Games of Decentralized Inventory Management

by

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DEDICATION

This dissertation is dedicated to my parents who love me unconditionally and support me throughout my life in every possible way.

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ABSTRACT

Any decentralized retail or wholesale system of competing entities requires a benefit sharing arrangement when competing entities collaborate after their demands are realized. For instance, consider a distribution system similar to the observed behavior of independent car dealerships. If a dealership does not have in stock the car requested by a customer, it might consider acquiring it from a competing dealer. Such behavior raises questions about competitive procurement strategies that achieve system optimal outcomes. This dissertation consists of three main bodies of work contained respectively in chapters 2, 3, and 4. In the first work – chapter 2, we examine a decentralized system that adopts an ex-post agreed transfer payment approach proposed by Anupindi et al. (2001). In particular, we state a set of conditions on cost parameters and distributions that guarantee uniqueness of pure strategy Nash equilibrium. In the second work – chapter 3, we introduce a multilevel graph framework that links decentralized inventory distribution models as a network of stochastic programming with recourse problems. This framework depicts independent retailers who maximize their individual expected profits, with each retailer independently procuring inventory in the ex-ante stage in response to forecasted demand and anticipated cooperative recourse action of all retailers in the system. The graph framework clarifies the modeling connection between problems in a taxonomy of decentralized inventory distribution models. This unifying perspective links the past work and shades

light on future research directions. In the last work – chapter 4, we examine and recast the biform games modeling framework as two-stage stochastic programming with recourse. Biform games modeling framework addresses two-stage games with competitive first stage and cooperative second stage without ex-ante agreement on profit sharing scheme. The two-stage stochastic programming view of biform games is demonstrated on examples from all the known examples regarding operational decision problems of competing firms from the literature. It allows an "old" mathematical methodology to showcase its versatility in modeling combined competitive and cooperative game options. In short, this dissertation provides important insights, clarifications, and strategic limitations regarding collaborations in decentralized distribution system.

1. INTRODUCTION

Given the uncertainty of business environments, many manufacturers and service providers consider strategies that permit cooperation with their competitors in production, material replenishment, warehouse sharing, and transportation. The analysis of strategic operations management problems as game problems has gained momentum in the last 10-15 years and today constitutes one of the primary modeling tools for research in this area. My research is focused on exploring multi-stage competitive and cooperative games encountered in supply chain and inventory management. Modeling operations management cost sharing problems is an important basis for analyzing firm's investments and strategic commitment decisions. This dissertation consists of three main bodies of work (– chapters 2, 3, and 4) addressing this important topic.

Chapter 2 examines a generic distribution system like an independent car dealerships. If a dealership does not have in stock the car requested by a customer, it might consider acquiring it from a competing dealer. Such behavior requires procurement strategies that try to achieve system optimal (first-best) outcomes. We analyzed the existence and uniqueness of pure strategy Nash equilibrium (PSNE) for a decentralized system that adopts a transfer payment approach proposed by Anupindi et al. (2001). In particular, we state a set of conditions on cost parameters and distributions that guarantee uniqueness of PSNE and discuss its consequences. We also examine a situation with incomplete information and expand the scope of the earlier models by relaxing the assumption of satisfying local demand first. That is, we allow the retailers to transship their inventory regardless of the local demand status if such transshipment increases retailers profit, and observe that this model extension does not affect our results relative to the more restrictive case. The paper (Suakkaphong and Dror, 2010a) composed of the first chapter will appear in TOP - Journal of the Spanish Society of Statistics and Operations Research in the near future. This work serves as a launching point for two subsequent chapters.

In Chapter 3, using stochastic programming with recourse model as a building block, we constructed a unifying taxonomy for a large variety of decentralized inventory games. This framework depicts independent retailers who maximize their individual expected profits, with each retailer independently procuring inventory in the ex ante stage in response to forecasted demand and anticipated inventory decisions of the other retailers. In the ex post stage, in response to the realized demand and competitors' chosen procurement levels, each retailer exercises a recourse action. For instance, retailers may coordinate inventory swaps to satisfy shortage with overage with profits shared between collaborating retailers. Our framework provides a unifying parsimonious view through a single methodological prism for a large variety of problems studied in isolation in the past. We posit that this taxonomy framework will enable manufactures and service providers state, understand, and solve their strategic problems. Perhaps equally importantly, as recourse options are laid out, the graph framework clarifies the modeling connection between problems in a taxonomy of decentralized inventory distribution models. This unifying perspective links the past work and sheds light on future research directions.

Chapter 4 applies stochastic programming to a special class of decision games called biform games. Biform games modeling framework addresses two-stage games with competitive first stage and cooperative second stage without ex-ante agreement on profit sharing scheme. For instance, a manufacturing firm might have to decide upfront regarding her production capacity based on beliefs about her customers' demand and the capacity installed by her competitor. Later, their decisions are about forming coalition(s) that generate the highest surplus and are likely to deliver the best value to each of the firms. Presumably, the resulting surplus (the participants' realized payoff) has to be shared fairly. This type of two-stage problems was modeled in Brandenburger and Stuart (2007). We show that the methodology proposed by Brandenburger and Stuart (2007) can be cast as a special case of a two-stage stochastic programming with recourse model. The two-stage stochastic programming view of biform games is demonstrated on examples from all the known examples regarding operational decision problems of competing firms from the literature. It allows an "old" mathematical methodology to showcase its versatility in modeling combined competitive and cooperative game options. Two working papers (Suakkaphong and Dror, 2010b,c) composed of chapters 3 and 4 have been submitted for publication.

Chapter 5 of this dissertation highlights the major research contributions, and indicates a number of promising future research directions.

2. MANAGING DECENTRALIZED INVENTORY AND TRANSSHIPMENT

2.1. Introduction

Anupindi et al. (2001) proposed a framework for a study of a single-commodity decentralized inventory system. Decentralized inventory systems are common in a broad range of supply chain networks and impact many aspects of a daily commercial behavior. However, it is not obvious how and if one can “engineer” an operational process that causes a decentralized system’s performance to match an optimal performance of a centralized setting. In order to induce the participants to behave collectively as well as a single commercial entity, in many cases one resorts to some form of transfer payments. For example, a case of a car dealership acquiring a car for its customer from a neighboring dealership requires that all parties be appropriately compensated. To understand the conditions for a stable first-best solution that allow competing entities to collaborate rationally, we have to carefully examine the solution’s sensitivity to all relevant levers.

This paper studies a single-commodity multi-player inventory procurement and storage operations in a decentralized two-stage decision system very similar to one described by Anupindi et al. (2001), referred from now on as ABZ. To avoid confusion and for the sake of self containment, we deviate at times from ABZ’s notation. A precursor to this line of research is an extensive body of work that we draw on as we progress with the analysis. Our analysis reveals that the stable solution proposed in ABZ is sensitive to assumptions that lead to a proof of uniqueness of first-best Nash equilibrium (Nash equilibrium that corresponds to a first-best solution). We prove that the conditions stated in ABZ are not sufficient to assure uniqueness of first-best Nash. Moreover, we assert that relaxing the assumption of satisfying local demand first has no impact on the results regarding unique Nash, its sensitivity to all players playing Nash, and the impact of complete information assumption.

For completeness, we provide below a short review of basic game theory terminology.

2.1.1. Game Theory Terminology

Since the cooperation of retailers can be viewed as a cooperative game with transferable utility, we restate some basic cooperative game terminology. Let $N = \{1, \ldots, n\}$ be a set of retailers. In our inventory game, retailers are players. The set N is referred to as the *grand coalition.* A nonempty subset $S \subseteq N$ is a *coalition.* There are $2^N - 1$ different coalitions that can be formed. A *characteristic function* v is a set function such that $v(\emptyset) = 0$ and associates a real number $v(S) \in \mathbb{R}$ with each subset $S \subseteq N$. We can think of the characteristic function as an amount of profit that retailers who are members of S generate as a result of forming a coalition S to transship products only among themselves. The pair (N, v) denotes a cooperative game. The decision on how the profits are shared is called an *allocation rule*. An allocation rule α ($\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$) determines an allocation of profit to each individual retailer. An allocation that enables stable cooperation (no subset of retailers has an incentive to withdraw from the grand coalition) is called a *core allocation*. The *core* is a set of core allocations. An allocation α is in the core of game (N, v) if $\sum_{i \in S} \alpha_i \geq v(S)$ for all $S \subseteq N$ and $\sum_{i \in N} \alpha_i = v(N)$.

In addition to cooperative game terminology, the decentralized inventory system requires an understanding of basic competitive game terminology. Let $\mathbb{S}_i$ be set of inventory strategies available for player i and a strategy $s_i \in \mathbb{S}_i$ denote a strategy carried out by a player i. In our inventory game, a set of strategies is a non-negative amount of inventory ordered by each retailer in the first stage. A *payoff function* u_i of player i associates a real number $u_i(s_1, \ldots, s_n) \in \mathbb{R}$ with the strategies $s_1, \ldots, s_n$ chosen by players individually. We can think of the payoff function as an amount of profit that retailer i expects to get over an infinite sequence of repeated decentralized inventory games as a function of the chosen strategies. We use a tuple $(\mathbb{S}_1, \ldots, \mathbb{S}_n; u_1, \ldots, u_n)$ to denote a competitive inventory game. The strategies $(s_1^*, \ldots, s_n^*)$ are a *Nash equilibrium* if, for each player i, s_i^* is player i's best response to the strategies $(s_1^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*)$ chosen by other players; that is, s_i^* solves $\max_{s_i \in S_i} u_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*)$. Unique pure strategy Nash equilibrium (PSNE) provides arguably a rational prediction of what options players may pursue whenever unique PSNE exists in a game.

2.1.2. The Outline

The remainder of this paper is organized as follows. Section 2.2 describes the basic setup of decentralized distribution systems (DDS). Section 2.3 reexamines the existence and uniqueness of first-best Nash equilibrium for the DDS that adopts the ABZ's transfer payment approach. We then state a set of conditions on cost parameters and distributions that guarantee uniqueness of first-best Nash equilibrium. The implications of failure in satisfying the necessary conditions is examined next. To reiterate the importance of Nash equilibrium, Section 2.4 presents the effect of non-Nash strategy on DDS. For completeness, Section 2.5 illustrates the situation when the players' complete information assumption does not hold. That is, we examine the transferred payment approach with respect to incentive compatibility property. Both Section 2.4 and Section 2.5 provide insights into strategic limitation of implementing collaboration in DDS. Section 2.6 expands the scope of the previous model by relaxing the assumption of satisfying local demand first. We assume that retailer is allowed to transship her inventory regardless of the local demand status if such transshipment increases her profit. The observations in Sections 2.3, 2.4, and 2.5 still hold true in the extended model. As the main contribution, this paper provides important insights and clarifications regarding collaboration assumptions in DDS. The more involved technical details and proofs are presented in the Appendix.

2.2. Model Description

Suppose competing independent retailers faced random demands in DDS. In the first-stage, inventories are ordered based on anticipated demands, and retailers may end up with excess demand or supply. In the second stage, these retailers use pooling of residual, i.e., excess demand at one retailer's local inventory can be satisfied from surplus transshipped from other retailers' local inventories.

In contrast to the decentralized system, in the centralized inventory system the goal of both inventory decisions and transshipment decisions is to maximize the expected profit of the overall system. An optimal solution for a centralized inventory system is referred to as the *first-best solution* and the maximum expected profit of the overall distribution system as the *first-best profits.* A first-best solution does not consider how the profit will be shared among retailers. Subsequently, the total profit of a decentralized system cannot exceed the first-best profit. To achieve first-best profits in the decentralized system, ABZ proposes a set of conditions and claims that these conditions result in first-best profit and unique pure strategy Nash equilibrium (PSNE). Note that uniqueness of PSNE is crucial for the stability of the decentralized system. We examine ABZ's conditions in detail in the next section and show that they do not necessarily result in a unique PSNE. We remedy this shortcoming by presenting sufficient conditions that guarantee uniqueness of PSNE.

2.2.1. Past Assumptions and Claims

ABZ assumes that each retailer will choose to satisfy local demand from the local stock before sharing the residual demand with other retailers. It also introduces the notion of *claims* for units stored in centralized warehouse. Claims indicate ownership for each unit of inventory. The claim holder pays for inventory holding cost of the unit and can decide on where the unit will be transshipped to. For simplicity, we exclude the option of shared warehouses.

We first restate the ABZ's conditions that allow the DDS to achieve the first-best profit. These conditions are: (1) profit allocation must be in the *core* of *a snapshot allocation game* (defined below); (2) a *PSNE* must exist in the first stage; and (3) the first-stage inventory decisions must result in the same inventory ordering quantities as the centralized system.

The snapshot allocation game (SAG) occurs in the second-stage of the DDS. The SAG is defined as a transshipment game with respect to retailers' inventory ordering quantity $\vec{X}$ and a demand realization $\vec{D}$. The characteristic function of SAG is defined as a maximum achievable excess profit from transshipment (in addition to what is achieved without pooling of residuals) to be shared among retailers who join the transshipment coalition $S \subseteq N$. Since the SAG game is superadditive, the largest possible excess profit is the excess profit achievable by the grand coalition N. A profit allocation is in the core of SAG game if no subset of retailers receives smaller amount of excess profit than they can earn on their own.

For the SAG, a profit *allocation rule based on dual prices* for the solution of transshipment profit maximization problem is in the core. This well-known classical result is based on the works of Shapley and Shubik (1975), and Samet and Zemel (1984). A transshipment game or a transportation game is an extension of an assignment game. It belongs to a class of games called linear programming games (LP-games) that has been extensively studies in the past (Samet and Zemel, 1984; Sánchez-Soriano et al., 2001). The dual prices are determined for each unit of excess local demand and for each unit of excess local supply.

The second condition for the solution of DDS is an existence of a PSNE in the first stage. The first-stage game can be defined as a tuple $(\mathbb{S}_1, \ldots, \mathbb{S}_n; \tilde{u}_1, \ldots, \tilde{u}_n)$. A retailer i chooses a strategy $s_i = X_i$ from her set of available strategies $\mathbb{S}_i \subset \mathbb{R}_+$. Hence, the set of game strategies $\mathbb{S} = \mathbb{S}_1 \times \mathbb{S}_2 \times \cdots \times \mathbb{S}_n \subset \mathbb{R}^n_+$ represents a set of possible nonnegative amounts of inventory ordered by retailers in the first stage. The function $\tilde{u}_i$ is an individual payoff function. The value of $\tilde{u}_i$ ($\tilde{u}_i : \mathbb{S} \to \mathbb{R}$) is equal to the sum of expected profit earned by retailer i in the first stage and an expected profit to be allocated to retailer i in the second stage for the excess demand and surplus distribution. ABZ claims that there exists a PSNE for the decentralized inventory game if the second-stage profit function of each retailer is simultaneously continuous in inventory levels at all retailers, unimodal in each retailer's own inventory level, and the demand distribution function belongs to the class of Polya Frequency Functions of order 2 (PF2) (say, normal, exponential, or uniform distribution among others).

We will return to this point later in the paper.

The last condition for the solution of DDS requires that the first-stage independent inventory decisions result in the same inventory levels as for the centralized system. The purpose of this condition is to ensure that DDS achieves the first-best profit. However, an allocation based on dual price does not necessarily imply that the retailers order the same inventory levels as in the centralized system. For that reason, ABZ constructs an allocation rule based on a scheme of ex-post side payments between the retailers, restated below and in Section 2.2.2, that is claimed to satisfy all three rules. ABZ further claims that with their proposed allocation rule, the existence of a unique first-best solution implies the existence of a unique PSNE.

2.2.2. The ABZ Decentralized Distribution Model

Say a group of retailers selects ex ante an allocation rule α for the SAG that attains first-best profit but ex post is not necessarily in the core of SAG. Thus, the solution to the second-stage transshipment game (the SAG) may not be enforceable (rational). To counter this fact, ABZ creates a new allocation $\tilde{\alpha}$ by adding side payments to allocation α. The calculation rule of side payments is based on another allocation — $\alpha'(\vec{X}, \vec{D})$. Allocation $\alpha'(\vec{X}, \vec{D})$ is required to be in the core of SAG and is computed in turn for each demand realization, but may not necessarily be an allocation based on dual prices of the transshipment game. Now assume that there is a unique PSNE for an allocation α. The side payments are equal to the difference between the allocation

α and the allocation $\alpha'(\vec{X}^*, \vec{D})$, evaluated at that unique PSNE $\vec{X}^*$ (= unique first best $\vec{X}^*$). Hence, when the modified allocation $\tilde{\alpha}$ is used, an allocation in the core of SAG in the second stage is obtained by making side payments (adjusting α) based on α^x. The solution of the expected profit maximization in the first-stage remains unchanged.

For instance, consider a fractional allocation that is calculated by, first, combining retailers overall profits (both local and transshipment profit), then re-distributing those profits using previously agreed fractions, and finally, paying each retailer that fractional amount minus her local profit. The agreed fractions could be of any value but overall add up to one. In practice, each such fraction might depend on the retailer's bargaining power. At some realizations of demand, the fractional allocation may not be in the core of SAG. But, the fractional allocation attains first-best profit because it encourages retailers to order the same inventory level as a centralized system. ABZ constructs a new allocation by modifying the fractional allocation using side payments. The side payments are fixed to the difference between the allocation based on dual prices of transshipment game and the fractional allocation, evaluated at the PSNE of the game that uses the modified allocation. ABZ claims that this can be done because the allocation based on dual prices of the corresponding transshipment game is always in the core of SAG regardless of the first-stage inventory decision. They claim that these side payments retain the modified allocation in the core of SAG while preserving PSNE and the first-best profit when PSNE inventory level is ordered. ABZ claims are more formally restated below.

Consider a case of $n > 1$ retailers who make independent inventory stocking decision but agree to cooperate on second-stage transshipment decision. Let r_i, c_i, and v_i, where $i = 1, \ldots, n$, represent unit revenue, unit cost, and unit salvage value of a retailer i, respectively. Let t_{ij} represent the transshipping cost from retailer i to retailer j.

In the first stage, each retailer makes decision on her inventory level. Let the vector $\vec{X} = (X_1, \ldots, X_n)$ denote the levels of inventory ordered in the first stage where X_i is in a given nonempty compact convex subset; $\underline{X}_i \leq X_i \leq \overline{X}_i$, $[\underline{X}_i, \overline{X}_i] \subset \mathbb{R}_+$. Then, the demand represented by the vector $\vec{D} = (D_1, \ldots, D_n)$ is realized at all retailers. Retailer i sells $B_i = \min\{X_i, D_i\}$ units and may have $H_i = \max\{X_i - D_i, 0\}$ unit surplus or $E_i = \max\{D_i - X_i, 0\}$ unit shortage.[1] The profit expected at each retailer is $J_i(\vec{X}) = E_{\vec{D}}(P_i(\vec{X}, \vec{D}))$ where $P_i(\vec{X}, \vec{D}) = [r_i B_i + v_i H_i - c_i X_i] + \alpha_i(\vec{X}, \vec{D})$.

The function $\alpha_i(\vec{X}, \vec{D})$ is the profit allocated to retailer i as a result of transshipment game. For instance, the transshipment profit created by a 2-retailer system is $(r_1 - v_2 - t_{2,1}) \min\{E_1, H_2\} + (r_2 - v_1 - t_{1,2}) \min\{E_2, H_1\}$. Therefore, this amount will be shared between the two retailers such that $\alpha_1(\vec{X}, \vec{D}) + \alpha_2(\vec{X}, \vec{D}) = (r_1 - v_2 - t_{2,1}) \min\{E_1, H_2\} + (r_2 - v_1 - t_{1,2}) \min\{E_2, H_1\}$.

At this point, for completeness, we introduce three definitions (see also ABZ):

Definition 2.1. *For a given inventory level $\vec{X}$ and demand realization $\vec{D}$, the* com-

[1] In this section as in ABZ, we assume that retailers must satisfy their local demand first. A more general model that relaxes this assumption is discussed in Section 2.6.

bined profit *is represented by*

$$P_N^c(\vec{X},\vec{D}) \;=\; \sum_{i\in N}[r_iB_i+v_iH_i-c_iX_i]+W_N(\vec{X},\vec{D})$$

where $W_N(\vec{X},\vec{D})$ *is the optimal value of transshipment problem represented by:*

$$\begin{aligned} W_N(\vec{X},\vec{D}) = \quad & \max_{\vec{y}} \; \sum_{i\in N}\sum_{j\in N, j\neq i}(r_j-v_i-t_{i,j})y_{i,j} \\ & \text{s. t.} \quad \sum_{j\in N, j\neq i} y_{i,j} \le H_i \textit{ for all } i\in N \\ & \qquad\;\; \sum_{i\in N, i\neq j} y_{i,j} \le E_j \textit{ for all } j\in N \\ & \qquad\;\; \textit{for all } y_{i,j}\ge 0. \end{aligned} \tag{2.1}$$

The quantity $y_{i,j}$ *represents the number of units of product transshipped from retailer* i *to retailer* j *in the second stage.*

The expected combined profit is $J_N^c(\vec{X}) = E_{\vec{D}}(P_N^c(\vec{X},\vec{D}))$. *The* first-best solution $\vec{X}^{c^*}$ *is the solution that maximizes the expected combined profit assuming that retailers make centralized decisions in both stages. The* first-best profit $J_N^c(\vec{X}^{c^*})$ *is the expected combined profit when the first-best solution* $\vec{X}^{c^*}$ *is played. We assume* $J_N^c(\vec{X}^{c^*})\ge 0$.

Definition 2.2. *Let the* fractional allocation $\alpha_i^f(\vec{X},\vec{D})$ *be defined as* $\alpha_i^f(\vec{X},\vec{D}) = \gamma_i P_N^c(\vec{X},\vec{D}) - [r_iB_i+v_iH_i-c_iX_i]$, *where* γ_i *is a fraction agreed by all retailers such that* $\sum_{i\in N}\gamma_i = 1$ *and for all* i, $\gamma_i\in(0,1)$. *Note that* $\alpha_i^f(\vec{X},\vec{D})$ *can be negative.*

Definition 2.3. *Let the* dual allocation $\alpha_i^d(\vec{X},\vec{D})$ *be defined as the allocation based on dual price of transshipment game. That is*

$$\alpha_i^d(\vec{X},\vec{D}) = \lambda_iH_i+\delta_iE_i \textit{ and } \sum_{i\in N}\alpha_i^d(\vec{X},\vec{D}) = W_N(\vec{X},\vec{D}).$$

The dual prices λ_i and δ_j are obtained from the optimal solution of the dual of problem (2.1).

Theorem 2.1. (Theorems 5.1 and 5.2, Corollary 5.1 in ABZ). *Consider a modified fractional allocation rule that allocates the residual profits to player $i \in N$ as follows:*

$$\begin{aligned}
\alpha_i^m(\vec{X}, \vec{D}) &= \alpha_i^f(\vec{X}, \vec{D}) + \alpha_i^d(\vec{X}^{c^*}, \vec{D}) - \alpha_i^f(\vec{X}^{c^*}, \vec{D}) \\
&= \gamma_i P_N^c(\vec{X}, \vec{D}) - [r_i B_i + v_i H_i - c_i X_i] \\
&\quad + \lambda_i^{c^*} H_i^{c^*} + \delta_i^{c^*} E_i^{c^*} \\
&\quad - \gamma_i P_N^c(\vec{X}^{c^*}, \vec{D}) + [r_i B_i^{c^*} + v_i H_i^{c^*} - c_i X_i^{c^*}],
\end{aligned}$$

where $\vec{X}^{c^*}$ is the first-best solution. Then the PSNE using $\alpha_i^m(\vec{X}, \vec{D})$ is first-best and the $\alpha_i^m(\vec{X}^{c^*}, \vec{D})$ allocation values are in the core of the transshipment game.

Proof. Consider when $\vec{X} = \vec{X}^{c^*}$.

$$\begin{aligned}
\alpha_i^m(\vec{X}^{c^*}, \vec{D}) &= \gamma_i P_N^c(\vec{X}^{c^*}, \vec{D}) - [r_i B_i^{c^*} + v_i H_i^{c^*} - c_i X_i^{c^*}] \\
&\quad + \lambda_i^{c^*} H_i^{c^*} + \delta_i^{c^*} E_i^{c^*} \\
&\quad - \gamma_i P_N^c(\vec{X}^{c^*}, \vec{D}) + [r_i B_i^{c^*} + v_i H_i^{c^*} - c_i X_i^{c^*}] \\
&= \lambda_i^{c^*} H_i^{c^*} + \delta_i^{c^*} E_i^{c^*} \\
&= \alpha_i^d(\vec{X}^{c^*}, \vec{D})
\end{aligned}$$

The allocation $\alpha_i^m(\vec{X}^{c^*}, \vec{D})$ is equal to the allocation $\alpha_i^d(\vec{X}^{c^*}, \vec{D})$. Recall that the allocation $\alpha_i^d(\vec{X}, \vec{D})$ is always in the core of the transshipment game (Samet and Zemel, 1984). Hence, the allocation $\alpha_i^m(\vec{X}^{c^*}, \vec{D})$ will also be in the core of the transshipment game when the inventory levels $\vec{X}^{c^*}$ are ordered by all retailers.

Now, we check if the first-best solution $\vec{X}^{c^*}$ is the Nash equilibrium solution. We reduce the function $P_i(\vec{X}, \vec{D})$ as follows:

$$
\begin{aligned}
P_i(\vec{X}, \vec{D}) &= [r_iB_i + v_iH_i - c_iX_i] + \alpha_i^m(\vec{X}, \vec{D}) \\
&= [r_iB_i + v_iH_i - c_iX_i] + \gamma_i P_N^c(\vec{X}, \vec{D}) - [r_iB_i + v_iH_i - c_iX_i] \\
&\quad + \alpha_i^d(\vec{X}^{c^*}, \vec{D}) - \alpha_i^f(\vec{X}^{c^*}, \vec{D}) \\
&= \gamma_i P_N^c(\vec{X}, \vec{D}) + \alpha_i^d(\vec{X}^{c^*}, \vec{D}) - \alpha_i^f(\vec{X}^{c^*}, \vec{D}).
\end{aligned}
$$

The expected payoff of retailer i is then:

$$
J_i(\vec{X}) = \gamma_i E_{\vec{D}}(P_N^c(\vec{X}, \vec{D})) + E_{\vec{D}}(\alpha_i^d(\vec{X}^{c^*}, \vec{D})) - E_{\vec{D}}(\alpha_i^f(\vec{X}^{c^*}, \vec{D})).
$$

The value $E_{\vec{D}}(\alpha_i^d(\vec{X}^{c^*}, \vec{D}))$ and $E_{\vec{D}}(\alpha_i^f(\vec{X}^{c^*}, \vec{D}))$ are essentially constants as $\vec{X}^{c^*}$ only depends on $\vec{D}$. Thus, the vector $\vec{X}$ that maximizes $J_i(\vec{X})$ is the same as $\vec{X}$ that maximizes $E_{\vec{D}}(P_N^c(\vec{X}, \vec{D}))$. Therefore, the first-best solution $\vec{X}^{c^*}$ is the Nash equilibrium solution and the first-best profit can be achieved using this allocation $\alpha_i^m(\vec{X}, \vec{D})$. $\square$

2.3. Existence and Uniqueness of First-Best Nash Equilibrium

Corollary 5.1 of ABZ assumes that there exists a PSNE based on condition described in their Theorem 4.2. We show that there exists a PSNE for a game that uses ABZ's allocation rule under weaker conditions. We also show that under certain conditions the uniqueness of the maximum of the expected centralized profit function implies the uniqueness of PSNE. Such conditions were not discussed in ABZ. Finally, we examine situations with multiple PSNE.

2.3.1. Conditions for the Existence of PSNE

We first discuss the conditions for the existence of PSNE by examining retailer i's expected profit function $J_i(\vec{X})$ when ABZ's allocation rule is used. In ABZ's work, the conditions for the existence of PSNE include: (1) the expected profit function $J_i(\vec{X})$ for retailer i is simultaneously continuous in $\vec{X}$, and (2) $J_i(\vec{X})$ is unimodal in X_i for every $\vec{X}_{N\setminus i}$.

Proposition 2.1. *There exists a PSNE for DDS that adopts ABZ's allocation rule.*

Proof. If $\vec{X}$ is a first-best solution, then $\vec{X}$ is also a PSNE. The reason for it is that the best-response to other retailers' playing first-best strategies is to play first-best strategy. Thus, if there exists a first-best solution, then there must exist a PSNE. According to Weierstrass's theorem, if a function $f : C \to \mathbb{R}$ is continuous and its domain is a compact subset C of $\mathbb{R}^n$, then there are vectors in C that maximize the function f. That is, if the inventory domain constitutes a compact subset of $\mathbb{R}^n$ and the expected centralized profit $J_N^c(\vec{X})$ is continuous in $\vec{X}$, then there exists a first-best solution and it follows that there exists a PSNE.

In our setting, each retailer's inventory level falls in a closed and bounded interval of $\mathbb{R}$, hence the domain is a compact subset of $\mathbb{R}^n$. We proceed to check whether the expected centralized profit $J_N^c(\vec{X})$ is continuous in $\vec{X}$

Recall that

$$P_N^c(\vec{X}, \vec{D}) = \sum_{i \in N} r_i B_i + v_i H_i - c_i X_i + W_N(\vec{X}, \vec{D})$$

where $W_N(\vec{X}, \vec{D})$ is the profit from transshipment as defined in (2.1). For any given $\vec{D}$, $W_N(\vec{X}, \vec{D})$ is continuous in $\vec{X}$ because there is no fixed cost related to transshipment profits. In addition, there is no fixed cost related to local profits at any retailers. As a result, the centralized profit $P_N^c(\vec{X}, \vec{D})$ is continuous in $\vec{X}$.

According to Kolmogorov and Fomin (1970, p.109), a real function continuous on a compact metric space $\mathbb{R}$ is uniformly continuous on $\mathbb{R}$. In our case, for all i, X_i are defined on nonempty compact convex subsets of $\mathbb{R}$ and $P_N^c(\vec{X}, \vec{D})$ is continuous in $\vec{X}$. Hence, $P_N^c(\vec{X}, \vec{D})$ is uniformly continuous in $\vec{X}$, and it follows that $J_N^c(\vec{X})$ is continuous in $\vec{X}$. $\square$

2.3.2. The Uniqueness of PSNE

When using ABZ's allocation rule, it is important to ascertain that the PSNE/first-best inventory level of DDS is unique because the side payment calculation is based on the value of the PSNE/first-best inventory level. ABZ does not provide a direct proof of the uniqueness of PSNE, but states (in Theorem 5.3 of ABZ) that if the distribution system exhibits a unique first-best solution, then the PSNE under their allocation rule is unique. This, however is not true. We provide a counterexample with multiple PSNE after first stating sufficient conditions that imply the uniqueness of PSNE.

Lemma 2.1. *Suppose that $J_N^c(\vec{X})$ is strictly quasi-concave in X_i for each $i \in N$. If there is a unique point $\vec{X}^*$ where $J_N^c(\vec{X})$ is* strictly *increasing in X_i for $X_i < X_i^*$, and $J_N^c(\vec{X})$ is* strictly *decreasing in X_i for $X_i > X_i^*$ for all $i \in N$, then $\vec{X}^*$ is the unique PSNE that corresponds to the first-best solution.*

Proof. Recall that the expected profit function for player i is:

$$J_i(\vec{X}) \;\; = \;\; \gamma_i J_N^c(\vec{X}) + E_{\vec{D}}(\alpha_i^d(\vec{X}^{c^*}, \vec{D})) - E_{\vec{D}}(\alpha_i^f(\vec{X}^{c^*}, \vec{D})).$$

Both $E_{\vec{D}}(\alpha_i^d(\vec{X}^{c^*}, \vec{D}))$ and $E_{\vec{D}}(\alpha_i^f(\vec{X}^{c^*}, \vec{D}))$ do not depend on $\vec{X}$. Hence, a player i's strategy X_i that maximizes her expected profit function, also maximizes the expected centralized profit $J_N^c(\vec{X})$.

Because there is a unique point $\vec{X}^*$ where $J_N^c(\vec{X})$ is strictly increasing in X_i for $X_i < X_i^*$, and $J_N^c(\vec{X})$ is strictly decreasing in X_i for $X_i > X_i^*$ for all $i \in N$, no other points are local maxima. Therefore, the point $\vec{X}^*$ is a global maximum, i.e., a unique first-best solution.

The point $\vec{X}^*$ is also a unique PSNE because (i) no player has an incentive to deviate from it, and (ii) for any other point, say at $\vec{X}^\circ \neq \vec{X}^*$, each player i would be better off not playing X_i°, given that other players play $\vec{X}^\circ_{N\setminus i}$.

In other words, we can state that uniqueness of first-best solution implies uniqueness of Nash equilibrium if the conditions in Lemma 2.1 are satisfied. $\square$

A pertinent question is what are the demand distributions and cost parameters that would satisfy the conditions in Lemma 2.1, that is, assure a unique PSNE. Strict quasi-concavity in X_i for each $i \in N$ does not necessarily imply such unique PSNE point. However, given strict quasi-concavity in X_i but not necessarily in $\vec{X}$, there are only two cases that allow for multiple PSNE points.

The first case is the existence of multiple strict local maxima. At each strict local maximum $\vec{X}^*$, there is a neighborhood of $\vec{X}^*$ such that $J_N^c(\vec{X})$ is strictly increasing for an $\vec{X}$ in this neighborhood, $\vec{X} < \vec{X}^*$ in each component, and strictly decreasing for an $\vec{X}$ in this neighborhood $\vec{X} > \vec{X}^*$ in each component. Each of these local maxima is strictly quasi-concave in X_i's separately but not necessarily in $\vec{X}$. Thus, each such strict local maximum corresponds to PSNE. To ensure that there are no multiple strict local maxima, it is sufficient that $J_N^c(\vec{X})$ be weakly quasi-concave in $\vec{X}$. These sufficiency conditions are discussed more formally in the three paragraphs following Lemma2 and in Proposition 2.

The other case of $J_N^c(\vec{X})$ with multiple PSNE is when $J_N^c(\vec{X})$ has what we call a "ridge".

Definition 2.4. *Let $f : \vec{X} \to \mathbb{R}$ be a continuous function. There exists a ridge for function f if, for an $|\vec{\epsilon}| > 0$ and a point $\vec{X}^* = (X_1^*, \ldots, X_n^*)$ with*

$X_i^* = \arg\max_{X_i} f(X_i, \vec{X}^*_{N\setminus i})$ *for all $i \in N$, there is a point $\vec{X}' = \vec{X}^* + \vec{\epsilon}$ such that*

$X'_i = \arg\max_{X_i} f(X_i, \vec{X}'_{N\setminus i})$ *for all* $i \in N$.

The example below demonstrates the "role" of a ridge.

Example 2.1. *Assume two retailers with symmetric cost parameters* $r_i = 10$, $c_i = 1.2$, $v_i = -1$ *for* $i = 1, 2$, *and* $t_{1,2} = t_{2,1} = 2$. *Suppose the retailers agree on using ABZ's allocation rule with* $\gamma_i = 0.5$. *Let the demand be known and fixed at 50 for both retailers. This system has a unique first-best solution at* $(X_1^{c^*}, X_2^{c^*}) = (50, 50)$ *with the first-best profit of $880. The unique first-best solution is one of the PSNE. This system has infinite number of PSNE, e.g. (49,51) with expected profit of ($439,$439), (48,52) with expected profit of ($438,$438), (47,53) with expected profit of ($437,$437), and so on. This centralized profit* $J_N^c(\vec{X})$ *is strictly quasi-concave in* X_i *for each* $i \in N$ *and quasi-concave in* $\vec{X}$. *However, there is a ridge along the line* $X_1 + X_2 = 100$. $J_N^c(\vec{X})$ *is strictly increasing on the left-hand side, and strictly decreasing on the right-hand side for each retailer. In this case, every point* X_i^* *on the ridge line (see Figure 2.1) corresponds to PSNE.*

Note that weak local maxima (flat plateau) is just another instance of a ridge. However, not all points on a ridge are a local maxima. When $J_N^c(\vec{X})$ is differentiable everywhere, we can also describe a ridge as the case where best-response correspondences of two retailers describe the same graph.

The existence of a ridge violates the conditions of Lemma 2.1, more specifically, the uniqueness of $\vec{X}^*$. In the case of strict quasi-concave in X_i's and single maximum,

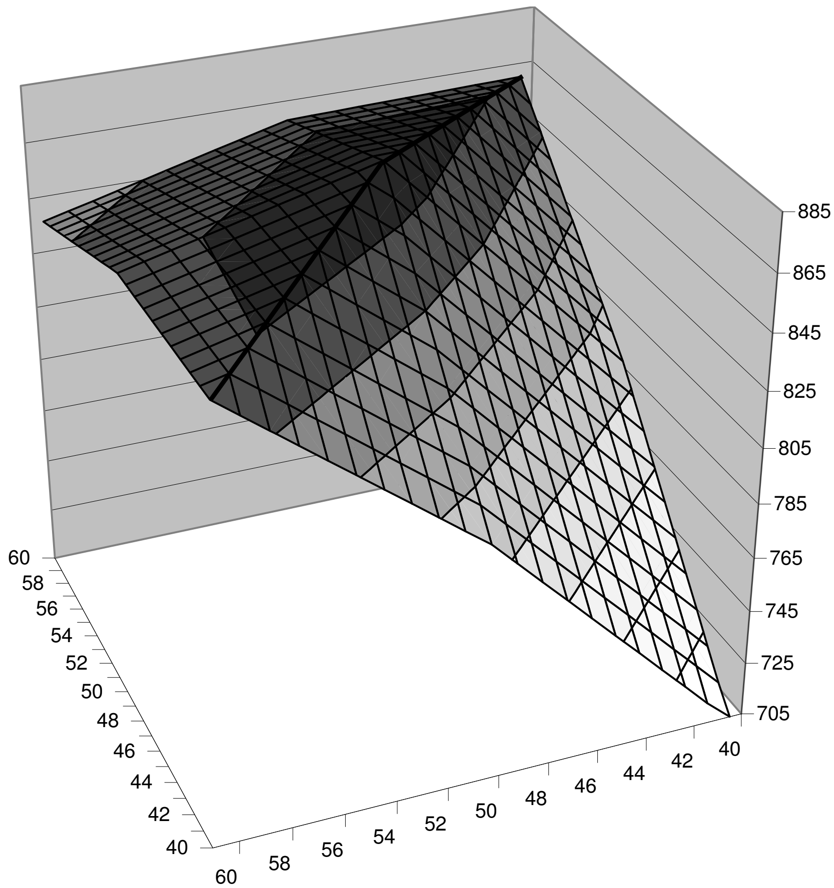

Figure 2.1. Centralized profit $J_N^c(\vec{X})$ with ridge

the following conditions (Lemma 2.2 and the subsequent discussion) will eliminate the possible existence of a ridge, thus guaranteeing the uniqueness of PSNE.

Lemma 2.2. *Given $J_N^c(\vec{X})$ strictly quasi-concave in X_i, let $\overline{D}_i$ and $\underline{D}_i$ represent a highest and lowest possible demand for retailer i, and $\overline{X}_i$ and $\underline{X}_i$ be the upper and lower bound of inventory level for retailer i. There exists a ridge if*

(a) There are at least two retailers, i and j, such that $\overline{X}_i > \overline{D}_i$ and $\underline{X}_j < \underline{D}_j$, and

(b) There exists a PSNE at $\vec{X}^* = (X_1^*, \ldots, X_n^*)$ where $X_i^* > \overline{D}_i$ and $X_j^* < \underline{D}_j$.

The proof is given in the Appendix.

In view of Lemma 2.2, we first restrict the strategy space to be within the demand distribution range. That is $\underline{D}_i \leq \underline{X}_i < \overline{X}_i \leq \overline{D}_i$ for all $i \in N$. Second, we restrict the implicit best-response correspondence of each retailer. For instance, consider $J_N^c(\vec{X})$ that is differentiable everywhere. An implicit best-response correspondence of retailer i is:

$$\frac{\partial J_N^c(\vec{X})}{\partial X_i} = 0.$$

Denote a best-response correspondence of retailer i as $Br_i(\vec{X}_{N\backslash i})$. That is,

$$Br_i(\vec{X}_{N\backslash i}) : (\mathbb{X}_1 \times \cdots \times \mathbb{X}_{i-1} \times \mathbb{X}_{i+1} \times \cdots \times \mathbb{X}_n) \rightarrow \mathbb{X}_i, \text{ where } \mathbb{X}_i = \left[\underline{X}_i, \overline{X}_i\right].$$

Define $\hat{Br}_i(X_j) = Br_i(X_j, \vec{X}_{N\backslash\{i,j\}})$ and $\hat{Br}_j(X_i) = Br_j(X_i, \vec{X}_{N\backslash\{i,j\}})$ for any two retailers i and j and any fixed $\vec{X}_{N\backslash\{i,j\}}$. We can plot $\hat{Br}_i(X_j)$ on a two dimensional plane (X_i, X_j) as a graph where $X_i = \hat{Br}_i(X_j)$. On the same plane, we can also plot an inverse function of $\hat{Br}_j(X_i)$ as a graph where $X_i = \hat{Br}_j^{-1}(X_j)$. If the two functions describe the same graph in some neighborhood, then there is a ridge. Thus, to assure a unique PSNE we require that for any pair (i, j) the two graphs cross only once within the strategy space.

In summary, if (i) $J_N^c(\vec{X})$ is strictly quasi-concave in each X_i, (ii) weakly quasi-concave in $\vec{X}$, and (iii) there is no ridge present for $J_N^c(\vec{X})$, then there is a unique PSNE. We now discuss demand distributions and cost parameters that satisfy (i) and (ii). First, we characterize $P_N^c(\vec{X}, \vec{D})$ and demand distributions. Then, we characterize the cost parameters.

Proposition 2.2. *If the following statements are satisfied then $J_N^c(\vec{X})$ is strictly quasi-concave in each X_i and weakly quasi-concave in $\vec{X}$.*

(a) The demand density function $f(\vec{D})$ is strictly *log-concave in $\vec{D}$.*

(b) $P_N^c(\vec{X}, \vec{D})$ is weakly *log-concave in $(\vec{X}, \vec{D})$.*

(c) $P_N^c(\vec{X}, \vec{D})$ is strictly *log-concave in X_i for all $i \in N$.*

Proof. Since (strictly) log-concave implies (strictly) quasi-concave, we require that $J_N^c(\vec{X})$ be strictly log-concave in each X_i and weakly log-concave in $\vec{X}$.

According to Prékopa (1973, Theorem 6), if the integrand is log-concave in its argument (in our case a vector $(\vec{X}, \vec{D})$) and the domain of integration is a convex subset of $\mathbb{R}^N$, then the integral is log-concave.

Recall that

$$J_N^c(\vec{X}) = \int_\Omega P_N^c(\vec{X}, \vec{D}) f(\vec{D}) d\vec{D}$$

where Ω is the support of $f(\vec{D})$ – the probability density function of demand. First, the expected centralized profit $J_N^c(\vec{X})$ is weakly log-concave in $\vec{X}$ if $P_N^c(\vec{X}, \vec{D}) f(\vec{D})$ is log-concave in $(\vec{X}, \vec{D})$. We can achieve that by requiring $f(\vec{D})$ to be log-concave in $\vec{D}$ and $P_N^c(\vec{X}, \vec{D})$ to be log-concave in $(\vec{X}, \vec{D})$ (as stated in assumption (b)) because log-concavity is preserved under multiplication. Notice that "strict" is not required at this point.

Second, the expected centralized profit $J_N^c(\vec{X})$ is strictly log-concave in X_i for all $i \in N$ if $P_N^c(\vec{X}, \vec{D}) f(\vec{D})$ is strictly log-concave in $(X_i, \vec{D})$.

Let $f(\vec{D})$ be strictly log-concave in $\vec{D}$ (as stated in assumption (a)) and $P_N^c(\vec{X},\vec{D})$ be (weakly) log-concave in $(\vec{X},\vec{D})$ (as stated in assumption (b)). Assumption (b) implies that $P_N^c(\vec{X},\vec{D})$ is (weakly) log-concave in $(X_i,\vec{D})$. Define a function $G(X_i,\vec{D}) = P_N^c(\vec{X}_{N\setminus i}, X_i, \vec{D})f(\vec{D})$. Note that $\vec{X}_{N\setminus i}$ is fixed.

We separate our analysis to 2 cases. The first case is for any two points $A = (X_i^A, \vec{D}^A)$ and $B = (X_i^B, \vec{D}^B)$ such that $\vec{D}^A \neq \vec{D}^B$. Clearly, $G(\lambda A + (1-\lambda)B) > G(A)^\lambda G(B)^{(1-\lambda)}$ because of strict log-concavity of $f(\vec{D})$. The second case is for any two points $A = (X_i^A, \vec{D}^A)$ and $B = (X_i^B, \vec{D}^B)$ such that $X_i^A \neq X_i^B$ and $\vec{D}^A = \vec{D}^B$. In this case, if $P_N^c(\vec{X},\vec{D})$ is strictly concave in X_i (as stated in assumption (c)), then $G(\lambda A + (1-\lambda)B) > G(A)^\lambda G(B)^{(1-\lambda)}$ by strict concavity of $P_N^c(\vec{X},\vec{D})$ in X_i. □

There are many probability density functions that are strictly log-concave, such as normal distribution and exponential distribution. (See Bagnoli and Bergstrom, 2005.)

At this point, we examine the conditions on cost parameters so that $P_N^c(\vec{X},\vec{D})$ be weakly log-concave in $(\vec{X},\vec{D})$ and strictly log-concave in X_i for all $i \in N$. Consider $P_N^c(\vec{X},\vec{D})$ for a given $\vec{D}$ as X_i changes as depicted in Figure 2.2. We assume through out this analysis that salvage value v_i is less than unit cost c_i for all $i \in N$. In addition, to avoid degeneracy for any two retailers i and j, $r_i - t_{j,i} \neq c_j$, $r_j - t_{i,j} \neq c_i$, and $v_i > v_j - t_{i,j}$ (it does not pay to transship in terms of the salvage value).

The graph in Figure 2.2 has four regions. Figure 2.2 illustrates the case where the peak is between the third and the fourth region. However, such graph does not

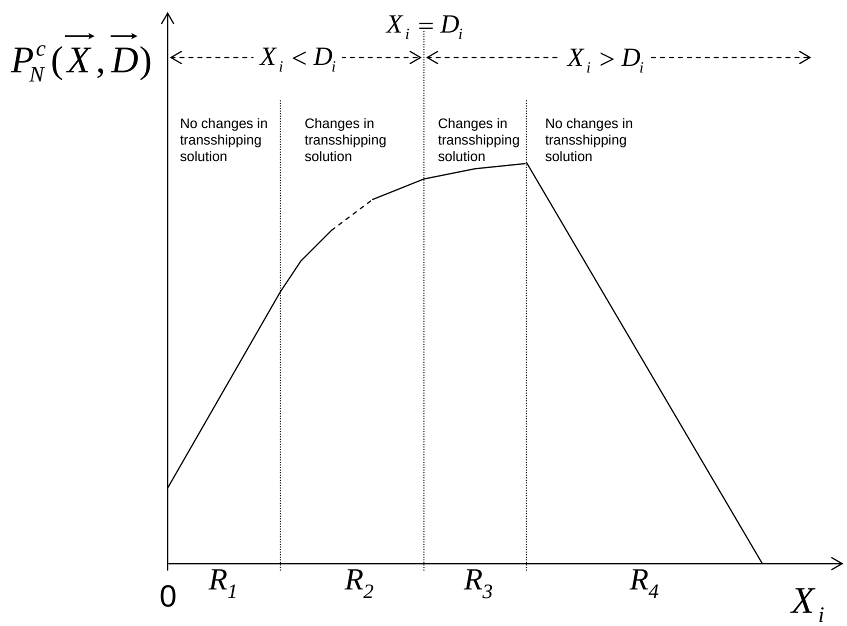

Figure 2.2. Centralized profit function

necessarily have all four regions for all possible demand realizations. For instance, when the system has a large amount of surplus, the first and third regions may vanish, and the peak would be between the second and fourth regions.

Consider the first region $R_1 = \left\{X_i : X_i + \sum_{j \in N, j \neq i} y^*_{j,i} \leq D_i\right\}$. The demand D_i at retailer i is larger than the inventory position X_i. Given the excess demand and supply in the system, an optimal transshipping solution y^* is found. Some excess inventory at other retailer j may be transshipped to retailer i. The sum of units transshipped to retailer i is $\sum_{j \in N, j \neq i} y^*_{j,i}$. This first region constitutes the case when an increase in X_i will not change the optimal transshipping pattern, i.e., no additional excess inventory at other retailers can be transshipped to retailer i, and after transshipment, there are still unsatisfied demands. The slope of the profit function (marginal profit) in this region is constant and equal to $\rho^\circ := r_i - c_i$.

The second region $R_2 = \left\{X_i : X_i \leq D_i \text{ and } X_i + \sum_{j \in N, j \neq i} y^*_{j,i} > D_i\right\}$ follows the first region. In the second region, as X_i increases, the optimal transshipping solution changes. Assume that we increase X_i by a small $\epsilon > 0$. The ϵ at retailer i will generate centralized profit at the rate of ρ°. Moreover, the ϵ amount of inventory from some other retailers is freed and is reallocated to other retailers with shortage or be disposed at salvage rate of $v_j, j \neq i$ of the j^{th} original owners of the ϵ amount of inventory. The marginal profit in this region cannot be greater than ρ° because of the following:

Consider a point in this second region, say at $\vec{X'}$ where $X'_i = X_i + \epsilon', \epsilon' > 0$. The marginal profit at this point is $\rho' = \rho^\circ - \frac{\partial W_N(\vec{X'}, \vec{D})}{\partial E_i}$. Recall that surplus $H_i = \max\{X_i - D_i, 0\}$ and shortage $E_i = \max\{D_i - X_i, 0\}$ for all $i \in N$. An increase of ϵ amount of inventory to retailer i makes E_i smaller. The question is whether $\frac{\partial W_N(\vec{X'}, \vec{D})}{\partial E_i}$ can be negative. If it is negative, then the marginal profit in the second region will be greater than ρ°. An increase in E_i induces a change in transshipping solution only if it is better. So, the lower bound of $\frac{\partial W_N(\vec{X'}, \vec{D})}{\partial E_i}$ is 0. Hence $\frac{\partial W_N(\vec{X}, \vec{D})}{\partial E_i}$ is nonnegative.

Using the similar approach, let the next point, say at $\vec{X''}$ where $X''_i = X'_i + \epsilon'', \epsilon'' > 0$, have the marginal profit $\rho'' = \rho^\circ - \frac{\partial W_N(\vec{X''}, \vec{D})}{\partial E_i}$. Obviously, $\frac{\partial W_N(\vec{X''}, \vec{D})}{\partial E_i} \geq \frac{\partial W_N(\vec{X'}, \vec{D})}{\partial E_i}$ because of the nature of the transshipment profit maximization. Therefore, $\rho'' \leq \rho'$. So, we can be certain that the graph in this region is strictly log-concave.

The third region is $R_3 = \left\{X_i : X_i > D_i \text{ and } X_i \leq D_i + \sum_{j \in N, j \neq i} y^*_{i,j}\right\}$. The retailer i is now a transshipment source in the transportation problem. This region, if exists, will also be strictly log-concave because of the nondegeneracy requirement $r_j - t_{i,j} \neq c_i$ and because the transshipment solution only changes when it is profitable, i.e., $r_j - v_i - t_{i,j} > 0$ for $i \neq j$. The best transshipment solution should address the retailers that generate greater transshipment profit before addressing the retailers that generate smaller profit. However, note that the slope in this region $\frac{\partial W_N(\vec{X},\vec{D})}{\partial H_i}$ might be negative near the fourth region such that $r_j - c_i - t_{i,j} < 0$, while the transshipment profit $r_j - v_i - t_{i,j} > 0$.

The last region is $R_4 = \left\{X_i : X_i > D_i \text{ and } X_i > D_i + \sum_{j \in N, j \neq i} y^*_{i,j}\right\}$. We encounter the last region when an increase in inventory of retailer i does not change an optimal transshipment solution and overall profit declines. The centralized profit decreases at the rate of $v_i - c_i \leq 0$ assuming that local excess inventory is disposed only at local retailer because the salvage value minus transshipment cost at any other retailer is below the local salvage value. Note that the slope of the fourth region is always lower than the slope of the third region because $r_j - c_i - t_{i,j} > v_i - c_i$ for any retailer j involved in the third region.

Not discussed here is the transition between the second region and the third region. It is presented in the Appendix where the following proposition is proven.

Proposition 2.3. *If the marginal profits from own selling at each one of the retailers are more than or equal to the marginal profits from units sold through transshipment, then the profit function is strictly log-concave in each X_i.*

We know that if Proposition 2.3 is satisfied $P_N^c(\vec{X}, \vec{D})$ is concave (and therefore log-concave) in D_i for all $i \in N$. This is because the profit increases as D_i increases (up to the sum of all X_i). The profit increases at the highest rate of $r_i - c_i$ from sales at local retailer i. Then rate of increase in profit declines as the transshipment solution changes. When the demand is higher than the inventory in the system, the profit $P_N^c(\vec{X}, \vec{D})$ is stable.

So far we restrict the cost parameters so that $P_N^c(\vec{X}, \vec{D})$ is strictly log-concave in each X_i and log-concave in each D_i. Next, we show the requirement for log-concavity of $P_N^c(\vec{X}, \vec{D})$.

Proposition 2.4. *$P_N^c(\vec{X}, \vec{D})$ is (weakly) log-concave in $(\vec{X}, \vec{D})$ if it is more profitable to satisfy local demand first.*

Proof. Because (weak) concavity implies (weak) log-concavity, we prove that $P_N^c(\vec{X}, \vec{D})$ is log-concave in $(\vec{X}, \vec{D})$ by showing that $P_N^c(\vec{X}, \vec{D})$ is concave in $(\vec{X}, \vec{D})$ when the condition in Proposition 2.3 is satisfied.

The transportation problem is submodular in the vector of its sources and sinks (Theorem 3.4.1 of Topkis (1998)). Submodularity implies concavity. Hence, the transportation problem is concave in its vector of supply and demand. (See also Lemma 2 of Karaesmen and van Ryzin, 2004).

Recall that

$$P_N^c(\vec{X}, \vec{D}) = \sum_{i \in N} r_i B_i + v_i H_i - c_i X_i + W_N(\vec{X}, \vec{D})$$

where $W_N(\vec{X}, \vec{D})$ is the optimal value of the transshipment problem defined in (1).

The only difference between $P_N^c(\vec{X}, \vec{D})$ and the transportation problem is that our setting requires retailers to satisfy local demand first. Hence, $P_N^c(\vec{X}, \vec{D})$ is not necessarily concave if local demand does not generate profit as much as transshipping to other retailers. A restriction on cost parameters is required to make $P_N^c(\vec{X}, \vec{D})$ coincide with the transportation problem.

Consider when the condition to satisfy local demand first is relaxed. The right-hand side of constraints becomes X_i and D_j instead of $\max\{X_i - D_i, 0\}$ and $\max\{D_j - X_j, 0\}$. The profit function $P_N^c(\vec{X}, \vec{D})$ is changed such that $\sum_{i \in N} r_i B_i + v_i H_i - c_i X_i$ is removed since it would be included in the transportation problem. In this case, $P_N^c(\vec{X}, \vec{D})$ is concave in $(\vec{X}, \vec{D})$.

To cause the solution of this new setting coincide with the solution of the original problem, its transshipment pattern is required to achieve the maximum overall profit when retailers satisfy local demand first. That is, when the local profit is greater than the profit from any transshipment pairs. $\square$

In summary, using ABZ's allocation rule, there exists a unique PSNE for DDS if Propositions 2.3 and 2.4 are satisfied, the strategy space is limited to domain of demand distributions, and density function of demand is strictly log-concave.

When the conditions of Propositions 2.3 or 2.4 are omitted, multiple PSNE may exist as shown in the following example.

Example 2.2. *When* $r_1 = 5.4$, $r_2 = 5.6$, $c_1 = 3.2$, $c_2 = 1.2$, $v_1 = 4$, $v_2 = -1$, $t_{1,2} = 0$, $t_{2,1} = 2$, *and* $\gamma_i = 0.5$. *The demands are independent and uniformly distributed in the range of [49,51]. Assume that the strategy space for inventory levels is bounded in [48,52]. This system has a unique first-best solution at (52,50) with expected profit of \$330.35. However, it has an infinite number of PSNE, e.g. (48,52), (48.5,51.5), etc., with the combined expected profit of \$328.53 as shown in Figure 2.3.*

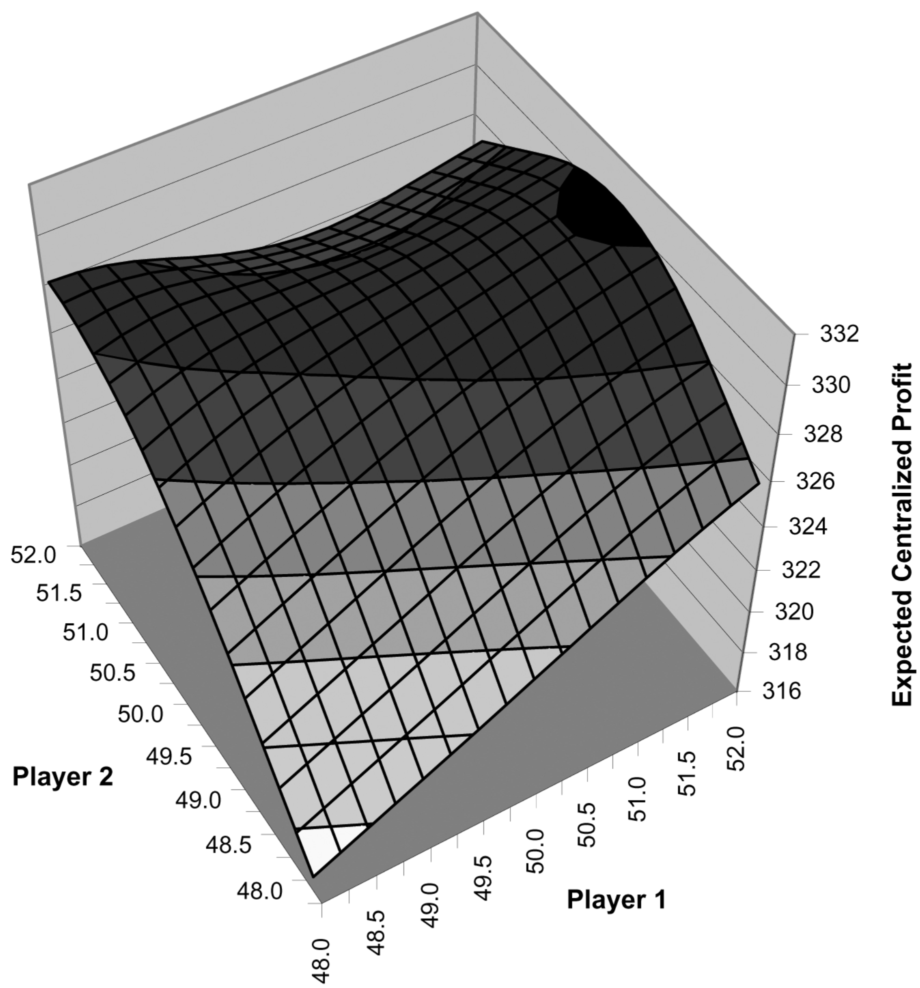

Figure 2.3. Decentralized distribution system with the unique first-best and non-unique PSNE

The uniqueness of PSNE for ABZ's allocation rule is important because the side payment calculations are based on the value of the unique PSNE/first-best inventory level. For instance, consider an n-retailer's DDS that has two different PSNE/first-best inventory levels, $\vec{X}^{*A}$ and $\vec{X}^{*B}$ where

$$\begin{aligned}\vec{X}^{*A} &= \left\{X_1^{*A}, X_2^{*A}, \ldots, X_n^{*A}\right\} \\ \vec{X}^{*B} &= \left\{X_1^{*B}, X_2^{*B}, \ldots, X_n^{*B}\right\}.\end{aligned}$$

What PSNE/first-best inventory level should be used for calculating the side payments? Retailer 1 may prefer the side payment at A and therefore choose inventory level X_1^{*A}. On the other hand, retailer 2 may be better off at B and choose inventory level X_2^{*B}. In this case, the resulting inventory levels are not a member of PSNE, and the first-best expected profit is not achieved.

2.4. Observations Related to Non-Nash Strategy

2.4.1. Modifying ABZ's Allocation – Practical Remedy for Non-Nash Strategy

With regards to any second-stage allocation rule, it is reasonable to require that an allocation of profit from transshipment should only be shared by retailers who participate in the transshipment. This, however, does not apply to the fractional allocation $\alpha_i^f(\vec{X}, \vec{D})$ because it is allowed to be a non-core allocation of the transshipment game as intended in ABZ. Recall that $\alpha_i^f(\vec{X}, \vec{D})$ may be negative. The retailer with negative $\alpha_i^f(\vec{X}, \vec{D})$ would be better off not joining the coalition for transshipment. With above understanding, we modify the allocation $\alpha_i^m(\vec{X},\vec{D})$ as follows:

$$\alpha_i^{\tilde{m}}(\vec{X},\vec{D}) = \begin{cases} \alpha_i^f(\vec{X},\vec{D}) + \alpha_i^d(\vec{X}^{c^*},\vec{D}) - \alpha_i^f(\vec{X}^{c^*},\vec{D}) & \\ +\frac{1}{\phi}\sum_{j\in N\backslash\Phi_{\vec{X},\vec{D}}}[\alpha_j^f(\vec{X},\vec{D}) + \alpha_j^d(\vec{X}^{c^*},\vec{D}) & \\ -\alpha_j^f(\vec{X}^{c^*},\vec{D})], & \text{if } i \in \Phi_{\vec{X},\vec{D}} \\ & \\ 0 & \text{otherwise} \end{cases}$$

where $\Phi_{\vec{X},\vec{D}}$ is a set of retailers who are involved in the transshipment solution after realization, and ϕ is the number of retailers in $\Phi_{\vec{X},\vec{D}}$. The amount that would have been paid to retailers not involved in the transshipment solution is distributed equally among retailers who are involved in the transshipment solution. Note that this allocation $\alpha_i^{\tilde{m}}(\vec{X},\vec{D})$ is always in the core of the transshipment game if the first-best solution $\vec{X}^{c^*}$ is played because the value of $\alpha_j^d(\vec{X}^{c^*},\vec{D})$ is zero for all retailer $j \in N\backslash\Phi_{\vec{X},\vec{D}}$.

Proposition 2.5. *The Nash equilibrium solution using the allocation rule $\alpha_i^{\tilde{m}}(\vec{X},\vec{D})$ is not necessarily the first-best solution.*

Proof. We derive the payoff function $P_i(\vec{X},\vec{D})$ as:

$$P_i(\vec{X},\vec{D}) = \begin{cases} \gamma_i P_N^c(\vec{X},\vec{D}) + \alpha_i^d(\vec{X}^{c^*},\vec{D}) - \alpha_i^f(\vec{X}^{c^*},\vec{D}) & \\ +\frac{1}{\phi}\sum_{j\in N\backslash\Phi_{\vec{X},\vec{D}}}[\alpha_j^f(\vec{X},\vec{D}) + \alpha_j^d(\vec{X}^{c^*},\vec{D}) & \\ -\alpha_j^f(\vec{X}^{c^*},\vec{D})], & \text{if } (i \in \Phi_{\vec{X},\vec{D}}) \\ & \\ [r_iB_i + v_iH_i - c_iX_i] & \text{otherwise} \end{cases}$$

We further reduce it to $P_i(\vec{X},\vec{D}) = \gamma_i P_N^c(\vec{X},\vec{D}) + \alpha_i^d(\vec{X}^{c^*},\vec{D}) - \alpha_i^f(\vec{X}^{c^*},\vec{D}) + Q_i(\vec{X},\vec{D})$ where

$$Q_i(\vec{X},\vec{D}) = \begin{cases} \frac{1}{\phi}\sum_{j\in N\backslash\Phi_{\vec{X},\vec{D}}}[\alpha_j^f(\vec{X},\vec{D}) + \alpha_j^d(\vec{X}^{c^*},\vec{D}) & \\ -\alpha_j^f(\vec{X}^{c^*},\vec{D})] & \text{if } (i \in \Phi_{\vec{X},\vec{D}}) \\ & \\ -\alpha_i^f(\vec{X},\vec{D}) - \alpha_i^d(\vec{X}^{c^*},\vec{D}) + \alpha_i^f(\vec{X}^{c^*},\vec{D}) & \text{otherwise} \end{cases}$$

The best-response correspondence (the inventory ordering strategy) of each player is a strategy that maximizes the expected payoff $J_i(\vec{X}) = E_{\vec{D}}(P_i(\vec{X}, \vec{D}))$. It is not the same as maximizing the centralized profit $P_N^c(\vec{X}, \vec{D})$ because $E_{\vec{D}}(Q_i(\vec{X}, \vec{D}))$ is not constant as in (9). Hence, with the allocation $\alpha_n^{\tilde{m}}(\vec{X}, \vec{D})$, the first-best solution will not necessarily be the one selected. $\square$

Proposition 2.5 points out a potential misuse with respect to allocation $\alpha_i^m(\vec{X}, \vec{D})$. Consider the case that one (or more) retailer chooses to order inventory that does not correspond to the first-best strategy. In that case, for the allocation $\alpha_i^m(\vec{X}, \vec{D})$, the corresponding SAG game is likely to have an empty core because the additional SAG profit may be shared with retailers who do not participate in the corresponding transshipment solution.

2.4.2. Effect of Non-Nash Strategy

If we assume a complete information DDS, then we expect retailers to order the first-best inventory levels in the first-stage and expect that the modified allocation $\alpha_i^m(\vec{X}, \vec{D})$ is in the core of the transshipment game. However, as Aumann (1997a) states, "polls and laboratory experiments indicate that people often fail to conform to some of the basic assumptions of rational decision theory." Aumann and Maschler (1995) also claim that "unlike the situations treated in classical game theory, a participant in a real-life conflict situation usually lacks information on the strategies that are available to him and to his opponent, on the actual outcomes and their utility to each of the participants, and on the amount of information that the other participants possess." With that in mind, we examine in this section the sensitivity of allocation $\alpha_i^m(\vec{X}, \vec{D})$ to non-Nash strategies.

To analyze the effect of playing a non-PSNE strategy, we need to examine the issue of implementation and demand realizations. In centralized inventory game literature, it is proven that some cost games only have a nonempty core for the expected cost game (Hartman and Dror, 2005). For any specific demand realization, the core of the game is likely to be empty. Fortunately, DDS do not have that pitfall because the core of the second-stage transshipment game is always nonempty regardless of demand realization. The allocation based on dual prices always belongs to the core. But, there is another issue to consider. When retailers agreed on the profit allocation ex-ante, the profit allocation in the core of SAG is calculated in an expectation. Once the demand is realized at every retailer, the retailers would reevaluate the allocation (ex-post) given the actual demand and inventory levels chosen in the first stage. If the chosen inventory levels are at unique PSNE, ABZ's allocation is guaranteed to be in the core. Otherwise, it is not. What are the implications if a retailer chooses to order non-PSNE inventory level in the first-stage? In such case, we examine if allocation $\alpha_i^m(\vec{X}, \vec{D})$ stays in the core of the realized transshipment game.

Example 2.3. *Consider the numerical example 3 provided in ABZ. Recall that cost structure is as follows: for* $i = 1, 2,$ $r_i = 10, c_i = 1.2, v_i = -1, t_{1,2} = 1,$ *and* $t_{2,1} = 2.$ *The demands are assumed to be independent and uniformly distributed between [0,100]* *at each separate retailer. The inventory position based on the allocation $\alpha_i^m(\vec{X},\vec{D})$ with $\gamma_i = 0.5$ is (76.81,62.35), i.e., the (first-best) Nash equilibrium inventory levels $X_1^{c^*} = 76.81$ and $X_2^{c^*} = 62.35$. Let us assume that an extreme scenario happens such that retailer 1 does not order any units of inventory. Let us also assume that retailer 2 orders 62.35 units of inventory. Pick arbitrary 75 and 70 for the demand realized at retailers 1 and 2, respectively. In this case, there is no transshipment since both retailers face shortages. Note that the resulting profit $548.68 is generated by retailer 2 alone. Given the allocation $\alpha_i^m(\vec{X},\vec{D})$ with $\gamma_i = 0.5$ as proposed in ABZ, retailer 1 would receive $337.06 from retailer 2. As a result, retailer 2 would want to break from cooperation as she can do better on her own. Thus, the allocation $\alpha_i^m(\vec{X},\vec{D})$ is not always in the core of the corresponding transshipment game. Consequently, the following observation is stated without proof.*

Observation 2.1. *At non-PSNE inventory position, the allocation $\alpha_i^m(\vec{X},\vec{D})$ is not always in the core of the transshipment game when $\vec{X} \neq \vec{X}^{c^*}$. That is, there exists $S \subseteq N$ such that*

$$\sum_{i \in S} \alpha_i^m(\vec{X},\vec{D}) < W_S(\vec{X},\vec{D})$$

and

$$\sum_{i \in N} \alpha_i^m(\vec{X},\vec{D}) = W_N(\vec{X},\vec{D})$$

where $W_S(\vec{X},\vec{D})$ is the maximum amount of profit that could be generated during the transshipment game by coalition S given inventory position $\vec{X}$ and demand $\vec{D}$, *and $W_N(\vec{X}, \vec{D})$ is the maximum amount of profit that could be generated during the transshipment game by the grand coalition N.*

Proof. Recall that

$$\sum_{i \in S} \alpha_i^m(\vec{X}, \vec{D}) \;=\; \sum_{i \in S} \alpha_i^f(\vec{X}, \vec{D}) + \sum_{i \in S} \alpha_i^d(\vec{X}^{c^*}, \vec{D}) - \sum_{i \in S} \alpha_i^f(\vec{X}^{c^*}, \vec{D}).$$

Consider the case where inventory level $X_i = X_i^{c^*}$ with demand realization $D_i > X_i$ for all $i \in S$, and inventory level $X_j < X_j^{c^*}$ with demand realization $D_j > X_j$ for all $j \in N \backslash S$. In this case, there is no transshipment for this realization as every retailer has shortage and $W_S(\vec{X}, \vec{D}) = 0$. Assume also that $D_j < X_j^{c^*}$ for all $j \in N \backslash S$ which means that if retailers in $N \backslash S$ ordered the PSNE inventory level, they would have surplus and there would be transshipment. Then,

$$\sum_{i \in S} \alpha_i^f(\vec{X}^{c^*}, \vec{D}) > \sum_{i \in S} \alpha_i^f(\vec{X}, \vec{D}).$$

In that case, it is possible like in example above that $D_i - X_i^{c^*} > X_j^{c^*} - D_j$ for $i \in S$ and $j \in N \backslash S$ such that

$$\sum_{i \in S} \alpha_i^d(\vec{X}^{c^*}, \vec{D}) < \sum_{i \in S} \alpha_i^f(\vec{X}^{c^*}, \vec{D}) - \sum_{i \in S} \alpha_i^f(\vec{X}, \vec{D}).$$

As a result, we will have $\sum_{i \in S} \alpha_i^m(\vec{X}, \vec{D}) < 0 \leq W_S(\vec{X}, \vec{D})$ and that $\alpha_i^m(\vec{X}, \vec{D})$ proposed by ABZ is not in the core of the transshipment game. $\square$

Note that the discussion in this section has a flavor of *open-loop* strategies. In noncooperative setting, open-loop strategies are functions of calendar time alone, as oppose to closed-loop strategies which are functions of calendar time and the history of play until that date. According to Fudenberg and Tirole (1991), "If the players can condition their strategies on other variables in addition to calendar time, they may prefer not to use open-loop strategies in order to react ... to possible deviations by their rivals from the equilibrium strategies." In our setting, we assume that the second-stage transshipment profit allocation rule is decided ex-ante among retailers. So, we can consider the open-loop noncooperative strategy of retailer i as to cooperate with the grand coalition and receive the payoff of α_i^m in the second-stage transshipment game. This strategy might not be optimal if some retailers do not choose PSNE inventory position and retailer i might be better off breaking from the grand coalition if she has an option to do so in response to other retailers' deviations. Moreover, we can say that the two-stage strategy tuple (Ordering PSNE inventory level in the first stage, Cooperating with grand coalition in the second stage) is not a subgame-perfect equilibrium because retailers do not respond optimally to unanticipated strategy deviations.

On the other hand, consider DDS that uses dual allocation rule. The two-stage open-loop strategy tuple of this game (Ordering PSNE inventory level in the first stage, Cooperating with grand coalition in the second stage) is a subgame-perfect equilibrium because retailers still respond optimally since the dual allocation is always in the core of the transshipment game.

2.5. Allocation Rules and Incentive Compatibility

The above DDS is modeled based on a number of assumptions. The assumption discussed in this section is that of complete information. We assume that retailers share their information of unit revenue r_i, unit cost c_i, unit salvage value v_i, transshipping cost $t_{i,j}$, and distribution of demand D_i in the first stage. This information is considered common knowledge and the retailers have a right to order any inventory levels they prefer. In this two-stage game, such complete information assumption might be difficult to verify, especially information related to distribution of demand. It might be difficult to check whether a retailer lies about her distribution of demand. However, it is important to know whether a retailer has an incentive to lie.

Consider retailer A who shares her information with a number of other retailers with an agreement to cooperate in the transshipments stage. Assume that before ordering the inventory in the first stage based on demand parameters μ_A, σ_A, she learns that her demand has changed so that $\tilde{\mu}_A > \mu_A$ and $\tilde{\sigma}_A < \sigma_A$. She faces a number of options. We consider only the three options below.

(a) She chooses not to inform others about the change in demand distribution parameters and maximize her expected profit (say, using a newsvendor model) given that she will not join any coalition in the second stage.

(b) She chooses not to inform others about the change. She, then, assumes that the other retailers will choose their inventory levels based on the first-best/PSNE solution $\vec{X}^{c^*}$ calculated using the original demand distribution. Hence, she will choose an inventory level X_i that maximizes her expected profit, including profit from transshipment, based on her private updated demand distribution.

(c) She chooses to inform others about the change and chooses an optimization model that maximizes her expected profit assuming that the other retailers will choose their inventory levels based on the first-best/PSNE solution $\vec{X}^{c^*}$ calculated using the updated demand distribution.

We assume for cases (b) and (c) that all retailers will share all shortage/surplus for transshipment and omit the possibility that some retailers may hold back shortage/surplus. This option was discussed in Granot and Sošić (2003) that considers DDS as a three-stage model. Retailers choose inventory levels in the first stage. After demand is realized, each retailer fulfills her local demand and at the second stage decides how much shortage/surplus she should share with the other retailers. At the third stage, the collaborated transshipment decisions take place. The main result of Granot and Sošić (2003) is that dual allocation rules may induce retailers not to share their shortage/surplus. Granot and Sošić (2003) analyze allocation rules on completely sharing property, value preserving property, and efficient property. An allocation rule is called completely sharing if it induces all the retailers to share their total residual supply/demand with other retailers, it is called value preserving if it induces all the retailers to share their residual supply/demand in amounts that do not result in a decrease in the total transshipment profit, and it is called efficient if the full amount of transshipment profit is allocated to retailers. Based on their model, Granot and Sošić (2003) propose a fractional allocation rule that is efficient value preserving, is a Nash equilibrium profile, and induces a first-best solution, but may not always be in the core of the third stage transshipment game. (See Granot and Sošić, 2003, Theorem 13.) We assume that all retailers are binded ex-ante by contract to share all of their shortage/surplus for transshipment. Such ex-ante contractual agreement is reasonable; it induces maximum total profit from transshipment for each demand realization. This is different than the option of reneging after the demand is realized allowed in Granot and Sošić (2003).

For the cases (b) and (c), we ask whether retailer A has an incentive to keep private her new distribution information. This implies testing whether a proposed allocation satisfies *incentive compatibility* property. In mechanism design, when a competitive game has incentive compatibility property, incentive for every player of telling the truth is of higher utility than telling a lie. Note that the first stage of our game is competitive in nature, none of the retailers are obligated to disclose their information. In our DDS, the allocation with incentive compatibility property should encourage retailers to choose option (c) described above. On the other hand, if options (a) or (b) are better than option (c), then retailer A has an incentive not to share the information about her true distribution of demand.

Unfortunately, the allocation proposed in ABZ does not have the incentive compatibility property.

Observation 2.2. *Assume that all retailers will share all shortage/surplus for transshipment. If the demand distribution is not of common knowledge, DDS that adopts ABZ's allocation rules is not necessarily incentive compatible.*

Proof. An incentive compatible allocation required that, for individual player, telling the truth is more profitable to her than telling a lie. Consider a decentralized distribution system that adopts ABZ allocation rules. Let the true cumulative demand distribution be $F^{true}(\vec{D})$. Assume this $F^{true}(\vec{D})$ is not a common knowledge and is only known to retailer i. Thus, the distribution of demand assumed by all the players but i is $F^{false}(\vec{D})$. We prove that the profit for i is greater or equal under i's knowing $F^{true}(\vec{D})$ and the rest of players assuming $F^{false}(\vec{D})$ than when all players know $F^{true}(\vec{D})$.

Let the first-best Nash equilibrium inventory level for the decentralized distribution system with cumulative demand distribution $F^{true}(\vec{D})$ and $F^{false}(\vec{D})$ be $\vec{X}^{true} = (\vec{X}^{true}_{N\setminus i}, X^{true}_i)$ and $\vec{X}^{false} = (\vec{X}^{false}_{N\setminus i}, X^{false}_i)$, respectively.

Let $\vec{X}^{response} = (\vec{X}^{false}_{N\setminus i}, X^{response}_i)$ be an inventory level when only retailer i has the knowledge of true demand distribution $F^{true}(\vec{D})$ while all other retailers assume the demand distribution is $F^{false}(\vec{D})$. Essentially, X^{true}_i and $X^{response}_i$ are two point on the best response function of retailer i. We prove that, there exists $F^{false}(\vec{D})$ such that $J_i^{response} - J_i^{true} > 0$, where

$$\begin{aligned}
J_i^{true} &= \max_{X_i} \int_{\mathbb{R}_+^N} P_i((\vec{X}_{N\backslash i}^{true}, X_i), \vec{D}) dF^{true}(\vec{D}) \\
&= \int_{\mathbb{R}_+^N} P_i(\vec{X}^{true}, \vec{D}) dF^{true}(\vec{D}) \\
&= \int_{\mathbb{R}_+^N} \left[r_i B_i^{true} + v_i H_i^{true} - c_i X_i^{true}\right] + \alpha_i^m(\vec{X}^{true}, \vec{D}) dF^{true}(\vec{D}) \\
&= \int_{\mathbb{R}_+^N} \left[r_i B_i^{true} + v_i H_i^{true} - c_i X_i^{true}\right] + \alpha_i^d(\vec{X}^{true}, \vec{D}) dF^{true}(\vec{D})
\end{aligned}$$

and

$$\begin{aligned}
J_i^{response} &= \max_{X_i} \int_{\mathbb{R}_+^N} P_i((\vec{X}_{N\backslash i}^{false}, X_i), \vec{D}) dF^{true}(\vec{D}) \\
&= \int_{\mathbb{R}_+^N} P_i(\vec{X}^{response}, \vec{D}) dF^{true}(\vec{D}) \\
&= \int_{\mathbb{R}_+^N} r_i B_i^{response} + v_i H_i^{response} - c_i X_i^{response} + \alpha_i^m(\vec{X}^{response}, \vec{D}) dF^{true}(\vec{D}) \\
&= \int_{\mathbb{R}_+^N} \gamma_i P_N^c(\vec{X}^{response}, \vec{D}) + \alpha_i^d(\vec{X}^{false}, \vec{D}) - \alpha_i^f(\vec{X}^{false}, \vec{D}) dF^{true}(\vec{D}) \\
&= \int_{\mathbb{R}_+^N} \gamma_i P_N^c(\vec{X}^{response}, \vec{D}) + \alpha_i^d(\vec{X}^{false}, \vec{D}) - \gamma_i P_N^c(\vec{X}^{false}, \vec{D}) \\
&\quad + \left[r_i B_i^{false} + v_i H_i^{false} - c_i X_i^{false}\right] dF^{true}(\vec{D}).
\end{aligned}$$

So, we have

$$\begin{aligned}
J_i^{response} - J_i^{true} &= \int_{\mathbb{R}_+^N} \left[\gamma_i P_N^c(\vec{X}^{response}, \vec{D}) - \gamma_i P_N^c(\vec{X}^{false}, \vec{D})\right] \\
&+ \left[\left[r_i B_i^{false} + v_i H_i^{false} - c_i X_i^{false}\right] + \alpha_i^d(\vec{X}^{false}, \vec{D})\right] \\
&- \left[\left[r_i B_i^{true} + v_i H_i^{true} - c_i X_i^{true}\right] + \alpha_i^d(\vec{X}^{true}, \vec{D})\right] dF^{true}(\vec{D}).
\end{aligned} \tag{2.2}$$

Consider the case with sufficiently small γ_i, so we can disregards the contribution of the term $\left[\gamma_i P_N^c(\vec{X}^{response}, \vec{D}) - \gamma_i P_N^c(\vec{X}^{false}, \vec{D})\right]$. Recall that X_i^{true} maximizes centralized profit and, at the same time, maximizes retailer i profit given ABZ allocation, but does not necessarily maximize retailer i profit given allocation based on dual price as shown on the third line of (2.2). Hence, if there is an inventory level $X_i^{false} = X_i^{\#}$ that maximizes retailer i profit given allocation based on dual price as shown on the second line of (2.2), then $J_i^{response} - J_i^{true} > 0$. With the knowledge of X_i^{false}, retailer i may create $F^{false}(\vec{D})$, share this false information about the distribution to other retailers, and enjoy higher profit than sharing truthful information. Hence, it is not guaranteed that telling the truthful cumulative distribution function will always give highest profit to retailer i. $\square$

Example 2.4. *We apply the idea to the numerical example 3 provided in ABZ that we mention earlier in Section 2.4. Recall that cost structure is as follows: for* $i = 1, 2$, $r_i = 10, c_i = 1.2, v_i = -1, t_{1,2} = 1,$ *and* $t_{2,1} = 2$. *Retailers agree on using ABZ's allocation rule with* $\gamma_i = 0.5$. *Let the distribution of demand assumed by retailer 2 be independent and uniform on [0,100] for both players, while retailer 1 knows that the true distribution of demand is uniform on [0,110] for herself and [0,100] for retailer 2. If retailer 1 communicates the true distribution of demand to retailer 2, then retailers 1 and 2 will order 84.20 and 62.00 units, and receive \$406.99 and \$375.16, respectively. If retailer 1 does not communicate the true distribution of demand to retailer 2, then retailers 1 and 2 will order 79.31 and 62.35 units, and receive \$407.23 and \$374.15, respectively. (Note that retailer 2 would not know that her expected* *profit is \$374.15) We can see that retailer 1 would choose to not communicate the true demand distribution to retailer 2.*

2.6. Relaxing the Assumption on Satisfying Local Demand First

Prior to this section, we assumed that retailer must satisfy local demand first and dispose (salvage) excess inventory only locally. However, in the real market, an independent retailer might choose to transship products to other retailers if doing so is more profitable to her. For example, retailer A makes \$5 profit per units when selling locally but earns \$7 when she transships her inventory to retailer B. Then, retailer A would transship her inventory to retailer B before satisfying her local demand. Similarly, if the salvage value at retailer A is lower than the salvage value at retailer B minus transshipment cost between them, then the retailer A would transship her excess inventory for disposal at retailer B. In this section, we model DDS by relaxing assumptions on satisfying local demand first and on disposing excess inventory only at local retailer. Then, we examine whether ABZ's allocation can still achieve the first-best profit. Thus, we extend the "range" of our analysis.

As before, competitive independent retailers face random demands. In the first stage, inventories are independently ordered based on anticipated demands. Assume that all inventory is stored locally. In the second-stage, these retailers use pooling of stocks, i.e., any demand at one retailer can be satisfied from inventory transshipped from other retailers. We assume that retailers will cooperate and make profit maximizing centralized decision to transship inventory to satisfy all demands in the system.

Assume N retailers as before. Let r_i, c_i, and v_i, where $i = 1, \ldots, n$, represent unit revenue, unit cost, and unit salvage value of a retailer i, respectively. Let $t_{i,j}$ represent the transshipping cost from retailer i to retailer j for all $i, j \in N$. In the first stage, each retailer makes decision on her inventory level. Let the vector $\vec{X} = (X_1, \ldots, X_n)$ denote the levels of inventory ordered in the first stage by all retailers. Then, the demand represented by the vector $\vec{D} = (D_1, \ldots, D_n)$ is realized at all retailers. Note that this decentralized distribution system reduces to the prior DDS if $r_i - c_i > r_j - c_i - t_{i,j}$ and $v_i - c_i > v_j - c_i - t_{i,j}$ for all $i, j \in N$.

Now consider a two-retailer case; if retailer 2 can order inventory at a cost c_2 that is higher than cost $c_1 + t_{1,2}$ of obtaining transshipment from retailer 1, one might misinterpret that retailer 2 would not order at all and let retailer 1 order for her. This is incorrect because the second-stage game is a cooperative game. The profit made from sales at retailer 2 must be shared with retailer 1 according to an agreed allocation rule. Retailer 2's share of profit per unit from transshipment might be lower than the profit per unit when retailer 2 sells from her own local inventory. Thus, the behavior of retailers depends on the allocation rules that they agreed on.

Assume that these retailers agree to allocate profit that is a result of transshipment game using allocation rule α^x. The profit expected at each retailer i is

$$J_i(\vec{X}) = E_{\vec{D}}(\alpha_i^x(\vec{X}, \vec{D}))$$

such that $\sum_{i \in N} \alpha_i^x(\vec{X}, \vec{D}) = \hat{W}_N(\vec{X}, \vec{D})$. The transshipment problem $\hat{W}_N(\vec{X}, \vec{D})$ is represented by:

$$
\begin{aligned}
\hat{W}_N(\vec{X}, \vec{D}) = \quad \max_{\vec{y}} \quad & \sum_{i \in N} \sum_{j \in N} (r_j - c_i - t_{i,j}) y_{i,j} + \sum_{i \in N} \sum_{j \in N} (v_j - c_i - t_{i,j}) z_{i,j} \\
\text{s.t.} \quad & \sum_{j \in N} (y_{i,j} + z_{i,j}) = X_i \text{ for all } i \in N \\
& \sum_{i \in N} y_{i,j} \leq D_j \text{ for all } j \in N \\
& \text{for all } y_{i,j}, z_{i,j} \geq 0.
\end{aligned}
$$

The quantity $y_{i,j}$ represents the number of units of product transshipped from retailer i to sell by retailer j in the second stage. The quantity $z_{i,j}$ represents the number of units of product transshipped from retailer i to dispose at retailer j in the second stage. Note that $\hat{W}_N(\vec{X}, \vec{D})$ is continuous in $\vec{X}$ because there is no fixed cost related to local profits or transshipment profits at any retailers.

At this point, we examine the conditions that result in the decentralized distribution system achieving the first-best profit. Similar to the previous model, we need (1) profit allocation to be in the core of the transshipment game, (2) a PSNE in the first stage, and (3) the first-stage inventory decisions to result in the same inventory levels as the centralized system. Consider ABZ's allocation which is a combination of fractional allocation and allocation based on dual prices of transshipment game. The local profit term in fractional allocation is zero because the expected profit only relies on the profit allocated from transshipment. The allocation based on dual prices of transshipment game $\alpha_i^d(\vec{X},\vec{D})$ is:

$$\alpha_i^d(\vec{X},\vec{D}) = \lambda_i X_i + \delta_i D_i.$$

Using ABZ's allocation rule, the side payment moves the profit allocation based on fractional rule to the allocation according to the dual price of the transshipment game. Therefore, ABZ's allocation will be in the core of the transshipment game if Nash equilibrium inventory level is chosen by each individual retailer as previously discussed in Section 2.4. The characteristic of the transshipment problem $\hat{W}_N(\vec{X},\vec{D})$ is still the same as before. The dual price for inventory disposed at the salvage value will be assigned to the original owner of the inventory, not the retailer where the inventory is disposed because there are no limits on the disposal capacity, a dual price of zero will be assigned to the retailer where the inventory is disposed.

We examine the individual expected profit $J_i(\vec{X})$ assuming that ABZ's allocation rule is applied. We know that if there exists a unique PSNE, then the first-stage inventory decisions result in the same inventory levels as the centralized system.

In terms of the existence of a PSNE, recall that if there exists a first-best solution, then there exists a PSNE. In this setting, the strategy space for inventory level is nonempty compact convex subsets of a Euclidean space. Moreover, there exists a first-best solution if $J_N^c(\vec{x})$ is continuous in $\vec{X}$.

In this case, the centralized profit $P_N^c(\vec{X},\vec{D})$ is equivalent to the transshipment profit $\hat{W}_N(\vec{X},\vec{D})$. The transshipment profit function is uniformly continuous because there is no fixed cost related to transshipment profits. It follows that $J_N^c(\vec{x})$ is continuous in $\vec{X}$.

In terms of the uniqueness of PSNE, the relaxed model has fewer requirements for uniqueness. There exists a unique PSNE if (i) there is no ridge present for $J_N^c(\vec{X})$ (see Section 2.3.2 for restriction on cost parameters), (ii) each demand density function is *strictly* log-concave, and (iii) nondegeneracy cost parameters are assumed, i.e., salvage value is less than unit cost, and for any two retailers i and j, $r_i - t_{j,i} \neq c_j$ and $r_j - t_{i,j} \neq c_i$.

We no longer have to require conditions for strict log-concavity of $P_N^c(\vec{X}, \vec{D})$ on each X_i and (weakly) log-concave in $(\vec{X}, \vec{D})$ as previously described in Propositions 2.3 and 2.4. This is because the solution of transportation problem already has those properties.

2.7. Discussion

DDSs with cooperative transshipment are undoubtedly an important research area. Retailers, such as a car dealership, find this type of practice attractive because it improves customer satisfaction, reduces excess inventory, and may potentially generate higher profit than traditional DDS without transshipment.

Past literature on DDS with cooperative transshipment adopted two key assumptions when analyzing DDS with cooperative transshipment. First, they assumed complete information because the demand distributions and accurate cost parameters of all retailers are crucial for calculation of optimal inventory. The complete information assumption is technically feasible in today's advanced information systems and supply chain management software. However, in practice voluntary complete information sharing arrangement among competing players is somewhat questionable. In Section 2.5, we emphasized that noncooperative incentives may result if the assumption of complete information does not hold. For instance, a player may be induced to lie about her true demand distribution parameters. In future research it would be interesting to examine different profit allocation rules with regards to properties such as incentive compatibility. Furthermore, if sharing of demand and pricing information is not completely free of charge, it might be valuable to consider mechanism design that ensures truth telling.

Another key assumption is that all players are individually rational in making their competitive decision in the first stage but are confined by an agreed ex-post transshipment profit allocation rule in the second stage. This is assumed even in the case that such profit allocation rule may not necessarily be the best course of action they could take in the second stage. For some allocation rules that are always in the core of the transshipment game, e.g., dual price allocation rule, this assumption generally holds, and the accuracy of expected profit calculation is not affected. But for other allocation rules such as ABZ rule, we show in Section 2.4 that if some players do not play PSNE strategy, the allocation may not necessarily be in the core of the transshipment game. Therefore, a number of players might be better-off without cooperation in the transshipment stage. When the cooperative outcome does not hold, the expected profit calculation is no longer straight forward. Further analysis is needed in this case.

With respect to ABZ's model, we have noted that a number of assumptions may influence its validity. For instance, the uniqueness of PSNE/first-best profit will be achieved only if demand distribution and cost parameters are restricted as we discussed in Section 2.3. It would be of interest to see an empirical study and compare real-life performance of this model to a traditional newsboy model with ad-hoc transshipment arrangement.

DDS is applicable to many industries and supply chain settings, not limited to just car dealerships. For instance, the operation of independent lumber companies in Scandinavia (that motivates the work of Sandsmark, 2009) and the pooling of spare parts inventories at air carrier companies in Brussels (Wong et al., 2006) are all of similar flavor. Since most realistic situations involve random variability in demand, even with the best forecasting technique, the likelihood of correctly matching demand with supply is not very promising. Shared resources and capabilities help companies cope effectively with unexpected or unusual demands for products and services.

2.8. Appendix

Proof of Lemma 2.2. The general idea is to show that, given condition (a), there is a point in the neighborhood that has equal or higher expected centralized profit at $\vec{X}^*$ that satisfies condition (b).

If only one of the retailers, i or j, changes her inventory by a small ϵ, $|\epsilon| > 0$, then the expected centralized profit $J_N^c(\vec{X})$ decreases because $\vec{X}^*$ was her best response given all other retailers choose $\vec{X}^*_{N\setminus\{i,j\}}$. In addition, ϵ would not change the situation of a retailer having shortage or overage.

For a positive ϵ, consider a neighborhood point $\vec{X}^{1\circ} = (X_i^* + \epsilon, X_j^* - \epsilon, \vec{X}^*_{N\setminus\{i,j\}})$ that represents the point after both retailers i and j change their inventories. Without finding the optimal transshipment pattern, one can calculate the lower bound $\underline{J}_N^c(\vec{X}^{1\circ})$ of expected centralized profit given that an increase in retailer i is transshipped to retailer j. Hence, almost surely (with probability 1)

$$\underline{J}_N^c(\vec{X}^{1\circ}) - J_N^c(\vec{X}^*) = (r_j - c_i - t_{i,j})\epsilon - (r_j - c_j)\epsilon = (c_j - c_i - t_{i,j})\epsilon$$

For a negative ϵ, consider another neighborhood point $\vec{X}^{2\circ} = (X_i^* + \epsilon, X_j^* - \epsilon, \vec{X}^*_{N\setminus\{i,j\}})$. The lower bound $\underline{J}_N^c(\vec{X}^{2\circ})$ of expected centralized profit given that a decrease in retailer i reduces transshipment to retailer j. In addition, retailer j increase in inventory is used towards her local demand. Hence, almost surely

$$\underline{J}_N^c(\vec{X}^{2\circ}) - J_N^c(\vec{X}^*) = -(r_j - c_i - t_{i,j})(-\epsilon) + (r_j - c_j)(-\epsilon) = (c_j - c_i - t_{i,j})\epsilon$$

Notice that if $(c_j - c_i - t_{i,j}) > 0$, a neighborhood point from adding a positive epsilon can improve upon $J_N^c(\vec{X}^*)$. On the other hand, if $(c_j - c_i - t_{i,j}) < 0$, a neighborhood point from adding a negative epsilon can improve upon $J_N^c(\vec{X}^*)$. Therefore regardless of cost parameters, there always exists a point in the neighborhood that has equal or higher expected centralized profit at $\vec{X}^*$. Hence, the ridge always exists. $\square$

Proof of Proposition 2.3. Let the set of X_i in the second region be

$$R_2 = \left\{ X_i : X_i \leq D_i \text{ and } X_i > D_i - \sum_{j \in N, j \neq i} y^*_{j,i} \right\}$$

and the set of X_i in the third region be

$$R_3 = \left\{ X_i : X_i > D_i \text{ and } X_i \leq D_i + \sum_{j \in N, j \neq i} y^*_{i,j} \right\}.$$

To ensure that the profit function is strictly log-concave, we need the minimum slope in the second region to be greater than or equal to the maximum slope in the third region. That is,

$$\rho^\circ - \max_{X_i \in R_2} \frac{\partial W_N(\vec{X}, \vec{D})}{\partial E_i} \geq \max_{X_i \in R_3} \frac{\partial W_N(\vec{X}, \vec{D})}{\partial H_i}. \tag{2.3}$$

We know that $\rho^\circ = r_i - c_i$. The upper bound of $\max_{X_i \in R_2} \frac{\partial W_N(\vec{X}, \vec{D})}{\partial E_i}$ is $\max_k (r_i - v_k - t_{k,i}) + (v_k - c_k)$; that is the most profit per unit made from transshipping to retailer i. So, the lower bound of the left-hand-side term is $r_i - c_i - \max_k (r_i - c_k - t_{k,i})$.

For the right-hand-side, the upper bound of $\max_{X_i \in R_3} \frac{\partial W_N(\vec{X}, \vec{D})}{\partial H_i}$ is $\max_j (r_j - v_i - t_{i,j}) + (v_i - c_i)$, i.e., the most profit made from transshipping from retailer i. So, the upper bound of the right-hand-side term is $\max_j (r_j - c_i - t_{i,j})$.

Rearrange (2.3), to get

$$\begin{aligned} r_i - c_i - \max_k (r_i - c_k - t_{k,i}) &\geq \max_j (r_j - c_i - t_{i,j}) \\ r_i - c_i &\geq \max_k (r_i - c_k - t_{k,i}) + \max_j (r_j - c_i - t_{i,j}). \end{aligned} \tag{2.4}$$

Hence, to ensure that the profit function is strictly log-concave we require the cost parameters to imply that, for any retailer, it is more profitable to satisfy local demand from its own inventory, rather than to ship from other retailer and transshipping its own inventory to sell at yet another retailer. □

If (a) there exist retailers j and k such that $r_i - c_i < (r_j - c_i - t_{i,j})$ and $r_i - c_i < (r_i - c_k - t_{k,i})$, then the lower bound of the left-hand-side term of (2.4) is a negative value and the slope in the third region is strictly positive and greater than $r_i - c_i$. So, we might have a downward slope in the second region and upward slope in the third region as shown in Figure 2.4. In such case, the profit function is neither concave nor quasi-concave (hence, not log-concave).

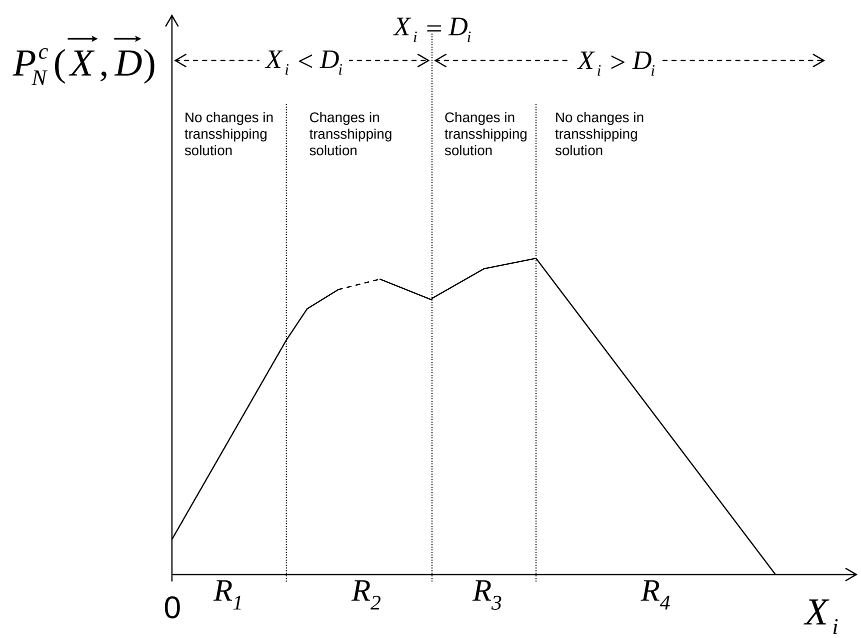

Figure 2.4. Non-concave centralized profit function

However, if (b) there exist retailers j and k such that $r_i - c_i < (r_j - c_i - t_{i,j})$ but $r_i - c_i \geq (r_i - c_k - t_{k,i})$, then we have a upward slope in the second region but the slope in the third region might be higher. The profit function is log-concave (and quasi-

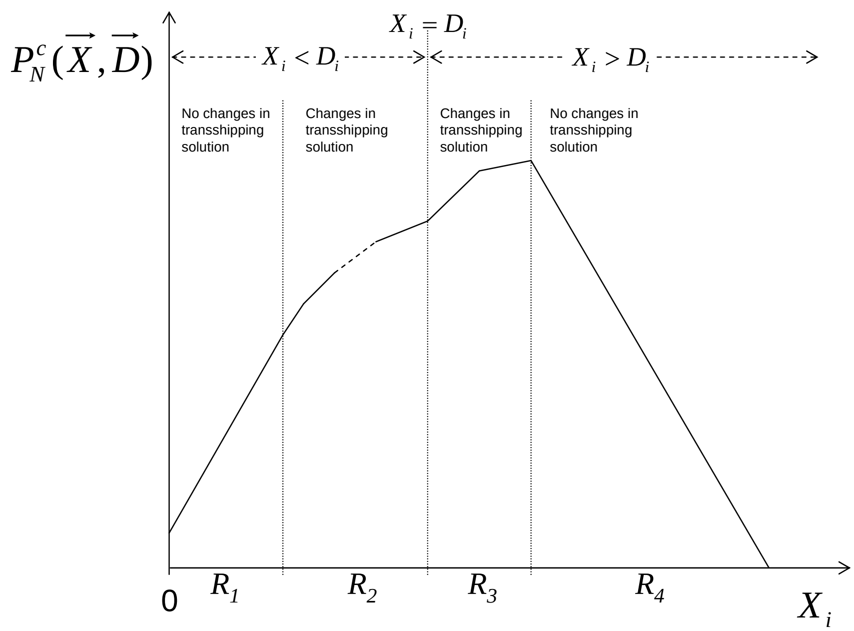

Figure 2.5. Quasi-concave centralized profit function

concave) as shown in Figure 2.5. For the two cases: (c) $r_i - c_i \geq \max_j (r_j - c_i - t_{i,j})$ and $r_i - c_i < \max_k (r_i - c_k - t_{k,i})$, and (d) $r_i - c_i \geq \max_j (r_j - c_i - t_{i,j})$ and $r_i - c_i \geq \max_k (r_i - c_k - t_{k,i})$, the strict log concavity in each X_i depends on the cost parameters and could go either way.

3. STOCHASTIC PROGRAMMING FRAMEWORK FOR DECENTRALIZED INVENTORY WITH TRANSSHIPMENT

3.1. Introduction

The query in this paper pertains to a generic collection (a group) of independent retailers, like independent car dealerships, fresh strawberry stands, aviation parts merchants, and others, who decide, each one independently, about their individual procurement strategy for a single product of uncertain demand. The retailers' initial procurement decisions account for all the available options beyond the point of learning the actual demand values. For instance, a retailer can anticipate cooperation ex post with other retailers by means of a transshipment mechanism (inventory swaps), etc.

We propose to view this basic setting and all its problems through the prism of traditional stochastic programming methodology. More to the point, we view the inventory procurement options of a retailer competing with similar retailers as a strategic decision of how much to order at the outset (ex ante) without knowing the exact nature of her demand. Nevertheless, it is the presumed that the retailer has the option of communicating and collaborating with her competitor(s) at a later stage and ex post responding in a variety of ways to one's true realized demand.

Stochastic programming is a mathematical methodology for solving optimization problems with time dependent stochastic variables representing uncertainty of future events. The methodology was introduced by Dantzig (1955); Beale (1955); Walkup and Wets (1967) among others. For basic exposition, definitions, concepts, and additional references, see among other introductory books the text by Birge and Louveaux (1997).

Stochastic programming model envisions two (or more) decision stages: Given a probability space (Ω, Ξ, P) and certain known initial information, e.g., cost parameters, distances, and as in our case, demand uncertainty parameters, the *first-stage decisions* $x \in X$ are taken at a cost $f^1(x)$ without full information regarding the random variable $\tilde{\xi}$ (the demand). Subsequently, after executing x, the uncertainty event - the true value of $\tilde{\xi}$ is revealed (the instantiation of $\tilde{\xi}$ is denoted by ξ). At that point, another set of decisions $y \in Y$ is taken at a cost $f^2(y, x, \xi)$ as a response to the value ξ and the first stage decision x.

The common objective is to minimize the expected cost; this process is modeled in a form of an optimization problem for determining x that minimizes the following:

$$\min_x \quad f^1(x) + E_{\tilde{\xi}}[Q(x, \xi)], \text{ s.t.} \quad x \in X$$

where X is the feasible set for x and the *recourse function* (in linear form) is:

$$\begin{aligned} Q(x,\xi) = \min_{y} \quad & f^2(y,x,\xi) \\ \text{s.t.} \quad & t_i^2(x,\xi) + g_i^2(y,\xi) \leq 0, i = 1,\ldots,\bar{m}_2 \\ & t_i^2(x,\xi) + g_i^2(y,\xi) = 0, i = \bar{m}_2+1,\ldots,m_2. \\ & y \in Y. \end{aligned}$$

The common assumptions for SP model are: the functions $f^2(\cdot,\cdot,\xi)$, $t_i^2(\cdot,\xi)$, and $g_i^2(\cdot,\xi)$ be continuous for any value ξ. It is also assumed that $Q(x,\xi)$ is measurable for all $x \in X$ and $\xi \in \Xi$, and the probabilistic description of the random variables $\tilde{\xi}$ is available in a form of probability distributions with known parameters.

As many stochastic programming professionals know well, for a majority of real-life problems, after the realization of the random variables, the second-stage decision domain Y can be very large and computationally "unmanageable". To mitigate this problem, one has to resort to restrictions in the recourse options by limiting the recourse policies in the second stage. We have to consider carefully what restrictions lead to computational resolution of the strategy domain and potential real-life implementation of effective solutions. This is an inseparable part of a real-life operational system design. In this exposition we present a systematic exploration process (layout of a taxonomy) of the different recourse restrictions that are applicable in the case of decentralized inventory with transshipment. The presumed aim of imposing recourse restrictions in the second stage of our stochastic programming model is conceptual and mathematical "tractability" that retains a meaningful modeling interpretation.

A loose meaning of mathematical tractability is being able to prescribe an algorithmic approach that allows for the computation of solutions (either exact or approximate) in a reasonable time. Not every real-life decentralized inventory procurement problem with all possible decision options allows for computational solvability. There are important trade-offs between allowing a richer set of recourse options and being able to compute a solution. For instance, allowing retailer i to consider any subset of retailers for cooperation ex post brings forth from the outset the issue of real-life computability. To illustrate our concept of restricted recourse aimed to improve computational tractability more clearly at a price of sacrificing optimality we point to stochastic demand problem in Dror et al. (1989), where a recourse restriction is imposed on vehicle routing options versus Dror (1993) where such recourse restriction is removed. Thus, one of the aims of this paper's taxonomy is also to expose the reader to the modeling options connected through recourse assumption that allow to consider computational tractability in tandem with the expressiveness of the modeling representation.

Our contribution can be summarized as follows: First, we propose a single, explicit, well understood, methodological prism of stochastic programming with recourse for an important large family of supply chain problems that in the past have been modeled using ad-hock methods or modeling points of view. The proposed taxonomy graph provides a natural framework for a systematic examination of operational options at different junctions, their mathematical consequences/models, and available solution properties. At the same time we indicate modeling limitations and open research venues for a number of models examined in the past. In addition, we contribute some results and questions related to the adequate pricing of the swapped items and the existence of Walrasian equilibria for the salvage goods market.

Decentralized inventory systems have been studied by a considerable number of various authors under many specific assumptions. At a risk of being too "narrow" with the citations of related work, we only mention the more recent papers, since the vast body of literature does not allow in a single paper to "do" justice to all pertinent contributions.

For a start, Anupindi et al. (2001) analyze two-stage systems with an assumption that retailer's local demand must be satisfied first before she can transship overage to other retailers. In addition, Anupindi et al. (2001) assume that the retailers do not hold back shortage or overage when transshipment decisions are made. Retailers participation is non-binding, hence Anupindi et al. (2001) restrict their focus on profit allocation schemes that are stable. The allocation schemes they studied include allocation based on dual prices and modified fractional allocation (see Section 3.3.2). Granot and Sošić (2003) followed by removing the assumption that the retailers fully contribute their shortage or overage. They model their problem version as a "three-stage" problem. Other papers on a related topic of product substitution (Lippman and McCardle, 1997; Netessine and Rudi, 2003) assume that only fraction of customers will accept transshipped/substituted goods. In this context Lippman and McCardle (1997) analyze both a duopoly and an n-firm (oligopoly) model of inventory competition. In such setting, Netessine and Rudi (2003) show conditions for uniqueness of Nash equilibrium. In terms of second stage transshipment, Stuart (2005) considers duopoly model of inventory competition, but assumes stable price competition in the second stage. This differs from the transshipment price coordination as described by Rudi et al. (2001), that aims to set transshipment prices that maximize systems' expected profit. This issue of transshipment prices maximizing systems' expected profit is further taken up in Hu, Duenyas, Kapuscinski (2007). However our interest is in a market clearing prices as in Walrasian equilibrium price. Apart from decentralized inventory system, stochastic programming methodology has also been used to analyze centralized inventory system with demand uncertainty (Chen and Zhang, 2009; Ozen et al., 2009). Since this paper puts forth a unifying framework for a large variety of decentralized procurement models, we point to another paper that has proposed a more comprehensive framework for both procurement and capacity decisions under the heading of *newsvendor networks* (Van Mieghem and Rudi (2002)). Our goal is less ambitious than that of Van Mieghem and Rudi (2002) since we present a more detailed unifying view of basic, single period problems, without extending it to dynamic settings. We do not presume that the cited papers are fully inclusive but just a representative sample. For a more comprehensive review and listing of references for the broader area of inventory games and cooperation, see Nagarajan and Sošić (2008).

3.2. The Model

Consider a decentralized inventory distribution system such that for each retailer the (individual) generic stochastic programming model corresponds to a two stage stochastic program. The details of the decisions, random variables, and the process are as follows: Let $N = \{1, \ldots, n\}$ be the index set of the retailers ($n \geq 2$) and consider a decentralized inventory distribution system of n competing retailers. Each of the retailers responds to an uncertain demand vector $\vec{d} = (d_1, \ldots, d_n)$ that, in the second stage, takes on a (realized) value $\vec{d'} = (d'_1, \ldots, d'_n) \in (D_1 \times \ldots \times D_n)$, where each D_i, $i \in N$ is a given nonempty subset of $\mathbb{R}_+$ (say, contiguous intervals of the nonnegative real line). Unit costs, consumer selling prices, salvage values, and the transshipment cost matrix is assumed fixed for now (deterministic), and known to all retailers. Let r_i, c_i, v_i, $i \in N$, represent unit revenue, unit cost, and unit salvage value of a retailer $i, (i \in N)$, and t_{ij} denote the transshipping cost from retailer i to retailer j.

In the first stage, each retailer makes an individual inventory procurement decision. Let $x_i \in X_i$ denote the inventory quantity ordered by retailer i in the first stage. Assume that X_i is a given nonempty compact convex subset (say, an interval) of $\mathbb{R}_+$ and that $D_i \subseteq X_i$. From the perspective of retailer i, her objective is to choose an order quantity x_i that maximizes her expected profit, assuming that she is aware of other (similar) retailers in the system who are making the same type of decisions. Let $\Pi_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$ denote the expected profit for retailer

i when each retailer $k \in N$ respectively orders a nonnegative inventory quantity x_k. Let $\Pi_i^*(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)$ denote the optimized value of $\Pi_i(\cdot)$ for retailer i in response to the inventory values $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ selected by the other $n-1$ retailers. That is,

$$\Pi_i^*(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n) = \max_{x_i \in X_i} -c_i x_i + \mathbb{E}_{\vec{d}}\left[Q_i(\vec{x}, \vec{d'})\right], \tag{3.1}$$

where $\mathbb{E}_{\vec{d}}\left[Q_i(\vec{x}, \vec{d'})\right]$ is retailer i's second-stage expected profit. In terms of the chronology of events, the following unfolds: First, retailer i orders x_i at a cost of c_i per unit. Simultaneously, other retailers also individually order their inventories $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$. Next, after observing demand realization (denoted as $\vec{d'}$), retailer i responds to her local demand and the realized demand of the other retailers and decides on recourse action. That is, she decides how to respond to the demand that actually occurred in the system.

In an aside note; the default degenerate problem version, not considered here, is the case when excess demand is lost and excess inventory is simply salvaged – the classical newsvendor problem.

To relate the first-stage decision problem to Cournot oligopoly, we can think of (3.1) as retailer i's utility for choosing inventory procurement strategy x_i', that is the optimal solution of the program in (3.1), given that other retailers choose $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Formally, we can write retailer i's utility function as $\Pi_i(x_1, \ldots, x_n) = -c_i x_i + \mathbb{E}_{\vec{d}}\left[Q_i(\vec{x}, \vec{d})\right]$.

We use a tuple $(X_1, \ldots, X_n; \Pi_1, \ldots, \Pi_n)$ to denote this competitive inventory game. The inventory procurement strategies $(x_1^*, \ldots, x_n^*)$ are a *pure strategy Nash equilibrium* (PSNE) if, for each retailer i, x_i^* is retailer i's best response to the strategies $(x_1^*, \ldots, x_{i-1}^*, x_{i+1}^*, \ldots, x_n^*)$ chosen by other retailers; that is, x_i^* solves

$$\max_{x_i \in X_i} \Pi_i(x_1^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_n^*).$$

Solving (3.1) for PSNE is not our main focus here. Note however that a system of n stochastic programming models must be solved simultaneously to obtain the PSNE(s) — the equilibrium inventory procurement strategy.

Gaining a better understanding of the different potential connections between the real-life second-stage expected profits options in (3.1) and their mathematical formulations is the main objective of this paper. In a decentralized inventory management operations each retailer is faced with a wide range of decision options. Say a retailer is committed to local demand first (thus, restricting her recourse range) and after demand is realized ends up with overage. At this point, there might exist a variety of alternatives for her to consider. She may contact other retailers, some of whom are short, to form a coalition for the redistribution (transshipment) of excess inventory that results in a more profitable matching of overage with shortage. Such a coalition has to decide (ex ante or ex post) on how the additional realized profit ought to be shared. The retailer may choose to announce the price she is willing to sell her overage. She may decide to keep her valuation confidential and ask for bids, or consider announcing her overage information and valuation. She may have an ex-ante "understanding" with other retailers in regards to overage and shortage and examine her other alternatives – say the option of defection. To keep the second-stage model mathematically tenable, we systematically describe a "progression" (a taxonomy) of recourse restrictions for the resulting mathematical structures of $Q_i(\vec{x}, \vec{d'})$ reflecting the different assumptions. Two stage stochastic programming with recourse methodology provides a framework for such analysis with a plethora of options that represents different restrictions on the form of recourse. Not all recourse options (restriction) have to represent/account for economically rational profit maximizing impetus. One might also incorporate restrictions that reflect an industry norm, legal business context, cultural/behavioral environment, etc.

3.3. Second-Stage Models

In this section, we introduce the second-stage models using a "recourse tree" representing a taxonomy for the different recourse restriction options. It is a synthesis of earlier works and new analysis/directions.

At the outset we note that the second-stage profit function $Q_i(\vec{x}, \vec{d'})$ has generally 2 parts: local revenue $L_i(\vec{y}_i, \vec{x}, \vec{d'})$ and (a cooperative) transshipment profit $T_i(\vec{y}_i, \vec{y}_{-i}, \vec{x}, \vec{d'})$, where $\vec{y}_i$ is retailer i's recourse decision and $\vec{y}_{-i}$ represents all other retailers' recourse decisions. As before, price parameters are given and known by all retailers. These local revenue and transshipment profits are maximized with respect to recourse decision $\vec{y}_i \in \vec{Y}_i$. That is,

$$Q_i(\vec{x}, \vec{d}') = \max_{\vec{y}_i \in \vec{Y}_i} \quad L_i(\vec{y}_i, \vec{x}, \vec{d}') + T_i(\vec{y}_i, \vec{y}_{-i}, \vec{x}, \vec{d}').$$

In decentralized inventory system, if there are no restrictions imposed on retailer's recourse actions $\vec{Y}_i$, and the analytic form of the T_i function, the recourse decision vector $\vec{y}_i$ will be composed of a large number of recourse variables and the T_i function may be "intractable". Each restriction limits the number and the "range" of components of $\vec{y}_i$ and results in a functional (implicit or explicit) form for T_i. The T_i functions capture the different assumptions of recourse options.

3.3.1. Basic Options for Local Demand

We start the taxonomy with two basic assumptions on local demand (see Figure 3.1 below.)

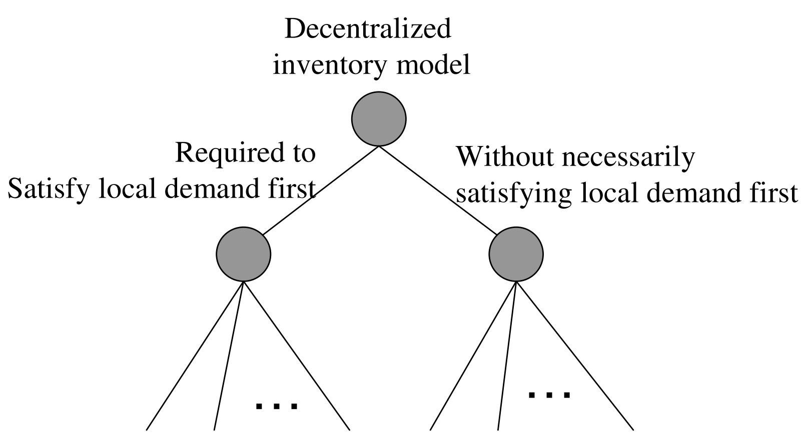

Figure 3.1. Recourse tree structure (2 levels)

<u>Restriction on Satisfying the Local Demand First</u>

Imposing a policy that requires each retailer to adhere to local demand before anything else represents a basic recourse restriction. In many number of cases, such policy is natural, especially if the profitability in a long run, or even business' existence, is heavily dependent on the loyalty of local/regular customers – say a restaurant known for certain specialty items on its manu. If it sales out of these items noticeably by responding to the demand of a neighboring upscale restaurant's customers, it might be undermining its own operation. In other cases adhering to local demand might be the accepted business culture.

Let l_i denote the quantity of locally satisfied demand. When every retailer has to satisfy local demand before considering collaboration regarding the excess inventory (transshipping if any), with other retailers, the local revenue is modeled as $L_i(\vec{x}, \vec{d'}) = r_i l_i + v_i(x_i - l_i)$, where $l_i = \min(x_i, d'_i)$. Thus, l_i is not a component of a second-stage decision vector $\vec{y}_i$.

Note that $v_i(x_i - l_i)$ is included in the local profit L_i because retailer i can always guarantee for herself a salvage value v_i per unit before transshipment. Hence, $T_i(\cdot)$ is an amount, in addition to salvage value, earned by retailer i based on her recourse action.

<u>No Commitment on Satisfying Local Demand First</u>

Relaxing the adherence to local demand, retailer i will transship her inventory to any subset of the retailers that will result in the highest profit for herself. Thus, the amount of locally satisfied demand l_i is a component of a second-stage recourse decision vector $\vec{y}_i$. In essence, the amount of locally satisfied demand is decided only after more profitable transshipments have been addressed. Say, a Christmas tree operation in mid December might not hesitate shipping a fraction of its trees to competitors in lieu of higher profits. Obviously, the feasible region of l_i is in the interval $[0, \min(x_i, d'_i)]$. For instance, with an assumption that all other retailers' recourse decisions in $\vec{y}_i$ are similarly restricted, we can model the second-stage profit function as:

$$
\begin{aligned}
Q_i(\vec{x}, \vec{d'}) = \max_{l_i} \quad & r_i l_i + v_i(x_i - l_i) + T_i(l_i, \vec{l}_{-i}, \vec{x}, \vec{d'}) \\
\text{s.t.} \quad & l_i \in [0, \min(x_i, d'_i)]
\end{aligned}
$$

where $\vec{l}_{-i}$ denotes all other retailers' quantities of locally satisfied demand.

Surprisingly, we were unable to find any literature on this important topic. It would be interesting to examine empirical data and comment on the managerial policies prevalent in such cases. This is clearly a topic for further research.

3.3.2. Restrictions Related to Transshipment

This section discusses the restrictions related to the transshipment stage (see Figure 3.2 below.) Note that the related (third level) nodes in Fig. 2 are connected to both nodes in the level above since imposing transhipment policy restriction is applicable to both second level nodes. That is, we could have transshipment restrictions if local demand has to be satisfied first or when allowing for shipments to nonlocal customers irrespective of local demand. The restrictions type may depend on responding to local demand first or independent of it. For instance, transhipment agreements between neighboring independent car-dealerships might not be binding but a consequence of developing "good will reciprocal assets" between any given pair of dealerships. Empirical data would be of help when constructing adequate models for these cases. There exists a considerable economic literature on reciprocity games but its influence on the OR/OM literature is yet to materialize.

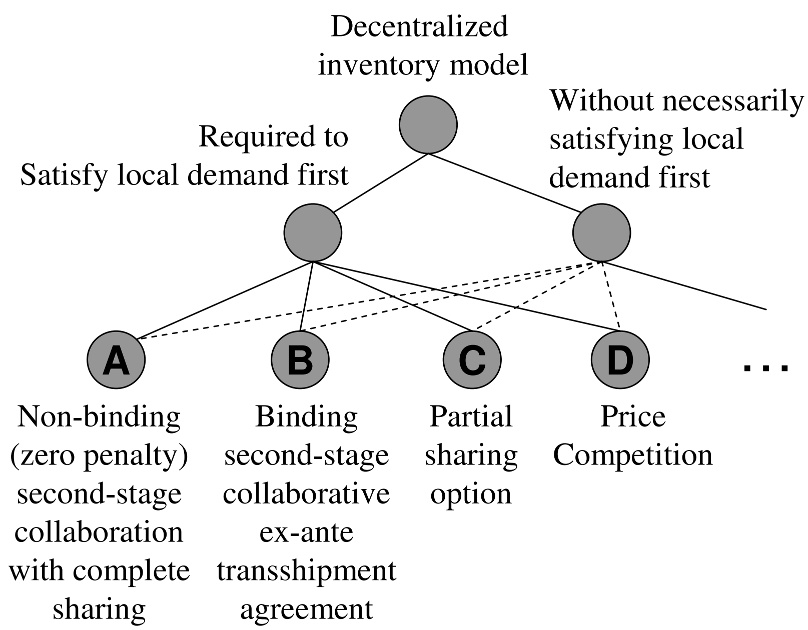

Figure 3.2. Recourse tree structure (3 levels)

<u>Non-Binding Second-Stage Collaboration with Complete Sharing (Node A)</u>

Examining Fig. 2, third level nodes from left to right we start with node A – the case where retailers have a non-binding ex-ante agreement about a centralized optimal second-stage transshipment policy sharing all shortage and overage and an allocation rule α for sharing the resulting profits.

Consider the following: After we know the demand and satisfy local demand, retailer i in collaboration with some proper subset of retailers can improve her profit over her share in the centralized transhipment profit based on the allocation rule α. In such case, since the ex ante collaborative agreement is not binding, a rational retailer will consider defecting in the second-stage and, subsequently, retailer's second-stage transshipment profit $T_i(\vec{y}_i, \vec{y}_{-i}, \vec{x}, \vec{d'})$ will depend on the subset of retailers who bind together for transshipment. The corresponding recourse model ought to capture the recourse decision space that includes all possible subcoalitional outcomes. That is, by assuming that all other retailers' recourse decisions in $\vec{y}_i$ are similarly restricted, the second-stage profit function is:

$$Q_i(\vec{x}, \vec{d'}) = \max_{S \ni i} \quad L_i(\vec{x}, \vec{d'}) + T_i^{\alpha}(S, \vec{x}, \vec{d'}) \tag{3.2}$$
$$\text{s.t.} \quad S \subseteq N$$

where $T_i^{\alpha}(S \ni i, \vec{x}, \vec{d'})$ is an amount, according to the allocation rule α allocated to retailer i when applied to a subcoalition $S \ni i$. Clearly, the coalitions S whose members extract their maximal profits are dependent on a realization $\vec{d'}$ and it is not well understood how they they form. The existence of S would be an outcome of the expected behavior of rational agents. At this point we are not aware of any empirical studies on this topic.

Note that node A assumes complete sharing of shortage and surplus in the second stage transshipment. That is, retailers do not hold back shortage or overage in the second stage. The local revenue function can be stated as $L_i(\vec{x}, \vec{d'}) = r_i \min(x_i, d'_i) + v_i \max(x_i - d'_i, 0)$.

A special case of the above is the case where $T_i^\alpha(N, \vec{x}, \vec{d'})$ is a profit function of a game (N, W) that is balanced (has a nonempty core) given $\vec{x}, \vec{d'}$, with the characteristic function $W(S, \vec{x}, \vec{d'}) = \sum_{i \in S} T_i^\alpha(S, \vec{x}, \vec{d'})$ for $S \subseteq N$ (for more recent work on stability of such games see Baiou and Balinski, 2002). To ensure that all (rational) retailers voluntarily collaborate in the centralized transshipment without actually imposing it with a contract, one must select ex ante an allocation rule α that is in the core of the balanced cooperative game (N, W). That is, no subcoalition is strictly better off on their own. We also assume that in the game (N, W) any $i \in N$ prefers a larger coalition set for the same payoff. If an allocation rule α is in the core, the optimal solution to (3.2) is $S^* = N$ and the second-stage profit function can be reduced to

$$\begin{aligned} Q_i(\vec{x}, \vec{d'}) &= L_i(\vec{x}, \vec{d'}) + T_i^\alpha(N, \vec{x}, \vec{d'}) \\ &= r_i \min(x_i, d'_i) + v_i \max(x_i - d'_i, 0) + T_i^\alpha(N, \vec{x}, \vec{d'}). \end{aligned} \tag{3.3}$$

Assuming that in the corresponding (N, W) any $i \in N$ prefers a larger coalition set for the same payoff, imposes a recourse restriction that has computational implications. We do not have to examine all proper subsets $S \subset N$ in our $Q_i(\cdot, \cdot)$ functional. However, reversing this restriction might introduce considerable conceptual and computational difficulties and could perhaps lead to new interesting research findings.

Some instances of the model corresponding to node A were analyzed in Anupindi et al. (2001) and Suakkaphong and Dror (2010a). A short synopsis of their results exposes the generic issues that have to be addressed as a part of selecting a procurement strategy.

We start with α allocation that satisfies the core conditions. Since $T_i^{\alpha}(N, \vec{x}, \vec{d'})$ corresponds to the profit function of a cooperative game with a given $\vec{x}, \vec{d'}$, the core conditions require that:

$$\begin{aligned} \sum_{i \in N} T_i^{\alpha}(N, \vec{x}, \vec{d'}) &= W(N, \vec{x}, \vec{d'}) \\ \sum_{i \in S} T_i^{\alpha}(N, \vec{x}, \vec{d'}) &\geq W(S, \vec{x}, \vec{d'}) = \sum_{i \in S} T_i^{\alpha}(S, \vec{x}, \vec{d'}), \text{ for all } S \subseteq N \end{aligned} \tag{3.4}$$

$$\begin{aligned} \text{where} \quad W(S, \vec{x}, \vec{d'}) = \quad & \max_{\vec{y}} \sum_{i \in S} \sum_{j \in S, j \neq i} (r_j - v_i - t_{ij}) y_{ij} \\ \text{s. t.} \quad & \sum_{j \in S, j \neq i} y_{ij} \leq \max\{x_i - d'_i, 0\} \text{ for all } i \in S \\ & \sum_{i \in S, i \neq j} y_{ij} \leq \max\{d'_j - x_j, 0\} \text{ for all } j \in S \\ & \text{for all } y_{ij} \geq 0. \end{aligned} \tag{3.5}$$

The optimal transshipping pattern y^*_{ij} for (3.5) represents the number of units of product transshipped from retailer i to retailer j in the coalition S. Note that each coefficient term in the objective includes a negative salvage value $(-v_i)$, since, as mentioned before, a retailer with excess inventory is expected to receive at least the salvage value v_i per unit from transshipment. Hence, there is no double counting of salvage.

We state a number of results based on Anupindi et al. (2001) and Suakkaphong and Dror (2010a).

Theorem 3.1. (Anupindi et al., 2001, Theorem 4.1) *The core of the second-stage transshipment game is nonempty. In particular, the allocation based on dual prices of transshipment problem* $W(N, \vec{x}, \vec{d'})$ *is an allocation in the core.*

This result is based on a classic work of Shapley and Shubik (1975) (expanded on by Samet and Zemel, 1984 and Sánchez-Soriano et al., 2001).

In terms of system performance, a decentralized solution is compared with the performance of the "first-best" solution – the centralized inventory decision.

Definition 3.1. *The* expected combined profit *(first-best profit) is represented by:*

$$\begin{aligned}\Pi^{c^*}(\vec{x}^{c^*}) &= \max_{\vec{x}\in\vec{X}} \Big\{ -\sum_{i\in N} c_i x_i + \mathbb{E}_{\vec{d}} \Big[W(N, \vec{x}, \vec{d'}) \\ &\quad + \sum_{i\in N} r_i \min(x_i, d'_i) + v_i \max(x_i - d'_i, 0) \Big] \Big\}\end{aligned}$$

The first-best solution $\vec{x}^{c^*}$ *is the solution that maximizes the expected combined profit for the set* N *of retailers (*$\vec{x}^c$ *and* $\vec{x}^{c^*}$ *stand for "centralized inventory decision" and "optimal centralized inventory decision" or first-best respectively;* Π^c *and* Π^{c^*} *being the centralized inventory profit and optimal centralized inventory profit respectively).*

Generally, if the decentralized inventory decisions coincide with the first-best solution, the first-best profit can be achieved. However, allocations based on dual prices of transshipment do not necessarily result in a profit value equal to the first-best solution (Anupindi et al. (2001)). Therefor Anupindi et al. (2001) propose another core allocation rule (see also Suakkaphong and Dror, 2010a).

For notational expediency we omit referring to the set N.

Definition 3.2. *Let the* fractional allocation $\alpha_i^f(\vec{x}, \vec{d'})$ *be defined as*

$$\alpha_i^f(\vec{x}, \vec{d'}) = \gamma_i \Pi^c(\vec{x}, \vec{d'}) - [r_i \min(x_i, d'_i) + v_i \max(x_i - d'_i, 0) - c_i X_i],$$

where $\Pi^c(\vec{x}, \vec{d'}) = W(\vec{x}, \vec{d'}) + \sum_{i \in N} -c_i x_i + r_i \min(x_i, d'_i) + v_i \max(x_i - d'_i, 0)$*, and* $\gamma_i \geq 0$ *is a fraction determined by the retailers such that* $\sum_{i \in N} \gamma_i = 1$*. Note that* $\alpha_i^f(\vec{x}, \vec{x})$ *can be negative. If we allow to consider* $\gamma_i \leq 0$ *for some i's, the interpretation of fractional allocation changes since. The option of subsidizing a facility in order to rise the collective profit can be considered in future research.*

Theorem 3.2. (Anupindi et al., 2001, Theorems 5.1 and 5.2, Corollary 5.1) *Consider a modified fractional allocation rule that allocates the residual profits to player* $i \in N$ *as follows:*

$$\alpha_i^m(\vec{x}, \vec{d'}) \quad = \quad \alpha_i^f(\vec{x}, \vec{d'}) + \alpha_i^d(\vec{x}^{c^*}, \vec{d'}) - \alpha_i^f(\vec{x}^{c^*}, \vec{d'})$$

where $\vec{x}^{c^*}$ *is the first-best solution. Then the pure strategy Nash equilibrium based on core allocation rule* $\alpha_i^m(\vec{x}, \vec{d'})$ *corresponds to a first-best solution.*

Two observations (*Suakkaphong and Dror (2010a)*):

Observation 3.1. *At non-pure strategy Nash equilibrium inventory position, when* $\vec{x} \neq \vec{x}^{c^*}$ *the allocation* $\alpha_i^m(\vec{x}, \vec{d'})$ *is not always in the core of the transshipment game.*

Observation 3.2. *If the demand distribution of $\vec{d}$ is not of common knowledge, the decentralized inventory system that adopts ABZ allocation rules is not incentive compatible.*

Suakkaphong and Dror (2010a) propose sufficient conditions for uniqueness of pure strategy Nash equilibrium (PSNE). The sufficient conditions include: (i) the expected centralized profit function has to have a unique global maximum, and (ii) there is no "ridge" present in the expected centralized profit function as plotted on $\vec{x}$.

Binding Second-Stage Collaborative Ex-ante Transshipment Agreement (Node B)

Binding ex-ante second-stage collaborative transshipment agreement to include all overage and shortage represents a different recourse restriction in the two-stage stochastic programming model. With a binding agreement, each retailer must participate in collaborative transshipment solution and share their overage/shortage. Retailers cannot consider proper subcoalitions. Hence, the second-stage profit function is modeled as (3.3). We can state the following:

Theorem 3.3. *If $T_i^{\alpha}(N, \vec{x}, \vec{d'})$ for all $i \in N$ increases as $W(\vec{x}, \vec{d'})$ increases (monotonic in W) and $Y_{ij} \ni y_{ij}$ is a closed set, then there exist a transshipment pattern y_{ij} that maximizes every individual retailer's profit for a realization $\vec{d'}$. Hence,*

$$x_i^* = argmax_{x_i \in X_i} \left\{ -c_i x_i + \mathbb{E}_{\vec{d}} \left[r_i \min(x_i, d_i') + v_i \max(x_i - d_i', 0) + W(\vec{x}, \vec{d'}) \right] \right\} \quad (3.6)$$

Since the above result is straight-forward, we omit the proof. Allocation rules that satisfy the above theorem include a proportional allocation $\alpha_i^p(\vec{x}, \vec{d'}) = \gamma_i W(\vec{x}, \vec{d'})$

where γ_i is a fraction reflecting retailer's contribution or negotiation power such that $\sum_{i \in N} \gamma_i = 1$ and for all i, $\gamma_i \in (0, 1)$.

Partial Sharing Option (Node C)

The case of retailers being free to hold back any part of shortage or overage assets in the second stage in order to increase their individual share of the transshipment profits. In this case, the second-stage recourse decision vector $\vec{y}_i$ will include the amounts of excess individual demand e_i and individual supply h_i that i frees up for transshipment (note that $e_i \times h_i = 0$ for all i).

Given an ex-ante allocation rule α, let $T_i^\alpha(h_i, \vec{h}_{-i}, e_i, \vec{e}_{-i}, S, \vec{x}, \vec{d'})$ be the profit allocated to retailer i from transshipment (in addition to salvage value) when (i) retailer i contributes h_i units of excess supply and e_i units of excess demand, (ii) all other retailers contribute $\vec{h}_{-i}$ of excess supply and $\vec{e}_{-i}$ of excess demand, and (iii) retailer i is a member of subcoalition S.

The second-stage profit function can be stated as:

$$
\begin{aligned}
Q_i(\vec{x}, \vec{d'}) = \max_{h_i, e_i, S \ni i} \quad & r_i l_i + v_i(x_i - l_i) + T_i^\alpha(h_i, \vec{h}_{-i}, e_i, \vec{e}_{-i}, S, \vec{x}, \vec{d'}) \\
\text{s.t.} \quad & S \subseteq N \\
& h_i \le x_i - l_i \\
& e_i \le d_i - l_i \\
& h_i, e_i \ge 0
\end{aligned}
\tag{3.7}
$$

where $l_i = \min(x_i, d'_i)$.

Note that in order for (7) to be a meaningful expression, the resulting subset S has to be a maximizing subset for all its members. Otherwise, (7) is ill defined.

This problem was examined in Granot and Sošić (2003) under the heading of a three-stage model. First, retailers decide on their procurement levels. Then, after demand is realized and each retailer responds to her local demand, she determines how much shortage/overage to share with the other retailers. Finally, the collaborated transshipment decisions take place. An allocation rule is *value preserving* (Granot and Sošić, 2003) if it induces all retailers to share their shortage/overage in amounts that do not result in a decrease in the total additional profit, and it is *complete sharing* if it induces all retailers to share all of their (total) shortage/overage. (Granot and Sošić, 2003) main findings are summarized below.

Theorem 3.4. (Granot and Sošić, 2003, Theorem 9) *There are no completely sharing or value-preserving allocation rules for transshipment games based on dual prices.*

Hence, transshipment dual prices are not an appropriate shortage/overage sharing mechanism in this case. Retailers may not necessarily choose $h_i = x_i - l_i$ and $e_i = d_i - l_i$, for all $i \in N$ in (3.7). In the case of two retailers, both retailers will actually choose $h_i = 0$ and $e_i = 0$. Consequently, an allocation rule based on dual price does not lead to the first-best outcome.

Granot and Sošić (2003, Theorem 11) observe (based on results from Young (1985) and Housman and Clark (1998)) that, *there are no completely sharing core allocation rules for transshipment games with four or more retailers*. This suggests that for any allocation function $T_i^\alpha(h_i, \vec{h}_{-i}, e_i, \vec{e}_{-i}, S, \vec{x}, \vec{d'})$ in (3.7), for $S = N$, retailers may not necessarily choose $h_i = x_i - l_i$ and $e_i = d_i - l_i$, for $i \in N$. Moreover, Granot and Sošić (2003, Theorem 12) observe that: *there are no value-preserving core allocation rules for transshipment games with six or more retailers.*

To sidestep the functional difficulties of this issue, the individual subset choice $S \ni i$ is removed from the second-stage decision options. That is, all retailers are bind to cooperate in transshipment while still having the option for partial sharing of shortage/overage. The restricted second-stage profit function can now be stated as:

$$
\begin{aligned}
Q_i(\vec{x}, \vec{d'}) = \max_{h_i, e_i} \quad & r_i l_i + v_i(x_i - l_i) + T_i^\alpha(h_i, \vec{h}_{-i}, e_i, \vec{e}_{-i}, \vec{x}, \vec{d'}) \\
\text{s. t.} \quad & h_i \leq x_i - l_i \\
& e_i \leq d_i - l_i \\
& h_i, e_i \geq 0
\end{aligned}
$$

where $l_i = \min(x_i, d'_i)$.

By manipulating a fractional allocation rule, Granot and Sošić (2003) proposed another allocation, i.e., $T_i^\alpha(h_i, \vec{h}_{-i}, e_i, \vec{e}_{-i}, \vec{x}, \vec{d'})$, that leads to the first-best outcome. This allocation is not necessarily in the core of the corresponding game (N, W) where

$$W(\vec{h}, \vec{e}, S, \vec{x}, \vec{d'}) = \sum_{i \in S} T_i^\alpha(h_i, \vec{h}_{-i}, e_i, \vec{e}_{-i}, S, \vec{x}, \vec{d'}) \text{ for } S \in N.$$

Theorem 3.5. (Granot and Sošić, 2003, Theorem 13) *The allocation rule defined by*

$$\alpha_i^g(h_i, \vec{h}_{-i}, e_i, \vec{e}_{-i}, \vec{x}, \vec{d'}) = \gamma_i \Pi^c(\vec{x}, \vec{d'}) - [r_i l_i + v_i(x_i - l_i) - c_i x_i], \tag{3.8}$$

with $\gamma_i \in (0,1), \sum_{j\in N}\gamma_j = 1$, *is an efficient value-preserving allocation rule which induces the inventory levels in a first-best solution to be a Nash equilibrium profile.*

The individual rationality is satisfied if retailers choose γ_is that are proportional to the expected individual profit. In an infinitely repeated game setting with a large enough discount factor, Huang and Sosic (2010b) show that the dual allocations result in in a subgame perfect Nash equilibrium with all retailers fully sharing their overage and shortage.

Price Competition (Node D)

In a less structured version of the transshipment one does not assume that the individual retailers value their shortage and overage uniformly using the fixed transshipment matrix (t_{ij}) and their r_i, v_i values and cooperate in transhipment. One can view the decentralized inventory system as a market of overage/shortage with price competition (Figure 3.3). We examine two cases in terms of timing pricing and roles of retailers. For the pricing, the model assumes that the prices are either set ex ante, or set ex post. In terms of the roles of retailers, the prices can be set by either sellers or buyers or both.

Ex-ante price competition (node G)

Consider the case when the prices are set ex ante by the sellers (node G). Other cases of retailers' roles can be modeled similarly. Price τ_{ij} for unit overage is set by retailer i for each retailer $j \in N, j \neq i$. A simpler version is the case with a policy

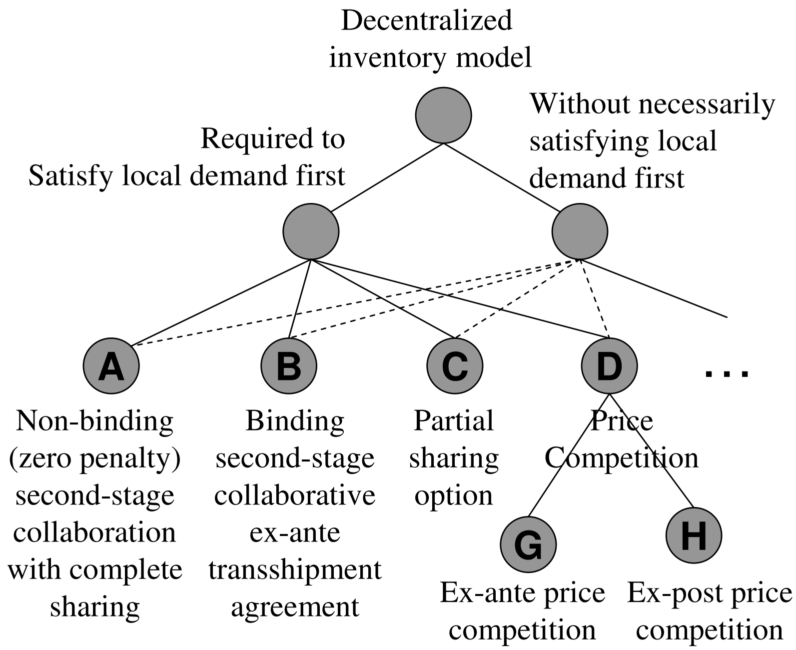

Figure 3.3. Recourse tree structure (4 levels)

$\tau_{ij} = \tau_i$, for all $j \in N, j \neq i$. In ex-ante price competition, the τ_{ij}'s are first-stage decision variables. That is, the first-stage expected profit of retailer i is expressed as

$$\begin{aligned} &\Pi_i^*((x_1, \vec{\tau}_1), \ldots, (x_{i-1}, \vec{\tau}_{i-1}), (x_i^*, \vec{\tau}_i^*), (x_{i+1}, \vec{\tau}_{i+1}), \ldots, (x_n, \vec{\tau}_n)) \\ &\qquad = \max_{x_i \in X_i, \vec{\tau}_i \in \vec{\mathbb{T}}_i} -c_i x_i + \mathbb{E}_{\vec{d}} \left[Q_i(\vec{x}, \vec{\tau}, \vec{d}') \right], \end{aligned} \tag{3.9}$$

where $\vec{\tau}_i = (\tau_{i1}, \tau_{i2}, \ldots, \tau_{in})$, $\vec{\tau} = (\vec{\tau}_1, \vec{\tau}_2, \ldots, \vec{\tau}_n)$. Because these prices $\vec{\tau}$ are set ex ante, the second stage decision model can be stated as,

$$
\begin{aligned}
Q_i(\vec{x},\vec{\tau},\vec{d'}) = \max_{(y_{i1},\ldots,y_{in}),(y_{1i},\ldots,y_{ni})} \quad & r_i l_i + v_i(x_i - l_i) \\
& + \sum_{j\in N} (\tau_{ij} - t_{ij} - v_i) y_{ij} + \sum_{j\in N} (r_i - \tau_{ji}) y_{ji} \\
\text{s. t.} \quad & \sum_{j\in N} y_{ij} = x_i - l_i \\
& \sum_{j\in N} y_{ji} = d_i - l_i \\
& \vec{y} \in \vec{\mathbb{Y}},
\end{aligned}
\tag{3.10}
$$

where $l_i = \min(x_i, d_i')$, complete sharing of residual is assumed, and $\mathbb{Y}$ represents the set of transshipment acceptable to all retailers. Model (3.10) assumes that the retailers who have overage pay for the transshipping cost. In the two-retailer case, it can be shown that $\vec{\mathbb{Y}} = (\mathbb{Y}_{12}, \mathbb{Y}_{21}) = ([0, x_1 - l_1], [0, x_2 - l_2])$ when $\tau_{12} \in [t_{12} + v_1, r_2]$ and $\tau_{21} \in [t_{21} + v_2, r_1]$ because the solution to (3.10) for retailer 1 is also the optimal solution for retailer 2. The transshipment price coordination for 2-retailer system as described by Rudi et al. (2001) aims on setting a transshipment price that maximizes system expected profit.

However, to define $\mathbb{Y}$ formally for n-retailer system, further assumptions are required because another retailer k may find that retailer i's proposed transshipment pattern $\vec{y}$ is not maximizing her expected profit and may decide not to participate (cooperate) in the transshipment.

To solve this dilemma, Huang and Sošić (2010a) suggest an approach that employs a central coordinator (depot) who unilaterally decides on the overall transshipment for all the retailers. That is, first, the transshipment prices τ_{ij} are separated into two elements: τ_{i0} which is the price retail i asks for transshipping to the depot and τ_{0j} which is the price retail j will pay for transshipping from the depot. The depot is responsible for the actual transshipment cost $t_{ij} = t_{i0} + t_{0j}$.

Huang and Sošić (2010a) do not explicitly indicate the "managerial" objective of the depot. They simply impose a constraint on non-negative operating profit for the depot. That is, $\tau_{0j} - \tau_{i0} \geq t_{ij}$; they denote this condition by (ID). They also denote seller i's rational constraint $\tau_{i0} - v_i \geq 0$ by (IRS), and buyer j's rational constraint $r_j - \tau_{0j} \geq 0$ by (IRB). Huang and Sošić (2010a) propose a pricing scheme such that (ID), (IRS), and (IRB) are satisfied, and the depot will be "neutral", i.e., will end up with zero profit for any demand realizations and inventory positions.

Theorem 3.6. (Huang and Sošić, 2010a, Proposition 8) *Let* $v_m + t_{m0} = \max_i\{v_i + t_{i0}\}$ *and denote a transshipment price agreement by*

$$\tau_{i0} = v_m + t_{m0} - t_{i0} \quad \forall i, \tag{3.11}$$

$$\tau_{0j} = v_m + t_{m0} + t_{0j} \quad \forall j. \tag{3.12}$$

This price agreement makes the central depot neutral and satisfies (ID), (IRS), and (IRB).

Such pricing scheme has the property that the retailers do not have to consider sharing the depot's operating profit. However, this pricing scheme is not well engineered because it does not have a clear objective; it does not maximize expected profit of an individual retailer, expected total profit of the system, transshipment profit of individual retailer, nor the realized total profit of the system. Furthermore, the depot will have no incentive for proposing an efficient transshipment patterns because its operating cost is zero independent of the transshipment solution.

As an alternative to Huang and Sošić (2010a), assume that given demand realization, the central depot maximizes its operating surplus. That is,

$$\begin{aligned}
\vec{\mathbb{Y}} = \quad & \arg\max_{\vec{y}} \sum_{i\in N}\sum_{j\in N, j\neq i} (\tau_{0j} - t_{ij} - \tau_{i0}) y_{ij} \\
\text{s.t.} \quad & \sum_{j\in N, j\neq i} y_{ij} \leq \max\{x_i - d_i', 0\} \text{ for all } i \in N \\
& \sum_{i\in N, i\neq j} y_{ij} \leq \max\{d_j' - x_j, 0\} \text{ for all } j \in N \\
& \text{for all } y_{ij} \geq 0.
\end{aligned}$$

First, consider a case when $\vec{\mathbb{Y}}$ is not a singleton. The depot is indifferent among the transshipment solutions, but each retailer is not necessarily indifferent among these solutions since these solutions impact the calculation of the first-stage individual expected profit. Hence, the depot is required to use a non-random algorithm to rank the transshipment solutions. One option for resolving this dilemma is to minimize the maximum difference between the best-off and the worst-off retailers. That is,

$$\begin{aligned}
\vec{\mathbb{Y}}^* = \arg\min_{\vec{y}\in\vec{\mathbb{Y}}} \quad & \Bigg\{ \max_i \{ \sum_{j\in N} (\tau_{ij} - v_i) y_{ij} + \sum_{j\in N} (r_i - \tau_{ji}) y_{ji} \} \\
& - \min_k \{ \sum_{j\in N} (\tau_{kj} - v_k) y_{kj} + \sum_{j\in N} (r_k - \tau_{jk}) y_{jk} \} \Bigg\}. \qquad (3.13)
\end{aligned}$$

In the case that $\vec{\mathbb{Y}}^*$ is still not a singleton, to reach a unique solution other ranking methods will have to be used on top of this method, say a lexicographical ranking. The depot is ex-ante required to communicate explicitly the ranking method used.

Now, consider the case with $\vec{\mathbb{Y}}$ a singleton. All retailers will have to agree to this transshipment pattern because of the ex-ante constraint. In this case, the optimal value of (3.13) can be greater than 0. The model shown in (3.9) assumes that the central depot's operating profit is shared by retailers. If the central depot's operating profit is shared, it is not clear how to incorporate this profit in the first-stage expected profit calculations. Clarifying this issue is left for future studies.

Ex-post price competition (node H)

Now, consider the case when the prices are set ex post (node H). The price competition happens ex post with respect to market demand and supply after demand realization and after satisfying each local demand. That is, the first-stage expected profit is defined as in (3.1) while the second-stage profit is defined as:

$$Q_i(\vec{x}, \vec{d'}) = r_i \min(x_i, d'_i) + v_i \max(0, x_i - d'_i) + U_i^*(\vec{x}, \vec{d'}, \vec{p}^*, \vec{q}^*) \tag{3.14}$$

where $U_i^*(\vec{x}, \vec{d'}, \vec{p}^*, \vec{q}^*)$ represents retailer i's utility (profit) given that the transshipment decision depends on the exchange market equilibrium price $\vec{p}^*$ and quantity $\vec{q}^*$. The above model assumes that there exists a market equilibrium such that the market will be cleared. That is, shortage/overage that may not be traded/exchanged with profit will leave the market. Retailers are price takers. Before we formally define $U_i^*(\vec{x}, \vec{d'}, \vec{p}^*, \vec{q}^*)$, we briefly discuss the setting of the exchange market.

The equilibrium that we are referring to here is a "general equilibrium", or sometimes, referred to as "Walrasian equilibrium." In our case, all n retailers are consumers who consume $2n$ types of commodities. These commodities are overage and shortage at n retailers. Because of varying cost parameters at each retailer, it is insufficient to model overage and shortage as 2 types of commodities. Let the amount of retailer i's *initial endowment* be denoted by $\vec{w}_i = (w_i^{h1}, \ldots, w_i^{hn}, w_i^{e1}, \ldots, w_i^{en})$.

$$\begin{aligned} w_i^{hi} &= \max(x_i - d_i, 0), \quad w_i^{hj} = 0 \\ w_i^{ei} &= \max(d_i - x_i, 0), \quad w_i^{ej} = 0 \text{ for all } j \neq i. \end{aligned}$$

Let $\vec{q}_i = (q_i^{h1}, \ldots, q_i^{hn}, q_i^{e1}, \ldots, q_i^{en})$ denote the amount of retailer i's *consumption bundle*. An allocation $\vec{q} = (\vec{q}_1, \ldots, \vec{q}_n)$ is a collection of n consumption bundles describing the result of exchange for n retailers. A *feasible allocation* is one that satisfies $\sum_{i \in N} \vec{q}_i \leq \sum_{i \in N} \vec{w}_i$. Each retailer i acts as if she was solving the following problem:

$$\begin{aligned} \max_{\vec{q}_i} \quad & U_i(\vec{q}_i) \\ \text{s.t.} \quad & \vec{p}\vec{q}_i \leq \vec{p}\vec{w}_i \end{aligned}$$

where $\vec{p}^T = (p^{h1}, \ldots, p^{hn}, p^{e1}, \ldots, p^{en})$ is transpose of a vector of market prices and the retailer i's utility is defined as:

$$\begin{aligned} U_i(\vec{q}_i) = \max_{\vec{y}} \quad & \sum_{k \in N} \sum_{j \in N, j \neq k} (r_j - v_k - t_{kj}) y_{kj} \\ \text{s.t.} \quad & \sum_{k \in N, k \neq j} y_{jk} \leq q_i^{hj} \text{ for all } j \in N \\ & \sum_{k \in N, k \neq j} y_{kj} \leq q_i^{ej} \text{ for all } j \in N \\ & \text{for all } y_{kj} \geq 0. \end{aligned}$$

That is, if retailer i is in control of consumption bundle $\vec{q}_i$, she will maximize the profit generated from that consumption bundle by transshipping overage to shortage in the most efficient way. The Walrasian equilibrium is a pair $(\vec{p}^*, \vec{q}^*)$ such that $\sum_{i \in N} \vec{q}_i(\vec{p}^*, \vec{p}^* \vec{w}_i) \leq \sum_{i \in N} \vec{w}_i$. Retailer i's utility is written as:

$$
\begin{aligned}
U_i^*(\vec{x}, \vec{d}', \vec{p}^*, \vec{q}^*) = \max_{\vec{q}_i} \quad & U_i(\vec{q}_i) \\
\text{s.t.} \quad & \vec{p}^* \vec{q}_i \leq \vec{p}^* \vec{w}_i
\end{aligned}
$$

where $\vec{w}_i$ is a function of $\vec{x}$ and $\vec{d}'$. The above decentralized inventory model as defined by (3.1), (3.14), and (3.15) requires that there is only one Walrasian equilibrium, or if there are multiple Walrasian equilibria, all equilibria must have the same profit value in order to compute expected profit.

We know that all retailers have the same form of utility function $U_i(\vec{q}_i)$ which is continuous, (weakly) concave, and non-decreasing in $\vec{q}_i$. Hence, given a certain price $\vec{p}$, it is possible to have a set $\Lambda = \{\vec{q}^{*1}, \ldots, \vec{q}^{*k}\}$, $|\Lambda| > 1$, of consumptions that maximize individual retailer's utility with $U_i(\vec{q}_i^{*1}) = \ldots = U_i(\vec{q}_i^{*k})$ for all $i \in N$. Thus, uniqueness of Walrasian equilibrium cannot be guaranteed.

It will be interesting to examine for this setting the conditions under which Walrasian equilibria have the same profit value. That is, if there is a set $Z = \{(\vec{p}^{*1}, \vec{q}^{*1}), \ldots, (\vec{p}^{*k}, \vec{q}^{*k})\}$, $|Z| > 1$, of Walrasian equilibria, then $U_i^*(\vec{x}, \vec{d}', \vec{p}^{*1}, \vec{q}^{*1}) = \ldots = U_i^*(\vec{x}, \vec{d}', \vec{p}^{*k}, \vec{q}^{*k})$ for all $i \in N$. This is another issue left for future studies.

Because the utility function is weakly increasing in $\vec{q}_i$, it lacks several useful properties. For instance, Walrasian equilibrium allocation may not be a member of the core of the corresponding transshipment game. We note that when overall shortage $\sum_{j\in N} q_i^{ej}$ is less than overall overage $\sum_{j\in N} q_i^{hj}$, an increase in overage q_i^{hj} does not increase the utility, and vice versa. To guarantee that the utility function is strictly increasing in $\vec{q}$, we can add a small perturbation to the retailers' preference (utility function) and a requirement that retailers not trade without profit. To accomplish this, we add an epsilon value to overage and shortage. This will make retailers prefer to hold on to overage and shortage rather than trade them with zero increase in profit (utility). The new utility function is written as:

$$
\begin{aligned}
U_i(\vec{q}_i) = \max_{\vec{y}} \qquad & \sum_{k\in N} \sum_{j\in N, j\neq k} (r_j - v_k - t_{kj}) y_{kj} + \epsilon \left\{ \sum_{j\in N} (q_i^{hj} + q_i^{ej}) \right\} \\
\text{s. t.} \quad & \sum_{k\in N, k\neq j} y_{jk} \leq q_i^{hj} \text{ for all } j \in N \\
& \sum_{k\in N, k\neq j} y_{kj} \leq q_i^{ej} \text{ for all } j \in N \\
& \text{for all } y_{kj} \geq 0.
\end{aligned}
$$

The above utility function is strictly increasing in $\vec{q}_i$.

Let $W(S, \vec{x}, \vec{d'})$ represent the transshipment profit of coalition S for a given $\vec{x}$ and $\vec{d'}$.

$$
\begin{aligned}
W(S, \vec{x}, \vec{d'}) = \quad \max_{\vec{y}} \quad & \sum_{i\in S} \sum_{j\in S, j\neq i} (r_j - v_i - t_{ij}) y_{ij} \\
& + \epsilon \left\{ \sum_{i\in S} (\max\{x_i - d'_i, 0\} + \max\{d'_i - x_i, 0\}) \right\} \\
\text{s. t.} \quad & \sum_{j\in S, j\neq i} y_{ij} \leq \max\{x_i - d'_i, 0\} \text{ for all } i \in S \\
& \sum_{i\in S, i\neq j} y_{ij} \leq \max\{d'_j - x_j, 0\} \text{ for all } j \in S \\
& \text{for all } y_{ij} \geq 0.
\end{aligned}
\tag{3.15}
$$

Theorem 3.7. *Assume that retailers are not trading without profit. All Walrasian equilibrium allocations that correspond to Walrasian equilibria in decentralized inventory system are members of the core of the second-stage transshipment game* (N, W) *where* W *is defined in (3.15).*

The proof of this new result is based on Jehle and Reny (2000) in their Theorem 5.6.

To conclude this subsection we note the similarities between the motivation of this model and that of Stuart (2005). However, Stuart (2005) assumes that their market consists of buyers and sellers, while in our market there is no distinction between buyer and seller (of overages). In addition, Stuart's unit cost and salvage value are constant for all retailers (producer). That is, $c_i = c, v_i = v$. Moreover, their transshipping cost is zero, i.e., $t_{ij} = 0$. Finally, their demand is a function of price while our demand is exogenous. As a result, in Stuart (2005) the problem is to find the market clearing price $\tau_{ij} = \tau$.

3.4. Restrictions Related to Transshipment for Disposal

Up to this point, we assumed that the overage without profitable demand is disposed at a local salvage value v_i. In terms of scope, we now modify our assumptions to allow retailers to transship their overage for disposal at retailers in a manner that generates the greatest salvage net revenue. The corresponding analysis depends on the retailers' agreements with respect to sharing the salvage value. For instance, if participating retailers have an ex-ante agreement regarding their share in the salvage net revenue resulting from transshipping for disposal, or retailers compete on price of disposal, the model will change. This section discusses the restrictions related to local demand (see Figure 3.4 below.)

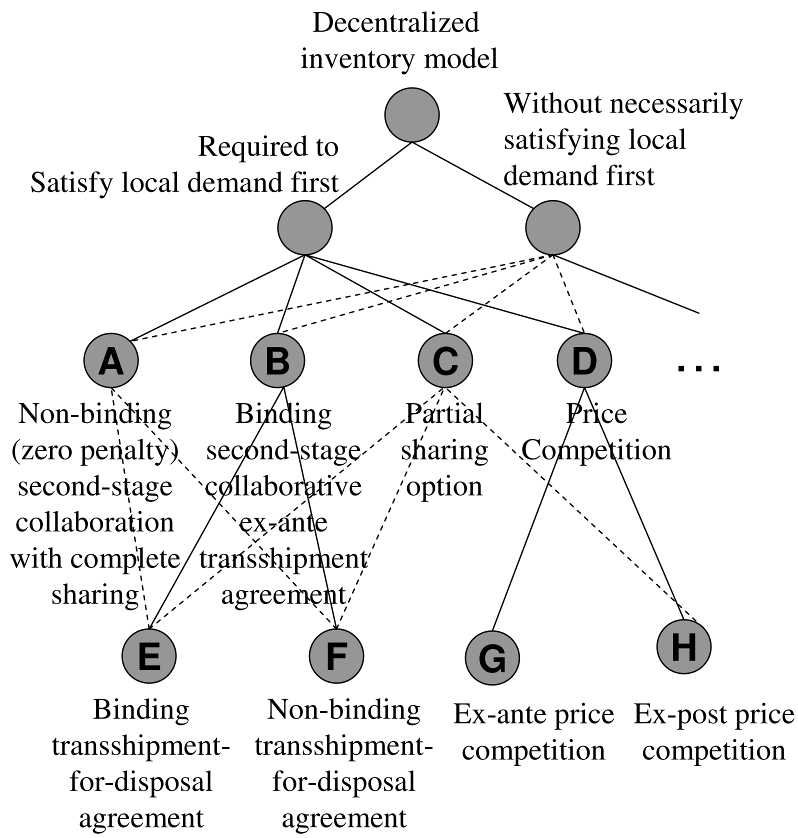

Figure 3.4. Recourse tree structure (4 levels)

3.4.1. Binding Transshipment for Disposal Agreement (Node E)

Related literature (Anupindi et al., 2001; Granot and Sošić, 2003; Suakkaphong and Dror, 2010a) generally assumes that the overage that is not transshipped (has no "profitable" demand) is disposed at a local salvage value v_i, $v_i \geq v_j - t_{ij}$ for all $i, j \in N$. However, one may consider the case when retailers have options to transship their overage to dispose it (salvage) at a retailer that generates the greatest net revenue. Specifically, we examine some models related to transshipment pattern (solution) and profit sharing for salvage options. Say, we are dealing with agricultural products and an option of processing unsold volumes for other useful purposes (compost, biofuel, etc.).

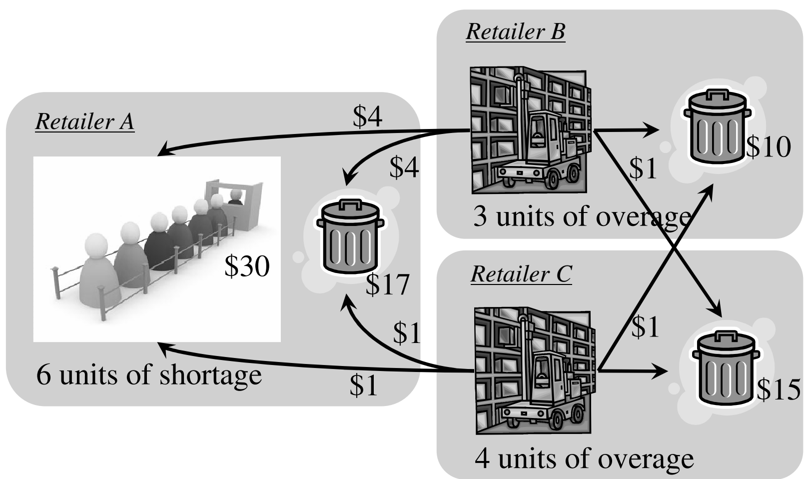

Figure 3.5. Example

Consider an example shown in Figure 3.5. This 3-retailer system has the following parameters: $(r_A, r_B, r_C) = (30, 30, 30)$, $(t_{AB}, t_{BA}, t_{AC}, t_{CA}, t_{BC}, t_{CB}) = (4, 4, 1, 1, 1, 1)$, $(v_A, v_B, v_C) = ((17, 10, 15)$. Suppose a demand realization instance with retailer A having 6 units of shortage, and retailers B and C having 3 and 4 units of overage, respectively. If the overage cannot be transshipped for disposal, with the 3-retailers cooperating ex post we solve the following centralized transshipment problem:

$$
\begin{aligned}
W(N, \vec{x}, \vec{d'}) = \quad & \max_{\vec{y}} \sum_{i \in N} \sum_{j \in N, j \neq i} (r_j - v_i - t_{ij}) y_{ij} \\
\text{s. t.} \quad & \sum_{j \in N, j \neq i} y_{ij} \leq \max\{x_i - d'_i, 0\} \text{ for all } i \in N \\
& \sum_{i \in N, i \neq j} y_{ij} \leq \max\{d'_j - x_j, 0\} \text{ for all } j \in N \\
& \text{for all } y_{ij} \geq 0.
\end{aligned}
\tag{3.16}
$$

where N={A,B,C} and y_{ij} is the amount transshipped from retailer i to j. The optimal solution requires the transshipment of 3 units from B to A, 3 units from C to A and the disposal of 1 unit of overage at C. The corresponding total revenue (optimal value of the above problem) is \$90. Because this model computes the revenue in addition to the salvage value, the actual revenue from transshipment, including the local salvage revenue, is \$90 + (3 × \$10) + (4 × \$15) = \$180.

Now we examine how this revenue is shared. Consider two allocation rules: one based on dual prices and the other based on an equal surplus sharing rule. That is:

Rule I: Retailers agree ex ante to share transshipment profit based on dual price of the transshipment problem. The allocation based on dual prices of transshipment is defined as

$$
\alpha_i^d(\vec{x}, \vec{d'}) = \lambda_i \max\{x_i - d'_i, 0\} + \delta_i \max\{d'_i - x_i, 0\} \tag{3.17}
$$

where λ_i and δ_i represent dual prices of $W(N, \vec{x}, \vec{d'})$. The dual prices are determined for each unit of shortage as well as each unit of overage. Assume also that they are not allowed to hold back shortage or overage. (We will discuss this assumption further in

Section 3.3.2.) The amounts of revenue shared to each retailer are \$84, \$36, and \$60 to A, B, and C, respectively. Note that the dual price for overage at B and shortage at A are \$2 and \$14, respectively. Since sharing based on dual price is in the core of this "cooperative game", no retailer or a subset of retailers is better-off by deviating from the centralized transshipment solution. The transshipment is stable.

Rule II: Assume that these retailers have a binding ex-ante agreement using a simple equal surplus sharing rule, that is, the profit between two retailers involved in transshipment are equally split. The corresponding revenue amounts allocated to each retailer are \$45, \$54, and \$81 to A, B, and C, respectively. Note that the allocation rule based on an equal surplus sharing is not necessarily in the core of the corresponding cooperative transshipment game for all realizations and usually requires a binding agreement for retailers to cooperate.

Below we compare the above two solution concepts in the case when retailers are allowed to transship for disposal at other locations.

First, consider a case when an increase in salvage net revenue from transshipping for disposal does not have to be shared between participating retailers, implying that the owner of overage receives full amount of salvage value less the transshipping cost. The salvage net revenue at retailer i can be updated to a new value that reflects the highest possible salvage value less transshipping cost. That is, $v_i' = \max(v_i, \max_{j \in N, j \neq i}(v_j - t_{ij}))$. All other parts of the model stay the same.

In that case we update the salvage values to $(v_A', v_B', v_C') = (17, 14, 16)$ – the disposal locations do not receive any share from disposal revenues. The solution is to transship 2 units from B to A, and 4 units from C to A. B disposes 1 unit of overage. In terms of revenue, the total revenue (optimal value of the above problem) is \$76. The actual revenue from transshipment, including the local salvage revenue, is $\$76 + (3 \times \$14) + (4 \times \$16) = \$182$.

We examine how this revenue is shared using the two allocation rules: dual prices and an equal surplus sharing rule: (i) assume that the retailers agree ex ante to share transshipment profit based on dual price of the transshipment problem. The amounts of revenue shared to each retailer are \$72, \$42, and \$68 to A, B, and C, respectively; (ii) assume that these retailers have a binding ex-ante agreement using an equal surplus sharing rule. The amounts of revenue shared to each retailer are \$38, \$54, and \$90 to A, B, and C, respectively.

Non-Binding Transshipment for Disposal Agreement (node F)

Now consider a case when an increase in salvage net revenue from transshipping for disposal is shared between participating retailers using dual price. The salvage net revenue at retailer i can be updated to a new value that reflects the highest possible salvage value less transshipping cost. That is, $v_i' = \max(v_i, \max_{j \in N, j \neq i}(v_j - t_{ij}))$. All other parts of the model stay the same.

This case can be justified by the fact that zero dual price value is linked with an infinite quantity (≥ 0 non-negativity constraint). Because an infinite quantity can be disposed at any retailer, the disposal locations (retailers) will not receive share of disposal revenue when this rule based on dual price is agreed upon.

If there is no ex-ante agreement on sharing disposal revenue, the transshipment and disposal of overage can be seen as a competitive market. Because of high competition between the disposal locations (retailers), the disposal locations will not receive share of disposal revenue when this rule based on dual price is agreed upon. Thus, the salvage net revenue at retailer i can also be updated to a new value that reflects the highest possible salvage value less transshipping cost.

Description	Revenue	A	B	C
No transshipment for disposal, Dual allocation 3(B $\rightarrow$ A), 3(C $\rightarrow$ A),1(C $\rightarrow$ C disposal)	180	84	36	60
No transshipment for disposal, Equal surplus allocation 3(B $\rightarrow$ A), 3(C $\rightarrow$ A),1(C $\rightarrow$ C disposal)	180	45	54	81
Allow transshipment for disposal, Dual allocation 2(B $\rightarrow$ A), 4(C $\rightarrow$ A),1(B $\rightarrow$ C disposal)	182	72	42	68
Allow transshipment for disposal, Equal surplus allocation, Disposal locations do not receive disposal revenue 2(B $\rightarrow$ A), 4(C $\rightarrow$ A),1(B $\rightarrow$ C disposal)	182	38	54	90
Allow transshipment for disposal, Equal surplus allocation, 2(B $\rightarrow$ A), 4(C $\rightarrow$ A),1(B $\rightarrow$ C disposal)	182	41	50	91

Table 3.1. Transshipment solutions.

Next consider the case when an increase in salvage net revenue from transshipping for disposal is shared between participating retailers using a rule of equal partition of surplus. The solution of a problem in (3.16) using updated salvage value $v_i' = \max(v_i, \max_{j \in N, j \neq i}(v_j - t_{ij}))$ is also an optimal solution that generates highest transshipment profit. The solution is to transship 2 units from B to A, and 4 units from C to A. B disposes 1 unit of overage. The actual revenue from transshipment is \$182. However, the solution of (3.16) does not immediately provide the allocation amount for each retailer. The following model shows another equivalent setting for this case.

$$
\begin{aligned}
\hat{W}(N,\vec{x},\vec{d'}) = \quad & \max_{\vec{y}} \quad \sum_{i\in N}\sum_{j\in N}(r_j - v_i - t_{i,j})y_{i,j} + \sum_{i\in N}\sum_{j\in N}(v_j - v_i - t_{i,j})z_{i,j} \\
& \text{s. t.} \quad \sum_{j\in N}(y_{i,j} + z_{i,j}) = \max\{x_i - d'_i, 0\} \text{ for all } i \in N \\
& \qquad\quad \sum_{i\in N} y_{i,j} \le \max\{d'_j - x_j, 0\} \text{ for all } j \in N \\
& \qquad\quad \text{for all } y_{i,j}, z_{i,j} \ge 0.
\end{aligned}
$$

With this setting, retailer k will receive $\sum_{j\in S} 0.5\{(r_j - v_k - t_{k,j})y_{k,j} + (v_j - v_k - t_{k,j})z_{k,j}\} + \sum_{i\in S} 0.5\{(r_k - v_i - t_{i,k})y_{i,k} + (v_k - v_i - t_{i,k})z_{i,k}\}$. The amounts of revenue shared to each retailer are \$41, \$50, and \$91 to A, B, and C, respectively. In summary, allowing transshipment of salvage expands the recourse options and must be handled with some care in the second-stage model. Is it a third stage? It does not need to be considered a third stage if the expected values are rolled back to influence the first stage strategic procurement decisions. The literature on this topic, if there is any, is very skimp. Perhaps this is an indication that further work is due.

3.5. Discussion and Conclusion

Metaphorically, this paper is about a single strawberry stand operator. Her customers' demand for fresh strawberries is uncertain and some days she orders too many and some days too few. Other nearby strawberry stands are in the same situation. The strawberry stand operators know of each other and when they see shortage or overage is about to occur it is common to inquire who is overstocked and would like to reduce her inventory. In this paper we discuss the potential options of trade between the strawberry stand operators in response to their customers' demands. It is a case of an endogenous trade in response to exogenous demand. The basic strawberry stand problem serves as a symbolic micro-cosmos for a very large family of problems. How many fresh strawberry boxes should "I" order? Who to contact in a case of shortage, who in a case of overage? Should "I" form/join an agreement between strawberry stand operators for overages and shortages? Who should be allowed to join? How are the costs of such strawberry exchange split between the members? Where/how should "I" dispose of the unsold strawberries, etc. The list is long.

This paper presents the strawberry stand operator problem as a taxonomy graph that systematically links the different decisions as graph nodes representing operating assumptions and analytical resolutions. Is is based on the premise that strawberry stand operator is maximizing her expected profit.

Why should we, or anyone else, care? Because we want to understand as much as possible what is involved in doing this trade. It has a tremendous economic impact on everyone.

What can we learn from the taxonomy graph and the analysis of the linked nodes? First of all it depicts a clear picture of interrelations of assumptions and decision options. Secondly, using the umbrella of stochastic programming with recourse methodology, we learn what answers to expect, and are these answers computable, offer a stable business configuration, present the best decision options, or that we not yet know how certain situations will play out.

The analytic methodology is that of a two-stage stochastic programming with recourse. This does not mean it is a two stage actions problem. It only means that under the second stage heading, after learning the demand, we move all the implications in its expected value projection of the actual recourse operational costs and profits. We first order the strawberries, then we see how the customers react, then we know if we ought to seek more strawberries, ship some strawberries to competitors, or decide of salvaging the left-over strawberries in a least costly manner. After the first order all else is the recourse – the second stage.

Our basic approach is that of a two-stage stochastic programming optimization. The second stage reflects a recourse option each retailer has when facing a given demand. In general, the variety of potential recourse options is too large for exact computational analysis. Thus, the modeling considers restrictions and approximations to real-life options. The presumed aim of stating recourse restrictions in the second stage of the stochastic programming model is the communication of an "appropriate" problem description and the mathematical "tractability" that retains a meaningful modeling interpretation. We describe a progression of a number of recourse restrictions and the subsequent mathematical expressions of second-stage profit function reflecting different modeling assumptions.

Past studies on decentralized inventory system are mapped onto our taxonomy. Anupindi et al. (2001) work (mapped to node A) analyzed two-stage systems under the assumption that each retailer's local demand must be satisfied first before she can transship overage to other retailers. The contract between retailers is assumed to be non-binding, hence Anupindi et al. (2001) focus on profit allocation schemes that are stable. They also assume that all retailers do not hold back shortage or overage when transshipment decision is made. The allocation schemes that they studied include allocation based on dual prices and modified fractional allocation. When the contract is binding, the system may use the second-stage profit allocation scheme that is unstable such as fractional allocation as proportional allocation. This case is mapped to node B. Granot and Sošić (2003) (mapped to node C) removed the assumption that all retailers do not hold back shortage or overage. In this setting with four or more retailers, first-best profit cannot be guaranteed.

Node D represents the system with price competition. The works of Rudi et al. (2001) and Huang and Sošić (2010a), ex ante price competition, were discussed and mapped to node G (a child node of node D). However, we noted that this setting is not practical for more than two retailers. Another approach with price competition ex post is offered. It is mapped to node H. We use Walrasian equilibrium to explain the individual second-stage profit behavior.

Fourth level nodes can be added under each of the third-level nodes to map the different assumptions on transshipment for disposal as we show at node E and F. The different assumptions will lead to different second-stage profit models. Moreover, when relaxing the assumption on satisfying local demand first, additional models can be generated and analyzed as represented by dotted lines connecting the second node in the second level to the third-level nodes in all taxonomy figures.

Other problem instances reflecting retailer behaviors can also be mapped into this taxonomy. For example, we may add a node in the third level for adversarial actions among retailers (node I in Figure 3.6). Consider a group of retailers $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$. When they agree (ex ante or ex post) to transship their overage, their individual expected profits increase. However, retailer $\boldsymbol{a}$ might find that if she deviates from transshipping agreement, it will raise the cost for $\boldsymbol{b}$ and/or $\boldsymbol{c}$ by a certain dollar amounts. In the long run, one of them might be driven out of the market with profit implications for the remaining retailers. This behavior is called "Raising rivals' costs" and has been around for a long time. It was formally introduced in economics literature by Salop and Scheffman (1983).

A number of extensions to the proposed taxonomy are ripe for future research. For node A, past studies focused on core allocations. We are not aware of analysis when allocation α is not in the core. If allocation α is not in the core, at some realization, a subset of retailers would have an incentive to deviate from the grand coalition.

For node G (ex ante price competition), further study is required on how to model the operation of the depot (central coordinator for transshipments). If the depot is

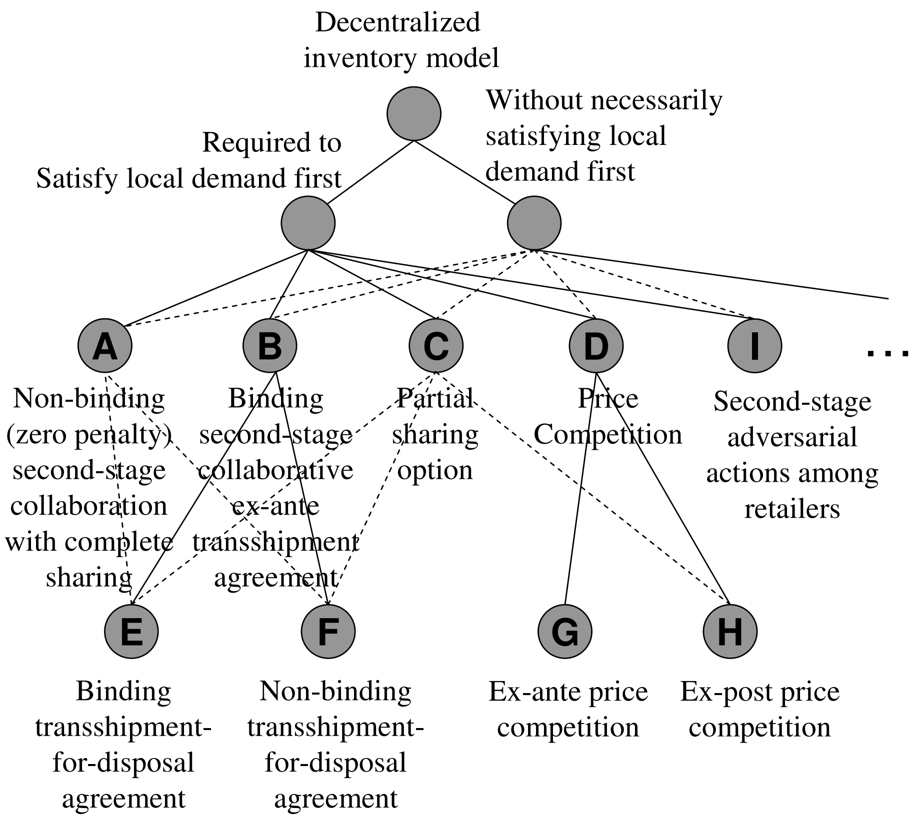

Figure 3.6. Recourse tree structure (4 levels)

a profit maximizing entity, the profit at the central depot should be shared among retailers. The profit sharing scheme will effect the inventory decision in the first stage.

For node H (ex post price competition), it would be interesting to examine the conditions for unique Walrasian equilibrium value and the affect of equilibrium on inventory decision. In addition, one may study the effect of ex post price competition under the case of partial sharing (node C).

Furthermore, (new nodes) one may look at the ex-post price competition under the assumption that retailers do not necessarily satisfying local demand first and may transship their overage for salvage at other retailers. That is, different extensions of taxonomy can be produced from a combination of existing nodes.

We suggest that this taxonomy can serve as a general framework for analyzing decentralized inventory system. We have left out from this discussion the algorithmic and computational issues related to solutions for the stochastic programming with recourse model of this framework.

4. BIFORM GAME: REFLECTION AS A STOCHASTIC PROGRAMMING PROBLEM

4.1. Introduction

Consider a two-stage decision problem faced by a single firm or a number of firms regarding her/their production capacity – a game. In the first stage each firm (a player in the game) is choosing an upfront strategy based on the firm's belief regarding a strategy chosen by her opponent(s) and beliefs regarding a subsequent effect of the chosen strategies on the second-stage problem/game. For instance, a manufacturing firm might have to decide upfront about her production capacity based on beliefs about her customers' demand and the capacity installed by her competitor.

Second-stage decisions are about forming coalition(s) that are likely to deliver the best value to each of the firms. For instance, a number of firms and their customers may form a coalition that cooperates in order to generate a surplus. Presumably, the resulting surplus (the participants' realized payoff) is shared fairly – the coalition is stable.

The objective of each firm in the first stage is to choose a strategy (say a production capacity), that will maximize her expected payoff based on her beliefs (anticipation) regarding the firm's likely share of the profits in each coalition. The actual payoff is realized after the second-stage cooperative game is played.

This kind of two-stage problems was modeled in Brandenburger and Stuart (2007) where the format of a biform game was introduced for combining competitive games with cooperative games in one formal model. Nash equilibrium is guiding the first-stage competitive decisions. For the second-stage cooperative game decisions core solutions are used. Core solutions (a closed and bounded convex polyhedral set that is sometimes empty) generally do not promise a unique outcome, thus, Brandenburger and Stuart (2007) set forth the notion of a confidence index to allow/facilitate for the computation of the second stage expected payoff reflected back in the first-stage problem. In their (2007) paper, and in Stuart (2005, 2007a,b), Chatain and Zemsky (2007), various examples are presented that demonstrate the "utility" of the biform game model.

In this paper, we show that the methodology proposed by Brandenburger and Stuart (2007) can be cast as a special case of a two-stage stochastic programming with recourse model. Stochastic programming is a mature mathematical methodology (dating back to Dantzig, 1955, and Beale, 1955) for solving optimization problems with time dependent stochastic variables representing uncertainty of future events. It has a vast support literature, rich in methodological and analytical ideas. It has been successfully used to model and solve decision problem such as newsvendor problem, manufacturing capacity planning, and numerous other problems (Birge and Louveaux, 1997). Although stochastic programming is traditionally used for analyzing problems faced by a single decision maker, it can be combined with Nash equilibrium concept to model a game faced by multiple decision makers as demonstrated in Suakkaphong and Dror (2010b).

The remainder of this paper is organized as follows. Section 4.2 presents various examples of biform games from Brandenburger and Stuart (2007); Stuart (2005, 2007a,b); and Chatain and Zemsky (2007), recast as two-stage stochastic programming problems. Section 4.3 formally states our two-stage stochastic programming reflection of the biform model. Section 4.4 examine an example of a two stage game with an empty core in its cooperative stage. Section 4.5 examines the contribution of our work vis a vis the biform games approach and provides insights into strategic limitation of our model.

4.2. Examples

This section restates the biform games examples (five in total) found in Brandenburger and Stuart (2007), Stuart (2007b), and Chatain and Zemsky (2007) as two-stage stochastic programming with recourse models. It is an exhausting listing for the sake of completeness that might at times "look" repetitious. For that reason four of the more straight forward examples – Negative-Advertising Game (Brandenburger and Stuart, 2007), Repositioning Game (Brandenburger and Stuart, 2007), Buyer Asymmetry in Monopoly (Stuart, 2007a), and Inventory Competition with Deterministic Demand (Stuart, 2005) – are placed in Appendix.

Example 4.1. *Branded Ingredient Game* (Brandenburger and Stuart, 2007, Example 2.1).

In the Branded Ingredient Game a supplier of a single item must decide upfront whether or not to inscribe her logo on the item (the product) at a cost of \$1 that consequently has an effect on the end consumers' willingness-to-pay. There are two manufacturing firms, Firms 1 and 2, both competing for this item. The supplier can supply the product to at most one of the firms. The supplier's production cost for the item is $c_p = \$1$. There are numerous end consumers (buyers), each interested in purchasing a single unit of the product from one of the two competing manufacturing firms. The buyers uniformly express a willingness-to-pay $p_1 = \$9$ for the item sold by Firm 1, and $p_2 = \$3$ for the item sold by Firm 2. If the supplier decides to inscribe her logo on the item at an upfront cost of $c_l = \$1$, the buyers' willingness-to-pay for the item from Firm 2 will increase to $p_2' = \$7$, but the buyer's willingness-to-pay for Firm 1's item will not change and remain at \$9.

The sequence of "moves" in the game is as follows: First, the supplier produces a single unit of the product at a cost of $c_p = \$1$. Then, the supplier chooses between the option of branding the product with her insignia (a logo) at an additional cost of $c_l = \$1$ or not branding the product. Hence, the total cost of the product $c = c_p + c_l$ if the supplier decides to inscribe the logo on her product. After the supplier's decision is executed, Firm 1 and Firm 2 compete for the product. The firm that wins the competition sells the product to her buyer at a price that equals to the buyer's willingness-to-pay.

Consider the case when the supplier does not brand the product; thus saving \$1. Firm 1 will consider paying the supplier at most \$9 – her buyers' willingness-to-pay. Similarly, Firm 2 will be willing to pay at most \$3. Therefore, Firm 1 would win this competition by offering to pay a price anywhere in the open interval of $(3, 9)$ dollars for the product. In this case, the supplier's profit will lie in the open interval $(2, 8)$ $(3 - 1 = 2$ and $9 - 1 = 8)$. That is, the supplier is guaranteed a profit of no less than \$2. The remaining surplus of \$6 will be shared between the supplier and Firm 1 based on the outcome of their price negotiation/bargaining.

Now, what happens if the supplier brands the product at an upfront cost of \$1? Firm 1 is still willing to pay the supplier at most \$9 since her buyers' willingness-to-pay for the product remains at \$9. Firm 2, however, is willing to pay at most \$7 due to the increase in willingness-to-pay of her buyers. Firm 1 would still win this product competition by offering the supplier a price anywhere between \$7 and \$9. Thus, the supplier will end up with profit in the open interval $(5, 7)$ $(7 - 2 = 5$ and $9 - 2 = 7)$. That is, the supplier is guaranteed a profit of no less than \$5. The remaining surplus of \$2 will be shared between the supplier and Firm 1 based on the outcome of their bargaining.

Brandenburger and Stuart (2007) model the above supplier's decision problem as a biform game. The key assumption in the biform game model is that the supplier's parameter value $\alpha, (\alpha \in [0, 1])$ represents the supplier's belief in the likelihood of earning the maximum possible profit in each strategy respectively. That is, when choosing not to invest in the logo inscription, α represents the belief for a likelihood of \$8 profit, and $1 - \alpha$ the belief in \$2 profit. Brandenburger and Stuart (2007) refer to α as the confidence index. "An optimistic player i will have a confidence index α^i close to one, indicating that player i anticipates capturing most of the value to be divided in the residual bargaining" and "Roughly speaking, it indicates how well player i anticipates doing in the resulting cooperative games."

Below, we recast the Branded Ingredient Game example as a two-stage stochastic programming problem in a straight forward fashion. The supplier's first-stage decision is captured by the binary variable $x \in \{0, 1\}$ reflecting the options in the upfront investment. Let the random variable ξ' represent a fraction of residual profit "captured" by the supplier (less the amount the supplier is guaranteed to receive), that corresponds to the bargaining power of the supplier in relation to the two firms. Since a bargaining outcome cannot usually be determined a priori, ξ' reflects the uncertainty of the bargaining outcome. The supplier cannot be certain whether she will receive maximum possible profit (when $\xi = 1$), or just the amount she is guaranteed to receive (when $\xi = 0$), or some profit between the two extremes. In the second stage, after the first-stage investment decision x has been acted upon, the two firms compete for the product. Following the competition outcome the winning firm and the supplier negotiate/bargain for their share of the total profit (revenue less the production and investment cost). That is, after the instantiation of ξ' is determined through bargaining, the supplier will then receive the guaranteed amount and in addition she will receive ξ times the residual profit.

Formally, let $v(x, \xi)$ represent the second-stage profit achievable by the supplier for a given first-stage decision x and realized fraction ξ - an indication of her bargaining power. The stochastic integer programming problem can be stated as

$$\max_{x \in \{0,1\}} \quad -c_p - c_l x + E_{\xi'}[v(x, \xi')]$$

That is, the supplier is selecting an x that maximizes the negative value of production and investment in the logo plus the expectation of her second-stage profit. The second-stage profit $v(x, \xi)$ can be rewritten as

$$\begin{aligned} v(x, \xi) &= \min(p_1, p_2 + 4x) + \xi(\max(p_1, p_2 + 4x) - \min(p_1, p_2 + 4x)) \\ &= \xi \max(p_1, p_2 + 4x) + (1 - \xi) \min(p_1, p_2 + 4x) \\ &= \xi \max(9, 3 + 4x) + (1 - \xi) \min(9, 3 + 4x) = 9\xi + (1 - \xi)(3 + 4x) \end{aligned} \tag{4.1}$$

The term $4x$ communicates the increase in buyer's willingness-to-pay $(p_2' - p_2)$ for the product of Firm 2. Due to the competition between the two firms, the supplier is guaranteed to receive (before accounting for production and logo cost of $-c_p - c_l x$) at least $\min(p_1, p_2 + 4x)$ but no more than the amount the winning firm can generate, that is $\max(p_1, p_2 + 4x)$. The instantiation of supplier's bargaining power ξ' determines the supplier's final revenue.

In terms of probability distribution of ξ'; it depends on the supplier's belief system. It can be uniform, truncated normal, based on supplier's bargaining history, etc.

One of the belief systems can be in the form of confidence index α proposed by Brandenburger and Stuart (2007) for the biform model. That is, consider the confidence index $\alpha \in [0,1]$. The confidence index can be stated as a probability distribution of ξ' as follows:

$$Pr\{\xi' = 1\} = \alpha, \quad Pr\{\xi' = 0\} = 1 - \alpha$$

From (4.1), we know that $v(x,0) = 3 + 4x$ and $v(x,1) = 9$. Hence, the two-stage stochastic integer program of the branded ingredient game, assuming the confidence index α, is

$$\begin{aligned} & \max_{x \in \{0,1\}} -c_p - c_l x + Pr\{\xi' = 0\} v(x,0) + Pr\{\xi' = 1\} v(x,1) \\ = & \max_{x \in \{0,1\}} -1 - x + (1-\alpha)(3+4x) + 9\alpha \end{aligned}$$

That is, the supplier should invest upfront in a logo if her confidence index α is less than 0.75, and should not invest if α is greater than 0.75. She is indifferent between the two options when $\alpha = 0.75$.

The Branded Ingredient Game is modeled in the biform game framework by Brandenburger and Stuart (2007) for two reasons; (a) the first-stage decision involves supplier's individual decision reflecting the competition between the two firms, and (b) the second-stage decision is viewed as a cooperative game $(N, V^x(S)), S \subseteq N$, where $V^x(S)$ is a characteristic function (the profit) of the cooperatives game as a result of the first-stage decision x as follows:

$$\begin{aligned} N &= \{\text{Supplier, Firm 1, Firm 2}\}, \\ V^x(S) &= \$0 \text{ for } S \subset N, |S| \leq 1, \\ V^x(\{\text{Firm 1, Firm 2}\}) &= \$0, \\ V^x(\{\text{Supplier, Firm 1}\}) &= p_1, \\ V^x(\{\text{Supplier, Firm 2}\}) &= p_2 + 4x, \\ V^x(N) &= \max(p_1, p_2 + 4x). \end{aligned}$$

Cores of these games are the payoff values such that the supplier receives an amount $v(x, \xi)$ in the interval $(\min(p_1, p_2 + 4x), \max(p_1, p_2 + 4x))$, Firm 1 receives an amount in the interval $(\$0, V^x(N) - v(x, \xi))$, and Firm 2 receives \$0.

Example 4.2. *An Innovation Game* (Brandenburger and Stuart, 2007, Example 2.2).

In Innovation Game there are two firms, each with capacity of producing two units of current-generation product at zero unit cost. The two firms decide simultaneously and independently whether or not to invest $c_1 = c_2 = \$5$ in an upgrade of their current-generation product to produce a new-generation product. The production capacity is independent of the upgrade decision. The market for the products consists of three buyers, each willing to purchase one unit of current-generation (old technology) or new-generation product (new technology) at $p_{old} = \$4$ and $p_{new} = \$7$, respectively. We assume that the buyers prefer the new-generation product. This can be substituted by an assumption that second-stage decisions maximize the collective surplus/profit.

The sequence of events is as follows: In the first stage, Firms 1 and 2 independently (simultaneously) choose between the two options; invest at a cost $c_1 = c_2 = \$5$ and produce two units of the new-generation product, or stay the course and produce two units of the current-generation product. After the products are manufactured, the three buyers enter the market setting up the second-stage problem. The supply in the second stage could be zero, two, or four units of the new-generation product and the complementary number of the current-generation product in each case depending on the investment strategies chosen by Firm 1 and 2. The demand for the products does not depend on the production technology and stays at three units.

First, consider the case when both firms choose to remain with their current-generation technology. In that case they produce 0 units of the new-generation product and 4 units total of the current-generation product. Since their marginal production cost is zero, the total surplus in the market is $3 \times p_{old} = \$12$. Because supply exceeds demand, the two competing firms will not receive any share of the surplus, and each buyer will receive a surplus of \$4 per the one unit the buyer purchased. Observe that this setting can be modeled directly as a cooperative game $(N, V^{\vec{x}=(0,0)}(S)), S \subseteq N$, where $V^{\vec{x}=(0,0)}(S)$ is a characteristic function (the surplus) of the cooperative game as a result of the first-stage decision not to invest $\vec{x} = (0, 0)$.

$V^{\vec{x}=(0,0)}(N) = 12$. This game has a unique (a singleton) core solution as described above in terms of shares of surplus.

Next, consider the case when only one of the firms chooses to invest \$5 and upgrade her products while the other firm does not. In this case the market supply consists of two new-generation product units and two units of the current-generation product. Since the buyers would rather purchase the new-generation product than purchase the current-generation product, the "third" buyer will have to take one of the two units of the current-generation product. The total revenue for the products sold is $(2 \times p_{new}) + (1 \times p_{old}) = \$18$. (The result stays the same when we assume that the products-buyers matching is done to maximize group's surplus.) Given that the "third" buyer will definitely receive \$4 of surplus, buyers "1" and "2" are unlikely to settle for anything less than \$4 of surplus. Moreover, because all buyers compete for the two units of the new-generation product, the first two buyers will receive no more than \$4 share of surplus. Hence, the upgrading firm, say Firm 1, will receive $(2 \times (p_{new} - \$4)) = \$6$ share of surplus and will end up with $\$6 - c_1 = \$1$ profit. The firm that did not invest in an upgrade does not receive a share of the surplus.

Finally, consider the case when both firms choose to invest in the new technology. The total revenue is $(3 \times p_{new}) = \$21$. The two firms compete away for three buyers, therefore not only each firm does not receive a share of the surplus, the firms end up with \$5 loss each while each buyer captures a surplus of \$7 per unit purchased. The argument for this equilibrium outcome is that the firms cannot undo their \$5 investments, therefore, they can only minimize loss. When one firm asks for, say \$2.5, for each unit (zero loss), the other firm may ask for \$2.4, and so forth.

The three above cases can be modeled as four different cooperative games as follows:

$(N, V^{\vec{x}=(0,0)}(S)), (N, V^{\vec{x}=(0,1)}(S)), (N, V^{\vec{x}=(1,0)}(S))$, and $(N, V^{\vec{x}=(1,1)}(S))$. Each game has a single-point core solution. Hence, each second-stage game is deterministic as a result of first-stage decisions by two firms.

Consider two approaches for recasting this problem as mathematical programming problem. In the first approach ignore the competitive equilibrium and augment the problem with a probability distribution that mimics (represents) the belief regarding the likelihood of the opponent investing in the technological upgrade – the new-generation product. In this approach the likelihood of opponent's \$5 investment decision is modeled as an uncertain event in the two-stage stochastic programming model. The second approach models this game as a system of two deterministic 0,1 integer linear programs, one for each firm, followed by Nash equilibrium strategy resolution.

<u>The first approach:</u> Because of symmetry we only analyze Firm 1. The first-stage decision of Firm 1 is denoted by a binary decision variable $x_1 \in \{0, 1\}$ indicating no investment and \$5 investment, respectively. Let ξ' be a random variable with two potential values representing Firm 2's decision ($\xi' \in \{0, 1\}$).

Let $v_1(x_1, \xi)$ be the second-stage profit achievable by Firm 1 based on its first-stage decision x_1 and Firm 2's r.v. instantiation ξ. The two-stage stochastic programming problem of Firm 1 can be written as

$$\max_{x_1 \in \{0,1\}} \quad -c_1 x_1 + E_{\xi'}[v_1(x_1, \xi')] \tag{4.2}$$

That is, Firm 1 is selecting x_1 that maximizes the negative value of investment plus her second-stage expected profit.

Now assume that Firm 1 believes that Firm 2 will invest with probability $\beta > 0$. We can rewrite problem (4.2) as:

$$\max_{x_1 \in \{0,1\}} \quad -c_1 x_1 + \beta(v_1(x_1, 1)) + (1 - \beta)(v_1(x_1, 0)) \tag{4.3}$$

Firm 1's second-stage profit has the following form:

$$v_1(x_1, \xi) = \begin{cases} \$0 & \text{if } x_1 = 0 \text{ and } \xi = 0 \\ \$0 & \text{if } x_1 = 0 \text{ and } \xi = 1 \\ \$6 & \text{if } x_1 = 1 \text{ and } \xi = 0 \\ \$0 & \text{if } x_1 = 1 \text{ and } \xi = 1 \end{cases}$$

Thus, the problem in (4.3) can be reduced to

$$\max_{x_1 \in \{0,1\}} \quad -\$5 x_1 + (1 - \beta)(v_1(x_1, 0))$$

That is, Firm 1 ought to invest upfront in the new-generation product if she believes that β is less than 1/6, and stay put if β is greater than 1/6. Firm 1 is indifferent between the two options when $\beta = 1/6$.

Second approach: Consider recasting this example as a 0,1 integer linear programming problem – a special (degenerate) case of two-stage stochastic programming with recourse. The first-stage decision of each firm is denoted by a binary decision variable $x_i \in \{0, 1\}, i = 1, 2$, indicating no investment and \$5 investment, respectively.

In the second stage, after the first-stage investment decision $\vec{x} = (x_1, x_2)$ has been implemented, the market settles.

Let $v_i(\vec{x})$ represent the second-stage profit achievable by firm i for a given first-stage decision $\vec{x}$. Firm i's 0,1 integer programming problem can be written as

$$\max_{x_i \in \{0,1\}} \qquad -cx_i + v_i(\vec{x})$$

That is, firm i selects x_i that maximizes the negative value of investment plus her second-stage profit. Firm i's $(i \in \{1,2\}, i \neq j)$ second-stage profit is:

$$v_i(\vec{x}) = \begin{cases} \$0 & \text{if } x_i = 0 \text{ and } x_j = 0 \\ \$0 & \text{if } x_i = 0 \text{ and } x_j = 1 \\ \$6 & \text{if } x_i = 1 \text{ and } x_j = 0 \\ \$0 & \text{if } x_i = 1 \text{ and } x_j = 1 \end{cases}$$

The end result of this model is the Battle of the Sexes game (Luce and Raiffa , 1957; Fudenberg and Tirole, 1991) as shown in Figure 4.1. In this game, there are two pure

		Firm 2	
		Not invest	Invest
Firm 1	Not invest	0, 0	0, 1
	Invest	1, 0	-5, -5

Figure 4.1. Innovation game payoffs

strategy Nash equilibria (x_1, x_2), namely $(0, 1)$ and $(1, 0)$, and one mixed strategy Nash equilibrium asking each firm to invest (choose $x_i = 1$) with probability $1/6$.

Example 4.3. *A Coordination Game* (Brandenburger and Stuart, 2007, Example 5.2).

In this example there are three manufacturers - players in the game. With their current technology each manufacturer's profit is \$2 (after subtracting the production costs). Each player independently and simultaneously with the other two decides prior to production whether to invest in a new technology at the cost of $c_1 = c_2 = c_3 = \$1$. After their individual investment decisions have materialized, the players choose whether to form a coalition with at least one of the two of the other players. Nontrivial coalitions are either pairs or a threesome. A player who does not invest in new technology will earn \$2 regardless of coalition membership. A player who invested in new technology and is a member of a coalition with at least one member who also invested in new technology will earn \$4 (symbiotic effect). A player who invested but is not a member of a coalition with at least one of the members who invested, will only earn \$2. That is, "the new technology costs \$1 more per player, and is worth \$2 more per player, provided at least two players adopt it" (Brandenburger and Stuart, 2007).

Let the first-stage decision of player i, ($i \in \{1, 2, 3\}$), be represented by a binary decision variable $x_i \in \{0, 1\}$ indicating respectively no action or investment in new technology. The coordination game fits the biform game framework of Brandenburger and Stuart (2007) because the first stage is a non-cooperative game and the second stage is a cooperative game $(N, V^{\vec{x}}(S)), S \subseteq N$, where $V^{\vec{x}}(S)$ is a characteristic function of the cooperative game as a result of the first-stage decisions $\vec{x} = (x_1, x_2, x_3)$

of the three players, where

$$
\begin{aligned}
N &= \{\text{Player 1, Player 2, Player 3}\}, \\
V^{\vec{x}}(S) &= \begin{cases} 2 \times |S| + 2 \times \sum_{i \in S} x_i & \text{if } \sum_{i \in S} x_i \geq 2, \text{for all } S \subseteq N \\ 2 \times |S| & \text{otherwise, for all } S \subseteq N \end{cases}
\end{aligned}
$$

Notice that each instance of the second-stage cooperative game has a single-point core. The core of games $(N, V^{\vec{x}=(0,0,0)}(S))$, $(N, V^{\vec{x}=(1,0,0)}(S))$, $(N, V^{\vec{x}=(0,1,0)}(S))$, and $(N, V^{\vec{x}=(0,0,1)}(S))$ is an allocation in which each player gets \$2. The core of the game $(N, V^{\vec{x}=(1,1,1)}(S))$ is an allocation in which each player gets \$4. The core of the game $(N, V^{\vec{x}=(1,1,0)}(S))$ represents an allocation in which players 1 and 2 get \$4 each, and player 3 gets \$2, similarly for $(N, V^{\vec{x}=(1,0,1)}(S))$ and $(N, V^{\vec{x}=(0,1,1)}(S))$. Hence, the second stage in each game is deterministic as a result of first-stage decisions by firms.

Analogous to Example 4.2, we propose two ways of recasting this problem as 0,1 integer programming problems: (1) Ignoring the competitive equilibrium, we can model the problem from the perspective of one of the players (since they are symmetric). We can add a probability distribution that represents the player's belief regarding the likelihood of the other two players' investment in the new technology. That is, the other two players' \$1 investment decision is modeled as a random variable with a known distribution (representing a particular belief) in a two-stage stochastic programming model. (2) We can state this game as a system of three 0,1 programs (deterministic), one for each player. Then, the problem of determining Nash equilibrium strategy can be resolved.

Consider the <u>first approach</u>. First-stage decision of Player 1 is denoted by a binary decision variable $x_1 \in \{0, 1\}$ indicating no investment and \$1 investment, respectively. Let ξ_2', ξ_3' be random variables representing Players 2 and 3 decision values, where ξ_2', $\xi_3' \in \{0, 1\}$.

Let $v_1(x_1, \xi_2, \xi_3)$ represent the second-stage profit achievable by Player 1 given her first-stage decision x_1, Player 2's instantiation of the r.v. $= \xi_2$ (her decision), and Player 3's instantiation value of the r.v. $= \xi_3$ (her decision). The two-stage stochastic programming problem of Player 1 can be written as

$$\max_{x_1 \in \{0,1\}} \quad -c_1 x_1 + E_{\xi_2', \xi_3'}[v_1(x_1, \xi_2', \xi_3')]$$

That is, Player 1 is selecting x_1 that maximizes the negative value of investment plus her second-stage expected profit. Suppose that Player 1 believes that players 2 and 3 will choose to invest with a probability $\beta_2 > 0$ and $\beta_3 > 0$, respectively. Say ξ_2' and ξ_3' are independent. We can rewrite the above problem as:

$$\begin{aligned} \max_{x_1 \in \{0,1\}} \quad & -c_1 x_1 + \beta_2 \beta_3 (v_1(x_1, 1, 1)) + \beta_2 (1 - \beta_3)(v_1(x_1, 1, 0)) \\ & +(1 - \beta_2)\beta_3 (v_1(x_1, 0, 1)) + (1 - \beta_2)(1 - \beta_3)(v_1(x_1, 0, 0)) \end{aligned} \tag{4.4}$$

The second-stage \$ profit of Player 1 can be written as:

$$v_1(x_1, \xi_2, \xi_3) = \begin{cases} 4 & \text{if } x_1 = 1 \text{ and } \xi_2 + \xi_3 \geq 1 \\ 2 & \text{otherwise} \end{cases}$$

The problem in (4.4) can be reduced to

$$\max_{x_1 \in \{0,1\}} \quad -x_1 + x_1(2\beta_2 + 2\beta_3 - 2\beta_2\beta_3)$$

The result is that Player 1 should invest upfront in the new technology if she believes that $2\beta_2 + 2\beta_3 - 2\beta_2\beta_3$ is greater than 1, and not invest if $2\beta_2 + 2\beta_3 - 2\beta_2\beta_3$ is less than 1. She is indifferent between the two options when $2\beta_2 + 2\beta_3 - 2\beta_2\beta_3$ equals 1. By introducing different distribution assumptions on ξ_2' and ξ_3' we can model this way a plethora of problem versions.

Consider the <u>second approach</u> that recasts this same example as a binary integer programming problem – a special (degenerate) case of two-stage stochastic programming with recourse. Player i first-stage decision is represented by a binary decision variable $x_i \in \{0, 1\}, i = 1, 2, 3,$ indicating no investment and \$1 investment, respectively. In the second stage, after the first-stage investment decision $\vec{x} = (x_1, x_2, x_3)$ has taken place, the corresponding cooperative game is played.

Let $v_i(\vec{x})$ denote the second-stage earnings for player i given first-stage decisions $\vec{x}$. The binary integer programming problem of player i can be written as

$$\max_{x_i \in \{0,1\}} \qquad -cx_i + v_i(\vec{x})$$

That is, player i will choose x_i that maximizes her second-stage earning. From the core solution of the cooperative game $(N, V^{\vec{x}}(S))$, the second-stage profit of Player i can be written as

$$v_i(\vec{x}) = \begin{cases} \$4 & \text{if } x_1 + x_2 + x_3 \geq 2 \text{ and } x_i = 1 \\ \$2 & \text{otherwise} \end{cases}$$

Observe that in this game there are two (pure-strategy) Nash equilibria. The two equilibria prescribe either for all to invest, or for all not to invest.

Example 4.4. *Creating Monopoly Power* (Stuart, 2007b) .

In this example the players' set consists of one seller and twelve buyers ($b = 12$). Each buyer $j \in \{1, \dots, b\}$ has a willingness-to-pay value of w_j dollars for one product unit from the seller, where $w_j = 14 - j, j = 1, \dots, b$. Thus, $w_1 > w_2 > \dots > w_b > 0$. The seller has to decide upfront on her production capacity; she may install up to b units of capacity. Assume that the seller has a marginal capacity cost of c_c per unit and zero marginal production cost — $c_p = 0$.

The sequence in this game is as follows: In the first stage, the seller chooses production capacity of $x \in \{0, 1, 2, \dots, b\}$ for her facility. The capacity installation costs are linear in c_c (= \$1 per unit), thus, the seller's total first-stage cost equals $c_c x$. In the second stage the twelve buyers and the seller form a coalition that is likely to maximize their surplus and the seller manufactures at zero cost any number $y \leq x$ of product.

Similar to previous examples, the second stage can be represented by a cooperative game. Let $B = \{1, 2, \dots, b\}$ be the set of buyers and $N = \{\text{Seller}\} \cup B$. In this cooperative game, given first-stage capacity decision x, the seller and a subset of buyers form a coalition $S \subseteq N$. The seller produces y units of product, and the coalition generates a surplus $V^x(S)$.

Let $\chi_S(j)$ be the characteristic function of S such that $\chi_S(j) = 1$ when $j \in S$, and $\chi_S(j) = 0$ when $j \notin S$. Hence, for each integer $x \in \{0, 1, 2, \dots, b\}$, the characteristic function of the game $(N, V^x(S)), S \subseteq N$ is given by:

$$V^x(S) = \begin{cases} \$0 & \text{if } S \subseteq B, \\ & \text{ or } S = \{Seller\}, \\ & \\ -c_p \min\{x, |S|-1\} + \sum_{j=1}^{R} \chi_S(j) w_j & \text{otherwise} \end{cases} \quad (4.5)$$

where R is the index of the buyer in S who has lowest willingness-to-pay but is still able to buy the product. That is,

$$R = \max\{r : \textstyle\sum_{j=1}^{r} \chi_S(j) \leq \min(x, |S|-1)\}$$

$$V^x(N) = \textstyle\sum_{j=1}^{x}\{w_j - c_p\} \text{ or essentially } = \sum_{j=1}^{x} w_j \quad \text{since } c_p = 0.$$

Now, let $v^x_{seller}, v^x_1, \ldots, v^x_b$, represent the allocation of $V^x(N)$ to the seller and each buyer, respectively. The conditions for nonempty core require that

$$\sum_{i \in N} v^x_i = V^x(N) \quad (4.6)$$

$$\sum_{i \in S} v^x_i \geq V^x(S) \quad (4.7)$$

For this game, the core is nonempty because $V^x(\cdot)$ is superadditive and $V^x(S) = 0$ for all $S \subseteq B$. ($v^x_{seller} = V^x(N)$ and $v^x_i = 0, i = 1, \ldots, b$, is an allocation satisfying (4.6) and (4.7).) The observation below states conditions for nonempty core that are in this case equivalent to (4.6), (4.7).

Observation 4.1. (Stuart, 2007b, Proposition 1) *The core of the monopoly game* $(N, V^x(S))$ *is not empty if (4.8), (4.9), and (4.10) are satisfied.*

$$v^x_{seller} + \textstyle\sum_{j=1}^{b} v^x_j = V^x(N), \quad (4.8)$$

$$0 \leq v^x_j \leq V^x(N) - V^x(N \backslash \{j\}) \text{ for } j \in B, \quad (4.9)$$

$$V^x(N) - \textstyle\sum_{j=1}^{b} \{V^x(N) - V^x(N \backslash \{j\})\} \leq v^x_{seller} \leq V^x(N). \quad (4.10)$$

Proof. It is clear that conditions (4.8), (4.9), and (4.10) are necessary for nonempty core and (4.8) implies (is the same as) (4.6). Therefore, we only need to show that (4.8), (4.9), and (4.10) are also sufficient for establishing a nonempty core. That is, we have to show that (4.8), (4.9), and (4.10) imply (4.7). Because $V^x(S) = 0$ for all $S \subseteq B$, we only need to consider $S \subseteq N, S \ni$ Seller.

From (4.8) and (4.9), we have

$$\sum_{i \in S} v_i^x = V^x(N) - \sum_{j \in N \backslash S} v_j^x \geq V^x(N) - \sum_{j \in N \backslash S} \{V^x(N) - V^x(N \backslash \{j\})\} \tag{4.11}$$

The term $\sum_{j \in N \backslash S} \{V^x(N) - V^x(N \backslash \{j\})\}$ is the sum of marginal contribution of each buyer j in $N \backslash S$. From (4.5) we have

$$\sum_{j \in N \backslash S} \{V^x(N) - V^x(N \backslash \{j\})\} = \left(\sum_{j \in N \backslash S, j \leq x} w_j \right) - w_{x+1} \min(t, b - x) \tag{4.12}$$

$$V^x(N) - V^x(S) = \left(\sum_{j \in N \backslash S, j \leq x} w_j \right) - (w_{x+1} + \ldots + w_{\min(x+t, b)})$$

where $t = |T|$ and $T = \{j : j \in N \backslash S, j \leq x\}$. Thus,

$$\sum_{j \in N \backslash S} \{V^x(N) - V^x(N \backslash \{j\})\} \leq V^x(N) - V^x(S)$$

Substitute this in (4.11) to reach $\sum_{i \in S} v_i^x \geq V^x(S)$ for all $S \subseteq N$. Therefore, all conditions (4.8), (4.9), and (4.10) are necessary and sufficient for nonempty core. $\square$

Below, we recast this game as a two-stage stochastic programming with recourse problem. The first-stage decision of the seller is represented by the integer variable $x \geq 0$ indicating the production capacity that requires an upfront investment.

Let the random variable ξ' represent a fraction of residual profit captured by the seller reflecting the seller's bargaining power in relation to the buyers. The random variable ξ' highlights the uncertainty in the bargaining outcome. In the second stage, after the first-stage investment decision x has taken place, the twelves buyers compete for the products. Following the competition outcome the winning buyers and the seller bargain on how to distribute the total profit (revenue less the sum of the marginal capacity costs) between them. That is, after the instantiation $\xi \in [0,1]$ of the r.v. ξ' is determined through bargaining, then the seller will receive the guaranteed amount and in addition she will receive ξ times the residual profit. In this case, the guaranteed amount is the lower bound in (4.10) and the residual profit is the difference between the upper bound and lower bound.

Formally, with a slight change of notation let $v_{seller}(x, \xi)$ represent the second-stage profit achievable by the seller for a given first-stage decision x and instantiation value ξ. The stochastic programming problem can be stated as

$$\begin{aligned} f_{seller}(x^*) = \max_x \quad & -c_c x + E_{\xi'}[v_{seller}(x, \xi')] \\ \text{s.t.} \quad & 0 \leq x \leq b \end{aligned}$$

The second-stage profit $v_{seller}(x, \xi)$ is

$$\begin{aligned} v_{seller}(x, \xi) &= V^x(N) - \sum_{j=1}^{b} \{V^x(N) - V^x(N \backslash \{j\})\} \\ & \quad + \xi \sum_{j=1}^{b} \{V^x(N) - V^x(N \backslash \{j\})\} \\ &= V^x(N) + (\xi - 1) \sum_{j=1}^{b} \{V^x(N) - V^x(N \backslash \{j\})\} \text{ for } j = 1, \ldots, b. \end{aligned}$$

From (4.12) we have

$$\sum_{j=1}^{b} \{V^x(N) - V^x(N\backslash\{j\})\} = \begin{cases} \sum_{j=1}^{x} \{w_j - w_{x+1}\} & \text{if } x < b, \\ \sum_{j=1}^{x} \{w_j\} & \text{if } x = b. \end{cases}$$

$$v_{seller}(x, \xi) = \begin{cases} \sum_{j=1}^{x} \{w_j\} + (\xi - 1) \sum_{j=1}^{x} \{w_j - w_{x+1}\} & \text{if } x < b, \\ \sum_{j=1}^{x} \{w_j\} + (\xi - 1) \sum_{j=1}^{x} \{w_j\} & \text{if } x = b. \end{cases} \tag{4.13}$$

In terms of probability distribution of ξ'; it depends on the seller's belief system. Again consider the confidence index $\alpha \in [0, 1]$ as in Brandenburger and Stuart (2007). It can be stated as a probability distribution function of ξ' as follows:

$$Pr\{\xi' = 1\} = \alpha, \quad Pr\{\xi' = 0\} = 1 - \alpha$$

From (4.13), we know that $v_{seller}(x, 1) = \sum_{j=1}^{x} \{w_j\}$ and

$$v_{seller}(x, 0) = \begin{cases} (w_{x+1})x & \text{if } x < b, \\ 0 & \text{if } x = b. \end{cases}$$

Hence, the two-stage integer stochastic program of the monopoly power game, assuming the confidence index α, is

$$\begin{aligned} f_{seller}(x^*) &= \max_{0 \le x \le b} \quad -c_c x + E_{\xi'}[v_{seller}(x, \xi')] \\ &= \max_{0 \le x \le b} \quad -c_c x + [Pr\{\xi' = 0\} \times v_{seller}(x, 0)] + [Pr\{\xi' = 1\} \times v_{seller}(x, 1)] \\ &= \max_{0 \le x \le b} \quad -c_c x + [(1 - \alpha) v_{seller}(x, 0)] + [\alpha v_{seller}(x, 1)]\,. \end{aligned}$$

Given that there are $b = 12$ buyers, each with willingness-to-pay $w_j = 14 - j$, capacity cost $c_c = 1$, and zero production cost, the second-stage profit for $\xi = 1$ is $v_{seller}(x, 1) = 14x - \sum_{j=1}^{x} j = 13.5x - 0.5x^2$, and for $\xi = 0$ is $v_{seller}(x, 0) = 13x - x^2$ for $x < b$.

We can restate the above as an integer programming problem:

$$f_{seller}(x^*) = \max_{0 \leq x < b} \quad -x + (1-\alpha)(13x - x^2) + \alpha(13.5x - 0.5x^2); x \text{ is integer} \tag{4.14}$$

The optimal solution to (4.14) requires for the seller to install the production capacity $x = (12 + 0.5\alpha)/(2 - \alpha)$. If the seller is pessimistic ($\alpha = 0$), she should install 6-unit production capacity in the first stage. If the seller is optimistic ($\alpha = 1$), she should install 12-unit production capacity. This is the same amount as for a market with perfect price discrimination. If her belief is at midpoint ($\alpha = 0.5$), she should install 8-unit production capacity. Notice that if an allocation chosen for the second-stage cooperative game is based on the Shapley value, the seller will also choose 12-unit production capacity (see Stuart, 2007b, Proposition 4).

Another informative paper, Stuart (2007a), analyzes a situation that resembles the monopoly setting of Stuart (2007b) but it removes the restriction on the number of units that each buyer is willing to buy and modifies each buyer's willingness-to-pay for each unit she is willing to buy. The main result of Stuart (2007a) is that the seller prefers symmetric buyers. We include a stochastic programming model for Stuart (2007a) in the Appendix and note that Example 4.4 can be modeled as a special case of Stuart (2007a).

Example 4.5. *Horizontal Scope* (Chatain and Zemsky, 2007).

In this example, there are $m > 2$ potential suppliers that may enter the outsourcing market at a fixed setup cost $c > 0$. In the outsourcing market, there are $n > 0$ buyers (one can think of buyers as manufacturers or specialized construction companies), each outsourcing two tasks, A and/or B. We follow the setting described in Chatain and Zemsky (2007) and assume that a fraction p of the buyers are outsourcing one A task and one B task. Let set AB contain these type of buyers. $|AB| = pn$. In addition, sets AA and BB that contain buyers who outsource two A tasks and two B tasks, respectively. For simplicity $|AA| = |BB| = (1-p)n/2$.

Each supplier may serve any number of buyers. Each supplier's value creation depends on her organization design. She may choose an organization design that specializes in either A or B at the highest value, or provide a combination of A and B, each at a lower value. Let $d_i \in [0, 1]$ denote the organization design decision variable of supplier i. The value created by performing one task A and one task B for one buyer, respectively, are:

$$V_A(d_i) = 1 - td_i^2;\ \ V_B(d_i) = 1 - t(1-d_i)^2,$$

where $0 < t < 1$ reflects an organizational tradeoff parameter. When a buyer contracts only one supplier to perform both of her tasks, resulting in a client-specific scope economies, a surplus of $r \geq 0$ is also created in addition to the value from performing the tasks as described above.

We can see that, if the supplier wants to specialize in task A, she should choose $d_i = 0$. On the other hand, if the supplier wants to specialize in task B, she should choose $d_i = 1$. According to Chatain and Zemsky (2007), "The parameter t gives the importance of organization tradeoffs: The greater the organizational tradeoffs, the more the surplus falls as a supplier's organization diverges from the optimal design for a given task."

The sequence in this game is as follows: In the first stage, each supplier i must decide upfront whether or not to enter the market. Moreover, if she enters the market, she must also choose an organization design d_i that will maximize her expected profit. In the second stage, set $\mathcal{B} = \{1, \ldots, n\}$ of buyers enter the outsourcing market and cooperative game is played.

This example can be modeled as a two-stage mixed-integer programming problem. The first-stage decisions of supplier i are represented by two variables: a binary variable $x_i \in \{0, 1\}$ indicating her decision to enter the outsourcing (second-stage) market, and a variable $d_i \in [0, 1]$ indicating her chosen organization design. In the second stage, after the first-stage decisions values for x_i and d_i for all $i \in \{1, \ldots, m\}$ have been set, the market responds optimally.

The second-stage decision can be seen as a cooperative game $(N, V^{(\vec{x},\vec{d})}(S)), S \subseteq N$, where $V^{(\vec{x},\vec{d})}(S)$ is a characteristic function of the cooperative game that is a function of the first-stage decisions $(\vec{x}, \vec{d})$. Let $\mathcal{F} = \{1, \ldots, m\}$ be a set of suppliers, $\mathcal{B} = AA \cup BB \cup AB$ be a set of buyers, and $N = \mathcal{F} \cup \mathcal{B}$.

First, consider the case when there is only one buyer and a subset of suppliers. The most surplus would be generated by having the buyer outsource to the suppliers who create the most value. This would depend on the type of buyer j as shown below.

Let $R \subseteq \mathcal{F}$. For for all $R \subseteq \mathcal{F}$, the game characteristic function is

$$V^{(\vec{x},\vec{d})}(\{j\} \cup R) = \begin{cases} 2 - 2t\min_{i\in R}(d_i)^2 + r & \text{if } j \in AA, \\ 2 - 2t\min_{i\in R}(1-d_i)^2 + r & \text{if } j \in BB, \\ \max\{V_{AB1}^{(\vec{x},\vec{d})}(\{j\} \cup R), V_{AB2}^{(\vec{x},\vec{d})}(\{j\} \cup R)\} & \text{if } j \in AB, \end{cases}$$

where $V_{AB1}^{(\vec{x},\vec{d})}(\{j\} \cup R)$ represents the value when a buyer of AB type outsources both tasks A and B to only one supplier, and $V_{AB2}^{(\vec{x},\vec{d})}(\{j\} \cup R)$ represents the value when a buyer of AB type outsources to two different suppliers.

$$\begin{aligned} V_{AB1}^{(\vec{x},\vec{d})}(\{j\} \cup R) &= 2 + r - t\min_{i\in R}(d_i^2 + (1-d_i)^2) \\ V_{AB2}^{(\vec{x},\vec{d})}(\{j\} \cup R) &= 2 - t\min_{i\in R}(d_i)^2 - t\min_{i\in R}(1-d_i)^2. \end{aligned}$$

Going back to n buyers, since each supplier has unlimited capacity, when a number of buyers are members of coalition S, each buyer in S will outsource to the supplier that suits her best. In fact, if each supplier i chooses a distinct value for d_i, at least two, at most three suppliers will be active outsourcing providers. Two of these active suppliers who will serve buyers in AA and BB, respectively, are the suppliers with the lowest d_i organization design, possibly 0, and the highest d_i organization design, possibly 1. One other supplier i might be active if $r - t(d_i^2 + (1-d_i)^2) > 0$ and $i = \arg\min_{i\in\mathcal{F}}(d_i^2 + (1-d_i)^2)$. One can assume that this supplier i has $d_i = 0.5$ since it minimizes $d_i^2 + (1-d_i)^2$.

Because the above is true for every $j \in \mathcal{B}$, the characteristic function of the cooperative game for multiple buyers is

$$V^{(\vec{x},\vec{d})}(S) = \begin{cases} 0 & \text{if } S \subseteq \mathcal{B}, \text{ or } S \subseteq \mathcal{F}, \\ \sum_{j\in(S\cap\mathcal{B})} V^{(\vec{x},\vec{d})}(\{j\} \cup (S \cap \mathcal{F})) & \text{otherwise.} \end{cases}$$

For the grand coalition N the value is

$$V^{(\vec{x},\vec{d})}(N) = \sum_{j\in\mathcal{B}} V^{(\vec{x},\vec{d})}(\{j\}\cup\mathcal{F}) \tag{4.15}$$

Chatain and Zemsky (2007, Proposition 1) shows that there always exist core allocations because the set of players N can be split into two nonempty, disjoint sets $\mathcal{F}$ and $\mathcal{B}$ such that $V^{(\vec{x},\vec{d})}(\mathcal{F}) = 0$ and $V^{(\vec{x},\vec{d})}(\mathcal{B}) = 0$. In addition, the value of the grand coalition is the sum of the value created by each buyer cooperating with all suppliers as shown in (4.15). In fact, an allocation with $v_j = V^{(\vec{x},\vec{d})}(\{j\}\cup\mathcal{F})$ for $j \in \mathcal{B}$ and $v_i = 0$ for all $i \in \mathcal{F}$ is always in the core of this game as shown below.

$$\begin{aligned}
\sum_{k\in N} v_k &= \sum_{k\in\mathcal{B}} V^{(\vec{x},\vec{d})}(\{k\}\cup\mathcal{F}) = V^{(\vec{x},\vec{d})}(N) \\
\sum_{k\in S} v_k &= \sum_{k\in(S\cap\mathcal{B})} V^{(\vec{x},\vec{d})}(\{k\}\cup\mathcal{F}) \text{ for all } S\subseteq N \\
&\geq \sum_{k\in(S\cap\mathcal{B})} V^{(\vec{x},\vec{d})}(\{k\}\cup(S\cap\mathcal{F})) = V^{(\vec{x},\vec{d})}(S)
\end{aligned}$$

Now, consider a polyhedral set that describes the core of this game. For any supplier $i \in \mathcal{F}$, the core allocation value v_i satisfies the following:

$$0 \leq v_i \leq V^{(\vec{x},\vec{d})}(N) - V^{(\vec{x},\vec{d})}(N\backslash i). \tag{4.16}$$

This means that supplier i may receive up to her marginal contribution. Consider the case that she is the supplier with $d_i = 1$, then all buyers in BB would outsource to her and thus generate a surplus value of say a dollars. If supplier i is excluded from the grand coalition, all buyers in BB would outsource to the supplier with the second best d_i value and generate b dollars surplus. Thus, $a - b$ is supplier i marginal contribution. If supplier i is not one of the three active suppliers, she will receive 0 – her marginal contribution. Similarly, if each supplier does not choose a distinct value of d_i, e.g., there are many suppliers with $d_i = 1, 0$, and 0.5, then the chosen supplier i will receive her marginal contribution = 0.

Let the random variable $\xi' \in [0, 1]$ represent a fraction of profit captured by the supplier i reflecting her bargaining power in relation to the buyers. The supplier will receive the guaranteed amount plus ξ times the surplus. In this case, the guaranteed amount is the lower bound in (4.16) and the surplus is the difference between the upper bound and lower bound.

Formally, let $v_i(\vec{x}, \vec{d}, \xi')$ represent the surplus allocated to supplier i for a given first-stage collective decisions of all suppliers $\vec{x}$ and $\vec{d}$, and instantiation ξ. The stochastic mixed-integer programming problem can be stated as

$$\begin{aligned} f_i(x_i^*, d_i^*) = \max_{x_i, d_i} \quad & -cx_i + x_i E_{\xi'}[v_i(\vec{x}, \vec{d}, \xi')] \\ \text{s. t.} \quad & x_i \in \{0, 1\} \\ & 0 \leq d_i \leq 1 \end{aligned}$$

The second-stage surplus $v_i(\vec{x}, \vec{d}, \xi)$ is

$$v_i(\vec{x}, \vec{d}, \xi) \;\; = \;\; \xi \left(V^{(\vec{x}, \vec{d})}(N) - V^{(\vec{x}, \vec{d})}(N \backslash i) \right)$$

Assume that the probability distribution of ξ' is given by the confidence index $\alpha \in [0, 1]$. Thus, $v_i(\vec{x}, \vec{d}, 0) = 0$ and $v_i(\vec{x}, \vec{d}, 1) = V^{(\vec{x}, \vec{d})}(N) - V^{(\vec{x}, \vec{d})}(N \backslash i)$. Hence, the stochastic mixed-integer program of the horizontal scope game, assuming the confidence index α, is

$$\begin{aligned} f_i(x_i^*, d_i^*) &= \max_{x_i \in \{0,1\}, d_i \in [0,1]} \quad -cx_i + x_i E_{\xi'}[v_i(\vec{x}, \vec{d}, \xi')] \\ &= \max_{x_i \in \{0,1\}, d_i \in [0,1]} \quad -cx_i + x_i \left[Pr\{\xi' = 0\} \times v_i(\vec{x}, \vec{d}, 0) \right] \\ &\qquad\qquad +x_i \left[Pr\{\xi' = 1\} \times v_i(\vec{x}, \vec{d}, 1) \right] \\ &= \max_{x_i \in \{0,1\}, d_i \in [0,1]} \quad -cx_i + x_i \left(\alpha v_i(\vec{x}, \vec{d}, 1) \right). \end{aligned}$$

The optimal x_i and d_i depend on other suppliers' first-stage decisions. Therefore, Nash Equilibrium analysis is required to explain the result of this game. Based on Chatain and Zemsky (2007), the set of possible Nash equilibria includes; (i) no supplier enters the market, (ii) single supplier i enters the market with $d_i = 0.5$, (iii) two suppliers i, j enter the market with $d_i = 0$ and $d_j = 1$, (iv) three suppliers i, j, k enter the market with $d_i = 0$, $d_j = 1$, $d_k = 0.5$, and (v) two suppliers i, j enter the market with $d_i = 0$ and $d_j > 0.5$, or $d_i = 1$ and $d_j < 0.5$.

Definitely, a supplier should not enter the outsourcing market if her expected share of surplus is less than the fixed setup cost c. This explains why many possible equilibria do not include all three types of suppliers. For instance, when two suppliers with $d_i = 0$ and $d_j = 1$ are already in the market, supplier k's expected shared of surplus is 0 because she may not be chosen, and if she is chosen, there is another supplier who can substitute her at the same value-added rate as hers. For complete pure strategy Nash equilibrium analysis, see Chatain and Zemsky, 2007.

Note that for this example, stochastic programming with recourse model does not directly facilitate solving the problem for Nash equilibria. Nevertheless, modeling this case as a stochastic programming with a recourse model provides a new distinct view of the problem that allows for computational solution procedures grounded in stochastic programming.

4.3. Model

In this section, we describe the general form for reformulation of biform game as a two-stage stochastic programming problem. That is, we state the biform definition of Brandenburger and Stuart (2007) and recast it as a stochastic program.

Let $N = \{1, \ldots, n\}$ be a set of players. Let X_i be set of first-stage competitive strategies available for player i and a strategy $x_i \in X_i$ denotes a strategy carried out by a player i. In biform game, X_i is a finite set. A *payoff function* f_i of player i associates a real number $f_i(x_1, \ldots, x_n) \in \mathbb{R}$ with the strategies $x_1, \ldots, x_n$ chosen by players individually. We use a tuple $(X_1, \ldots, X_n; f_1, \ldots, f_n)$ to denote a first-stage competitive game. The strategies $(x_1^*, \ldots, x_n^*)$ are a *Nash equilibrium* if, for each player i, x_i^* is player i's best response to the strategies $(x_1^*, \ldots, x_{i-1}^*, x_{i+1}^*, \ldots, x_n^*)$ chosen by other players; that is, x_i^* solves $\max_{x_i \in X_i} f_i(x_1^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_n^*)$. Unique pure strategy Nash equilibrium (PSNE) provides arguably a rational prediction of what options players may pursue whenever unique PSNE exists in a game. To describe the payoff function f_i of player i, we need to state the second-stage cooperative game and the biform game's sequence.

The set N is referred to as the *grand coalition*. A nonempty subset $S \subseteq N$

is a *coalition*. There are $2^N - 1$ different nonempty coalitions that may form. A *characteristic function* $V^{(x_1,\ldots,x_n)}(S)$ is a set function such that $V^{(x_1,\ldots,x_n)}(\emptyset) = 0$ and associates a real number $V^{(x_1,\ldots,x_n)}(S) \in \mathbb{R}$ with each subset $S \subseteq N$. We can think of the characteristic function as a surplus that players who are members of S generate as a result of forming a coalition S in the second-stage cooperative game given that strategies $(x_1, \ldots, x_n)$ were played in the first stage. The pair $(N, V^{(x_1,\ldots,x_n)})$ denotes a second-stage cooperative game.

We limit our analysis to the class of second-stage games that satisfy *superadditivity* property, i.e., $V^{(x_1,\ldots,x_n)}(S \cup \{i\}) \geq V^{(x_1,\ldots,x_n)}(S) + V^{(x_1,\ldots,x_n)}(\{i\})$ for every $x_1, \ldots, x_n \in X^1 \times \ldots \times X^n$ and for all $S \subseteq N$ and $i \notin S$.

The decision on how the profits are shared is called an *allocation rule*. An allocation that enables stable cooperation (no subset of players has an economic incentive to withdraw from the grand coalition) is called a *core allocation*. An allocation rule v ($v = (v_1, \ldots, v_n) \in \mathbb{R}^n$) determines an allocation of surplus to each individual player. The *core* is a set of core allocations. An allocation v is in the core of game $(N, V^{(x_1,\ldots,x_n)})$ if $\sum_{i \in S} v_i \geq V^{(x_1,\ldots,x_n)}(S)$ for all $S \subseteq N$ and $\sum_{i \in N} v_i = V^{(x_1,\ldots,x_n)}(N)$. The core, if nonempty, is in general a nontrivial (not a singleton) closed convex polyhedral set. Moreover, cooperative games may have an empty core.

Assume that the second-stage game $(N, V^{(x_1,\ldots,x_n)})$, as part of a biform game, has a non-empty core for every $x_1, \ldots, x_n$. When player i computes $f_i(\cdot)$ and chooses her first-stage strategy, she must consider the plausible set of stable v_i values. Let

$C(N, V^{(x_1,\ldots,x_n)})$ denote the core of the second-stage cooperative game. Define

$$\begin{aligned}\underline{v_i}^{(x_1,\ldots,x_n)} &= \min\{v_i \,\big|\, \text{allocation rule } v = (v_1, ..., v_n) \in C(N, V^{(x_1,\ldots,x_n)})\} \quad (4.17)\\ \overline{v_i}^{(x_1,\ldots,x_n)} &= \max\{v_i \,\big|\, \text{allocation rule } v = (v_1, ..., v_n) \in C(N, V^{(x_1,\ldots,x_n)})\}.\end{aligned}$$

Brandenburger and Stuart (2007) proposes a way to include the second-stage allocation into $f_i(\cdot)$ by using a confidence index $\alpha \in [0, 1]$. That is, player i believes that she will earn $\overline{v_i}^{(x_1,\ldots,x_n)}$ and $\underline{v_i}^{(x_1,\ldots,x_n)}$ with confidence α_i and $1-\alpha_i$, respectively. Given the proposed computation method, Brandenburger and Stuart (2007) define the biform game model as following:

Definition 4.1. *(Brandenburger and Stuart, 2007, Definition 4.1) An* n-player biform game *is a collection* $(X_1, \ldots, X_n; V; \alpha_1, \ldots, \alpha_n)$ *where*

(a) for each $i = 1, \ldots, n$, X_i *is a finite set;*

(b) V *is a map from* $X_1 \times \ldots \times X_n$ *to the set of maps from* $P(N)$*, a power set of* N*, to the reals, with* $V^{(x_1,\ldots,x_n)}(\emptyset) = 0$ *for every* $x^1, \ldots, x^n \in X^1 \times \ldots \times X^n$ *and*

(c) for each $i = 1, \ldots, n, 0 \leq \alpha_i \leq 1$.

For a general two-stage stochastic programming model, the *first-stage decisions* $x \in X$ are taken at a cost $c^1(x)$ without full information regarding the random variable $\tilde{\xi}$. Subsequently, after executing x, the uncertainty event the true value of $\tilde{\xi}$, ξ is revealed. At that point, the *second-stage decisions* $y \in Y$ may be taken at a cost $c^2(y, x, \xi)$ as a response to the value ξ and the first stage decision x. The objective is to minimize the expected cost (equivalently, maximize expected profit); this process is modeled as an optimization problem of determining x that minimizes the following:

$$\min_{x \in X} \qquad c^1(x) + E_{\tilde{\xi}}[Q(x, \xi)]$$

where X is the feasible set for x and the *recourse function* is:

$$Q(x, \xi) = \min_{y \in Y} \qquad c^2(y, x, \xi).$$

We can describe the first-stage competitive payoff of biform game using stochastic programming as:

$$f_i(x_1, \ldots, x_{i-1}, x_i^*, x_{i+1}, \ldots, x_n) = \max_{x_i \in X_i} \qquad -c^1(x_i) + E_{\xi'}[v_i(\vec{x}, \xi')]$$

where $c^1(x_i)$ is player i's cost of her first-stage decision x_i and $E_{\xi'}[v_i(\vec{x}, \xi')]$ is the expectation of her second-stage revenue with respect to uncertainty represented by random variable ξ'. Uncertainty exists due to a number of reasons. First, in the biform game setting, players do not make ex-ante agreement on the surplus allocation rule to be used in the second-stage game. After the first-stage decisions have taken place, there could be many ways to share surplus fairly among players (if the second-stage game has a nontrivial core). Player i usually does not know for certain the chosen surplus allocation rule. Second, there are cases that there is no stable solution for sharing the surplus among players in the grand coalition (the second-stage game has an empty core), it can be anticipated that players may form stable subcoalitions. Again, Player i may not know for certain her share of surplus from each stable subcoalition. We can only assume that she wants to cooperate in the subcoalition that maximizes her expected share of the surplus.

In the biform framework literature, the focus has been directed to the case of a nontrivial core. The biform game model assumes that the player's parameter value $\alpha \in [0,1]$ represents her belief in the likelihood of earning the maximum possible share of surplus. The probability distribution of ξ' has usually been described as: $Pr\{\xi' = 1\} = \alpha, \quad Pr\{\xi' = 0\} = 1 - \alpha$.

Observation 4.2. *For a given biform game* $(X_1, \ldots, X_n; V; \alpha_1, \ldots, \alpha_n)$*, if the second-stage game* $(N, V^{(x_1,\ldots,x_n)})$ *for all* $(x^1, \ldots, x^n) \in X^1 \times \ldots \times X^n$ *always has a non-empty core, then the first-stage competitive payoff using stochastic programming problem representation is*

$$\begin{aligned} & f_i(x_1, \ldots, x_{i-1}, x_i^*, x_{i+1}, \ldots, x_n) \\ & \qquad = \max_{x_i \in X_i} -c^1(x_i) + \alpha \overline{v_i}^{(x_1,\ldots,x_n)} + (1-\alpha)\underline{v_i}^{(x_1,\ldots,x_n)} \end{aligned} \tag{4.18}$$

Probability distribution of ξ' can also be uniform, truncated normal, based on supplier's bargaining history, etc. Regardless of distribution, given that random variable $\xi' \in [0,1]$, we always have $E_{\xi'}[\xi'] = \hat{\alpha}$. That is, one can translate any distribution of $\xi' \in [0,1]$ to a single number – confidence index – in the biform game computation. However, when an uncertainty ξ' in the surplus captured by player i in the cooperative stage is not expressed simply by the difference between the upper and the lower values but is correlated with (a function of), say, demand uncertainty, or other intrinsic system variable, then $E_{\xi'}[\xi']$ is not likely to be same as α – the biform's belief value. Biform framework is not general enough to model this type of uncertainty, while stochastic programming methodology can model it just fine.

When the core is a single point, the confidence index plays no role in the computation of the payoff because $\overline{v_i}^{(x_1,\dots,x_n)} = \underline{v_i}^{(x_1,\dots,x_n)}$.

All games in the examples given in the previous section satisfy the non-empty core condition as stated above. These games are superadditive and have 2 types of players such that each player type on her own has zero value. This family of games is called "market games" (Shapley and Shubik, 1969) and constitutes an important subset of assignment games that always have a nonempty core.

4.4. Emptiness of the Core

So far we considered only the case with a nonempty core in the second-stage cooperation phase. However, the existence of a nonempty core cannot be guaranteed. In this section, with an aid of an example, we examine an empty-core second-stage scenario. That is, what issues need to be addressed if second-stage game has an empty core. Recall that each combination of first-stage decisions $(x_1, \dots, x_n) \in X^1 \times \dots \times X^n$ generates a specific second-stage cooperative game $(N, V^{(x_1,\dots,x_n)})$. When the first-stage game requires binary decisions ($X_i = \{0, 1\}$ for all $i \in N$) there could be as many as $2^n - 1$ nontrivial different second-stage cooperative games. It is possible that some of these games will have an empty core. In this section we illustrate how we can extend the use of the confidence index to games with empty core and compute the expected surplus for each player.

In order to use the confidence index of Brandenburger and Stuart (2007), the upper bound and lower bound value of the potential profit share for each of the players must be provided. For the game with non-empty core, the bounds are defined as (4.17) and used in (4.18) to calculate the expected second-stage share of surplus. For the game with an empty core, the lower bound calculation is more direct; it is the value if the player i chooses to operate alone – $V^{(x_1,\dots,x_n)}(\{i\})$. That is, we can always require that the lower bound satisfies the "individual rationality" constraint, $v_i \geq V^{(x_1,\dots,x_n)}(\{i\})$. There are a number of ways to define the corresponding upper bound. Since we cannot assume that the grand coalition will form (the core is empty), for games in which superadditivity holds, the value $\overline{v_i}^{(x_1,\dots,x_n)} = V^{(x_1,\dots,x_n)}(N) - \sum_{j \in N \setminus \{i\}} V^{(x_1,\dots,x_n)}(\{j\})$ can still serve as an upper bound. However, the confidence index as in Brandenburger and Stuart (2007), though technically still applicable, has a questionable interpretation in this case.

Given that one can always guarantee that the allocation rule satisfies the individual rationality constraint for games in which superadditivity holds, the focus is shifted to the estimation of the upper bound. If the core is empty then we assume that some subset of players, say a subset $T \subseteq N$, may form a "stable" subcoalition and restrict our analysis to the reduced game (subgame) for $T \subseteq N$. Given that there might be a number of candidate stable reduced games for $i \in N$, we examine a stable subcoaltion selection options for player i that maximize her expected share of the game's surplus.

For game $(N, V^{(x_1,\ldots,x_n)})$, let a subgame of $(N, V^{(x_1,\ldots,x_n)})$ be game $(T, V^{T,(x_1,\ldots,x_n)})$, where $\emptyset \neq T \subseteq N$ and $V^{T,(x_1,\ldots,x_n)}(S) = V^{(x_1,\ldots,x_n)}(S)$ for all $S \subseteq T$. The coalition T is stable when its core is non-empty, that is $C(T, V^{T,(x_1,\ldots,x_n)}) \neq \emptyset$. Now we can define the new bounds as before.

$$\begin{aligned}
\underline{v_i}^{T,(x_1,\ldots,x_n)} &= \min\{v_i \,\big|\, \text{allocation rule } v = (v_j)_{j\in T} \in C(T, V^{T,(x_1,\ldots,x_n)})\} \\
\overline{v_i}^{T,(x_1,\ldots,x_n)} &= \max\{v_i \,\big|\, \text{allocation rule } v = (v_j)_{j\in T} \in C(T, V^{T,(x_1,\ldots,x_n)})\}.
\end{aligned}$$

These bounds are the upper and lower bounds of the surplus share of player i given the nonempty core of the subgame $(T, V^{(x_1,\ldots,x_n),T})$. The bounds are used to calculate the expected share of surplus in the two-stage stochastic programming model as follows:

$$f_i(x_1,\ldots,x_{i-1},x_i^*,x_{i+1},\ldots,x_n) = \max_{x_i\in X_i} \quad -c^1(x_i) + Q_i(x_1,\ldots,x_n)$$

where

$$\begin{aligned}
Q_i(x_1,\ldots,x_n) = \max_{T\subseteq N} &\quad \alpha\overline{v_i}^{T,(x_1,\ldots,x_n)} + (1-\alpha)\underline{v_i}^{T,(x_1,\ldots,x_n)} \\
\text{s. t.} &\quad C(T, V^{T,(x_1,\ldots,x_n)}) \neq \emptyset.
\end{aligned}$$

We illustrate the issues in this case using the example below.

Example 4.6. *A game with an empty core.*

Consider the case of a business owner who approaches three advertising firms, each with its cable channel, for production and broadcasting of three advertisement programs, one per each firm. Each firm has to decide independently whether or not to accept the job offer. The business owner is willing to pay \$17 to each firm for her job. Once the decisions to accept the jobs are made, the firms may cooperate and reduce their cost of creating the required artworks. For simplicity we do not introduce competition in the case.

The players in this game are the three advertising firms. The first-stage decision space for $i \in \{1, 2, 3\}$, is $X_i = \{0, 1\}$ (a binary variable), indicating that i is turning down the job or respectively accepting the job offer.

Assuming away the broadcasting costs, let $c(S)$ denote the total cost of creating advertisements artworks for firms in coalition S. When Firm 1 takes the job on her own without collaboration, her costs are $c(\{1\}) = \$15$. When Firm 2 takes the job without collaboration, her costs are $c(\{2\}) = \$18$, and similarly for Firm 3 her costs are $c(\{3\}) = \$20$. If $S = \{\text{Firms 1 and 2}\}$ their costs are $c(\{1, 2\}) = \$30$, representing a combined saving of \$2. The following is the complete cost schedule for advertisement(s) collaboration costs, assuming all firms take their assigned jobs. Note that the costs are subadditive.

$$c(\{1\}) = 15; \qquad c(\{2\}) = 18; \qquad c(\{3\}) = 20;$$

$$c(\{1, 2\}) = 30; \qquad c(\{1, 3\}) = 31; \qquad c(\{2, 3\}) = 32;$$

$$c(\{1, 2, 3\}) = 46.6.$$

Given that the business owner is paying \$17 to each firm that completes its job, we can compute the characteristic functions of the second-stage cooperative games for any given first-stage decisions as shown in Figure 4.2. For instance, $V^{(1,1,1)}(\{1,2\}) = (17 \times 2) - c(\{1,2\}) = 4$.

$x_3 = 0$

	$x_2 = 0$	$x_2 = 1$
$x_1 = 0$	$V^{(0,0,0)}(\text{N}) = 0$ $V^{(0,0,0)}(\{1,2\}) = 0$ $V^{(0,0,0)}(\{2,3\}) = 0$ $V^{(0,0,0)}(\{3,1\}) = 0$ $V^{(0,0,0)}(\{1\}) = 0$ $V^{(0,0,0)}(\{2\}) = 0$ $V^{(0,0,0)}(\{3\}) = 0$	$V^{(0,1,0)}(\text{N}) = -1$ $V^{(0,1,0)}(\{1,2\}) = -1$ $V^{(0,1,0)}(\{2,3\}) = -1$ $V^{(0,1,0)}(\{3,1\}) = 0$ $V^{(0,1,0)}(\{1\}) = 0$ $V^{(0,1,0)}(\{2\}) = -1$ $V^{(0,1,0)}(\{3\}) = 0$
$x_1 = 1$	$V^{(1,0,0)}(\text{N}) = 2$ $V^{(1,0,0)}(\{1,2\}) = 2$ $V^{(1,0,0)}(\{2,3\}) = 0$ $V^{(1,0,0)}(\{3,1\}) = 2$ $V^{(1,0,0)}(\{1\}) = 2$ $V^{(1,0,0)}(\{2\}) = 0$ $V^{(1,0,0)}(\{3\}) = 0$	$V^{(1,1,0)}(\text{N}) = 4$ $V^{(1,1,0)}(\{1,2\}) = 4$ $V^{(1,1,0)}(\{2,3\}) = -1$ $V^{(1,1,0)}(\{3,1\}) = 2$ $V^{(1,1,0)}(\{1\}) = 2$ $V^{(1,1,0)}(\{2\}) = -1$ $V^{(1,1,0)}(\{3\}) = 0$

$x_3 = 1$

	$x_2 = 0$	$x_2 = 1$
$x_1 = 0$	$V^{(0,0,1)}(\text{N}) = -3$ $V^{(0,0,1)}(\{1,2\}) = 0$ $V^{(0,0,1)}(\{2,3\}) = -3$ $V^{(0,0,1)}(\{3,1\}) = -3$ $V^{(0,0,1)}(\{1\}) = 0$ $V^{(0,0,1)}(\{2\}) = 0$ $V^{(0,0,1)}(\{3\}) = -3$	$V^{(0,1,1)}(\text{N}) = 1$ $V^{(0,1,1)}(\{1,2\}) = -1$ $V^{(0,1,1)}(\{2,3\}) = 2$ $V^{(0,1,1)}(\{3,1\}) = -3$ $V^{(0,1,1)}(\{1\}) = 0$ $V^{(0,1,1)}(\{2\}) = -1$ $V^{(0,1,1)}(\{3\}) = -3$
$x_1 = 1$	$V^{(1,0,1)}(\text{N}) = 3$ $V^{(1,0,1)}(\{1,2\}) = 2$ $V^{(1,0,1)}(\{2,3\}) = -3$ $V^{(1,0,1)}(\{3,1\}) = 3$ $V^{(1,0,0)}(\{1\}) = 2$ $V^{(1,0,0)}(\{2\}) = 0$ $V^{(1,0,0)}(\{3\}) = -3$	$V^{(1,1,1)}(\text{N}) = 4.4$ $V^{(1,1,1)}(\{1,2\}) = 4$ $V^{(1,1,1)}(\{2,3\}) = 2$ $V^{(1,1,1)}(\{3,1\}) = 3$ $V^{(1,1,1)}(\{1\}) = 2$ $V^{(1,1,1)}(\{2\}) = -1$ $V^{(1,1,1)}(\{3\}) = -3$

Figure 4.2. Characteristic functions of the second-stage cooperative games

Note that only the cooperative game $(N, V^{(1,1,1)}(S))$ (in the 8th box corresponding to $\vec{x} = (1,1,1)$) has an empty core. That is, only when all firms decide to take the job, the second-stage game's core is empty.

Now we examine the potential coalitional outcome and individual values assuming that each firm wants to participate in a subcoalition that maximizes her expected share of surplus.

Firm 1 has three stable options: forming $\{1,2\}$, $\{1,3\}$, or $\{1\}$. For subcoalition $\{1,2\}$, subgame $(\{1,2\}, V^{\{1,2\},(x_1,x_2,x_3)=(1,1,1)})$ is stable with $\overline{v_1}^{\{1,2\},(1,1,1)} = 5$ and $\underline{v_1}^{\{1,2\},(1,1,1)} = 2$. Notice that $\overline{v_1}^{\{1,2\},(1,1,1)}$ is greater than $V^{\{1,2\},(1,1,1)}$ which is 4. This is because Firm 2 would lose \$1 when she operates alone. For subcoalition $\{1,3\}$, $\overline{v_1}^{\{1,3\},(1,1,1)} = 6$ and $\underline{v_1}^{\{1,3\},(1,1,1)} = 2$. For subcoalition $\{1\}$, $\overline{v_1}^{\{1\},(1,1,1)} = \underline{v_1}^{\{1\},(1,1,1)} =$

2.

Firm 2 has three stable options: forming $\{1,2\}$, $\{2,3\}$, or $\{2\}$. For subcoalition $\{1,2\}$, subgame $(\{1,2\}, V^{\{1,2\},(x_1,x_2,x_3)=(1,1,1)})$ is stable with $\overline{v_2}^{\{1,2\},(1,1,1)} = 2$ and $\underline{v_2}^{\{1,2\},(1,1,1)} = -1$. For subcoalition $\{2,3\}$, $\overline{v_2}^{\{2,3\},(1,1,1)} = 5$ and $\underline{v_2}^{\{2,3\},(1,1,1)} = -1$. For subcoalition $\{2\}$, $\overline{v_2}^{\{2\},(1,1,1)} = \underline{v_2}^{\{2\},(1,1,1)} = -1$.

Firm 3 has three stable options: forming $\{1,3\}$, $\{2,3\}$, or $\{3\}$. For subcoalition $\{1,3\}$, subgame $(\{1,3\}, V^{\{1,3\},(x_1,x_2,x_3)=(1,1,1)})$ is stable with $\overline{v_3}^{\{1,3\},(1,1,1)} = 1$ and $\underline{v_3}^{\{1,3\},(1,1,1)} = -3$. For subcoalition $\{2,3\}$, $\overline{v_3}^{\{2,3\},(1,1,1)} = 3$ and $\underline{v_3}^{\{2,3\},(1,1,1)} = -3$. For subcoalition $\{3\}$, $\overline{v_3}^{\{3\},(1,1,1)} = \underline{v_3}^{\{3\},(1,1,1)} = -3$.

Assume that the confidence indices of Firms 1, 2, and 3 are α, β, and γ, respectively. We can model a biform game for Firm 1 as

$$f_1(x_2, x_3) = \max_{x_1 \in \{0,1\}} Q_1(x_1, x_2, x_3)$$

where

$$\begin{aligned} Q_1(x_1, x_2, x_3) = \max_{T \subseteq \{1,2,3\}} \quad & \alpha \overline{v_1}^{T,(x_1,x_2,x_3)} + (1-\alpha)\underline{v_1}^{T,(x_1,x_2,x_3)} \\ \text{s.t.} \quad & C(T, V^{T,(x_1,x_2,x_3)}) \neq \emptyset. \end{aligned}$$

Consider the case of full participation. From the perspective of Firm 1, the value of the three subcoalitions $\{1,2\}$, $\{1,3\}$, or $\{1\}$ are: $5\alpha + 2(1-\alpha)$, $6\alpha + 2(1-\alpha)$, and 2. Subcoalition $T^* = \{1,3\}$ is the best choice for Firm 1 as it has the highest expected profit of $Q_1(1,1,1) = 6\alpha + 2(1-\alpha)$.

For Firms 2 and 3, we find that the best option for both is to form the subcoalition

$\{2,3\}$ with $Q_2(1,1,1) = 5\beta - 1(1-\beta)$ and $Q_3(1,1,1) = 3\gamma - 3(1-\gamma)$. If Firms 2 and 3 actually decide to form such a coalition, they will affect Firm 1's realized profit. Perhaps one is allowed to assume that Firm 1's projection of a coalitional formation might be captured in α. For instance, Firm 1 may consider counter offers to either Firm 2 or Firm 3. Thus, the negotiation and coalitional formation result can be reflected in the confidence index α.

Figure 4.3 shows the final payoffs to Firms 1,2, and 3 respectively.

	$x_2 = 0$	$x_2 = 1$
$x_1 = 0$	0, 0, 0	0, –1, 0
$x_1 = 1$	2, 0, 0	$5\alpha + 2(1-\alpha)$, $2\beta - (1-\beta)$, 0

$x_3 = 0$

	$x_2 = 0$	$x_2 = 1$
$x_1 = 0$	0, 0, –3	0, $5\beta - (1-\beta)$, $3\gamma - 3(1-\gamma)$
$x_1 = 1$	$6\alpha + 2(1-\alpha)$, 0, $\gamma - 3(1-\gamma)$	$6\alpha + 2(1-\alpha)$, $5\beta - (1-\beta)$, $3\gamma - 3(1-\gamma)$

$x_3 = 1$

Figure 4.3. Payoffs of game with empty core

With the payoffs matrix shown in Figure 4.3, we can compute the pure strategy Nash equilibrium for any known α, β, and γ. For instance, when $\beta > 1/3$ and $\gamma > 3/4$, the pure strategy Nash equilibrium is for all three firms to take the jobs. However, when $\beta < 1/6$ and $\gamma < 1/2$, the pure strategy Nash equilibrium is such that only Firm 1 takes the job.

4.5. Discussion

First, let us make it clear that we believe that biform game modeling framework of combining both the competitive and cooperative games for the purpose of evaluation of firms' economic interactions is a good way of modeling games of this kind.

Robert Aumann notes in *Introductory Remarks* (Cooperation: Game-Theoretic Approaches, 1997, Hart and Mas-Colell eds. p. 5-6), that the cooperative approach and the non-cooperative approach are different ways of looking at the same game rather than the analysis of different games. *"The noncooperative theory is strategy oriented. It studies what we expect the players to do in the game. The cooperative theory, on the other hand, studies the outcomes we expect. In the cooperative approach we look directly at the space of outcomes, not the nitty-gritty of how one gets there.... In the cooperative theory we are interested in what the players can achieve; thus we ask how coalitions can form, what coalitions will form and how coalitions that form divide what they achieve."*

Biform games position these competitive vis a vis cooperative concerns very well into one single framework.

What we set forward in this paper is a recasting of the biform games in a stochastic programming perspective. The main message of this paper is that we can reproduce the main strengths/contributions of biform game methodology using stochastic programming with recourse methodology. It works and offers a number of advantages. The easy low hanging fruit is that of modeling firms'/players' beliefs about their bargaining power. With stochastic programming a whole range of probability distributions can be directly used to represent such bargaining power and beliefs instead of one parameter weighted average between the upper bound and lower bound of the core of the cooperative profit. Nevertheless, similar to the confidence indexes of biform games, using probability distributions may not be "mutually consistent" among players. The sum of players expected share of the second-stage cooperative profit may not necessarily be equal to the expected grand coalition's profit. In some realizations, some players would end up with an outcome lower than that anticipated. However, stochastic programming offers richer options. Another advantage lies in the direct decoupling of the strategic first stage from the different options of subsequent potential cooperations. This part comes through directly and very clearly in the stochastic programming problem representation.

Solving stochastic programming with recourse problems has a relatively long tradition. Since 1950ties stochastic programming has seen a steady progress in related mathematics and its computational implementations. This methodological knowledge ought to serve well in the proposed modeling approach when bundling competition with cooperation at a firm's level such as biform games.

Stating the advantages we also note that there are several issues awaiting future empirical and theoretical developments. First, experimental work could/should be conducted to study what probability distributions may better describe second-stage real-life profit sharing schemes. What are the appropriate allocation schemes? Such experimental results would be particularly useful when second stage empty core is encountered. Probability distributions representing belief in a value share might consider incorporating knowledge regarding the effects of blocking coalitions and other frictions when forming coalitions. This would allow for a more realistic comparison and evaluation of the alternative strategic decisions.

Finally, it is not clear whether solving for pure Nash equilibrium is always easier in the stochastic programming setting but certainly it is no more difficult than in the biform game representation.

4.6. Appendix

Example 4.7. *Negative-Advertising Game* (Brandenburger and Stuart, 2007, Example 5.1)

Consider three firms and two buyers. Each of the firms has the capacity of producing one unit of a product at no cost. In addition, one of the firms (say Firm 1), can decide whether to engage in negative advertising hurting the sales potential of the other two firms. Each of the two buyers is willing to buy one unit of the product without discriminating between the three firms at a price of \$2 if no action is taken by Firm 1. However, if Firm 1 engages in negative advertising, each buyer will scale down her willingness-to-pay to \$1 for a unit of a product from Firms 2 or Firm 3, while her willingness-to-pay for Firm 1's product will remain as before – \$2. Also assume in this case that the buyers prefer the product of Firm 1 to the products of the other firms. (Similarly to Example 2, this preference assumption can be replaced by the assumption that the second-stage decisions maximize collective surplus.)

The sequence of events in this game is as follows: In the first stage, Firm 1 decides whether to engage in negative advertising. In the second stage, after Firm 1's decision has been played-out, the market settles as above.

Now consider the case when Firm 1 does not engage in negative advertising. A surplus of \$4 is generated by two buyers buying from two of the three firms. The three firms compete away for two buyers, therefore no firm receives any part of the surplus, leaving each buyer with a surplus of \$2.

Next, consider the case when Firm 1 engages in negative advertising. Both buyers would rather purchase Firm 1's product than purchase from the other two firms, but one of the two buyers, say Buyer 2, will have to purchase one unit from either Firm 2 or Firm 3. The total surplus in this case is \$2 + \$1 = \$3. Given that Buyer 2 will definitely receive \$1 of surplus, Buyer 1 will not settle for anything less than \$1. Moreover, because both buyers compete for Firm 1's product, Buyer 1 will not receive more than \$1 of surplus. Hence, Firm 1 will receive \$1, and each buyer will receive \$1.

Both of the above cases are deterministic. This example can be recast in a straight forward fashion as a two-stage binary integer programming problem. The first-stage Firm 1 decision is represented by a binary decision variable $x_1 \in \{0, 1\}$ indicating no action and negative advertising, respectively. In the second stage, after the first-stage decision x_1 has been acted on, the market responds optimally.

Similar to Example 4.1, the negative-advertising game is modeled in the biform game framework in Brandenburger and Stuart (2007) because the first-stage decision involves only Firm 1's individual competitive decision, and the second-stage decision can be seen as a cooperative game $(N, V^{x_1}(S)), S \subseteq N$, where $V^{x_1}(S)$ is a characteristic function of the cooperative game that is only a function of the first-stage decision x_1 of Firm 1 (allowing us to denote it as $V^{x_1}(S)$ without added confusion). Let $F = \{\text{Firm 1, Firm 2, Firm 3}\}, \quad B = \{\text{Buyer 1, Buyer 2}\}$, and $N = F \cup B$;

$$\begin{aligned} V^{x_1}(S) &= 0 \text{ for } S \subseteq F \text{ or } S \subseteq B, \\ V^{x_1=0}(S \cup T) &= 2 \times \min\{|S|, |T|\} \text{ for } S \subseteq F \text{ and } T \subseteq B, \\ V^{x_1=1}(S \cup T) &= \begin{cases} \min\{|S|, |T|\} & \text{if } S \subseteq F \backslash \{\text{Firm 1}\} \text{ and } T \subseteq B, \\ \max\{0, \min\{|S|, |T|\} - 1\} & \\ \quad + 2 \times \min\{1, |T|\} & \text{if } S \subseteq F, S \ni \text{Firm 1, and } T \subseteq B \end{cases} \end{aligned}$$

The core of the game $(N, V^{x_1=0}(S))$ is an allocation (a singleton) in which each firm gets \$0 and each buyer gets \$2. The core of the game $(N, V^{x_1=1}(S))$ is an allocation (a singleton) in which Firm 1, Buyers 1 and 2, each gets \$1, and Firms 2 and 3, each get \$0.

Let $v_1(x_1)$ represent the second-stage profit achievable by Firm 1 for first-stage decision x_1. The binary integer programming problem for Firm 1 can be written as: $\max_{x_1} v_1(x_1); \quad \text{s.t. } x_1 \in \{0, 1\}$.

That is, Firm 1 will choose the x_1 value that maximizes her second-stage profit. From the core solution of $(N, V^{x_1}(S))$, the second-stage profit of Firm 1 can be written as $v_1(x_1) = 0$ if $x_1 = 0$ and $v_1(x_1) = 1$ if $x_1 = 1$.

In summary; it is optimal for Firm 1 to engage in negative-advertising in the first stage and receive \$1 in the second stage.

Example 4.8. *A Repositioning Game* (Brandenburger and Stuart, 2007, Example 5.3)

In the Repositioning Game example there are 3 manufacturing firms and 2 buyers. Each firm has the capacity of producing one unit of a product. Each buyer wants to buy one unit of the product. The production costs for each of the firms are $c_1 = \$1, c_2 = \$4$, and $c_3 = \$7$ respectively. Since the three products represent different quality, the willingness-to-pay for each of the respective product units by two identical buyers are respectively $p_1 = \$8, p_2 = \$11$, and $p_3 = \$14$. Notice that each firm can generate \$7 surplus. Now, assume that only Firm 2 has an option of repositioning her product at a cost of $c_r = \$1$ and at the same time reducing the production cost to $c_2' = \$3$ while raising the buyers' willingness-to-pay for her product to $p_2' = \$12$. Buyers have no preference among products from the three firms.

The sequence of decisions in this game can be viewed differently. We consider two sequence options, whether Firm 2 is committing to the repositioning of its product or not, followed by (a) the buyers contract what products they buy before production starts – an upfront sell, or (b) each of the three firms commits to production before buyers make any purchasing decision.

First, consider game sequence (a). In the first stage, Firm 2 chooses between the two options; invest at a cost $c_r = \$1$ to reposition her product ($x_2 = 1$), or to do nothing ($x_2 = 0$). After Firm 2's investment decision x_2 has taken place, the two buyers enter the market and make decision on contracting with two of three firms for production and delivery. In this case, we are faced with the second-stage cooperative game with five players. The firm without a contract does not incur production cost and gets \$0.

The first-stage decision by Firm 2 is represented by a binary variable $x_2 \in \{0, 1\}$ indicating no investment, and $c_r = \$1$ repositioning investment, respectively. The second stage is a cooperative game $(N, V_a^{x_2}(S)), S \subseteq N$, where the sets are $F = \{\text{Firm 1, Firm 2, Firm 3}\}$, $B = \{\text{Buyer 1, Buyer 2}\}$, $N = F \cup B$, and the characteristic functions are denoted as

$$
\begin{aligned}
V_a^{x_2}(S) &= 0 \text{ for } S \subseteq F \text{ or } S \subseteq B, \\
V_a^{x_2=0}(S \cup T) &= 7 \times \min\{|S|, |T|\} \text{ for } S \subseteq F \text{ and } T \subseteq B, \\
V_a^{x_2=1}(S \cup T) &= \begin{cases} 7 \times \min\{|S|, |T|\} & \text{if } S \subseteq F \backslash \{\text{Firm 2}\} \\ & \text{ and } T \subseteq B, \\ 7 \times \max\{0, \min\{|S|, |T|\} - 1\} & \\ \quad + 9 \times \min\{1, |T|\} & \text{if } S \subseteq F, S \ni \text{Firm 2}, \\ & \text{and } T \subseteq B \end{cases}
\end{aligned}
$$

Assume that Firm 2 chooses not to invest in repositioning in the first stage; then the grand coalition in the second-stage will generate \$14 surplus. Since the three firms compete away for two buyers, therefore in equilibrium each buyer will receive \$7, and the firms receive \$0.

Now, assume that Firm 2 chooses to invest in repositioning in the first-stage. The grand coalition in the second-stage can generate \$16 surplus. At the core allocation of this game (a singleton), Firm 2 will receive \$2 (or \$1 after deducting repositioning cost). The other two firms will receive \$0. Each of the buyers will receive \$7.

Let $v_2^a(x_2)$ represent the second-stage profit achievable by Firm 2 given her first-stage decision x_2. The two-stage 0-1 programming problem of Firm 2 can be stated as

$$\max_{x_2 \in \{0,1\}} \quad -c_r x_2 + v_2^a(x_2)$$

That is, Firm 2 will choose x_2 that maximizes her second-stage earning. From the core solution of the corresponding cooperative game, the second-stage profit of Firm 2 can be written as: $v_2^a(x_2) = 0$ if $x_2 = 0$ and $v_2^a(x_2) = 1$ if $x_2 = 1$. Hence, it is optimal for Firm 2 to invest in repositioning and receive \$1 net profit.

Now, consider the game sequence (b) where each of the three firms pays for production cost and manufactures her product before buyers make any purchasing decision. In the first stage, Firm 2 chooses between the two options; to invest at a cost $c_r = \$1$ to reposition her product, or do nothing. After the firm's decision is executed, all three firms pay for their respective production costs c_1, c_2, and c_3. In the second stage, the two buyers enter the market and make decision on which two of three firms they will buy from. The firm i who has no buyer will end up with loss equal to c_i.

In this case the second stage is a cooperative game $(N, V_b^{x_2}(S)), S \subseteq N$, where

$F = \{\text{Firm 1, Firm 2, Firm 3}\}, B = \{\text{Buyer 1, Buyer 2}\}, N = F \cup B$, and

$$\begin{aligned} V_b^{x_2}(S) &= \$0 \text{ for } S \subseteq F \text{ or } S \subseteq B, \\ V_b^{x_2=0}(S \cup T) &= \begin{cases} \sum_{i \in S} p_i & \text{if } |S| \leq |T|, S \subseteq F \text{ and } T \subseteq B, \\ \max_{i \in S} p_i & \text{if } |S| > |T|, |T| = 1, S \subseteq F \text{ and } T \subseteq B, \\ p_3 + p_2 & \text{if } |S| > |T|, |T| = 2, S \subseteq F \text{ and } T \subseteq B, \end{cases} \end{aligned}$$

For $V_b^{x_2=1}(S \cup T)$, we replace p_2 by p_2' in all three cases. Notice that the value of each (sub) coalition is now determined by the selling prices p_1, p_2, and p_3 because the production cost incurred in stage one.

If Firm 2 chooses not to invest in repositioning in the first stage then the value of grand coalition in the second stage is $p_3 + p_2 = \$25$. In the core of this game $(N, V_b^{x_2=0}(S))$, Firms 1, 2, and 3 will receive \$0, \$3 and \$6, respectively. In equilibrium each buyer will receive \$8. The argument for this (an equilibrium argument) is that the firms cannot undo their investments, therefore, they can only minimize loss. When Firm 1 asks for, say \$1 (zero loss) so her buyer will receive \$7, Firm 2 may ask for \$3 (\$1 loss – better than \$4 loss if she loses the competition) so her buyer will receive \$8 and drives Firm 1 out from the competition. So on and so forth.

If Firm 2 chooses to invest in repositioning in the first-stage then the grand coalition in the second-stage will generate $p_3 + p_2' = \$26$. In the core allocation of this game, Firms 1, 2, and 3 will receive \$0, \$4 and \$6, respectively. Each buyer will receive \$8.

Let $v_2^b(x_2)$ represent the second-stage income achievable by Firm 2 for her first-stage decision x_2. The two-stage 0-1 programming problem of Firm 2 can be written as

$$\max_{x_2 \in \{0,1\}} \quad -c_r x_2 - c_2 - (c_2' - c_2)x_2 + v_2^b(x_2)$$

Note that the production costs c_2 and c_2' are part of the first-stage expression. From the core solution of the corresponding cooperative game, the second-stage profit of Firm 2 can be written as $v_2^b(x_2) = 3$ if $x_2 = 0$ and $v_2^b(x_2) = 4$ if $x_2 = 1$. Therefore it is optimal for Firm 2 to invest in repositioning with \$0 net profit.

Example 4.9. *Buyer (asymmetry) in monopoly* (Stuart, 2007a)

Similarly to Example 4.4, the players' set consist of one seller and b buyers. Each buyer $j \in \{1, \ldots, b\}$ is willing to buy q_j units of product from the seller. The seller has to decide upfront on her production capacity; she may install an integer capacity of $x \in X = \{0, 1, \ldots, D\}$, where $D > 1$ is the cumulative demand. Assume that the seller has a linear capacity cost of c_c per unit and a marginal production cost per unit equal c_p.

A buyer may be willing to pay a few dollars less for the second or third unit than the first unit. That is, we do not assume that the buyer's willingness-to-pay for a unit of product is constant but only that it is non-increasing. We describe the buyers' willingness-to-pay using the aggregate demand curve which is a sorted sequence of willingness-to-pay $w_i, i = 1, \ldots, D$, such that $w_1 \geq \ldots \geq w_D > 0$.

Each buyer is associated with the aggregate demand curve by a collection of functions $\{\psi_j\}_{j=1}^b$ such that $\psi_j(i) = 1$ if point (unit) i on the aggregate demand curve is associated with buyer j, and $\psi_j(i) = 0$ otherwise. Moreover, $\sum_{j=1}^{b} \psi_j(i) = 1$ for all $i = 1, \ldots, D$ – the total cumulative demand is accounted for.

Decisions sequence in the game is as follows: In the first stage, the seller chooses to install x-unit capacity at a cost of c_c per unit – the first-stage cost for the seller is $c_c x$. In the second stage, a subset S of buyers and the seller form a coalition that will maximize their surplus and the seller manufactures any number $y \leq x$ of product units at a cost of c_p per unit.

As in all other examples, the second stage setting can be described as a cooperative game. Let m be the seller, $B = \{1, \ldots, b\}$ be the set of buyers and $N = \{m\} \cup B$. In this game, for a given first-stage capacity decision x, the seller and a subset of buyers form a coalition $S \subseteq N$, the seller produces y units of product, and the coalition generates a surplus $V^x(S)$. In this example, we have $c_c = 0$, $c_p = 0$, and $w_i > 0$ for all $i = 1, \ldots, D$. Hence, for each integer $x \in X = \{0, 1, \ldots, D\}$, the characteristic function of the game $(N, V^x(S)), S \subseteq N$ is given by:

$$V^x(S) \quad = \quad \begin{cases} \$0 & \text{if} S \subseteq B, \text{ or } S = \{m\}, \\ \sum_{i=1}^{R^x(S)} \sum_{j \in S \setminus \{m\}} \psi_j(i) w_i & \text{otherwise} \end{cases} \tag{4.19}$$

where $R^x(S)$ is the index of the unit ($\leq D$) which is associated to the buyer with the lowest willingness-to-pay for a manufactured unit given that at most x units can be produced. That is, $V^x(N) = \sum_{i=1}^{x} w_i$ and $R^x(S) = \max\{r : \sum_{i=1}^{r} \sum_{j \in S \setminus \{m\}} \psi_j(i) \leq \min(x, \sum_{i=1}^{D} \sum_{j \in S \setminus \{m\}} \psi_j(i)\}$.

Now, let $v_m, v_1, \ldots, v_b$, represent the allocation of $V^x(N)$ to the seller and each buyer, respectively. Similar to Example 4.4, the core allocation of this game is nonempty because $V^x(\cdot)$ is superadditive and $V^x(S) = 0$ for all $S \subseteq B$. Observation 4.1 still holds true because $\sum_{j\in N\setminus S}\{V^x(N) - V^x(N\setminus\{j\})\} \leq V^x(N) - V^x(S)$.

To explain this fact, consider $V^x(N\setminus\{j\})$ for $j \in N\setminus S$ which can be derived from (4.19). Since w_i's have a descending order, by removing buyer j, some q_j units of product formerly assigned to buyer j are free and reassigned to another buyers, who were previously assigned to the position without real units, which are $(x+1), (x+2), \ldots, (x+q_j)$. Now consider $V^x(N\setminus\{j,k\})$ for $j,k \in N\setminus S$. In this case, $q_j + q_k$ units formerly assigned to buyers j and k are reassigned which include $(x+1), (x+2), \ldots, (x+q_j+q_k)$. Clearly, $V^x(N) - V^x(N\setminus\{j,k\}) \geq [V^x(N) - V^x(N\setminus\{j\})] + [V^x(N) - V^x(N\setminus\{k\})]$. The same reduction can show that

$$\sum_{j\in N\setminus S}\{V^x(N) - V^x(N\setminus\{j\})\} \leq V^x(N) - V^x(S).$$

Below, we recast this game as a two-stage stochastic programming problem. The first-stage decision of the seller is represented by the integer variable x indicating the production capacity that requires an upfront investment.

Let the random variable $\xi' \in [0,1]$ represent a fraction of residual profit captured by the seller reflecting the seller's bargaining power in relation to the buyers. In the second stage, after the first-stage investment decision x has taken place, buyers in the coalition compete for the products. Following the competition outcome the winning buyers and the seller bargain on how to share the total profit (revenue less the sum of the marginal capacity costs). That is, after the instantiation value $\xi \in [0,1]$ of the r.v. ξ' is determined through bargaining, the seller will then receive the guaranteed amount and in addition she will receive ξ times the residual profit. In this case, the guaranteed amount is the lower bound in (4.10) and the residual profit is the difference between the upper bound and lower bound.

Formally, let $v_m(x,\xi)$ represent the second-stage profit achievable by the seller for a given first-stage decision x and instantiation ξ. The stochastic integer programming problem can be stated as

$$f_m(x^*) = \max_{0\leq x\leq D} \qquad E_{\xi'}[v_m(x,\xi')]$$

The second-stage profit $v_m(x,\xi)$ is

$$\begin{aligned} v_m(x,\xi) &= V^x(N) - \sum_{j=1}^{b}[V^x(N) - V^x(N\backslash\{j\})] + \xi(\sum_{j=1}^{b}[V^x(N) - V^x(N\backslash\{j\})]) \\ &= V^x(N) - (1-\xi)(\sum_{j=1}^{b}[V^x(N) - V^x(N\backslash\{j\})]) \text{ for } j = 1,\ldots,b. \end{aligned}$$

From (4.19) we have

$$v_m(x,\xi) = \begin{cases} \sum_{i=1}^{x} w_i - (1-\xi)\sum_{j\in P_x}\sum_{i=x+1}^{R^x(N\backslash\{j\})}\sum_{l\in B\backslash\{j\}}[\psi_l(i)w_i] & \text{if } x < D, \\ \sum_{i=1}^{x} w_i - (1-\xi)\sum_{i=1}^{x} w_i & \text{if } x = D. \end{cases}$$

where P_x is the set of included buyers defined as $P_x = \{j \in B : \sum_{i=1}^{x}\psi_j(i) > 0\}$. That is, the buyers in P_x are able to buy from the seller, given that at most, x units can be produced ($|P_x| \leq x$).

Assume that the probability distribution of ξ' is given by the confidence index $\alpha \in [0,1]$. We have

$$\begin{aligned} v_m(x,1) &= \sum_{i=1}^{x}\{w_i\} \\ v_m(x,0) &= \begin{cases} \sum_{i=1}^{x}\{w_i\} - \sum_{j\in P_x}\sum_{i=x+1}^{R^x(N\backslash\{j\})}\sum_{l\in B\backslash\{j\}}[\psi_l(i)(w_i)] & \text{if } x < D, \\ 0 & \text{if } x = D. \end{cases} \end{aligned}$$

Hence, the two-stage stochastic integer program of the monopoly power game, assuming the confidence index α, is

$$\begin{aligned} f_m(x^*) = \max_{0\le x\le D} &\quad E_{\xi'}[v_m(x,\xi')] \\ = \max_{0\le x\le D} &\quad [Pr\{\xi'=0\}\times v_m(x,0)] + [Pr\{\xi'=1\}\times v_m(x,1)] \\ = \max_{0\le x\le b} &\quad [(1-\alpha)v_m(x,0)] + [\alpha v_m(x,1)]\,. \end{aligned}$$

Suppose there are 7 buyers. The first buyer wants 7 units, the second buyer 6 units, ..., and the seventh buyer 1 unit, hence $D = 28$. Each buyer is willing to pay $w_i = 10$. The second-stage profit for $\xi = 1$ is $v_m(x,1) = 10x$; for $\xi = 0$ is $v_m(x,0) = 10x - \sum_{j\in P_x}\sum_{i=x+1}^{R^x(N\setminus\{j\})}\sum_{l\in B\setminus\{j\}}[10\psi_l(i)]$ for $x < D$. We can rewrite the corresponding integer programming problem as

$$f_m(x^*) = \max_{0\le x<D} \quad (1-\alpha)(10x - \sum_{j\in P_x}\sum_{i=x+1}^{R^x(N\setminus\{j\})}\sum_{l\in B\setminus\{j\}}[10\psi_l(i)]) + \alpha(10x)$$

If the seller is pessimistic ($\alpha = 0$), she should install 22-unit production capacity in the first stage. If the seller is optimistic ($\alpha = 1$), she should install 28-unit production capacity. If her belief is at midpoint ($\alpha = 0.5$), she should install 23-unit production capacity.

In this model, Stuart (2007a) shows that the seller prefer symmetric buyers, i.e., every buyer is willing to buy the same number of units at the same price. Example 4.4 can be modeled as a special case of Stuart (2007a).

Example 4.10. *Inventory Competition with Deterministic Demand* (Stuart, 2005)

In an inventory competition game with deterministic demand, a system of n players consists of a set $B = \{1, \ldots, b\}$ of buyers and a set $M = \{1, \ldots, m\}, (|M| \geq 2)$ of sellers. Similar to Example 4.4, each buyer $j \in \{1, \ldots, b\}$ has a willingness-to-pay value of w_j dollars for one product unit from sellers, where $w_1 > w_2 > \ldots > w_b > 0$. Each seller $i \in M$ has to decide upfront on her production capacity; she may install $x_i \geq 0$ units of capacity. Assume that every seller has a marginal capacity cost of c_c per unit and a marginal production cost of c_p.

The sequence in this game is as follows: In the first stage, each seller chooses production capacity of $x_i \geq 0$ for her facility. The capacity installation costs are linear in c_c, thus, the seller's total first-stage cost equals $c_c x_i$. In the second stage the buyers and the sellers form a coalition that is likely to maximize their surplus and the seller manufactures at a cost of c_p any number $y_i \leq x_i$ of product.

The second stage can be represented by a cooperative game. Let $N = M \cup B$. Given competitive first-stage capacity decisions $(x_1, x_2, \ldots, x_m)$, a subset of buyers and sellers form a coalition $S \subseteq N$. The sellers in S produces $\sum_{i \in S \cap M} y_i$ units of product, and the coalition generates a surplus $V^{\vec{x}}(S)$.

Let $x_S = \sum_{i \in S \cap M} x_i$, $B_S = \{j \in S \cap B : w_j \geq c_p\}$ and $d_s = |B_S|$. The characteristic function of the game $(N, V^{\vec{x}}(S)), S \subseteq N$ is given by:

$$V^{\vec{x}}(S) \quad = \quad \begin{cases} \$0 & \text{if } S \subseteq B, \text{or } S \subseteq M, \\ \sum_{j=1}^{R} \chi_S(j)(w_j - c_p) & \text{otherwise} \end{cases}$$

where $\chi_S(j)$ be the characteristic function of S such that $\chi_S(j) = 1$ when $j \in S$, and $\chi_S(j) = 0$ when $j \notin S$ and R is the index of the buyer in S who has lowest willingness-to-pay but is still able to buy the product. That is,

$$R = \max\{r : \textstyle\sum_{j=1}^{r} \chi_S(j) \leq \min(x_S, d_S)\}$$

$$V^x(N) = \textstyle\sum_{j=1}^{\min(x_N, d_N)} \{w_j - c_p\}$$

assuming that all $w_j \geq c_p$.

The core allocation is nonempty because $V^{\vec{x}}(\cdot)$ is superadditive and $V^{\vec{x}}(S) = 0$ for all $S \subseteq B$ or $S \subseteq M$.

Below, we recast this game as a two-stage stochastic programming problem. The first-stage decision of each seller is represented by the integer variable $x_i \geq 0$ indicating the production capacity that requires an upfront investment.

Let a random variable ξ_i' represent a fraction of residual profit captured by seller i reflecting her bargaining power in relation to the buyers. In the second stage, after the first-stage investment decisions $\vec{x}$ has taken place, the buyers compete for the products (or sellers compete for the buyers if supply is greater than the demand). Following the competition outcome the winning buyers and sellers bargain on how to distribute the total profit (revenue less the sum of the marginal costs) between them. That is, after the instantiation $\xi_i \in [0, 1]$ of the r.v. ξ_i' is determined through bargaining, Seller i will then receive the guaranteed amount and in addition she will receive ξ_i times the residual profit.

Formally, let $v_i(\vec{x}, \xi_i)$ represent the second-stage profit achievable by Seller i for a given first-stage decision x and instantiation value ξ_i. The stochastic programming problem can be stated as

$$f_i(\vec{x}_{-i}, x_i^*) = \max_{x_i \geq 0} \quad -c_c x_i + E_{\xi_i'}[v_i(\vec{x}, \xi_i')]$$

where the second-stage profit $v_i(\vec{x}, \xi_i') = \underline{v_i}^{\vec{x}} + \xi_i'(\overline{v_i}^{\vec{x}} - \underline{v_i}^{\vec{x}})$,

$$\underline{v_i}^{\vec{x}} = \begin{cases} x_i(w_{x_N+1} - c_p) & \text{if } x_N < d_N, \\ 0 & \text{if } x_N \geq d_N, \text{ and} \end{cases} \tag{4.20}$$

$$\overline{v_i}^{\vec{x}} = \begin{cases} x_i(w_{x_N} - c_p) & \text{if } x_N \leq d_N \text{ and } x_i \neq x_N, \\ \sum_{j=1}^{x_N}(w_j - c_p) & \text{if } x_N \leq d_N \text{ and } x_i = x_N, \\ 0 & \text{if } x_N > d_N \text{ and } x_i \neq x_N, \\ \sum_{j=1}^{d_N}(w_j - c_p) & \text{if } x_N > d_N \text{ and } x_i = x_N. \end{cases} \tag{4.21}$$

We derive (4.20) and (4.21) from Stuart (2005, Lemma 1 and Lemma 2). When $x_i = x_N$, the market is monopoly. In terms of probability distribution of ξ_i'; it depends on seller i's belief system. Again consider the confidence index $\alpha_i \in [0, 1]$. It can be stated as a probability distribution function of ξ_i' as $Pr\{\xi_i' = 1\} = \alpha_i$, and $Pr\{\xi_i' = 0\} = 1 - \alpha_i$.

From (4.20) and (4.21), the two-stage integer stochastic program of the Inventory Competition with Deterministic Demand game, assuming seller i's confidence index α_i, is

$$\begin{aligned} f_i(\vec{x}_{-i}, x_i^*) &= \max_{x_i \geq 0} \quad -c_c x_i + E_{\xi_i'}[v_i(\vec{x}, \xi_i')] \\ &= \max_{x_i \geq 0} \quad -c_c x_i + \left[Pr\{\xi_i' = 0\} \times \underline{v_i}^{\vec{x}}\right] + \left[Pr\{\xi_i' = 1\} \times \overline{v_i}^{\vec{x}}\right] \\ &= \max_{x_i \geq 0} \quad -c_c x_i + \left[(1 - \alpha_i)\underline{v_i}^{\vec{x}}\right] + \left[\alpha_i \overline{v_i}^{\vec{x}}\right]. \end{aligned}$$

For instance, when there are $b = 3$ buyers with willingness-to-pay $(w_1, w_2, w_3) = (14, 10, 6)$, and $m = 2$ with zero capacity cost and production cost. The second-stage profit of Seller 1 for $\xi_1 = 1$ is $\overline{v_1}^{\vec{x}} = x_1 w_{x_1+x_2}$ if $0 \leq x_1 \leq 3 - x_2, x_2 > 0$. The second-stage profit of Seller 1 for $\xi_1 = 0$ is $\underline{v_1}^{\vec{x}} = x_1(w_{x_1+x_2+1})$ if $0 \leq x_1 < 3 - x_2$ and 0 otherwise. The Nash equilibrium results depend on the actual value of α_1 and α_2.

5. FUTURE WORK AND CONCLUSIONS

Research Contributions

In the last ten years the applications of both competitive and cooperative game theory in supply chain analysis has become an acceptable practice because it provides important insights and understanding of the operational behavior of the participating firms/players.

This dissertation is about strategic inventory procurement decisions and cooperation. That is, we examine multi-stage competitive and cooperative interactions by profit maximizing agents as games in decentralized inventory management arena. The pertinent questions focus on modeling collaborative cost sharing and response to demand uncertainty.

Since real-life depiction of sales in most markets pictures random variability in demand with numerous options for the different independent byers/players' decisions, even with the best forecasting techniques, perfect matching of demand with supply is not likely to occur in practice. Sharing resources and capabilities help firms cope more effectively with the uncertain demands for products and services. A number of studies under various assumptions have recently attempted to shed light on the problem of strategic procurement decisions in supply chain inventory management. This dissertation studies decentralized inventory management games that reflect on a number of aspects of these managerial problems. Throughout this work, we assume individual rationality of the players.

In Chapter 2, we examine a decentralized system that adopts a predetermined transfer payment approach proposed by Anupindi et al. (2001). We analyze the existence and the uniqueness of pure strategy Nash equilibrium (PSNE) for a decentralized system that adopts the transfer payment approach. With respect to Anupindi et al. (2001)'s model, we note that assumptions on demand distribution and cost parameters influence its validity. Our examination sets forth the conditions on the cost parameters and distributions that guarantee uniqueness of PSNE. The past literature on decentralize inventory management assumed complete information of the demand distributions and the cost parameters of all retailers. In practice, however, voluntary complete information sharing arrangement among competing players is questionable. We examine a situation with incomplete information and conclude that in the case stated in Anupindi et al. (2001) cooperative incentives might no longer hold without the assumption of complete information. In this case we also expand the scope of the earlier models by relaxing the assumption of satisfying local demand first. In Chapters 3 and 4, we advocate a modeling prism of stochastic programming with recourse methodology as a way to address decentralized inventory procurement games.

In Chapter 3, using stochastic programming with recourse model as a building block, we construct a unifying taxonomy for a large variety of decentralized inventory games. This framework depicts independent retailers who maximize their individual expected profits, with each retailer independently procuring inventory in the ex-ante stage in response to forecasted demand and anticipated inventory decisions of the other retailers. In the ex-post stage, in response to the realized demand and competitors' chosen procurement levels, each retailer exercises a recourse action. For instance, retailers may coordinate inventory swaps to satisfy shortage with overage with profits shared between collaborating retailers. Our basic approach is that of a two-stage stochastic programming optimization. Our framework provides a unifying parsimonious view through a single methodological prism for a large variety of problems studied in isolation in the past. We posit that this taxonomy framework will enable manufactures and service providers state, understand, and solve their strategic problems. Perhaps equally importantly, as recourse options are laid out, the graph framework clarifies the modeling connection between problems in a taxonomy of decentralized inventory distribution models. This unifying perspective links the past work and sheds light on future research directions. However, we have left out from this work the algorithmic and computational issues related to solutions for the stochastic programming with recourse.

Chapter 4 applies stochastic programming to a special class of decision games called biform games. Biform games modeling framework addresses two-stage games with competitive first stage and cooperative second stage without ex-ante agreement on profit sharing scheme. Biform games position these competitive and cooperative concerns into one single modeling framework. For instance, a manufacturing firm might have to decide upfront regarding her production capacity based on beliefs about her customers' demand and the capacity installed by her competitor. Later, their decisions are about forming coalition that generates the highest surplus and are likely to deliver the best value to each of the firms. Presumably, the resulting surplus (the participants' realized payoff) has to be shared fairly. This type of two-stage problems was modeled in Brandenburger and Stuart (2007). We show that the methodology proposed by Brandenburger and Stuart (2007) can be cast as a special case of a two-stage stochastic programming with recourse model. The two-stage stochastic programming view of biform games is demonstrated on all the known examples regarding operational decision problems of competing firms in the literature, including Brandenburger and Stuart (2007); Stuart (2005, 2007a,b); and Chatain and Zemsky (2007). With stochastic programming, a whole range of probability distributions can directly be used to represent firms' bargaining power and beliefs instead of one parameter weighted average between the upper bound and lower bound of the core of the cooperative profit as proposed in the biform framework.

Future Work

There are several issues awaiting future empirical and theoretical developments related to decentralized inventory management game.

In chapter 2, in the decentralized inventory system that adopts the allocation rule proposed by Anupindi et al. (2001), a player may have incentives to revel her true demand distribution parameters. In future research, we would like to consider different profit allocation rules with regards to incentive compatibility property and some other important properties. If sharing of demand and pricing information is not completely free of charge, it might be valuable also to consider mechanism design that ensures truth telling.

In Chapter 3, a number of extensions to our proposed taxonomy are ripe for future research. First, analysis of non-core allocation and its effect on stochastic modeling should be studied. If an allocation is not in the core, at some realizations, a subset of retailers would have an incentive to deviate from the grand coalition. Second, it would be of interest to examine in more details options of coordination of the second-stage cooperative game by a third party.

In Chapter 4, experimental work could/should be conducted to study what probability distributions may better describe second-stage real-life profit sharing schemes. Such experimental results would be particularly useful when second-stage empty core is encountered. Moreover, probability distributions representing beliefs about profit/surplus allocation might consider incorporating knowledge regarding the effects of blocking coalitions and other frictions when forming coalitions. This would allow for a more realistic comparison and evaluation of the alternative strategic decisions.

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