Intrinsic Geometric Flows on Manifolds of Revolution

by
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ABSTRACT

An intrinsic geometric flow is an evolution of a Riemannian metric by a two-tensor. An extrinsic geometric flow is an evolution of an immersion of a manifold into Euclidean space. An extrinsic flow induces an evolution of a metric because any immersed manifold inherits a Riemannian metric from Euclidean space. In this paper we discuss the inverse problem of specifying an evolution of a metric and then seeking an extrinsic geometric flow which induces the given metric evolution. We limit our discussion to the case of manifolds that are rotationally symmetric and embeddable with codimension one. In this case, we reduce an intrinsic geometric flow to a plane curve evolution. In the specific cases we study, we are able to further simplify the evolution to an evolution of a function of one variable. We provide soliton equations and give proofs that some soliton metrics exist.

Chapter 1

INTRODUCTION

1.1 Intrinsic Geometric Flows

In 1904, Henri Poincaré wondered if it could be possible for a three-dimensional manifold to have trivial fundamental group but not be homeomorphic to the three-dimensional sphere [40]. In 2000, the Clay Institute listed the Poincaré conjecture among it's seven "Millennium Problems"[16].

In 1982 Richard Hamilton, motivated by Eells and Sampson [18], suggested a program to solve the problem via geometric partial differential equations [25]. Hamilton introduced an evolution equation, known as the Ricci flow, which evolves a Riemannian metric in time according to (negative two times) its Ricci tensor. Hamilton used the Nash-Moser inverse function theorem to prove the equation had short-time existence. He showed that if a solution had positive Ricci curvature initially, then the flow had long-time existence and, when suitably rescaled, would converge (at $t = \infty$) to a metric with constant Ricci curvature. In other words, he showed that any three-dimensional manifold with positive Ricci curvature was diffeomorphic to one with constant Ricci curvature. In the case of a simply-connected, compact three-manifold, this implies it is diffeomorphic to the three-sphere.

If we allow for negative Ricci curvatures, however, things aren't so straightforward. It is known, for instance, that a "peanut-shaped" three-manifold can develop a singularity in finite time [42] [3] [4]. This is where the proof stalled until Perelman overcame the remaining obstacles to complete the program using a modified version of the Ricci flow know as Ricci flow with surgeries [37] [38] [39]. The Ricci flow with surgeries allows the Ricci flow to continue until the singularity occurs. Then the problem region is cut out and replaced with a more manageable region. With Perelman's

work, the Poincaré conjecture, and the broader Thurston's geometrization conjecture are proven. A complete proof can be found in [9] or [33].

The Ricci flow is an example of an intrinsic geometric flow. An intrinsic geometric flow is an evolution of a Riemannian metric by a symmetric two-tensor.

After the promise of the Ricci flow was demonstrated, it was only natural to ask if there are any other intrinsic flows which might prove useful. A natural tensor to consider would be the Einstein tensor since finding Einstein metrics is a very important in studying Riemannian geometry and topology of manifolds. The problem with flow by (two times) the Einstein tensor is that it is not parabolic, which means that there may not be short-time existence. The Yamabe flow is another natural choice. Proposed by Hamilton in [24], the Yamabe flow was introduced as a method to prove the Yamabe conjecture, originally proven in [32]. The Yamabe conjecture states that any metric on a compact manifold of dimension three or higher is conformally equivalent to one with constant scalar curvature. The Yamabe flow preserves the conformal class of a metric and is expected to evolve a manifold toward one with constant scalar curvature.

Another intrinsic flow the cross curvature flow introduced by Chow and Hamilton [13]. The cross curvature flow is the evolution of the metric by two times the cross curvature tensor. The cross curvature flow is only parabolic on a positively curved or negatively curved manifold.

1.2 Extrinsic Geometric Flows

An extrinsic flow is the evolution of an embedding of a manifold into Euclidean space by a (usually normal) vector field. In 1974 the Gauss curvature flow was proposed by Firey [20] as a model for the erosion of stones. Mean curvature flow was used in geometric measure theory to find minimal surfaces [7].

In the case of the Gauss curvature flow, Firey proved that a convex surface that

is symmetric about the origin shrinks to a point, but becomes asymptotically spherical in the process. He conjectured that the symmetry assumption is not necessary. Andrews proved Firey's conjecture in [2]. It was pointed out by Chow and Hamilton that in the case of a three-dimensional hypersurface of four-dimensional Euclidean or Minkowski space, the cross curvature flow is the same as the Gauss curvature flow. This relationship can be seen easily from calculations in Chapter 3.

The mean curvature flow continues to be an important research area. In 1986, Huisken [26] proved a result analogous to Hamilton's. That is, a convex hypersurface will evolve into a round sphere under the (properly rescaled) mean curvature flow. In mean curvature flow, singularities occur in much the same way as they do in the Ricci flow. With the mean curvature flow, however, it is known how to formulate a weak solution which continues past these singularities. It was proven simultaneously in [11] and [19] that a weak flow exists and continues through the singularity. This is accomplished by viewing a hypersurface as a level set of a function. There is an evolution of that function such that all of its level sets evolve by the mean curvature flow. The key to this approach is that if two hypersurfaces are separate initially, they will remain separate for the duration of the flow.

1.3 Relating Intrinsic and Extrinsic Flows

The observation that the cross curvature flow, an intrinsic flow, can be realized as Gauss curvature flow begs the question: might it be possible to realize other intrinsic flows as extrinsic flows? In other words, given an initial metric and an intrinsic geometric flow with short-time existence, is it possible to find an initial embedding and an evolution of embeddings so that the induced metric at all times solves the given flow? As it turns out, the answer to that question was known before the question itself. When John Nash proved his famous isometric embedding theorem [34], he demonstrated that not only is there an isometric embedding of any Riemannian

manifold into Euclidean space, but also given a (short) smooth path through the space of metrics, there is a corresponding smooth path through the space of embeddings. Those results are nicely summarized by Ben Andrews in [1].

Let g_F be the metric induced by the embedding F and g_{F+V} be the metric induced by the embedding $F + V$, then we have the following:

Theorem 1.1. *A compact manifold M of dimension n has a $\mathcal{C}^\infty$ embedding F into $\mathbb{R}^N$, where $N = \frac{n(n+5)}{2}$. If h is a $\mathcal{C}^k$ symmetric $(2,0)-$tensor field on M with $k \geq 3$, which is sufficiently small in $\mathcal{C}^3$, then there exists a $\mathcal{C}^k$ map $V : M \to \mathbb{R}^N$ such that $g_{F+V} = g_F + h$. By g_F we mean the metric induced by F and by g_{F+V}, we mean the metric induced by $F + V$.*

Remark. Solutions to this embedded flow may exist on a smaller time interval than solutions to the corresponding intrinsic flow.

1.4 Summary of Paper

We briefly discuss three important simplifications which will be used throughout this paper:

1. We assume that our metrics are only defined up to diffeomorphism.

2. We assume that our manifold is a manifold of revolution.

3. We assume that the generating curve for our manifold is the graph of a function.

If two metrics have the relationship that one is the pull-back of the other by a diffeomorphism, then we may say that they have the same *Riemannian structure*. The space of Riemannian structures is defined as the space of Riemannian metrics modulo this relation. See [17] and [5] for more information on this Berger-Ebin decomposition. It can simplify things greatly to consider a geometric problem on the space of Riemannian structures instead of Riemannian metrics. That is, instead

of looking for a metric that satisfies a certain condition, look for one that differs by a diffeomorphism from one that does. In this way, Mathias Gunther [22] [23] simplified Nash's proof of the isometric embedding theorem. It was also using this technique that Dennis DeTurck simplified Hamilton's proof of short-time existence of the Ricci flow[15]. What this means in terms of intrinsic geometric flows is that one may add the Lie derivative of the metric along any smooth vector field to the evolution equation. The solution to this modified flow will have the same Riemannian structure as the one without the Lie derivative. In Chapter 3, we will see that adding an arbitrary tangential vector field to the evolution of the embedding correspongds to changing an evolving metric by a time-dependent diffeomorphism. By adding an arbitrary tangential vector field to our flows, we will simplify the evolution equation.

Though it is interesting to note that any intrinsic flow that has short-time existence can be realized as an extrinsic flow, the analysis in working in such a setting might be very complicated. To simplify things, we consider the case when the manifold in question a manifold of revolution generated by a plane curve. Theorem 1.1 doesn't apply in this case because its codimension needs to be large. Rubinstein and Sinclair [41] considered the Ricci flow on three-manifolds of revolution and constructed a numerical simulation whereby the intrinsic Ricci flow was estimated, then with the estimation of the metric at each time, an embedding was found that induced the metric. We take a different approach. Instead of solving the intrinsic flow, we find an evolution on the embedding itself that induces the given intrinsic flow. By considering only manifolds of revolution, we reduce the intrinsic flow to a plane curve evolution.

One last simplification is that we will make an assumption that the generating curve for a manifold is the graph of a function. By choosing the appropriate tangential vector field discussed in the previous section, we can reduce a geometric flow to a PDE on a real function instead of on a metric, or equivalently, a system of PDE's. Our geometric flow will reduce to a PDE, and our soliton equation will reduce to a second order ODE. This allows us to prove their existence using elementary analy-

sis. For example, local existence of soliton metrics will be almost immediate from undergraduate ODE theory [8].

The problem we will mostly focus on will be of proving the existence of rotationally symmetric solitons. Solitons are solutions to geometric flows which are self-similar in time up to a change of scale. If the change of scale is positive, the soliton called *expanding*. If it is unchanging, the soliton is knowns as *steady*. If it is negative, the soliton is called a *shrinking* soliton. Solitons are very important in geometric flows because they tend to describe asymptotic limits at singularities. In [28], Thomas Ivey proves the existence of Ricci expanding and steady rotationally symmetric solitons in dimension three and higher using an intrinsic proof. Viewing an intrinsic flow as a manifold of revolution offers an alternative way of proving these existence results. We offer an alternate proof of Ivey's result in Chapter 4, and prove an analogous result for the Yamabe flow in Chapter 5 and flow by the Einstein tensor in Chapter 6.

In Chapter 2 we provide a very brief background on some of the important concepts we will use and specify our notation.

In Chapter 3 we perform some of the important calculations which are needed in the chapters that follow.

In Chapter 4 we deal with the Ricci flow specifically. Our main result is an alternate proof of the following:

Theorem 4.1. *For each non-negative real number ϵ, and for integers $n \geq 2$, there exists a complete rotationally symmetric metric g on $\mathbb{R}^n$ such that the following conditions hold:*

1. *g has constant sectional curvature 1 at the origin.*

2. *g is a Ricci soliton metric whose scale at time $t = 0$ is changing by a factor of ϵ.*

In Chapter 5 we turn our attention to the Yamabe flow. The main result is the following result:

Theorem 5.1. *For each non-negative real number ϵ, and for integers $n \geq 2$, there exists a complete rotationally symmetric metric g on $\mathbb{R}^n$ such that the following conditions hold:*

1. *g has constant sectional curvature 1 at the origin.*
2. *g is a Yamabe soliton metric whose scale at time $t = 0$ is changing by a factor of ϵ.*

In Chapter 6 we first consider a flow by a metric's Einstein tensor and prove the following:

Theorem 6.1. *For each non-negative real number ϵ, and for integers $n \geq 4$, there exists a complete rotationally symmetric metric g on $\mathbb{R}^n$ such that the following conditions hold:*

1. *g has constant sectional curvature 1 at the origin.*
2. *g is a soliton metric for the flow by twice the Einstein tensor whose scale at time $t = 0$ is changing by a factor of ϵ.*

We also analyze the mean curvature flow and the Gauss curvature flow and construct an equation for flow by an general tensor.

After reducing our soliton equations to second order ODE, we can solve them numerically and graph them. In Chapter 7 we provide graphs of expanding, steady and shrinking solitons for various flows and discuss some interesting observations. Those figures are produced using the Matlab code from Appendix A.

In Chapter 8, we will consider negatively curved three-dimensional metrics embedded in four-dimensional Minkowski space. As Chow and Hamilton point out, the

cross curvature flow on these metrics is induced by the Gauss curvature flow. We consider specifically those manifolds which are warped products of two-dimensional hyperbolic space with the real line. Again, we reduce this intrinsic geometric flow to a PDE on a function. Since short-time existence of the cross curvature flow on a non-compact manifold is unknown, we can only conjecture that such a PDE has short-time existence with Neumann conditions. We discuss a strategy to prove that if the PDE has existence with Neumann conditions, that under appropriate rescaling all metrics converge to hyperbolic three-space in the limit.

Finally, in Chapter 9 we discuss some possible future research directions.

Chapter 2

BACKGROUND AND NOTATION

2.1 Riemannian Geometry

Here we give a brief review of the fundamental concepts of Riemannian geometry that we will be using. For an excellent introduction, see [31] (the author also recommends [6] and [36]). We will use coordinate notation and Einstein summation notation. For example, a Riemannian metric will be denoted g_{ij}. A metric is *rotationally symmetric* if it is of the form

$$g = \eta^2(s)ds^2 + \phi^2(s)d\theta^2,$$

where $d\theta^2$ is the standard round metric on an $(n-1)$-sphere. We will always use the Riemannian connection, so we define the Christoffel symbols by

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}\left(\frac{\partial}{\partial x^i}g_{jl} + \frac{\partial}{\partial x^j}g_{il} - \frac{\partial}{\partial x^l}g_{ij}\right).$$

Then let $\vec{X} = X^i\frac{\partial}{\partial x^i}$ and $\vec{Y} = Y^i\frac{\partial}{\partial x^i}$ be vector fields where $\frac{\partial}{\partial x^i}$ are coordinates. Then the covariant derivative of $\vec{Y}$ with respect to $\vec{X}$ is

$$\nabla_{\vec{X}}\vec{Y} = (\vec{X}(Y^k) + X^iY^j\Gamma_{ij}^k)\frac{\partial}{\partial x^k}.$$

We often abbreviate $\nabla_{\frac{\partial}{\partial x^i}}$ by ∇_i. ∇ can be uniquely extended to each tensor bundle T_l^kM using the following rules:

1. If f is a function, then $\nabla_{\vec{X}}f = \vec{X}f$.

2. ∇ obeys the product rule with respect to tensor products:

$$\nabla_{\vec{X}}(F \otimes G) = (\nabla_{\vec{X}}F) \otimes G + F \otimes (\nabla_{\vec{X}}G).$$

3. ∇ respects traces. If "tr" denotes the trace on any pair of indices,

$$\nabla_{\vec{X}}(tr\vec{Y}) = tr(\nabla_{\vec{X}}\vec{Y}).$$

The *Lie derivative* of the metric along a vector field $\vec{W}$ is given by

$$\mathcal{L}_{\vec{W}}g(\vec{Y},\vec{Z}) = g(\nabla_{\vec{Y}}\vec{W},\vec{Z}) + g(\vec{Y},\nabla_{\vec{Z}}\vec{W}).$$

The $(3,1)$ *Riemannian curvature tensor* is defined as follows:

$$R_{ijk}{}^{l}\frac{\partial}{\partial x^l} = \nabla_i\nabla_j\frac{\partial}{\partial x^k} - \nabla_j\nabla_i\frac{\partial}{\partial x^k}.$$

The $(4,0)$ version is simply $R_{ijkl} = g_{lm}R_{ijk}{}^{m}$. The $(2,0)$ *Ricci tensor* is the trace of the $(3,1)$ Riemannian curvature tensor on the first and last indices. Namely,

$$R_{jk} = R_{ijk}{}^{i}.$$

The *scalar curvature* is the trace of the Ricci tensor $R = g^{ij}R_{ij}$. The *Einstein tensor* is defined to be $E_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$. For any two coordinates $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial x^j}$ in the tangent space of M, the *sectional curvature* of the plane spanned by $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial x^j}$ is denoted $K(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})$ and defined to be

$$K\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right) = \frac{R_{ijji}}{g_{ii}g_{jj} - g_{ij}^2}.$$

2.2 Euclidean Submanifolds

We consider the geometry of an immersion $F : M \to \mathbb{R}^N$ of an $n-$dimensional manifold M into $\mathbb{R}^N$ for some $N > n$. Then there is a canonical metric on M called the *induced metric* which is the pull-back of the standard Euclidean inner product along F:

$$g_{ij} = \left\langle\frac{\partial F}{\partial x^i},\frac{\partial F}{\partial x^j}\right\rangle.$$

The *second fundamental form*, denoted II, is a symmetric, $\mathcal{C}^\infty$, bilinear map

$$\mathcal{T}_p(M)\times\mathcal{T}_p(M) \longrightarrow \mathcal{N}_p(M),$$

where $\mathcal{T}_p(M)$ is the tangent space to M at a point p, and $\mathcal{N}_p(M)$ is the normal space to M at p. The second fundamental form is defined as the projection of the second derivative onto the normal space. Namely:

$$\mathrm{II}_{ij} = \mathrm{II}\left(\frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\right) = \sum_{k=1}^{N-n} \left\langle \frac{\partial^2 F}{\partial x^i \partial x^j}, \vec{n}^k \right\rangle \vec{n}^k,$$

where $\vec{n}^k$ is an orthonormal basis for the normal space. Let $\vec{n}$ be some section of the normal bundle, then

$$0 = \frac{\partial}{\partial x^i}\left\langle \frac{\partial F}{\partial x^j}, \vec{n} \right\rangle = \left\langle \frac{\partial^2 F}{\partial x^i \partial x^j}, \vec{n} \right\rangle + \left\langle \frac{\partial F}{\partial x^j}, \frac{\partial \vec{n}}{\partial x^i} \right\rangle = \left\langle \mathrm{II}_{ij}, \vec{n} \right\rangle + \left\langle \frac{\partial F}{\partial x^j}, \frac{\partial \vec{n}}{\partial x^i} \right\rangle.$$

This relation,

$$\left\langle \mathrm{II}_{ij}, \vec{n} \right\rangle = -\left\langle \frac{\partial F}{\partial x^j}, \frac{\partial \vec{n}}{\partial x^i} \right\rangle,$$

is known as the *Weingarten Equation*. From the second fundamental form, we construct the Riemannian curvature tensor using the *Gauss Equation* as follows:

$$R_{ijkl} = \left\langle \mathrm{II}_{il}, \mathrm{II}_{jk} \right\rangle - \left\langle \mathrm{II}_{ik}, \mathrm{II}_{jl} \right\rangle.$$

In the case where $N = n+1$, the normal space to the manifold is one-dimensional and therefore the second fundamental form is equivalent to a symmetric $(2,0)$ tensor, which will be denoted h_{ij} (to be precise, $\mathrm{II}_{ij} = h_{ij}\vec{n}$, where $\vec{n}$ is a normal vector field). The *shape operator* is the $(1,1)$ version of the second fundamental form:

$$s^i_j = g^{ik} h_{jk}.$$

Its eigenvalues are called *principal curvatures*. The *Gauss curvature* is defined as the determinant of the shape operator. The *mean curvature* is defined as the trace of the shape operator

Remark. Sometimes the mean curvature is defined as $1/n$ tr(s), the mean of the principal curvatures, hence the name.

Notice that in this case, the Gauss equation becomes

$$R_{ijkl} = h_{il}h_{jk} - h_{ik}h_{jl}.$$

When we take the appropriate trace on each side, we see that this equation implies that the second fundamental form and the Ricci tensor are simultaneously diagonalizable. This fact will be used throughout this paper.

2.3 Solitons

In the field of geometric flows, solitons are a kind of generalized fixed point. Specifically, the evolution might not be zero, but the manifold is evolving only by rescaling and reparameterization.

Definition 2.1. A soliton of a geometric flow

$$\frac{\partial}{\partial t}g_{ij} = 2T_{ij}$$

is a solution to the flow which is of the form:

$$g(t) = \sigma(t)\psi_t^*(g_0),$$

where $\sigma(t)$ is some function of t and ψ_t is a one-parameter family of diffeomorphisms. We refer to the parameter σ in this equation as the *scale*.

Equivalently (a good explanation why these two definitions are equivalent can be found in[12]), we can define a soliton in the following way:

Definition 2.2. A soliton of a geometric flow

$$\frac{\partial}{\partial t}g_{ij} = 2T_{ij}$$

is a metric of the form

$$T_{ij}(0) = \mathcal{L}_{\vec{X}}g_0 + 2\epsilon g_0$$

for some $\vec{X}$ and ϵ. The parameter ϵ is the change of scale at time $t = 0$.

Solitons, also referred to as self-similar solitons, are important to geometric flows because they often describe the asymptotic behavior of the metric at a singularity of a parabolic flow. We name a soliton according to the flow it solves so, for instance, a soliton for for the Ricci flow will be called a *Ricci soliton*.

Chapter 3
CALCULATIONS

3.1 Relating Intrinsic and Extrinsic Flows

We begin by exploring the relationship between a variation of an embedding and the variation of the induced metric. The following theorem is similar to one found in [26].

Proposition 3.1. *Let F be an immersion of a manifold into $\mathbb{R}^{\mathbb{N}}$ for some integer N. Let F evolve by the equation*

$$\frac{\partial F}{\partial t} = \vec{V} + \vec{W},$$

where $\vec{V}$ is a normal vector field and $\vec{W} = W^i \frac{\partial F}{\partial x^i}$ is tangential. Then the induced metric will evolve by

$$\frac{\partial}{\partial t} g_{ij} = -2\left\langle \vec{V}, \mathrm{II}_{ij} \right\rangle + (\mathcal{L}_{\vec{W}} g)_{ij}.$$

Proof. We calculate:

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= \frac{\partial}{\partial t}\left\langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \right\rangle = \left\langle \frac{\partial(\vec{V}+\vec{W})}{\partial x^i}, \frac{\partial F}{\partial x^j} \right\rangle + \left\langle \frac{\partial F}{\partial x^i}, \frac{\partial(\vec{V}+\vec{W})}{\partial x^j} \right\rangle, \\
&= \left\langle \frac{\partial \vec{V}}{\partial x^i}, \frac{\partial F}{\partial x^j} \right\rangle + \left\langle \frac{\partial F}{\partial x^i}, \frac{\partial \vec{V}}{\partial x^j} \right\rangle + \left\langle \frac{\partial \vec{W}}{\partial x^i}, \frac{\partial F}{\partial x^j} \right\rangle + \left\langle \frac{\partial F}{\partial x^i}, \frac{\partial \vec{W}}{\partial x^j} \right\rangle, \\
&= -2\left\langle \vec{V}, \frac{\partial^2 F}{\partial x^i \partial x^j} \right\rangle + \left\langle \nabla_i \vec{W}, \frac{\partial F}{\partial x^j} \right\rangle + \left\langle \frac{\partial F}{\partial x^i}, \nabla_j \vec{W} \right\rangle, \\
&= -2\left\langle \vec{V}, \mathrm{II}_{ij} \right\rangle + (\mathcal{L}_{\vec{W}} g)_{ij}.
\end{aligned}$$

□

3.2 The Plane Curve Evolution

Suppose that we have an n-dimensional manifold of revolution M generated by a plane curve $\gamma(s) = (x(s), y(s))$ with $y > 0$. (Here we do not assume that s is arclength).

Choose coordinates that include $\frac{\partial}{\partial s}$ along with standard spherical coordinates on an $(n-1)$ sphere. We also assume that there is some rotationally symmetric $(2,0)$ tensor whose $(1,1)$ version looks like the following:

$$T_i^j = \begin{pmatrix} \lambda_s & 0 & 0 & 0 & 0 \\ 0 & \lambda_\theta & 0 & 0 & 0 \\ 0 & 0 & \lambda_\theta & 0 & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & 0 & \lambda_\theta \end{pmatrix}.$$

Theorem 3.2. *Let $\gamma(s,t) = (x(s,t), y(s,t))$ evolve by the following system:*

$$\begin{aligned} \frac{\partial x}{\partial t} &= \int \left(\frac{|\gamma'|^2}{x'} \left(\frac{K_s}{K_\theta} \lambda_\theta - \lambda_s \right) \right) ds + \frac{\lambda_\theta}{K_\theta} \frac{y'}{|\gamma'|} + Z(s)x', \\ \frac{\partial y}{\partial t} &= -\frac{\lambda_\theta}{K_\theta} \frac{x'}{|\gamma'|} + Z(s)y', \end{aligned}$$

where $Z(s)$ is an arbitrary function of s that extends smoothly to M and K_s and K_θ are the principal curvatures. Then the induced metric on M will evolve by

$$\frac{\partial}{\partial t} g_{ij} = -2T_{ij} + (\mathcal{L}_Z g)_{ij},$$

where $Z = Z(s)\frac{\partial}{\partial s}$.

We can simplify Proposition 3.1 in the case where the manifold in question is a manifold of revolution generated by a curve $\gamma(s) = (x(s), y(s))$. For now, we assume that the manifold is three-dimensional. The embedding is given by

$$F : (S_1, S_2) \times [0, 2\pi] \times [0, \pi] \to \mathbb{R}^4,$$

where

$$F(s, \theta, \phi) = (x(s), y(s)\cos(\theta)\sin(\phi), y(s)\sin(\theta)\sin(\phi), y(s)\cos(\phi)).$$

We use the following orthogonal (but not orthonormal) basis for the tangent space:

$$\begin{aligned}
\frac{\partial F}{\partial s} &= (x', y'\cos(\theta)\sin(\phi), y'\sin(\theta)\sin(\phi), y'\cos(\phi)),\\
\frac{\partial F}{\partial \theta} &= (0, -y\sin(\theta)\sin(\phi), y\cos(\theta)\sin(\phi), 0),\\
\frac{\partial F}{\partial \phi} &= (0, y\cos(\theta)\cos(\phi), y\sin(\theta)\cos(\phi), -y\sin(\phi)).
\end{aligned}$$

Then the metric is given by

$$g_{ij} = \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \right\rangle = \begin{pmatrix} |\gamma'|^2 & 0 & 0 \\ 0 & y^2\sin^2(\phi) & 0 \\ 0 & 0 & y^2 \end{pmatrix}.$$

We calculate the connection with $\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial}{\partial i}g_{jl} + \frac{\partial}{\partial j}g_{il} - \frac{\partial}{\partial l}g_{ij}\right)$.

Therefore,

$$\begin{aligned}
\Gamma^s_{ss} &= \frac{1}{2|\gamma'|^2}(\partial_s|\gamma'|^2) = \frac{1}{2}\partial_s \ln(|\gamma'|^2),\\
\Gamma^s_{\theta\theta} &= \frac{-yy'\sin^2(\phi)}{|\gamma'|^2},\\
\Gamma^\phi_{\theta\theta} &= -\sin(\phi)\cos(\phi),\\
\Gamma^s_{\phi\phi} &= \frac{-y'y}{|\gamma'|^2},\\
\Gamma^\theta_{s\theta} &= \frac{y'}{y},\\
\Gamma^\phi_{s\phi} &= \frac{y'}{y},\\
\Gamma^\theta_{\theta\phi} &= \cot(\phi),
\end{aligned}$$

and all other Christoffel symbols are zero or determined by symmetry.

At this point, we calculate the second fundamental form using the Weingarten formula. To find the second fundamental form, we must choose a unit normal vector field. Without loss of generality, we choose

$$\vec{n} = \frac{1}{|\gamma'|}(y', -x'\cos(\theta)\sin(\phi), -x'\sin(\theta)\sin(\phi), -x'\cos(\phi)).$$

It has the following derivatives:

$$\begin{aligned}
\frac{\partial \vec{n}}{\partial s} &= \left(\frac{\partial}{\partial s}\frac{1}{|\gamma'|}\right)|\gamma'|\vec{n} + \frac{1}{|\gamma'|}(y'', -x''\cos(\theta)\sin(\phi), -x''\sin(\theta)\sin(\phi), -x''\cos(\phi)),\\
\frac{\partial \vec{n}}{\partial \theta} &= \frac{1}{|\gamma'|}(0, x'\sin(\theta)\sin(\phi), -x'\cos(\theta)\sin(\phi), 0),\\
\frac{\partial \vec{n}}{\partial \phi} &= \frac{1}{|\gamma'|}(0, -x'\cos(\theta)\cos(\phi), -x'\sin(\theta)\cos(\phi), x'\sin(\phi)).
\end{aligned}$$

We calculate the second fundamental form:

$$\begin{aligned}
h_{ss} &= -\frac{\partial \vec{n}}{\partial s}\cdot\frac{\partial F}{\partial s}\\
&= \frac{-1}{|\gamma'|}(-y'x''\cos^2(\theta)\sin^2(\phi) - y'x''\sin^2(\theta)\sin^2(\phi) - y'x''\cos^2(\phi) + y''x'),\\
&= \frac{y'x'' - y''x'}{|\gamma'|},\\
h_{\theta\theta} &= -\frac{\partial \vec{n}}{\partial \theta}\cdot\frac{\partial F}{\partial \theta} = \frac{-1}{|\gamma'|}(-yx'\sin^2(\theta)\sin^2(\phi) - x'y\cos^2(\theta)\sin^2(\phi))\\
&= \frac{yx'\sin^2(\phi)}{|\gamma'|},\\
h_{\phi\phi} &= -\frac{\partial \vec{n}}{\partial \phi}\cdot\frac{\partial F}{\partial \phi} = \frac{-1}{|\gamma'|}(-yx'\cos^2(\theta)\cos^2(\phi) - yx'\sin^2(\theta)\cos^2(\phi) - yx'\sin^2(\phi))\\
&= \frac{yx'}{|\gamma'|}.
\end{aligned}$$

We have:

$$h_{ij} = \frac{1}{|\gamma'|}\begin{pmatrix} y'x'' - y''x' & 0 & 0\\ 0 & yx'\sin^2(\phi) & 0\\ 0 & 0 & yx' \end{pmatrix}.$$

Using the metric to raise an index gives the shape operator:

$$s_i^j = \begin{pmatrix} \frac{y'x''-y''x'}{|\gamma'|^3} & 0 & 0\\ 0 & \frac{x'}{y|\gamma'|} & 0\\ 0 & 0 & \frac{x'}{y|\gamma'|} \end{pmatrix}.$$

The eigenvalues of this tensor are called the principal curvatures. Denote the principal curvatures by $K_s = \frac{y'x''-y''x'}{|\gamma'|^3}$, $K_\theta = \frac{x'}{y|\gamma'|}$

Remark. The sign of these depends on our arbitrary choice of normal vector. Choosing the opposite unit vector would reverse the signs of these, but Theorem 3.2 won't depend on the choice of normal vector.

The normal vector field is determined by a vector field in $\mathbb{R}^2$, which is normal to γ. Specifically,

$$\vec{V} = V(s)\frac{(y'(s), -x'(s))}{|\gamma'|}.$$

Similarly, $\vec{W}$ is determined by a two-dimensional vector field tangential to γ;

$$\vec{W} = W(s)\gamma'.$$

Since it is clear how to extend $\vec{W}$ to the entire manifold of revolution, it makes sense to talk about $(\mathcal{L}_{\vec{W}})g_{ij}$. Recall that the Lie derivative of the metric along a vector field $\vec{W}$ is given by

$$\mathcal{L}_{\vec{W}}g(\vec{Y},\vec{Z}) = g(\nabla_{\vec{Y}}\vec{W},\vec{Z}) + g(\vec{Y},\nabla_{\vec{Z}}\vec{W}).$$

We find the requisite covariant derivatives:

$$\begin{aligned}
\nabla_s\vec{W} &= \left(\frac{\partial W(s)}{s} + W(s)\Gamma^s_{ss}\right)\frac{\partial}{\partial s} = \left(\frac{\partial W(s)}{\partial s} + W(s)\frac{1}{2}\partial_s \ln(|\gamma'|^2)\right)\frac{\partial}{\partial s},\\
\nabla_\theta\vec{W} &= \left(W(s)\Gamma^\theta_{s\theta}\right)\frac{\partial}{\partial\theta} = W(s)\frac{y'}{y}\frac{\partial}{\partial\theta},\\
\nabla_\phi\vec{W} &= \left(W(s)\Gamma^\phi_{s\phi}\right)\frac{\partial}{\partial\phi} = W(s)\frac{y'}{y}\frac{\partial}{\partial\phi}.
\end{aligned}$$

Returning to our vector field $\vec{W} = W(s)\frac{\partial}{\partial s}$, we have

$$\mathcal{L}_{\vec{W}}g_{ij} = \begin{pmatrix} 2\left(\frac{\partial W(s)}{\partial s} + W(s)\frac{1}{2}\partial_s\ln(|\gamma'|^2)\right)g_{ss} & 0 & 0\\ 0 & 2W(s)\frac{y'}{y}g_{\theta\theta} & 0\\ 0 & 0 & 2W(s)\frac{y'}{y}g_{\phi\phi}\end{pmatrix}.$$

Notice that if we raise an index of this two-tensor, the 2nd and 3rd eigenvalues are equal. This is due to the rotational symmetry of our metric. Further, by rotational symmetry, we see what these tensors are in arbitrary dimension: the 2nd through

the n-th eigenvalues will be equal. We therefore return to arbitrary dimension at this point and it is clear how to proceed.

We consider a variation of γ,

$$\frac{\partial\gamma}{\partial t} = \vec{V} + \vec{W} = V(s)\frac{(y'(s), -x'(s))}{|\gamma'|} + W(s)(x'(s), y'(s)).$$

This is a system of two PDE:

$$\begin{aligned}\frac{\partial x}{\partial t} &= \frac{V(s)y'}{|\gamma'|} + W(s)x', \\ \frac{\partial y}{\partial t} &= \frac{-V(s)x'}{|\gamma'|} + W(s)y'.\end{aligned}$$

We assume that the metric g_{ij} is evolving according to the equation

$$\frac{\partial}{\partial t}g_{ij} = -2T_{ij},$$

where

$$T_i^j = \begin{pmatrix} \lambda_s & 0 & 0 & 0 & 0 \\ 0 & \lambda_\theta & 0 & 0 & 0 \\ 0 & 0 & \lambda_\theta & 0 & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & 0 & \lambda_\theta \end{pmatrix}$$

in our basis. According to Proposition 3.1, it must be true that

$$-2T_i^j = -2V(s)s_i^j + g^{jk}\mathcal{L}_{\vec{W}}g_{ik}.$$

This gives us the following system:

$$\begin{aligned}-2\lambda_s &= -2V(s)K_s + 2\left(\frac{\partial W(s)}{\partial s} + W(s)\frac{1}{2}\partial_s \ln(|\gamma'|^2)\right), \\ -2\lambda_\theta &= -2V(s)K_\theta + 2W(s)\frac{y'}{y}.\end{aligned}$$

This system has the general solution

$$\begin{aligned}\vec{W} &= \frac{x'}{|\gamma'|^2}\int\left(\frac{|\gamma'|^2}{x'}\left(\frac{K_s}{K_\theta}\lambda_\theta - \lambda_s\right)\right)ds, \\ \vec{V} &= \frac{y'}{|\dot{\gamma}|}\int\left(\frac{|\gamma'|^2}{x'}\left(\frac{K_s}{K_\theta}\lambda_\theta - \lambda_s\right)\right)ds + \frac{\lambda_\theta}{K_\theta}.\end{aligned}$$

By Proposition 3.1, if we add an arbitrary tangential vector field $\vec{Z} = Z(s)\frac{\partial F}{\partial s}$, this adds $\mathcal{L}_{\vec{Z}} g_{ij}$ to the evolution of the metric. This completes the proof of Theorem 3.2.

3.3 Simplifications

The ability to add an arbitrary tangential vector field gives us the freedom to try and find better forms for the evolution in Theorem 3.2. For example, we could choose $Z(s)$ so that $\frac{\partial}{\partial t}|\gamma'| = 0$. This would mean that if γ were initially parameterized by arclength, it would remain so. Another option is to assume that γ is initially the graph of a function: $\gamma(s) = (s, y(s))$. Then one could choose Z so that $\frac{\partial x}{\partial t} = 0$. This reduces the system to a single equation in y.

We assume that γ is the graph of a function, where y is the independent variable and x is the dependent variable. For much of what follows, we will assume that

$$\gamma = (x(s), s),$$

and Z will be chosen so that $\frac{\partial y}{\partial t} = 0$. Using Theorem 3.2 with $y = s$, $y' = 1$, and $Z = \frac{\lambda_\theta}{K_\theta}\frac{x'}{|\gamma'|}$, we get

$$\frac{\partial x}{\partial t} = \int \left(\frac{|\gamma'|^2}{x'} \left(\frac{K_s}{K_\theta}\lambda_\theta - \lambda_s \right) \right) ds + \frac{\lambda_\theta}{K_\theta}|\gamma'|.$$

Strictly speaking, this isn't a PDE, but an integro-differential equation. We can easily differentiate both sides with respect to s to rid ourselves of the integral but something else is true also. When we substitute K_s, K_θ, λ_s, and λ_θ into this equation, we tend to observe a repetition of the term

$$\frac{x''}{1 + (x')^2}.$$

This suggests the change of variables

$$\psi = \tan^{-1}(x'),$$

since $\psi' = \frac{x''}{1+(x')^2}$. Geometrically, ψ is the angle γ' makes with the $y-$axis.

Remark. The function $\psi(s)$ is an odd function due to the symmetry of the manifold. We will use this fact often.

It is easier to simplify Equation 3.3 with specific flows, since the λ_s and λ_θ terms are known explicitly, so more detail will follow in the later sections.

3.4 Some Important Tensors

Here we compute some of the important curvature tensors. The Riemannian curvature tensor is calculated using the Gauss formula as follows:

$$\begin{aligned} R_{s\theta\theta s} &= h_{ss}h_{\theta\theta}, \\ R_{s\phi\phi s} &= h_{ss}h_{\phi\phi}, \\ R_{\theta\phi\phi\theta} &= h_{\theta\theta}h_{\phi\phi}. \end{aligned}$$

The $(1,1)$ Ricci tensor is $R^i_j = g^{ik}R_{jk} = g^{ik}g^{lm}R_{jlmk}$.

Proposition 3.3. *In coordinates that include $\frac{\partial}{\partial s}$ along with standard spherical coordinates on an $n-1$ sphere, the $(1,1)$ Ricci tensor on a manifold of revolution is:*

$$R^j_i = \begin{pmatrix} (n-1)K_sK_\theta & 0 & 0 & 0 \\ 0 & K_sK_\theta + (n-2)K_\theta^2 & 0 & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & 0 & K_sK_\theta + (n-2)K_\theta^2 \end{pmatrix}.$$

The scalar curvature is the trace:

Proposition 3.4. *The scalar curvature on a manifold of revolution is:*

$$R = 2(n-1)K_sK_\theta + (n-1)(n-2)K_\theta^2.$$

The Einstein tensor is the difference between the two:

$$E_i^j = R_i^j - \frac{1}{2} R \delta_i^j.$$

The cross curvature tensor is defined to be

$$X_{ij} = \left(\frac{\det E_{ij}}{\det g_{ij}} \right) V_{ij},$$

where V_{ij} is the inverse of E^{ij}, the $(0,2)$ Einstein tensor. For $n = 3$, the Einstein tensor has eigenvalues $-K_2K_3$, $-K_1K_3$, and $-K_1K_2$. We calculate the cross curvature tensor:

Proposition 3.5. *In a basis in which the Ricci tensor is diagonal, the cross curvature tensor on a three-manifold is:*

$$X_{ij} = \begin{pmatrix} K_1^2 K_2 K_3 & 0 & 0 \\ 0 & K_1 K_2^2 K_3 & 0 \\ 0 & 0 & K_1 K_2 K_3^2 \end{pmatrix}.$$

We substitute $K_1 = K_s$ and $K_2 = K_3 = K_\theta$ to reflect the rotational symmetry. We will discuss the cross curvature tensor in Chapter 8.

We also point out that it makes sense to define a (codimension-one) "mean curvature tensor" and a "Gauss curvature tensor" as (negative one half) the 2-tensor by which the metric evolves when a manifold evolves by the respective flow. Under the mean curvature flow, the embedding evolves by H times a unit normal vector, where H is the trace of the second fundamental form, $H = K_s + (n-1)K_\theta$. Using Propositon 3.1, we see that in this case, the metric evolves by

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij} &= -2(K_s + (n-1)K_\theta) \begin{pmatrix} K_s & 0 & 0 & 0 \\ 0 & K_\theta & 0 & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & 0 & K_\theta \end{pmatrix}, \\ &= -2R_{ij} + -2 \begin{pmatrix} K_s^2 & 0 & 0 & 0 \\ 0 & K_\theta^2 & 0 & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & 0 & K_\theta^2 \end{pmatrix}. \end{aligned}$$

It makes sense that the Ricci tensor appears in the mean curvature tensor given the observed qualitative similarities between the two flows [12]. The relationship holds even for general (not necessarily rotationally symmetric) hypersurfaces as well. In general, if an $n-$manifold is embedded in $\mathbb{R}^{n+1}$ and the embedding evolves according to the mean curvature flow, then the induced metric evolves according to

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij} + -2\begin{pmatrix} K_1^2 & 0 & 0 & 0 \\ 0 & K_2^2 & 0 & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & 0 & K_2^2 \end{pmatrix}.$$

This is an intrinsic tensor in the case of a three-manifold. In that case, we write an intrinsic flow which is induced by the mean curvature flow:

$$\frac{\partial}{\partial t}g_{ij} = -2\left(R_{ij} + \frac{1}{\det E}X_{ik}g^{kl}X_{lj}\right).$$

The Gauss curvature flow is the evolution of an embedding by the determinant of the shape operator times a unit normal vector field. Using Propostion 3.1, we see that under the Gauss curvature flow, the metric evolves by

$$\frac{\partial}{\partial t}g_{ij} = -2\begin{pmatrix} K_1^2K_2K_3\dots K_n & 0 & 0 & 0 \\ 0 & K_1K_2^2K_3\dots K_n & 0 & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & 0 & K_1K_2K_3\dots K_n^2 \end{pmatrix}.$$

As observed in [13], in three dimensions this is the cross curvature tensor.

3.5 Solitons

We now consider the soliton equations. A steady soliton can be found by setting the normal component of the equation equal to zero. So a steady soliton is one that solves

$$\frac{y'}{|\gamma'|}\int\left(\frac{|\gamma'|^2}{x'}\left(\frac{K_s}{K_\theta}\lambda_\theta - \lambda_s\right)\right)ds + \frac{\lambda_\theta}{K_\theta} = 0.$$

Note that we do not know a priori that a rotationally symmetric manifold is embeddable with co-dimension one, so it is possible that this equation will not find all rotationally symmetric steady solitons (only ones that are hypersurfaces of $\mathbb{R}^{n+1}$; we make an assumption that our solitons satisfy this constraint). We assume

$$\gamma_t(s) = f(t)\gamma_0(s).$$

In most of the analysis that follows, we will assume that our generating curves are graphs of functions, $\gamma(s) = (x(s), s)$. We also allow an arbitrary tangential vector field Z to be added, and thus:

$$\begin{aligned} \left.\frac{\partial x}{\partial t}\right|_{t=0} &= cx + Zx', \\ \left.\frac{\partial y}{\partial t}\right|_{t=0} &= cs + Z, \end{aligned}$$

where $c = \left.\frac{\partial f}{\partial t}\right|_{t=0}$. We can restrict the equation to $t = 0$ because if a metric is a soliton initially, it will remain so [12]. In order to reduce this system to a single PDE, choose $Z = -cs$ so that $\frac{\partial y}{\partial t} = 0$. Then

$$\left.\frac{\partial x}{\partial t}\right|_{t=0} = cx - csx'.$$

Differentiate both sides with respect to s:

$$\left.\frac{\partial x'}{\partial t}\right|_{t=0} = cx' - cx' - csx'' = -csx''.$$

We again make the substitution $\psi = \tan^{-1}(x')$, so

$$\left.\frac{\partial \psi}{\partial t}\right|_{t=0} = \frac{1}{1+(x')^2}\frac{\partial x'}{\partial t} = \frac{-csx''}{1+(x')^2} = -cs\psi'.$$

Given an evolution equation in terms of ψ, we can substitute $-cs\psi'$ for $\frac{\partial \psi}{\partial t}$ to get a soliton equation. The symmetry of the manifold in question requires that ψ be an odd function. So we will impose the boundary conditions $\psi(0) = 0$. We also choose the condition $\psi'(0) = 1$ which determines the curvature at the origin. A solution to

this ODE is a soliton whose sectional curvature all equal 1 at the "vertex" and whose scale is changing by a factor of c at $t = 0$. The choice of $c = 0$ gives a steady solition, while $c > 0$ gives an expanding soliton and $c < 0$ gives a shrinking solition.

Chapter 4
THE RICCI FLOW

In this chapter we find the evolution of the plane curve γ so that the manifold of revolution generated by γ evolves by the Ricci flow. Using this evolution, we provide an alternate proof of the existence of steady and expanding solitons, originally proved by Ivey in [28].

Theorem 4.1. *For each non-negative real number ϵ, and for integers $n \geq 2$, there exists a complete rotationally symmetric metric g on $\mathbb{R}^n$ which arises as a manifold of revolution inside $\mathbb{R}^{n+1}$, such that the following conditions hold:*

1. *g has constant sectional curvature 1 at the origin.*
2. *g is a Ricci soliton metric whose scale at time $t = 0$ is changing by a factor of ϵ.*

We will take the Ricci flow to mean any evolution equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \mathcal{L}_Z g_{ij},$$

where Z is any smooth time-dependent vector field on M. Solutions to this flow differ from solutions to the traditional Ricci flow equation:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij},$$

by only an evolving diffeomorphism. Once again, we assume that our manifold is generated by a curve $\gamma(s) = (x(s), y(s))$. By Theorem 3.2, the evolution of the curve is

$$\begin{aligned}\frac{\partial x}{\partial t} &= \int \left(\frac{|\gamma'|^2}{x'} \left(\frac{K_s}{K_\theta} \lambda_\theta - \lambda_s \right) \right) ds + \frac{\lambda_\theta}{K_\theta} \frac{y'}{|\gamma'|} + Z(s)x', \\ \frac{\partial y}{\partial t} &= -\frac{\lambda_\theta}{K_\theta} \frac{x'}{|\gamma'|} + Z(s)y',\end{aligned}$$

where $\lambda_s = (n-1)K_sK_\theta$ and $\lambda_\theta = K_sK_\theta + (n-2)K_\theta^2$ by Proposition 3.3, and Z is yet to be determined. Substituting, we have

$$\frac{\partial x}{\partial t} = \int \left(\frac{|\gamma'|^2}{x'} \left(K_s^2 - K_sK_\theta \right) \right) ds + (K_s + (n-2)K_\theta) \frac{y'}{|\gamma'|} + Z(s)x', \quad (4.1)$$

$$\frac{\partial y}{\partial t} = -(K_s + (n-2)K_\theta) \frac{x'}{|\gamma'|} + Z(s)y'. \quad (4.2)$$

Without the integral term in the evolution of x and the Z term in both, the evolution equation would be

$$\begin{aligned}\frac{\partial x}{\partial t} &= (K_s + (n-2)K_\theta) \frac{y'}{|\gamma'|}, \\ \frac{\partial y}{\partial t} &= -(K_s + (n-2)K_\theta) \frac{x'}{|\gamma'|},\end{aligned}$$

which is

$$\frac{\partial F}{\partial t} = H\vec{v},$$

the mean curvature flow on an $(n-1)$-dimensional manifold of revolution in $\mathbb{R}^n$.

Notice that the integrand vanishes when the manifold is isotropic. Furthermore, if we picture a three-dimensional "peanut-shaped" manifold, we can see that the integrand is highest on the neck and vanishes in places where the manifold looks like a sphere locally. So we see that the integral term gives the stretching along the axis of rotation, while otherwise, the manifold behaves exactly like the two-dimensinoal mean curvature flow.

Remark. For dimension at least 3, we expect a neckpinch to occur on the "peanut-shaped" manifold, since the evolution of y is negative on the neck. We also see that a neckpinch will not occur on a two-dimensional peanut.

4.1 The Round Two-Torus and $\mathbb{S}^1 \times \mathbb{S}^{n-1}$

By round torus, we mean the manifold of revolution generated by a circle in the upper half-plane. Some round two-tori are embeddable into $\mathbb{R}^3$. Under the Ricci flow, such a torus will evolve into a flat two-torus as $t \to \infty$. The flat two-torus is not smoothly embeddable into $\mathbb{R}^3$. So we wonder, does it lose embeddability immediately? Does it remain embeddable for all finite time? Or does it lose embeddability at some intermediate time? We consider an example and give evidence that it is initially embeddable, but loses embeddability immediately. We conjecture that the same would happen for all round two-tori. Take γ to be the curve

$$\gamma(s) = (\sin(s), 2 + \cos(s)) \text{ for } -\pi \leq s \leq \pi.$$

Then

$$\begin{aligned} K_s &= \frac{y'x'' - y''x'}{|\gamma'|^3} = 1, \\ K_\theta &= \frac{x'}{y} = \frac{\cos(s)}{2 + \cos(s)}. \end{aligned}$$

We use Equations 4.1 and 4.2 with $Z = 0$ to find the evolution of this curve at $t = 0$:

$$\begin{aligned} \frac{\partial x}{\partial t} &= \int_0^s \left(\frac{1}{\cos(\sigma)} - \frac{1}{2 + \cos(\sigma)} \right) d\sigma + \left(1 + (n-2)\frac{\cos(s)}{2 + \cos(s)} \right) (-\sin(s)), \\ \frac{\partial y}{\partial t} &= -\left(1 + (n-2)\frac{\cos(s)}{2 + \cos(s)} \right) \cos(s). \end{aligned}$$

The first problem occurs at $s = \pm\frac{\pi}{2}$, where $\frac{\partial x}{\partial t}$ approaches $\pm\infty$ due to the integral term $\int_0^s \frac{1}{\cos(\sigma)} d\sigma$. This means that the velocity vector becomes infinite at the leftmost and rightmost points on the circle. We interpret this as a loss of differentiability of the embedding. The problem term $\int_0^s \frac{1}{\cos(\sigma)} d\sigma$ can be defined by symmetry for $|s| > \frac{\pi}{2}$. For example, when $s > \frac{\pi}{2}$, we could substitute $\int_0^{-s+\pi} \frac{1}{\cos(\sigma)} d\sigma$ for $\int_0^s \frac{1}{\cos(\sigma)} d\sigma$. Then we can define the evolution of our curve for all $s \in [-\pi, \pi]$ except $\pm\frac{\pi}{2}$ and now we observe another problem. The vector field is not continuous. At $s = -\pi$, we have

$$\begin{aligned}\frac{\partial x}{\partial t} &= \int_0^{-\pi} \frac{-1}{2+\cos(\sigma)} d\sigma, \\ \frac{\partial y}{\partial t} &= 1,\end{aligned}$$

whereas at $s = \pi$, we have

$$\begin{aligned}\frac{\partial x}{\partial t} &= \int_0^{\pi} \frac{-1}{2+\cos(\sigma)} d\sigma, \\ \frac{\partial y}{\partial t} &= 1.\end{aligned}$$

For the vector field to be continuous, it should happen that the evolution at these two points is equal, however, we can see that the evolution of x differs by a sign at the two points and is not zero. So even if we ignored the loss of differentiability at $s = \frac{\pi}{2}$, there would still be a "tearing apart" at $s = \pi$. Notice that both problems arise from the integral term which doesn't depend on n, and therefore this phenomenon should be the same for all round $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ embedded in $\mathbb{R}^{n+1}$.

We conjecture the following:

Conjecture 4.2. *If a metric on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ is embeddable in $\mathbb{R}^{n+1}$ it will not remain so on any time interval $[0, \epsilon)$ with $\epsilon > 0$.*

4.2 Comment on a Level-Set Method

It was shown in [19] and [11] that the mean curvature flow can be formulated as the evolution of a function whereby all level sets are evolving by the mean curvature flow. This is a valuable technique because we can use viscosity solutions to continue the flow through singularities. The level set technique hinges on the fact that if two level sets are not touching initially they never will. We describe an example using the Ricci flow where two manifolds which don't intersect at $t = 0$ collide at some small time

later. Consider the peanut shape described in the previous section. By examining Equations 4.1 and 4.2 we see that at the caps of the peanut the velocity is outward. The Ricci flow causes the peanut to expand outward in a direction parallel to the axis of symmetry. Now suppose the peanut is contained in a slightly larger sphere. The sphere shrinks. Therefore, as long as the sphere is close enough to the peanut, they collide soon after $t = 0$.

4.3 Reducing to a Single PDE

We apply some of the simplifications discussed in Chapter 3 to reduce the Ricci flow equations to a single PDE under certain circumstances.

Proposition 4.3. *Under a certain change of coordinates and choice of Z, Equations 4.1 and 4.2 can be written as*

$$\frac{\partial \psi}{\partial t} = -\cos^2(\psi)\left(\frac{(\psi')^2}{\tan(\psi)} - \frac{\psi'}{s} + \psi'' + \frac{(n-2)\psi'}{s\cos^2(\psi)} - \frac{(n-2)\tan(\psi)}{s^2}\right). \tag{4.3}$$

Proof. First, we assume that $\gamma = (x(s), s)$, and choose

$$Z = (K_s + (n-2)K_\theta)\frac{x'}{|\gamma'|}.$$

This reduces the Ricci flow system to

$$\begin{aligned} \frac{\partial x}{\partial t} &= \int \left(\frac{|\gamma|^2}{x'}\left(K_s^2 - K_s K_\theta\right)\right) ds + (K_s + (n-2)K_\theta)\frac{(1+(x')^2)}{|\gamma'|}, \\ \frac{\partial y}{\partial t} &= 0. \end{aligned}$$

Make the substitutions

$$\begin{aligned} |\gamma'| &= (1+(x')^2)^{1/2}, \\ K_s &= \frac{x''}{(1+(x')^2)^{3/2}}, \\ K_\theta &= \frac{x'}{s(1+(x')^2)^{1/2}}. \end{aligned}$$

Then

$$\frac{\partial x}{\partial t} = \int \left(\frac{1}{x'} \left(\frac{x''}{1+(x')^2} \right)^2 - \frac{x''}{(1+(x')^2)s} \right) ds + \frac{x''}{1+(x')^2} + (n-2)\frac{x'}{s}.$$

Since the term x (undifferentiated) never appears, we may differentiate both sides with respect to s to eliminate the integral and give a second order PDE in terms of x'.

$$\frac{\partial x'}{\partial t} = \frac{1}{x'} \left(\frac{x''}{1+(x')^2} \right)^2 - \frac{x''}{(1+(x')^2)s} + \frac{\partial}{\partial s} \left(\frac{x''}{1+(x')^2} + (n-2)\frac{x'}{s} \right).$$

We make the substitution

$$\psi = \tan^{-1}(x'),$$

which is suggested by the repetition of the term $\frac{x''}{1+(x')^2}$. Then $\psi' = \frac{x''}{1+(x')^2}$.

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= \frac{\frac{\partial x'}{\partial t}}{1+(x')^2} = \cos^2(\psi)\frac{\partial x'}{\partial t}, \\ &= \cos^2(\psi) \left(\frac{(\psi')^2}{\tan(\psi)} - \frac{\psi'}{s} + \psi'' + \frac{(n-2)\psi'}{s\cos^2(\psi)} - \frac{(n-2)\tan(\psi)}{s^2} \right). \end{aligned}$$

□

4.4 The Existence of Steady and Expanding Ricci Soltions

Here we provide a proof of Theorem 4.1. In the previous section we have seen that if a curve $\gamma = (x(s), s)$ is evolving according to Equation 4.3 where $\psi = \tan^{-1}(x')$, then the $n-$dimensional manifold of revolution generated by γ will evolve by the Ricci flow. Here we derive a soliton equation. We have seen in Chapter 3 that if γ generates a soliton then $\frac{\partial \psi}{\partial t} = -cs\psi'$. We make this substitution to get a Ricci soliton equation:

$$-cs\psi' = \cos^2(\psi)\left(\psi'' + \frac{(\psi')^2}{\tan(\psi)} - \frac{\psi'}{s} + \frac{(n-2)\psi'}{s\cos^2(\psi)} - \frac{(n-2)\tan(\psi)}{s^2}\right).$$

Rearranging, we have the following:

Proposition 4.4. *Let $\psi(s)$ solve the equation*

$$\psi'' = \frac{-cs\psi'}{\cos^2(\psi)} - \frac{(\psi')^2}{\tan(\psi)} + \frac{\psi'}{s} - \frac{(n-2)\psi'}{s\cos^2(\psi)} + \frac{(n-2)\tan(\psi)}{s^2}, \tag{4.4}$$

with conditions $\psi(0) = 0$, $\psi'(0) = 1$, then there is a rotationally symmetric soliton metric whose scale is changing by c at $t = 0$ and whose sectional curvatures at the origin are all 1.

We take the boundary conditions $\psi(0) = 0$, since ψ should be an odd function, and $\psi'(0) = 1$, which causes all solution manifolds to have curvature 1 at the origin. Notice that the right hand side of this equation vanishes at $s = 0$ even though most of the individual terms approach positive or negative infinity. It is important that $\psi''(0)$ is zero, since ψ is an odd function. We assume $c \geq 0$, which is the condition for a soliton to be expanding or steady.

Claim 4.5. *If 4.4 does not have a solution on all $\mathbb{R}$ then it has a solution on a neighborhood $s \in (-S, S)$ and $\psi \to \pm\frac{\pi}{2}$ as $s \to \pm S$.*

Proof. By standard ODE theory, Equation 4.4 has existence in a neighborhood of $s = 0$. Observe the obstructions to global existence: ψ becomes zero or $\pi/2$ at $s = S$ for some $S \in (0, \infty)$. If at any $s > 0$, $\psi'(s) =$, then

$$\psi''(s) = \frac{(n-2)\tan(\psi)}{s^2} > 0.$$

Therefore, ψ' is never negative when $s > 0$. So $\psi = 0$ is impossible for $s > 0$. (Recall the boundary conditions $\psi(0) = 0, \psi'(0) = 1$.) Hence there is some $S \in (0, \infty)$ such that $\psi \to \frac{\pi}{2}$ as $s \to S$. Since ψ must be an odd function, it is also true that $\psi \to -\frac{\pi}{2}$ as $s \to -S$. □

If Equation 4.4 has a solution on all $\mathbb{R}$, then we have found a soliton. In that case there is $\psi(s)$ for all $s \in \mathbb{R}$ and substituting back $x' = \tan(\psi)$, with any boundary condition we like, we obtain $x(s)$ for all s. Then $(x(s), s)$ will be the generating curve for a soliton. It may not always be, however that there is a solution on all $\mathbb{R}$. Consider, for example, the cigar soliton. This soliton is generated by a curve which has $x' \to \infty$ as $s \to s_0 < \infty$. Hence, we expect Equation 4.4 not to have a solution on all s for $n = 2$ and $c = 0$, but the resulting curve still generates a complete soliton metric. We now show that this will always happen if 4.4 does not have a solution on all $\mathbb{R}$.

Claim 4.6. *If Equation 4.4 does not have a solution on all $\mathbb{R}$, the curve given by its solution still generates a complete soliton metric.*

Proof. We assume that there is some positive $S < \infty$ beyond which 4.4 can not be solved. However, if the arclength of the curve from 0 to S, $\int_0^S \sqrt{1 + (x')^2} ds = \int_0^S \frac{1}{\cos(\psi)} ds$, is infinite, then γ still generates a complete manifold. Dividing both sides of Equation 4.4 by ψ' gives

$$\frac{\partial}{\partial s} \ln \psi' = \frac{-cs}{\cos^2(\psi)} + \frac{n-2}{s\cos(\psi)} \left(\frac{\sin(\psi)}{s\psi'} - \frac{1}{\cos(\psi)} \right) - \frac{\psi'}{\tan(\psi)} + \frac{1}{s}.$$

Upon observation, the right hand side is negative for large enough ψ' and ψ close enough to $\frac{\pi}{2}$, which shows that ψ' is bounded. Therefore,

$$\lim_{s \to S} \frac{\cos(\psi)}{S - s} = \lim_{s \to S} \sin(\psi)\psi' = \lim_{s \to S} \psi'(s) < C < \infty.$$

So then $\frac{1}{\cos(\psi)} > \frac{k}{S-s}$ (where $k = 1/C$) for s close to S. Integrating both sides, we see that $\int_0^S \frac{1}{\cos(\psi)} ds = \int_0^S \sqrt{1 + (x')^2} ds$ is infinite. This shows that the length of the curve $\gamma = (x(s), s)$ is infinite, and hence that the manifold it generates is complete. □

Therefore, even if the soliton equation doesn't have global existence, its solution still produces a complete, non-compact soliton. This completes the proof of Theorem 4.1.

Chapter 5

THE YAMABE FLOW

Here we assume that γ is of the form $g = (x(s), s)$, and derive an ODE whose solution guarantees a Yamabe steady, shrinking, or expanding soliton depending on the choice of a parameter in that equation. Explicitly, we prove the following:

Theorem 5.1. *For each non-negative real number ϵ, and for integers $n \geq 2$, there exists a complete rotationally symmetric metric g on $\mathbb{R}^n$ such that the following conditions hold:*

1. *g has constant sectional curvature 1 at the origin.*
2. *g is a Yamabe soliton metric whose scale at time $t = 0$ is changing by a factor of ϵ.*

We will take the Yamabe flow to mean any evolution equation

$$\frac{\partial}{\partial t} g_{ij} = -R g_{ij} + \mathcal{L}_Z g_{ij}, \tag{5.1}$$

where Z is any smooth time-dependent vector field on M.

Proposition 5.2. *If a curve $\gamma(x(s), s)$ evolves by the system*

$$\begin{aligned} \frac{\partial x}{\partial t} &= \int \frac{1}{x'} \left((n-1) \frac{(x'')^2}{(1+(x')^2)^2} + \frac{(n-4)(n-1)}{2} \frac{x' x''}{s(1+(x')^2)} \right. \\ &\quad \left. - \frac{(n-1)(n-2)}{2} \frac{(x')^2}{s^2} \right) ds \\ &\quad + (n-1) \frac{x''}{(1+(x')^2)} + \frac{(n-1)(n-2)}{2} \frac{x'}{s}, \end{aligned}$$

then the manifold of revolution generated by γ evolves by Equation 5.1 for some Z.

Proof. By Theorem 3.2 we begin with the system

$$\begin{aligned}\frac{\partial x}{\partial t} &= \int\left(\frac{|\gamma'|^2}{x'}\left(\frac{K_s}{K_\theta}\lambda_\theta-\lambda_s\right)\right)ds+\frac{\lambda_\theta}{K_\theta}\frac{y'}{|\gamma'|}+Z(s)x',\\ \frac{\partial y}{\partial t} &= -\frac{\lambda_\theta}{K_\theta}\frac{x'}{|\gamma'|}+Z(s)y',\end{aligned}$$

where $\lambda_s=\lambda_\theta=R/2$ and by Probposition 3.4,

$$R=2(n-1)K_sK_\theta+(n-1)(n-2)K_\theta^2.$$

So

$$\begin{aligned}\frac{\partial x}{\partial t} &= \int\left(\frac{|\gamma'|^2}{x'}\left((n-1)K_s^2+\frac{(n-1)(n-2)}{2}K_sK_\theta\right.\right.\\ &\quad\left.\left.-\ (n-1)K_sK_\theta-\frac{(n-1)(n-2)}{2}K_\theta^2\right)\right)ds\\ &\quad+\frac{y'}{|\gamma'|}\left((n-1)K_s+\frac{(n-1)(n-2)K_\theta}{2}\right)+Z(s)x',\\ \frac{\partial y}{\partial t} &= -\frac{x'}{|\gamma'|}\left((n-1)K_s+\frac{(n-1)(n-2)K_\theta}{2}\right)+Z(s)y'.\end{aligned}$$

Assume $y=s$ and $y'=1$ and choose $Z(s)=\frac{x'}{|\gamma'|}\left((n-1)K_s+\frac{(n-1)(n-2)K_\theta}{2}\right)$. Then

$$\begin{aligned}\frac{\partial x}{\partial t} &= \int\left(\frac{|\gamma'|^2}{x'}\left((n-1)K_s^2+\frac{(n-1)(n-2)}{2}K_sK_\theta\right.\right.\\ &\quad\left.\left.-(n-1)K_sK_\theta-\frac{(n-1)(n-2)}{2}K_\theta^2\right)\right)ds\\ &\quad+\left((n-1)K_s+\frac{(n-1)(n-2)}{2}K_\theta\right)|\gamma'|,\\ &= \int\left(\frac{|\gamma'|^2}{x'}\left((n-1)K_s^2+\frac{(n-4)(n-1)}{2}K_sK_\theta-\frac{(n-1)(n-2)}{2}K_\theta^2\right)\right)ds\\ &\quad+\left((n-1)K_s+\frac{(n-1)(n-2)}{2}K_\theta\right)|\gamma'|.\end{aligned}$$

Make the substitutions

$$\begin{aligned} K_s &= \frac{y'x'' - y''x'}{|\gamma'|^3} = \frac{x''}{(1+(x')^2)^{3/2}}, \\ K_\theta &= \frac{x'}{y|\gamma'|} = \frac{x'}{s(1+(x')^2)^{1/2}}, \end{aligned}$$

to get

$$\begin{aligned} \frac{\partial x}{\partial t} &= \int \frac{1}{x'} \left((n-1)\frac{(x'')^2}{(1+(x')^2)^2} + \frac{(n-4)(n-1)}{2}\frac{x'x''}{s(1+(x')^2)} \right. \\ &\quad \left. -\frac{(n-1)(n-2)}{2}\frac{(x')^2}{s^2} \right) ds \\ &\quad +(n-1)\frac{x''}{(1+(x')^2)} + \frac{(n-1)(n-2)}{2}\frac{x'}{s}. \end{aligned}$$

□

Proposition 5.3. *Under a certain change of coordinates and choice of Z, the equation in Proposition 5.2 can be written as*

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= \cos^2(\psi)\left((n-1)\frac{(\psi')^2}{\tan(\psi)} + \frac{(n-1)(n-4)}{2}\frac{\psi'}{s} - (n-1)(n-2)\frac{\tan(\psi)}{s^2} \right. \\ &\quad \left. +(n-1)\psi'' + \frac{(n-1)(n-2)}{2}\frac{\psi'}{\cos^2(\psi)s} \right). \end{aligned}$$

Proof. We make the substitution $\psi = \tan^{-1}(x')$. Rewrite the system in terms of ψ:

$$\begin{aligned}
\frac{\partial\psi}{\partial t} &= \frac{\partial}{\partial t}\tan^{-1}(x') = \frac{\frac{\partial x'}{\partial t}}{1+(x')^2} = \frac{\frac{\partial}{\partial s}\frac{\partial x}{\partial t}}{1+(x')^2}, \\
&= \frac{1}{1+(x')^2}\left((n-1)\frac{(x'')^2}{x'(1+(x')^2)^2} + \frac{(n-4)(n-1)}{2}\frac{x''}{s(1+(x')^2)}\right. \\
&\quad \left. -\frac{(n-1)(n-2)}{2}\frac{x'}{s^2}\right) \\
&\quad +\frac{1}{1+(x')^2}\left(\frac{\partial}{\partial s}\left((n-1)\frac{x''}{(1+(x')^2)} + \frac{(n-1)(n-2)}{2}\frac{x'}{s}\right)\right), \\
&= \cos^2(\psi)\left((n-1)\frac{(\psi')^2}{\tan(\psi)} + \frac{(n-1)(n-4)}{2}\frac{\psi'}{s} - \frac{(n-1)(n-2)}{2}\frac{\tan(\psi)}{s^2}\right. \\
&\quad \left. +(n-1)\psi'' + \frac{(n-1)(n-2)}{2}\frac{\psi'}{\cos^2(\psi)s} - \frac{(n-1)(n-2)}{2}\frac{\tan(\psi)}{s^2}\right), \\
&= \cos^2(\psi)\left((n-1)\frac{(\psi')^2}{\tan(\psi)} + \frac{(n-1)(n-4)}{2}\frac{\psi'}{s} - (n-1)(n-2)\frac{\tan(\psi)}{s^2}\right. \\
&\quad \left. +(n-1)\psi'' + \frac{(n-1)(n-2)}{2}\frac{\psi'}{\cos^2(\psi)s}\right).
\end{aligned}$$

□

We showed in Chapter 3 that if we wish to obtain a soliton equation, we substitute $-cs\psi'$ for $\frac{\partial\psi}{\partial t}$. By doing this and solving for ψ'', we get the following:

Proposition 5.4. *A solution to the equation*

$$\psi'' = \frac{-cs\psi'}{(n-1)\cos^2(\psi)} - \frac{(\psi')^2}{\tan(\psi)} - \frac{(n-4)\psi'}{2s} + \frac{(n-2)\tan(\psi)}{s^2} - \frac{(n-2)\psi'}{2s\cos^2(\psi)},$$

for a given c corresponds to a soliton solution to the Yamabe flow. The parameter c determines which kind of soliton it is.

We use boundary conditions $\psi(0) = 0$ and $\psi'(0) = 1$ and assume that $c \geq 0$. Similar to the main proof in Chapter 4, we have the following claims:

Claim 5.5. *If the equation in Proposition 5.4 does not have a solution on all $\mathbb{R}$ then it has a solution on a neighborhood $s \in (-S, S)$ and $\psi \to \pm\frac{\pi}{2}$ as $s \to \pm S$.*

Proof. By standard ODE theory, there is a solution on some neighborhood of 0. The equation will only have a solution so long as $0 < \psi < \frac{\pi}{2}$. We illustrate that ψ' is never negative by pointing out that if ψ' were zero, then

$$\psi'' = \frac{(n-2)\tan(\psi)}{s^2} > 0.$$

Because of the boundary conditions, this implies that ψ is always increasing, and can never be 0 (except at $s = 0$). Therefore, if the equation in Proposition 5.4 doesn't have a solution on all $\mathbb{R}$ it is because there is some $S \in (0, \infty)$ such that $\psi \to \frac{\pi}{2}$ as $s \to S$. By symmetry, $\psi \to -\frac{\pi}{2}$ as $s \to -S$ □

Again, there are two cases: if the equation in Proposition 5.4 has a solution for all $\mathbb{R}$, we are done. If not, we show that the curve is still infinite, and hence the manifold it generates is complete.

Claim 5.6. *If the equation in Proposition 5.4 does not have a solution on all* $\mathbb{R}$*, the curve given by its solution still generates a complete soliton metric.*

Proof. Similarly to the proof of Theorem 4.1, if we show that ψ' is bounded as $s \to S$, we will have shown that the curve γ is of infinite length. This will imply that the manifold generated by γ is complete. Dividing both sides of the Equation 5.4 by ψ' gives

$$\begin{aligned}\frac{\partial}{\partial s}\ln\psi' &= \frac{-cs}{(n-1)\cos^2(\psi)} + \frac{(n-2)}{s\cos(\psi)}\left(\frac{-1}{2\cos(\psi)} + \frac{\sin(\psi)}{\psi' s}\right) \\ &\quad - \frac{\psi'}{\tan(\psi)} - \frac{(n-4)}{2s}.\end{aligned}$$

Upon observation, the right hand side is negative for large enough ψ' and ψ close enough to $\frac{\pi}{2}$. This shows that the length of the curve $\gamma = (x(s), s)$ is infinite, and hence that the manifold it generates is complete. □

This completes the proof of Theorem 5.1.

Chapter 6
OTHER FLOWS

In previous chapters, we have derived soliton equations on the Ricci and Yamabe flows by assuming that the soliton is a manifold of revolution in $\mathbb{R}^{n+1}$ and generated by the graph of a function where the dependent variable runs parallel to the axis of revolution. In this section we use the same method to derive soliton equations for flow by the Einstein tensor, the mean curvature flow, and the Gauss curvature flow. Since the procedure has already been used several times, we will be brief.

6.1 Flow by the Einstein Tensor

Consider the evolution of the metric by 2 times the Einstein tensor:

$$\frac{\partial}{\partial t} g_{ij} = 2R_{ij} - Rg_{ij}. \tag{6.1}$$

This flow is not parabolic, so short-time existence is not known. We can, however formulate an equation for rotationally symmetric Einstein solitons. We prove the following:

Theorem 6.1. *For each non-negative real number ϵ, and for integers $n \geq 4$, there exists a complete rotationally symmetric metric g on $\mathbb{R}^n$ such that the following conditions hold:*

1. *g has constant sectional curvature 1 at the origin.*
2. *g is a soliton metric for the flow in Equation 6.1 whose scale at time $t = 0$ is changing by a factor of ϵ.*

Proof. In the case of the Ricci flow and Yamabe flow, we began with Equation 3.2, assume $y = s$, choose Z so that $\frac{\partial y}{\partial t} = 0$, substitute the eigenvalues of the Einstein

tensor for λ_s and λ_θ, plug in the principal curvatures K_s and K_θ, differentiate both sides, make the substitution $\psi = \tan^{-1}(x')$, and finally substitute $-cs\psi'$ for $\frac{\partial \psi}{\partial t}$ to obtain a soliton equation. However, in this case there is a simplifying observation. Since the Einstein tensor is a linear combination of the Yamabe and Ricci tensors, we can just subtract the two ψ evolutions.

$$\begin{aligned}
\frac{\partial \psi}{\partial t} &= \cos^2(\psi)\left((n-1)\frac{(\psi')^2}{\tan(\psi)} + \frac{(n-1)(n-4)}{2}\frac{\psi'}{s} - (n-1)(n-2)\frac{\tan(\psi)}{s^2}\right. \\
&\quad \left. +(n-1)\psi'' + \frac{(n-1)(n-2)}{2}\frac{\psi'}{s\cos^2(\psi)}\right) \\
&\quad -\cos^2(\psi)\left(\psi'' + \frac{(\psi')^2}{\tan(\psi)} - \frac{\psi'}{s} + \frac{(n-2)\psi'}{s\cos^2(\psi)} - \frac{(n-2)\tan(\psi)}{s^2}\right), \\
&= \cos^2(\psi)\left((n-2)\psi'' + (n-2)\frac{(\psi')^2}{\tan(\psi)} + \frac{(n-2)(n-3)}{2}\frac{\psi'}{s}\right. \\
&\quad \left. -(n-2)^2\frac{\tan(\psi)}{s^2} + \frac{(n-2)(n-3)}{2}\frac{\psi'}{s\cos^2(\psi)}\right).
\end{aligned}$$

Now we substitute $-cs\psi'$ for $\frac{\partial \psi}{\partial t}$ and solve for ψ'':

$$\psi'' = \frac{-cs\psi'}{\cos^2(\psi)(n-2)} - \frac{(n-3)}{2}\frac{\psi'}{s} + (n-2)\frac{\tan(\psi)}{s^2} - \frac{(\psi')^2}{\tan(\psi)} - \frac{(n-3)}{2}\frac{\psi'}{s\cos^2(\psi)}. \quad (6.2)$$

The proof, just like the proofs of Theorems 5.1 and 4.1, is completed by dividing both sides of the equation by ψ' and showing that the right side is negative for large values of ψ' and for ψ close to $\frac{\pi}{2}$. Notice however, that in this case, the proof fails if $n = 3$ since the term $-\frac{(n-3)}{2}\frac{\psi'}{s\cos^2(\psi)} + (n-2)\frac{\tan(\psi)}{s^2}$ must be negative. Numerically, we see (weak) evidence that the Einstein flow does not have this kind of soliton for dimension 3 (compare figures 7.3 and 7.4). □

Remark. By the results of Berger and Ebin [5], there can not be compact steady solitons for this flow other than Einstein metrics.

6.2 The Mean Curvature Flow

It may seem strange to use this technique to analyze extrinsic flows, however our coordinate change can shed insight into the manifold of revolution case. If a submanifold (of revolution) is evolving by the mean curvature flow,

$$\frac{\partial F}{\partial t} = H\vec{v},$$

where $\vec{v}$ is a unit normal vector field and H is the trace of the shape operator, we can find the evolution of the metric using Proposition 3.1. We see that $\lambda_s = K_s^2 + (n-1)K_sK_\theta$ and $\lambda_\theta = K_sK_\theta + (n-1)K_\theta^2$. Then recall the system

$$\begin{aligned}
\frac{\partial x}{\partial t} &= \int \left(\frac{|\gamma'|^2}{x'} \left(\frac{K_s}{K_\theta}\lambda_\theta - \lambda_s \right) \right) ds + \frac{\lambda_\theta}{K_\theta}\frac{y'}{|\gamma'|} + Z(s)x', \\
\frac{\partial y}{\partial t} &= -\frac{\lambda_\theta}{K_\theta}\frac{x'}{|\gamma'|} + Z(s)y'
\end{aligned}$$

from Theorem 3.2. Notice that once we plug in λ_s and λ_θ, the integrand is zero. This leaves us with

$$\begin{aligned}
\frac{\partial x}{\partial t} &= (K_s + (n-1)K_\theta)\frac{y'}{|\gamma'|} + Z(s)x', \\
\frac{\partial y}{\partial t} &= -(K_s + (n-1)K_\theta)\frac{x'}{|\gamma'|} + Z(s)y'.
\end{aligned}$$

Again we can turn this into an evolution in x alone by assuming $y = s$, which means

$$\begin{aligned}
K_s &= \frac{x''}{(1+(x')^2)^{3/2}}, \\
K_\theta &= \frac{x'}{s(1+(x')^2)^{1/2}}.
\end{aligned}$$

The equation is

$$\frac{\partial x}{\partial t} = \frac{x''}{1+(x')^2} + (n-1)\frac{x'}{s}.$$

Differentiating both sides gives

$$\frac{\partial x'}{\partial t} = \frac{\partial}{\partial s}\left(\frac{x''}{1+(x')^2} \right) + (n-1)\frac{x''}{s} - (n-1)\frac{x'}{s^2}.$$

We will again make the substitution $\psi = \tan^{-1}(x')$

$$\frac{\partial \psi}{\partial t} = \cos^2(\psi)\left(\psi'' + (n-1)\frac{\psi'}{s\cos^2(\psi)} - (n-1)\frac{\tan(\psi)}{s^2}\right).$$

To derive the soliton equation, substitute $-cs\psi'$ in for $\frac{\partial \psi}{\partial t}$ and solve for ψ'':

$$\psi'' = -\frac{cs\psi'}{\cos^2(\psi)} - (n-1)\frac{\psi'}{s\cos^2(\psi)} + (n-1)\frac{\tan(\psi)}{s^2}. \tag{6.3}$$

6.3 The Gauss Curvature Flow

We derive the equation in ψ for the Gauss curvature flow,

$$\frac{\partial F}{\partial t} = \det(s_i^j)\vec{v}.$$

The procedure is exactly the same as for all other flows covered thus far. We begin with the system

$$\begin{aligned}
\frac{\partial x}{\partial t} &= \int \left(\frac{|\gamma'|^2}{x'}\left(\frac{K_s}{K_\theta}\lambda_\theta - \lambda_s\right)\right) ds + \frac{\lambda_\theta}{K_\theta}\frac{y'}{|\gamma'|} + Z(s)x', \\
\frac{\partial y}{\partial t} &= -\frac{\lambda_\theta}{K_\theta}\frac{x'}{|\gamma'|} + Z(s)y',
\end{aligned}$$

where now $\lambda_s = K_s^2 K_\theta^{(n-1)}$ and $\lambda_\theta = K_s K_\theta^n$. As with the mean curvature flow, the integrand is zero, and the system becomes:

$$\begin{aligned}
\frac{\partial x}{\partial t} &= K_s K_\theta^{(n-1)}\frac{y'}{|\gamma'|} + Z(s)x', \\
\frac{\partial y}{\partial t} &= -K_s K_\theta^{(n-1)}\frac{x'}{|\gamma'|} + Z(s)y'.
\end{aligned}$$

Making substitutions, we turn this into an equation in x alone.

$$\frac{\partial x}{\partial t} = \frac{x''}{1+(x')^2}\frac{(x')^{n-1}}{(1+(x')^2)^{(n-1)/2}s^{n-1}}.$$

In terms of ψ,

$$\frac{\partial x}{\partial t} = \psi'\frac{\sin^{n-1}(\psi)}{s^{n-1}}.$$

So

$$\frac{\partial x'}{\partial t} = \psi'' \frac{\sin^{n-1}(\psi)}{s^{n-1}} + (\psi')^2(n-1)\frac{\cos^{n-1}(\psi)}{s^{n-1}} - \psi'(n-1)\frac{\sin^{n-1}(\psi)}{s^n},$$

and

$$\frac{\partial \psi}{\partial t} = \cos^2(\psi)\left(\psi'' \frac{\sin^{n-1}(\psi)}{s^{n-1}} + (\psi')^2(n-1)\frac{\cos^{n-1}(\psi)}{s^{n-1}} - \psi'(n-1)\frac{\sin^{n-1}(\psi)}{s^n}\right).$$

The soliton equation is obtained by substituting $-cs\psi'$ for $\frac{\partial \psi}{\partial t}$ and solving for ψ'':

$$\psi'' = -\frac{cs\psi'}{\cos^2(\psi)\sin^{n-1}(\psi)} - (n-1)\frac{(\psi')^2\cos^{n-1}(\psi)}{\sin^{n-1}(\psi)s} + (n-1)\frac{\psi'}{s}. \tag{6.4}$$

6.4 A General Form

We derive a general form. Begin with Equation 6.6, and take derivatives with respect to s of both sides:

$$\frac{\partial x'}{\partial t} = \frac{|\gamma'|^2}{x'}\left(\frac{K_s}{K_\theta}\lambda_\theta - \lambda_s\right) + \frac{\partial}{\partial s}\left(\frac{\lambda_\theta}{K_\theta}|\gamma'|\right).$$

Again, assume $\gamma = (x(s), s)$ which means

$$\begin{aligned} |\gamma'| &= (1+(x')^2)^{1/2} \\ K_s &= \frac{x''}{(1+(x')^2)^{3/2}} \\ K_\theta &= \frac{x'}{s(1+(x')^2)^{1/2}}, \end{aligned}$$

so we have

$$\begin{aligned} \frac{\partial x'}{\partial t} &= \frac{x''s\lambda_\theta}{(x')^2} - \frac{(1+(x')^2)\lambda_s}{x'} + \frac{\partial \lambda_\theta}{\partial s}\frac{s(1+(x')^2)}{x'} + \lambda_\theta \frac{\partial}{\partial s}\left(\frac{s}{x'}(1+(x')^2)\right), \\ &= \frac{x''s\lambda_\theta}{(x')^2} - \frac{(1+(x')^2)\lambda_s}{x'} + \frac{\partial \lambda_\theta}{\partial s}\frac{s(1+(x')^2)}{x'} + \lambda_\theta \frac{1}{x'}(1+(x')^2) \\ &\quad -\lambda_\theta \frac{sx''}{(x')^2}(1+(x')^2) + 2\lambda_\theta s x'', \\ &= \lambda_\theta s x'' - \frac{(1+(x')^2)\lambda_s}{x'} + \frac{\partial \lambda_\theta}{\partial s}\frac{s(1+(x')^2)}{x'} + \lambda_\theta \frac{1}{x'}(1+(x')^2) \end{aligned}$$

and

$$\frac{x'}{1+(x')^2}\frac{\partial x'}{\partial t} = \lambda_\theta s\frac{x'x''}{1+(x')^2} - \lambda_s + s\frac{\partial \lambda_\theta}{\partial s} + \lambda_\theta.$$

Substituting

$$\frac{\frac{\partial x'}{\partial t}}{(1+(x')^2)} = -cs\frac{x''}{1+(x')^2},$$

we have the following:

Theorem 6.2. *Let T_{ij} be a tensor which is simultaneously diagonalizable with the Ricci tensor and has eigenvalues $\lambda_s, \lambda_\theta, \cdots, \lambda_\theta$ on a rotationally symmetric manifold. Assume the equation*

$$-cs\tan(\psi)\psi' = \lambda_\theta s\tan(\psi)\psi' - \lambda_s + s\frac{\partial \lambda_\theta}{\partial s} + \lambda_\theta \tag{6.5}$$

has a solution. Then there is a soliton metric for the flow $\frac{\partial g_{ij}}{\partial t} = -2T_{ij}$ which is generated by the curve $\gamma = (x(s), s)$, where $x' = \tan(\psi)$.

Usually Equation 6.5 is second order because $\frac{\partial \lambda_\theta}{\partial s}$ contains ψ''. Notice that we do not guarantee completeness. To prove the resulting metric is complete, one must either show that the solution to 6.5 exists on all $\mathbb{R}$ or go through a proof similar to the proofs of Theorems 4.1 and 5.1.

Remark. It is often easier to plug the specific eigenvalues into the equation

$$\frac{\partial x}{\partial t} = \int\left(\frac{|\gamma'|^2}{x'}\left(\frac{K_s}{K_\theta}\lambda_\theta - \lambda_s\right)\right)ds + \frac{\lambda_\theta}{K_\theta}|\gamma'| \tag{6.6}$$

first then find a second order ODE in ψ for a soliton rather than to try and use a general form. We derive the general form here for completeness.

Chapter 7

NUMERICS

In this section, we numerically solve the soliton equations 4.4, 5.4, 6.2, 6.3, and 6.4 and briefly discuss the results. We use the matlab code in Appendix A which can be called using the command :

```
soliton (n, ct, flow)
```

where n is the dimension, ct is the instant change of scale (the c parameter in the soliton equations), and flow is one of 'Ricci', 'Mean', 'Gauss', 'Yamabe', or 'Einstein'. This code produces a graph of the generating curve γ for the specified soliton and its reflection about the $x-$axis. For example, the two-dimensional steady Ricci soliton known as the cigar, is generated by the command

```
soliton (2, 0, 'Ricci')
```

which produces the image shown in Figure 7.1

7.1 Steady Solitons

Letting $c = 0$, we obtain differential equations whose solutions are steady solitons. These tend to be either asymptotic to something resembling a paraboloid or the cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$. Numerically, it is difficult to distinguish between a complete soliton which is asymptotic to the cylinder and an incomplete one, since either way, the numerics become unstable as $\psi \to \infty$. We provide here the images of steady solitons, however we only claim they exist as complete manifolds when we have proven so in other sections.

7.2 Expanding Solitons

For $c > 0$, we obtain an expanding soliton. Expanding solitons tend to be shaped like shuttlecocks and tend to be asymptotic to cones. The following are three-dimensional solitons with $c = 2$ for various flows: the Ricci flow (Figure 7.9) the Yamabe flow (Figure 7.10) flow by the Einstein tensor (Figure 7.11), the mean curvature flow (Figure 7.12) and the Gauss curvature flow (Figure 7.13).

To illustrate the convergence of a family of expanding solitons to the Bryant steady soliton, we plot a series of three-dimensional expaning solitons and the three-dimensional Bryant soliton on the same graph. The expanding solitons all converge to a conic shape away from the origin.

7.3 Shrinking Solitons

By setting $c < 0$, a solution to our differential equation would imply the existence of a shrinking soliton. The most well-known is the round sphere. By setting $c = -(n-1)$ in the case of the Ricci flow, we obtain a round sphere. Indeed, we see this in the numerics.

Note that we only see half of the sphere because of our assumption that $\gamma = (x(s), s)$. This raises the question of whether we are able to create other compact solitons by gluing a mirror image onto the manifold. Ivey has answered this question by proving that there are no non-compact, non-trivial shrinking Ricci solitons [27], which means that if we were to artificially complete our shrinkers by gluing the mirror image onto the manifold, the resulting metric would not be C^∞. Nevertheless, it would be a shrinking, Ricci soliton. Another way of saying this is that for every negative c, not equal to $-(n-1)$, there is a rotationally symmetric shrinking soliton metric in a neighborhood of the origin which cannot be extended to a complete Ricci soliton. We are unable to eliminate the possiblity that there are non-compact shrinking solitons which are asymptotic to a cylinder.

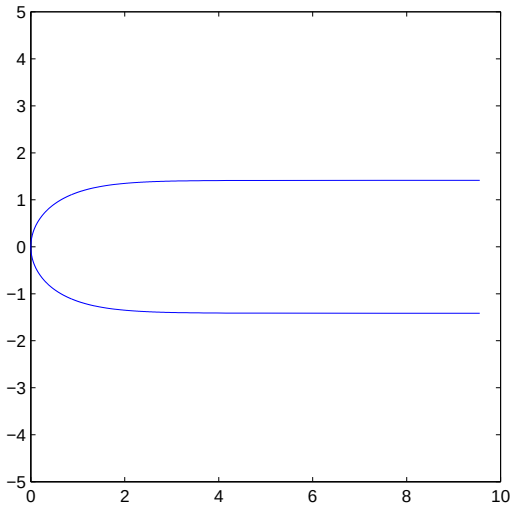

FIGURE 7.1. The two-dimensional steady Ricci soliton known as the cigar.

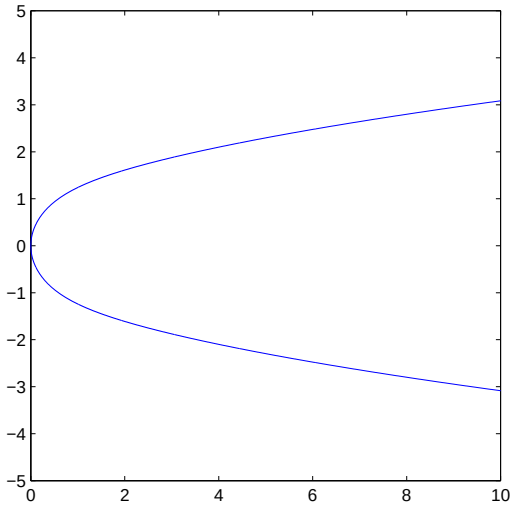

FIGURE 7.2. A three-dimensional Ricci steady soliton, known as the Bryant soliton.

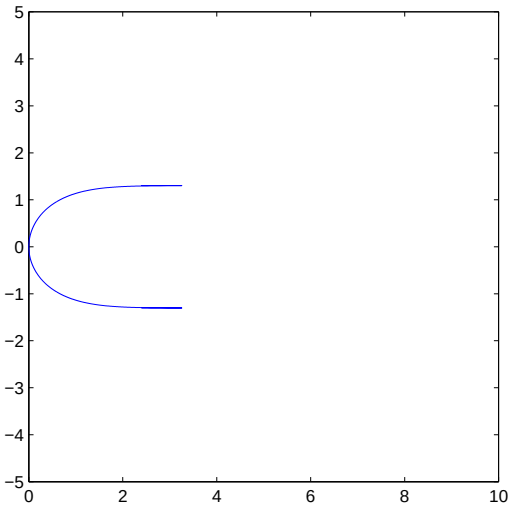

FIGURE 7.3. A three-dimensional Einstein steady soliton.

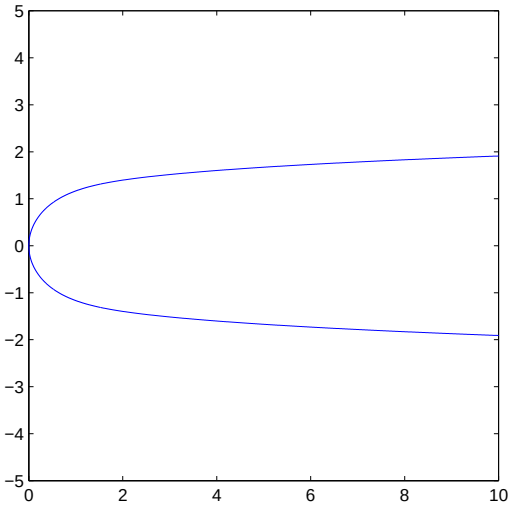

FIGURE 7.4. A four-dimensional Einstein steady soliton.

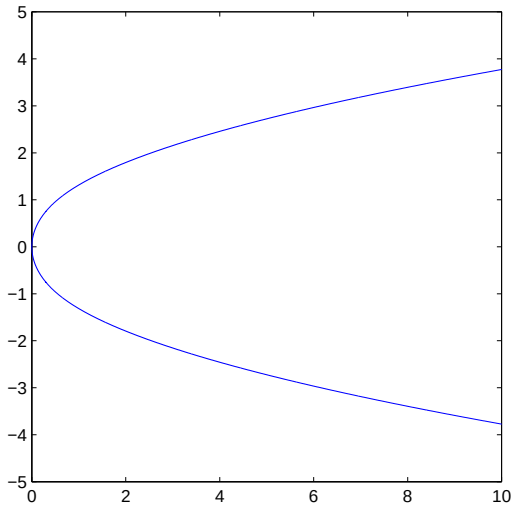

FIGURE 7.5. A three-dimensional mean curvature flow steady soliton.

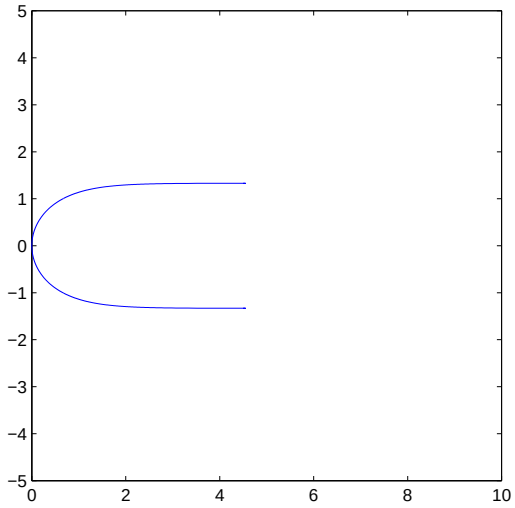

FIGURE 7.6. A three-dimensional Gauss steady soliton.

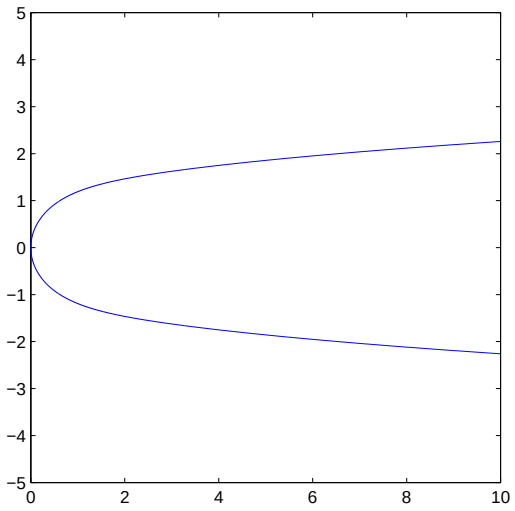

FIGURE 7.7. A three-dimensional Yamabe steady soliton.

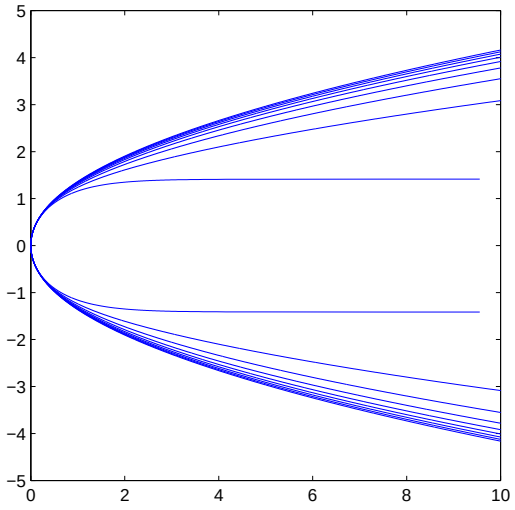

FIGURE 7.8. Ricci Bryant solitons in dimensions 2 (innermost) through 10 (outermost).

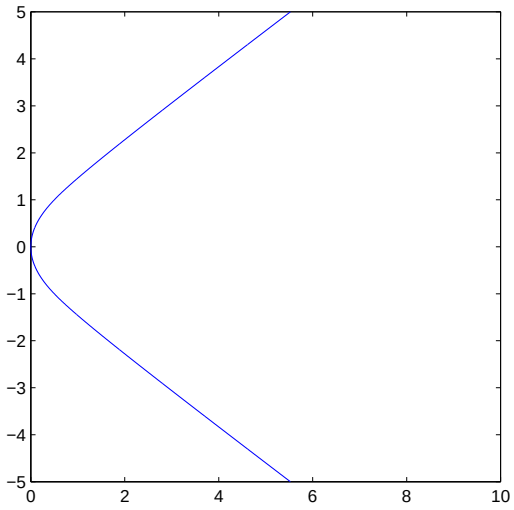

FIGURE 7.9. A three-dimensional expanding Ricci soliton.

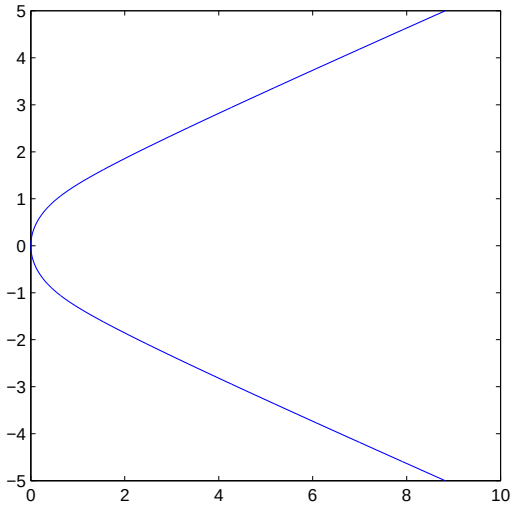

FIGURE 7.10. A three-dimensional expanding Yamabe soliton.

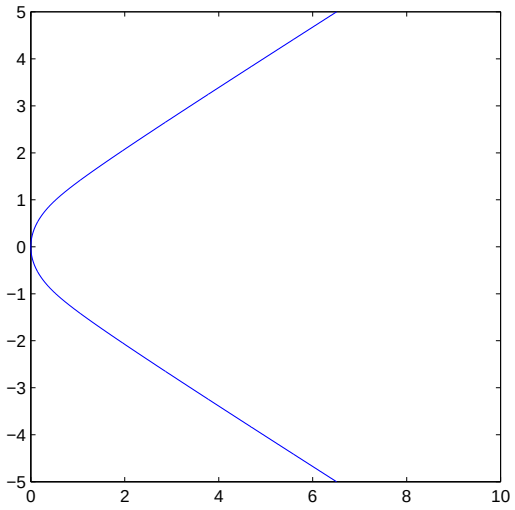

FIGURE 7.11. A three-dimensional expanding Einstein soliton.

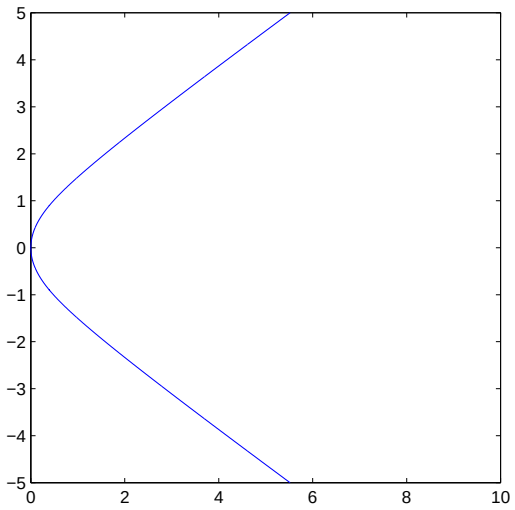

FIGURE 7.12. A three-dimensional expanding mean curvature flow soliton.

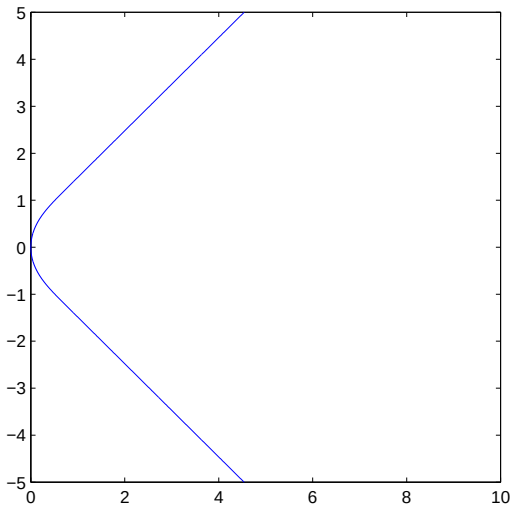

FIGURE 7.13. A three-dimensional expanding Gauss curvature flow soliton.

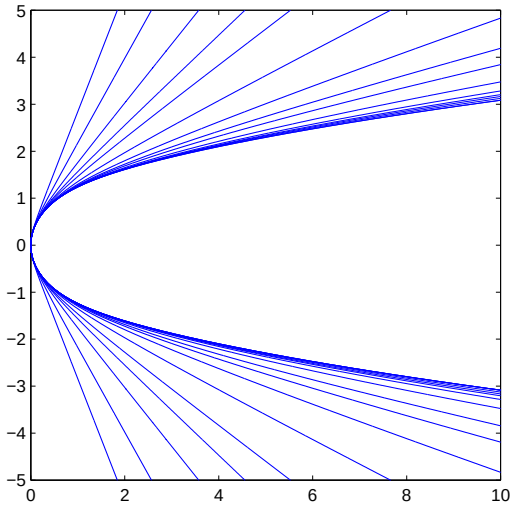

FIGURE 7.14. A sequence of expanding three-dimensional Ricci solitons and the steady soltion. The outermost one corresponds to c=20. The innermost one is the steady, c=0

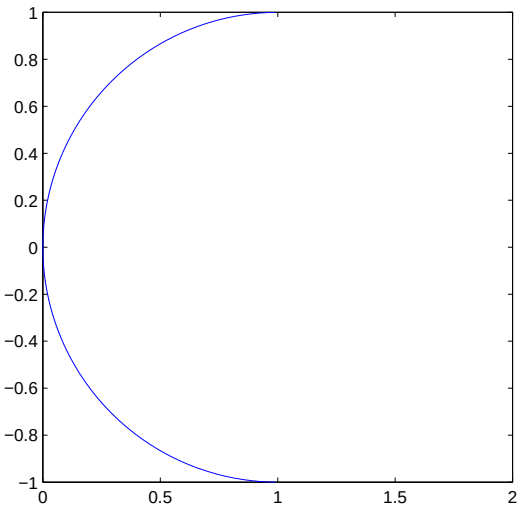

FIGURE 7.15. By setting $n = 3$, and $c = -2$, we obtain the round three-sphere. We only see half because we are assuming the generating curve is the graph of a function

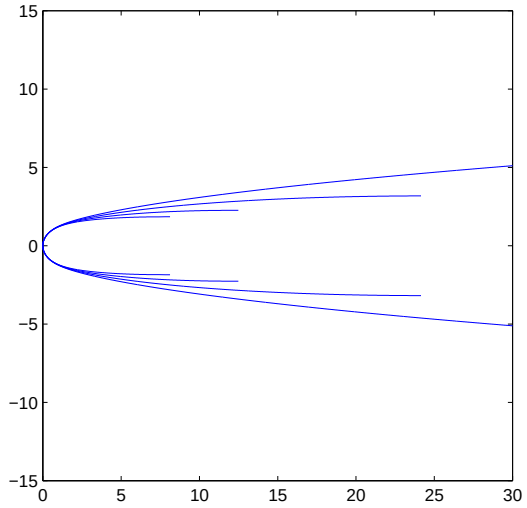

FIGURE 7.16. The three-dimensional Ricci steady soliton on the outside, followed by the shrinkers $c = -0.1$, $c = -0.2$, and $c = -0.3$ progressing inward.

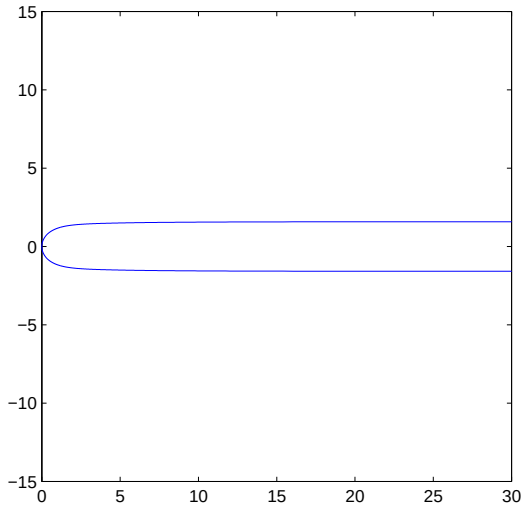

FIGURE 7.17. The three-dimensional Yamabe shrinker $c = -0.4$. It appears to be asymptotic to a cylinder.

Chapter 8

COMMENTS ON THE CROSS CURVATURE FLOW

In [13], Chow and Hamilton introduced the cross curvature flow to study negatively curved three-manifolds. In that paper, they conjecture that under proper rescaling, the cross curvature flow should evolve a negatively curved metric on a three-manifold into a hyperbolic metric (a metric with constant negative sectional curvature). In this chapter, we consider a very special case of non-compact, negatively curved three-manifolds and discuss a strategy for proving that they approach hyperbolic metrics at $t = \infty$ under proper rescaling. This chapter involves pseudo-Riemannian geometry and we recommend [35] as a reference.

Consider a situation wherein the ambient manifold is Minkowski four-space with signature $(+, +, -, +)$. We consider the cross curvature flow of a three-dimensional submanifold $\mathcal{M}$ of this space. It has been noted in [13] that in this case the cross curvature flow is induced by the Gauss curvature flow. Here we briefly illustrate why. Recall from Proposition 3.5 that the eigenvalues of the cross curvature tensor are $-K_1^2K_2K_3, -K_1K_2^2K_3$, and $-K_1K_2K_3^2$. We can use Proposition 3.1 to find the eigenvalues of that tensor by which the metric evolves under the Gauss curvature flow. By that equation,

$$\frac{\partial}{\partial t}g_{ij} = -2\det(s)h_{ij},$$

which has eigenvalues $-2K_1^2K_2K_3, -2K_1K_2^2K_3$, and $-2K_1K_2K_3^2$. Since that shape operator and the cross curvature tensor are simultaneously diagonalizable, we can conclude that the metric will evolve by the cross curvature flow:

$$\frac{\partial}{\partial t}g_{ij} = 2X_{ij}.$$

Let $\mathcal{M}$ be by a curve $\gamma(s) = (x(s), y(s))$. Define an embedding:

$$F(s,\phi,\theta) = (y(s)\sinh(\phi)\cos(\theta), y(s)\sinh(\phi)\sin(\theta), y(s)\cosh(\phi), x(s)).$$

Notice that for a fixed s, this is an embedding of a two-dimensional hyperboloid into the three-dimensional Minkowski space $(\cdot,\cdot,\cdot,x(s))$. Our metrics will be warped products of two-dimensional hyperboloids. Notice also that in the case when γ is a hyperbola, our induced metric will be hyperbolic three-space. We make the following assumptions on γ:

- $y(s) > 0$. Otherwise the curvature of $\mathcal{M}$ would be infinite.
- $(x')^2 - (y')^2 > 0$. This guarantees that the induced metric is positive definite.
- $y'x'' - x'y'' < 0$. This guarantees that the sectional curvature is negative.
- We assume that γ can be re-parameterized as the graph of a function $\gamma(s) = (s, y(s))$.
- For convenience, we assume that $y > |x|$ so that we may use convenient coordinates.
- We require $y^2 - x^2$ be bounded above zero and below infinity. This will force the curvatures to be bounded above zero and below infinity.
- $\frac{\partial}{\partial s}\sqrt{y^2 - x^2}$ should approach zero at positive and negative infinities.

We choose the following orthogonal basis for the tangent space of $\mathcal{M}$:

$$\begin{aligned}
\frac{\partial F}{\partial s} &= (y'(s)\sinh(\phi)\cos(\theta), y'(s)\sinh(\phi)\sin(\theta), y'(s)\cosh(\phi), x'(s)),\\
\frac{\partial F}{\partial \phi} &= (y(s)\cosh(\phi)\cos(\theta), y(s)\cosh(\phi)\sin(\theta), y(s)\sin(\phi), 0),\\
\frac{\partial F}{\partial \theta} &= (-y(s)\sinh(\phi)\sin(\theta), y(s)\sinh(\phi)\cos(\theta), 0, 0).
\end{aligned}$$

The induced metric is found using the standard Lorentzian metric on Minkowski space with signature $(+,+,-,+)$:

$$g = \left((x')^2 - (y')^2\right) ds^2 + y^2 \sinh^2(\phi) d\theta^2 + y^2 d\phi.$$

Lemma 8.1. *The sectional curvatures are*

$$\begin{aligned} K(\frac{\partial}{\partial s}, \frac{\partial}{\partial \theta}) = K(\frac{\partial}{\partial s}, \frac{\partial}{\partial \phi}) &= \frac{x'(y'x'' - x'y'')}{((x')^2 - (y')^2)^2}, \\ K(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}) &= \frac{-(x')^2}{y((x')^2 - (y')^2)}. \end{aligned}$$

Proof. In order to calculate the curvatures of $\mathcal{M}$, we need the second fundamental form, which is found taking derivatives of a unit normal vector field. We make a choice for the unit normal

$$\vec{n} = \frac{(x' \sinh(\phi)\cos(\theta), x' \sinh(\phi)\sin(\theta), x' \cosh(\phi), y')}{\sqrt{(x')^2 - (y')^2}}.$$

It has the following derivatives:

$$\begin{aligned} \frac{\partial \vec{n}}{\partial s} &= \frac{\partial}{\partial s}\left(\frac{1}{\sqrt{(x')^2 - (y')^2}}\right)(x' \sinh(\phi)\cos(\theta), x' \sinh(\phi)\sin(\theta), x' \cosh(\phi), y') \\ &\quad + \frac{(x'' \sinh(\phi)\cos(\theta), x'' \sinh(\phi)\sin(\theta), x'' \cosh(\phi), y'')}{\sqrt{(x')^2 - (y')^2}}, \\ \frac{\partial \vec{n}}{\partial \theta} &= \frac{(-x' \sinh(\phi)\sin(\theta), y' \sinh(\phi)\cos(\theta), 0, 0)}{\sqrt{(x')^2 - (y')^2}}, \\ \frac{\partial \vec{n}}{\partial \phi} &= \frac{(x' \cosh(\phi)\cos(\theta), y' \cosh(\phi)\sin(\theta), y' \sinh(\phi), 0)}{\sqrt{(x')^2 - (y')^2}}. \end{aligned}$$

We now use the Weingarten relations to find the second fundamental form:

$$\begin{aligned} h\left(\frac{\partial F}{\partial s}, \frac{\partial F}{\partial s}\right) &= -\left\langle \frac{\partial \vec{n}}{\partial s}, \frac{\partial F}{\partial s} \right\rangle = \frac{y'x'' - x'y''}{\sqrt{(x')^2 - (y')^2}}, \\ h\left(\frac{\partial F}{\partial \theta}, \frac{\partial F}{\partial \theta}\right) &= -\left\langle \frac{\partial \vec{n}}{\partial \theta}, \frac{\partial F}{\partial \theta} \right\rangle = \frac{-x'y \sinh^2(\phi)}{\sqrt{(x')^2 - (y')^2}}, \\ h\left(\frac{\partial F}{\partial \phi}, \frac{\partial F}{\partial \phi}\right) &= -\left\langle \frac{\partial \vec{n}}{\partial \phi}, \frac{\partial F}{\partial \phi} \right\rangle = \frac{-x'y}{\sqrt{(x')^2 - (y')^2}}. \end{aligned}$$

Raising an index, we have the shape operator:

$$S_i^j = \begin{pmatrix} \frac{y'x''-x'y''}{((x')^2-(y')^2)^{\frac{3}{2}}} & 0 & 0 \\ 0 & \frac{-x'}{y\sqrt{(x')^2-(y')^2}} & 0 \\ 0 & 0 & \frac{-x'}{y\sqrt{(x')^2-(y')^2}} \end{pmatrix}.$$

We denote the principle curvatures $K_s = \frac{y'x''-x'y''}{((x')^2-(y')^2)^{\frac{3}{2}}}$ and $K_\theta = \frac{-x'}{y\sqrt{(x')^2-(y')^2}}$. The sectional curvatures are the products of these with opposite sign:

$$\begin{aligned} K(\frac{\partial}{\partial s}, \frac{\partial}{\partial \theta}) &= K(\frac{\partial}{\partial s}, \frac{\partial}{\partial \phi}) = -K_s K_\theta = \frac{x'(y'x''-x'y'')}{((x')^2-(y')^2)^2}, \\ K(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}) &= -K_\theta^2 = \frac{-(x')^2}{y((x')^2-(y')^2)}. \end{aligned}$$

Here it is evident that the condition $y'x'' - x'y'' < 0$ implies that $\mathcal{M}$ is negatively curved. □

Theorem 8.2. *If $\gamma = (s, y(s))$, and y evolves by*

$$\frac{\partial y}{\partial t} = \frac{y''}{y^2\left(1-(y')^2\right)^2},$$

then $\mathcal{M}$ will evolve by

$$\frac{\partial}{\partial t} g_{ij} = 2X_{ij} + \mathcal{L}_V g_{ij}$$

for some time-dependent vector field V.

Proof. We assume that the submanifold $\mathcal{M}$ is evolving by the Gauss curvature flow plus a tangential vector field V:

$$\frac{\partial F}{\partial t} = -\det(s)\vec{n} + V(s)\frac{\partial}{\partial s} = \frac{(x')^3y'' - y'(x')^2x''}{y^2\left((x')^2-(y')^2\right)^{\frac{5}{2}}} + V(s)\frac{\partial}{\partial s}.$$

By the symmetry of the manifold, this evolution is equivalent to assuming that our generating curve γ is evolving by

$$\frac{\partial \gamma}{\partial t} = \frac{-\det(s)}{\sqrt{(x')^2-(y')^2}}(y', x') + V(s)(x', y').$$

We decompose this into the evolutions of x and y, which gives rise to a system:

$$\frac{\partial x}{\partial t} = \frac{-\det(s)y'}{\sqrt{(x')^2-(y')^2}} + Vx', \tag{8.1}$$

$$\frac{\partial y}{\partial t} = \frac{-\det(s)x'}{\sqrt{(x')^2-(y')^2}} + Vy'. \tag{8.2}$$

We choose V in such a way that the evolution of x is zero and reduce the system to a single PDE. Without loss of generality, we may assume that γ can be written as the graph of a function, otherwise the induced pseudo-metric would not be positive definite. This means that $V = \frac{\det(s)y'}{x'\sqrt{(x')^2-(y')^2}}$, $x = s$, $x' = 1$, and $\det(s) = \frac{-y''}{y^2(1-(y')^2)^{\frac{5}{2}}}$. Rewriting:

$$\frac{\partial y}{\partial t} = \frac{-\det(s)}{\sqrt{1-(y')^2}} + \frac{\det(s)(y')^2}{\sqrt{1-(y')^2}} = -\det(s)\frac{1-(y')^2}{\sqrt{1-(y')^2}} = \frac{y''}{y^2\left(1-(y')^2\right)^2}.$$

□

Lemma 8.3. *Under a change of coordinates, Equations 8.1 and 8.2 can be rewritten as*

$$\frac{\partial u}{\partial t} = \left(\frac{u'}{u}\tanh(s)+1\right)^2 \frac{uu''-2(u')^2+u^2}{u(u^2-(u')^2)^2},$$

$$\frac{\partial v}{\partial t} = 0.$$

Proof. We change the coordinate system into the following coordinates:

$$u = \sqrt{y^2-x^2},$$

$$v = \tanh^{-1}(\frac{x}{y}),$$

$$y = u\cosh(v),$$

$$x = u\sinh(v).$$

These coordinates are more natural, mainly because we expect the cross curvature flow to evolve a negatively curved manifold into hyperbolic space (with a proper

renormalization), and in these coordinates, hyperbolic space is represented by the equation $u = c$. where c is some positive constant. We now calculate relevant

quantities in these new coordinates.

$$
\begin{aligned}
x' &= u'\sinh(s) + u\cosh(s),\\
y' &= u'\cosh(s) + u\sinh(s),\\
x'' &= u''\sinh(s) + 2u'\cosh(s) + u\sinh(s),\\
y'' &= u''\cosh(s) + 2u'\sinh(s) + u\cosh(s),\\
yy' &= uu'\cosh^2(s) + u^2\sinh(s)\cosh(s),\\
xx' &= uu'\sinh^2(s) + u^2\sinh(s)\cosh(s),\\
yx' &= uu'\sinh(s)\cosh(s) + u^2\cosh^2(s),\\
xy' &= uu'\sinh(s)\cosh(s) + u^2\sinh^2(s),\\
xx' - yy' &= uu',\\
xy' - yx' &= u^2,\\
(x')^2 - (y')^2 &= u^2 - (u')^2,\\
\frac{x'}{y} &= \frac{u'}{u}\tanh(s) + 1,\\
x''y' &= u'u''\cosh(s)\sinh(s) + 2(u')^2\cosh^2(s) + uu'\sinh(s)\cosh(s)\\
&\quad + uu''\sinh^2(s) + 2uu'\sinh(s)\cosh(s) + u^2\sinh^2(s),\\
x'y'' &= u'u''\cosh(s)\sinh(s) + 2(u')^2\sinh^2(s) + uu'\sinh(s)\cosh(s)\\
&\quad + uu''\cosh^2(s) + 2uu'\sinh(s)\cosh(s) + u^2\cosh^2(s),\\
x''y' - x'y'' &= 2(u')^2 - uu'' - u^2,\\
K(s,\theta) = K(s,\phi) &= \frac{x'}{y}\frac{x''y' - x'y''}{((x')^2 - (y')^2)^2} = \left(\frac{u'}{u}\tanh(s) + 1\right)\frac{2(u')^2 - uu'' - u^2}{(u^2 - (u')^2)^2},\\
K(\theta,\phi) &= -\left(\frac{x'}{y\sqrt{(x')^2 - (y')^2}}\right)^2 = -\left(\frac{u'}{u}\tanh(s) + 1\right)^2\frac{1}{u^2 - (u')^2},\\
\det(s) &= \left(\frac{x'}{y}\right)^2\frac{x''y' - x'y''}{((x')2 - (y')^2)^{\frac{5}{2}}}\\
&= \left(\frac{u'}{u}\tanh(s) + 1\right)^2\frac{2(u')^2 - uu'' - u^2}{(u^2 - (u')^2)^{\frac{5}{2}}}.
\end{aligned}
$$

We now translate the evolution described by Equations 8.1 and 8.2 into the new

coordinate system

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{y\frac{\partial y}{\partial t} - x\frac{\partial x}{\partial t}}{u} + Wu' = \frac{-\det(s)}{u\sqrt{(x')^2-(y')^2}}(yx' - xy') + Wu',\\ \frac{\partial v}{\partial t} &= \frac{y\frac{\partial x}{\partial t} - x\frac{\partial y}{\partial t}}{u^2} + Wv' = \frac{-\det(s)}{u^2\sqrt{(x')^2-(y')^2}}(yy' - xx') + Wv'.\end{aligned}$$

Choose

$$W = \frac{\det(s)}{u^2v'\sqrt{(a')^2-(b')^2}}(bb' - aa')$$

so that $\frac{\partial v}{\partial t} = 0$. We also may choose that our curve γ is parametrized by a parameter s in such a way that $v = s$. So

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{-\det(s)}{u\sqrt{(x')^2-(y')^2}}(yx' - xy') + \frac{\det(s)u'}{u^2\sqrt{(x')^2-(y')^2}}(yy' - xx'),\\ &= \frac{\det(s)}{u^2\sqrt{(x')^2-(y')^2}}(u'(yy' - xx') - u(yx' - xy')),\\ &= \frac{\det(s)}{u^2\sqrt{u^2-(u')^2}}(u(u')^2 - u^3) = \frac{-\det(s)}{u}\sqrt{u^2-(u')^2},\end{aligned}$$

which further reduces to

$$\frac{\partial u}{\partial t} = \left(\frac{u'}{u}\tanh(s) + 1\right)^2 \frac{uu'' - 2(u')^2 + u^2}{u(u^2-(u')^2)^2}. \tag{8.3}$$

So we can reduce the cross curvature flow to Equation 8.3. □

We expect that under the cross curvature flow a negatively curved metric will approach a flat manifold [13], but we'd like to show that under an appropriate renormalization, it will approach a negatively curved metric of constant curvature. To do this we use the K-renormalized cross curvature flow introduced in [30], abbreviated KXCF. This flow is defined as

$$\frac{\partial}{\partial t}g_{ij} = 2X_{ij} - 2K^2 g_{ij}. \tag{8.4}$$

for some negative constant K. A hyperbolic metric of constant curvature K is a fixed point of this flow. We expect all negatively curved manifolds to converge to such a

metric. It was proven in [30] that such a fixed point is asymptotically stable. We prove the following:

Proposition 8.4. *If $\gamma = (x, y)$ is evolving according to*

$$\frac{\partial u}{\partial t} = -K^2 u + \left(\frac{u'}{u}\tanh(s) + 1\right)^2 \frac{uu'' - 2(u')^2 + u^2}{u(u^2 - (u')^2)^2}, \tag{8.5}$$

where K is some negative constant, $u = \sqrt{y^2 - x^2}$, $v = s = \tanh^{-1}(\frac{x}{y})$, and prime is differentiation with respect to s, then the manifold generated by γ is evolving according to the KXCF.

Proof. It was proven in [30] that if g is evolving by the KXCF, $\psi(t) = Ae^{2K^2 t}$, $\tilde{g} = \psi g$, and $\tilde{t} = \int_0^t \psi^2(\tau)d\tau$, then

$$\frac{\partial \tilde{g}}{\partial \tilde{t}} = 2\tilde{X}_{ij}.$$

We show that with the same changes of scale, Equation 8.5 is transformed into Equation 8.3. First, note that if $\tilde{\gamma} = \sqrt{\psi}\gamma$ then the induced metrics on the generated manifolds have the relationship $\tilde{g} = \psi g$. Therefore, we can assume that $\tilde{x} = \sqrt{\psi}x$, $\tilde{y} = \sqrt{\psi}y$, $\tilde{u} = \sqrt{\psi}u$, and $\tilde{v} = v = s$. Then (assuming u evolves by Equation 8.5):

$$\begin{aligned}
\frac{\partial \tilde{u}}{\partial \tilde{t}} &= \frac{\partial(\sqrt{\psi}u)}{\partial t}\frac{\partial t}{\partial \tilde{t}} = \frac{1}{\psi^2}\left(\frac{\partial \sqrt{\psi}}{\partial t}u + \sqrt{\psi}\frac{\partial u}{\partial t}\right), \\
&= \psi^{-3/2}K^2 u + \psi^{-3/2}\left(-K^2 u + \left(\frac{u'}{u}\tanh(s) + 1\right)^2 \frac{uu'' - 2(u')^2 + u^2}{u(u^2 - (u')^2)^2}\right), \\
&= \psi^{-3/2}\left(\frac{u'}{u}\tanh(s) + 1\right)^2 \frac{uu'' - 2(u')^2 + u^2}{u(u^2 - (u')^2)^2}, \\
&= \left(\frac{\tilde{u}'}{\tilde{u}}\tanh(s) + 1\right)^2 \frac{\tilde{u}\tilde{u}'' - 2(\tilde{u}')^2 + \tilde{u}^2}{\tilde{u}(\tilde{u}^2 - (\tilde{u}')^2)^2}.
\end{aligned}$$

□

With a metric of constant negative sectional curvature κ, $u = |\kappa|^{-1/2}$, so a metric of constant curvature K is a fixed point of Equation 8.5 because then

$$\frac{\partial u}{\partial t} = -|K|^{3/2} + \frac{|K|^{-1}}{|K|^{-1/2}|K|^{-2}} = 0.$$

It is evident from Equation 8.4 that another metric of constant negative curvature not equal to K will approach one with constant curvature K. We illustrate this in our extrinsic context by integrating and finding the exact solution.

Proposition 8.5. *If γ generates a manifold of constant negative curvature, then it will approach a curve that generates a manifold of constant curvature K as $t \to \infty$.*

Proof. If the metric has constant curvature, then u is constant, and Equation 8.5 becomes the following ODE:

$$\frac{du}{dt} = -K^2 u + \frac{1}{u^3}.$$

Integrating:

$$\begin{aligned}
\int \frac{u^3}{-K^2u^4+1} du &= t + C, \\
-\frac{\ln|-K^2u^4+1|}{4K^2} &= t + C, \\
K^2u^4 - 1 &= Ce^{-4K^2t}, \\
u(t) &= \left(\frac{1}{K^2} + Ce^{-4K^2t}\right)^{1/4}.
\end{aligned}$$

Solving for initial conditions we get

$$u(t) = \left(\frac{1}{K^2} - \frac{1}{K^2}e^{-4K^2t} + u_0^4 e^{-4K^2t}\right)^{1/4}. \tag{8.6}$$

We see that as $t \to \infty$, $u \to |K|^{-1/2}$, which generates a manifold of constant curvature K. □

Short-time existence of the cross curvature flow on non-compact manifolds is not currently known. We conjecture the following:

Conjecture 8.6. *Equation 8.5 has short time existence with condition $\frac{\partial u}{\partial s} \to 0$ as $s \to \pm\infty$.*

We further conjecture that any initial curve will approach the constant $u = |K|^{-1/2}$ as $t \to \infty$. A possible approach is to use the constant curvature solutions as barrier functions. Namely, we have the following:

Conjecture 8.7. *If* $c \leq u(s,0) \leq C$, *then*

$$\left(\frac{1}{K^2} - \frac{1}{K^2}e^{-4K^2t} + c^4 e^{-4K^2t}\right)^{1/4} \leq u(s,t) \leq \left(\frac{1}{K^2} - \frac{1}{K^2}e^{-4K^2t} + C^4 e^{-4K^2t}\right)^{1/4}$$

for all $t \geq 0$.

This would show that any manifold generated by a curve γ meeting the conditions at the beginning of this chapter converges to one of constant negative curvature under a rescaling of the cross curvature flow.

Chapter 9

FUTURE DIRECTIONS

We have seen that the method of embedding seems to be a fruitful method of analyzing rotationally symmetric solitons. This is mostly because it often reduces the complexity from a system of partial differential equations to a single second order ordinary differential equation. A logical next step would be to study singularity formation in rotationally symmetric manifolds. The expected behavior of a singularity is that, under a proper rescaling in space, a soliton would form in the limit. It s conceivable that the simplified equations might be easier to analyze.

Another possible research direction would be to try and generalize these calculations to more general types of embeddings. For example, there should be a large class of doubly warped product metrics which are embeddable with codimension 2. Is this enough symmetry that the evolution equations are still simpler than in their intrinsic form? Even if the metric is rotationally symmetric, it may not be embeddable with codimension 1. It can, however, be embedded with codimension 2. A future direction might be to derive these equations. We could also ask about the general case. We saw in Chapter 1 that every intrinsic flow with short-time existence can be induced from an evolution of an embedded submanifold of some (possibly enormous) Euclidean space. There are questions that could be asked about this case as well. For example, can we find an evolution of the submanifold which lasts as long as the flow?

Appendix A

MATLAB CODE

```
function dummy = soliton(n, ct, flow)

format long e;
deltax = 0.001;
xmax = 5;
x = 0;
y = 0;
yprime = 1;
ypprime = 0;


while x(length(x)) <= xmax & y < pi/2




    xi = x(length(x)) + deltax;
    x = [x, xi];
    yprimei = yprime(length(yprime))
              +deltax*ypprime(length(ypprime));

    yprime =[yprime, yprimei];
    yi = y(length(y)) + deltax*yprimei;


    y = [y, yi];
    switch flow
        case 'Ricci'
         ypprimei = (-ct*xi*yprimei)/(cos(yi)^2)
          - ((n-2)*yprimei)/(xi*cos(yi)^2)
          + ((n-2)*tan(yi))/(xi^2) - (yprimei^2/tan(yi))
          + yprimei/xi;


        case 'Mean'
```

```
        ypprimei = (-ct*xi*yprimei)/(cos(yi)^2)
         - ((n-1)*yprimei)/(xi*cos(yi)^2)
         + ((n-1)*tan(yi))/(xi^2);

        case 'Gauss'
         ypprimei = (-ct*(xi)^n*yprimei)
          /((sin(yi)^(n-1))*(cos(yi)^2))
         + ((n-1)*yprimei)/(xi)
         - ((n-1)*yprimei^2)/(tan(yi));

        case 'Yamabe'
         ypprimei = (-ct*xi*yprimei)/((n-1)*cos(yi)^2)
          - ((n-2)*yprimei)/(2*(cos(yi)^2)*xi)
          + ((n-2)*tan(yi))/(xi^2) - (yprimei^2)/tan(yi)
          - (yprimei*(n-4))/(2*xi);


        case 'Einstein'
         ypprimei = (-ct*xi*yprimei)/((n-2)*cos(yi)^2)
          - (yprimei*(n-3))/(2*xi)+ ((n-2)*tan(yi))/(xi^2)
          - (yprimei^2)/tan(yi)
           -((n-3)*yprimei)/(2*(cos(yi)^2)*xi);

        otherwise
            disp('Unrecognized Flow, options are Ricci,
             Mean, Gauss, Yamabe, or Einstein')
            break
    end
    ypprime = [ypprime, ypprimei];



end

bprime = 0;
b = 0;
for i = 1:length(x)
    bprime(i) = tan(y(i));
    b(i) = trapz(bprime)*deltax;
end
plot (0,0)
```

```
hold on
%plot(x, bprime)
%plot(-x, -bprime)
plot(b, x)
plot(b,-x)


hold off
axis ([0 2*xmax -xmax xmax])
%axis([-xmax xmax -xmax xmax])
axis square;

end
```

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