k_2-1}\Lambda^{k_3}\tilde{v}^{k_4} + \right. \\ \left. k_3 u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3-1}\tilde{v}^{k_4} + k_4 u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4-1}\right) \end{aligned} \tag{4.48}$$

or even simpler

$$\sum_{|k|=1}^{\infty} C_k\Big(\frac{3}{8} + \frac{1}{4}k_1 + \frac{1}{4}k_2 + \frac{1}{4}k_3 + \frac{1}{8}k_4\Big) u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4} = \Big(\sum_{|i|>1} B^{|i|} u^{i_1}\tilde{\alpha}^{i_2}\tilde{\beta}^{i_3}\Lambda^{i_4}\tilde{v}^{i_5}\Big)$$

$$\left(1 + \sum_{|k|=1}^{\infty} C_k \left(k_1 u^{k_1-1}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4} + k_2 u^{k_1}\tilde{\alpha}^{k_2-1}\Lambda^{k_3}\tilde{v}^{k_4} + k_3 u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3-1}\tilde{v}^{k_4} + k_4 u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4-1}\right)\right) \quad (4.49)$$

We now insert the expression (4.45) into (4.48) to get

$$\sum_{|k|=1}^{\infty} C_k\left(\frac{3}{8} + \frac{1}{4}k_1 + \frac{1}{4}k_2 + \frac{1}{4}k_3 + \frac{1}{8}k_4\right)u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4} = \left(\sum_{|i|>1} B^{|i|}u^{i_1}\tilde{\alpha}^{i_2}\left(\sum_{|l|=1}^{\infty} C_l u^{l_1}\tilde{\alpha}^{l_2}\Lambda^{l_3}\tilde{v}^{l_4}\right)^{i_3}\Lambda^{i_4}\tilde{v}^{i_5}\right) \left(1 + \sum_{|k|=1}^{\infty} C_k \left(k_1 u^{k_1-1}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4} + k_2 u^{k_1}\tilde{\alpha}^{k_2-1}\Lambda^{k_3}\tilde{v}^{k_4} + k_3 u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3-1}\tilde{v}^{k_4} + k_4 u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4-1}\right)\right) \quad (4.50)$$

We compute the coefficients of the order N by observing the following inequality

$$\sum_{|k|=1}^{N} C_k\left(\frac{3}{8} + \frac{1}{4}k_1 + \frac{1}{4}k_2 + \frac{1}{4}k_3 + \frac{1}{8}k_4\right)u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4} \prec \left(\sum_{|i|>1}^{N} B^{|i|}u^{i_1}\tilde{\alpha}^{i_2}\left(\sum_{|l|=1}^{N-1} C_l u^{l_1}\tilde{\alpha}^{l_2}\Lambda^{l_3}\tilde{v}^{l_4}\right)^{i_3}\Lambda^{i_4}\tilde{v}^{i_5}\right) \left(1 + \sum_{|k|=1}^{N-1} C_k \left(k_1 u^{k_1-1}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4} + k_2 u^{k_1}\tilde{\alpha}^{k_2-1}\Lambda^{k_3}\tilde{v}^{k_4} + k_3 u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3-1}\tilde{v}^{k_4} + k_4 u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4-1}\right)\right) \quad (4.51)$$

We remark that (4.51) is an inequality since the polynomial on the right will ordinary include some terms of degree greater than N. Taking $N = 1$ in (4.51) immediately gives that $C_{1,0,0,0} = C_{0,1,0,0} = C_{0,0,1,0} = C_{0,0,0,1} = 0$ since the left hand side is of degree at least two. We see again that C_k are inductively determined as polynomials in coefficients B^k where $k \leq N$ and C_k where $k < N$ as well as that all C_k are positive (thus we will not write absolute values of coefficients C_k).

Assume, by inductive hypothesis, that $C_k \leq B^{|k|}$ for $|k| < N$. We want to show $C_k \leq B^{|N|}$ for $|k| = N$.

From (4.51) and inductive hypothesis it follows

$$\sum_{|k|=1}^{N} C_k(\frac{3}{8}+\frac{1}{4}k_1+\frac{1}{4}k_2+\frac{1}{4}k_3+\frac{1}{8}k_4)u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4} \prec \left(\sum_{|i|>1}^{N} B^{|i|}u^{i_1}\tilde{\alpha}^{i_2}(\sum_{|l|=1}^{N-1} B^{|l|}u^{l_1}\tilde{\alpha}^{l_2}\Lambda^{l_3}\tilde{v}^{l_4})^{i_3}\Lambda^{i_4}\tilde{v}^{i_5}\right) \left(1+\sum_{|k|=1}^{N-1} B^{|k|}\left(k_1u^{k_1-1}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4}+k_2u^{k_1}\tilde{\alpha}^{k_2-1}\Lambda^{k_3}\tilde{v}^{k_4}+ k_3u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3-1}\tilde{v}^{k_4}+k_4u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4-1}\right)\right). \quad (4.52)$$

Recall $\sum_{|i|>1}^{\infty} B^{|i|}u^{i_1}\tilde{\alpha}^{i_2}\tilde{\beta}^{i_3}\Lambda^{i_4}\tilde{v}^{i_5}$ is a real-analytic function in the vicinity of SBC (origin) in all its arguments therefore it can be made arbitrary small by choosing the sufficiently small neighborhood of SBC. Note also

$$\sum_{|l|=1}^{N-1} B^{|l|}u^{l_1}\tilde{\alpha}^{l_2}\Lambda^{l_3}\tilde{v}^{l_4} \prec \sum_{|l|=1}^{\infty} B^{|l|}u^{l_1}\tilde{\alpha}^{l_2}\Lambda^{l_3}\tilde{v}^{l_4} = -1+\frac{1}{(1-Bu)(1-B\tilde{\alpha})(1-B\Lambda)(1-B\tilde{v})}$$

so it can be arbitraty small for the small enough $|u|, |\tilde{\alpha}|, |\Lambda|$, and $|\tilde{v}|$. Therefore we have in turn

$$\left|\left(\sum_{|i|>1}^{N} B^{|i|}u^{i_1}\tilde{\alpha}^{i_2}(\sum_{|l|=1}^{N-1} B^{|l|}u^{l_1}\tilde{\alpha}^{l_2}\Lambda^{l_3}\tilde{v}^{l_4})^{i_3}\Lambda^{i_4}\tilde{v}^{i_5}\right)\right| \le \epsilon\cdot\min(u^2,\tilde{\alpha}^2,\Lambda^2,\tilde{v}^2) \quad (4.53)$$

where $\epsilon \ll 1$. Combining (4.52) and (4.53) we obtain

$$\sum_{|k|=1}^{N} C_k(\frac{3}{8}+\frac{1}{4}k_1+\frac{1}{4}k_2+\frac{1}{4}k_3+\frac{1}{8}k_4)u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4} \prec \epsilon\left((u^2+\tilde{\alpha}^2+\Lambda^2+\tilde{v}^2)+ \sum_{|k|=1}^{N} B^{|k|}\left(k_1+k_2+k_3+k_4)u^{k_1}\tilde{\alpha}^{k_2}\Lambda^{k_3}\tilde{v}^{k_4}\right)\right). \quad (4.54)$$

From (4.54) we obtain for $|k| = N$

$$C_k \le \epsilon\frac{k_1+k_2+k_3+k_4}{\frac{3}{8}+\frac{1}{4}k_1+\frac{1}{4}k_2+\frac{1}{4}k_3+\frac{1}{8}k_4}B^{N-1} \le B^N \quad (4.55)$$

which complete the proof of the inductive step and in turn the proof of the convergence. □

4.2.3. The Proof of the Main Proposition (4.1.1)

(a) Series expansion: We firstly find the expansion of the quotient $\frac{u_1}{u_2}$ in terms of variables $(u, \Lambda, \tilde{v})$. We observe that in notation of subsection (4.2.1)

$$\frac{u_1}{u_2} = \frac{F_1}{F_2} = \tan(\frac{1}{5}\tan^{-1}(5\tilde{\alpha}) - \frac{2}{5}\tan^{-1}(5\tilde{\beta}) + \frac{\pi}{4}). \tag{4.56}$$

Therefore to obtain formal expansion (4.8) for $\frac{u_1}{u_2}$, it is sufficient to obtain formal power series expansion for $\tilde{\alpha}$ and $\tilde{\beta}$ in variables u,Λ, and $\tilde{v}$. The expansion (4.31), we obtained in proposition (4.2.1.) for the stable manifold, is the key ingredient we use to obtain expansions for $\tilde{\alpha}$ and $\tilde{\beta}$.

We start by rewriting $h_1 := \frac{v_1^2}{4} - \frac{1}{u_1}$, $h_2 := \frac{v_2^2}{4} - \frac{1}{u_2}$ as

$$\frac{u_1 v_1^2}{u_2 v_2^2} = \frac{(1 + h_1 u_1)}{(1 + h_2 u_2)},$$

which is in notation used in subsection (4.2.1.) equivalent to

$$\frac{S_3 S_2^2}{C_3 C_2^2} = \frac{(1 + urh_1 S_3)}{(1 + urh_2 C_3)}, \tag{4.57}$$

where

$$S_2 := \sin(\frac{2}{5}\tan^{-1}(5\tilde{\alpha}) + \frac{1}{5}\tan^{-1}(5\tilde{\beta}) + \frac{\pi}{4}), \quad C_2 := \cos(\frac{2}{5}\tan^{-1}(5\tilde{\alpha}) + \frac{1}{5}\tan^{-1}(5\tilde{\beta}) + \frac{\pi}{4});$$
$$S_3 := \sin(\frac{1}{5}\tan^{-1}(5\tilde{\alpha}) - \frac{2}{5}\tan^{-1}(5\tilde{\beta}) + \frac{\pi}{4}), \quad C_3 := \cos(\frac{1}{5}\tan^{-1}(5\tilde{\alpha}) - \frac{2}{5}\tan^{-1}(5\tilde{\beta}) + \frac{\pi}{4}).$$

We rewrite (4.57) as a formal power series by expanding r from (4.28), which was given as,

$$\frac{1}{2} = \frac{1}{rS_3} + \frac{1}{rC_3} + \mathcal{K}(r, \tilde{\alpha}, \tilde{\beta}, \Lambda),$$

and $S_2,\ C_2,\ S_3, C_3$ in terms of $\tilde{\alpha},\ \tilde{\beta}$.

$$10\tilde{\alpha} + \sum_{|k|>1} \bar{\mathcal{H}}^{|k|}(\tilde{\alpha}, \tilde{\beta}) = 4(h_1 - h_2)u + \sum_{|k|>1} \tilde{\mathcal{H}}^{|k|}_{h_1,h_2}(u, \tilde{\alpha}, \tilde{\beta}, \Lambda, \tilde{v}). \tag{4.58}$$

Note that now k stands for the general multi-index. In the sum on the left $k \in \mathbb{N}_0^2$ while in the sum on the right $k \in \mathbb{N}_0^5$. We also note that the coefficients of

$\tilde{\mathcal{H}}^{|k|}_{h_1,h_2}(u,\tilde{\alpha},\tilde{\beta},\Lambda,\tilde{v})$ depend on h_1 and h_2 but they are always followed with the same power of u. This fact follows directly from the expansion.

We now insert the expansion for the stable manifold (4.31) into equation (4.58) in order to eliminate the variable $\tilde{\beta}$ and obtain

$$10\tilde{\alpha} + \sum_{|k|>1} \mathcal{H}^{|k|}(u,\tilde{\alpha},\Lambda,\tilde{v}) = 4(h_1 - h_2)u + \sum_{|k|>1} \tilde{\mathcal{H}}^{|k|}_{h_1,h_2}(u,\tilde{\alpha},\Lambda,\tilde{v}).$$

or equivalently

$$\tilde{\alpha} = \frac{2}{5}(h_1 - h_2)u + \sum_{|k|>1} \hat{\mathcal{H}}^{|k|}_{h_1,h_2}(u,\tilde{\alpha},\Lambda,\tilde{v}). \tag{4.59}$$

We computed the stable manifold in the proposition 4.2.1 with the specific aim of using it to eliminate variable $\tilde{\beta}$ from the equation (4.59).

The obtained equation (4.59) is analytic in all its argument. We use the analytic version of implicit function theorem to obtain the expansion of $\tilde{\alpha}$ in terms of variables u,Λ and $\tilde{v}$ from the equation (4.59). Assume $\tilde{\alpha}$ is given as a formal power series

$$\tilde{\alpha} = \sum_{|k|=1}^{\infty} \mathcal{P}^{|k|}(u,\Lambda,\tilde{v}) \tag{4.60}$$

where $\mathcal{P}^{|k|}(u,\Lambda,\tilde{v}) = \sum_{|k|} c_k u^{k_1}\Lambda^{k_2}\tilde{v}^{k_3}$ with undetermined coefficients c_k. We insert (4.60) into (4.59) to get

$$\mathcal{P}^{|k|}(u,\Lambda,\tilde{v}) = \mathcal{Q}^{|k|}_{h_1,h_2}(u,\Lambda,\tilde{v}) \tag{4.61}$$

where homogeneous polynomial $\mathcal{Q}^{|k|}_{h_1,h_2}$ involves coefficients of polynomials $\mathcal{H}^{|l|}$, $\tilde{\mathcal{H}}^{|l|}_{h_1,h_2}$ for $|l| \leq |k|$ and $P^{|l|}$ for $|l| < |k|$ since the sums in (4.61) begin with quadratic terms. A particular coefficient c_k is therefore given uniquely by recursion as a polynomial in c_l, $|l| < |k|$ and coefficients of polynomials $\mathcal{H}^{|l|}$, $\tilde{\mathcal{H}}^{|l|}_{h_1,h_2}$ for $|l| \leq |k|$, starting from $c_{1,0,0} = \frac{2}{5}(h_1 - h_2)$, $c_{0,1,0} = 0$, $c_{0,0,1} = 0$.

An expansion for $\tilde{\beta}$ in variables u,Λ and $\tilde{v}$ now follows immediately by inserting expansion (4.60) into the expansion (4.31) for the stable manifold.

Finally the expansion (4.8) follows by expanding (4.56) as a power series and inserting expansions for $\tilde{\alpha}$ and $\tilde{\beta}$.

(b) Convergence: We now show that the power series expansion of $\frac{u_1}{u_2}$ in variables u, Λ and $\tilde{v}$ obtained in part (a) is convergent under stated hypothesis of the Main Proposition.

Assume, for the moment, that the formal series (4.60) for $\tilde{\alpha}$ constructed above is convergent in in the small enough neighborhood of $(u, \Lambda, \tilde{v}) = (0, 0, 0)$. Observe that the expansion for $\tilde{\beta}$ is obtained by inserting expansion (4.60) into the local manifold (4.31) which is real analytic in the small enough neighborhood of the SBC. The convergence of expansion for $\tilde{\beta}$ (in variables $u, \Lambda, \tilde{v}$) then follows from the well-known lemma that the composition of real analytic functions is real analytic the proof of which can be found on page 33 of [KP1]. Without loss of generality we state this lemma in the case when the origin is the center of the analytical domain.

Lemma 4.2.2. *If $f_1, f_2, ..., f_m$ are real analytic in some neighborhood of the point $\mathbf{0} \in \mathbb{R}^k$ and g is real analytic in some neighborhood of the point $(f_1(\mathbf{0}), f_2(\mathbf{0}), ..., f_m(\mathbf{0})) \in \mathbb{R}^m$, then $g(f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_m(\mathbf{x}))$ is real analytic in a neighborhood of $\mathbf{0}$.* □

Therefore, if we apply the previous lemma in our case, there exists sufficiently small u (recall $\Lambda := \frac{u}{\hat{u}}$ and $\tilde{v} := u^{\frac{1}{2}}\hat{v}$ so basically the size of u entirely controls the size of our analytic neighborhood) such that

$$|\tilde{\alpha}| < \frac{\sqrt{3}}{5} \text{ and } |\tilde{\beta}| < \frac{\sqrt{3}}{5}.$$

In turn

$$|\frac{1}{5}\tan^{-1}(5\tilde{\alpha})| < \frac{\pi}{12} \text{ and } |\frac{1}{5}\tan^{-1}(5\tilde{\beta})| < \frac{\pi}{12}$$

thus

$$|\frac{1}{5}\tan^{-1}(5\tilde{\alpha}) - \frac{2}{5}\tan^{-1}(5\tilde{\beta})| < \frac{\pi}{4}$$

which finally implies

$$|\tan(\frac{1}{5}\tan^{-1}(5\tilde{\alpha}) - \frac{2}{5}\tan^{-1}(5\tilde{\beta}))| < 1$$

and the convergence of

$$\begin{aligned}\frac{u_1}{u_2} &= \tan(\tfrac{1}{5}\tan^{-1}(5\tilde{\alpha}) - \tfrac{2}{5}\tan^{-1}(5\tilde{\beta}) + \tfrac{\pi}{4}) \\ &= \Big(1 + \tan(\tfrac{1}{5}\tan^{-1}(5\tilde{\alpha}) - \tfrac{2}{5}\tan^{-1}(5\tilde{\beta}))\Big)\Big(1 - \tan(\tfrac{1}{5}\tan^{-1}(5\tilde{\alpha}) - \tfrac{2}{5}\tan^{-1}(5\tilde{\beta}))\Big)^{-1},\end{aligned}$$

by one more application of the lemma about real analyticity of composition of the real analytic functions since we are just inserting the convergent expansion for $\tilde{\alpha}$ and $\tilde{\beta}$ into real analytic function.

So in order to complete the proof of convergence we must only show the convergence of the series expansion (4.60) for $\tilde{\alpha}$ which we assumed so far.

Let u be such that $|uh_1| \ll 1$ and $|uh_2| \ll 1$ then the equation (4.58) is a real analytic. From lemma 4.2.2. it follows that (4.59) is also real analytic equation since it is obtained from (4.58) by inserting real analytic expansion (4.31) for $\tilde{\beta}$. The expansion (4.60) obtained from (4.59) is now real analytic as well by the real analytic version of the implicit function theorem that can be found in [KP1] or [KP2]. We supply the proof for the sake of completeness of the argument.

We use majorant method to prove that (4.60) is convergent. We rewrite the equation (4.59) as

$$\tilde{\alpha} = \sum_{|k|>0}(a_{k,0} + a_{k,1}\tilde{\alpha})u^{k_1}\Lambda^{k_2}\tilde{v}^{k_3} + \sum_{|k|\geq 0, i\geq 2} a_{k,i}\tilde{u}^{k_1}\Lambda^{k_2}\tilde{v}^{k_3}\tilde{\alpha}^i \tag{4.62}$$

where $a_{k,i}$ are known coefficients. Let

$$\mathcal{M}(\tilde{u}, \tilde{\alpha}, \Lambda, \tilde{v}) = \sum_{|k|>0}(b_{k,0} + b_{k,1}\tilde{\alpha})u^{k_1}\Lambda^{k_2}\tilde{v}^{k_3} + \sum_{|k|\geq 0, i\geq 2} b_{k,i}\tilde{u}^{k_1}\Lambda^{k_2}\tilde{v}^{k_3}\tilde{\alpha}^i \tag{4.63}$$

be a majorant of the right hand side of (4.59) existence of which is ensured by the analyticity of the equation in question. We claim that the expansion

$$\tilde{\alpha}_{\mathcal{M}} = \sum_{|k|=1}^{\infty} C_k u^{k_1}\Lambda^{k_2}\tilde{v}^{k_3} \tag{4.64}$$

obtained from the equation

$$\tilde{\alpha} = \mathcal{M}(\tilde{u}, \tilde{\alpha}, \Lambda, \tilde{v}) \tag{4.65}$$

is majorant for (4.60). This follows from the fact that we can apply the same procedure to find the formal solution to the majorant equation (4.65). To show that $|c_k| \leq C_k$ we observe that the only difference in the way we compute these coefficients, is that polynomial for C_k involve only positive terms. More precisely if we substitute $\sum_{|k|>1} c_k u^{k_1} \Lambda^{k_2} \tilde{v}^{k_3}$ into (4.62) we obtain the explicit recurrence relationship

$$\begin{aligned} c_{k+e_j} &= a_{k+e_j,0} + \textstyle\sum_{|l|>0,\ |\bar{l}|>0,\ l+\bar{l}=k+e_j} a_{\bar{l},1} c_l \\ &+ \textstyle\sum_{|\bar{l}|\geq 0,\ i\geq 2,\ |\iota^i|>0,\ \iota^1+...+\iota^i=k+e_j} Mon(\iota_1, ..., \iota_i) a_{\bar{l},i} c_{\iota_1} \cdot ... \cdot c_{\iota_i} \end{aligned} \tag{4.66}$$

where

$$e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1),$$

and $Mon(\iota^1, ..., \iota^i)$ is suitable positive monomial coefficient. It is now clear that in the case when the $a_{k,i}$ are replaced by $b_{k,i}$ all terms on the right hand side of (4.66) are positive which proves $|c_k| \leq C_k$ for all $k \in \mathbb{N}_0^3$. Consequently if the series (4.63) is convergent then the series (4.60). It just remain to find a majorant such that the equation (4.65) has the analytic solution.

Observe that from analyticity of the right hand side of (4.62) there exists positive constants K and B so that $|a_{k,i}| \leq KB^{|k|+i}$ for all $k \in \mathbb{N}_0^3$ and $i \in \mathbb{N}_0$. From this and the fact that $a_{\mathbf{0},0} = a_{\mathbf{0},1} = 0$ it follows that we can take for the majorant

$$\begin{aligned} \mathcal{M}(\tilde{u}, \tilde{\alpha}, \Lambda, \tilde{v}) = & -K - KB\tilde{\alpha} + \textstyle\sum_{k\geq 0, i\geq 0} KB^{|k|+i} u^{k_1} \Lambda^{k_2} \tilde{v}^{k_3} \tilde{\alpha}^i \\ = & \ -K - KB\tilde{\alpha} + K(1-B\tilde{u})^{-1}(1-B\Lambda)^{-1}(1-B\tilde{v})^{-1}(1-B\tilde{\alpha})^{-1}. \end{aligned}$$

For this choice of majorant, $\tilde{\alpha}_{\mathcal{M}} = \sum_{|k|=1}^{\infty} C_k u^{k_1} \Lambda^{k_2} \tilde{v}^{k_3}$ is a solution of the equation

$$\tilde{\alpha} = -K - KB\tilde{\alpha} + K(1-C\tilde{u})^{-1}(1-B\Lambda)^{-1}(1-B\tilde{v})^{-1}(1-B\tilde{\alpha})^{-1}$$

equivalently

$$(1-B\tilde{\alpha})[K + (1+KB)\tilde{\alpha}] = K(1-B\tilde{u})^{-1}(1-B\Lambda)^{-1}(1-B\tilde{v})^{-1}$$

which is quadratic in $\tilde{\alpha}$,

$$B(1+KB)\tilde{\alpha}^2 - \tilde{\alpha} + K\left((1-B\tilde{u})^{-1}(1-B\Lambda)^{-1}(1-B\tilde{v})^{-1} - 1\right) = 0$$

so it can be solved explicitly as

$$\tilde{\alpha} = \frac{1 \pm \sqrt{1 - 4BK(1+BK)\left((1-B\tilde{u})^{-1}(1-B\Lambda)^{-1}(1-B\tilde{v})^{-1} - 1\right)}}{2B(1+KB)}.$$

The solution is clearly analytic at $(\tilde{u}, \Lambda, \tilde{v}) = (0, 0, 0)$. $\square$

5. Concluding Remarks

Ever since Sundman's work on construction of global power series solutions for the three-body problem and the central role played by regularization of binary collision singularities, the study of collision singularities in celestial mechanics became a subject in its own right. The early successes were followed by disappointing conclusions of Siegel's research [SM] about regularizability of triple-collisions. Never the less geometric study of singularities [McG1] provided important insights in the nature of N-body problem. As Rene Thom once said, it is not possible to fully understand a mathematical problem without understanding it singularities. During all this time the question of regularization transformation for Simultaneous Binary Collision singularity remanded hopelessly difficult and until now open for investigation.

This work provide complete but not entirely satisfactory answer to the question of regularization transformations for the SBC singularities. It also opens some new questions.

The key step of this work was considering the systems of the lesser complexity experiencing this kind of phenomena. Consequently the chapters 2-4 are arranged in ascending order with respect to the complexity of the problem considered in them.

In chapter two, we considered integrable system of two two-body problems for which we new everything but for which the regularization transformation was elusive. It helped us identify the problem of the unified time transformation which in turn led us to definition of the new variable Y, the local integral of regularized equations of motion that was the piece we missed to construct regularizing transformation. The final results of this chapter are summarized in the Theorem 1.

We then considered, in chapter 3, restricted four-body problem which is the simplest of problems in which there is interaction between colliding pairs. This problem is non-integrable and despite the simple nature of interaction among colliding pairs time-transformation necessary for regularization led us to the distance-ratio problem. Ultimately solution of this problem required the sophisticated apparatus of asymptotic analysis as well as careful proofs of convergence of certain power series. The regularization results obtained in this chapter are summarized in Theorem2.

Finally, in chapter four we tried to follow the program we developed in chapters 2 and 3 in an attempt to settle the question of regularization transformation once for all. Our original approach to the distance-ratio problem could not be carried through due to the complicated nature of interaction between colliding pairs. We used McGehee transforms to conduct the local analysis of SBC singularity and solve the distance ratio problem which was needed for the completion of the regularization transformation. Unfortunately our solution, Theorem 3, is not as satisfactory as Theorem 1 and Theorem 2 because it does not regularize nearby single binary collisions. This is due not because of the deficiency of our transformation but rather due to the complexity of the singularity set. The study of the singularity set remains at this point open for investigation.

There are no principal obstacles for extending our results to the planar problems except increased complexity of the computations. It is also possible to generalize the results to the three dimensional space with the understanding that Levi-Civita transformation had to be substituted by Kustaanheimo-Stiefel transformation and that all computations due to topological obstructions must me carried as if the problem was embedded in the four dimensional space.

A. General Collision Manifolds

In the Appendix A, *Collision Manifold* of the n-body problem for a *general collision singularity* is constructed using a modification of McGehee's transformation introduced in [Wa]. Some elementary properties of the flow are also discussed. An alternative cumbersome construction using original McGehee's transformation was done before [El2] but proposition A.0.1 has not been previously discussed in literature.

Recall the equations of motion of n-body problem given in the introduction

$$m_k \frac{d^2\mathbf{q}_k}{dt^2} = \frac{\partial U}{\partial \mathbf{q}_k}, \ \ k = 1, .., n \tag{A.1}$$

where U is the potential function,

$$U = \sum_{1 \le j < i \le n} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|}.$$

Define the $3n \times 3n$ mass matrix $M = \operatorname{diag}(m_1, m_1, m_1; ...; m_n, m_n, m_n)$. Let $\mathbf{q} = (\mathbf{q}_1, ..., \mathbf{q}_n)^T$ and the momentum vector $\mathbf{p} = (\mathbf{p}_1, ..., \mathbf{p}_n)^T = (m_1 {\cdot} \dot{\mathbf{q}}_1, ..., m_n {\cdot} \dot{\mathbf{q}}_n)^T$, be coordinates in the phase space. The equations of motion now take the following form.

$$\frac{d\mathbf{q}}{dt} = M^{-1}\mathbf{p}, \quad \frac{d\mathbf{p}}{dt} = \nabla U. \tag{A.2}$$

Recall the known energy integrals of the n-body problem

$$T - U = H. \tag{A.3}$$

As in introduction, let

$$\Omega = \{\Omega_1, ..., \Omega_k\}, \quad |\Omega_1| + ... + |\Omega_k| = n$$

be a partition of the set $N = \{1, 2, ..., n\}$ into k subsets $k < n$. The *singularity set* of partition Ω was defined as

$$\Delta_\Omega = \{\mathbf{q} \in \mathbb{R}^{3n} |\ \mathbf{q}_i = \mathbf{q}_j \text{ iff } i, j \in \Omega_l \text{ for some } \Omega_l \in \Omega\}.$$

Let $\Delta := \cup_{\Omega}\Delta_{\Omega}$. A solution of the system (A.2) experience *general collision singularity* as $t \to t_2^-$ if

$$\lim_{t \to t_2^-} \mathbf{q}(t) = (\mathbf{L}_1, ..., \mathbf{L}_k) \in \Delta.$$

Following [W], we introduce $(\mathbf{F}, \mathbf{G}, \tau)$ as follows:

$$u^{-1} := 2U. \tag{A.4}$$

Suppose $\mathbf{q}_j \to \mathbf{L}_i$ where $i = 1, ..., k$ and $j \in \Omega_i$ as $t \to t_2^-$. Define

$$\mathbf{F}_j := \frac{\mathbf{q}_j - \mathbf{L}_i}{u}, \quad \mathbf{G}_j := u^{\frac{1}{2}}\mathbf{p}_j \tag{A.5}$$

and

$$\mathbf{F} = (\mathbf{F}_1, ...\mathbf{F}_n)^T; \quad \mathbf{G} = (\mathbf{G}_1, ...\mathbf{G}_n)^T. \tag{A.6}$$

We also introduce new time as

$$d\tau := u^{-\frac{3}{2}}dt. \tag{A.7}$$

The equations of motion in new coordinates and new time become

$$\begin{aligned} \frac{du}{d\tau} &= -2(M^{-1}\mathbf{G}, \nabla U(u, \mathbf{F}))u, \\ \frac{d\mathbf{F}}{d\tau} &= M^{-1}\mathbf{G} + 2(M^{-1}\mathbf{G}, \nabla U(u, \mathbf{F}))\mathbf{F}, \\ \frac{d\mathbf{G}}{d\tau} &= \nabla U(u, \mathbf{F}) - (M^{-1}\mathbf{G}, \nabla U(u, \mathbf{F}))\mathbf{G}, \end{aligned} \tag{A.8}$$

where

$$U(u, \mathbf{F}) = U(\mathbf{F}) + \mathcal{K}(u, \mathbf{F}) \tag{A.9}$$

and

$$U(\mathbf{F}) := \sum_{i=1}^{k} \left(\sum_{j,l \in \Omega_i,\ j<l} \frac{m_l m_j}{|\mathbf{F}_l - \mathbf{F}_j|} \right), \tag{A.10}$$

$$\mathcal{K}(u, \mathbf{F}) := u \cdot \sum_{j \in \Omega_{i_1},\ l \in \Omega_{i_2},\ j<l,\ i_1 \neq i_2} \frac{m_l m_j}{|\mathbf{L}_{i_2} - \mathbf{L}_{i_1} - u(\mathbf{F}_l - \mathbf{F}_j)|}. \tag{A.11}$$

We note U($\mathbf{F}$) is the part of potential function coming from the interaction among particles within the same cluster. On the other hand $\mathcal{K}(u,\mathbf{F})$ is the part of the potential function coming from the interaction of different collapsing clusters. $\nabla U(u,\mathbf{F})$, $(,)$ represent the gradient vector of $U(u,\mathbf{F})$ with respect to $\mathbf{F}$ and the inner product of two $3n$-vectors, respectively. In new variables $(u,\mathbf{F},\mathbf{G})$ integral of energy and the identity $u^{-1} = 2U$ become

$$(\mathbf{G}, M^{-1}\mathbf{G}) = 1 + 2uH,$$
$$\tfrac{1}{2} = U(u,\mathbf{F}). \tag{A.12}$$

We call the solution of (A.8) satisfying conditions (A.12) proper solutions. Every proper solution correspond to a solution of (A.2) and vice verse. Condition (A.12) implies that $U(u,\mathbf{F})$ is real analytic in all its variables on the image of the set Ω_Δ. Thus for any given initial state $(\mathbf{q}_0,\mathbf{p}_0,t_0)$ of the system (A.2) the time interval of existence of corresponding proper solution of equation (A.8) is $\tau = (-\infty,\infty)$. Even the singular solutions of (A.2) are transfered into the regular solutions of (A.8) by the means of time transform. The time transform slows down the flow so that it takes infinitely long for a solution of (A.8) to reach the point corresponding to a singularity of the equation (A.2). Never the less, the study of (A.8) helps us to identified important properties of (A.2).

In the new coordinates the *general collision singularity* corresponds to $u = 0$. By setting $u = 0$ into (A.12) we obtain so called "*Collision Manifold*".

$$(\mathbf{G}, M^{-1}\mathbf{G}) = 1,$$
$$\tfrac{1}{2} = U(\mathbf{F}). \tag{A.13}$$

The Collision Manifold is pasted to each energy level manifold as a boundary. The corresponding differential equations obtain from (A.8) by inserting $u = 0$ describe fictitious flow of (A.2) on it. The flow is defined by

$$\frac{d\mathbf{F}}{d\tau} = M^{-1}\mathbf{G} + 2(M^{-1}\mathbf{G}, \nabla U(\mathbf{F}))\mathbf{F}, \tag{A.14}$$

$$\frac{d\mathbf{G}}{d\tau} = \nabla U(\mathbf{F}) - (M^{-1}\mathbf{G}, \nabla U(\mathbf{F}))\mathbf{G}. \tag{A.15}$$

In the sequel, we list some properties of the flow on the *Collision Manifold*.

Consider the flow of the so called decoupled n-body problem where there is no interaction among different collapsing clusters, i.e. $K(u, \mathbf{F}) = 0$, on the zero $H = 0$ energy surface. The flow is defined by

$$\frac{du}{d\tau} = -2(M^{-1}\mathbf{G}, \nabla U(\mathbf{F}))u, \tag{A.16}$$

$$\frac{d\mathbf{F}}{d\tau} = M^{-1}\mathbf{G} + 2(M^{-1}\mathbf{G}, \nabla U(\mathbf{F}))\mathbf{F}, \tag{A.17}$$

$$\frac{d\mathbf{G}}{d\tau} = \nabla U(\mathbf{F}) - (M^{-1}\mathbf{G}, \nabla U(\mathbf{F}))\mathbf{G}. \tag{A.18}$$

The zero energy integral manifold of the decoupled n-body problem is given as

$$\begin{aligned} (\mathbf{G}, M^{-1}\mathbf{G}) &= 1, \\ \tfrac{1}{2} &= U(\mathbf{F}) \end{aligned} \tag{A.19}$$

Proposition A.0.1. *The flow of the decoupled n-body problem, given by (A.16-A.18), restricted to the zero energy surface (A.19) is equivalent to the fictitious flow, given by (A.14-A.15), on Collision Manifold (A.13).*

Proof:[$\Rightarrow$] Let $(u(\tau), \mathbf{F}(\tau), \mathbf{G}(\tau))$ be a solution of (A.16-A.18) satisfying the relation (A.19) and some initial condition $(u(\tau_0), \mathbf{F}(\tau_0), \mathbf{G}(\tau_0))$. Then $(\mathbf{F}(\tau), \mathbf{G}(\tau))$ must satisfy equations (A.14-A.15) since they are identical to (A.17-A.18) and relation (A.13) which is identical to (A.19).

[$\Leftarrow$] Suppose now $(\hat{\mathbf{F}}(\tau), \hat{\mathbf{G}}(\tau))$ is a solution of (A.14-A.15) satisfying relation (A.13) and the initial condition $(\hat{\mathbf{F}}(\tau_0), \hat{\mathbf{G}}(\tau_0)) = (\mathbf{F}(\tau_0), \mathbf{G}(\tau))$. Then it must satisfy (A.17-A.18) restricted to (A.19). Note that system (A.16-A.18) actually decouples to the equation (A.16) and system (A.17-A.18). We put $(\hat{\mathbf{F}}(\tau), \hat{\mathbf{G}}(\tau))$ into the (A.16) and solve it for initial condition $u(\tau_0)$ to obtain $\hat{u}(\tau)$. Then the solution $(\hat{u}(\tau), \hat{\mathbf{F}}(\tau), \hat{\mathbf{G}}(\tau))$ of must coincide with $(u(\tau), \mathbf{F}(\tau), \mathbf{G}(\tau))$ by the theorem about uniqueness of ODEs. □

We now recall the definition of the *gradient-like* flow.

Definition A.0.1. A flow on a manifold is said to be gradient-like with respect to the function ν if and only if ν is is strictly increasing except on the rest points. Function ν is called Lyapunov function.

Proposition A.0.2. *The flow on the Collision Manifold is gradient-like with respect to a function* $\nu = (\mathbf{F}, \mathbf{G})(\mathbf{F}, M\mathbf{F})^{-\frac{1}{4}}$.

Proof: By direct computations using equations of motion (A.14),(A.15), relations (A.13) and Euler identity $(\nabla U(\mathbf{F}), \mathbf{F}) = -U(\mathbf{F})$ we have

$$\frac{d\nu}{d\tau} = \frac{1}{2}(\mathbf{F}, M\mathbf{F})^{-\frac{1}{4}} \left(1 - \frac{(\mathbf{F}, \mathbf{G})^2}{(\mathbf{F}, M\mathbf{F})}\right)$$

Using the Schwartz inequality (for M-norm) and the identity $(\mathbf{G}, M^{-1}\mathbf{G}) = 1$

$$|(\mathbf{F}, \mathbf{G})| \leq (\mathbf{F}, M\mathbf{F})^{\frac{1}{2}}(\mathbf{G}, M^{-1}\mathbf{G})^{\frac{1}{2}} = (\mathbf{F}, M\mathbf{F})^{\frac{1}{2}}$$

or equivalently

$$\frac{(\mathbf{F}, \mathbf{G})^2}{(\mathbf{F}, M\mathbf{F})} \leq 1$$

Thus

$$\begin{aligned} \frac{d\nu}{d\tau} &= \tfrac{1}{2}(\mathbf{F}, M\mathbf{F})^{-\frac{1}{4}} \left(1 - \tfrac{(\mathbf{F},\mathbf{G})^2}{(\mathbf{F},M\mathbf{F})}\right) \\ &\geq \tfrac{1}{2}(\mathbf{F}, M\mathbf{F})^{-\frac{1}{4}}(1-1) \geq 0 \end{aligned}$$

with the equality if and only if $\mathbf{F} = \lambda M^{-1}\mathbf{G}$ for some constant λ. Since ν is differentiable except at the discrete set of times,it is non-decreasing. If $\frac{d\nu}{d\tau} = 0$ for some non-trivial time interval, then $\mathbf{F} = \lambda M^{-1}\mathbf{G}$ for that interval. Substituting into (A.14) we get $\frac{d\mathbf{F}}{d\tau} = 0$. From $\mathbf{F} = \lambda M^{-1}\mathbf{G}$ also follows $\frac{d\mathbf{G}}{d\tau} = \frac{1}{\lambda} M \frac{d\mathbf{F}}{d\tau} = 0$ and the solution curve is stationary. □

The following proposition, in which notation ω is used to denote the ω-limit set, addresses the question of the fate of the collision orbits.

Proposition A.0.3. *Let* $(\mathbf{q}(t), \mathbf{p}(t))$ *be a solution of (A.2) satisfying initial conditions* $(\mathbf{q}_0, \mathbf{p}_0)$ *so that* $\lim_{t\to t_2^-} \mathbf{q}(t) \in \Delta$. *Let* $(u(\tau), \mathbf{F}(\tau), \mathbf{G}(\tau))$ *be the function obtained via (A.4,A.5) from* $(\mathbf{q}(t), \mathbf{p}(t))$ *and* τ *obtained via (A.7) from* t, *then* $\omega(u(\tau), \mathbf{F}(\tau), \mathbf{G}(\tau))$ *is a non-empty subset of the rest points of the vector field (A.8) on Collision Manifold.*

Proof: The proof will consists of three steps. In the first step we show that

$$\lim_{\tau\to\infty} u(\tau) = 0$$

implies that $\lim_{\tau\to\infty} \nu(\tau) = \nu^* < 0$ where as before $\nu := (\mathbf{F}, \mathbf{G})(\mathbf{F}, M\mathbf{F})^{-\frac{1}{4}}$. We then show that solution curve $(u(\tau), \mathbf{F}(\tau), \mathbf{G}(\tau))$ tents to a compact set as $\tau \to \infty$ which in turn implies that $\omega(u(\tau), \mathbf{F}(\tau), \mathbf{G}(\tau)) \neq 0$. This completes step two. Finally in step three, if $(u_1, \mathbf{F}_1, \mathbf{G}_1) \in \omega(u(\tau), \mathbf{F}(\tau), \mathbf{G}(\tau))$ then $\nu(\mathbf{F}_1, \mathbf{G}_1) = \nu^*$ for all τ and by Proposition A.0.2 it follows that $(\mathbf{F}_1, \mathbf{G}_1)$ is a rest point.

By direct computation using the equations of motion (A.8) and identities (A.12) we get

$$\frac{d\nu}{d\tau} = \frac{\frac{1}{2} + 2uH + K(u, \mathbf{F}) + \mathbf{F}\nabla K(u, \mathbf{F})}{(\mathbf{F}, M\mathbf{F})^{\frac{1}{4}}} - \frac{\nu^2}{2(\mathbf{F}, M\mathbf{F})^{\frac{3}{4}}}. \tag{A.20}$$

From (A.20) it follows that $\frac{d\nu}{d\tau} > 0$ on $\nu = 0$ and sufficiently small u since $K(u, \mathbf{F}) = o(u)$ and $\mathbf{F}\nabla K(u, \mathbf{F}) = o(u^2)$. Therefore if $\nu < 0$ for all $\tau > \tau_0$ there exists $\hat{\nu} < 0$ such that

$$\nu(\tau) \leq \hat{\nu} < 0 \tag{A.21}$$

for all $\tau > \tau_0$.

Observe also that by the Schwartz inequality

$$|\nu| = |(\mathbf{F},\mathbf{G})|(\mathbf{F},M\mathbf{F})^{-\frac{1}{4}} \leq \frac{(\mathbf{F},M\mathbf{F})^{\frac{1}{2}}(\mathbf{G},M^{-1}\mathbf{G})^{\frac{1}{2}}}{(\mathbf{F},M\mathbf{F})^{\frac{1}{4}}}. \tag{A.22}$$

We insert $(\mathbf{G},M^{-1}\mathbf{G}) = 1 + 2uH$ into (A.22) to get

$$|\nu| \leq (\mathbf{F},M\mathbf{F})^{\frac{1}{4}}(1+2uH)^{\frac{1}{2}}$$

which in turn implies

$$|\nu| \leq (\mathbf{F},M\mathbf{F})^{\frac{1}{4}}(1+\epsilon) \tag{A.23}$$

where ϵ is arbitrary small as $u(\tau) \to 0$. Therefore by combining (A.21) and (A.23) we obtain

$$|\hat{\nu}| \leq (\mathbf{F},M\mathbf{F})^{\frac{1}{4}} \tag{A.24}$$

or equivalently

$$\frac{1}{|\hat{\nu}|} \geq \frac{1}{(\mathbf{F},M\mathbf{F})^{\frac{1}{4}}} \tag{A.25}$$

We use (A.21) and (A.25) to obtain an estimate for $u(\tau)$. Namely by direct computations

$$\frac{d}{d\tau}\ln(u(\mathbf{F},M\mathbf{F})^{\frac{1}{2}}) = \frac{\nu}{(\mathbf{F},M\mathbf{F})^{\frac{3}{4}}} \leq \frac{\hat{\nu}}{\hat{\nu}^3}. \tag{A.26}$$

Therefore

$$u(\tau) \leq \frac{1}{\hat{\nu}^2}e^{\frac{\hat{\nu}}{\hat{\nu}^3}\tau}. \tag{A.27}$$

On the another hand from (A.20) and (A.23) we find

$$\frac{d\nu}{d\tau} \geq -Bu(\tau) \tag{A.28}$$

for some positive constant B. By inserting (A.27) into (A.28) and integrating with respect to τ we find for all sufficiently large τ that $u(\tau) \geq u(\tau_1) - \epsilon$ for all $\tau \geq \tau_1$ where τ_1 is some fixed time which in turn shows that

$$\limsup_{\tau\to\infty} \nu(\tau) = \lim_{\tau\to\infty} \nu(\tau) = \nu^*.$$

This completes the step one of the proof.

From (A.20) we see that $\frac{d\nu}{d\tau} > 0$ is large for large $K(\mathbf{F}) := \frac{K(u,F)}{u}$. Therefore the solution curve $(u(\tau), \mathbf{F}(\tau), \mathbf{G}(\tau))$ tents to a compact set $u = 0$, $\nu = \nu^*$ and $K(\mathbf{F}) \leq B$ for some positive constant B as $\tau \to \infty$ which in turn implies that $\omega(u(\tau), \mathbf{F}(\tau), \mathbf{G}(\tau)) \neq 0$.

Finally , if $(u_1, \mathbf{F}_1, \mathbf{G}_1) \in \omega(u(\tau), \mathbf{F}(\tau), \mathbf{G}(\tau))$ then $\nu(\mathbf{F}_1, \mathbf{G}_1) = \nu^*$ for all τ and by Proposition A.0.2 it follows that $(\mathbf{F}_1, \mathbf{G}_1)$ is a rest point.

$\square$

B. Rate of Approach of Collisions

In Appendix B we prove a classical result about the rate of approach of collisions needed for the proofs of the lemmas 3.2.1 and 4.1.1. Although the result is well-known for general collision singularities [Sa2], we restrict our consideration to the case of the collinear four-body problem. The proof is a simple consequence of the properties of the flow on the collision manifold proved in Appendix A.

Proposition B.0.1. *Let* $\vec{\varphi}(t), t \in (t_1, t_2)$ *be a solution of equation (4.3) in* $\mathcal{U}_{K,\rho}$*. Furthermore assume that* $\lim_{t \to t_2^-}(u_1(t), u_2(t), \hat{u}(t)) = L \in \Delta_{12,34}$*. There exists a constant* B_1 *such that*

$$U \quad \sim \quad B_1(t-t_2)^{-\frac{2}{3}} \tag{B.1}$$

as $t \to t_2^-$*.*

Proof: We consider the system (A.8) obtained from the system (4.3) via the coordinate transforms (A.4-A.5) and the time transform (A.7). Note that to show $U \sim B_1(t-t_2)^{-\frac{2}{3}}$ as $t \to t_2^-$ it is sufficient to show

$$u \sim B(t-t_2)^{\frac{2}{3}} \tag{B.2}$$

for some constant B as $t \to t_2^-$. By direct computations using the vector field (A.8) we find (in the notation of the Appendix A)

$$\frac{d}{d\tau}\ln(u(\mathbf{F}, M\mathbf{F})^{\frac{1}{2}}) = \frac{\nu}{(\mathbf{F}, M\mathbf{F})^{\frac{3}{4}}}. \tag{B.3}$$

Now recall from the proposition (A.0.3) that $\nu \to \nu^* < 0$ and $(\mathbf{F}, M\mathbf{F})^{\frac{3}{4}} \to b > 0$ as $\tau \to \infty$. Therefore by integrating (B.3) we obtain for some $\nu_1, \nu_2 < 0$ and $b_1, b_2 > 0$

$$b_1 e^{\nu_1 \tau} < u(\mathbf{F}, M\mathbf{F})^{\frac{1}{2}}(\tau) < b_2 e^{\nu_2 \tau} \tag{B.4}$$

and in turn

$$\hat{b}_1 e^{\nu_1 \tau} < u(\tau) < \hat{b}_2 e^{\nu_2 \tau} \tag{B.5}$$

where $\hat{b}_1, \hat{b}_2 > 0$. Therefore

$$u(\tau) \sim b e^{\hat{\nu}\tau} \tag{B.6}$$

as $\tau \to \infty$ for some positive constant b. We use (B.6) and the time transformation (A.7) to obtain

$$b e^{\frac{3}{2}\hat{\nu}\tau} d\tau \sim dt. \tag{B.7}$$

We integrate (B.7) from t to t_2 (for t sufficiently close to t_2) to get

$$\tau \sim \frac{2}{3\hat{\nu}} \ln \frac{3\hat{\nu}}{2b}(t - t_2). \tag{B.8}$$

We now insert (B.8) into (B.6) to obtain (B.2), which competes the proof of the proposition. □

Corollary B.0.1. *Let $\vec{\varphi}(t), t \in (t_1, t_2)$ be a solution of equation (4.3) in $\mathcal{U}_{K,\rho}$. Furthermore assume that $\lim_{t \to t_2^-}(u_1(t), u_2(t), \hat{u}(t)) = L \in \Delta_{12,34}$. There exist constants $B_2,\ B_3$ such that*

$$u_i \sim B_2(t - t_2)^{\frac{2}{3}} \tag{B.9}$$

$$v_i \sim B_3(t - t_2)^{-\frac{1}{3}} \tag{B.10}$$

as $t \to t_2^-$ where $i = 1, 2$.

Proof: To prove asymptotic formula (B.9) recall

$$u_1 := x_2 - x_1$$

thus

$$u_1 = u \frac{x_2 - L_1 + L_1 - x_1}{u} = u(F_2 - F_1)$$

where $u^{-1} := 2U$, $F_2 := \frac{x_2 - L_1}{u}$ and $F_1 := \frac{x_1 - L_1}{u}$ as in Appendix A. Since

$$\lim_{t \to t_2^-} (F_2 - F_1) = B$$

where B is a constant then $u_1 \sim u$ as $t \to t_2^-$ and the asymptotic formula (B.9) (for $i = 1$) follows from (B.1). The proof for $i = 2$ is identical

To show (B.10), we recall

$$\frac{dv_i}{dt} = -\frac{2}{u_i^2} + \frac{\partial \mathcal{K}}{\partial u_i}.$$

Since $\frac{\partial \mathcal{K}}{\partial u_1}$ is analytic in $\mathcal{U}_{K,\rho}$ it is bounded, thus

$$\frac{dv_i}{dt} \sim -\frac{2}{u_i^2} \text{ as } t \to t_2^-. \tag{B.11}$$

We insert (B.9) into (B.11) to get

$$\frac{dv_i}{dt} \sim -\frac{2}{B_2^2}(t - t_2)^{-\frac{4}{3}} \text{ as } t \to t_2^-. \tag{B.12}$$

(B.10) follows by integrating (B.12) which is justifiable by Tauberian theorems [Sa2]. □

Finally, we supply the proofs for the 3.2.1 and 4.1.1 which are well-documented in the literature. Actually, the statement and the proof of the 3.2.1 are almost identical to that of the 4.1.1 except for the fact that the vector field (3.1) is somewhat simpler than (4.3), thus we focus on later. Let us recall the statement of the 4.1.1

Lemma B.0.1. *Let $\vec{\varphi}(t), t \in (t_1, t_2)$ be a solution of equation (4.3) in $\mathcal{U}_{K,\rho}$. Furthermore assume that $\lim_{t \to t_2^-}(u_1(t), u_2(t), \hat{u}(t)) = L \in \Delta_{12,34}$. Then the following limits exist and are finite:*

(i) $\lim_{t \to t_2^-} \hat{v}(t)$;

(ii) $\lim_{t \to t_2^-} h_1(t)$, $\lim_{t \to t_2^-} h_2(t)$;

(iii) $\lim_{t \to t_2^-} \frac{u_1(t)}{u_2(t)}$.

Proof: (i) Recall $\frac{d^2\hat{u}}{dt^2} = \frac{d\hat{v}}{dt} = \frac{\partial\mathcal{K}}{\partial\hat{u}}$, where $\mathcal{K}$ and consequently $\frac{\partial\mathcal{K}}{\partial\hat{u}}$ are real-analytic in $\mathcal{U}_{K,\rho}$ implying the finiteness of $\lim_{t\to t_2^-}\frac{d^2\hat{u}}{dt^2}$. Since both $\hat{u}(t)$ and $\frac{d^2\hat{u}}{dt^2}$ have a finite limit as $t \to t_2^-$, by Tauberian Theorem [Sa2], it follows that $\hat{v}(t) := \frac{d\hat{u}}{dt}$ has a finite limit as $t \to t_2^-$.

(ii) From (4.5) and the mean-value theorem it follows for $t \in (t_1, t_2)$

$$|h_1(t) - h_1(t_2)| \leq \frac{dh_1}{dt}(t^*)(t - t_2) = v_1(t^*)\frac{\partial\mathcal{K}}{\partial u_1}(t^*)(t - t_2),\ t^* \in (t, t_2) \tag{B.13}$$

$\frac{\partial\mathcal{K}}{\partial\hat{u}}$ is bounded on $\mathcal{U}_{K,\rho}$ since it is real-analytic. On another hand $v_1(t^*) \sim (t-t_2)^{-\frac{1}{3}}$ on collision orbits when $t \to t_2^-$ by Corollary B.0.1. Therefore for some positive bound B we have

$$|h_1(t) - h_1(t_2)| \leq B(t - t_2)^{-\frac{1}{3}}(t - t_2),$$

thus the limit $\lim_{t\to t_2^-} h_1(t) = h(t_2)$ exists. To show that this limit is finite it is enough to show that $h_1(t)$ is uniformly bounded for all $t \in (t_1, t_2)$. We integrate the equation

$$\frac{dh_1}{dt} = v_1\frac{\partial\mathcal{K}}{\partial u_1}$$

to get

$$h_1(t) = h_1(t^*) + \int_{t^*}^{t} v_1(s)\frac{\partial\mathcal{K}}{\partial u_1}ds \tag{B.14}$$

where $t^* \in (t_1, t_2)$ is some fixed moment of time. Recall now that $\frac{\partial\mathcal{K}}{\partial u_1}$ is real-analytic on $[t_1, t_2]$ and u_1 bounded. Integration by parts shows that the right-hand side is bounded. The proof for h_2 is identical.

(iii) Recall that in (4.3) we had

$$\frac{dv_1}{dt} = -\frac{2}{u_1^2} + \frac{\partial\mathcal{K}}{\partial u_1} \tag{B.15}$$

where $\mathcal{K}$ and consequently $\frac{\partial\mathcal{K}}{\partial u_1}$ are real-analytic in $\mathcal{U}_{K,\rho}$. Multiplying the both sides of (B.15) by v_1 we get

$$\frac{d}{dt}\left(\frac{1}{2}v_1^2\right) = -\frac{2}{u_1^2}\frac{du_1}{dt} + \frac{\partial\mathcal{K}}{\partial u_1}\frac{du_1}{dt} \tag{B.16}$$

which implies

$$\frac{1}{2}v_1^2 = \frac{2}{u_1} + \mathcal{G}_1 \tag{B.17}$$

and $\mathcal{G}_1$ is real-analytic on all orbits. Similarly, we obtain

$$\frac{1}{2}v_2^2 = \frac{2}{u_2} + \mathcal{G}_2, \tag{B.18}$$

where $\mathcal{G}_2$ is real-analytic on all orbits.

The functions $u_i(t),\ i = 1, 2$ are smooth for $t \in (t_1, t_2)$. Therefore by Caushy's mean value theorem we have

$$\lim_{t \to t_2} \frac{u_1(t)}{u_2(t)} = \frac{v_1(t^*)}{v_2(t^*)}, \tag{B.19}$$

where $t^* \in (t, t_2)$ and t is sufficiently close to t_2. Now, (B.17) implies

$$\left(\frac{v_1(t)}{v_2(t)}\right)^2 = \frac{u_2}{u_1}(1 + \frac{u_1}{2}\mathcal{G}_1)(1 + \frac{u_2}{2}\mathcal{G}_2)^{-1} \tag{B.20}$$

equivalently

$$\left(\frac{v_1(t)}{v_2(t)}\right)^2 = \frac{u_2}{u_1}(1 + o(t)), \tag{B.21}$$

where $\lim_{t \to t_2^-} o(t) = 0$. Substituting (B.21) into (B.19) yields

$$\left(\frac{u_1(t)}{u_2(t)}\right)^2 \frac{u_1(t^*)}{u_2(t^*)} = 1 + o(t^*). \tag{B.22}$$

Observe now that by (B.17) and (B.18) both u_1 and u_2 tend to 0 monotonically as $t \to t_2^-$. Therefore (B.22) proves (iii). □

References

[A1] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, (1989).

[A2] V. I. Arnold, *Ordinary Differential Equations*, Springer-Verlag, (1992).

[A3] V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, (1978).

[AK] V. I. Arnold, V. V. Kozlov, A. I. Neishtadt, and E. Khukhro , *Mathematical Aspects of Classical and Celestial Mechanics*, Springer-Verlag, (2006).

[B] E. Belbruno, *On simultaneous double collision in the collinear four-body problem*, J. Differential Equations **52** (1984), 415-431.

[BH] N. Bleistein and R. Handelsman, *Asymptotic Expansions of Integrals*, Dover (1986).

[D1] R. Devaney, *Triple collision in the planar isosceles three body problem*, Invet. Math. **60** , 249-267.

[D2] R. Devaney, *Singularities in Classical Mechanical Systems*, Ergodic Theory and Dynamical Systems, Birkhauser, (1981), 211-333.

[Ea] R. Easton, *Regularization of vector fields by surgery*, J. Diff. Eqs. **10** (1971), 92-99.

[El1] M. S. ElBialy, *Collision-ejection manifold and collective analytic continuation of simultaneous binary collisions in the planar n-body problem*, J. Math. Analysis and Applications **203** (1996), 55-77.

[El2] M. S. ElBialy, *Collision singularities in celestial mechanics*, SIAM J. Math. Anallysis **21**(6) (1990), 1563-1593.

[El3] M. S. ElBialy, *Simultaneous binary collisions in the collinear N-body problem*, J. Diff. Equations **102**(2) (1993), 209-235.

[El4] M. S. ElBialy, *The flow of the N-body problem near a simultaneous-binary-collision singularity and integrals of motion on the collision manifold*, Arch. Rational Mech. Anal. **134** (1996), 303-340.

[H] E. Hille, *Ordinary Differential Equations in the Complex Domain*, Dover, (1976).

[KP1] S. Krantz and H. Parks, *A Primer of Real Analytic Functions*, Birkhauser, (2002).

[KP2] S. Krantz and H. Parks, *The Implicit Function Theorem*, Birkhauser, (2002).

[M1] J. Moser, *Regularization of Kepler's problem and the averaging method on a manifold*, Comm. Pure and Appl. Mathematics **23** (1970), 609-636.

[M2] J. Moser, *Stable and Random Motions in Dynamical Systems*, Princeton University Press, (2001).

[McG1] R. McGehee, *Triple collision in collinear three-body problem*, Invent. Math. **27** (1973), 191-227.

[McG2] R. McGehee, *Singularities in classical celestial mechanics*, Proc. Internat. Congress of Mathematicians, Helsinki, Finland, 1978.

[Mo] R. Moeckel, *Orbits near triple collision in the three-body problem*, Indiana Univ. Math. J. **32** (1983), 221-224.

[MH] K. Meyer and G. Hall, *Introduction to Hamiltonian Dynamical Systems and the N-body Problem*, Springer-Verlag, (1991).

[MM] J. N. Mather and R. McGehee, *Solution for collinear four-body problem which becomes unbounded in finite time*, Lecture Notes in Physics, Vol **3**, Springer-Verlag, (1975), 573-597.

[Po] H. Pollard, *Mathematical Introduction to Celestial Mechanics*, Prentice-Hall, (1966).

[Pn] P. Painleve, *Leçons sur la théorie analytique des équations différentielles*, Harmann (1987).

[PW1] P. Punoševac and Q. Wang, *Regularization of simultaneous binary collisions in some gravitational systems*, to appear.

[PW2] P. Punoševac and Q. Wang, *Regularization of simultaneous binary collisions in the collinear four-body problem*, to appear.

[S] C. L. Siegel, *Der Dreiestoβ*, Ann. Math. **42** (1941), 127-168.

[Sa1] D. G. Saari, *The manifold structure for collision and for hyperbolic-parabolic orbits in the n-body problem*, J. Differential Equations **55** (1984), 300-329.

[Sa2] D.G. Saari, *Collisions, Rings and Other Newtonian n-body Problems*, AMS **104**, (2005).

[SL] C. Simo and E. Lacomba, *Regularization of simultaneous binary collisions in the n-body problem*, J. Differential Equations **98** (1992), 241-259.

[SM] C. L. Siegel and J. K. Moser, *Lecture on Celestial Mechanics*, Springer-Verlag, (1971).

[Sp] H. Sperling, *On the real singularities of the n-body problem*, J. Reine Angew. Math. **245** (1970), 15-40.

[SS] E. L. Stiefel and G. Scheifele, *Linear and Regular Celestial Mechanics*, Springer-Verlag, (1971).

[Su] K. F. Sundman, *Mémoire sur le probleme des trios corps*, Acta. Math. **36** (1913), 105-179.

[Wa] Q. D. Wang, *The global solution of N-body problem*, Celest. Mech. **50** (1991), 73-88.

[Wi] A. Wintner, *The Analytic Foundations of Celestial Mechanics*, Princeton University Press, (1947).

[X] Z-H. Xia, *The existence of non-collision singularity in Newtonian systems*, Ann. of Math. **135**(3) (1992), 411-468.

[Z] H. von Zeipel, *Sur les singularitś du probléme des n Corps*, Ark. Mat. Astronomi och Fysik **32** (1908), 1-4.