# Phenomenological Analysis of Heterotic Strings: Non-abelian Constructions and Landscape Studies

by

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As members of the Dissertation Committee, we certify that we have read the dissertation prepared by Vaibhav Hemant Wasnik entitled “Phenomenological Analysis of Heterotic Strings: Non-abelian Constructions and Landscape Studies” and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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Vaibhav Hemant Wasnik

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# Table of Contents

Table of Contents—*Continued*

# List of Figures

## Abstract

String theory offers the unique promise of unifying all the known forces in nature. However, the internal consistency of the theory requires that spacetime have more than four dimensions. As a result, the extra dimensions must be compactified in some manner and how this compactification takes place is critical for determining the low-energy physical predictions of the theory. In this thesis we examine two distinct consequences of this fact.

First, almost all of the prior research in string model-building has examined the consequences of compactifying on so-called “abelian” orbifolds. However, the most general class of compactifications, namely those on non-abelian orbifolds, remains almost completely unexplored. This thesis focuses on the low-energy phenomenological consequences of compactifying strings on non-abelian orbifolds. One of the main interests in pursuing these theories is that they can, in principle, naturally give rise to low-energy models which simultaneously have N=1 supersymmetry along with scalar particles transforming in the adjoint of the gauge group. These features, which are exceedingly difficult to achieve through abelian orbifolds, are exciting because they are the key ingredients in understanding how grand unification can emerge from string theory.

Second, the need to compactify gives rise to a huge “landscape” of possible resulting low-energy phenomenologies. One of the goals of the landscape program in string theory is then to extract information about the space of string vacua in the form of statistical correlations between phenomenological features that are otherwise uncorrelated in field theory. Such correlations would thus represent features of string theory that hold independently of a vacuum-selection principle. In this thesis, we study statistical correlations between two features which are likely to be central to any potential description of nature at high-energy scales: gauge symmetries and spacetime supersymmetry. We analyze correlations between these two kinds of symmetry within the context of perturbative heterotic string vacua, and find a number of striking features. We find, for example, that the degree of spacetime supersymmetry is strongly correlated with the probabilities of realizing certain gauge groups, with unbroken supersymmetry at the string scale tending to favor gauge-group factors with larger rank. We also find that nearly half of the heterotic landscape is nonsupersymmetric and yet tachyon-free at tree level; indeed, less than a quarter of the tree-level heterotic landscape exhibits any supersymmetry at all at the string scale.

# 1. Introduction

The Standard Model has had great success in explaining experimental data. The lagrangian of the model is invariant under a $SU(3) \otimes SU(2) \otimes U(1)$ local gauge symmetry. The fields in the models fall into representations of the gauge group. In addition to this, the parameters of the model have to be put in by hand to match experimental data.

The observed three generations of quarks and leptons have been successfully incorporated into the model. CP invariance in the bulk of processes as well as the breaking of CP symmetry in certain processes can be explained through a choice of certain parameters in the theory. Nuclear $\beta$ decay, neutral current scattering processes, color confinement are some of the phenomena which can be explained by the structure of the theory and/or choice of certain parameters in the theory.

Research attempts have been made to find correlations between the seemingly uncorrelated parameters in the theory. Grand Unified Theories were one of the outcomes of such research attempts. The Standard Model is made up of three different simple groups and the coupling constants of each of these simple portions are different. It has been suggested that the three groups $SU(3)$, $SU(2)$, $U(1)$ are part of a larger group like $SO(10)$ or $SU(5)$, which contain $SU(3) \otimes SU(2) \otimes U(1)$ as a subgroup. These models are Grand Unified Theory (GUT) models. The scalar GUT Higgs boson breaks the GUT symmetry to the Standard Model at the GUT scale. Hence, the low energy theory is the theory of the Standard Model. The couplings of the three simple groups of the Standard Model run and equal each other at the GUT scale. (For an easy to understand review of the basics of GUTs, please refer to Ref. [1]).

The Standard Model has a problem which needs an explanation. This is the problem of the quadratic divergence of the Standard Model Higgs boson mass. This problem is also called as the "Hierarchy Problem". To understand this issue consider the diagram in Fig.1(a). The diagram in Fig.1(a) comes from the Yukawa coupling term $\lambda\bar{\psi}\psi\phi$, which is the vertex at both ends of the diagram. $\phi$ represents the Standard Model Higgs and $\psi$ represents the fermions in the theory. This diagram represents the one loop correction to the Higgs mass. The diagram scales as $\Lambda^2$, where $\Lambda$ is the scale of a higher energy theory (example: GUTs) at which new physics starts appearing. Since the Standard Model is supposed to describe physics below the weak scale, the amplitudes cannot scale as positive powers of $\Lambda$. This contribution should be canceled.

Supersymmetry is a leading candidate for explaining the cancellation of the above amplitude. Supersymmetry is an extended symmetry of the Poincaré algebra. Poincaré algebra comprises of the following commutation relations

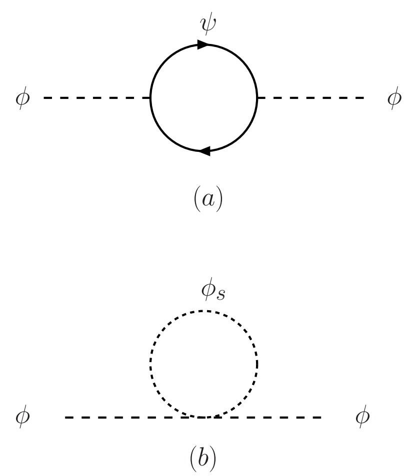


Figure 1.1. Higgs Divergence and its Cancellation

$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho}L_{\mu\sigma} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho} + \eta_{\mu\sigma}L_{\nu\rho}) \quad (1.1)$$

$$[L_{\mu\nu}, P_\rho] = \eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu \quad (1.2)$$

$$[P_\nu, P_\mu] = 0. \quad (1.3)$$

$L_{\mu\nu}$'s are the generators of rotation, $P_\nu$ is the generator of translation. $\eta$ is the Minkowski metric. All the generators $L_{\mu\nu}, P_\nu$ above are bosonic. This algebra can be extended by introducing new fermionic generators, because of the results of the Coleman-Mandula theorem (Ref. [2]). These generators, (henceforth referred to as $Q_\alpha, Q_{\dot\beta}$) have appropriate commutation relations with the rotation operators of the Poincaré algebra so as to qualify as spin 1/2 operators.

The generators associated with supersymmetry transformations are made up of a left handed Weyl spinor $Q_\alpha$ and a right handed Weyl spinor $Q_{\dot\beta}$, where $\alpha, \dot\beta \in \{1, 2\}$ are spinor indices.

The anti-commutation relation which plays the main part in cancellation of the Standard Model Higgs mass divergence is (Ref. [3])

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2{\sigma^\mu}_{\alpha\dot{\beta}} P_\mu. \tag{1.4}$$

The $\sigma^\mu$'s are the Pauli matrices. $\bar{Q}_{\dot{\beta}} = {Q^*}_\beta$. A lagrangian constructed out of such a theory has a symmetry in which bosons transform into fermions (and vice-versa) because of the spin 1/2 nature of the Q operator. The supersymmetric lagrangian has a $\lambda^2\phi^2{\phi_s}^2$ ($\phi$ is the Standard Model Higgs, whereas $\phi_s$ is the superpartner of the fermion field) term which produces the loop diagram in Fig.1(b). This contribution comes with an opposite sign to the Standard Model contribution from Fig.1(a) and hence cancels the quadratic divergence of the Standard Model Higgs mass.

The Standard Model gauge couplings should run and equal each other at the GUT scale in order for unification to make any sense whatsoever. It is however seen that this is not true unless a minimal $N = 1$ supersymmetric extension of the Standard Model is considered. The theory at the GUT scale would then be a $N = 1$ supersymmetric GUT theory.

The Standard Model can describe the physics below the weak scale with the help of dimension four or less operators. However gravity requires the presence of higher dimensional operators at lower energies. There has always been a strong desire from within the scientific community to express gravity in an unified framework with other known forces of nature.

A major research area over the past many decades has worked with this idea of unification of gravity with other known forces of nature. This area of research constitutes the domain of string theory.

The basics of string theory will be described in the sections below, please consult Ref. [4] for a comprehensive treatment.

## 1.1. String theory

String theories are theories in which the fundamental entity is a string instead of a particle. As the string propagates through spacetime, it describes a worldsheet instead of a world line. The string can be open or closed. Open and closed strings can be studied independently of each other in spite of hypothesized duality relations between the two (Ref. [5]) . We will be considering only closed strings in this dissertation. Also we would be considering the case of flat spacetime only. Gravitons will appear as excitations of string theory which later act as background fields. Please refer to Ref. [4] and references therein for further understanding. The worldsheet can be described by two co-ordinates, $\sigma$ corresponding to a position along the length of the string and $\tau$ which is analogous to the time co-ordinate. We will be considering the bosonic string in this section. The position of the string in spacetime is given by the co-ordinate $X^\mu(\sigma, \tau)$.

The string action can be written as

$$S = T\int \sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu d^2\sigma \tag{1.5}$$

where $T$ is the tension in the string, $h^{\alpha\beta}$ is the metric on the two dimensional string worldsheet, $h$ is the determinant of $h^{\alpha\beta}$ and $d^2\sigma = d\sigma d\tau$.

To understand the above equation, we can imagine the worldsheet of the string to be a two dimensional manifold, with metric $h^{\alpha\beta}$. Then, the above equation would correspond to the area of the worldsheet in spacetime. The classical path traveled by the string is the one that minimizes the above action.

However it is also possible to write the area of the worldsheet as

$$S = T\int \sqrt{(\frac{\partial X^\mu}{\partial\tau}.\frac{\partial X_\mu}{\partial\tau})(\frac{\partial X^\mu}{\partial\sigma}.\frac{\partial X_\mu}{\partial\sigma}) - (\frac{\partial X^\mu}{\partial\tau}\frac{\partial X_\mu}{\partial\sigma})^2}d^2\sigma. \tag{1.6}$$

Since eqn (1.5) and eqn (1.6) define the same quantity, the physics of the system should be independent of the extra field $h^{\alpha\beta}$. $h^{\alpha\beta}$ has three independent components. Two of these components can be eliminated through a reparametrization of $\sigma$ and $\tau$. With this reparametrization $h^{\alpha\beta}$ can be put in the form, $h^{\alpha\beta} = \Lambda(\sigma,\tau)\eta^{\alpha\beta}$. Now, it can be easily checked that equation (1.5) is invariant under the identification $h^{\alpha\beta} \rightarrow \Lambda(\sigma,\tau)h^{\alpha\beta}$. To see this note that $\sqrt{h}$ is the square root of the determinant of $h_{\alpha\beta}$ and hence transforms as $\sqrt{h} \rightarrow [\Lambda(\sigma,\tau)]^{-1}\sqrt{h}$. This symmetry is known as the conformal symmetry.

The classical equation of motion $\frac{\delta S}{\delta h^{\alpha\beta}} = 0$ gives us

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}h_{\alpha\beta}h^{\alpha'\beta'}\partial_{\alpha'}X^\mu\partial_{\beta'}X_\mu = 0. \tag{1.7}$$

Conformal invariance is the statement that $(T_\alpha)^\alpha = 0$.

Conformal invariance and reparametrization invariance together imply that we can make a choice $h^{\alpha\beta} = \eta^{\alpha\beta}$. With such a choice the equation of motion for $X^\mu(\sigma,\tau)$ becomes the wave equation

$$[\partial_\sigma\partial^\sigma - \partial_\tau\partial^\tau]X^\mu(\sigma,\tau) = 0. \tag{1.8}$$

For closed strings we have the boundary condition $X^\mu(\sigma+\pi,\tau) = X^\mu(\sigma,\tau)$ (going once around a closed string corresponds to $\sigma$ changing from $\sigma$ to $\sigma+\pi$). With this boundary condition the solution to the above equation becomes

$$X^\mu{}_L = x^\mu + \frac{1}{2}l^2p^\mu(\tau-\sigma) + \frac{i}{2}l\sum_{n\neq 0}\frac{1}{n}\alpha_n{}^\mu e^{-2in(\tau-\sigma)} \tag{1.9}$$

$$X^\mu{}_R = x^\mu + \frac{1}{2}l^2p^\mu(\tau+\sigma) + \frac{i}{2}l\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)} \tag{1.10}$$

where $l = 1/\pi T^{1/2}$, ${\alpha_n}^\mu$ is the amplitude contribution of the $e^{-2in(\tau-\sigma)}$ factor to ${X^\mu}_L$, with an analogous definition for $\tilde{\alpha}_n$. ${X^\mu}_R$ implies a function of $(\tau+\sigma)$ and ${X^\mu}_L$ implies a function of $(\tau-\sigma)$, such that $X^\mu(\sigma,\tau) = {X^\mu}_L + {X^\mu}_R$. We will be using labels (L,R) to represent a co-ordinate system made up of $(\tau-\sigma, \tau+\sigma)$ instead of $(\tau,\sigma)$. $x^\mu$ and $p^\mu$ are the center of mass position and momentum of the string, which appear as the position and momentum of the low energy particle. Since string excitations are identified as particles at lower energies, we should have the following commutation relationship

$$[x^\mu, p^\mu] = i\hbar. \tag{1.11}$$

Quantizing the world sheet theory gives us the following commutation relationships

$$[X^\mu, P^\mu] = i\hbar. \tag{1.12}$$

Here $P^\mu$ is the canonical momentum corresponding to $X^\mu$. Substituting for $X^\mu$ and $P^\mu$ (Ref. [4]) gives us the following commutation relations

$$[{\alpha_m}^\mu, {\alpha_n}^\nu] = im\delta_{m+n}\eta^{\mu\nu} \tag{1.13}$$

$$[\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = im\delta_{m+n}\eta^{\mu\nu}. \tag{1.14}$$

Here, $\delta_{m+n} = 0$ if $m+n \neq 0$, else $\delta_{m+n} = 1$.

$$[{\alpha_m}^\mu, \tilde{\alpha}_n^\nu] = 0. \tag{1.15}$$

Define

$$L_m = \frac{T}{2}\int_0^\pi e^{-2im\sigma} T_{LL} d^2\sigma = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\alpha_n \tag{1.16}$$

$$\bar{L}_m = \frac{T}{2}\int_0^\pi e^{2im\sigma} T_{RR} d^2\sigma = \frac{1}{2}\sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n}\tilde{\alpha}_n. \tag{1.17}$$

Here the stress energy tensor is considered in the L(R) co-ordinate system i.e. $T_{LL} = T_{\sigma-\tau,\sigma-\tau}$ and $T_{RR} = T_{\sigma+\tau,\sigma+\tau}$ as explained before. Any state of the theory $|\phi\rangle$ should obey $L_m|\phi\rangle = 0$, $\bar{L}_m|\phi\rangle = 0$ for $m \neq 0$. $\alpha_{m-n}$ and $\alpha_n$ only commute for $m \neq 0$. For $m = 0$ we have a normal ordering ambiguity in the definition of $L_0$ (Ref. [4]). For $m = 0$, $(L_0 - a)|\phi\rangle = 0$, $(\bar{L}_0 - \bar{a})|\phi\rangle = 0$ instead should be obeyed. Here $a, \bar{a}$ are normal ordering constants. These relations are the quantum equivalent of $T_{\alpha\beta} = 0$.

Unitarity of the theory, requires the absence of states with $\langle\phi|\phi\rangle < 0$, which are even massless at the string scale. This constraints the dimensions of the theory and the value of $a$. It sets the spacetime dimension of the theory to be $D = 26$ and $a = 1$.

## 1.2. Superstring theory

In the previous section the elements of bosonic string theory were presented. The theory did not have fermionic excitations. Also, bosonic string theory has tachyons in its spectrum. String theories with worldsheet supersymmetry produce fermions. These theories are called superstring theories. The subset of these theories which are spacetime supersymmetric do not have tachyons.

The bosonic string theory was presented as a two dimensional worldsheet theory, with the space time fields $X^\mu(\sigma,\tau)$ living on the worldsheet. The $X^\mu$'s are scalars of the two dimensional worldsheet theory. The Lorentz symmetry is a global symmetry of the two dimensional worldsheet theory. $X^\mu(\sigma,\tau)$ transforms as a vector under this Lorentz symmetry. Let us introduce fields $\psi^\mu(\sigma,\tau)$ on the worldhseet. Let them transform as spinors of the two dimensional theory, but transform as vectors according to the Lorentz symmetry. The action in the $h^{\alpha\beta} = \eta^{\alpha\beta}$ gauge could be written as

$$S = \int d^2\sigma(\partial^\alpha X^\mu \partial_\alpha X_\mu - \bar{\psi}^\mu \gamma^\alpha \partial_\alpha \psi_\mu) \tag{1.18}$$

which in turn could be written as

$$S = \int d^2\sigma(\partial^\alpha X^\mu \partial_\alpha X_\mu - i\psi^{\mu\dagger}\gamma^0 \gamma^\alpha \partial_\alpha \psi_\mu). \tag{1.19}$$

The action is worldsheet supersymmetric. The two gamma matrices are

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{1.20}$$

and

$$\gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{1.21}$$

For the $\psi$ fields to be worldsheet spinors, we need the following commutations relations

$$\{\psi^\mu(\sigma), \psi^\nu(\sigma')\} = \pi\eta^{\mu\nu}\delta^2(\sigma - \sigma'). \tag{1.22}$$

The matrix corresponding to the $\gamma^5$ in four dimensions is called $\gamma = \gamma^0\gamma^1$ in the two dimensional theory

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.23}$$

Since a fermion in two dimensions has two components, it can be written as $(\psi_-, \psi_+)$. We can choose this to be a real spinor as $i\gamma^\alpha\partial_\alpha$ is real.

Hence if $\partial_\sigma - \partial_\tau = \partial_L$ and $\partial_\sigma + \partial_\tau = \partial_R$ eqn(1.19) becomes

$$S = \int d^2\sigma(\partial^\alpha X^\mu \partial_\alpha X_\mu - \psi^\mu{}_R \partial_R \psi_{\mu R} - \psi^\mu{}_L \partial_L \psi_{\mu L}). \tag{1.24}$$

The above lagrangian is worldsheet supersymmetric with $X^\mu$ and $\psi^\mu$ transforming into each other. Here $\psi_L$ implies a function of $\sigma - \tau$ and $\psi_R$ implies a function of $\sigma + \tau$.

In order to preserve conformal symmetry at the quantum level we need the dimension of spacetime to be ten.

Let us also mention that there is a residual symmetry which appears even after reparametrization invariance and conformal invariance together have been used to set $h_{\alpha\beta}$ to $\eta_{\alpha\beta}$. Further transformations on the worldsheet co-ordinates are possible which do not change this choice of $h_{\alpha\beta}$. This residual gauge symmetry can be used to fix two $X^\mu$ and two $\psi^\mu$ fields. This gets rid of ghosts in the theory. The theory however becomes a theory with $SO(8)$ Lorentz symmetry.

The requirement $T_{\alpha\beta} = 0$, becomes $L_m|\phi\rangle = 0$ for $m > 0$, $(L_0 - a)|\phi\rangle = 0$, $\bar{L}_m|\phi\rangle = 0$ for $m > 0$ and $(\bar{L}_0 - \bar{a})|\phi\rangle = 0$ for the quantum theory (Ref. [4]). The boundary conditions for the closed superstring fields have the form $X^\mu(\sigma+\pi,\tau) = X^\mu(\sigma,\tau)$ and $\psi^\mu{}_R(\sigma+\pi,\tau) = \mp\psi^\mu{}_R(\sigma,\tau)$ with $\psi^\mu{}_L(\sigma+\pi,\tau) = \mp\psi^\mu{}_L(\sigma,\tau)$. To understand this note that the lagrangian should be periodic when going from $\sigma$ to $\sigma+\pi$. Periodicity would allow $\psi^\mu{}_{L,R}(\sigma+\pi) = e^{iw}\psi^\mu{}_{L,R}(\sigma)$ ($w$ is a real number). But reality of the fermions only allows the subset $\psi^\mu{}_{L,R}(\sigma+\pi,\tau) = \mp\psi^\mu{}_{L,R}(\sigma,\tau)$

$\psi^\mu{}_{L,R}(\sigma+\pi,\tau) = \psi^\mu{}_{L,R}(\sigma,\tau)$ boundary conditions are termed as Ramond (R) boundary conditions.

$\psi^\mu{}_L(\sigma+\pi,\tau) = \psi^\mu{}_L(\sigma,\tau)$ gives

$$\psi^\mu{}_L = \frac{1}{\sqrt{2}} \sum_{n\in Z} d_n{}^\mu e^{-2in(\tau-\sigma)} \tag{1.25}$$

and $\psi^\mu{}_R(\sigma+\pi,\tau) = \psi^\mu{}_R(\sigma,\tau)$ gives

$$\psi^\mu{}_R = \frac{1}{\sqrt{2}} \sum_{n\in Z} \tilde{d}^\mu_n e^{-2in(\tau+\sigma)}. \tag{1.26}$$

$\tilde{d}^\mu_n$, $d_n{}^\mu$ are the amplitude contributions (which become string excitations upon quantization) from $e^{-2in(\tau-\sigma)}$, $e^{-2in(\tau+\sigma)}$ to $\psi^\mu{}_L$, $\psi^\mu{}_R$ respectively.

$\psi^\mu{}_{L,R}(\sigma+\pi,\tau) = -\psi^\mu{}_{L,R}(\sigma,\tau)$ boundary conditions are termed as Neveu-Schwarz (NS) boundary conditions.

$\psi^\mu{}_L(\sigma+\pi,\tau) = -\psi^\mu{}_L(\sigma,\tau)$ gives

$$\psi^\mu{}_L = \frac{1}{\sqrt{2}} \sum_{r\in Z+\frac{1}{2}} b_r{}^\mu e^{-2ir(\tau-\sigma)} \tag{1.27}$$

${\psi^\mu}_R(\sigma+\pi,\tau) = -{\psi^\mu}_R(\sigma,\tau)$ gives

$$ {\psi^\mu}_R = \frac{1}{\sqrt{2}} \sum_{r\in Z+\frac{1}{2}} \tilde{b}^\mu_r e^{-2ir(\tau+\sigma)}. \tag{1.28} $$

$\tilde{b}^\mu_n$, ${b_n}^\mu$ are the amplitude contributions (which become string excitations upon quantization) from $e^{-2in(\tau-\sigma)}$, $e^{-2in(\tau+\sigma)}$ to ${\psi^\mu}_L$, ${\psi^\mu}_R$ respectively.

Hence, we can have the following combinations R-R, R-NS, NS-R, NS-NS for superstrings, corresponding to the $(\psi_R)$ - $(\psi_L)$ boundary conditions.

It should be noted that Lorentz invariance forces all ten left moving fermions (right moving fermions) to have the same boundary condition. That is either all of the left (right) movers are of NS type or R type.

The Ramond excitations are of special interest. Because of equation (1.22), their commutation relations after quantization become

$$ \{{d^\nu}_m, {d^\mu}_n\} = \delta_{m+n}\eta^{\mu\nu}. \tag{1.29} $$

For $m = n = 0$, the commutation relations become

$$ \{{d^\nu}_0, {d^\mu}_0\} = \eta^{\mu\nu}. \tag{1.30} $$

These are just the gamma matrix commutation relationships for spacetime Lorentz algebra. So we see that two dimensional fermions which transform as space time vectors can produce spacetime fermionic excitations upon quantization.

Even though it has not been proved here, it can be shown that this theory with worldsheet supersymmetry can produce a theory with excitations, which form a representation of spacetime supersymmetry under certain situations. We will show this for the case of Type II strings in a later section.

## 1.3. Modular invariance and GSO constraints

Until now we have described string theories as being two dimensional conformally invariant field theories. The excitations of these strings are seen as particles of the low energy theory. The string states satisfy $(L_0 - a)|\phi\rangle = 0$, $L_m|0\rangle = 0$ (for $m > 0$) as well as $(\bar{L}_0 - a)|\phi\rangle = 0$, $\bar{L}_m|0\rangle = 0$ (for $m > 0$) as talked about before. However there are additional constraints that the string states have to satisfy as will be talked about below.

Let us consider a one loop closed string spacetime propogation amplitude. This is depicted pictorially by Fig.1.2. The amplitude can be expressed as a path integral. For a closed bosonic string the path integral takes the form (for more information please refer to Ref. [4])

$$ Z(t) = \int DX^\mu(\sigma,\tau) Dh_{\alpha\beta}(\sigma,\tau) e^{-(\frac{1}{2}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu)}. \tag{1.31} $$

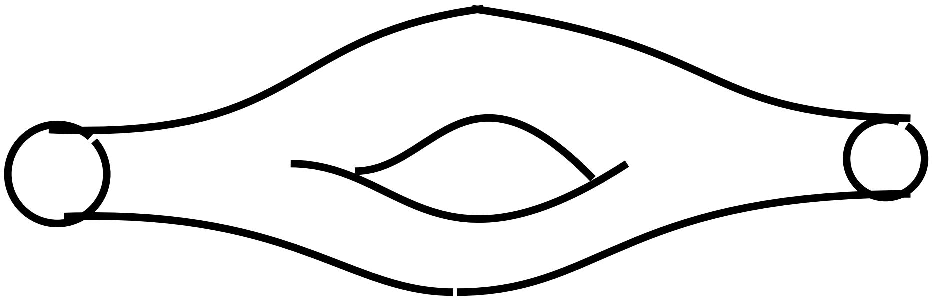

FIGURE 1.2. Toroidal string diagram involving a closed string going to a closed string

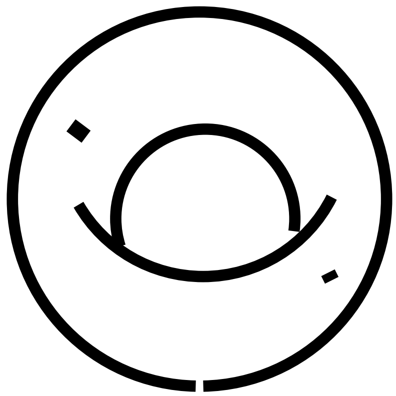

FIGURE 1.3. Toroidal string diagram involving a string going to a string where a conformal transformation has transformed the incoming and outgoing strings into points on the torus

By a conformal transformation $h_{\alpha,\beta} = e^{\phi} h_{\alpha,\beta}$, Fig.1.2 can be converted into the the compact diagram of Fig.1.3.

The points in the figure correspond to vertex operators (Ref. [4]) of the incoming and outgoing strings. The one loop amplitude in the case of a string can hence be reduced to a toroidal amplitude, corresponding to the possibility of a string returning to its original configuration after $\tau$-time $t$. As far as the story on the string worldsheet goes, the torus can be imagined as a rectangle with length 1 unit along the $\sigma$ direction and length $t$ along the $\tau$ direction. The torus can be defined by identifying the opposite sides of the rectangle, as in Fig.1.4.

Integrating over $t$ would yield a summation over all possible worldsheet toroidal configurations in the path integral.

Additional symmetries arise because of defining a conformal theory on a torus. These symmetries constitute a symmetry called as modular invariance. This symmetry is made up of the following transformations $t \to t + 1$ and $t \to -\frac{1}{t}$. The string

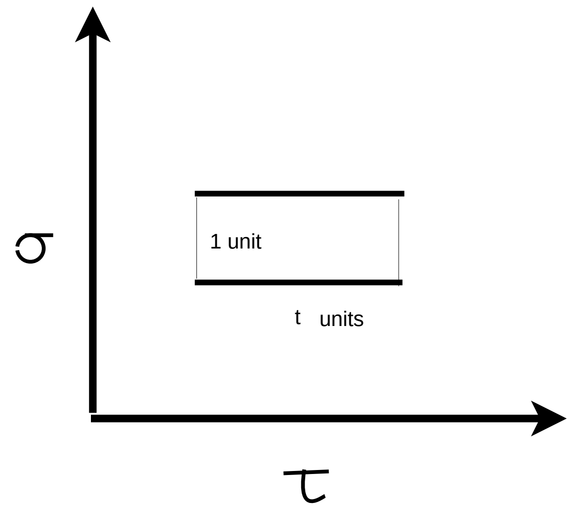


FIGURE 1.4. Representation of the torus in worldsheet space

amplitude should be invariant under these transformations. This puts constraints called as GSO (Green-Schwarz-Olive) constraints on the states allowed to propagate in a string theory.

## 1.4. Construction of states

As was explained before the requirement of conformal invariance imposes the constraints $(L_0 - a)|\phi\rangle = 0$ and $L_m|\phi\rangle = 0$ (for $m > 0$) as well as $(\bar{L}_0 - \bar{a})|\phi\rangle = 0$ and $\bar{L}_m|\phi\rangle = 0$ (for $m > 0$) on the states of the theory. $a, \bar{a}$ are normal ordering constants, which can be understood as being related to the anomaly of the two dimensional conformal symmetry. In gauge theories the contribution of the fermions to the gauge anomaly is additive. We have a similar situation in the case of the two dimensional conformal symmetry. A fermion with an NS boundary condition contributes $1/24$ to $a$, whereas a R boundary condition fermion contributes $-1/48$ to the conformal anomaly $a$. A two dimensional boson contributes $-1/24$ to $a$.

It was mentioned before that all the left moving (right moving) fermions have the same boundary condition, because of lorentz symmetry. Hence in all four different combinations of string boundary conditions are possible, depending on the various combinations of the left moving and right moving boundary conditions.

### 1.4.1. The type 0 string

For these strings both left and right moving worldsheet fermions have the same boundary conditions. Either both of them are R type or both of them are NS type. When both are NS type then $a, \bar{a}$ equal -1/2. In this case modular invariance imposes the boundary condition $N_R - N_L = even$, where $N_R$ ($N_L$) are the number of excitations of the right (left) woldsheet fermions that make up a state.

The vacuum $|0\rangle_L \bigotimes |0\rangle_R$ easily becomes one possible state of the system. The mass formula $\alpha' M^2 = 2(L_0 + \bar{L}_0 + a + \bar{a})$ ($\alpha' = \frac{1}{M_{string}{}^2}$. $M_{string} = g_{string} M_{Planck}$. Here $M_{string}$ implies the string scale. $M_{Planck}$ is the Planck Scale. $g_{string}$ is the string coupling constant. Please refer to Ref. [4] for an explanation of these terms), tells us that for this state $\alpha' M^2 = -2$. Hence this state is a tachyon.

$b^{\mu}{}_{-1/2}|0\rangle_L \bigotimes b^{\nu}{}_{-1/2}|0\rangle_R$ written as $V_8 \bigotimes V_8$ (meaning that the state is a cross product of two $SO(8)$ vectors) is another possible state from this sector. This state is massless. It can be split into the graviton $g_{\mu\nu}$, an antisymmetric tensor $b_{\mu\nu}$ and a scalar $\phi$.

Let us now consider states from the fermionic sector where both left movers and right movers are of the R type. The $a$ and $\bar{a}$ are both equal to 0 in this case. The modular invariance constraints translate into $N_L - N_R$ being either even or odd. This is merely saying that $N_L$ and $N_R$ can be even or odd. As was explained before the $d^{\mu}{}_0$'s obey $\{d^{\mu}{}_0, d^{\nu}{}_0\} = \eta^{\mu\nu}$. Hence these excitations produce the spinors of the Lorentz group. Because $N_L$ and $N_R$ can be even or odd, spinors of both chirality are possible in this case. Hence the possible string excitations can be written as $\{S_8, C_8\}_L \bigotimes \{S_8, C_8\}_R$. The $\{S_8, C_8\}$ implies $SO(8)$ spinors of both chirality. $N_L - N_R = even$ produce a Type 0A string theory and $N_L - N_R = odd$ produce a Type 0B string theory.

### 1.4.2. The type II string

All four combinations of left moving and right moving boundary conditions give us the type II string.

If both left moving and right moving worldsheet fermions are of the NS type, then modular invariance adds in an extra constraint to $N_L - N_R = even$, $N_R = odd$. $a$ and $\bar{a}$ both equal -1/2 in this case. The first constraint allows both the tachyon and the massless gravity multiplet of type 0 string. However, the second constraint gets rid of the tachyon.

If both left and right movers are of the Ramond type, then $a$ as well as $\bar{a}$ equal 0. Modular invariance adds in the constraints $N_L - N_R$ being odd as well as even just as the type 0 case. However we have an extra constraint that $N_R$ should be odd. Hence we end up with the massless $\{S_8, C_8\}_L \times \{S_8\}_R$ in this case.

If left movers are NS type and right movers are R type then we have $a = -1/2$ and $\bar{a} = 0$. In this case modular invariance imposes the following constraints: $N_L - N_R$ is even and $N_R$ is odd. Hence the right movers are $S_8$ lorentz spinors. In this case we can have a massless state $b^{\mu}{}_{-1/2}|0\rangle_L \bigotimes \{S_8\}_R$. This state is really a $V_8 \bigotimes S_8$ state, where $V_8$ means a SO(8) vector. This state is called a gravitino and is the supersymmetric partner of the graviton which is part of the gravity multiplet.

If the right movers are of NS type and left movers are of R type then we have $\bar{a} = -1/2$ and $a = 0$. Modular invariance imposes the constraints: $N_L - N_R$ is even or odd and $N_R$ is odd. Another way of saying this is that $N_R$ is odd, while $N_L$ can be either even or odd. Hence, this produces the massless states $\{S_8, C_8\}_L \bigotimes \{V_8\}_R$.

Collecting all the above massless states $V_8 \bigotimes V_8$, $\{S_8, C_8\}_L \bigotimes \{S_8\}_R$ , $V_8 \bigotimes S_8$, $\{S_8, C_8\}_L \bigotimes \{V_8\}_R$, we can visualize a much better grouping $(V_8 \bigoplus \{C_8, S_8\})_L \bigotimes (V_8 \bigoplus S_8)_R$. We see that there are two spacetime supersymmetries above. One of them exchanges $V_8$ and $S_8$ among the right movers, the other exchanges $V_8$ and $\{C_8, S_8\}$ among the left movers.

Exchange of $V_8$ and $S_8$ for left movers is similar to an identical exchange for the right movers and this string is called the Type IIA string. Exchange of $V_8$ and $C_8$ on the other hand is opposite to the $V_8$ and $S_8$ exchange for right movers and such a string is called the Type IIB string.

The theory lacks tachyons. Even though we have shown the emergence of a massless supersymmetric spectrum , it can be shown that the spectrum is supersymmetric for all possible masses (Ref. [4]).

## 1.5. Heterotic strings

In the last section we considered superstrings, which produced fermions, however superstring theories do not produce gauge bosons. Heterotic strings are a construction which produce this element of the particle spectrum.

Let us review a few concepts from the last two sections. The bosonic strings, have twenty six left moving and twenty six right moving bosonic fields living on the string worldsheet. The action for the bosonic string in the $h^{\alpha\beta} = \eta^{\alpha\beta}$ gauge is

$$S = \int d^2\sigma (\partial^L X^{\mu}{}_L \partial_L X_{\mu_L} + \partial^R X^{\mu}{}_R \partial_R X_{\mu_R}). \tag{1.32}$$

Here $\partial^L$ means that we take a derivate with respect to $\sigma - \tau$ . $\partial^R$ means that we take a derivate with respect to $\sigma + \tau$. The $X^{\mu}$ are worldsheet bosons and obey a $SO(24)_L \times SO(24)_R$ symmetry, with respect to the worldsheet physics, which in reality is the Lorentz symmetry of space time physics ($SO(24)$ and not $SO(26)$ because of residual gauge symmetries as was talked about in the previous section). As far as the worldsheet physics goes $D$ is just the number of bosonic excitations, twenty six of these excitations are required in order to prevent the presence of ghosts in the spectrum of the theory. Left movers and right movers are separate fields. In the above lagrangian we have twenty six left moving and twenty six right moving fields.

We may as well have had just twenty six left moving or twenty six right moving worldsheet bosonic fields in our theory by themselves.

Let us consider a two dimensional theory with twenty six such left moving fields as below

$$S_1 = \int d^2\sigma (\partial^L X^i{}_L \partial_L X_{iL}). \tag{1.33}$$

Please note that we have removed the spacetime index $\mu$ and instead used the index $i$ in the above equation. There was apriori no reason for the worldsheet bosons $X$ to be spacetime vectors. We can also choose the fields $X$ to be spacetime scalars. The above action is self consistent in itself as far as the two dimensional theory goes.

We also studied the worldsheet supersymmetric string action in the $h^{\alpha\beta} = \eta^{\alpha\beta}$ gauge in the previous section

$$S_2 = \int d^2\sigma (\partial^\alpha X^\mu \partial_\alpha X_\mu - \psi^\mu{}_R \partial_R \psi_{\mu R} - \psi^\mu{}_L \partial_L \psi_{\mu L}). \tag{1.34}$$

Here the number of dimensions necessary for elimination of ghost states equals ten. As far as the worldsheet theory goes, the $X^\mu$ and $\psi^\mu$ are just worldsheet bosonic and fermionic fields. We can take either left moving or right moving $X^\mu$ and $\psi^\mu$ fields and still get a consistent worldsheet theory. Let us consider a right moving superstring theory with the action

$$S_3 = \int d^2\sigma (\partial^R X^\mu{}_R \partial_R X_{\mu R} - \psi^\mu{}_R \partial_R \psi_{\mu R}). \tag{1.35}$$

The number of $\mu$'s (spactime dimensions) in the above lagrangian equals ten.

We could combine the actions in eqn (1.33) and eqn (1.35) to describe a theory with ten $X^\mu{}_R$ fields, ten $\psi^\mu{}_R$ fields, sixteen $X^i{}_L$ fields and ten $X^\mu{}_L$ fields. Hence we would get the action shown below

$$S_4 = \int d^2\sigma (\partial^L X^i{}_L \partial_L X_{iL} + \partial^L X^\mu{}_L \partial_L X_{\mu L}) + \int d^2\sigma (\partial^R X^\mu{}_R \partial_R X_{\mu R} - \psi^\mu{}_R \partial_R \psi_{\mu R}). \tag{1.36}$$

Since none of the theories described by eqns (1.33) and (1.35) lagrangians contain ghosts states, the theory described by the above action should not contain ghosts either. The above string is called a heterotic string.

### 1.5.1. Free fermionic construction

A boson compactified to radius $\sqrt{2\alpha'}$ in a two dimensional field theory at the Planck scale can be described by a free fermion in a two dimensional field theory. A compactified boson implies the boson $\phi$ is identified with $\phi + 2\pi R$ in the configuration space of the scalar field $\phi$. This causes the spectrum of the bosonic excitations to be discrete, which allows for a one to one correspondence with a discrete fermionic spectrum (Ref. [4]). We can use the above correspondence to write sixteen of the (L) bosons as sixteen (L) complex fermions. These fermions are complex, unlike the real fermions $\psi^\mu$. Hence the heterotic string action becomes

$$S = \int d^2\sigma(-\psi^i{}_L\partial_L\psi_{iL}) + \int d^2\sigma\partial^L X^\mu{}_L\partial_L X_{\mu L} + \int d^2\sigma(\partial^R X^\mu{}_R\partial_R X_{\mu R} - \psi^\mu{}_R\partial_R\psi_{\mu R}). \tag{1.37}$$

The lagrangian does not produce ghost states on quantization as talked before. The number of the $\psi^i$ fields equal sixteen and the number of space time dimensions $\mu$ equal ten. We can compactify six of these dimensions to reach four dimensions. If we compactify to a radius $\sqrt{2\alpha'}$ again, the action becomes

$$\begin{aligned} S \;=\; & \int d^2\sigma(-\psi^i{}_L\partial_L\psi_{iL}) + \int d^2\sigma\partial^L X^\mu{}_L\partial_L X_{\mu L} \\ & + \int d^2\sigma(\partial^R X^\mu{}_R\partial_R X_{\mu R} - \psi^\mu{}_R\partial_R\psi_{\mu R}) \\ & + \int d^2\sigma(\partial^R X^j{}_R\partial_R X_{jR} - \psi^j{}_R\partial_R\psi_{jR}). \end{aligned} \tag{1.38}$$

Since the $\psi_i$'s don't carry lorentz indices and are complex their boundary conditions can be written as

$$\psi_i(\sigma+\pi,\tau) = -e^{-2\pi i V^\sigma{}_i}\psi_i(\sigma,\tau). \tag{1.39}$$

$V^\sigma{}_i$ is a phase factor. Collection of $V^\sigma{}_i$ for $i \in [0,32]$ gives us the $V^\sigma$ vector. The above boundary condition can be used to define the mode expansions for the fermions. For example, the mode expansion for the (R) fermions becomes

$$\psi_R{}^i(\sigma,\tau) = \sum_{n=1}^{\infty}(b_{n+V^\sigma{}_i-\frac{1}{2}}e^{-i(n+V^\sigma{}_i-1/2)(\sigma+\tau)} + d^\dagger{}_{n-V^\sigma{}_i-1/2}e^{i(n-V^\sigma{}_i-1/2)(\sigma+\tau)}). \tag{1.40}$$

The number operator is defined as

$$N_i = \sum_{n=1}^{\infty}(b^\dagger{}_{n+V^\sigma{}_i-\frac{1}{2}}b_{n+V^\sigma{}_i-\frac{1}{2}} - d^\dagger{}_{n-V^\sigma{}_i-1/2}d_{n-V^\sigma{}_i-1/2}), \tag{1.41}$$

the $b^i$'s produce fermions and $d^i$'s produce anti-fermions.

As mentioned before the requirement of modular invariance imposes additional constraints on the allowed spectrum. On a torus we have boundary conditions along the $\tau$ direction also. This boundary condition is

$$\psi_i(\sigma, \tau + \pi) = -e^{-2\pi i {V^\tau}_i} \psi_i(\sigma, \tau) \tag{1.42}$$

${V^\tau}_i$ is a phase factor. Collection of ${V^\tau}_i$ for $i \in [0, 32]$ gives us the $V^\tau$ vector. If $V^\sigma$ is a boundary condition vector then $V^\sigma + V^\tau$ is also a boundary condition vector along the $\sigma$ direction as can be easily seen from the construction of a torus.

We can have a collection of linearly independent $V$ vectors labeled as $V_i$, and linear combinations of these $V$ vectors can act as boundary conditions along the $\sigma$ and $\tau$ directions, giving possible string excitations. Modular invariance of the partition function , imposes the following constraints on the string excitations (Ref. [6])

$$V_i \cdot N_{\alpha V} = \sum_j k_{ij} \alpha_j + s_i - V_i \cdot \bar{\alpha V}. \tag{1.43}$$

The equality above is true to mod 1. Here $\alpha V = \alpha_k V_k$ is one possible sector producing string excitations similar to $V^\sigma$ above. The $N_{\alpha V}$ is the number vector for a particular string excitation. $s_i$ is the first element of the thirty two dimensional vector $V_i$. The $k_{ij}$'s are the parameters of the string model and satisfy (Ref. [6])

$$k_{ij} + k_{ji} = V_i . V_j \qquad (mod \quad 1). \tag{1.44}$$

The $k_{ij}$ also have to be mod m, where m is the number which when multiplied by $V_j$ produces a vector with every element an integer. In addition we also have the following constraints (Ref. [6])

$$k_{ii} + k_{i0} + s_i - \frac{1}{2} V_i \cdot V_i = 0 \qquad (mod \quad 1) \tag{1.45}$$

$$s_i = V_i^{2+3l} + V_i^{3+3l} + V_i^{4+3l}. \tag{1.46}$$

The above equation is evaluated at mod 1, where $l$ takes values 0,1,2. A choice of linearly independent $V_i$'s and $k_{ij}$ obeying above constraints are an example of an abelian string model.

We should also mention here that all elements of the $V_i$ vectors and the elements of the constructed sectors $\alpha V$ should lie in the interval $[-\frac{1}{2}, \frac{1}{2})$. The $k_{ij}$'s should also lie in the interval $[-\frac{1}{2}, \frac{1}{2})$.

### 1.5.2. Example of an abelian heterotic string model

Let us consider the simple example of a model which is made up of only one $V$ vector. This is the model with $V_0$ vector below

$$V_0 \quad = \quad \big((-\frac{1}{2})(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})^3 | (-\frac{1}{2})^{22}\big). \tag{1.47}$$

The 3 and 22 to which the blocks in the above equation are raised, refer to the presence of that many muliptles of the blocks in $V_0$ respectively. Because this vector corresponds to each fermion going to itself as you travel once around the closed string, this vector should exist in any heterotic string model.

This model produces the **0** sector, corresponding to a $V$ vector with all of its entries equal to zero. This sector has $\bar{a} = -1/2$ and $a = -1$. The GSO constraints for this sector become

$$V_0 \cdot N = \sum_{i=0}^{i=9} -\frac{N^R{}_i}{2} - \sum_{i=10}^{i=32} -\frac{N^L{}_i}{2} = -\frac{1}{2}. \tag{1.48}$$

The state with $\alpha' M^2 = -2$, ($\alpha' M^2 = 2(L_0 + \bar{L_0} + a + \bar{a})$ (Ref. [4])) is $|0\rangle_R \bigotimes b^\mu{}_{-1/2}|0\rangle_L$. This state is tachyonic. Another state $b^\mu{}_{-1/2}|0\rangle_R \bigotimes \alpha_\nu|0\rangle_L$ is also allowed by the GSO constraints. It is a massless state and is just the gravity multiplet as can be easily seen.

Another possible state is $b^\mu{}_{-1/2}|0\rangle_R \bigotimes b^i{}_{-1/2} b^j{}_{-1/2}|0\rangle_L$. This state has a lorentz vector index as well as internal indices. This state fits the description of a gauge boson. A gauge boson should always be in the adjoint representation of a gauge group as is well known. The symmetry generators of the internal fermion action $S = -\int d^2\sigma \bar{\psi^i}_L \partial_L \psi^i{}_L$ are given by $\bar{\psi^i}_L \psi^i{}_L$. With these charge vectors it is easy to see that the states $b^\mu{}_{-1/2}|0\rangle_R \bigotimes b^i{}_{-1/2} b^j{}_{-1/2}|0\rangle_L$ fall into the adjoint representation of the $SO(44)$ gauge group.

## 1.6. Discussion

In this section we started with introducing Grand Unified Theories as a possibility for unifying the three simple groups of the Standard Model in to a single group. We talked about the hierarchy problem and talked about supersymmetry being a possible solution to the same. N=1 supersymmetric extension of the Standard Model was mentioned as a extension of the Standard Model which helped with the unifications of the Standard Model gauge couplings at the GUT scale. Any theory beyond the Standard Model should address ways of getting N=1 supersymmetric GUT theories as an important area of research. String theory was motivated as a theory for describing the physics beyond the Standard Model.

The basics of bosonic string theory were first introduced. This theory does not produce gauge bosons or fermions and contains spacetime tachyons. Next superstring theory which possesses worldsheet supersymmetry was introduced as a way of getting spacetime fermions in string theory. This theory can have spacetime supersymmetry under certain conditions. Modular invariance in string theory was motivated. The massless spectrum of Type 0 and Type II theory was constructed. Type II theories are theories with spacetime supersymmetry. Even though superstring theories are successful in producing fermions and can produce supersymmetry, they don't produce gauge bosons. Heterotic string theory which produces this facet of string theory was also presented. The basics of the free fermionic construction were presented and an abelian heterotic string model was presented.

Even though abelian heterotic string theories can produce gauge bosons, they don't produce adjoint scalars, which are required for GUT phenomenology. Non-abelian orbifolds are a construction that have a potential for producing adjoint scalars. We describe the construction and properties of non-abelian orbifolds in chapter 2. The chapter illustrates this construction and phenomenology of the $D_4$ non-abelian orbifold. We also present a general proof for absence of chiral fermions in possible phenomenologically pleasing GUT theories obtained through non-abelian orbifold construction.

As was mentioned in the introduction, string theories are consistent only in space-time dimensions larger than four. The extra dimensions of the theory have to be compactified. String compactification on abelian orbifolds was talked about in the introduction, in the case of heterotic strings. However, there are other ways of compactifying string theories. It has been conjectured that the number of such possible compactifications is of the order of $10^{500}$. It has also been suggested that possibly the universe just selects one vaccum out of these $10^{500}$ and there is no explicit vacuum selection principle. Hence under such an assumption a statistical study of correlations between various facets of these vacuua is worthy of research. In chapter 3, we consider correlations between supersymmetry and gauge symmetry on the lanscape of the heteortic string vacuua. We conclude with trying to answer the question of the naturalness of occurence of supersymmetry in the heterotic string landscape in chapter 4.

# 2. Non-abelian orbifolds

## 2.1. Introduction

We have talked about the importance of $N = 1$ supersymmetric Grand Unified theories in the introduction. The introduction also talked about heterotic string theory being a leading candidate for explanining beyond the Standard Model physics. Hence the research on getting $N = 1$ GUTs from heterotic string theory should be an important area of research.

As was discussed in the introduction, heterotic string compactifications to four dimensions produces gauge bosons, fermions, as well as spacetime supersymmetry (under certain conditions). However $N = 1$ supersymmetry in free fermionic heterotic string construction using abelian orbifolds does not allow the simultaneous appearance of adjoint scalars. This is a hinderance to N=1 SUSY GUTs production from heterotic string theory. Abelian constructions also produce rank twenty two gauge groups, without rank reduction which are uncomfortably large.

String compactification on non-abelian orbifolds is an attempt to remedy these problems. They have a potential for producing string models with chiral fermions, with adjoint scalars and gauge groups of smaller ranks.

Section 2.2 reviews the construction of non-abelian orbifolds. Section 2.2.1 is a review of the general formalism for constructing non-abelian orbifolds (Ref. [7], Ref. [8] and Ref. [9]). Section 2.2.2 presents a short review of the free fermionic construction for constructing abelian orbifolds. Section 2.2.3 presents an extension of the free fermionic construction technique to construct non-abelian orbifolds. This is a review of the work presented in Ref. [10]. In section 2.2.4, we present a specific case of non-abelian orbifolds, the case of $D_4$ non-abelian orbifolds. The $D_4$ case is developed further in sections 2.2.5 and 2.2.6. Section 2.3 presents a summary of the construction and analysis of the models that were explicitly constructed. Section 2.4 lists the general inferences for $D_4$. Section 2.4.1 and 2.4.2 talk about gauge groups and N=1 supersymmetry realizations in $D_4$ constructions. Section 2.4.3 talks about the most important result for $D_4$: The absence of asymmetry between chiral and chiral conjugate fermions. Section 2.5 outlines a general proof for absence of chiral fermions in phenomenologically interesting non-abelian orbifold constructions. The appendix lists a few models which were explictly constructed.

For the pioneering work on orbifolds please refer to references Ref. [7] and Ref. [8]. The first paper on non-abelian orbifolds in literature is Ref. [9]. The work in Ref. [11] and Ref. [12] talks about the need for non-abelian orbifolds in constructing heterotic string models which are interesting from a phenomenological perspective. The paper that seriously discussed non-abelian orbifolds in reference to string phenomenology is Ref. [10].

## 2.2. Review of non-abelian orbifolds

### 2.2.1. General formalism

Let us denote the fields living on the string worldsheet by $\psi$ (need not be fermions). Also let us assume that the string worldsheet lagrangian is invariant under the group transformation

$$\psi \rightarrow g\psi \tag{2.1}$$

where $g$ is an element of a group $\mathbf{G}$. Because of the above relation we can consider the following boundary conditions for a closed string on a torus.

$$\psi(\sigma + \pi, \tau) = g\psi(\sigma, \tau) \tag{2.2}$$

$$\psi(\sigma, \tau + \pi) = h\psi(\sigma, \tau) \tag{2.3}$$

where $g, h \in \mathbf{G}$. Evaluation of loop amplitudes imposes the constraints of modular invariance on the possible excitations of the string.

In theory, the classical string lagrangian could be invariant under any Unitary group. The vacuum choice would however singe out a subgroup $\mathbf{G}$. The string excitations should be invariant under the whole group $\mathbf{G}$. Now there are two possibilities either $gh = hg$ or $gh \neq hg$. Let us consider the first case. That is, the group $\mathbf{G}$ is abelian.

In such a case a string excitation from a sector $\psi(\sigma+\pi, \tau) \rightarrow g\psi(\sigma, \tau)$ when acted upon by $h$ gets transformed into

$$(h\psi(\sigma + \pi, \tau)) = hgh^{-1}h\psi(\sigma, \tau) = g(h\psi(\sigma, \tau)). \tag{2.4}$$

So we end up with a state in the $g$ sector itself. Hence, the method of constructing all the states in the theory boils down to the following prescription.

1) Choose an element $g$ from $\mathbf{G}$.

2) Only allow the states from sector $g$ which are invariant upon action by all elements present in $\mathbf{G}$.

The string partition function could be written as below

$$Z = \frac{1}{|G|} \sum_{g,h} Z_h{}^g \tag{2.5}$$

where $|G|$ refers to the number of elements in G.

If instead our orbifold group is non-abelian, then we have added features. Let us say that our non-abelian group $\mathbf{G}$ contains two elements c and d that don't commute. Let us say that we have a closed string with the boundary condition

$$\psi(\sigma + \pi, \tau) = c\psi(\sigma, \tau). \tag{2.6}$$

$$\psi(\sigma, \tau + \pi) = b\psi(\sigma, \tau). \tag{2.7}$$

$b \in G$ and $[b, c] = 0$. Now, the question is what happens when you act on this string state with the element d. The endpoints of the string get converted to $d\psi(\sigma + \pi, \tau)$ and $d\psi(\sigma, \tau)$. Hence multiplying both sides of above equation by d, we get

$$d\psi(\sigma + \pi, \tau) = dcd^{-1}d\psi(\sigma, \tau) \tag{2.8}$$

this gives us

$$Z(c, b) = Z(dcd^{-1}, dbd^{-1}). \tag{2.9}$$

So the string state actually gets converted to a state belonging to the $dcd^{-1}$ sector. Hence, when we construct states that are invariant under the whole non-abelian group we should consider states that are linear combinations of states from the $c$ sector (which are invariant under the action of group elements that commute with $c$) with the corresponding states in the $dcd^{-1}$ sector. To produce the entire spectrum we collect states from all possible sectors $c$.

To frame the above statement mathematically, let us say that our non-abelian group is $G$. Next we consider all possible maximal abelian subgroups $G_p$, $G_q$ etc. A maximal abelian subgroup is defined as a abelian subgroup $G_m$, to which if you add any other element of G not belonging to $G_p$, you get a non-abelian subgroup of $G$. Also let us define $|G_p|$ to be the number of elements in $G_p$. Then it is easy to verify that the partition function constructed by taking states from all possible sectors, which are invariant under the whole non-abelian group is (Ref. [10])

$$Z(G) = \frac{1}{|G|}[\sum_p |G_p|Z(G_p) - \sum_{p<q} |G_{pq}|Z(G_{pq}) + \sum_{p<q<r} |G_{ppr}|Z(G_{pqr})] \tag{2.10}$$

where $G_{pq}$ is the group formed by the intersection of $G_q$ and $G_q$ with a similar definition for $G_{pqr}$. In the expression above it should be noted that for each $Z$ on the R.H.S we are only including states that are invariant under the whole non-abelian group $G$. This chapter utilizes the methods of free fermionic construction to realize non-abelian orbifold constructions. The free-fermionic presented in the introduction is presented once again, to make contact with the concept of abelian orbifolds.

### 2.2.2. Free fermionic construction: short review

In the case of closed strings, the string worldsheet lagrangian symmetry allows for the following boundary conditions for the internal degrees of the freedom.

$$\psi^i(\tau + \pi, \sigma) = -e^{-2\pi\alpha V^i}\psi^i(\tau, \sigma) \tag{2.11}$$

$$\psi^i(\tau, \sigma + \pi) = -e^{-2\pi\beta V^i}\psi^i(\tau, \sigma) \tag{2.12}$$

$\alpha V$ and $\beta V$ correspond to possible linear combinations of n V vectors.

In order to make connection with the orbifold picture we label the symmetry induced by $V_i$ by $g$ and the symmetry induced by $V_j$ by $h$ ($g, h \in \mathbf{G}$, $[g, h] = 0$). The string partition function in the orbifold picture is given by

$$Z = \sum_{g \in G} Tr[(-1)^{F_g} q^{{H_g}^L} q^{{H_g}^R} P_g] = \frac{1}{|G|} \sum_{g,h \in G} Z(g, h) \tag{2.13}$$

where

$$P_g = \frac{1}{G} \sum_{h \in G} h \tag{2.14}$$

is the projection operator, which takes the value 0 for a state that is projected out and value 1 for a state that is projected in. $q = e^{2\pi\tau}$, $q^{H_g}$ refers to the patition function summation over all states that come from the sector $g$. $L, R$ refer to left and right movers. A $\alpha V$ would correspond to $g^m h^n$ with a similar expression for $\beta V$. Making contact with the free fermionic prescription described in the previous chapter we see that h can be taken to be

$$h = e^{2\pi i [V_i . N_{\alpha V} - k_{ij} \alpha_j - s_i - c V_i . \alpha V]}. \tag{2.15}$$

In order that we have a consistent string partition function evaluated at one loop, we impose the following constraints.

- Worldsheet supersymmetry

$$V^1 = V^{2+3i} + V^{3+3i} + V^{4+3i} \tag{2.16}$$

  The above equation is evaluated at mod 1. i takes values 0,1,2.

- Modular invariance

- Physically sensible projection: A space time boson contributes with weight 1, a space time fermion with weight -1 in the partition function.

The value of $s_i$ equals the first element of $V_i$. In addition we have the following GSO constraints

$$k_{ij} + k_{ji} = V_i . V_j \qquad (mod \quad 1). \tag{2.17}$$

The $k_{ij}$ also have to be mod m, where m is the number which when multiplied by $V_j$ produces a vector whose every element is an integer.

$$k_{ii} + k_{i0} + s_i - \frac{1}{2} V_i \cdot V_i = 0 \qquad (mod \quad 1). \tag{2.18}$$

The partition function then becomes

$$Z = \frac{1}{\Pi_{m_i}} \sum_{\alpha,\beta} Tr[(-1)^{\alpha_s} q^{H^R{}_{\alpha V}} q^{H^L{}_{\alpha V}} e^{2\pi i \beta_i [V_i.N_{\alpha V} - k_{ij}\alpha_j - s_i - V_i.\alpha V]}. \tag{2.19}$$

The states from the $\alpha V$ sector that are kept satisfy

$$V_i.N_{\alpha V} = k_{ij}\alpha_j - s_i - V_i.\alpha V \qquad (mod \quad 1). \tag{2.20}$$

### 2.2.3. Non-abelian orbifold extension

If instead our orbifold group is non-abelian, then we have added features. Let us say that our non-abelian group **G** contains two elements $c$ and $d$ that don't commute. Let us say that we have a closed string with the boundary condition

$$\psi(\sigma + \pi, \tau) = c\psi(\sigma, \tau). \tag{2.21}$$

Now, the question is what happens when you act on this string state with an element d. The endpoints of the string get converted to $d\psi(\sigma + \pi, \tau)$ and $d\psi(\sigma, \tau)$. Hence multiplying both sides of above equation by $d$, we get

$$d\psi(\sigma + \pi, \tau) = dcd^{-1} d\psi(\sigma, \tau). \tag{2.22}$$

Hence we get

$$Z(c, b) = Z(dcd^{-1}, dbd^{-1}). \tag{2.23}$$

$b \in \mathbf{G}$, $[b, c] = 0$. In order to make sense out of eqn (2.23) a state from the $c$ sector should have a corresponding conjugate state in sector $dcd^{-1}$. Hence, we can have the following situations.

#### If $c$ belongs to $G_p$ and $dcd^{-1}$ belongs to $G_q$

Since $p$ and $q$ are 2 different subgroups of the non-abelian group and since $Z_p(c, b) = Z_q(dcd^{-1}, dbd^{-1})$, we can choose the V vectors in p and q to be the same, even though the $\psi^i$ basis are different. Hence in order to satisfy $Z_p(c, b) = Z_q(dcd^{-1}, dbd^{-1})$, we choose the $k_{ij}$'s in both p and q to be the same.

#### If $c$ and $dcd^{-1}$ belong to the same maximal abelian subgroup $G_p$

This just means that if we have a state in sector $c$ it should also have a conjugate in sector $dcd^{-1}$. Let the V vector representing $c$ be $V_i$ and the V vector representing $b$ be $V_j$. Now let us say the action of d converts these vectors into

$$V_{dcd^{-1}} = \lambda_{ik} V_k \tag{2.24}$$

$$V_{dbd^{-1}} = \lambda_{jk} V_k. \tag{2.25}$$

In order to satisfy $Z(c,b) = Z(dcd^{-1}, dbd^{-1})$ we should have

$$e^{2\pi i (V_j \cdot N_{V_i} - k_{ji} + s_j + V_j \cdot V_i)} = e^{2\pi i ((\lambda_{jk} V_k) \cdot N_{\lambda_{ik} V_k} - \lambda_{ik} \lambda_{jl} \sum k_{lk} + \lambda_{jk} s_k + \lambda_{jk} V_k \cdot \lambda_{il} V_l)}. \tag{2.26}$$

The equation above should hold for all states. Hence it should also hold for the vacuum. Solving we get.

$$k_{ji} = \lambda_{jk} k_{kl} \lambda_{il} - \bigtriangledown_j \cdot \lambda_{il} V_l \qquad (mod \quad 1). \tag{2.27}$$

**Three different abelian orbifolds** $Z(G_p), Z(G_q)$ **and** $Z(G_r)$

Let us say we have a sector $c$, which occurs in each of the maximal abelian domains $G_p, G_q$ and $G_r$, but it is expressed in different bases. Also let us consider elements $b_1 \in G_p$, $b_2 \in G_q$, $b_1 b_2 \in G_r$. and $b_1 b_2$ does not belong to $G_p$ and $G_q$. Then for a particular state with number vector expressed in 3 different bases as $N_{\alpha_p V_p}$, $N_{\alpha_q V_q}$, $N_{\alpha_r V_r}$, where $\alpha_p V_p$, $\alpha_q V_q$ , $\alpha_r V_r$ are the $c$ sector expressed in the different abelian models. We should have

$$e^{V_{b_1} \cdot N_{\alpha_p V_p}} e^{V_{b_2} \cdot N_{\alpha_q V_q}} = e^{V_{b_1 b_2} \cdot N_{\alpha_r V_r}}. \tag{2.28}$$

hence we get the following constraint

$$(V_{b_1} \cdot \alpha_p V_p - k_{b_1,p} \alpha_p) + (V_{b_2} \cdot \alpha_q V_q - k_{b_2,q} \alpha_q) = (V_{b_1 b_2} \cdot \alpha_r V_r - k_{b_1 b_2, r} \alpha_r)(mod 1) \tag{2.29}$$

In deriving the above relation we have used the equality $s_{\alpha_p V_p} + s_{\alpha_q V_q} = s_{\alpha_r V_r}$ $(mod \quad 1)$.

### 2.2.4. $D_4$ example

In this section we consider a particular example of a non-abelian orbifold model. This orbifold is generated by the $D_4$ group. The group is defined by two elements $r$ and $\theta$ and the following relationship between them.

$$r^2 = \theta^4 = 1 \tag{2.30}$$

$$\theta^3 r = r\theta. \tag{2.31}$$

The maximal abelian subgroups are the following

$$G_r \in [1, r, r\theta^2, \theta^2] \tag{2.32}$$

$$G_{r\theta} \in [1, r\theta, r\theta^3, \theta^2] \tag{2.33}$$

$$G_\theta \;\in\; [1, \theta, \theta^2, \theta^3]. \tag{2.34}$$

The subgroup

$$G_{\theta^2} \;\in\; [1, \theta^2] \tag{2.35}$$

is common to all of the maximal abelian subgroups of $D_4$. Other than this group, we don't have any more subgroups which can be formed by intersecting the maximal abelian subgroups. Hence using equation(2.10), the partition function for the $D_4$ theory becomes

$$Z = \frac{1}{2}Z_r + \frac{1}{2}Z_{r\theta} + \frac{1}{2}Z_\theta - \frac{1}{2}Z_{\theta^2}. \tag{2.36}$$

The only irreducible representation of the $D_4$ group is two dimensional and the following.

$$\theta = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \;, \quad r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.37}$$

Please note that in order to realize the orbifold boundary conditions, we have to choose a basis for the internal fermions. A basis chosen in one maximal abelian subgroup (e.g. $G_r$) is not a proper basis for describing a string which is periodic up to elements from the maximal abelian subgroup $G_\theta$. It is this difficulty of choosing a single basis to describe strings coming from various domains (maximal abelian subgroups) that makes a non-abelian construction more difficult and interesting that a abelian construction.

It should also be noted that observable quantities are independent of the choice of basis used to describe the internal fermions $\psi^i$. For example in order to figure out the gauge group and the explicit way in which the string spectrum transforms under the gauge group, we need to construct charge current operators. In the basis in which $\theta$ is diagonal the charge operators are of the form $\bar{\psi^i}\psi^i$. These charge operators are obviously invariant under $\theta$. Under application by $r$, they transform as

$$\bar{\psi^i}\psi^i \longrightarrow \psi^{\bar{i+1}}\psi^{i+1}, \qquad \psi^{\bar{i+1}}\psi^{i+1} \longrightarrow \bar{\psi^i}\psi^i \tag{2.38}$$

(we are only talking about internal fermions represented by the two dimensional representation of $D_4$ in the above equations). Hence the charge operators invariant under the whole $D_4$ group are

$$\frac{1}{\sqrt{2}}[\psi^{\bar{i+1}}\psi^{i+1} + \bar{\psi^i}\psi^i]. \tag{2.39}$$

As far as measurements go, two states having the same charge vectors, same mass, space time helicity etc, should be considered degenerate. Hence the problem arising because of different $\psi^i$ bases for different maximal abelian subgroups, gets circumvented if we look for the gauge charges as a reference, when comparing two states coming from different domains.

Let us start with a string model with V vectors and $k_{ij}$ as below.

$$\begin{aligned}
V_0 &= \Big((-\frac{1}{2})^{10} \mid (-\frac{1}{2})^{22}\Big) , \\
V_1 &= \Big((-\frac{1}{2})(-\frac{1}{2},0,0)(-\frac{1}{2},0,0)(-\frac{1}{2},0,0) \mid (0)^{22}\Big) , \\
V_2 &= \Big((-\frac{1}{2})(-\frac{1}{2},0,0)(0,-\frac{1}{2},0)(0,-\frac{1}{2},0) \mid \\
& (-\frac{1}{2})^4,(0,0),(-\frac{1}{2},-\frac{1}{2}),(0,0),(-\frac{1}{2},-\frac{1}{2})^3,(0,0)^3\Big) ,
\end{aligned}$$

$$k_{ij} = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 \end{pmatrix} \tag{2.40}$$

The 2-dimensional representation of $D_4$ talked about before could help us to write the following $V_r$, $V_{r\theta}$ and $V_\theta$ vectors

$$\begin{aligned}
V_r &= \Big(-\frac{1}{2}(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}),(0,-\frac{1}{2},0),(0,-\frac{1}{2},0) \mid (-\frac{1}{2},0)^8,(-\frac{1}{2},-\frac{1}{2})^3\Big) \\
V_{r\theta} &= \Big(0,(-\frac{1}{2},-\frac{1}{2},0),(-\frac{1}{2},-\frac{1}{2},0),(0,0,0) \mid (-\frac{1}{2},0)^8,(0,0)^3\Big) \\
V_\theta &= \Big(-\frac{1}{2},(0,0,-\frac{1}{2}),(-\frac{1}{2},0,0),(0,-\frac{1}{2},0) \mid (\frac{1}{4},-\frac{1}{4})^8(-\frac{1}{2},-\frac{1}{2})^3\Big).
\end{aligned}$$

Please note the first 16 entries in each of the $V_r$, $V_{r\theta}$,$V_\theta$ are constructed using the two dimensional representation of $D_4$. Also note that the corresponding elements of $V_0, V_1, V_2$ are made up of two dimensional blocks with both entries the same. This just means that the original orbifold constructed from $V_0, V_1, V_2$ was compactified on $D_4$, hence elements that define that orbifold commute with $D_4$.

We can add $V_{\theta^2}$ to the list of $V_0, V_1, V_2$ as $\theta^2$ commutes with all elements of $D_4$.

Next, we come to the relationships between conjugate sectors. In the $r$ sector $r$ and $r\theta^2$ are conjugate to each other because

$$\theta r \theta^{-1} = r\theta^2. \tag{2.41}$$

Also we know that the $r$ domain is constructed out of the independent vectors $V_0, V_1, V_2, V_r, V_\theta^2$. Hence the application by $\theta$ changes these vectors in the following way

$$\begin{array}{rcl} V_0 & \rightarrow & V_0 \\ V_1 & \rightarrow & V_1 \\ V_2 & \rightarrow & V_2 \\ V_r & \rightarrow & V_r + V_{\theta^2} \\ V_\theta^2 & \rightarrow & V_\theta^2. \end{array}$$

So the $\lambda$ matrix could be written as

$$\lambda = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.42}$$

With these $\lambda_{ij}$ and $V's$ we get an extra constraint on the $k_{ij}$'s given by eqn(2.27).

In the $r\theta$ domain we get a similar condition. The domain is generated by the following vectors $V_0, V_1, V_2, V_{r\theta}, V_{r\theta^3}$. Under operation by $\theta$ the vectors get converted to

$$\begin{array}{rcl} V_0 & \rightarrow & V_0 \\ V_1 & \rightarrow & V_1 \\ V_2 & \rightarrow & V_2 \\ V_{r\theta} & \rightarrow & V_{r\theta} + V_{\theta^2} \\ V_{\theta^2} & \rightarrow & V_{\theta^2}. \end{array}$$

The $\lambda$ matrix remains the same as for the $r$ domain, as can be easily checked. We get extra GSO constraints for the $k_{ij}$'s of this domain from equation eqn(2.27)

In the $\theta$ domain the generating vectors are $V_0, V_1, V_2, V_\theta$. Under application by $r$ we get the following transformation.

$$V_0 \rightarrow V_0 \tag{2.43}$$
$$V_1 \rightarrow V_1 \tag{2.44}$$
$$V_2 \rightarrow V_2 \tag{2.45}$$
$$V_\theta \rightarrow 3V_\theta. \tag{2.46}$$

Please note that we have a factor of 3 in the last line because $\theta$ gets converted into $\theta^3$ on application by r. This gives the following $\lambda$ matrix below as can be easily checked

$$\lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}. \tag{2.47}$$

Hence we get an extra GSO constraint of the form eqn(2.27).

Next we see that $V_0, V_1, V_2, V_\theta^2$ occur in all three domains (in different bases though). Hence from eqn(2.29) we get the following GSO constraint.

$$(V_r \cdot \alpha_p V_p - k_{r,p}\alpha_p) + (V_\theta \cdot \alpha_p V_p - k_{\theta,p}\alpha_p) = (V_{r\theta} \cdot \alpha_p V_p - k_{r\theta,p}\alpha_p) \tag{2.48}$$

where $\alpha_p \cdot V_p$ is a sector constructed out of $V_0, V_1, V_2, V_\theta^2$ .

In addition to the GSO constraints above, we have to impose the GSO constraints from equations eqn(2.17) and eqn(2.18) in each domain separately.

### 2.2.5. An example

A possible model is the one shown below

The $V$ vectors and $k_{ij}$'s in the $r$ domain are

$$\begin{aligned}
V_0 &= \Big((-\frac{1}{2})^{10} \mid (-\frac{1}{2})^{22}\Big) , \\
V_1 &= \Big((-\frac{1}{2})(-\frac{1}{2},0,0)(-\frac{1}{2},0,0)(-\frac{1}{2},0,0) \mid (0)^{22}\Big) , \\
V_2 &= \Big((-\frac{1}{2})(-\frac{1}{2},0,0)(0,-\frac{1}{2},0)(0,-\frac{1}{2},0) \mid \\
&\quad (-\frac{1}{2})^4,(0,0),(-\frac{1}{2},-\frac{1}{2}),(0,0),(-\frac{1}{2},-\frac{1}{2})^3,(0,0)^3\Big) , \\
V_r &= \Big(-\frac{1}{2}(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}),(0,-\frac{1}{2},0),(0,-\frac{1}{2},0) \mid (-\frac{1}{2},0)^8,(-\frac{1}{2},-\frac{1}{2})^3\Big) \\
V_{\theta^2} &= \Big(0^{10} \mid -\frac{1}{2}^{16} 0^6\Big)
\end{aligned}$$

$$k_{ij} = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}. \tag{2.49}$$

The $V$ vectors and $k_{ij}$'s in the $r\theta$ domain are

$$\begin{aligned}
V_0 &= \Big((-\frac{1}{2})^{10} \mid (-\frac{1}{2})^{22}\Big) \ , \\
V_1 &= \Big((-\frac{1}{2})(-\frac{1}{2},0,0)(-\frac{1}{2},0,0)(-\frac{1}{2},0,0) \mid (0)^{22}\Big) \ , \\
V_2 &= \Big((-\frac{1}{2})(-\frac{1}{2},0,0)(0,-\frac{1}{2},0)(0,-\frac{1}{2},0) \mid \\
& \quad (-\frac{1}{2})^4,(0,0),(-\frac{1}{2},-\frac{1}{2}),(0,0),(-\frac{1}{2},-\frac{1}{2})^3,(0,0)^3\Big) \ , \\
V_{r\theta} &= \Big(0,(-\frac{1}{2},-\frac{1}{2},0),(-\frac{1}{2},-\frac{1}{2},0),(0,0,0) \mid (-\frac{1}{2},0)^8,(0,0)^3\Big) \\
V_{\theta^2} &= \Big(0^{10} \mid -\frac{1}{2}^{16} 0^6\Big)
\end{aligned}$$

$$k_{ij} = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}. \tag{2.50}$$

The $V$ vectors and $k_{ij}$'s in the $\theta$ domain are

$$\begin{aligned}
V_0 &= \Big((-\frac{1}{2})^{10} \mid (-\frac{1}{2})^{22}\Big) \ , \\
V_1 &= \Big((-\frac{1}{2})(-\frac{1}{2},0,0)(-\frac{1}{2},0,0)(-\frac{1}{2},0,0) \mid (0)^{22}\Big) \ , \\
V_2 &= \Big((-\frac{1}{2})(-\frac{1}{2},0,0)(0,-\frac{1}{2},0)(0,-\frac{1}{2},0) \mid \\
& \quad (-\frac{1}{2})^4,(0,0),(-\frac{1}{2},-\frac{1}{2}),(0,0),(-\frac{1}{2},-\frac{1}{2})^3,(0,0)^3\Big) \ , \\
V_{\theta} &= \Big(-\frac{1}{2},(0,0,-\frac{1}{2}),(-\frac{1}{2},0,0),(0,-\frac{1}{2},0) \mid (\frac{1}{4},-\frac{1}{4})^8(-\frac{1}{2},-\frac{1}{2})^3\Big) \\
V_{\theta^2} &= \Big(0^{10} \mid -\frac{1}{2}^{16} 0^6\Big)
\end{aligned}$$

$$k_{ij} = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{4} \end{pmatrix}. \tag{2.51}$$

Next we consider the construction of states. If a sector is constructed from $V$ vectors which are common to all domains, then a state coming from this sector should 'not' be projected out in each domain. It is only then, that this state would be considered invariant under the whole $D_4$ group. To explain this concretely let us consider the gauge bosons coming from the **0** sector. This sector is common to all the domains.

Let us consider the $r$ domain first. The constraints of invariance under the $G_r$ group (abelian GSO constraints) imposed on any state from the **0** sector, looks like below

$$
\begin{aligned}
V_0 \cdot N &= \Big( \sum_{0 \leq i \leq 9} (-\frac{1}{2} \cdot N_i) - \sum_{10 \leq i \leq 32} (-\frac{1}{2} \cdot N_i) \Big) = s_0 = -\frac{1}{2} \\
V_1 \cdot N &= \Big( \frac{N_0}{2} + \frac{N_1}{2} + \frac{N_4}{2} + \frac{N_7}{2} \Big) = s_1 = -\frac{1}{2} \\
V_2 \cdot N &= \Big( \frac{N_0}{2} + \frac{N_1}{2} + \frac{N_5}{2} + \frac{N_8}{2} - \frac{N_{10}}{2} - \frac{N_{11}}{2} - \frac{N_{12}}{2} - \frac{N_{13}}{2} - \frac{N_{16}}{2} \\
&\quad - \frac{N_{17}}{2}) \sum_{i=20, i=22, i=24} - \frac{N_i}{2} - \frac{N_{i+1}}{2} \Big) = s_2 = -\frac{1}{2} \\
V_r \cdot N &= \Big( \frac{N_0}{2} + \frac{N_1}{2} + \frac{N_2}{2}, + \frac{N_3}{2} + \frac{N_5}{2} + \frac{N_8}{2} - \sum_{5 \leq i \leq 12} \frac{N_{2i}}{2} \\
&\quad - \sum_{26 \leq i \leq 31} \frac{N_i}{2} \Big) = s_r = -\frac{1}{2} \\
V_{\theta^2} \cdot N &= - \sum_{10 \leq i \leq 25} \frac{N_i}{2} = s_{\theta^2} = 0.
\end{aligned}
$$

As far as the $r\theta$ and the $\theta$ domains go, the constraints for the $V_0, V_1, V_2, V_{\theta^2}$ remain the same. The extra constraint in the $r\theta$ domain is the constraints from the $V_{r\theta}$

$$
V_{r\theta} \cdot N = \Big( \frac{N_1}{2}, + \frac{N_2}{2} + \frac{N_3}{2} + \frac{N_4}{2} - \sum_{5 \leq i \leq 12} \frac{N_{2i}}{2} \Big) = s_{r\theta} = 0.
$$

The extra constraint in the $\theta$ domain, is the constraint from $V_\theta$

$$
V_\theta \cdot N = \Big( \frac{N_0}{2} + \frac{N_3}{2} + \frac{N_4}{2} + \frac{N_8}{2} + \sum_{5 \leq i \leq 12} (\frac{N_{2i}}{4} - \frac{N_{2i+1}}{4}) - \sum_{26 \leq i \leq 31} \frac{N_i}{2} \Big) = s_\theta = -\frac{1}{2}.
$$

Since the first sixteen left moving elements of the **V** vectors are constructed out of a two dimensional representation, the charge operators that are invariant under the whole $D_4$ group are

$$\frac{1}{\sqrt{2}}[\bar{\psi^{2i}}\psi^{2i}+\psi^{\bar{2i+1}}\psi^{2i+1}] \quad if \quad 5 \leq i \leq 12 \tag{2.52}$$

$$\bar{\psi^{i}}\psi^{i} \quad if \quad 26 \leq i \leq 31. \tag{2.53}$$

With this in mind, let us consider states which contain only the first sixteen left moving fermionic excitations.

Let us start with the $s_{invariant} = b^{10}_{-\frac{1}{2}}b^{11}_{-\frac{1}{2}} \mid 0\rangle$ state. This state is allowed by the constraints coming from $V_0, V_1, V_2, V_{\theta^2}$. This state is allowed by constraint coming from $V_\theta$ but not from constraints coming from $V_r$ and $V_{r\theta}$. Hence this state is not allowed.

Next let us consider the state $s_{non-invariant}= b^{10}_{-\frac{1}{2}}b^{12}_{-\frac{1}{2}} \mid 0\rangle$. This state satisfies the GSO constraints in the $r$ and $r\theta$ domain, but it does not satisfy the GSO constraints in the $\theta$ domain. The $D_4$ invariant state is $b^{10}_{-\frac{1}{2}}b^{12}_{-\frac{1}{2}} \mid 0\rangle + b^{11}_{-\frac{1}{2}}b^{13}_{-\frac{1}{2}} \mid 0\rangle$. It has the charge vector $[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0......]$. This is the projected state. So if we can find a state which is allowed in the $\theta$ domain and if the projected form of the state is $[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0......]$, then we know that the projected state with charge vector $[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0......]$ is allowed in every domain and hence should be an allowed state. Fortunately, the state $b^{10}_{-\frac{1}{2}}b^{13}_{-\frac{1}{2}} \mid 0\rangle$ is allowed in the $\theta$ domain. The corresponding $D_4$ invariant state is $b^{10}_{-\frac{1}{2}}b^{13}_{-\frac{1}{2}} \mid 0\rangle + b^{11}_{-\frac{1}{2}}b^{12}_{-\frac{1}{2}} \mid 0\rangle$. This state has a charge vector $[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0......]$. Hence the state with charge vector $[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0......]$ is allowed in each domain and hence is allowed in the $D_4$ model.

Also note that we didn't have such an argument in the case of the $s_{invariant}$ state. This is because the $D_4$ invariant state is $b^{10}b^{11} \mid 0\rangle$. And its charge vector is $[\frac{2}{\sqrt{2}}, 0, 0.....]$. We cannot find states which are allowed in the $r\theta$ or the $r$ domain which when $D_4$ projected give us a charge vector $[\frac{2}{\sqrt{2}}, 0, 0.....]$, as can be easily checked.

Let us call the two dimensional representation of $D_4$ contribution to a $V$ vector as a two dimensional block. We have labeled the states as $s_{invariant}$ and $s_{non-invariant}$ to suggest that the states are produced by excitations from the same two dimensional block or from different two dimensional blocks respectively.

We can easily see that if a non-invariant state is allowed in one domain, then the $D_4$ invariant state with corresponding charge vector is allowed in each domain. To see this note that the **number vector** from at least one two dimensional block of this state, should be $[\mp 1, 0]$. If this state is not allowed in any domain then the corresponding $[0, \mp 1]$ state is definitely allowed in that domain. You can easily see that the $D_4$ invariant states in both cases have the same charge vector. Hence the $D_4$ invariant state of a non-invariant state is always allowed by the orbifold construction.

Such a statement cannot be made about an invariant state. Only if such a state is allowed in each domain, would the $D_4$ invariant state be allowed.

As far as the states coming from excitations of the non two dimensional blocks go, they have to satisfy the GSO constraints in each domain. These excitations don't undergo any extra projection.

With this in mind, we can see that as far as the eight two dimensional blocks go we get a **SO(12)** $\otimes$ **SO(4)** = **SO(12)** $\otimes$ **SU(2)** $\otimes$ **SU(2)** group, realized at level two as can be easily checked. The non two dimensional portions of the gauge boson producing sector similarly generate the gauge group $SO(4)$ at level one.

Next let us consider the supersymmetry of the model. It is easy to see that a gravitino with right moving charge vector $(-\frac{1}{2}, \frac{1}{2}, 0, 0, -\frac{1}{2}, 0, 0, \frac{1}{2}, 0, 0)$ is the only gravitino allowed in each domain. Hence this model has **N=1** supersymmetry. So it suffices to list the scalars, as the fermions are just superpartners.

Since the **SO(10)** at level two is phenomenologically pleasing, we will only illustrate construction of states that are non-singlets under this group.

Let us first consider the sector common to all the domains

$$V_1 + V_2 + V_\theta^2 = \Big(0,0,0,0,-\frac{1}{2},-\frac{1}{2},0,-\frac{1}{2},-\frac{1}{2},0 \mid 0,0,0,0,-\frac{1}{2},-\frac{1}{2},0,0, -\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0,0,0,0,0\Big).$$

The constraints from $V_0, V_1, V_2, V_\theta^2$ allow a state of the form $[0,0,0,0,N_4,N_5,0$-$,N_7,N_8,0 \mid N_{10},N_{11},0,0,(1,1),0,0,(1,0)0,.....]$, such that $N_4+N_7 = N_5+N_8 = N_{10}+N_{11} = 1$. Hence eight states are possible. It can be easily seen that this is a non-invariant state. Hence it produces four states that are allowed upon projection by the complete $D_4$ orbifold. These are the highest weight states of a **(12,2,2)** representation of the **SO(12)** $\otimes$ **SU(2)** $\otimes$ **SU(2)** at level two.

We get scalars from the **0** sector as well. In abelian orbifold constructions, it is impossible to simultaneously get N=1 supersymmetry and scalars transforming in the adjoint representation of the gauge group. As far as $D_4$ case goes, if a state with left mover charge vector $[(1,0),(1,0),0000..]$ is not allowed in any domain, then a state with left mover charge vector $[(1,0),(0,1),0,0....]$ is definitely allowed, because it is a non-invariant state. The allowed $D_4$ projected state has the charge vector $[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0....]$ . Hence we see that we can end up with scalar adjoints for N=1. Altogether we get two scalar representations transforming as **(66,1,1)** and four scalar representations transforming as **(12,2,2)** from this sector.

2.2.6. A note on the possible representations of $V$ vectors for $D_4$

As was illustrated in the previous sections, eight two dimensional blocks were utilized in the construction of the $V$ vectors. We present here an argument as to why any other representation would not suffice.

Since $k_{i\theta}$ and $k_{\theta,\theta}$ should be mod 4, $V_\theta$, $V_r$ and $V_{r\theta}$ should contain either four or eight two dimensional blocks.

With this being said, we can easily see that four two dimensional blocks in the construction of $V$'s lead to contradictions. With four two dimensional blocks, we get the following form for $V_\theta^2$

$$V_\theta^2 = \Big((0)^{10} \mid (-\frac{1}{2})^8, (0)^{14}\Big).$$

This sector could produce the following possible charge vectors for the left moving portion of the gauge bosons $[(\frac{1}{2}, \frac{1}{2})^2, (\frac{1}{2}, -\frac{1}{2})^2, (0)^{14}]$, which is a **non-invariant** state. This state when $D_4$ projected ends up with a charge vector $[1, 1, 0, 0....]$. But this gauge boson also comes from the **0** sector. Hence we would end up with gauge boson degeneracies.

In order for this situation to not occur, the GSO constraints from the $V$'s common to each domain should not allow this non-invariant state. The GSO constraint from these $V$ vectors becomes

$$\begin{aligned}(V_i \cdot N)_L &= k_{i\theta^2} - V_i \cdot V_{\theta^2} \\ &= k_{\theta^2 i}\end{aligned}$$

where **L** means that we are considering the GSO constraint for the left movers. As discussed earlier, the counterparts of the two dimensional blocks of the $D_4$ vectors, for $V_i$'s are made up of two dimensional blocks, with both entries the same. Hence the non-invariant state above always contributes a value zero to the LHS of the above equation. Hence if $k_{\theta^2 i} \neq 0$, we can get rid of this gauge boson degeneracy problem. But since $k_{\theta^2 i} = 2k_{\theta i}$ and since $k_{\theta i}$ is either 0 or $\mp\frac{1}{2}$ mod 1, we will always end up with gauge boson degeneracies.

Hence only $D_4$ representations with $V$ vectors made up of eight two dimensional blocks are allowed.

If we were just considering abelian orbifold construction, degeneracies for gauge bosons would appear, if we had mistakenly used $V$ vectors which were not linearly independent. Since it is the projected gauge bosons and not the unprojected gauge bosons which are responsible for these gauge boson degeneracies, we could possibly conjecture that our choice of $V$ vectors in the $D_4$ orbifold construction were not independent to begin with. It is because all the $V$ vectors couldn't be described in a single basis for non-abelian constructions, that the possible linear dependence of $V$ vectors was not so obvious and this dependence of $V$ vectors was seen in the degeneracies of the projected gauge bosons.

## 2.3. Construction and analysis of models

A computer code was written to construct and analyze the $D_4$ models. Random $V$ vectors were chosen in each domain. Next, random $k_{ij}$'s were further chosen. The choice of $V$'s and $k_{ij}$'s was considered satisfactory if they satisfied the GSO constraints in each domain as well as GSO constraints between the domains. This process itself was very time consuming. When three vectors were present in the common sector (one $V_0$, one $V_1$ which was the supersymmetry producing sector and another $V$ vector), it took a month to produce 358791 consistent sets of $V$'s and $k_{ij}$'s. When four vectors were present in the common sector, just 25286 consistent sets of $V$'s and $k_{ij}$'s were produced.

These were analyzed for gauge group, supersymmetry, fermion and scalar irreps. Only distinct models were collected. Two models were considered distinct, if they had the same gauge group, same number of supersymmetries as well as same number of fermion and scalar irreps.

It was noticed that in the 358791 sets, only 57 sets produced distinct models. However in the 25286 sets, 227 distinct models were produced. This is because of the presence of an extra $V$ vector in the common sector. The extra constraints caused more distinct models to be scanned, as a varied space of $V$'s and $k_{ij}$'s had to be scanned.

Below we present some data obtained from the computer search below. The first subsection lists the data obtained from the 57 models followed by a sub-section that lists the data obtained from the 227 models talked about above. Only those properties that have a non-zero value are listed. For example we don't see any models with shatter twenty two. Hence shatter twenty two is not listed as an entry in the tables below.

It should be noted that even though certain correlations are observed (the absence of a model with $N = 4$ SUSY, for example), these correlations are not general trends, but are only seen because of the limited size of model space that we have scanned. As was noticed, the number of sets of $V$'s and $k_{ij}$'s produced dropped dramatically with the introduction of an extra $V$ vector in the common sector. However, this reduced set, produced far greater distinct models. We could in principle keep adding more and more $V$ vectors and run the codes for production of $V$'s and $k_{ij}$'s for longer periods of time to scan a bigger portion of the landscape. However, even if we could reach sufficiently constrained situations following the above procedures, we still reach limits because of the inherent non-randomness in production of $V$'s and $k_{ij}$'s , because of limitations imposed ( by the computational hardwares and softwares) in the production of randomness in $V$'s and $k_{ij}$'s.

Because of the limited time allowed and other constraints, the following data was obtained. Even though this data may not be too suggestive of deeper issues (not obvious without a computer simulation) in $D_4$ construction; The data still represents the analysis on the largest data set of non-abelian orbifold models ever constructed.

### 2.3.1. Some general analyses of the 57 models

| Gauge Group | Multiplicity |
|---|---|
| $SU_1(2)$ | 90 |
| $SU_1(4)$ | 9 |
| $SU_2(2)$ | 54 |

Table 2.1. Multiplicities of $SU$'s

| Gauge Group | Multiplicity |
|---|---|
| $SO_1(5)$ | 6 |
| $SO_1(7)$ | 6 |
| $SO_1(8)$ | 9 |
| $SO_1(9)$ | 6 |
| $SO_1(11)$ | 6 |
| $SO_1(12)$ | 9 |
| $SO_1(13)$ | 6 |
| $SO_1(15)$ | 6 |

Table 2.2. Multiplicities of $SO$'s at level 1

| Gauge Group | Multiplicity |
|---|---|
| $SO_2(8)$ | 24 |
| $SO_2(12)$ | 33 |
| $SO_2(16)$ | 12 |

Table 2.3. Multiplicities of $SO$'s at level 2

$U(1)$'s appeared only as $U(1)^3$ and were present in 9 of the 57 models.

| SUSY | Number of Models |
|---|---|
| $N = 0$ | 19 |
| $N = 1$ | 19 |
| $N = 2$ | 19 |
| $N = 4$ | 0 |

TABLE 2.4. Supersymmetry vs Number of Models

| Shatter | Number of Models |
|---|---|
| 2 | 3 |
| 3 | 9 |
| 4 | 18 |
| 5 | 9 |
| 6 | 9 |
| 7 | 3 |
| 9 | 6 |

TABLE 2.5. Shatter vs Number of Models

### 2.3.2. Some general analyses of the 227 models

| Gauge Group | Multiplicity |
|---|---|
| $SU_1(2)$ | 377 |
| $SU_1(4)$ | 50 |
| $SU_1(6)$ | 2 |
| $SU_2(2)$ | 292 |
| $SU_2(4)$ | 90 |

TABLE 2.6. Multiplicities of $SU$'s

An $Sp$ group appeared twice as $Sp_1(8)$ and $(F_4)_1$ appeared once in our search.

| Gauge Group | Multiplicity |
|---|---|
| $SO_1(5)$ | 98 |
| $SO_1(7)$ | 45 |
| $SO_1(8)$ | 28 |
| $SO_1(9)$ | 30 |
| $SO_1(10)$ | 9 |
| $SO_1(11)$ | 10 |
| $SO_1(12)$ | 5 |
| $SO_1(15)$ | 2 |

Table 2.7. Multiplicities of $SO$'s at level 1

| Gauge Group | Multiplicity |
|---|---|
| $SO_2(8)$ | 85 |
| $SO_2(10)$ | 49 |
| $SO_2(12)$ | 40 |
| $SO_2(16)$ | 11 |

Table 2.8. Multiplicities of $SO$'s at level 2

| Gauge Group | Multiplicity |
|---|---|
| $U(1)$ | 23 |
| $U(1)^2$ | 55 |
| $U(1)^3$ | 39 |
| $U(1)^4$ | 16 |
| $U(1)^5$ | 3 |
| $U(1)^6$ | 6 |
| $U(1)^7$ | 3 |

Table 2.9. Multiplicities of $U(1)$'s

| SUSY | Number of Models |
|---|---|
| $N = 0$ | 80 |
| $N = 1$ | 84 |
| $N = 2$ | 63 |
| $N = 4$ | 0 |

Table 2.10. Supersymmetry vs Number of Models

| Shatter | Number of Models |
|---|---|
| 2 | 1 |
| 3 | 9 |
| 4 | 36 |
| 5 | 47 |
| 6 | 45 |
| 7 | 52 |
| 8 | 21 |
| 9 | 6 |
| 10 | 4 |
| 11 | 6 |

TABLE 2.11. Shatter vs Number of Models

### 2.4. General inferences

In this section we will talk about some general properties of our models.

#### 2.4.1. Gauge group

In the case where only the **0** sector contributes to the production of gauge bosons, no invariant state contributes. However, all non-invariant states allowed by the GSO constraints from the $V$ vectors common to each domain, contribute to the gauge bosons. These gauge bosons realize a gauge group at level two. The biggest rank two group that could be produced is a rank eight $\mathbf{SO(16)}$. This group could obviously be broken down to a phenomenologically interesting $\mathbf{SO(10)}$, either by a Higgs mechanism or by adding an appropriate $V$ vectors to the original orbifold model, before it gets $D_4$ compactified.

#### 2.4.2. Supersymmetry

Next we look at ways of getting **N=1** supersymmetry. The $D_4$ partition function eqn(2.36), tells us that **N=1** supersymmetry requires $N_{\theta^2}$ be at least two. This is because the gravitinos of the $D_4$ model are subsets of the gravitinos of $\theta^2$ domain. $N_{\theta^2}$ equal to one, means that we couldn't have started with a model containing adjoint scalars to begin with, hence after $D_4$ compactification we couldn't get any adjoint scalars. Hence, we are left with the following configurations below.

| $N_r$ | $N_{r\theta}$ | $N_\theta$ | $N_{\theta^2}$ |
|---|---|---|---|
| 1 | 1 | 2 | 2 |
| 2 | 1 | 1 | 2 |
| 1 | 2 | 1 | 2 |
| 2 | 2 | 2 | 4 |

Also note that we shouldn't be ending up with a situation where **N=0** in any domain. Even though mathematically the partition function would give us one gravitino in the $D_4$ model, we wouldn't be getting equal number of bosons and fermions in the $D_4$ model.

If we can satisfy the above constraints with proper choices of $V$'s and $k_{ij}$, then we can get **N=1** supersymmetry, with possibility of adjoint scalars.

Generally if a invariant scalar state with left moving charge vector$[1, 1, 0, 0.......]$ is allowed in each domain, then it is seen that we don't end up with an adjoint scalar representation.

It was observed that scalar representations bigger than the scalar adjoint representation contain the scalar adjoint. In such a case we cannot get a separate scalar adjoint representation. Other than this case, $N = 1$ constructed from the table above gives us scalar adjoints.

### 2.4.3. Absence of asymmetry between chiral and chiral conjugate fermions

In order to get at least an SO(10) group at level two we choose our common sector vectors to contain at least five consecutive $(0,0)$ blocks in the left movers. Let us say that chiral fermions are coming from the domain containing the $V_r$ vector. We can choose the $V_r$ vector as being a fermion producing vector. That is $V_r$ produces the fermions.

The sector conjugate to $V_r$ is $V_{r\theta^2}$. In order to show that $V_r$ cannot be chiral, let us first assume that $V_r$ is chiral. Then a chiral state from $V_r$ sector needs a corresponding state in the $V_{r\theta^2}$ sector to form a $D_4$ invariant state. This would violate modular invariance in the $\theta$ domain or the $\mathbf{r}$ domain as shown through the free fermionic construction below.

The presence of an asymmetry between chiral and chiral conjugate fermions does require the presence of at least one V vector whose right moving part will have $-\frac{1}{2}$ entries only in positions where $V_r$ has zero entries and vice versa (except for the first entry), with the first entry of the V vector being $-\frac{1}{2}$, as can be easily checked.

Let us choose $V_0, V_1, V_r$ and V as below.

$$
\begin{aligned}
V_0 &= \left((-\frac{1}{2})^{10} \mid (-\frac{1}{2})^{22}\right) , \\
V_1 &= \left((-\frac{1}{2})(-\frac{1}{2},0,0)(-\frac{1}{2},0,0)(-\frac{1}{2},0,0) \mid (0)^{22}\right) , \\
V_r &= \left((-\frac{1}{2})(-\frac{1}{2},0,0)(0,-\frac{1}{2},0)(0,-\frac{1}{2},0) \mid (-\frac{1}{2},0)^{8}(0)^{6}\right) , \\
V &= \left((-\frac{1}{2})(0,-\frac{1}{2},0)(-\frac{1}{2},0,0)(0,0,-\frac{1}{2}) \mid (-\frac{1}{2},-\frac{1}{2})^{5}(0)^{12}\right) ,
\end{aligned}
$$

Now we will show that such a possibility is not consistent with the GSO constraints for $D_4$ orbifolds.

As was studied earlier the $\lambda$ matrix in the $\mathbf{r}$ could be written as

$$
\lambda_{ij} = \delta_{ij} \tag{2.54}
$$

if i and j are common sector vectors or $V_{\theta^2}$.

$$
\lambda_{rr} = 1, \lambda_{r\theta^2} = 1. \tag{2.55}
$$

The extra non-abelian constraints in the $\mathbf{r}$ don't impose extra constraints on the $k_{ij}$'s as $\triangle_i$ equals zero. However extra GSO constraints are imposed on $k_{ir}$

$$
\begin{aligned}
k_{ir} &= k_{ir} + k_{i\theta^2} - \triangle_i \times (...) \\
&= k_{ir} + k_{i\theta^2}.
\end{aligned} \tag{2.56}
$$

The above relations are true up to mod 1. The second line follows as $\triangle_i$ equals zero. This gives us $k_{i\theta^2}$ equal to zero. However the dot product of V and $V_{\theta^2}$ is 1 mod 2. This implies that $k_{\theta^2 i} = -\frac{1}{2}$ or $k_{\theta i} = \mp\frac{1}{4}$, which cannot be true as $k_{\theta i}$ has to be mod 2. If instead we had chosen V to give us an SO(12) group, we could get around this problem, as then the dot product of V and $V_{\theta^2}$ would be zero. But we would have another problem as the dot product of V and $V_r$ would then be mod 4, which is also not allowed.

## 2.5. General proof of absence of asymmetry between chiral and chiral conjugate fermions in non-abelian orbifolds

### 2.5.1. General form of the non-abelian group

The way that non-abelian orbifolds differ from abelian orbifolds is that acting with an element $a$ of the non-abelian group on a state belonging to a sector $b$, turns the state into a state of sector $a^{-1}ba$. If $a$ and $b$ don't commute then $b$ and $a^{-1}ba$ are different sectors.

Now two situations are possible. If $b$ and $a^{-1}ba$ commute then both sectors can be expressed by the same basis of internal fermions. If a **state** from $b$ gets transformed into a **state** in $a^{-1}ba$ under action by $a$, then action by $a$ can at the most just **interchange** the fermions. Otherwise, $b$ and $a^{-1}ba$ cannot be expressed in the same basis.

To be more explicit, action by $a$ can take $\psi_i$ to $\psi_I$ , $\psi_j$ to $\psi_J$ and vice versa, so that a state ${b_{-\frac{1}{2}}}^i {b_{-\frac{1}{2}}}^j$ belonging to sector $b$ gets transformed into a state ${b_{-\frac{1}{2}}}^I {b_{-\frac{1}{2}}}^J$ belonging to sector $a^{-1}ba$. However if $\psi_i$ is taken to $\psi_i + \psi_I$ and $\psi_j$ is taken to $\psi_j + \psi_J$ under action by $a$, then it would mean that we cannot transform the state ${b_{-\frac{1}{2}}}^i {b_{-\frac{1}{2}}}^j$ belonging to the $b$ sector into a state in the $a^{-1}ba$ sector. This is because any state in the $a^{-1}ba$ sector is of a form ${b_{-\frac{1}{2}}}^p {b_{-\frac{1}{2}}}^q$, because both $b$ and $a^{-1}ba$ are expressed by the same basis.

This just implies that in a basis in which $b$ is diagonal, $a$ has to be expressed by a matrix made up of rows in which only one entry is non-zero. i.e.

$$a_{ij} = a_i \delta_{i,j(i)} \tag{2.57}$$

where j(i) is an integer valued function of i. i.e. $a$ is just a permutation matrix.

In such a case, as has been stated in the $D_4$ case, the state from $b$ combines with the state from $a^{-1}ba$, to form a state invariant under $b$ and $a$ (at the most up to a phase, but that is physically fine, as all we want is that the scattering amplitudes should be invariant under $a$ and $b$). In our example above, the invariant state is ${b_{-\frac{1}{2}}}^i {b_{-\frac{1}{2}}}^j + {b_{-\frac{1}{2}}}^I {b_{-\frac{1}{2}}}^J$.

Also currents have to be constructed which are invariant under actions by $a$ and $b$. In the present case, the invariant currents are $\bar{\psi}_i\psi_i + \bar{\psi}_I\psi_I$, $\bar{\psi}_j\psi_j + \bar{\psi}_J\psi_J$ etc up to normalization constants.

In case we had a situation where action by $a$ took $\psi_i$ to $\psi_{i1}$, $\psi_{i1}$ to $\psi_{i2}$ and $\psi_{i2}$ to $\psi_i$, then we would have the invariant current as $\bar{\psi_i}\psi_i + \bar{\psi_{i1}}\psi_{i1} + \bar{\psi_{i2}}\psi_{i2}$ up to a normalization constant.

This mechanism causes the gauge group rank to be reduced and the gauge groups to be realized at higher levels as was shown explicitly in the $D_4$ case.

The other situation is when $b$ and $a^{-1}ba$ don't commute, i.e. they belong to different domains. In such case $b$ and $a^{-1}ba$ are expressed in completely different bases, which are not related to each other. The invariant states are formed by combination of states from $b$ and $a^{-1}ba$. The currents are a linear combination of currents from domain containing $b$ and from domain containing $a^{-1}ba$.

To be more explicit. Action by $a$ could take $\psi_i$ to $\Psi_I$ , $\psi_j$ to $\Psi_J$ and vice versa, so that a state $b_{-\frac{1}{2}}{}^i b_{-\frac{1}{2}}{}^j$ belonging to sector $b$ gets transformed into a state $B_{-\frac{1}{2}}{}^I B_{-\frac{1}{2}}{}^J$ belonging to sector $a^{-1}ba$.

The invariant currents in this case are $\bar{\psi_i}\psi_i + \bar{\Psi}_I\Psi_I$, $\bar{\psi_j}\psi_j + \bar{\Psi}_J\Psi_J$ etc up to normalization constants.

Note that the $\psi$, $\Psi$ distinction is explicitly mentioned above in order to stress the point that the fermionic bases in the two domains are independent.

Since these currents are constructed out of different bases, which are completely independent of each other, we don't have any shortening of roots in this case. The two sectors $b$ and $a^{-1}ba$ **should** have the same spectrum, and it is as if, we are counting the two sectors as giving one contribution. We have a phenomenology similar to abelian orbifolds in this case. Hence this scenario is not interesting.

### 2.5.2. Consequences of the elements of the non-abelian group having representations which are permutation matrices

As talked above the interesting cases arise when $b$ and $a^{-1}ba$ commute. If we consider two such n× n matrices $a$ and $b$, the requirement that $a$ and $b$ be diagonal or at most permutation matrices, implies that both $a^n$ and $b^n$ should be diagonal. To see this, please note that a permutation matrix has zero diagonal elements. The requirement that the matrix should be diagonalizable implies that two rows cannot have the non-zero elements at the same position. These two statements together imply that the $n^{th}$ power of a n-dimensional permutation matrix $b$, should be diagonal.

The basis in which $b$ is diagonal, $a^{-1}ba$ is $b$ with its entries permuted among itself (as $a$ is a permutation matrix), as can be easily checked. Hence in this basis $b$ commutes with $a^{-1}ba$. Since commutation relations are basis independent $b$ and $a^{-1}ba$ commute. Similarly, in the basis in which $a$ is diagonal $b$ is a permutation matrix. Hence, arguments expressed in this paragraph in turn imply $a$ and $b^{-1}ab$ commute with each other.

$a$ and $b^{-1}ab$ commuting with each other coupled with $a^{-1}ba$ commuting with $b$, implies

$$ab = \Pi ba \tag{2.58}$$

where $\Pi$ is a matrix that commutes with both $a$ and $b$. This implies that the only form that $\Pi$ can take is $e^{i\theta}\mathbf{1}$. Hence we get $ab = e^{i\theta}ba$. Choosing a basis such that $a$ is diagonal, we get the matrix equation below

$$a_{ii}b_{ij} = e^{i\theta}b_{ij}a_{jj}. \tag{2.59}$$

Hence $a_{ii} = e^{i\theta}a_{jj}$ if $b_{ij} \neq 0$. This implies that $e^{in\theta} = 1$, for a n-dimensional irreducible representation.

To summarize, in our non-abelian group the interesting cases arise, if for two elements $a$ and $b$, $ab = e^{i\theta}ba$. If the smallest irreducible representation of this group is n-dimensional then $a^n$ and $b^n$ commute with $a$ as well as $b$. Also $e^{ni\theta} = 1$.

Next we will consider situations in which not only does $a$ commute with $b^{-1}ab$ but also commutes with $c^{-1}ac$, such that $b$ does not commute with $a$, and $c$ does not commute with $a$.

The above paragraph implies that $ab = e^{i\theta_1}ba$, and $ac = e^{i\theta_2}ca$. In a basis in which $a$ is diagonal the first relationship implies that the minimum phase difference possible between non-identical elements of $a$ is $\theta_1$, whereas the second relationship implies this minimum phase difference equals to $\theta_2$. The two states are contradictory unless $\theta_1$ equals $\theta_2 = \theta$.

We should also notice that if $n\theta = 1$ for one domain, then as far as the gauge bosons coming from the $\mathbf{0}$ sector are concerned, the projected gauge bosons should be realized at level $n$. Since projected gauge bosons in each domain should be realized at the same level, $\theta$'s for each domain should equal each other. This in turn implies that $\theta$ can be considered a defining parameter of the non-abelian string model.

Up until now, we have shown that string model construction is possible only if two elements $a$ and $b$ of the non-abelian group obey $ab = e^{i\theta}ba$. $\theta$ can be considered as one of the defining parameters of the string model. We also have shown that for a n-dimensional irreducible representation $e^{ni\theta} = 1$. We will now show that $e^{mi\theta}$ cannot equal 1, for $m < n$.

To understand this note that if this was the case, that is, if $m\theta$ is an integer for m smaller than n, then we would have the entry at position $i$, being the same as the entry at position $i+m$. This would be the case for $V_a$ as well as $V_b$. In this case $\mathbf{0}$ will also have a state of the form $(0, 1, 0, 0, .. -1)_n$, wher the 1 and $-1$ are separated by $m$ entries. This would produce a state with all elements of the charge vector being zero, for charge operators which are invariant under the entire non-abelian group. This would produce unwanted degeneracies in the gauge bosons.

### 2.5.3. Constraints from N=1 supersymmetry and appearance of scalar adjoints, on the allowed gauge groups

Previously we had talked about how in the $D_4$ case, non-abelian orbifolds ensure that adjoint scalars are produced in string models that are N=1 supersymmetric. Let us discuss the general situation.

Let us represent the common sector vectors by $V_i$ and the non-abelian $V$ vectors by $V_{non-abelian}$. If the non-abelian group is represented by a n-dimensional representation, then the elements in the corresponding n-dimensional blocks of $V_i$ are either all $-\frac{1}{2}$ or all 0. The more the number of blocks with the same elements, the larger the higher level group. The corresponding n dimensional blocks in the $V_{non-abelian}$ can be written such that consecutive elements differ by $\theta$ ( If the non-abelian group is $ab = e^{i2\theta}ba$).

We will now show that, N=1 supersymmetric string models would end up with adjoint scalars only if the gauge bosons and the corresponding scalars come from a sector constructed out of the common $V_i$'s.

To see this, let us first consider the case in which gauge bosons are actually produced from the sectors constructed out of common $V_i$'s. Then it should be noted that the left movers of the adjoint gauge bosons contribute in the same way to the $V_i \cdot N$ portion of the GSO constraints. In an abelian orbifold some scalar excitations corresponding to gauge boson excitations get projected out if the string model is N=1 supersymmetric. The way non-abelian orbifolds get rid of this problem is as follows.

If the n dimensional blocks contribution to the $V_i$'s are of the form $[...(-1/2, -1/2...)_n(-1/2, -1/2...)_n(-1/2, -1/2...)_n...]$, then any sector constructed out of $V_i$'s would have a form $[...(-1/2, -1/2...)_n(-1/2, -1/2...)_n(-1/2, -1/2...)_n...]$ or $[...(0, 0...)_n(0, 0...)_n(0, 0...)_n...]$ . $V_{non-abelian}$ can be written in the form $[(-p/q,$-$-p/q + \theta, ....)_n, (-p/q, -p/q + \theta, ....)_n...]$. Hence if gauge bosons and corresponding scalars are constructed out of a sector made up of the $V_i$'s, then a number vector $N$ given by $[(1, 0..)_n(-1, 0, ...)_n]$ will contribute in the same way, to the GSO constraints for the $V_i$ vectors, as any number vector constructed with the 1's or -1's permuted within the respective n-dimensional blocks. However this will not be the case for the GSO constraints for the $V_{non-abelian}$ vector as can be easily seen.

For gauge bosons excitations for which the corresponding scalar excitation $N$ vectors get projected out (which is the situation that arises in N=1 supersymmetric abelian models), a permutation of the $N$ vector is included (This happens if for one entry in the $V_{non-abelian}$ n-dimensional block, there is another entry which differs from this entry by $\frac{1}{2}$ ). This is because a **specific** permutation of the $N$ vector (one that corresponds to two fermionic excitations of the **0** sector which cause a difference of $\frac{1}{2}$ among the corresponding elements of $V_{non-abelian}$) will satisfy all the GSO constraints even if the $N$ vector does not satisfy all the GSO constraints. Since the non-abelian invariant charge operators have a form $\sum_{i=1,n} \bar{\psi}_i \psi_i$, the $N$ vector and its permuted counterpart will evaluate similarly under a projected charge vector. This implies that the projected string model will have adjoint scalars.

If gauge bosons are constructed out of a sector containing $V_{non-abelian}$ vector, then finding permuted $N$ vectors which would produce a state analogous to a projected gauge boson, under invariance of the entire non-abelian group as talked about in the previous paragraph, would not be a possibility (one reason for this is the absence of two equal elements in an n-dimensional block). This would result in the absence of scalar adjoints.

If a sector with eight left movers being $-1/2$ and the other fourteen being 0 is used to construct gauge bosons, there is a possibility for the production of degenerate gauge bosons, as was talked in the case of $D_4$ orbifolds. Hence, the only sector that could produce gauge bosons contributing to the higher level gauge group, has to be the **0** sector. This just implies that only higher level $SU$ or $SO$ groups are possible through a non-abelian orbifold construction.

### 2.5.4. Absence of asymmetry between chiral and chiral conjugate fermions in interesting models

A n-dimensional block for $V_{non-abelian}$ can be written as $[-\frac{p}{qn}, -\frac{p}{qn} + \frac{l}{n}, -\frac{p}{qn} + \frac{2l}{n}, \ldots -\frac{p}{qn} + \frac{l(n-1)}{n}]$, where $p < q$. The first element is $-\frac{p}{qn}$, because both $a^n$ and $b^n$ are diagonal. If any element of the above n-dimensional block is greater in value than $\frac{1}{2}$, then the element should be subtracted by as many multiples of 1, until the element starts to lie in the interval $\in [-\frac{1}{2}, \frac{1}{2})$. Once we are done with this procedure, we should ensure that none of the elements should equal each other.

If, the vector $V_{non-abelian}$, could actually produce a fermion then, the sum of all the elements in the n-dimensional block would correspond to a charge eigenvalue of the invariant charge vector $\sum_{i=1}^{i=n} \bar{\psi}_i \psi_i$, where none of the fermions corresponding to the n-dimensional block are excited. This sum equals $(-\frac{p}{q} + \frac{l(n-1)}{2})$ as can be easily checked. In order to get a charge $\mp\frac{1}{2}$ (corresponding to interesting representations of SO and SU groups), $\frac{p}{q}$ should equal $\frac{1}{2}$ or 1.

If $\frac{p}{q}$ equal $\frac{1}{2}$, then the requirement that the difference between say $\frac{1}{2n}$ and $\frac{1}{2n} + \frac{l}{n}$ equal 1/2, would imply $\frac{l}{n}$ should be 1/2. This means that $n$ is an even number. Now let us see what this says about the sum of elements of the n-dimensional block. This sum equals $(-\frac{p}{q} + \frac{l(n-1)}{2})$. If n is even it says that in order for this sum to equal $\mp\frac{1}{2}$, $\frac{p}{q}$ should be an integer. Hence our initial assumption that $\frac{p}{q}$ equals $\frac{1}{2}$ was not right.

We have to end up with minimum rank four of a higher level gauge group. This forces the maximum dimension for a n-dimensional block to be five. Since $n$ is even it means, $n$ can only be 2 or 4.

$n = 2$ gives us a two dimensional representation with commutation relations of the form $ab = -ba$. Also $a^2$ and $b^2$ are diagonal. This implies that $D_4$ is a subgroup of such a non-abelian group. We have already shown that this does not give chiral fermions.

If $n = 4$ then $l$ can only take values 1 and 3. This is because $l = 2$ would produce two equal elements in the four dimensional block, which is not allowed as explained before. $l = 1$ and $l = 3$ produce the following block, $(-\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4})$.

If a $V$ vector containing this block produces fermions, then there are two options possible. Either the left movers vacuum energy is zero, or it is not. Since interesting gauge groups would only be possible if the number of n-dimensional blocks equals four or more, we see that a zero left mover vacuum energy would be impossible unless there is only one non-zero entry $-\frac{1}{2}$ in the four dimensional block. This contradicts the obtained form for the four dimensional block.

The other case is when the left mover vacuum energy is non-zero. In this case if states are constructed from the sector made up of $V_{non-abelian}$, then these states should involve at least one n-dimensional block where a non- $-\frac{1}{2}$ is excited. Since the $-\frac{1}{2}$ excitations are of zero energy, there are also states possible where, two fermions (one $-\frac{1}{2}$ excitation and one non- $-\frac{1}{2}$ excitations) from a n-dimensional block are excited. This would produce charge vectors which are not of the form $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$. Hence these states don't produce the required states required for chiral matter.

## 2.6. Discussion

In this chapter we reviewed the general formalism for construction of non-abelian orbifolds. Acting on a state coming from a particular sector with elements of the non-abelian group generally produces a state from a different sector. This is the crucial difference between abelian and non-abelian orbifolds. This property of non-abelian orbifolds gives rise to extra GSO constraints which were presented.

We illustrated the formalism for non-abelian orbifold model construction, by presenting the special case of $D_4$ non-abelian orbifolds. The formalism involved constructing maximal abelian subgroups of $D_4$. The states coming from the sectors of each maximal abelian subgroup (domain) had to obey the GSO constraint in that domain. The states also had to obey the extra GSO constraints which arose because of the non-abelian nature of $D_4$. Charge vectors had to be constructed in each domain which were invariant under the entire $D_4$ group. $D_4$ invariant states were also constructed in each domain. If a state arose from a sector which was common to more than one domain, then the corresponding $D_4$ invariant state had to be present in those domains, to be considered as a state of the theory. Else, it was projected out.

We showed how $D_4$ orbifolds realize N=1 SUSY with scalar adjoints. Some results obtained through an analysis of constructed models were also presented. The difficulties in producing a sizeable number of these models were also discussed. It was noted that the results don't reflect the state of affairs on the whole $D_4$ landscape.

However, it should also be noted that these results still represent the analysis on the largest data set of non-abelian orbifold models ever constructed.

We showed that $D_4$ constructions don't give chiral fermions and hence are not suitable for phenomenological studies. We concluded by outlining a proof for the absence of chiral fermions in GUT models which could have been interesting from a phenomenological viewpoint, obtained through non-abelian orbifold constructions.

## 2.7. Appendix: Actual models

Below we present a few $D_4$ models which were actually constructed by us.

### 2.7.1. Model 1:

The $V$ Vectors and the $k_{ij}$ are below. First the $V$ vectors of the $r$ domain are presented. $V_0$, $V_1$, $V_2$,$V_r$,$V_{\theta^2}$ are presented in the respective order, followed by the $k_{ij}$ for the r domain. Next the $V$ vectors for the $r\theta$ domain are presented. $V_0$, $V_1$, $V_2$,$V_{r\theta}$,$V_{\theta^2}$ are presented in the respective order, followed by the $k_{ij}$ for the $r\theta$ domain and finally, the $V$ vectors for the $\theta$ domain are presented. $V_0$, $V_1$, $V_2$,$V_\theta$ are presented in the respective order, followed by the $k_{ij}$ for the $\theta$ domain

The $V$ vectors and the $k_{ij}$ are mod 4 below.

```
( -2 -2 -2 -2 -2 -2 -2 -2 -2 -2    -2 -2 -2 -2 -2 -2 -2 -2 -2
                                   -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 )

( -2 -2  0  0 -2  0  0 -2  0  0     0  0  0  0  0  0  0  0  0
                                    0  0  0  0  0  0  0  0  0  0  0  0  0 )

(-2  0 -2  0 -2  0  0  0  0 -2    -2 -2 -2 -2  0  0 -2 -2 -2
                                    -2  0  0 -2 -2 -2 -2 -2 -2  0 -2 -2  0 )

( 0  0  0  0 -2 -2  0 -2  0 -2    -2  0 -2  0 -2  0 -2  0 -2
                                     0 -2  0 -2  0 -2  0  0  0  0  0  0  0 )

( 0  0  0  0  0  0  0  0  0  0    -2 -2 -2 -2 -2 -2 -2 -2 -2
                                    -2 -2 -2 -2 -2 -2 -2  0  0  0  0  0  0 )


  0 -2 -2 -2  0
 -2 -2 -2 -2  0
 -2  0 -2 -2  0
 -2  0 -2  0 -2
  0  0  0 -2  0

( -2 -2 -2 -2 -2 -2 -2 -2 -2 -2    -2 -2 -2 -2 -2 -2 -2 -2 -2 -2
                                    -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 )

( -2 -2  0  0 -2  0  0 -2  0  0     0  0  0  0  0  0  0  0  0  0
                                     0  0  0  0  0  0  0  0  0  0  0  0 )

( -2  0 -2  0 -2  0  0  0  0 -2    -2 -2 -2 -2  0  0 -2 -2 -2 -2
                                     0  0 -2 -2 -2 -2 -2 -2  0 -2 -2  0 )

(  0  0 -2 -2  0  0  0  0 -2 -2    -2  0 -2  0 -2  0 -2  0 -2  0
```

```
                                     -2  0 -2  0 -2  0 -2 -2 -2  0  0 -2 )

(  0  0  0  0  0  0  0  0  0  0    -2 -2 -2 -2 -2 -2 -2 -2 -2 -2
                                     -2 -2 -2 -2 -2 -2  0  0  0  0  0  0 )

  0 -2 -2 -2  0
 -2 -2 -2  0  0
 -2  0 -2 -2  0
 -2  0  0 -2 -2
  0  0  0 -2  0

( -2 -2 -2 -2 -2 -2 -2 -2 -2 -2   -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2
                                     -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 )

( -2 -2  0  0 -2  0  0 -2  0  0     0  0  0  0  0  0  0  0  0  0  0
                                       0  0  0  0  0  0  0  0  0  0  0 )

( -2  0 -2  0 -2  0  0  0  0 -2    -2 -2 -2 -2  0  0 -2 -2 -2 -2  0
                                       0 -2 -2 -2 -2 -2 -2  0 -2 -2  0 )

(  0  0 -2 -2 -2 -2  0 -2 -2  0     1 -1  1 -1  1 -1  1 -1  1 -1  1
                                     -1  1 -1  1 -1 -2 -2 -2  0  0 -2 )

  0 -2 -2 -2
 -2 -2 -2 -2
 -2  0 -2 -2
  0  0 -2  1
```

The spectrum for the above model is below. The scalars are just superpartners of the fermionic representations.

```
1 No of supersymmetries

[SO(12)_2] x [SU(2)_1] x [SU(2)_1] x [SU(2)_1] x [SU(2)_1] x [SU(2)_1]
x [SU(2)_1] x [SU(2)_2] x [SU(2)_2]

14 g.nss
0 g.n_u1

90 dimension predicted
76 number of gauge bosons

35 Fermion irreps:
66    -2-2 0 0-2 0 0-2 0 0   0 1 0 0 0 0    0    0    0    0    0    0    0    0
66    -2 2 0 0-2 0 0 2 0 0   0 1 0 0 0 0    0    0    0    0    0    0    0    0
48    -2 2 0 0 2 0 0-2 0 0   1 0 0 0 0 0    0    0    0    0    0    0    1    1
48    -2-2 0 0 2 0 0 2 0 0   1 0 0 0 0 0    0    0    0    0    0    0    1    1
24    -2 0 2 0 0 0 2-2 0 0   1 0 0 0 0 0    0    0    0    0    0    1    0    0
24    -2 0-2 0 0 0 2-2 0 0   1 0 0 0 0 0    0    0    0    0    1    0    0    0
24    -2-2 0 0 0 0-2 0 0-2   1 0 0 0 0 0    0    1    0    0    0    0    0    0
24    -2-2 0 0 0 0-2 0 0 2   1 0 0 0 0 0    1    0    0    0    0    0    0    0
64    -2-2 0 0 0 2 0 0 0 2   0 0 0 0 1 0    0    0    0    0    0    0    0    1
64    -2-2 0 0 0 2 0 0 0 2   0 0 0 0 0 1    0    0    0    0    0    0    0    1
64    -2-2 0 0 0-2 0 0 0-2   0 0 0 0 0 1    0    0    0    0    0    0    1    0
64    -2-2 0 0 0-2 0 0 0-2   0 0 0 0 1 0    0    0    0    0    0    0    1    0
12    -2 0 2 0 2 0 0 0 0 2   0 0 0 0 0 0    1    0    0    0    0    1    0    2
12    -2 0-2 0 2 0 0 0 0 2   0 0 0 0 0 0    1    0    0    0    1    0    0    2
 3    -2-2 0 0-2 0 0-2 0 0   0 0 0 0 0 0    0    0    0    0    0    0    0    2
 3    -2 2 0 0-2 0 0 2 0 0   0 0 0 0 0 0    0    0    0    0    0    0    0    2
```

```
 8    -2 0-2 0 0 0-2-2 0 0   0 0 0 0 0 0    0    0    0    0    0    1    1    1
 8    -2 0 2 0 0 0-2-2 0 0   0 0 0 0 0 0    0    0    0    0    1    0    1    1
 8    -2-2 0 0 0 0 2 0 0 2   0 0 0 0 0 0    0    1    0    0    0    0    1    1
 8    -2-2 0 0 0 0 2 0 0-2   0 0 0 0 0 0    1    0    0    0    0    0    1    1
12    -2 0 2 0 2 0 0 0 0-2   0 0 0 0 0 0    0    1    0    0    0    1    2    0
12    -2 0-2 0 2 0 0 0 0-2   0 0 0 0 0 0    0    1    0    0    1    0    2    0
 3    -2-2 0 0-2 0 0-2 0 0   0 0 0 0 0 0    0    0    0    0    0    0    2    0
 3    -2 2 0 0-2 0 0 2 0 0   0 0 0 0 0 0    0    0    0    0    0    0    2    0
 3    -2 2 0 0-2 0 0 2 0 0   0 0 0 0 0 0    0    0    0    0    0    2    0    0
16    -2 2 0 0 2 0 0-2 0 0   0 0 0 0 0 0    0    0    1    1    1    1    0    0
 3    -2 2 0 0-2 0 0 2 0 0   0 0 0 0 0 0    0    0    0    0    2    0    0    0
 4    -2 0-2 0 2 0 0 0 0 2   0 0 0 0 0 0    0    1    0    0    0    1    0    0
 4    -2 0-2 0 2 0 0 0 0-2   0 0 0 0 0 0    1    0    0    0    0    1    0    0
 4    -2 0 2 0 2 0 0 0 0 2   0 0 0 0 0 0    0    1    0    0    1    0    0    0
 4    -2 0 2 0 2 0 0 0 0-2   0 0 0 0 0 0    1    0    0    0    1    0    0    0
 3    -2 2 0 0-2 0 0 2 0 0   0 0 0 0 0 0    0    0    0    2    0    0    0    0
 3    -2 2 0 0-2 0 0 2 0 0   0 0 0 0 0 0    0    0    2    0    0    0    0    0
 3    -2 2 0 0-2 0 0 2 0 0   0 0 0 0 0 0    0    2    0    0    0    0    0    0
 3    -2 2 0 0-2 0 0 2 0 0   0 0 0 0 0 0    2    0    0    0    0    0    0    0
```

### 2.7.2. Model 2:

The $V$ Vectors and the $k_{ij}$ are below. First the $V$ vectors of the $r$ domain are presented. $V_0$, $V_1$, $V_2$,$V_r$,$V_{\theta^2}$ are presented in the respective order, followed by the $k_{ij}$ for the r domain. Next the $V$ vectors for the $r\theta$ domain are presented. $V_0$, $V_1$, $V_2$,$V_{r\theta}$,$V_{\theta^2}$ are presented in the respective order, followed by the $k_{ij}$ for the $r\theta$ domain and finally, the $V$ vectors for the $\theta$ domain are presented. $V_0$, $V_1$, $V_2$,$V_\theta$ are presented in the respective order, followed by the $k_{ij}$ for the $\theta$ domain

The $V$ vectors and the $k_{ij}$ are mod 4 below.

```
( -2 -2 -2 -2 -2 -2 -2 -2 -2 -2    -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2
                                     -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 )

( -2 -2  0  0 -2  0  0 -2  0  0     0  0  0  0  0  0  0  0  0  0  0
                                     0  0  0  0  0  0  0  0  0  0  0 )

( -2 -2 -2 -2  0  0 -2  0 -2  0    -2 -2  0  0  0  0  0  0  0  0 -2
                                     -2  -2 -2 -2 -2 -2  0  0  0  0 -2 )

(  0 -2  0 -2 -2 -2  0  0  0  0    -2  0 -2  0 -2  0 -2  0 -2  0 -2
                                    0 -2  0 -2  0  0  0  0  0  0  0 )

(  0  0  0  0  0  0  0  0  0  0    -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2
                                    -2 -2 -2 -2 -2  0  0  0  0  0  0 )

  0  0 -2 -2  0
  0  0  0 -2  0
 -2 -2 -2 -2  0
 -2  0  0  0 -2
  0  0  0 -2  0

 (-2 -2 -2 -2 -2 -2 -2 -2 -2 -2    -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2
                                    -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 )
```

```
 (-2 -2  0  0 -2  0  0 -2  0  0     0  0  0  0  0  0  0  0  0  0  0  0
                                     0  0  0  0  0  0  0  0  0  0 )

 (-2 -2 -2 -2  0  0 -2  0 -2  0    -2 -2  0  0  0  0  0  0  0  0 -2 -2
                                    -2 -2 -2 -2 -2  0  0  0  0 -2 )

 ( 0  0  0  0  0 -2 -2  0 -2 -2    -2  0 -2  0 -2  0 -2  0 -2  0 -2  0
                                    -2  0 -2  0 -2  0 -2 -2  0 -2 )

 ( 0  0  0  0  0  0  0  0  0  0    -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2
                                    -2 -2 -2 -2  0  0  0  0  0  0 )

  0  0 -2 -2  0
  0  0  0  0  0
 -2 -2 -2 -2  0
 -2  0 -2 -2 -2
  0  0  0 -2  0

( -2 -2 -2 -2 -2 -2 -2 -2 -2 -2    -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2
                                    -2   -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 )

( -2 -2  0  0 -2  0  0 -2  0  0     0  0  0  0  0  0  0  0  0  0  0
                                     0  0  0  0  0  0  0  0  0  0  0 )

( -2 -2 -2 -2  0  0 -2  0 -2  0    -2 -2  0  0  0  0  0  0  0  0 -2
                                    -2 -2 -2 -2 -2 -2  0  0  0  0 -2 )

( 0 -2  0 -2 -2  0 -2  0 -2 -2     1 -1  1 -1  1 -1  1 -1  1 -1  1
                                    -1  1 -1  1 -1 -2  0 -2 -2  0 -2 )

  0  0 -2 -2
  0  0  0 -2
 -2 -2 -2 -2
  0  0  0  1
```

The spectrum for the above model is below. The scalars are just superpartners of the fermionic representations.

```
1 No of supersymmetries

[SO(8)_2] x [SO(8)_2] x [SU(2)_1] x [SU(2)_1] x [SU(2)_1] x [SU(2)_1] x
[SU(2)_1] x [SU(2)_1]

14 g.nss
0 g.n_u1

74 dimension predicted
60 number of gauge bosons

35 Fermion irreps:
64    -2-2 0 0-2 0 0 2 0 0     0 0 1 0    0 0 1 0    0    0    0    0    0    0
64    -2-2 0 0 2 0 0-2 0 0     0 0 1 0    0 0 1 0    0    0    0    0    0    0
64    -2-2 0 0 2 0 0-2 0 0     0 0 0 1    0 0 0 1    0    0    0    0    0    0
64    -2-2 0 0-2 0 0 2 0 0     0 0 0 1    0 0 0 1    0    0    0    0    0    0
28    -2 2 0 0 2 0 0 2 0 0     0 1 0 0    0 0 0 0    0    0    0    0    0    0
28    -2 2 0 0-2 0 0-2 0 0     0 1 0 0    0 0 0 0    0    0    0    0    0    0
16    -2 0 2 0 0-2 0 2 0 0     0 0 0 1    0 0 0 0    0    0    0    1    0    0
```

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 16 | -2 0 2 0 0-2 0 2 0 0 | 0 0 1 0 | 0 0 0 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 16 | -2 0 2 0 0 2 0 2 0 0 | 0 0 0 1 | 0 0 0 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 16 | -2 0 2 0 0 2 0 2 0 0 | 0 0 1 0 | 0 0 0 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 16 | -2 0 2 0 2 0 0 0 0 2 | 0 0 0 1 | 0 0 0 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 16 | -2 0 2 0 2 0 0 0 0 2 | 0 0 1 0 | 0 0 0 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 16 | -2 0 2 0 2 0 0 0 0-2 | 0 0 0 1 | 0 0 0 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 16 | -2 0 2 0 2 0 0 0 0-2 | 0 0 1 0 | 0 0 0 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 64 | -2 0 0 2 0 2 0 2 0 0 | 1 0 0 0 | 0 0 1 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 64 | -2 0 0 2 0 2 0 2 0 0 | 1 0 0 0 | 0 0 0 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 64 | -2 0 0-2 0-2 0 2 0 0 | 0 0 0 1 | 1 0 0 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 64 | -2 0 0-2 0-2 0 2 0 0 | 0 0 1 0 | 1 0 0 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 28 | -2 2 0 0 2 0 0 2 0 0 | 0 0 0 0 | 0 1 0 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 28 | -2 2 0 0-2 0 0-2 0 0 | 0 0 0 0 | 0 1 0 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | -2 0-2 0 0 2 0 2 0 0 | 0 0 0 0 | 0 0 1 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 16 | -2 0-2 0 0 2 0 2 0 0 | 0 0 0 0 | 0 0 0 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| 16 | -2 0-2 0 0-2 0 2 0 0 | 0 0 0 0 | 0 0 1 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 16 | -2 0-2 0 0-2 0 2 0 0 | 0 0 0 0 | 0 0 0 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| 16 | -2 0-2 0 2 0 0 0 0-2 | 0 0 0 0 | 0 0 1 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 16 | -2 0-2 0 2 0 0 0 0-2 | 0 0 0 0 | 0 0 0 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| 16 | -2 0-2 0 2 0 0 0 0 2 | 0 0 0 0 | 0 0 1 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 16 | -2 0-2 0 2 0 0 0 0 2 | 0 0 0 0 | 0 0 0 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 16 | -2-2 0 0 2 0 0-2 0 0 | 0 0 0 0 | 0 0 0 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 3 | -2 2 0 0-2 0 0-2 0 0 | 0 0 0 0 | 0 0 0 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| 3 | -2 2 0 0-2 0 0-2 0 0 | 0 0 0 0 | 0 0 0 0 | 0 | 0 | 0 | 0 | 2 | 0 |
| 3 | -2 2 0 0-2 0 0-2 0 0 | 0 0 0 0 | 0 0 0 0 | 0 | 0 | 0 | 2 | 0 | 0 |
| 3 | -2 2 0 0-2 0 0-2 0 0 | 0 0 0 0 | 0 0 0 0 | 0 | 0 | 2 | 0 | 0 | 0 |
| 3 | -2 2 0 0-2 0 0-2 0 0 | 0 0 0 0 | 0 0 0 0 | 0 | 2 | 0 | 0 | 0 | 0 |
| 3 | -2 2 0 0-2 0 0-2 0 0 | 0 0 0 0 | 0 0 0 0 | 2 | 0 | 0 | 0 | 0 | 0 |

### 2.7.3. Model 3:

The $V$ Vectors and the $k_{ij}$ are below. First the $V$ vectors of the $r$ domain are presented. $V_0$, $V_1$, $V_2$,$V_r$,$V_{\theta^2}$ are presented in the respective order, followed by the $k_{ij}$ for the r domain. Next the $V$ vectors for the $r\theta$ domain are presented. $V_0$, $V_1$, $V_2$,$V_{r\theta}$,$V_{\theta^2}$ are presented in the respective order, followed by the $k_{ij}$ for the $r\theta$ domain and finally, the $V$ vectors for the $\theta$ domain are presented. $V_0$, $V_1$, $V_2$,$V_\theta$ are presented in the respective order, followed by the $k_{ij}$ for the $\theta$ domain

The $V$ vectors and the $k_{ij}$ are mod 4 below.

```
( -2 -2 -2 -2 -2 -2 -2 -2 -2 -2    -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2
                                      -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 )

( -2 -2  0  0 -2  0  0 -2  0  0     0  0  0  0  0  0  0  0  0  0  0
                                       0  0  0  0  0  0  0  0  0  0  0 )

( -2 -2 -2 -2 -2 -2 -2 -2 -2 -2    -2 -2  0  0 -2 -2 -2 -2  0  0 -2
                                      -2 -2 -2 -2 -2 -2  0 -2  0  0  0 )

(  0 -2 -2  0 -2 -2  0  0  0  0    -2  0 -2  0 -2  0 -2  0 -2  0 -2
                                       0 -2  0 -2  0  0  0  0  0  0  0 )

(  0  0  0  0  0  0  0  0  0  0    -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2
                                      -2 -2 -2 -2 -2  0  0  0  0  0  0 )

  0 -2 -2 -2  0
```

```
 -2 -2 -2 -2  0
 -2 -2 -2 -2  0
 -2  0  0  0 -2
  0  0  0 -2  0

( -2 -2 -2 -2 -2 -2 -2 -2 -2 -2    -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2
                                     -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 )

( -2 -2  0  0 -2  0  0 -2  0  0     0  0  0  0  0  0  0  0  0  0  0
                                      0  0  0  0  0  0  0  0  0  0  0 )

( -2 -2 -2 -2 -2 -2 -2 -2 -2 -2    -2 -2  0  0 -2 -2 -2 -2  0  0 -2
                                     -2 -2 -2 -2 -2 -2  0 -2  0  0  0 )

(  0  0  0  0 -2  0 -2 -2  0 -2    -2  0 -2  0 -2  0 -2  0 -2  0 -2
                                      0 -2  0 -2  0  0 -2  0 -2 -2 -2 )

(  0  0  0  0  0  0  0  0  0  0    -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2
                                     -2 -2 -2 -2 -2  0  0  0  0  0  0 )

  0 -2 -2  0  0
 -2 -2 -2  0  0
 -2 -2 -2 -2  0
  0 -2  0  0 -2
  0  0  0 -2  0

( -2 -2 -2 -2 -2 -2 -2 -2 -2 -2    -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2
                                     -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 )

( -2 -2  0  0 -2  0  0 -2  0  0     0  0  0  0  0  0  0  0  0  0  0
                                      0  0  0  0  0  0  0  0  0  0  0 )

( -2 -2 -2 -2 -2 -2 -2 -2 -2 -2    -2 -2  0  0 -2 -2 -2 -2  0  0 -2
                                     -2 -2 -2 -2 -2 -2  0 -2  0  0  0 )

(  0 -2 -2  0  0 -2 -2 -2  0 -2     1 -1  1 -1  1 -1  1 -1  1 -1  1
                                     -1  1 -1  1 -1  0 -2  0 -2 -2 -2 )

  0 -2 -2  0
 -2 -2 -2 -2
 -2 -2 -2 -2
 -2  0  0 -1
```

The spectrum for the above model is below. The scalars are just superpartners of the fermionic representations.

```
1 No of supersymmetries

[SO(12)_2] x [SO(11)_1] x [SU(2)_1] x [SU(2)_1] x [SU(2)_2]

14 g.nss
0 g.n_u1

130 dimension predicted
116 number of gauge bosons

25 Fermion irreps:
77    -2 2 0 0 2 0 0-2 0 0   2 0 0 0 0 0    0 0 0 0 0    0    0    0
66    -2-2 0 0-2 0 0-2 0 0   0 1 0 0 0 0    0 0 0 0 0    0    0    0
66    -2 2 0 0-2 0 0 2 0 0   0 1 0 0 0 0    0 0 0 0 0    0    0    0
```

| | | | | | | |
|---|---|---|---|---|---|---|
| 66 | -2-2 0 0 2 0 0 2 0 0 | 0 1 0 0 0 0 | 0 0 0 0 0 | 0 | 0 | 0 |
| 24 | -2 0 0-2-2 0 0 0 2 0 | 1 0 0 0 0 0 | 0 0 0 0 0 | 0 | 1 | 0 |
| 24 | -2 0 0 2-2 0 0 0-2 0 | 1 0 0 0 0 0 | 0 0 0 0 0 | 0 | 1 | 0 |
| 24 | -2 0 0-2-2 0 0 0-2 0 | 1 0 0 0 0 0 | 0 0 0 0 0 | 1 | 0 | 0 |
| 24 | -2 0 0 2-2 0 0 0 2 0 | 1 0 0 0 0 0 | 0 0 0 0 0 | 1 | 0 | 0 |
| 64 | -2 0-2 0 0-2 0-2 0 0 | 0 0 0 0 0 1 | 0 0 0 0 0 | 0 | 0 | 1 |
| 64 | -2 0 2 0 0 2 0-2 0 0 | 0 0 0 0 1 0 | 0 0 0 0 0 | 0 | 0 | 1 |
| 64 | -2 0 2 0 0 2 0-2 0 0 | 0 0 0 0 0 1 | 0 0 0 0 0 | 0 | 0 | 1 |
| 64 | -2 0-2 0 0-2 0-2 0 0 | 0 0 0 0 1 0 | 0 0 0 0 0 | 0 | 0 | 1 |
| 64 | -2 2 0 0 0 0 2 0 0-2 | 0 0 0 0 1 0 | 0 0 0 0 0 | 0 | 0 | 1 |
| 64 | -2 2 0 0 0 0-2 0 0 2 | 0 0 0 0 1 0 | 0 0 0 0 0 | 0 | 0 | 1 |
| 64 | -2 2 0 0 0 0-2 0 0 2 | 0 0 0 0 0 1 | 0 0 0 0 0 | 0 | 0 | 1 |
| 64 | -2 2 0 0 0 0 2 0 0-2 | 0 0 0 0 0 1 | 0 0 0 0 0 | 0 | 0 | 1 |
| 33 | -2 2 0 0 2 0 0-2 0 0 | 0 0 0 0 0 0 | 0 0 0 0 1 | 0 | 0 | 2 |
| 55 | -2-2 0 0 2 0 0 2 0 0 | 0 0 0 0 0 0 | 0 0 0 1 0 | 0 | 0 | 0 |
| 11 | -2-2 0 0-2 0 0-2 0 0 | 0 0 0 0 0 0 | 0 0 0 0 1 | 0 | 0 | 0 |
| 11 | -2 2 0 0-2 0 0 2 0 0 | 0 0 0 0 0 0 | 0 0 0 0 1 | 0 | 0 | 0 |
| 3 | -2-2 0 0-2 0 0-2 0 0 | 0 0 0 0 0 0 | 0 0 0 0 0 | 0 | 0 | 2 |
| 3 | -2 2 0 0-2 0 0 2 0 0 | 0 0 0 0 0 0 | 0 0 0 0 0 | 0 | 0 | 2 |
| 3 | -2-2 0 0 2 0 0 2 0 0 | 0 0 0 0 0 0 | 0 0 0 0 0 | 0 | 0 | 2 |
| 3 | -2-2 0 0 2 0 0 2 0 0 | 0 0 0 0 0 0 | 0 0 0 0 0 | 0 | 2 | 0 |
| 3 | -2-2 0 0 2 0 0 2 0 0 | 0 0 0 0 0 0 | 0 0 0 0 0 | 2 | 0 | 0 |

### 2.7.4. Model 4:

The $V$ Vectors and the $k_{ij}$ are below. First the $V$ vectors of the $r$ domain are presented. $V_0$, $V_1$, $V_2$,$V_r$,$V_{\theta^2}$ are presented in the respective order, followed by the $k_{ij}$ for the r domain. Next the $V$ vectors for the $r\theta$ domain are presented. $V_0$, $V_1$, $V_2$,$V_{r\theta}$,$V_{\theta^2}$ are presented in the respective order, followed by the $k_{ij}$ for the $r\theta$ domain and finally, the $V$ vectors for the $\theta$ domain are presented. $V_0$, $V_1$, $V_2$,$V_\theta$ are presented in the respective order, followed by the $k_{ij}$ for the $\theta$ domain

The $V$ vectors and the $k_{ij}$ are mod 4 below.

```
( -2 -2 -2 -2 -2 -2 -2 -2 -2 -2    -2 -2 -2 -2 -2 -2 -2 -2 -2
                                      -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 )

( -2 -2  0  0 -2  0  0 -2  0  0     0  0  0  0  0  0  0  0  0
                                      0  0  0  0  0  0  0  0  0  0  0  0  0 )

( -2 -2  0  0  0 -2  0  0 -2  0    -2 -2 -2 -2 -2 -2  0  0 -2
                                     -2  0  0 -2 -2 -2 -2 -2 -2  0  0 -2 -2 )

( -2 -2  0  0 -2 -2 -2 -2 -2 -2    -2  0 -2  0 -2  0 -2  0 -2
                                      0 -2  0 -2  0 -2  0  0  0  0  0  0  0 )

(  0  0  0  0  0  0  0  0  0  0    -2 -2 -2 -2 -2 -2 -2 -2 -2
                                     -2 -2 -2 -2 -2 -2 -2  0  0  0  0  0  0 )

  0 -2 -2 -2  0
 -2 -2 -2 -2  0
 -2  0 -2 -2  0
 -2 -2  0  0 -2
  0  0  0 -2  0

( -2 -2 -2 -2 -2 -2 -2 -2 -2 -2    -2 -2 -2 -2 -2 -2 -2 -2 -2 -2
                                     -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 )
```

```
( -2 -2  0  0 -2  0  0 -2  0  0     0  0  0  0  0  0  0  0  0  0
                                      0  0  0  0  0  0  0  0  0  0  0  0 )

( -2 -2  0  0  0 -2  0  0 -2  0    -2 -2 -2 -2 -2 -2  0  0 -2 -2
                                      0  0 -2 -2 -2 -2 -2 -2  0  0 -2 -2 )

(  0 -2 -2  0 -2  0 -2  0 -2 -2    -2  0 -2  0 -2  0 -2  0 -2  0
                                     -2  0 -2  0 -2  0 -2 -2 -2 -2 -2 -2 )

(  0  0  0  0  0  0  0  0  0  0    -2 -2 -2 -2 -2 -2 -2 -2 -2 -2
                                     -2 -2 -2 -2 -2 -2  0  0  0  0  0  0 )


  0 -2 -2 -2  0
 -2 -2 -2  0  0
 -2  0 -2 -2  0
 -2 -2 -2 -2 -2
  0  0  0 -2  0

( -2 -2 -2 -2 -2 -2 -2 -2 -2 -2    -2 -2 -2 -2 -2 -2 -2 -2 -2 -2
                                     -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 )

( -2 -2  0  0 -2  0  0 -2  0  0     0  0  0  0  0  0  0  0  0  0
                                      0  0  0  0  0  0  0  0  0  0  0  0 )

( -2 -2  0  0  0 -2  0  0 -2  0    -2 -2 -2 -2 -2 -2  0  0 -2 -2
                                      0  0 -2 -2 -2 -2 -2 -2  0  0 -2 -2 )

( -2  0 -2  0  0 -2  0 -2  0  0     1 -1  1 -1  1 -1  1 -1  1 -1
                                     1 -1  1 -1  1 -1 -2 -2 -2 -2 -2 -2 )

  0 -2 -2 -2
 -2 -2 -2 -2
 -2  0 -2 -2
  0  0  0  1
```

The spectrum for the above model is below. The scalars are just superpartners of the fermionic representations.

```
1 No of supersymmetries

[SO(12)_2] x [SO(8)_1] x [SU(2)_1] x [SU(2)_1] x [SU(2)_2] x [SU(2)_2]

14 g.nss
0 g.n_u1

106 dimension predicted
92 number of gauge bosons

27 Fermion irreps:
66    -2-2 0 0-2 0 0-2 0 0   0 1 0 0 0 0    0 0 0 0    0    0    0    0
66    -2-2 0 0 2 0 0 2 0 0   0 1 0 0 0 0    0 0 0 0    0    0    0    0
48    -2 2 0 0 2 0 0-2 0 0   1 0 0 0 0 0    0 0 0 0    0    0    1    1
48    -2 2 0 0-2 0 0 2 0 0   1 0 0 0 0 0    0 0 0 0    0    0    1    1
24    -2 0-2 0 2 0 0 0 2 0   1 0 0 0 0 0    0 0 0 0    0    1    0    0
24    -2 0 2 0 2 0 0 0 2 0   1 0 0 0 0 0    0 0 0 0    1    0    0    0
64    -2 0 0-2 0-2 0 2 0 0   0 0 0 0 1 0    0 0 0 0    0    0    0    1
```

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 64 | -2 0 0 -2 0 -2 0 2 0 0 | 0 0 0 0 0 1 | 0 0 0 0 | 0 | 0 | 0 | 1 |
| 64 | -2 0 0 2 0 2 0 2 0 0 | 0 0 0 0 1 0 | 0 0 0 0 | 0 | 0 | 1 | 0 |
| 64 | -2 0 0 2 0 2 0 2 0 0 | 0 0 0 0 0 1 | 0 0 0 0 | 0 | 0 | 1 | 0 |
| 24 | -2 2 0 0 0 -2 0 0 -2 0 | 0 0 0 0 0 0 | 1 0 0 0 | 0 | 0 | 0 | 2 |
| 3 | -2 -2 0 0 -2 0 0 -2 0 0 | 0 0 0 0 0 0 | 0 0 0 0 | 0 | 0 | 0 | 2 |
| 3 | -2 -2 0 0 2 0 0 2 0 0 | 0 0 0 0 0 0 | 0 0 0 0 | 0 | 0 | 0 | 2 |
| 8 | -2 0 2 0 2 0 0 0 -2 0 | 0 0 0 0 0 0 | 0 0 0 0 | 0 | 1 | 1 | 1 |
| 8 | -2 0 -2 0 2 0 0 0 -2 0 | 0 0 0 0 0 0 | 0 0 0 0 | 1 | 0 | 1 | 1 |
| 24 | -2 2 0 0 0 2 0 0 -2 0 | 0 0 0 0 0 0 | 0 0 1 0 | 0 | 0 | 2 | 0 |
| 24 | -2 2 0 0 0 2 0 0 -2 0 | 0 0 0 0 0 0 | 0 0 0 1 | 0 | 0 | 2 | 0 |
| 3 | -2 -2 0 0 -2 0 0 -2 0 0 | 0 0 0 0 0 0 | 0 0 0 0 | 0 | 0 | 2 | 0 |
| 3 | -2 -2 0 0 2 0 0 2 0 0 | 0 0 0 0 0 0 | 0 0 0 0 | 0 | 0 | 2 | 0 |
| 28 | -2 -2 0 0 -2 0 0 -2 0 0 | 0 0 0 0 0 0 | 0 1 0 0 | 0 | 0 | 0 | 0 |
| 32 | -2 2 0 0 2 0 0 -2 0 0 | 0 0 0 0 0 0 | 0 0 0 1 | 1 | 1 | 0 | 0 |
| 32 | -2 2 0 0 2 0 0 -2 0 0 | 0 0 0 0 0 0 | 0 0 1 0 | 1 | 1 | 0 | 0 |
| 8 | -2 2 0 0 0 -2 0 0 2 0 | 0 0 0 0 0 0 | 0 0 1 0 | 0 | 0 | 0 | 0 |
| 8 | -2 2 0 0 0 -2 0 0 2 0 | 0 0 0 0 0 0 | 0 0 0 1 | 0 | 0 | 0 | 0 |
| 8 | -2 2 0 0 0 2 0 0 2 0 | 0 0 0 0 0 0 | 1 0 0 0 | 0 | 0 | 0 | 0 |
| 3 | -2 -2 0 0 -2 0 0 -2 0 0 | 0 0 0 0 0 0 | 0 0 0 0 | 0 | 2 | 0 | 0 |
| 3 | -2 -2 0 0 -2 0 0 -2 0 0 | 0 0 0 0 0 0 | 0 0 0 0 | 2 | 0 | 0 | 0 |

# 3. Supersymmetry versus gauge symmetry on the heterotic landscape

## 3.1. Introduction

Recent developments in string theory suggest that there exists a huge "landscape" of self-consistent string vacua [27]. The existence of this landscape is of critical importance for string phenomenology since the specific low-energy phenomenology that can be expected to emerge from string theory depends critically on the particular choice of vacuum state. Detailed quantities such as particle masses and mixings, and even more general quantities and structures such as the choice of gauge group, number of chiral particle generations, and the magnitude of the supersymmetry-breaking scale, can be expected to vary significantly from one vacuum solution to the next. Thus, in the absence of some sort of vacuum-selection principle, it is natural to determine whether there might exist generic string-derived *statistical correlations* between different phenomenological features that would otherwise be uncorrelated in field theory [28]. In this way, one can still hope to extract phenomenological predictions from string theory.

To date, there has been considerable work in this direction [28, 29, 30, 31, 32, 33, 54, 35, 36, 37, 38]; for recent reviews, see Ref. [39]. Collectively, this work addresses questions ranging from the formal (such as the finiteness of the number of string vacua and the methods by which they may be efficiently scanned and classified) to the phenomenological (such as the value of the cosmological constant, the scale of supersymmetry breaking, and the statistical prevalence of the Standard Model gauge group and three chiral generations).

In this chapter, we shall undertake a statistical study of the correlations between two phenomenological features which are likely to be central to any description of nature at high energy scales: spacetime supersymmetry and gauge symmetry. Indeed, over the past twenty years, a large amount of theoretical effort has been devoted to studying string models with $\mathcal{N}$=1 spacetime supersymmetry. However, it is important to understand the implications of choosing $\mathcal{N}$=1 supersymmetry over other classes of string models (such as models with $\mathcal{N}$=2 or $\mathcal{N}$=4 supersymmetry, or even non-supersymmetric string models) within the context of the landscape. Moreover, since $\mathcal{N}$=1 supersymmetry plays a huge role in current theoretical efforts to extend the Standard Model, we shall also be interested in understanding the statistical prevalence of spacetime supersymmetry across the landscape and the degree to which the presence or absence of supersymmetry affects other phenomenological features such as the choice of gauge group and the resulting particle spectrum.

In this chapter, we shall investigate such questions within the context of the heterotic string landscape. There are several reasons why we shall focus on the heterotic landscape. First, heterotic strings are of tremendous phenomenological interest in their own right; indeed, these strings the framework in which most of the original work in string phenomenology was performed in the late 1980's and early 1990's. Second, heterotic strings have internal constructions and self-consistency constraints which are, in many ways, more constrained than those of their Type I (open) counterparts. Thus, they are likely to exhibit phenomenological correlations which differ from those that might be observed on the landscape of, say, intersecting D-brane models or Type I flux vacua. Finally, in many cases these perturbative supersymmetric heterotic strings are dual to other strings (*e.g.*, Type I orientifold models) whose statistical properties are also being analyzed in the literature. Thus, analysis of the perturbative heterotic landscape, both supersymmetric and non-supersymmetric, might eventually enable *statistical* tests of duality symmetries across the entire string landscape.

The first statistical study of the heterotic landscape appeared in Ref. [54]. This study, which focused exclusively on the statistical properties of non-supersymmetric ($\mathcal{N}$=0) tachyon-free heterotic string vacua, was based on a relatively small data set of four-dimensional heterotic string models [40] which were randomly generated using software originally developed in Ref. [53]. Since then, there have been several additional statistical examinations of certain classes of $\mathcal{N}$=1 supersymmetric heterotic strings [36, 37]. Together, such studies can therefore be viewed as providing heterotic analogues of the Type I statistical studies reported in Refs. [31, 32, 33].

Although the study we shall undertake here is similar in spirit to that of Ref. [54], there are several important differences which must be highlighted. First, as discussed above, we shall be focusing here on the effects of spacetime supersymmetry. Thus, we shall be examining models with all levels of spacetime supersymmetry ($\mathcal{N}$=0, 1, 2, 4), not just non-supersymmetric models, and examining how the level of spacetime supersymmetry correlates with gauge symmetry. Second, the current study is based on a much larger data set consisting of approximately $10^7$ heterotic string models which was newly generated for this purpose using an update of the software originally developed in Ref. [53]. This data set is thus approximately two orders of magnitude larger than that used for Ref. [54], and represents literally the largest set of distinct heterotic string models ever constructed. Indeed, for reasons we shall discuss in Sect. 3.3, we believe that data sets of this approximate size are probably among the largest that can be generated using current computer technology.

But perhaps most importantly, because our heterotic-string data set was newly generated for the purpose of this study, we are able to quote results that take into account certain subtleties concerning so-called "floating correlations". As discussed in Ref. [35], the problem of floating correlations is endemic to investigations of this type, and reflects the fact that not all physically distinct string models are equally likely to be sampled in any random search through the landscape. This thereby causes statistical correlations to "float" as a function of sample size. In Ref. [35], several methods were developed that can be used to overcome this problem, and it was shown through explicit examples that these methods allow one to extract correlations and statistical distributions which are not only stable as a function of sample size, but which also differ significantly from those which would have been naïvely apparent from a direct counting of generated models. We shall therefore employ these techniques in the current chapter, extracting each of our statistical results in such a way that they represent stable correlations across the entire heterotic landscape we are examining.

As with most large-scale statistical studies of this type, there are several limitations which must be borne in mind. First, our sample size is relatively small, consisting of only $\sim 10^7$ distinct models. However, although this number is miniscule compared with the numbers of string models that are currently quoted in most landscape discussions, we believe that the statistical results we shall obtain are stable as a function of sample size and would not change significantly as more models are added to the data sample. We shall discuss this feature in more detail in Sect. 3.3. Indeed, as mentioned above, data samples of the current size are likely to be the largest possible given current computer technology.

Second, the analysis in this chapter shall be limited to correlations between only two phenomenological properties of these models: their low-energy gauge groups, and their levels of supersymmetry. More detailed examinations of the particle spectra of these models will be presented in Ref. [57].

Finally, the models we shall be discussing are stable only at tree level. For example, the models with spacetime supersymmetry continue to have flat directions which have not been lifted. Even worse, the non-supersymmetric models (even though tachyon-free) will generally have non-zero dilaton tadpoles and thus are not stable beyond tree level. Despite these facts, each of the string models we shall be studying represents a valid string solution at tree level, satisfying all of the necessary string self-consistency constraints. These include the requirements of worldsheet conformal/superconformal invariance, modular-invariant one-loop and multi-loop amplitudes, proper spacetime spin-statistics relations, and physically self-consistent layers of sequential GSO projections and orbifold twists. Thus, although such models may not represent the sorts of truly stable vacua that we would ideally like to be studying, it is reasonable to hope that any statistical correlations we uncover are likely to hold even after vacuum stabilization. Indeed, since no stable perturbative non-supersymmetric heterotic strings have yet been constructed, this sort of analysis is currently the state of the art for large-scale statistical studies of this type, and mirrors the situation on the Type I side, where state-of-the-art statistical analyses [31, 32, 33] have also focused on models which are only stable at tree level. Eventually, once the heterotic model-building technology develops further and truly stable vacua can be analyzed, it will be interesting to compare those results with these in order to ascertain the degree to which vacuum stabilization might affect these other phenomenological properties.

This chapter is organized as follows. In Sect. 3.2, we describe the class of models that we shall be examining in this chapter. In Sect. 3.3, we summarize our method of analysis which enables us to overcome the problem of floating correlations in order to extract statistically meaningful correlations. In Sect. 3.4, we present our results concerning the prevalence of spacetime supersymmetry across the heterotic landscape, and in Sect. 3.5 we present our results concerning correlations between spacetime supersymmetry and gauge groups. Finally, our conclusions are presented in Sect. 3.6.

## 3.2. The models

The models we shall be examining in this chapter are similar to those studied in Ref. [54]. Specifically, each of the vacua we shall be examining in this chapter represents a weakly coupled critical heterotic string compactified to four large (flat) spacetime dimensions. In general, such a string may be described in terms of its left- and right-moving worldsheet conformal field theories (CFT's). For a string in four dimensions, these must have central charges $(c_R, c_L) = (9, 22)$ in order to enforce worldsheet conformal anomaly cancellation, and must exhibit conformal invariance for the left-movers and superconformal invariance for the right-movers. While any such CFT's may be considered, in this chapter we shall focus on those string models for which these internal worldsheet CFT's may be taken to consist of tensor products of free, non-interacting, complex (chiral) bosonic or fermionic fields.

As discussed in Ref. [54], this is a huge class of models which has been discussed and analyzed in many different ways in the string literature. On the one hand, taking these worldsheet fields as fermionic leads to the so-called "free-fermionic" construction [43] which will be our primary tool throughout this chapter. In the language of this construction, different models are achieved by varying (or "twisting") the boundary conditions of these fermions around the two non-contractible loops of the worldsheet torus while simultaneously varying the phases according to which the contributions of each such spin-structure sector are summed in producing the one-loop partition function. However, alternative but equivalent languages for constructing such models exist. For example, we may bosonize these worldsheet fermions and construct "Narain" models [44, 45] in which the resulting complex worldsheet bosons are compactified on internal lattices of appropriate dimensionality with appropriate self-duality properties. Furthermore, many of these models have additional geometric realizations as orbifold compactifications with appropriately chosen Wilson lines; in general, the process of orbifolding is quite complicated in these models, involving many sequential layers of projections and twists. All of these constructions generally overlap to a large degree, and all are capable of producing models in which the corresponding gauge groups and particle contents are quite intricate. Nevertheless, in all cases, we must ensure that all required self-consistency constraints are satisfied. These include modular invariance, physically sensible GSO projections, proper spin-statistics identifications, and so forth. Thus, each of these vacua represents a fully self-consistent string solution at tree level.

In order to efficiently survey the space of such four-dimensional string-theoretic vacua, we implemented a computer search based on the free-fermionic spin-structure construction [43]. Details of this study are similar to those of the earlier study described in Ref. [54], and utilize an updated version of the model-generating software that was originally written for Ref. [53]. In our analysis, we restricted our attention to those models for which our real worldsheet fermions can always be uniformly paired to form complex fermions, and therefore it was possible to specify the boundary conditions (or spin-structures) of these real fermions in terms of the complex fermions directly. We also restricted our attention to cases in which the worldsheet fermions exhibited either antiperiodic (Neveu-Schwarz) or periodic (Ramond) boundary conditions around the non-contractible loops of the torus. Of course, in order to build a self-consistent string model in this framework, these boundary conditions must satisfy tight constraints. These constraints are necessary in order to ensure that the one-loop partition function is modular invariant and that the resulting Fock space of states can be interpreted as arising through a physically sensible projection from the space of all worldsheet states onto the subspace of physical states with proper spacetime spin-statistics. Thus, within a given string model, it is necessary to sum over appropriate sets of untwisted and twisted sectors with different boundary conditions and projection phases.

Our statistical analysis consisted of an examination of over $10^7$ distinct vacua in this class. Essentially, each set of fermion boundary conditions and GSO projection phases was chosen randomly in each sector, subject only to the required self-consistency constraints. However, in our statistical sampling, we placed essentially no limits on the complexity of the orbifold twisting (*i.e.*, in the free-fermionic language, we allowed as many as sixteen linearly independent basis vectors). Thus, our statistical analysis included models of arbitrary intricacy and sophistication. We also made use of techniques developed specifically for analyzing string models generated in random searches, allowing for the mitigation of many of the effects of bias which are endemic to studies of this sort.

As part of our study, we generated string models with all degrees of spacetime supersymmetry ($\mathcal{N}$=0, 1, 2, 4) that can arise in four dimensions. For $\mathcal{N}$=0 models, we further demanded that supersymmetry be broken without introducing tachyons. Thus, the $\mathcal{N}$=0 vacua are all non-supersymmetric but tachyon-free, and can be considered as four-dimensional analogues of the ten-dimensional $SO(16) \times SO(16)$ heterotic string [46] which is also non-supersymmetric but tachyon-free. However, other than this, we placed no requirements on other possible phenomenological properties of these vacua such as their possible gauge groups, numbers of chiral generations, or other aspects of the particle content. We did, however, require that our string construction begin with a supersymmetric theory in which the supersymmetry may or may not be broken by subsequent orbifold twists. (In the language of the free-fermionic construction, this is tantamount to demanding that our fermionic boundary conditions include a superpartner sector, typically denoted $\mathbf{W}_1$ or $\mathbf{V}_1$.) This is to be distinguished from a potentially more general class of models in which supersymmetry does not appear at any stage of the construction. This is merely a technical detail in our construction, and we do not believe that this ultimately affects our results.

As with any string-construction method, the free-fermionic formalism contains numerous redundancies in which different choices of worldsheet fermion boundary conditions and/or GSO phases lead to identical string models in spacetime. Indeed, a given unique string model can have many different representations in terms of worldsheet constructions. For this reason, we judged string vacua to be distinct based on their spacetime characteristics — *i.e.*, their low-energy gauge groups and massless particle content.

| SUSY class | # distinct models |
|---|---|
| $\mathcal{N}$=0 (tachyon-free) | 4 946 388 |
| $\mathcal{N}$=1 | 3 772 679 |
| $\mathcal{N}$=2 | 492 790 |
| $\mathcal{N}$=4 | 1106 |
| Total: | 9 212 963 |

Table 3.1. The data set of perturbative heterotic strings analyzed in this chapter. For each level of supersymmetry allowed in four dimensions, we list the number of corresponding distinct models generated. As discussed in the text, models are judged to be distinct based on their spacetime properties (*e.g.*, gauge groups and particle content). All non-supersymmetric models listed here are tachyon-free and thus are four-dimensional analogs of the $SO(16) \times SO(16)$ string model in ten dimensions.

Given this, our ultimate data set of heterotic strings is as described in Table 3.1. Note that all non-supersymmetric models listed in Table 3.1 are tachyon-free, and thus are stable at tree level. We should mention that while generating these models, we also generated over a million distinct non-supersymmetric tachyonic vacua which are not even stable at tree level. We therefore did not include their properties in our analysis, and recorded their existence only as a way of gauging the overall degree to which the tree-level heterotic string landscape is tachyon-free. Also note that as the level of supersymmetry increases, the number of distinct models in our sample set decreases. This reflects the fact that relatively fewer of these models exist, so they become more and more difficult to generate. This will be discussed further in Sects. 3 and 4.

Of course, the free-fermionic construction realizes only certain points in the full model space of self-consistent heterotic string models. For example, since each worldsheet fermion is nothing but a worldsheet boson compactified at a specific radius, a larger (infinite) class of models can immediately be realized through a bosonic formulation by varying these radii away from their free-fermionic values. However, this larger class of models has predominantly only abelian gauge groups and rather limited particle representations. Indeed, the free-fermionic points typically represent precisely those points at which additional (non-Cartan) gauge-boson states become massless, thereby enhancing the gauge symmetries to become non-abelian. Thus, the free-fermionic construction naturally leads to precisely the set of models which are likely to be of direct phenomenological relevance.

We should note that it is also possible to go beyond the class of free-field string models altogether, and consider models built from more complicated worldsheet CFT's (*e.g.*, Gepner models). One could even go beyond the model space of critical string theories, and consider non-critical strings and/or strings with non-trivial background fields. Likewise, we may consider heterotic strings beyond the usual perturbative limit. However, although such models may well give rise to phenomenologies very different from those that emerge in free-field constructions, their spectra are typically very difficult to analyze and are thus not amenable to an automated statistical investigation.

## 3.3. Method of analysis

Each string model-construction technique provides a mapping between a space of internal parameters and a corresponding physical string model in spacetime. In the case of closed strings, for example, such internal parameters might include compactification moduli, boundary-condition phases, Wilson-line coefficients, or topological quantities specifying Calabi-Yau manifolds; in the case of open strings, by contrast, they might include D-brane dimensionalities and charges, wrapping numbers or intersection angles, fluxes, and the vevs of moduli fields. Regardless of the construction technique at hand, however, there is a well-defined procedure through which one can derive the spectrum and couplings of the corresponding model in spacetime.

Given this, one generally conducts a random search through the space of models by randomly choosing self-consistent values of these internal parameters, and then deriving the physical properties of the corresponding string models. Questions about statistical correlations are then addressed in terms of the relative abundances of models that emerge with different spacetime characteristics. Indeed, if $\{\alpha, \beta, \gamma, ...\}$ denote these different spacetime characteristics (or different combinations of these characteristics), then we are generally interested in extracting ratios of population abundances of the form $N_\alpha/N_\beta$, where $N_\alpha$ and $N_\beta$ are the numbers of models which exhibit physical characteristics $\alpha$ and $\beta$ across the landscape as a whole.

Clearly, we cannot survey the entire landscape, and thus we are forced to attempt to extract such ratios with relatively limited information. In particular, let us assume that our search has consisted of analyzing $D$ different randomly generated sets of internal parameters, ultimately yielding a set of different models in spacetime exhibiting varying physical characteristics. Let $M_\alpha(D)$ denote the number of distinct models which are found which exhibit characteristic $\alpha$. Our natural tendency is then to attempt to associate

$$\frac{N_\alpha}{N_\beta} \stackrel{?}{=} \frac{M_\alpha(D)}{M_\beta(D)} \tag{3.1}$$

for some sufficiently large value of $D$. While this relation might not hold exactly for relatively small values of $D$, the expectation is that we might be able to reach sufficiently large values of $D$ for which we might hope to extract reasonably accurate predictions for $N_\alpha/N_\beta$.

Unfortunately, as has recently been discussed in Ref. [35], Eq. (3.1) does not generally hold for any reasonable value of $D$ (short of exploring the full landscape). Indeed, the violations of this relation are striking, even in situations in which sizable fractions of the landscape are explored, and will ultimately doom any attempt at extracting population fractions in this manner. In the remainder of this section, we shall first explain why Eq. (3.1) fails. We shall then summarize the methods which were developed in Ref. [35] for circumventing these difficulties, and which we will be employing in the remainder of this chapter.

As stated above, each string model-construction technique provides a mapping between a space of internal parameters and a physical string model in spacetime. However, this mapping is not one-to-one, and there generally exists a huge redundancy wherein a single physical string model in spacetime can have multiple realizations or representations in terms of internal parameters. For this reason, the space of internal parameters is usually significantly larger than the space of obtainable distinct models.

The failure of this mapping to be one-to-one is critical because any random statistical study of the string landscape must ultimately take the form of a random exploration of the space of internal parameters that lead to these models. First, one must randomly choose a self-consistent configuration of internal parameters; only then can one derive and tabulate the spacetime properties of the corresponding model. But then we are faced with the question of determining whether spacetime models with multiple internal realizations should be weighted more strongly in our statistical analysis than models with relatively few realizations. In other words, we must decide whether our landscape *measure* should be based on internal parameters (wherein each model is weighted according to its number of internal realizations) or based on spacetime properties (wherein each physically distinct model is weighted equally regardless of the number of its internal realizations).

If we were to base our landscape measure on internal parameters, then these redundancies would not represent problems; they would instead become vital ingredients in our numerical analysis. However, if we are to perform statistics in the space of models in a physically significant way, it is easy to see that we are forced to count distinct models rather than distinct combinations of internal parameters. The reason for this is as follows. In many cases, these redundancies arise as the result of worldsheet symmetries (*e.g.*, mirror symmetries), and even though such symmetries may be difficult to analyze and eliminate analytically for reasonably complicated models, their associated redundancies are similar to the redundancies of gauge transformations and do not represent new physics. In other cases, such redundancies are simply reflections of the failures or limitations of a particular model-construction technique; once again, however, they do not represent new physics, but rather reflect a poor choice of degrees of freedom for our internal parameters, or a mathematical difficulty or inability to properly define their independent domains. Finally, such redundancies can also emerge because entirely different model-construction techniques can often lead to identical models in spacetime. Thus, two landscape researchers using different construction formalisms might independently generate random sets of models which partially overlap, but once again this does not mean that the models which are common to both sets should be double-counted when their statistical results are merged. Indeed, in all of these cases, redundancies in the mapping between internal parameters and spacetime properties do not represent differences of physics, but rather differences in the description of that physics. We thus must use spacetime characteristics (rather than the parameters internal to a given string construction) as our means of counting and distinguishing string models.

Many of these ideas can be illustrated by considering the $E_8 \times E_8$ heterotic string in ten dimensions. As is well known, this string model can be represented in many ways: as a $\mathbb{Z}_2$ orbifold of the $SO(32)$ supersymmetric string, as a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of the non-supersymmetric $SO(32)$ heterotic string, and so forth. Likewise, this model can be realized through an orbifold construction, through a free-fermionic construction, through a bosonic lattice construction, and through other constructions as well. Yet, there is only a single $E_8 \times E_8$ string model in ten dimensions. It is therefore necessary to tally distinct string models, and not distinct internal formulations, when performing landscape calculations and interpreting their results.

Unfortunately, this redundancy inherent in the mapping between internal parameters and their corresponding string models implies that in any random exploration of the space of models, certain string models are likely to be sampled much more frequently than other models. Thus, one must filter out this effect by keeping a record of each distinct model that has already been sampled so that each time an additional model is generated (*i.e.*, each time there is a new "attempt"), it can be compared against all previous models and discarded if it is not new. Although this is a memory-intensive and time-consuming process which ultimately limits the sizes of the resulting data sets that can be generated using current automated technology, this filtering can successfully be employed to eliminate model redundancies.

However, there remains the converse problem: because some models strongly dominate the random search, others effectively recede and are therefore extremely difficult to reach. They therefore do not tend to show up during the early stages of a random search, and tend to emerge only later in the search process after the dominant models have been more fully tallied. Indeed, as the search proceeds into its later stages, it is only the models with "rare" characteristics which increasingly tend to be generated, precisely because those models with "common" characteristics will have already been generated and tabulated. Thus, the proportion of models with "rare" characteristics tends to evolve rather dramatically as a function of time through the model-generation process.

This type of bias is essentially unavoidable, and has the potential to seriously distort the values of any numerical correlations that might be extracted from a random search through the landscape. In particular, as discussed in Ref. [35], this type of bias generally causes statistical correlations to "float" or evolve as a function of the sample size of models examined. Moreover, since one can ultimately explore only a limited portion of the landscape, there is no opportunity to gather statistics at the endpoint of the search process at which these correlations would have floated to their true values. This, then, is the problem of floating correlations.

Fortunately, as discussed in Ref. [35], there are several statistical methods which can be used in order to overcome this difficulty. These methods enable one to extract statistical correlations and distributions which are stable as a function of sample size and which, with some reasonable assumptions, represent the statistical results that would be obtained if the full space of models could be explored. We shall now describe the most important of these methods, since we shall be using this technique throughout the rest of this chapter.

In general, a model search proceeds as follows. One randomly generates a self-consistent set of internal parameters, and calculates the properties of the corresponding string model. One then compares this model against all models which have previously been generated: if the model is distinct, it is recorded and saved; if it is redundant, it is discarded. One then repeats this process. Early in the process, most attempts result in new distinct models because very few models have already been found. However, as the search proceeds, an increasing fraction of attempts fail to produce new models. This rise in the ratio of attempts per new model indicates that the space of models is becoming more and more explored. Thus, attempts per model can be used as a measure of how far into the full space of corresponding models our search has penetrated.

Therefore, if we are interested in extracting the ratio $N_\alpha/N_\beta$ for two physical characteristics $\alpha$ and $\beta$, as discussed above Eq. (3.1), the solution is *not* to extract this ratio through Eq. (3.1) because such a relation assumes that the spaces of $\alpha$-models and $\beta$-models are being penetrated at exactly the same rates during the random search process. Rather, the solution [35] is to keep a record not only of the models generated as the search proceeds, but also of the cumulative average *attempts* per model that are needed in order to generate these models. We then extract the desired ratio $N_\alpha/N_\beta$ through a relation of the form

$$\frac{N_\alpha}{N_\beta} = \frac{M_\alpha(d_\alpha)}{M_\beta(d_\beta)}\Bigg|_{\frac{d_\alpha}{M_\alpha(d_\alpha)}=\frac{d_\beta}{M_\beta(d_\beta)}} \tag{3.2}$$

where $d_\alpha$ and $d_\beta$ respectively represent the numbers of attempts that resulted in $\alpha$-models and $\beta$-models, regardless of whether the models in each class were distinct. Thus, we must essentially perform two independent search processes, one for $\alpha$-models and one for $\beta$-models, and we terminate these searches only when they have each reached the same degree of penetration as measured through their respective numbers of attempts per model $d_\alpha/M_\alpha$. The value of $N_\alpha/N_\beta$ obtained in this way should then be independent of the chosen reference value of $d_\alpha/M_\alpha$ for sufficiently large $d_\alpha/M_\alpha$. This method of extracting $N_\alpha/N_\beta$ is discussed more fully in Ref. [35], where the derivation and limitations of this method are outlined in detail.

Of course, in the process of randomly generating string models, we cannot normally control whether a random new model is of the $\alpha$- or $\beta$-type. Both will tend to be generated together, as part of the same random search. Thus, our procedure requires that we completely *disregard* the additional models of one type that might be generated in the process of continuing to generate the required, additional models of the other type. This is the critical implication of Eq. (3.2). Rather than let our model-generating procedure continue for a certain duration, with statistics gathered at the finish line as in Eq. (3.1), we must instead establish two separate finish lines for our search process, one for $\alpha$-models and one for $\beta$-models. Of course, these finish lines are not completely arbitrary, and must be chosen such they correspond to the same relative degree of penetration of the $\alpha$- and $\beta$-model spaces. Indeed, these finish lines must be balanced so that they correspond to points at which the same ratio of attempts per model has been reached. However, these finish lines will not generally coincide with each other, which requires that some data actually be disregarded in order to extract meaningful statistical correlations.

As discussed in Ref. [35], Eq. (3.2) will enable us to extract a value for the ratio $N_\alpha/N_\beta$ which is stable as a function of sample size only when the biases within the $\alpha$-model space are the same as those within the $\beta$-model space. In such cases, we can refer to the physical characteristics $\alpha$ and $\beta$ as being in the same universality class. However, for a given model-generation method (such as the free-fermionic construction which we shall be employing in this chapter), it turns out that many physical characteristics of interest $\{\alpha, \beta, ...\}$ have the property that they are in the same universality class. In the rest of this chapter, correlations for physical quantities will be quoted only when the physical characteristics being compared are in the same universality class. The above method is then used in order to extract these correlations.

### 3.4. Supersymmetry on the heterotic landscape

In this section, we begin our analysis of the structure of the heterotic string landscape. In so doing, we shall also provide an explicit example of the method described in Sect. 3. Our focus in this section is to determine the extent to which string models with different levels of unbroken supersymmetry ($\mathcal{N}$=0, 1, 2, 4) populate the tree-level four-dimensional heterotic landscape. For $\mathcal{N}$=0 models, we shall further distinguish between models which are tachyon-free at tree level, and those which are tachyonic. Note that these characteristics are all mutually exclusive and together span the entire landscape of heterotic string models in four dimensions. Thus, our goal is to achieve nothing less than a partitioning of the full set of tree-level heterotic string models according to their degrees of supersymmetry. (We stress that this analysis will be the only case in which unstable tachyonic $\mathcal{N}$=0 string models will be considered in this chapter.) We will then proceed in Sect. 5 to examine questions related to correlations between the numbers of unbroken supersymmetry generators and the corresponding gauge groups.

The landscape of four-dimensional heterotic strings is a relatively large and complex structure. It may therefore be useful, as an initial step, to quickly recall the much smaller "landscape" of *ten*-dimensional heterotic strings. In ten dimensions, the maximal allowed supersymmetry is $\mathcal{N}$=1, and thus our tree-level ten-dimensional landscape may be partitioned into only three categories: $\mathcal{N}$=1 models, $\mathcal{N}$=0 tachyon-free models, and $\mathcal{N}$=0 tachyonic models. Note that since the $\mathcal{N}$=0 tachyonic models are not even stable at tree level, the tree-level "landscape" actually consists only of models in the first two categories. However, for convenience, in this section we shall use the word "landscape" to describe the full set of heterotic vacuum solutions regardless of stability.

| SUSY class | % of 10D landscape | % of reduced 10D landscape |
|---|---|---|
| $\mathcal{N}$=0 (tachyonic) | 66.7 | 62.5 |
| $\mathcal{N}$=0 (tachyon-free) | 11.1 | 12.5 |
| $\mathcal{N}$=1 | 22.2 | 25.0 |

Table 3.2. Classification of the ten-dimensional tree-level heterotic "landscape" as a function of the number of spacetime supersymmetries and the presence/absence of tachyons at tree level. As always, models are judged to be distinct based on their gauge groups and particle contents. The full ten-dimensional heterotic landscape consists of nine distinct string models, while the landscape of models accessible through our random search methods is reduced by one model. In either case, we see that two thirds of the tachyon-free portion of the ten-dimensional landscape is supersymmetric. Thus unbroken supersymmetry tends to dominate the "landscape" consisting of ten-dimensional models which are stable at tree level.

As is well known [47], the full set of $D = 10$ heterotic strings consists of nine distinct string models: two are supersymmetric [these are the $SO(32)$ and $E_8 \times E_8$ models], one is non-supersymmetric but tachyon-free [this is the $SO(16) \times SO(16)$ string model [46]], and six additional models are non-supersymmetric and tachyonic. Expressed as proportions of a full ten-dimensional heterotic landscape, we therefore find the results shown in the middle column of Table 3.2. It is important to note, however, that not all of these models would be realizable through the methods we shall be employing in this chapter (involving a construction in which all degrees of freedom are represented in terms of complex worldsheet fermions). Indeed, one of the tachyonic non-supersymmetric models exhibits rank-reduction and thus would not be realizable in a random search of the sort we shall be conducting. Statistics for the corresponding "reduced" landscape of accessible models are therefore listed along the third column of Table 3.2; these are the statistics which will form the basis for future comparisons. Note that in either case, the tachyon-free portion of the ten-dimensional landscape is dominated by supersymmetric models. This suggests that breaking supersymmetry without introducing tachyons is relatively difficult in ten dimensions.

Our goal is to understand how this picture changes after compactification to four dimensions. Towards this end, one procedure might be to randomly generate a large set of string models, and see how many models one obtains of each type after a certain fixed time as elapsed. However, as discussed in Sect. 3.3, these percentages will generally float or evolve as a function of the total number of models examined. This behavior is shown in Fig. 3.1, and we see that while the non-supersymmetric percentages seem to be floating towards greater values, the supersymmetric percentages seem to be floating towards lesser values.

As discussed in Sect. 3.3, it is easy to understand the reason for this phenomenon. Clearly, as we continue to generate models randomly, an ever-increasing fraction of these models consists of models without supersymmetry. This in turn suggests that at any given time, we have already discovered a greater fraction of the space of supersymmetric models than non-supersymmetric models. This would explain why it becomes increasingly more difficult to randomly generate new, distinct supersymmetric models as compared with non-supersymmetric models, and why their relative percentages show the floating behavior illustrated in Fig. 3.1.

How then can we extract meaningful information? As discussed in Sect. 3.3, the remedy involves keeping track of not only the total numbers of distinct models found in each supersymmetric class, but also the total number of *attempts* which yielded a model in each class, even though such models were not necessarily new. This information is shown in Table 3.3 for our total sample of $\gtrsim 10^7$ models.

As we see from Table 3.3, the number of required attempts per model increases dramatically with the level of supersymmetry. This in turn implies, for example, that although we may have generated many fewer distinct $\mathcal{N}=4$ models than $\mathcal{N}=1$ models,

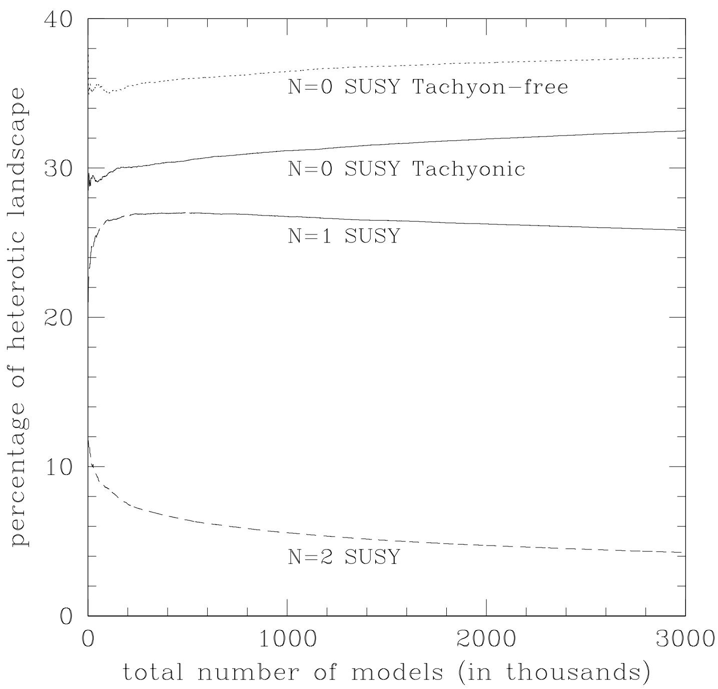


FIGURE 3.1. The numbers of distinct string models exhibiting different amounts of spacetime supersymmetry, plotted as functions of the total number of distinct string models examined. Models exhibiting $\mathcal{N}$=4 supersymmetry are too few to appear on this figure.

the full space of $\mathcal{N}$=4 models has already been penetrated much more fully than the space of $\mathcal{N}$=1 models. Thus, as we continue to generate more models, it should become relatively easier to generate non-supersymmetric models than supersymmetric models. If true, this would imply that the relative proportion of non-supersymmetric models should increase as we continue to generate more models, while the relative proportion of supersymmetric models should decrease. This is, of course, exactly what we have already seen in Fig. 3.1.

In order to extract final information concerning the relative sizes of these spaces, the procedure outlined in Sect. 3.3 instead requires that we do something different, and compare the numbers of distinct models generated in each class *at those points in our model-generating process when their corresponding numbers of attempts per model are equal*. It is only in this way that we can overcome the effects of floating correlations and extract stable relative percentages which do not continue to evolve as functions of the total sample size.

For example, let us consider the relative numbers of $\mathcal{N}$=1 and $\mathcal{N}$=2 models.

| SUSY class | # distinct models | # attempts | avg. attempts/model |
|---|---|---|---|
| $\mathcal{N}$=0 (tachyonic) | 1 279 484 | 3 810 838 | 2.98 |
| $\mathcal{N}$=0 (tachyon-free) | 4 946 388 | 18 000 000 | 3.64 |
| $\mathcal{N}$=1 | 3 772 679 | 24 200 097 | 6.41 |
| $\mathcal{N}$=2 | 492 790 | 13 998 843 | 28.41 |
| $\mathcal{N}$=4 | 1106 | 6 523 277 | 5 898.08 |
| Total: | 10 492 447 | 66 533 055 | 6.34 |

Table 3.3. This table expands on Table 3.1 by including the numbers of attempts to generate models in each class as well as the corresponding average numbers of attempts per distinct model. We also include information about the attempts which resulted in non-supersymmetric models whose spectra are tachyonic at tree level. It is apparent that the number of attempts per model increases rather dramatically as the level of supersymmetry increases, indicating that our heterotic string sample has penetrated further into the spaces of models with greater numbers of supersymmetries than into those with fewer.

Although we see from Table 3.3 that our full sample of $\gtrsim 10^7$ models contains approximately 7.66 times as many $\mathcal{N}$=1 models as $\mathcal{N}$=2 models, this is not the relative size of their corresponding model spaces because the $\mathcal{N}$=2 space of models has already been explored more fully than the $\mathcal{N}$=1 model space, with 6.41 attempts per $\mathcal{N}$=1 model compared with 28.41 attempts per $\mathcal{N}$=2 model. However, at an earlier point in our search, we found that it took an average of approximately 6.41 attempts to generate a new, distinct $\mathcal{N}$=2 model: this occurred when we had generated only approximately 90 255 models with $\mathcal{N}$=2 supersymmetry. This suggests that the space of $\mathcal{N}$=1 models is actually $3772679/90255 \approx 41.8$ times as large as the space of $\mathcal{N}$=2 models.

Moreover, we can verify that this ratio is actually stable as a function of sample size. For example, at an even earlier point in our search when we had generated only $\approx 2.22 \times 10^6$ $\mathcal{N}$=1 models, we found that an average of 3.64 attempts were required to generate a new, distinct $\mathcal{N}$=1 model. However, this same average number of attempts per model occurred in our $\mathcal{N}$=2 sample when we had generated only $\approx 53\,000$ $\mathcal{N}$=2 models. Thus, once again, the $\mathcal{N}$=1 and $\mathcal{N}$=2 model spaces appear to have a size ratio of $\approx 41.8 : 1$.

In this way, by comparing total numbers of models examined at equal values of attempts per model, we can extract the relative sizes of the spaces of models with differing degrees of supersymmetry and verify that these results are stable as functions of sample size (*i.e.*, stable as functions of the chosen value of attempts per model). Our results are shown in Table 3.4. As far as we can determine, the percentages quoted in Table 3.4 represent the values to which the percentages in Fig. 3.1 would float if we could analyze what is essentially the full landscape. However, short of examining the full landscape, we see that there is no single point at which these percentages would simultaneously appear in any finite extrapolation of Fig 3.1. Instead, it is only by comparing the numbers of models obtained at *different* points in our analysis that the true ratios quoted in Table 3.4 can be extracted.

| SUSY class | % of heterotic landscape |
|---|---|
| $\mathcal{N}$=0 (tachyonic) | 32.1 |
| $\mathcal{N}$=0 (tachyon-free) | 46.5 |
| $\mathcal{N}$=1 | 20.9 |
| $\mathcal{N}$=2 | 0.5 |
| $\mathcal{N}$=4 | 0.003 |

TABLE 3.4. Classification of the four-dimensional tree-level heterotic landscape as a function of the number of unbroken spacetime supersymmetries and the presence/absence of tachyons at tree level. This table is thus the four-dimensional counterpart of Table 3.2, which quoted analogous results for ten dimensions. Relative to the situation in ten dimensions, we see that compactification to four dimensions tends to *favor* breaking all spacetime supersymmetries without introducing tachyons at tree level.

Table 3.4 thus represents our final partitioning of the tree-level four-dimensional landscape according to the amount of supersymmetry exhibited. There are several rather striking facts which are evident from these results:

- First, we see that nearly half of the heterotic landscape is non-supersymmetric and yet tachyon-free.
- Second, we see that the supersymmetric portion of the heterotic landscape appears to account for less than one-quarter of the full four-dimensional heterotic landscape.
- Finally, models exhibiting extended ($\mathcal{N} \geq 2$) supersymmetries are exceedingly rare, representing less than one percent of the full landscape.

Of course, we stress once again that these results hold only for the *tree-level* landscape, *i.e.*, models which are stable at tree level only. It is not clear whether these results would persist after full moduli stabilization. However, assuming that they do, these results lead to a number of interesting conclusions.

The first conclusion is that the properties of the tachyon-free heterotic landscape as a whole are statistically dominated by the properties of string models which do *not* have spacetime supersymmetry. Indeed, the $\mathcal{N}$=0 string models account for over three-quarters of this portion of the heterotic string landscape. The fact that the $\mathcal{N}$=0 string models dominate the tachyon-free portion of the landscape suggests that breaking supersymmetry without introducing tachyons is actually *favored* over preserving supersymmetry for this portion of the landscape. Indeed, we expect this result to hold even after full moduli stabilization, unless an unbroken supersymmetry is somehow restored by stabilization.

The second conclusion which can be drawn from these results is that the supersymmetric portion of the landscape is almost completely comprised of $\mathcal{N}$=1 string models. Indeed, only 2% of the supersymmetric portion of the heterotic landscape has more than $\mathcal{N}$=1 supersymmetry. This suggests that the correlations present for the supersymmetric portion of the landscape can be interpreted as the statistical correlations within the $\mathcal{N}$=1 string models, with the $\mathcal{N}$=2 correlations representing a correction at the level of 2% and the $\mathcal{N}$=4 correlations representing a nearly negligible correction.

It is natural to ask what effects are responsible for this hierarchy. As was discussed in Sect. 3.3, two string models are considered distinct if any of their spacetime properties are found to be different. Two models which have the same number of unbroken spacetime supersymmetries must therefore differ in other features, such as their gauge groups and particle representations. Thus, if there exist more models with one level of supersymmetry than another, this must mean that there are more string-allowed configurations of gauge groups and particle representations with one level of supersymmetry than the other. Indeed, given the results of Table 3.4, our expectation is that increasing the level of supersymmetry will have the effect of decreasing the number of distinct models with a given gauge group, and possibly even the range of allowed gauge groups. We shall test both of these expectations explicitly in Sect. 3.5.

## 3.5. Supersymmetry versus gauge groups

Within the heterotic string, worldsheet self-consistency conditions arising from the requirements of conformal anomaly cancellation, one-loop and multi-loop modular invariance, physically sensible GSO projections, *etc.*, impose many tight constraints on the allowed particle spectrum. These constraints simultaneously affect not only the spacetime Lorentz structure of the theory (such as is involved in spacetime supersymmetry), but also the internal gauge structure of the theory. Thus, it is precisely within the context of string theory that we expect to find correlations between supersymmetries and gauge symmetries — features which would otherwise be uncorrelated in theories based on point particles.

In general, these correlations can lead to certain tensions in a given string construction. Models exhibiting large numbers of unbroken supersymmetries may be expected to have relatively rigid gauge structures, and vice versa. There are two specific types of correlations which we shall study. First, we shall analyze how the degree of supersymmetry affects the range of possible allowed gauge groups. For example, in extreme cases it may occur that certain gauge symmetries may not even be allowed for certain levels of spacetime supersymmetry. Second, even within the context of a fixed gauge group, we can expect the degree of spacetime supersymmetry to affect the range of allowed particle representations which can appear at the massless level. In other words, the number of distinct string models with a given fixed gauge group may be highly sensitive to the degree of spacetime supersymmetry.

Some of these features are already on display in the ten-dimensional heterotic "landscape". For example, no gauge group is shared between those ten-dimensional models with supersymmetry and those without. Moreover, in each case, there is only a single model with each allowed gauge group. Thus, in ten dimensions, the specification of the level of supersymmetry (and/or the gauge group) is sufficient to completely fix the corresponding particle spectrum.

Clearly, in four dimensions, things will be far more complex. In particular, we shall study three correlations in this section:

- First, we shall focus on the number of allowed gauge groups as a function of the degree of supersymmetry. We shall also study gauge-group multiplicities — *i.e.*, the probabilities that there exist distinct string models with the same gauge group but different particle spectra. This will be the focus of Sect. 5.1.

- Second, as a function of the degree of supersymmetry, we shall investigate "shatter" — *i.e.*, the degree to which our total (rank-22) gauge group is "shattered" into distinct irreducible factors, or equivalently the average rank of each irreducible gauge-group factor. This will be the focus of Sect. 5.2.

- Finally, as a function of the degree of supersymmetry, we shall study the probabilities of realizing specific (combinations of) gauge-group factors in a given string model. This will be the focus of Sect. 5.3.

As we shall see, these studies will find deep correlations which ultimately reflect the string-theoretic tension between supersymmetry and the string consistency conditions.

### 3.5.1. Numbers and multiplicities of unique gauge groups

We begin by studying the total numbers of distinct gauge groups which can be realized as a function of the number of unbroken supersymmetries in a given string model.

To do this, one direct approach can might be to classify models according to their numbers of unbroken spacetime supersymmetries, and tabulate the numbers of distinct gauge groups which appear as functions of the total number of models in each class. As we continue to generate more and more models, we then obtain the results shown in Fig. 3.2.

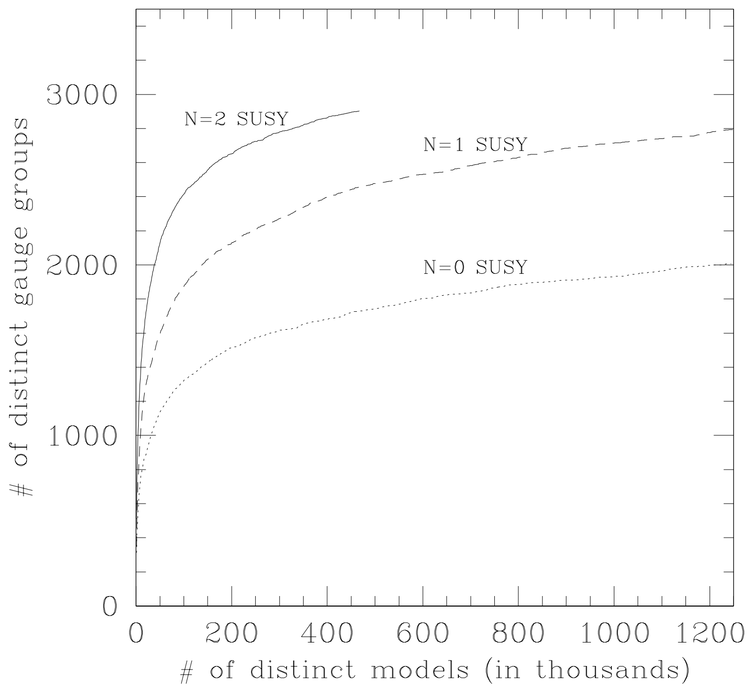


FIGURE 3.2. Numbers of distinct gauge groups obtained as functions of the number of distinct string models generated. Each curve corresponds to models with a different number of unbroken spacetime supersymmetries, with $\mathcal{N}$=0 signifying models which are non-supersymmetric but tachyon-free. We see that for a fixed sample size, models with more unbroken supersymmetries tend to exhibit a larger number of distinct gauge groups. (Note that models with $\mathcal{N}$=4 supersymmetry are too few to be shown in this plot.)

It is evident from Fig. 3.2 that for a fixed sample size, models with more unbroken supersymmetries tend to exhibit larger numbers of distinct gauge groups, or equivalently smaller numbers of model multiplicities per gauge group. For example, we see from Fig. 3.2 that when each class of models has reached a sample size of 500 000 models, the tachyon-free $\mathcal{N}$=0 models have a greater multiplicity per gauge group than $\mathcal{N}$=1 models by an approximate factor $\approx 1.4$, while the $\mathcal{N}$=2 models have a smaller multiplicity per gauge group than the $\mathcal{N}$=1 models by an approximate factor $\approx 0.8$. However, it is easy to understand this behavior. As the level of supersymmetry increases, there are more constraints on the possible particle spectra that can emerge for a given gauge group. This in turn implies that there are likely to be fewer ways for two models with the same gauge group to be distinct, which in turn implies that there is a greater chance that distinct models will be forced to exhibit distinct gauge groups. Thus, models exhibiting greater amounts of supersymmetry are likely, on average, to exhibit greater numbers of gauge groups amongst a fixed number of models.

Of course, as also evident from Fig. 3.2, the multiplicity of distinct models per gauge group exhibits a strong, floating dependence on the sample size. Therefore, in order to extract a stable ratio of multiplicity ratios — one which presumably represents the values of these ratios when extrapolated to the full landscape — we must employ the methods described in Sect. 3.3. We then obtain the results shown in the middle column of Table 3.5. Using these results in conjunction with the corresponding ratios of landscape magnitudes in Table 3.4, we can also calculate the relative numbers of distinct gauge groups realizable within each SUSY class of models. These results are shown in the final column of Table 3.5. Note that in each case, these quantities are quoted as ratios relative to their $\mathcal{N}$=1 values; this represents the most detailed information that can be extracted using the methods of Sect. 3.3.

| SUSY class | avg. multiplicity per gauge group | # of realizable gauge groups |
|---|---|---|
| $\mathcal{N}$=0 (tachyon-free) | 1.65 | 1.35 |
| $\mathcal{N}$=1 | 1.00 | 1.00 |
| $\mathcal{N}$=2 | 0.89 | 0.03 |

Table 3.5. The average relative multiplicities (distinct models per gauge group) and total numbers of realizable gauge groups, evaluated for heterotic string models with $\mathcal{N} = 0, 1, 2$ unbroken spacetime supersymmetries. In each case, these quantities are normalized to their $\mathcal{N}$=1 values.

We see from Table 3.5 that both the average multiplicities per gauge group and the total numbers of realizable gauge groups are monotonically decreasing functions of the number of unbroken supersymmetries. While this is to be expected on the basis of the arguments described above, we must realize that our class of $\mathcal{N}$=0 models does not consist of *all* non-supersymmetric models, but merely those which are tachyon-free. Thus, the requirement of avoiding tachyons could have turned out to be more stringent than the requirement of maintaining an unbroken supersymmetry, at least as far as generating a variety of gauge groups is concerned. This is indeed what happens in the *ten*-dimensional landscape, where there are fewer realizable gauge groups for non-supersymmetric tachyon-free models than for models with $\mathcal{N}$=1 supersymmetry. However, the results in Table 3.5 indicate that the opposite is true in $D = 4$.

Note that in Table 3.5, we do not quote results for the $\mathcal{N}$=4 portion of the heterotic landscape because the absolute numbers of models in this class are so small that no stable numerical results can be extracted relative to the other levels of supersymmetry. However, it is worth noting that literally each $\mathcal{N}$=4 model in our sample has a unique gauge group, so the *absolute* (rather than relative) gauge-group multiplicity in the $\mathcal{N}$=4 case is exactly 1.000. This only reinforces our general observation that increased levels of supersymmetry reduce the gauge-group multiplicity; indeed, we now see that the case of *maximal* supersymmetry appears to result in the *minimal* allowed gauge-group multiplicity. It is likely that this result can be proved analytically for the $\mathcal{N}$=4 landscape as a whole.

### 3.5.2. Shatter/average rank

Having studied the numbers of different possible gauge groups, we now turn our attention to the gauge groups themselves. Once again, our goal is to study how these gauge groups depend on the presence or absence of spacetime supersymmetry.

To begin the discussion, our focus in this section will be on what we call "shatter" [54]. Recall that the heterotic string models we are considering all have gauge groups with total rank 22. This stretches from models with gauge group $SO(44)$ all the way down to models with gauge groups of the form $U(1)^n \times SU(2)^{22-n}$ with potentially all values of $n$ in the range $0 \leq n \leq 22$. Following Ref. [54], we shall define the "shatter" for a given string model as the number of distinct irreducible gauge-group factors into which its total rank-22 gauge group has been shattered. Note that for this purpose, factors of $SO(4) \sim SU(2) \times SU(2)$ contribute two units to shatter. Since the total rank of the gauge group is fixed at 22 for such models, this means that shatter is also a measure of the average rank of the individual group group factors, with $\langle \text{rank} \rangle = 22/\text{shatter}$. Roughly speaking, shatter can also be taken as a measure of the degree of complexity needed for the construction of a given string model, with increasingly smaller individual gauge-group factors tending to require increasingly many non-overlapping sequences of orbifold twists and Wilson lines.

Given this definition of shatter, we may then calculate the distribution of shatter across the landscape of heterotic strings. We may calculate, for example, the relative probabilities that models with certain levels of shatter emerge across the landscape, and ask how these probability distributions vary with the amount of spacetime supersymmetry present in the model.

Our results are shown in Fig. 3.3. Once again, we stress that our raw data tends to evolve significantly as a function of the sample size of models considered. It is therefore necessary to employ the techniques described in Sect. 3 in order to extract stable results which should apply across the landscape as a whole. In practice, this requires a difficult and time-consuming process in which each of the data points shown in Figs. 3.3 for $\mathcal{N}=0, 1, 2$ has individually been extracted through the limiting procedure described in Sect. 3. Only then is an entire "curve" constructed for each level of supersymmetry, as shown.

For the $\mathcal{N}$=4 case, by contrast, our sample size is too small to permit stable results to be extracted. However, the fact that the attempts per model count in Table 3.3 is so large for the $\mathcal{N}$=4 models suggests that our $\mathcal{N}$=4 sample has already explored a significant fraction (and perhaps even most) of the corresponding landscape. The $\mathcal{N}$=4 curve in Fig. 3.3 thus represents a direct tally of our $\mathcal{N}$=4 sample set.

As evident from Fig. 3.3, certain features of these plots are independent of the level of spacetime supersymmetry. These therefore represent general trends which hold across the entire tachyon-free heterotic string landscape. For example, one general trend is a strong preference for models with relatively high degrees of shatter and correspondingly small average ranks for individual gauge-group factors — models exhibiting shatters near or in the teens clearly dominate. On the other hand, this preference for highly shattered gauge groups does *not* appear to extend to the limit of completely shattered models with shatter=22; indeed, the set of models with only rank-one gauge-group factors seems to represent a fairly negligible portion of the landscape regardless of the degree of supersymmetry. This indicates that most models in this class have gauge groups which contain at least one factor of rank greater than one.[1]

Another universal trend implied by (though not explicitly shown in) Fig. 3.3 is that string models with shatters of less than four accrue relatively little measurable amount of probability. Even in the $\mathcal{N}$=4 case, these models are thus actually quite rare across the landscape as a whole. In some sense, this too is to be expected, since there are many more ways of breaking a large gauge symmetry through orbifolds and non-trivial Wilson lines than of preserving it.

Despite these universal features, we see that spacetime supersymmetry nevertheless does have a significant effect on the shapes of these curves. In this regard, there are two features to note.

First, we observe that as the degree of unbroken supersymmetry increases, the *range* of probable shatter values also tends to increase, with probability shifting from models with high shatters to models with lower shatters. This is especially noticeable when comparing the distribution of the $\mathcal{N}$=2 and $\mathcal{N}$=4 models with those of the $\mathcal{N}$=0 and $\mathcal{N}$=1 models. These results indicate that models exhibiting smaller amounts of shatter (*i.e.*, models whose gauge-group factors have larger individual average ranks) become somewhat more probable as the level of supersymmetry increases. Ultimately, this correlation between unbroken supersymmetry and unbroken gauge symmetry emerges since both have their underlying origins in how our orbifold twists and Wilson lines are chosen.

Second, and perhaps more unexpectedly, we see that the degree of supersymme-

[1] Of course, we stress that this conclusion applies only for models in the free-fermionic class. In general, it is always possible to deform away from the free-fermionic limit by adjusting the internal radii of the worldsheet fields away from their free-fermionic values; in such cases, we expect all gauge symmetries to be broken down to $U(1)^{22}$. However, as noted earlier, the free-fermionic points typically represent precisely those points at which additional (non-Cartan) gauge-boson states become massless, thereby enhancing the gauge symmetries to become non-abelian. Thus, as discussed more fully in Sect. 2, the free-fermionic construction naturally leads to precisely the set of models which are likely to be of direct phenomenological relevance.

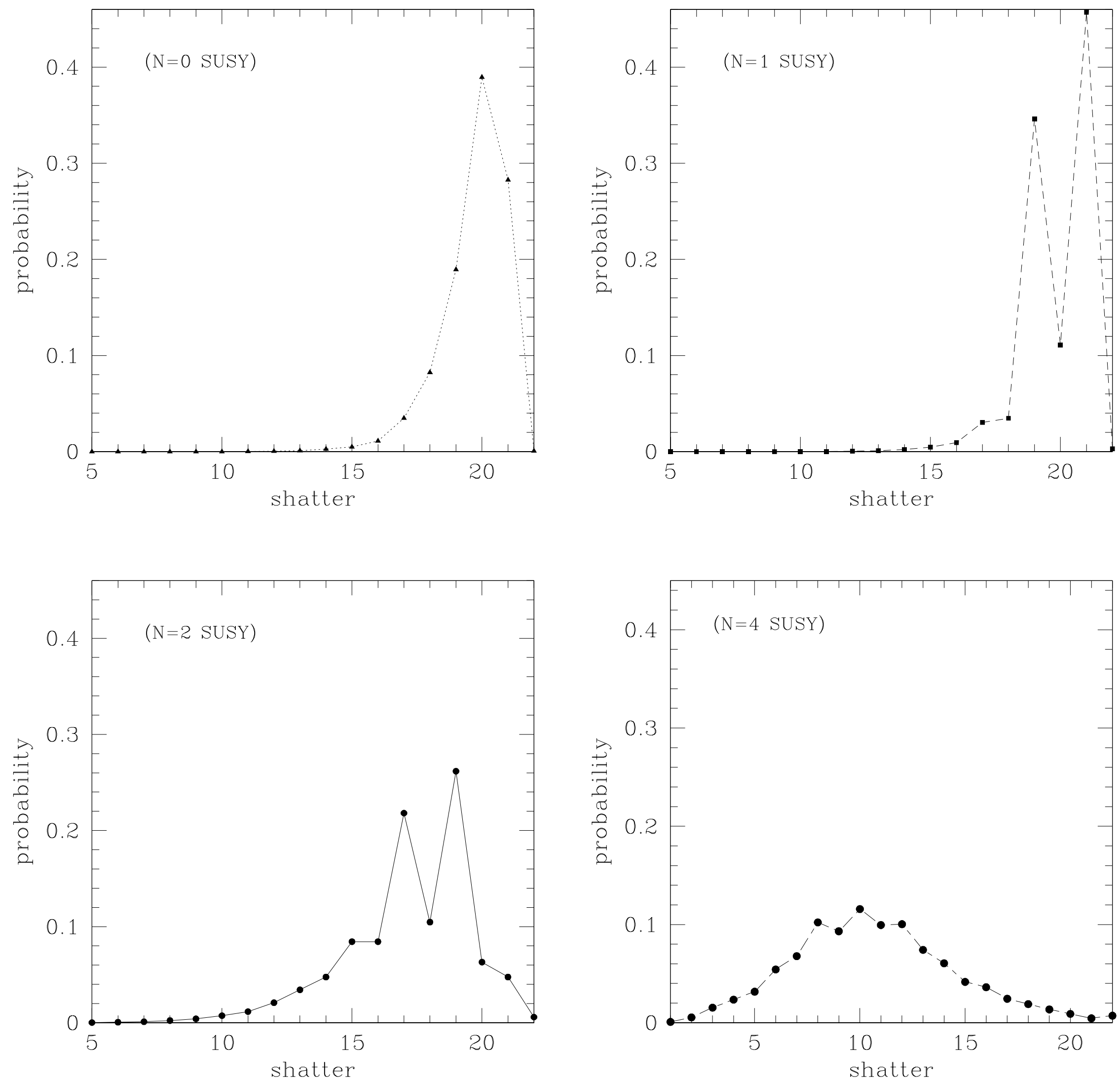


FIGURE 3.3. The absolute probabilities of obtaining distinct four-dimensional heterotic string models with different numbers of unbroken supersymmetries, plotted as functions of the degree to which their gauge groups are "shattered" into separate irreducible factors. The total value of the points (the "area under the curve") in each case is 1. Here $\mathcal{N}$=0 refers to models which are non-supersymmetric but tachyon-free.

try also affects the overall profiles of these curves. While the $\mathcal{N}$=0 curve is relatively smooth, exhibiting a single peak at shatter=20, these curves begin to experience even/odd oscillations as the degree of supersymmetry increases, with odd values of shatter significantly favored over even values when supersymmetry is present. The origins of this phenomenon are less apparent, and perhaps lie in the modular invariance and anomaly cancellation constraints which correlate the orders of the allowed twists leading to self-consistent string models. Interestingly, this even/odd behavior continues into the $\mathcal{N}$=4 case, although these oscillations are significantly less pronounced and flip sign, with evens now dominating over odds.

One notable feature of the $\mathcal{N}$=4 curve is its approximate reflection symmetry around shatter=10. It is unclear whether this is an exact symmetry which holds in situations with maximal supersymmetry, or whether this is merely an accident.

#### 3.5.3. Specific gauge-group factors

Finally, we turn to an analysis of the probabilities of realizing individual gauge-group factors. Just how likely is it, say, that a randomly chosen heterotic string model will exhibit an $SU(3)$ factor in its gauge group, and how does this probability correlate with the spacetime supersymmetry of the model?

Just as with previous questions, addressing this issue requires a detailed analysis along the lines discussed in Sect. 3. This is because the probabilities of realizing different gauge-group factors also float quite strongly as a function of sample size. As dramatic illustration of this fact, let us restrict our attention to models with $\mathcal{N}$=1 spacetime supersymmetry and calculate the probability that a given model will exhibit an $SU(3)$ gauge-group factor as a function of the number of models we have examined. We then obtain the result shown in Fig. 3.4, and it is clear that the percentage of models with $SU(3)$ gauge-group factors *floats* rather significantly as a function of the sample size. Indeed, on the basis of this information alone, it would be quite impossible to determine the final value to which this curve might float. Just as with previous examples, this floating behavior ultimately occurs because models with $SU(3)$ gauge-group factors are relatively difficult to generate using the construction methods we are employing; thus, they tend to emerge in increasing numbers only after other models are exhausted. As discussed more fully in Ref. [35], this does not imply that there are fewer of these models or that our construction method cannot ultimately reach them — all we can conclude is that they are less likely to be generated in a random search than other models, and thus they tend to emerge only later in the search process. Indeed, as we shall shortly see, models with $SU(3)$ gauge-group factors actually tend to dominate the landscape.

Therefore, in order to extract meaningful results, we again employ the methods discussed in Sect. 3. We then obtain the final percentages quoted in Table 4.2. We observe, in particular, that the probability of models with $\mathcal{N}$=1 supersymmetry exhibiting at least one $SU(3)$ gauge-group factor has actually risen all the way to 98%.

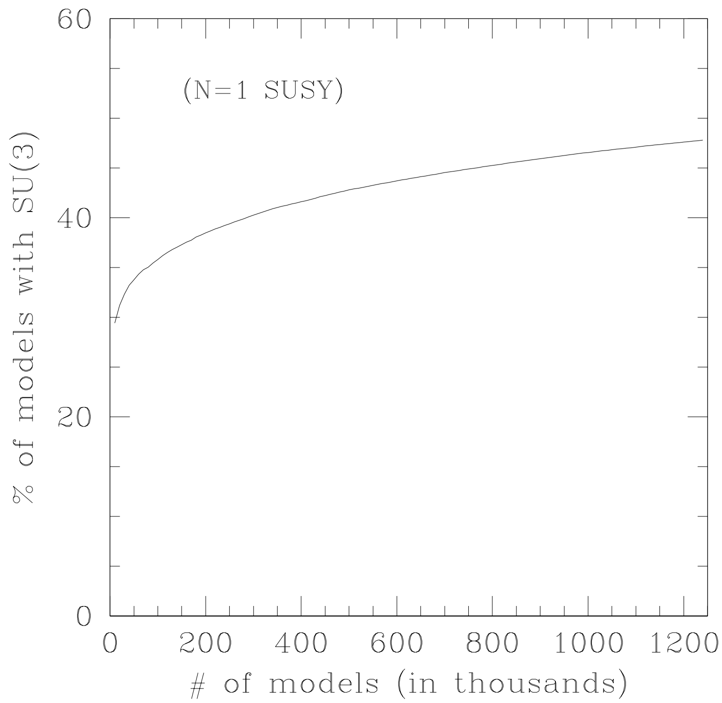


FIGURE 3.4. The percentage of distinct four-dimensional $\mathcal{N}$=1 supersymmetric heterotic string models exhibiting at least one $SU(3)$ gauge-group factor, plotted as a function of the number of models examined for the first 1.25 million models. We see that as we generate further models, $SU(3)$ gauge-group factors become somewhat more ubiquitous — *i.e.*, the fraction of models with this property *floats*. One must therefore account for this floating behavior using the methods described in Sect. 3 in order to extract meaningful information concerning the relative probabilities of specific gauge-group factors.

The fact that this probability has floated from nearly 55% to 98% only reinforces the importance of the analysis method presented in Sect. 3.3, and illustrates the need to properly account for floating correlations when quoting statistical results for such studies.

As we see from Table 4.2, supersymmetry can have quite sizable effects upon the probability of realizing specific groups. However, there are some general trends that hold for the full heterotic landscape. These trends include:

- A preference for $SU(n+1)$ over $SO(2n)$ groups for each rank $n$. Even though these two groups have the same rank, it seems that $SU$ groups are more common than the $SO$ groups for all levels of supersymmetry.

| gauge group | $\mathcal{N}$=0 | $\mathcal{N}$=1 | $\mathcal{N}$=2 | $\mathcal{N}$=4 |
|---|---|---|---|---|
| $U_1$ | 99.9 | 94.5 | 68.4 | 89.6 |
| $SU_2$ | 62.46 | 97.4 | 64.3 | 60.9 |
| $SU_3$ | 99.3 | 98.0 | 93.0 | 45.1 |
| $SU_4$ | 14.46 | 30.0 | 39.0 | 53.5 |
| $SU_5$ | 16.78 | 43.5 | 66.3 | 33.8 |
| $SU_{>5}$ | 0.185 | 1.7 | 10.6 | 73.0 |
| $SO_8$ | 0.482 | 1.6 | 6.2 | 21.1 |
| $SO_{10}$ | 0.084 | 0.2 | 1.6 | 18.7 |
| $SO_{>10}$ | 0.005 | 0.038 | 0.77 | 7.5 |
| $E_{6,7,8}$ | 0.0003 | 0.03 | 0.16 | 11.5 |

Table 3.6. Percentages of heterotic string models exhibiting specific gauge-group factors as functions of their spacetime supersymmetry. Here $SU_{>5}$ and $SO_{>10}$ collectively indicate gauge groups $SU(n)$ and $SO(2n)$ for any $n > 5$, while $\mathcal{N}$ refers to the number of unbroken supersymmetries at the string scale. Note that the $\mathcal{N}$=0 models are all tachyon-free.

- Groups with smaller rank are much more common than groups with larger rank. Once again, this also appears to hold for all levels of supersymmetry.
- Finally, the gauge-group factors comprising Standard-Model gauge group $G_{\rm SM} \equiv SU_3 \times SU_2 \times U_1$ are particularly common, much more so than those of any of its grand-unified extensions.

As we found in Sect. 4, the $\mathcal{N}$=0 string models dominate the tachyon-free portion of the heterotic landscape. Similarly, the $\mathcal{N}$=1 string models are the dominant part of the supersymmetric portion of the landscape. Nevertheless, it is interesting to examine the gauge-group probabilities across both of these portions of the landscape. These probabilities are easy to calculate by combining the results in Tables 3.4 and 4.2, leading to the results shown in Table 3.7.

Several features are immediately apparent from Table 3.7. First, gauge groups with larger ranks appear to be favored more strongly across the supersymmetric subset of the landscape than across the tachyon-free landscape as a whole. Since each of our heterotic string models in this class has a gauge group of fixed total rank, this preference for higher-rank gauge groups necessarily comes at the price of sacrificing smaller-rank gauge groups. Indeed, it often happens that this preference for larger-rank gauge groups actually precludes the appearance of any small-rank gauge groups whatsoever. Interestingly, the supersymmetric portion of the landscape seems to sacrifice $U(1)$ primarily and $SU(3)$ to a lesser extent. This is in contrast to

| gauge group | entire landscape | SUSY subset |
|---|---|---|
| $U_1$ | 98.00 | 93.89 |
| $SU_2$ | 73.22 | 96.62 |
| $SU_3$ | 98.85 | 97.88 |
| $SU_4$ | 19.42 | 30.21 |
| $SU_5$ | 25.37 | 44.03 |
| $SU_{>5}$ | 0.73 | 1.92 |
| $SO_8$ | 0.87 | 1.71 |
| $SO_{10}$ | 0.13 | 0.23 |
| $SO_{>10}$ | 0.02 | 0.06 |
| $E_{6,7,8}$ | 0.01 | 0.03 |

Table 3.7. Percentage of heterotic string models exhibiting specific gauge-group factors, quoted across the entire landscape of tachyon-free models (both supersymmetric and non-supersymmetric) as well as across only that subset of models with at least $\mathcal{N}\geq1$ spacetime supersymmetry. These results are derived from those of Table 4.2 using the landscape weightings in Table 3.4.

$SU(2)$, which is actually more strongly *favored* in the supersymmetric portion of the landscape than in the general tachyon-free landscape as a whole.

Second, the level of supersymmetry also appears to affect the probability distributions across the different possible gauge-group factors. The supersymmetric portion of the landscape has a much greater representation of the large rank groups. This suggests that the constraints placed on the string spectrum in order to preserve spacetime supersymmetry also have the effect of favoring larger gauge symmetries, a fact already noted in Sect. 5.2. In other words, there tends to be a decrease in the gauge-group multiplicity for highly shattered gauge groups which consist of only very small gauge-group factors, and thus the larger-rank gauge groups make up a larger proportion of the whole landscape. Indeed, this effect is particularly acute for that subset of the landscape exhibiting maximal $\mathcal{N}=4$ supersymmetry, where the larger-rank $SU$ gauge groups are particularly well represented.

## 3.6. Discussion

In this chapter, we have examined both the prevalence of spacetime supersymmetry across the heterotic string landscape and the statistical correlations between the appearance of spacetime supersymmetry and the gauge structure of the corresponding string models. Somewhat surprisingly, we found that nearly half of the heterotic landscape is non-supersymmetric and yet tachyon-free at tree level; indeed, less than a quarter of the tree-level heterotic landscape exhibits any supersymmetry at all at the string scale. Moreover, we found that the degree of spacetime supersymmetry is strongly correlated with the probabilities of realizing certain gauge groups, with unbroken supersymmetry at the string scale tending to favor gauge-group factors with larger rank.

There are several extensions to these results which are currently under investigation. For example, we would like to understand how the presence of supersymmetry affects the statistical appearance of the entire composite Standard-Model gauge group $G_{\rm SM} \equiv SU_3 \times SU_2 \times U_1$, and not merely the appearance of its individual factors. We would also like to understand how the presence or absence of supersymmetry affects other features which are equally important for the overall architecture of the Standard Model: these include the appearance of three chiral generations of quarks and leptons, along with a potentially correct set of gauge couplings and Yukawa couplings. This work has already been completed, and will be reported shortly [57].

Despite this progress, such studies have a number of intrinsic limitations which must continually be borne in mind. A number of these have been emphasized by us in recent articles (see, *e.g.*, the concluding sections of Refs. [54, 35]) and will not be repeated here. However, other limitations are particularly relevant for the results we have quoted here and thus deserve emphasis.

First, we must continually bear in mind that our study has been limited to models in which rank-cutting is absent. Thus, all of the four-dimensional heterotic string models we have examined exhibit a fixed maximal rank=22. This has the potential to skew the statistics of the different gauge-group factors. For example, it is possible that gauge-group factors with smaller ranks might be over-represented in this sample simply because the appearance of such groups is often the only way in which a given model can precisely saturate the total rank bound. By contrast, for models which can exhibit rank-cutting, this saturation would not be needed and it is therefore possible that lower-rank groups are consequently less abundant.

A second limitation of this study stems from the nature of performing random search studies in general. In Sect. 3.3, we summarized several methods by which the problematic issue of floating correlations can be transcended, and this has provided several examples of not only the need for such methods but also of the means by which they are implemented. As more fully discussed in Ref. [35], such problems are going to arise — and such methods are going to be necessary — whenever we attempt to extract statistical correlations from a large data set to which our computational access is necessarily limited. However, despite the apparent success of such methods, it is always a logical possibility that there exists a huge pool of string models with non-standard physical characteristics remaining just beyond our computational power to observe. The existence of such a pool of models would completely change the nature of our statistical results, to an extent which is essentially unbounded, yet we may miss this completely because of limited computational power. Although this becomes increasingly unlikely as our search through the landscape becomes larger and increasingly sophisticated, this nevertheless always remains a logical possibility which cannot be discounted.

But finally, perhaps the most serious limitation of our study is the fact that we are analyzing the statistical properties of string models which are not necessarily stable beyond tree level. Indeed, since none of our non-supersymmetric string models has a vanishing one-loop cosmological constant, these models in particular necessarily have non-zero dilaton tadpoles at one-loop order and thus become unstable. Even our supersymmetric models have flat directions which have not been lifted. Thus, as we have stressed throughout this chapter, the "landscape" we have examined in this chapter is at best a tree-level one. Despite this fact, however, it is important to realize that these models do represent self-consistent string solutions at tree level. Specifically, these models satisfy all of the constraints needed for worldsheet conformal/superconformal invariance, modular-invariant one-loop and multi-loop amplitudes, proper spacetime spin-statistics relations, and physically self-consistent layers of sequential GSO projections and orbifold twists. Indeed, since no completely stable perturbative heterotic strings have yet been constructed, this sort of analysis is currently the state of the art for large-scale statistical studies of this type. This mirrors the situation on the Type I side, where state-of-the-art statistical analyses have also focused on models which are only stable at tree level.

Nevertheless, we are then left with the single over-arching question: to what extent can we believe that the results we have found for this "tree-level" landscape actually apply to the true landscape that would emerge after all moduli are stabilized? The answer to this question clearly depends on the extent to which the statistical correlations we have uncovered here are likely to hold even after vacuum stabilization.

*A priori*, this is completely unknown. However, one surprising result of this chapter is the observation that the string self-consistency requirements themselves — even merely at tree-level — do *not* preferentially give rise to supersymmetric solutions at the string scale. Indeed, as we discussed in Sect. 4, less than a quarter of the tree-level heterotic landscape appears to exhibit any supersymmetry at all at the string scale. Thus, breaking supersymmetry without introducing tachyons is actually statistically *favored* over preserving supersymmetry, even at the string scale and even when the requirements of avoiding tachyons are implemented. Observations such as these tend to shift the burden of proof onto the SUSY enthusiasts, and perhaps reframe the question to one in which we might ask whether an unbroken supersymmetry is somehow *restored* by modulus stabilization. This seems unlikely, especially since most modern methods of modulus stabilization rely on breaking rather than introducing supersymmetry. In either case, however, this shows how the results of such studies — even though limited to only the tree-level landscape — can have the power to dramatically reframe the relevant questions. Indeed, once the technology for building heterotic string models develops further and truly stable vacua can be statistically analyzed in large quantities, it will be interesting to compare the statistical properties of those vacua with these in order to ascertain the degree to which vacuum stabilization might affect these other phenomenological properties.

Thus, it is our belief that such statistical landscape studies of this sort have their place, particularly when the results of such studies are interpreted correctly and in the proper context. As such, we hope that this study of the perturbative heterotic landscape may represent one small step in this direction.

# 4. Is SUSY Natural

## 4.1. Introduction

Most theoretical frameworks for physics beyond the Standard Model involve the introduction of supersymmetry (SUSY), and there are many reasons why this is so. First, supersymmetry solves the technical gauge hierarchy problem. Second, supersymmetry provides a dynamical mechanism for triggering electroweak symmetry breaking. Third, supersymmetry improves the accuracy of gauge coupling unification, and fourth, it provides a dark matter candidate. As a result, supersymmetry is truly ubiquitous in particle physics, appearing virtually everywhere — except the data.

However, through the years, lots of competing or alternative theories have been proposed. Some involve large extra dimensions and some involve small extra dimensions. Others do not explicitly involve extra dimensions at all, yet contain new, strongly coupled sectors. Likewise, especially over the past decade, the substance of phenomenological model-building has changed dramatically. Indeed, it is now quite common that a talk introducing a new particle-physics scenario will begin with a litany of sequential assumptions that would have sounded increasingly fantastic to the ear of a physicist a mere decade ago. We are made of open strings. And we live on a brane. And the brane lives in extra dimensions. And the brane is wrapped and intersects other branes. And the extra dimensions are warped. And the warping is severe and forms a throat. And the brane is falling into a throat. And so forth, and so on. Indeed, such scenarios now often form the backbone of cutting-edge model-building.

Admittedly, all of this may sound highly unnatural, and it is excusable to yearn for the simpler days of the MSSM and their cousins, the SUSY GUTs. But is SUSY itself truly natural? What does it mean to be "natural", anyway?

There are many different notions of naturalness that have appeared in the literature. For example, Dirac naturalness stipulates that an effective field theory (EFT) is natural if the dimensionless coefficients for all operators are $\sim \mathcal{O}(1)$ — *i.e.*, no exceedingly small or large numbers are allowed. In this sense, the large electroweak gauge hierarchy is unnatural, which is one of the biggest motivations for supersymmetry. Another notion is 't Hooft naturalness: even if such a number is small, it can be viewed as "natural" if its smallness is protected by a nearly unbroken symmetry.

But neither of these addresses the question of whether a theory, even if "natural" in the above sense, is *likely* to be right. How *likely* is SUSY to be the correct theory?

The word *likely* often causes us to shudder. Indeed, even though we constantly judge theories this way, we don't say this word aloud because the question of theoretical likelihoods seems more philosophical than scientific, especially when we have no data upon which to base our assessments. How likely relative to *what*? To all other theories that one can imagine? And who is doing the imagining? One might get very different responses depending on the identity of the unlucky proponent. Ultimately, we seem to be faced with a dead-end question. *How can one compare the likelihood of one theory against another?*

String theory provides a framework in which this question can be addressed in a mathematical way. This is because string theory provides a large set of possible vacuum solutions ("vacua", or string "models", collectively called "the landscape"), each of which corresponds to a different alternative universe with different physical laws. In this context, we can then place our likelihood question on firmer statistical footing, as originally advocated in Ref. [48]. *In the landscape of possible string solutions, how many of these solutions are supersymmetric? Is SUSY "natural" on this landscape, or relatively rare?*

This is the subject of this chapter.

## 4.2. Our Study: the Heterotic Landscape

In order to approach this question in a somewhat tractable way, we shall focus on the landscape associated with perturbative four-dimensional heterotic strings which are realizable using free-field constructions (such as those based on free worldsheet bosons or fermions). Our investigation will then focus on the fundamental question of determining the probability that such theories exhibit spacetime supersymmetry at the string scale. Note that this is different from previous discussions [49] in which a certain intrinsically supersymmetric framework [50] for the high-scale theory is assumed, and in which one asks about the likelihood that dynamical supersymmetry breaking is subsequently generated near the electroweak scale.

In a recent paper [51], we conducted a random exploration of the landscape associated with such models, using the free-fermionic construction method [52]. In this construction, each model can be described in terms of its left- and right-moving worldsheet conformal field theories. These conformal field theories consist of tensor products of non-interacting, free, complex fermionic fields. Different models are then achieved in this method by varying the boundary conditions of these worldsheet fermionic fields around the two non-contractible loops of the worldsheet torus. The sets of allowed boundary conditions for the fermionic fields are restricted by numerous string self-consistency conditions which also must be applied. These self-consistency conditions guarantee that the string partition function can be viewed as the trace over a Fock space corresponding to a self-consistent string model. Other perturbative self-consistency constraints (such as conformal and modular invariance) are also imposed.

This study was similar to earlier studies performed in Refs. [53, 54], and resulted in a data set of approximately $10^7$ distinct self-consistent four-dimensional heterotic string models, each of which is tachyon-free and hence stable at tree level. To the best of our knowledge, this is the largest set of distinct heterotic string models ever constructed. However, this study has some important limitations. For example, this study examined string models for which all real worldsheet fermions could be grouped together to form complex worldsheet fermions. This effectively reduces the number of consistent sets of fermion worldsheet boundary conditions, but allows for the utilization of many time-saving computational algorithms. The major phenomenological consequence of this restriction is that these string models have gauge groups with a fixed rank (namely twenty-two in four dimensions).

In our study, two models were considered distinct if their spacetime phenomenologies differ in some way. For example, two models were considered distinct if they differ in their amounts of unbroken supersymmetry, or in their gauge group or massless particle spectrum. Moreover, using data gathered from our sample set of models, a novel technique [55] was utilized in order to extract stable statistical results which do not vary as a function of sample size. As discussed in Ref. [55], this method transcends a mere statistical analysis of the models in our limited sample set, and yields results which are likely to be indicative of the corresponding landscape associated with such models as a whole. The techniques by which such results are extracted is discussed in Ref. [55].

## 4.3. Results

In this section, we shall describe the results of our study. As indicated above, the statistical methodology utilized in extracting these results is discussed in Ref. [55], and a full discussion of the results of Sects. 3.1 and 3.2 can be found in Ref. [51].

### 4.3.1. Prevalence of SUSY on the heterotic landscape

One of the first questions addressed in our study concerned the prevalence of spacetime supersymmetry in the heterotic landscape. In other words, what percentage of the heterotic string landscape consists of string models exhibiting an unbroken spacetime supersymmetry at the string scale? Clearly, answering this question is the first step towards addressing the overall issue of the naturalness of supersymmetry.

Our results [51] are reproduced in Table 4.1. Note that in quoting these results, we are explicitly focusing on only that portion of the heterotic string landscape which is stable at tree level. In other words, we are explicitly disregarding those non-supersymmetric portions of the landscape (which amount to approximately 32.1% of the total) for which tachyonic states exist at tree level.

Table 4.1 represents our final partitioning of the tree-level four-dimensional heterotic landscape according to its degree of supersymmetry. There are several rather striking facts which are evident from these results.

- First, we see that more than half of the stable heterotic landscape is non-supersymmetric and yet tachyon-free. Indeed, this proportion remains near half of the total even when the non-supersymmetric tachyonic models are included.

| SUSY class | % of heterotic landscape |
|---|---|
| $\mathcal{N}$=0 (tachyon-free) | 68.48 |
| $\mathcal{N}$=1 | 30.78 |
| $\mathcal{N}$=2 | 0.74 |
| $\mathcal{N}$=4 | 0.0044 |

Table 4.1. Classification of the four-dimensional heterotic landscape as function of the number of unbroken spacetime supersymmetries. We are explicitly focusing on string models which are stable (and thus tachyon-free) at tree level.

- Second, we see that the supersymmetric portion of the heterotic landscape appears to account for less than one-third of the full tachyon-free four-dimensional heterotic landscape.
- Finally, models exhibiting extended ($\mathcal{N} \geq 2$) supersymmetries are exceedingly rare, together representing less than one percent of the full landscape.

Of course, we stress once again that these results hold only for the *tree-level* landscape, *i.e.*, models which are stable at tree level only. *A priori*, it is not clear whether these results would persist after full moduli stabilization. However, it seems likely that they would, given that most modern methods of moduli stabilization (fluxes, superpotentials, *etc.*) tend to further break (rather than restore) spacetime supersymmetry. Indeed, under these assumptions, these results then lead to a number of interesting conclusions.

The first conclusion is that the properties of the tachyon-free heterotic landscape as a whole are statistically dominated by the properties of string models which do *not* have spacetime supersymmetry. Indeed, the $\mathcal{N}$=0 string models account for nearly three-quarters of this portion of the heterotic string landscape. The fact that the $\mathcal{N}$=0 string models dominate the tachyon-free portion of the landscape suggests that breaking supersymmetry without introducing tachyons is actually *favored* over preserving supersymmetry for this portion of the landscape. Indeed, we expect this result to hold even after full moduli stabilization (as has been argued within the context of Type I strings [56]), unless an unbroken supersymmetry is somehow restored by stabilization.

The second conclusion which can be drawn from these results is that the supersymmetric portion of the landscape is almost completely comprised of $\mathcal{N}$=1 string models. Indeed, only 2.4% of the supersymmetric portion of the heterotic landscape has more than $\mathcal{N}$=1 supersymmetry. This suggests that the correlations present for the supersymmetric portion of the landscape can be interpreted as the statistical correlations within the $\mathcal{N}$=1 string models, with the $\mathcal{N}$=2 correlations representing a correction at the level of 2% and the $\mathcal{N}$=4 correlations representing a nearly negligible correction.

In fact, the SUSY faction of the full string landscape may be even smaller than quoted here. One reason is that free-field string constructions (such as we are employing here) probably tend to artificially favor models with unbroken supersymmetry. Second, even when stabilized string models nevertheless exhibit spacetime SUSY at the string scale, there remains the difficult question of determining the statistical likelihood that this SUSY will survive all the way down to the electroweak scale [49]. At the very least, this should restrict the number of string models leading to weak-scale SUSY still further.

Thus, we conclude that weak-scale SUSY is rather *unnatural* from a string landscape perspective. On the one hand, this result shifts the burden of proof onto the SUSY enthusiasts, which represents a dramatic reframing of the underlying question of whether SUSY should exist at or above electroweak scale. But the conclusion that weak-scale SUSY is unnatural should not necessarily be viewed as a problem for string phenomenology. In fact, this result might even be considered good news: it implies that we will actually learn something about string theory and its preferred compactifications if/when weak-scale supersymmetry is actually discovered in upcoming collider experiments!

### 4.3.2. Correlations between SUSY and gauge groups

We shall now examine the effects of supersymmetry on the probability of realizing different gauge group factors in this landscape. What percentage of string models with a given level of supersymmetry will contain a given gauge group factor amongst its unbroken gauge group at the string scale? We shall also be interested in knowing the likelihood of realizing different gauge group factors for the full landscape as a whole.

Our results [51] are presented in Table 4.2. As can clearly be seen, supersymmetry has a profound effect upon the prevalence of different gauge group factors. Moreover, even independently of SUSY, there are some general trends which emerge from these results. These trends include:

- A preference for $SU(n+1)$ over $SO(2n)$ groups for each rank $n$. Even though these two groups have the same rank, it seems that $SU$ groups are more common than the $SO$ groups for all levels of supersymmetry.

- Groups with smaller rank are much more common than groups with larger rank. Once again, this also appears to hold for all levels of supersymmetry.

- Finally, the gauge-group factors comprising Standard-Model gauge group $G_{\text{SM}} \equiv SU(3) \times SU(2) \times U(1)$ are particularly common, much more so than those of any of its grand-unified extensions.

| gauge group | $\mathcal{N}$=0 | $\mathcal{N}$=1 | $\mathcal{N}$=2 | $\mathcal{N}$=4 | full landscape |
|---|---|---|---|---|---|
| $U_1$ | 99.9 | 94.5 | 68.4 | 89.6 | 98.0 |
| $SU_2$ | 62.46 | 97.4 | 64.3 | 60.9 | 73.2 |
| $SU_3$ | 99.3 | 98.0 | 93.0 | 45.1 | 98.9 |
| $SU_4$ | 14.46 | 30.0 | 39.0 | 53.5 | 19.4 |
| $SU_5$ | 16.78 | 43.5 | 66.3 | 33.8 | 25.4 |
| $SU_{>5}$ | 0.185 | 1.7 | 10.6 | 73.0 | 0.73 |
| $SO_8$ | 0.482 | 1.6 | 6.2 | 21.1 | 0.87 |
| $SO_{10}$ | 0.084 | 0.2 | 1.6 | 18.7 | 0.13 |
| $SO_{>10}$ | 0.005 | 0.038 | 0.77 | 7.5 | 0.021 |
| $E_{6,7,8}$ | 0.0003 | 0.03 | 0.16 | 11.5 | 0.011 |

Table 4.2. The percentage of heterotic string models exhibiting specific gauge group factors as functions of their spacetime supersymmetry. Here $SU_{>5}$ and $SO_{>10}$ collectively indicate gauge groups $SU(n)$ and $SO(2n)$ for any $n > 5$, while $\mathcal{N}$ refers to the number of unbroken supersymmetries at the string scale. Note that the $\mathcal{N}$=0 models are all tachyon-free. The rightmost column of this table is derived from the other columns using the landscape weightings in Table 4.1.

### 4.3.3. SUSY naturalness

At this point, we have presented both the unrestricted probability of finding different levels of supersymmetry on the heterotic landscape, and the restricted probability of finding different gauge group factors when a certain degree of supersymmetry is assumed. However, a more useful quantity might be the "inverse" of this last probability, namely the probability of finding different levels of supersymmetry given a specific gauge group factor. This would give an indication of how "natural" each level of supersymmetry is for each different possible gauge group factor.

In order to derive these probabilities, we can utilize the results presented above. First, let us recall some basic elements of probability theory. The results in the first four columns of Table 4.2 are necessarily conditional probabilities: each number in these columns represents the probability of finding a specific gauge group factor *given* a certain level of supersymmetry. Thus, if $A$ represents the occurrence of a specific gauge group factor and $B$ represents the occurrence of a specific level of supersymmetry, then the results presented in Table 4.2 are all of the form

$$P(A \mid B) \equiv \frac{P(A \cap B)}{P(B)} . \tag{4.1}$$

Of course, what we now seek is not $P(A \mid B)$, but the "inverse" $P(B \mid A)$. In general,

| SUSY | $U_1$ | $SU_2$ | $SU_3$ | $SU_4$ | $SU_5$ | $SU_{>5}$ | $SO_8$ | $SO_{10}$ | $SO_{>10}$ | $E_{6,7,8}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{N}=0$ | 69.80 | 58.41 | 68.79 | 50.98 | 45.29 | 17.33 | 37.98 | 43.68 | 16.21 | 1.85 |
| $\mathcal{N}=1$ | 29.68 | 40.94 | 30.51 | 47.53 | 52.78 | 71.56 | 56.66 | 46.75 | 55.38 | 83.00 |
| $\mathcal{N}=2$ | 0.51 | 0.65 | 0.69 | 1.48 | 1.92 | 10.65 | 5.25 | 8.95 | 26.84 | 10.59 |
| $\mathcal{N}=4$ | 0.004 | 0.002 | 0.002 | 0.012 | 0.006 | 0.44 | 0.11 | 0.63 | 1.57 | 4.57 |

Table 4.3. The percentage of string models with different levels of supersymmetry as a function of different gauge group factors. Thus, if we know that a given string model gives rise to a specific gauge group factor at the string scale, this table lists the corresponding probabilities that this model will have various levels of unbroken supersymmetry. This table can therefore be viewed as the "inverse" of Table 4.2.

the relationship between $P(B \mid A)$ and $P(A \mid B)$ is given as

$$P(B \mid A) ~=~ \frac{P(B)}{P(A)}\, P(A \mid B)~. \tag{4.2}$$

Fortunately, all of the probabilities needed within the right side of Eq. (4.2) are present in Tables 4.1 and 4.2. Specifically, $P(A)$ is given within the rightmost column of Table 4.2, while $P(B)$ is given in Table 4.1 and $P(A \mid B)$ is given in the rest of Table 4.2. Thus, given the presence of specific gauge group factor at the string scale, we can now determine the corresponding probability of finding different levels of unbroken supersymmetry at the string scale. Our results are presented in Table 4.3. Note that in calculating the results in Table 4.3, we have retained more significant digits than are explicitly shown in Tables 4.1 or 4.2.

By comparing these probabilities to the probabilities given in Table 4.1, it is possible to determine which gauge group factors tend to favor different levels of supersymmetry *beyond their expected representations on the landscape as a whole*. For example, we see from Table 4.1 that 68.48% of the stable heterotic landscape is non-supersymmetric. Thus, if a given gauge group factor is associated with models of which fewer than 68.48% are non-supersymmetric, then this gauge group factor can be said to preferentially *favor* supersymmetry.

These results have a number of dramatic implications.

- First, we observe that gauge group factors with large rank (greater than four) actually *favor* the appearance of unbroken supersymmetry.
- Second, we observe that the gauge group factors which comprise the Standard Model gauge group do *not* generally favor supersymmetry.
- Finally, we see that the $SU(n)$ gauge group factors (with $n > 5$) and the exceptional gauge groups $E_{6,7,8}$ overwhelmingly favor $\mathcal{N} = 1$ supersymmetry. This

preference is substantial, resulting from the combined effects of the individual probabilities contributing to Eq. (4.2).

We thus conclude that the heterotic string landscape appears to favor either the non-supersymmetric Standard Model gauge group or an $\mathcal{N} = 1$ SUSY GUT gauge group at the string scale. However, the opposite outcomes (namely the MSSM or a non-SUSY GUT gauge group) are significantly disfavored.

One important caveat is that the gauge group factors presented in Table 4.3 do *not* generally specify the gauge group fully. Indeed, these gauge group factors could be part of either a hidden sector or the visible sector. However, the gauge group factors listed in these tables are necessarily among those which are explicitly present in the full gauge group of the string model at the string scale.

## 4.4. Discussion

In this chapter, we have presented results concerning the prevalence of spacetime supersymmetry at the string scale and its possible statistical correlations with the unbroken gauge group which might also appear at the string scale. Since these two quantities (spacetime supersymmetry and unbroken gauge group) are completely independent in an ordinary quantum field theory based on point particles, these sorts of correlations represent true predictions of string theory and thereby provide one possible route towards answering the question as to whether supersymmetry is truly "natural" as a component of Beyond-the-Standard-Model physics.

As we have seen, the results of this calculation show that spacetime supersymmetry is *not* generically a feature of the low-energy limit of string theory. However, spacetime supersymmetry is actually statistically favored for certain gauge group factors.

There are some inherent limitations to these results which must continually be borne in mind. For example, these string models are generally unstable: the non-supersymmetric models generically have non-zero dilaton tadpoles, and the supersymmetric string models have flat directions. Thus, one might think that these results might change if only fully-stabilized models are considered. Unfortunately, no fully stable perturbative heterotic string models have ever been constructed. One could even argue that if the non-supersymmetry string models were required to be as "stable" as the supersymmetric string models (*e.g.*, only have some finite number of flat directions), then these results would also change. However, at this point in time, the string models considered in this study are state-of-the-art and are as stable as many of the other classes of string models which have been considered in other statistical studies.

Another issue facing this study concerns the extent to which these sorts of statistical correlations can be trusted, given that the full heterotic landscape has not been surveyed. However, this issue has been discussed in Ref. [55], and the methods developed there have been used in order to extract each of the results quoted here. Thus, to the best of our knowledge, the results quoted here are independent of the size of our sample of heterotic string models, and thus should persist across the landscape as a whole.

Finally, one could argue that the construction method utilized in this study necessarily only probes certain sections of the heterotic landscape. While this is true, these sections are the ones most likely to contain string models realizing non-abelian gauge symmetries. As such, these are the sections most likely to give rise to non-trivial low energy phenomenologies.

The interpretation of these results is also open to some debate. The probability-based definition of naturalness used in this chapter is not the traditional one, and may only hold relevance for a landscape study such as the one we are performing. However, this definition of naturalness has the advantage of being applicable in a wide variety of contexts, and does not resort to any aesthetic or theoretical prejudices concerning the parameters that appear in effective Lagrangians. As such, probability-based definitions of naturalness may have inherent advantages over other definitions.

There are several extensions to these results which are currently under investigation. For example, we would like to understand how the presence of supersymmetry affects the statistical appearance of the entire composite Standard-Model gauge group $G_{\text{SM}} \equiv SU(3) \times SU(2) \times U(1)$, and not merely the appearance of its individual factors. We would also like to understand how the presence or absence of supersymmetry affects other features which are equally important for the overall architecture of the Standard Model: these include the appearance of three chiral generations of quarks and leptons, along with a potentially correct set of gauge couplings and Yukawa couplings. This work will be reported elsewhere [57].

# 5. Conclusion

In this thesis we started with explaining the importance of getting N=1 SUSY GUTs for any theory of physics beyond the Standard Model. We talked about string theory being one of the leading contenders for physics beyond the Standard Model. We introduced the basics of string theory.

We talked about non-abelian orbifolds as being an important construction which would give us N=1 SUSY scalar adjoints (which are required for GUT theories) in string theory. The methods for non-abelian orbifold string model construction were reviewed. Construction of $D_4$ non-abelian orbifold string models were explicity presented. It was shown that $D_4$ models didn't produce chiral fermions and hence were unsuitable for phenomenology. Proof for absence of phenomenologically pleasing chiral GUT models in non-abelian orbifold constructions was also presented.

The results of statistical studies on heterotic string models were presented. Statistical methods that took into consideration the issue of floating correlations were reviewed. Correlations between appearance of degree of supersymmetry versus the gauge groups were studied. The relative abundance of various degrees of supersmmetry in the heterotic landscape were found. We found that nearly half of the heterotic landscape is nonsupersymmetric and yet tachyon-free at tree level; indeed, less than a quarter of the tree-level heterotic landscape exhibits any supersymmetry at all at the string scale. The relatives probabilities for finding different gauge groups as a function of degree of supersymmetry were also presented. It was found that the heterotic landscape favours either the non-supersymmetric Standard Model gauge group or an N=1 SUSY GUT gauge group at the string scale. The opposite outcomes (namely the MSSM or a non-supersymmetric GUT) are significantly disfavored.

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