SPECTRAL PROPERTIES OF THE RENORMALIZATION GROUP

by

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A Dissertation Submitted to the Faculty of the

Department of Mathematics

In Partial Fulfillment of the Requirements
For the Degree of

Doctor of Philosophy

In the Graduate College

The University of Arizona

2 0 1 0

The University of Arizona
Graduate College

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prepared by Mei Yin
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Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copies of the dissertation to the Graduate College.

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Statement by Author

Acknowledgments

I owe deep gratitude to my advisor Bill Faris for his continued help and support in my grad school years. I have learned a lot from him, not only about math, but also about life at large. He is such an amazing person. Without his guidance, this work would not have turned into a dissertation.

I am very grateful to my minor advisor Rabi Bhattacharya, and my committee members Tom Kennedy, Doug Pickrell, and Bob Sims for their kind and helpful suggestions and comments towards writing my thesis.

There are many other professors who have played an important role in my professional growth. I thank Dave Savitt, Philip Foth, Ken McLaughlin, Jan Wehr, John Palmer, Sean Fleming, Walter Piegorsch, Doug Ulmer, and Yi Hu for everything they have taught me, and for their confidence in me.

The Department of Mathematics at the University of Arizona has been an extremely nurturing and stimulating environment where I have been fortunate to spend the last four years. I will miss this place dearly after graduation.

Finally, I would like to thank my friends and family, especially my mom, for their active support in all my endeavors.

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Abstract

The renormalization group (RG) approach is largely responsible for the considerable success which has been achieved in developing a quantitative theory of phase transitions. This work investigates various spectral properties of the RG map for Ising-type classical lattice systems. It consists of four parts. The first part carries out some explicit calculations of the spectrum of the linearization of the RG at infinite temperature, and discovers that it is of an unusual kind: dense point spectrum for which the adjoint operators have no point spectrum at all, but only residual spectrum. The second part presents a rigorous justification of the existence and differentiability of the RG map in the infinite volume limit at high temperature by a cluster expansion approach. The third part continues the theme of the third part, and shows that the matrix of partial derivatives of the RG map displays an approximate band property for finite-range and translation-invariant Hamiltonians at high temperature. The last part justifies the differentiability of the RG map in the infinite volume limit at the critical temperature under a certain condition. In summary, the first part deals with special cases where exact computations can be done, whereas the remaining parts are concerned with a general theory and provide a mathematically sound base.

Chapter 1

INTRODUCTION AND BACKGROUND

1.1 Statistical mechanics

A central problem in mathematical physics is to study random systems of very high dimension. A classical setting for this is equilibrium statistical mechanics, for instance of particle systems or spin systems or alloys. The number of dimensions in such examples is huge, typically something like 10^{24}. For practical purposes, it might just as well be infinite. Another possible setting is that of random fields or quantum fields. In this case the number of dimensions, corresponding to the number of Fourier modes, is already infinite.

Equilibrium statistical mechanics in its modern form was formulated by Gibbs. He gave a probability model that was suitable for mathematical analysis. With considerable effort, the mathematical analysis of this kind of model could be accomplished in extremal situations, such as low density or high temperature. These techniques fail, however, for regions in which there are phase transitions and associated critical phenomena.

In the most elegant formulation of Gibbsian statistical mechanics, the system is considered in infinite volume. This bypasses, at least in part, complications due to boundary effects. A particular system is determined by a Hamiltonian H, which is a sum of infinitely many terms. The individual terms are described by a collection J of coupling constants, which may also be referred to as interactions. The idea is to make rigorous statements about the interactions. Formally, giving the interactions is the same as giving the Hamiltonian, but in infinite volume, the sum for the Hamiltonian will not converge. In the future, when we speak of a Hamiltonian, we are considering this as an informal way of talking about the interactions, or we are considering a finite volume approximation.

1.2 Critical phenomena

In his 1982 Nobel Prize lecture, Kenneth Wilson [24, page 583] described the context of the problem under consideration in this dissertation.

> There are a number of problems in science which have, as a common characteristic, that complex microscopic behavior underlies macroscopic effects.
>
> In simple cases the microscopic fluctuations average out when larger scales are considered, and the averaged quantities satisfy classical continuum equations. Hydrodynamics is a standard example of this, where atomic fluctuations average out and the classical hydrodynamic equations emerge. Unfortunately, there is a much more difficult class of problems where fluctuations persist out to macroscopic wavelengths, and fluctuations on all intermediate length scales are important too.
>
> In this last category are the problems of fully developed turbulent fluid flow, critical phenomena, and elementary-particle physics. The problem of magnetic impurities in nonmagnetic metals (the Kondo problem) turns out also to be in this category.

As a specific example of critical phenomena, Wilson [24, page 583] discussed the most obvious example, the water-steam transition.

A critical point is a special example of a phase transition. Consider, for example, the water-steam transition. Suppose the water and steam are placed under pressure, always at the boiling temperature. At the critical point—a pressure of 218 atm and temperature of 374^{o}C (Weast, 1981)—the distinction between water and steam disappears, and the whole boiling phenomenon vanishes. The principal distinction between water and steam is that they have different densities. As the pressure and temperature approach their critical values, the difference in density between water and steam goes to zero. At the critical point one finds bubbles of steam and drops of water intermixed at all size scales from macroscopic, visible sizes down to atomic scales. Away from the critical point, surface tension makes small drops or bubbles unstable; but as water and steam become indistinguishable at the critical point, the surface tension between the two phases vanishes. In particular, drops and bubbles near micron sizes cause strong light scattering, called "critical opalescence," and the water and steam become milky.

1.3 Universality

There is an astonishing empirical fact that certain exponents associated with critical phenomena are universal, in the sense that they depend only on overall features of the system, and not on a detailed specification of parameters. Here is a list of six such exponents:

- α describes how the specific heat diverges.
- β describes how the spontaneous magnetization converges to zero.

- γ describes how the magnetic susceptibility diverges.
- δ describes how the magnetization at critical temperature goes to zero as the magnetic field is turned off.
- ν describes how the correlation length diverges.
- η describes how the correlation function at critical temperature decays.

These exponents describe behavior at the critical point. For instance, the divergence to infinity of the specific heat as a function of temperature T is

$$C(T) \sim \frac{1}{|T - T_c|^{\alpha}},$$

where T_c is the critical temperature. At the critical point, the correlation as a function of separation r is given by

$$G(r) \sim \frac{1}{r^{d-2+\eta}},$$

where $d \geq 2$ is the space dimension. The other exponents have similar definitions. Furthermore, it is observed that there are exact nontrivial relations between the exponents, known as scaling laws:

Rushbrooke scaling law: $\alpha + 2\beta + \gamma = 2$.

Widom scaling law: $\gamma = \beta(\delta - 1)$.

Fisher scaling law: $\gamma = \nu(2 - \eta)$.

Josephson scaling law: $\nu d = 2 - \alpha$.

So the critical exponents in a universality class for fixed space dimension d, at first seemingly unrelated, should form a two-parameter family.

There are various universality classes. For $d = 2$ Ising systems, the values are

$$\alpha = 0, \qquad \beta = \frac{1}{8}, \qquad \gamma = \frac{7}{4},$$

$$\delta = 15, \qquad \nu = 1, \qquad \eta = \frac{1}{4}.$$

For $d = 3$ Ising systems, the exponents are only known from numerical computations. Fisher [7, page 660] shared his insights on universality in a lecture on the foundations of quantum field theory:

> And, beyond fluids and anisotropic ferromagnets, many other systems belong—more correctly their critical behavior belongs—to the "Ising universality class." Included are other magnetic materials (antiferromagnets and ferrimagnets), binary metallic alloys (exhibiting order-disorder transitions), certain types of ferroelectrics, and so on.
>
> For each of these systems there is an appropriate order parameter and, via Eq. (2), one can then define (and usually measure) the correlation decay exponent η which is likewise universal. Indeed, essentially any measurable property of a physical system displays a universal critical singularity.

1.4 Renormalization group

A historical breakthrough was the renormalization group (RG) as formulated by Wilson [24, page 584].

> The "renormalization-group" approach is a strategy for dealing with problems involving many length scales. The strategy is to tackle the problem in steps, one step for each length scale. In the case of critical

> phenomena, the problem, technically, is to carry out statistical averages over thermal fluctuations on all size scales. The renormalization-group approach is to integrate out the fluctuations in sequence, starting with fluctuations on an atomic scale and then moving to successively larger scales until fluctuations on all scales have been averaged out.

His work [23] has received extensive attention in the physics literature, both in the context of statistical mechanics and in the context of quantum field theory. Our focus will be on one approach, which is called the real-space RG. References in the physics literature for this topic may be found in the contribution of Niemeijer to the Domb and Green series [17, Volume 6].

In this approach to statistical mechanics, a renormalization group transformation consists of averaging over short distance blocks, followed by a rescaling that maps each block into a single lattice point. There are many possible methods of averaging, so there are many possible renormalization group transformations. Some popular candidates are decimation, majority rule, and the Kadanoff transformation. Away from the manifold of critical points, iterations of such a transformation should drive the system to a new system that is very far from critical. So one would expect the most interesting behavior to arise from the transformation applied to critical systems, or near-critical systems.

Among possible renormalization group transformations, there may be some that are good, in the sense that they are well-defined and have a fixed point with desirable properties. This fixed point should be a critical interaction, and the stable manifold of the fixed point should also consist of critical interactions.

In situations like the Ising system, the fixed point belongs to a stable manifold of codimension two in the space of Hamiltonians. The Hamiltonians on this stable manifold describe systems at a critical point, and iterations of the RG would take all points in this manifold to the fixed point. While there are infinitely many critical Hamiltonians, miraculously, their long distance properties are all described by the same fixed point Hamiltonian. This helped Wilson [24, page 592] realize the magical power of RG.

> This work showed me that a renormalization-group transformation, whose purpose was to eliminate an energy scale or a length scale or whatever from a problem, could produce an effective interaction with arbitrarily many coupling constants, without being a disaster. The renormalization-group formalism based on fixed points could still be correct, and furthermore one could hope that only a small finite number of these couplings would be important for the qualitative behavior of the transformations, with the remaining couplings being important only for quantitative computations.

This universality property is successfully explained by RG. Consider a good RG transformation. At the fixed point of interest, its linearization has spectrum contained in the unit disc except for two real eigenvalues which are strictly greater than 1. These eigenvalues are usually written in the form b^{y_T} and b^{y_H}, where b is the length rescaling factor.

Suppose we start with a Hamiltonian that is close to critical. The RG transformation will initially drive it toward the fixed point for a large number of iterations. When the transformed Hamiltonians are near the fixed point, their behavior will be governed by the linearization. In fact, all six of the critical exponents are related to the two eigenvalues of the linearization [12]:

$$\alpha = 2 - \frac{d}{y_T}, \qquad \beta = \frac{d - y_H}{y_T}, \qquad \gamma = \frac{2y_H - d}{y_T},$$

$$\delta = \frac{y_H}{d - y_H}, \qquad \nu = \frac{1}{y_T}, \qquad \eta = d + 2 - 2y_H.$$

For the $d = 2$ Ising model, $y_T = 1$ and $y_H = \frac{15}{8}$. This agrees with the exposition above. (More discussions may be found in [7] and [24].)

The practice in theoretical physics is to calculate approximate versions of this transformation. One neglects certain terms, or argues that certain terms become irrelevant. In order to do rigorous work in RG theory, one must apply expansion methods that include all terms, even to determine the RG transformation. Fortunately, if one thinks of the expansions defining the RG transformation as analogous to other expansions in statistical mechanics, then the expansions that define the RG map are not necessarily in the critical region. Therefore there is hope of using convergent expansions to define and analyze the properties of the RG transformation, to be followed by an analysis of the iterations of the transformation and of convergence properties near fixed points.

The mathematical work on RG transformation is extensive, but it is fair to say that many if not most questions remain unanswered. An important survey article by van Enter, Fernández, and Sokal [22] gives a widely accepted mathematical framework. This framework involves a mathematically precise notion of Gibbs state for a system with infinite many dimensions. These authors and others have shown that for some RG maps the image measure is not Gibbsian.

From now on, our discussion will mainly involve the RG for the case of the Ising model. The idea is to describe random functions σ on the spatial lattice $\mathbb{Z}^d$ with values in a space S. For the Ising model $S = \{-1, 1\}$ and is called the spin space. The random functions $\sigma : \mathbb{Z}^d \to S$ are described by a probability measure μ on the product space $S^{\mathbb{Z}^d}$. This measure has a formal density with respect to a reference product measure given by a multiple of $\exp(-H(\sigma))$. Here

$$H(\sigma) = \sum_{X \neq \emptyset} J(X) \prod_{x \in X} \sigma_x. \tag{1.1}$$

This is a divergent sum, so the expression for the density is only formal. Fortunately, when the coupling constants J are sufficiently local, in the sense that they belong to a certain Banach space, then it is possible to give a precise meaning to the notion that the probability measure μ is associated with the coefficients J. In this case the measure μ is called a Gibbs measure associated with J. (See [22] for a precise definition.) In general there may be more than one Gibbs measure μ associated with a given J, but this will typically happen only at special J for which there are multiple phases determined by "boundary conditions at infinity."

The RG action (integration followed by rescaling) maps a measure μ to another measure μ'. Suppose, as above, that μ arises from J, and furthermore suppose that μ is uniquely determined by J. Then μ' is also determined by J, and, if μ' is a Gibbs measure, then J' is determined by J. This map on coupling constants is the main concern of this dissertation.

1.5 Cluster expansion

A first attempt to analyze the RG for the Ising model was by expansion techniques at high temperature. This was in the work of Robert Israel [10], who used beautiful and ingenious techniques involving Banach algebras and conditional expectations. The Ising model at high temperature was also analyzed by Kashapov [11], using combinatorial techniques developed by Malyshev [14]. My dissertation also investigates the high temperature examples for the RG for the Ising model. However, this work uses a different technique—cluster expansion in a certain Banach space framework.

The subject of cluster expansion has been applied in various areas of mathematical physics. The techniques for performing such expansions have been developed by mathematical physicists over a period of many years. One important approach to a cluster expansion is by a sum indexed by connected graphs. The edges of the graphs correspond to various possible interaction terms. A key reference to the literature is the paper of Penrose [20] that clarified the relationship of sums over connected graphs to sums over trees. There are considerably fewer trees than connected graphs, so this gives a decisive simplification. Other more abstract techniques were developed by Kotecký and Preiss [13], and by other authors. Some references for this literature may be found, for instance, in the paper of Ueltschi [21], or a survey by Faris [5]. The cluster expansion technique depends on the exploitation of some sort of independence or locality properties to develop a representation. The combinatorial techniques mentioned above are then applied to this representation. For examples of the many successful applications of this framework, one can consult the book by Malyshev and Minlos [15].

The real interest of the RG is to define the transformation at intermediate temperature, in particular, the critical temperature. This is a considerably more difficult enterprise: one could worry about the issues raised by van Enter, Fernández, and Sokal [22], questioning whether the transformation is even defined. Fortunately, there is some hope for progress in this area due to the fact that the correlation length of the constrained system relevant to the definition of the RG transformation may well be finite, and may even sometimes be used as a small parameter. There is relatively little rigorous work in this direction. Pioneering efforts were made by Olivieri and his various collaborators [3] [18] [19]. Another approach in a similar spirit was developed in the important work of Haller and Kennedy [9].

1.6 Outline

A central theme of my dissertation is the existence and properties of the linearization of the RG transformation. The RG transformation may be thought of as a map of potentials that define the Gibbs state. The linearization then should be a linear transformation acting on such a Banach space.

The outline of my dissertation is the following:

Chapter 2: This part sets up the basic framework and introduces common notations that will be used throughout the dissertation.

Chapter 3: This part is inspired by Israel's 1979 paper [10], in which he found the eigenvalues of the linearization of the RG map at the trivial fixed point (zero interaction) corresponding to decimation and majority rule. We delve more into this matter and analyze the spectrum of these RG operations completely. It is of an unusual kind: dense point spectrum for which the adjoint operators have no point spectrum at all, but only residual spectrum. This may serve as a lesson in what one might expect in more general situations.

Chapter 4: Our basic assumption in this part is that the original interaction J is at high temperature. A cluster expansion is used to justify the existence of the renormalized interaction J' (and the partial derivatives $\frac{\partial J'(Z)}{\partial J(W)}$) in the infinite volume limit. The idea is that one starts with a J and finds a complicated but explicit expansion that defines a J' (and $\frac{\partial J'(Z)}{\partial J(W)}$). This expansion is derived from the formal expressions, but it is itself well-defined and convergent.

Chapter 5: Continuation of Chapter 4. The original interaction J is at high temperature. In addition, we assume that it is finite-range and translation-invariant.

We will show that the matrix of partial derivatives $\frac{\partial J'(Z)}{\partial J(W)}$ in this case displays an approximate band property.

Chapter 6: This part is inspired by Haller and Kennedy's 1996 paper [9], in which they studied the RG map of finite-range and translation-invariant Hamiltonians at the critical temperature. With the help of Dobrushin uniqueness condition and standard results on the polymer expansion, they discovered a condition that guarantees the existence of the infinite volume limit of the renormalized Hamiltonian. The present investigation extends their theoretical result. Under the same assumptions, we will show that the partial derivatives of the RG transformation exist.

1.7 Perspective

It seems appropriate to conclude this introduction with a quotation by Fisher [7, page 677] that hints at the broader implications of this subject.

> From this global standpoint, renormalization group theory represents a theoretical tool of depth and power. It first flowered luxuriantly in condensed matter physics, especially in the study of critical phenomena. But it is ubiquitous because of its potential for linking physical behavior across disparate scales; its ideas and techniques play a vital role in those cases where the fluctuations on many different physical scales truly interact. But it provides a valuable perspective—through concepts such as 'relevance,' 'marginality' and 'irrelevance,' even when scales are well separated!

Chapter 2

FRAMEWORK AND NOTATION

We consider renormalization group (RG) transformations for Ising-type lattice spin systems on $\mathbb{Z}^d$. Our original lattice is a set denoted by $\mathcal{L}$, and our image lattice is a set denoted by $\mathcal{L}'$ (both finite or countably infinite). $\mathcal{L}'$ indexes a partition of $\mathcal{L}$ into blocks, all with the same cardinality s. Thus for each site y in $\mathcal{L}'$, there is a corresponding block y^o that is a subset of $\mathcal{L}$. More generally, for each subset Y of $\mathcal{L}'$, there is a corresponding union of blocks Y^o that is a subset of $\mathcal{L}$. $\mathcal{L}$ is endowed with a metric d, and thus naturally induces a metric d' on $\mathcal{L}'$. For site x and subset X, we define the distance between x and X by

$$l(x, X) = \sup\{d(x, y) : y \in X\}. \tag{2.1}$$

For future purposes (in connection with finite-range), we also define the volume of X by

$$\mathrm{diam}(X) = \sup_{x \in X} l(x, X). \tag{2.2}$$

A spin variable $\sigma_x = \pm 1$ is assigned to each site x in $\mathcal{L}$, and a block spin variable $\sigma'_y = \pm 1$ is assigned to each site y in $\mathcal{L}'$. If X is a finite subset of the original lattice, then σ_X denotes the spin variable $\prod_{x \in X} \sigma_x$. Similarly, if Z is a finite subset of the image lattice, then σ'_Z denotes the block spin variable $\prod_{z \in Z} \sigma'_z$. The main physical properties of $\mathcal{L}$ are encoded in the Hamiltonian $H(\sigma) = -\sum_X J(X)\sigma_X$, where J is the original interaction defined on nonempty finite subsets of $\mathcal{L}$. Likewise, the main physical properties of $\mathcal{L}'$ are encoded in the Hamiltonian $H'(\sigma') = -\sum_Y J'(Y)\sigma'_Y$, where J' is the resulting interaction defined on nonempty finite subsets of $\mathcal{L}'$.

Here is the definition of the RG map for finite lattices $\mathcal{L}$ and $\mathcal{L}'$:

$$\frac{e^{\sum_Y J'(Y)\sigma'_Y}}{\sum_{\sigma'} e^{\sum_Y J'(Y)\sigma'_Y}} = \frac{\sum_\sigma \prod_{y\in\mathcal{L}'} T_y(\sigma,\sigma'_y) e^{\sum_X J(X)\sigma_X}}{\sum_\sigma e^{\sum_X J(X)\sigma_X}}, \tag{2.3}$$

where $\sum_\sigma$ and $\sum_{\sigma'}$ (normalized sums) denote the product probability measures on $\{+1,-1\}^{\mathcal{L}}$ and $\{+1,-1\}^{\mathcal{L}'}$, respectively, and $T_y(\sigma,\sigma'_y)$ denotes a specific RG probability kernel, which depends only on σ through the block corresponding to y, and satisfies both a symmetry condition,

$$T_y(\sigma,\sigma'_y) = T_y(-\sigma,-\sigma'_y), \tag{2.4}$$

and a normalization condition,

$$\sum_{\sigma'} T_y(\sigma,\sigma'_y) = 1 \tag{2.5}$$

for every σ and every y. Notice that because of (2.4) and (2.5),

$$\sum_\sigma T_y(\sigma,+1) = \sum_\sigma T_y(\sigma,-1) = 1. \tag{2.6}$$

Proposition 2.0.1. *The renormalized coupling constants J' are given by the expression*

$$J'(Z) = \sum_{\sigma'} \sigma'_Z \log(W(\sigma')), \tag{2.7}$$

where $W(\sigma')$ is the frozen block spin partition function given by

$$W(\sigma') = \sum_\sigma \prod_{y\in\mathcal{L}'} T_y(\sigma,\sigma'_y) e^{\sum_X J(X)\sigma_X}. \tag{2.8}$$

Proof. In order to write down an explicit expression of J', we use Fourier series on the group $\{+1,-1\}^{\mathcal{L}'}$. If $H'(\sigma') = -\sum_Y J'(Y)\sigma'_Y$, then $J'(Z) = \sum_{\sigma'} -H'(\sigma')\sigma'_Z$.

We see that

$$J'(Z) = \sum_{\sigma'} \sigma'_Z \log \left(\sum_{\sigma} \prod_{y \in \mathcal{L}'} T_y(\sigma, \sigma'_y) e^{\sum_X J(X)\sigma_X} \right)$$
$$+ \sum_{\sigma'} \sigma'_Z \log \left(\sum_{\sigma'} e^{\sum_Y J'(Y)\sigma'_Y} \right) - \sum_{\sigma'} \sigma'_Z \log \left(\sum_{\sigma} e^{\sum_X J(X)\sigma_X} \right). \quad (2.9)$$

An important observation here is that $\log\left(\sum_{\sigma'} e^{\sum_Y J'(Y)\sigma'_Y}\right)$ and $\log\left(\sum_{\sigma} e^{\sum_X J(X)\sigma_X}\right)$ are constants with respect to σ'_Z; thus, when summing over all possible image configurations σ', they both vanish. □

Proposition 2.0.2. *For every subset W of the original lattice and every subset Z of the image lattice, the partial derivative $\frac{\partial J'(Z)}{\partial J(W)}$ of the RG transformation is given by the expression*

$$\frac{\partial J'(Z)}{\partial J(W)} = \sum_{\sigma'} \sigma'_Z \frac{\sum_{\sigma} \prod_{y \in \mathcal{L}'} T_y(\sigma, \sigma'_y) e^{\sum_X J(X)\sigma_X} \sigma_W}{\sum_{\sigma} \prod_{y \in \mathcal{L}'} T_y(\sigma, \sigma'_y) e^{\sum_X J(X)\sigma_X}}. \quad (2.10)$$

Proof. We take the derivative of both sides of (2.7) with respect to $J(W)$. □

Remark. The above calculations are completely rigorous for finite lattices $\mathcal{L}$ and $\mathcal{L}'$, and may be interpreted in some more sophisticated limiting sense for countably infinite lattices $\mathcal{L}$ and $\mathcal{L}'$, according to standard techniques for the infinite volume limit in statistical mechanics, as will be shown in later chapters.

Chapter 3

SPECTRAL PROPERTIES OF THE RENORMALIZATION GROUP AT INFINITE TEMPERATURE

Many important physical properties emerge from spectral properties of the linearization of the RG map at a fixed point. The linearization acts on an infinite-dimensional Banach space of interactions. At a trivial fixed point (zero interaction), the spectral properties of the RG linearization can be worked out explicitly, without any approximation.

For illustration purposes, in this chapter, we assume all image blocks are cubicle with cardinality $s = b^d$. For each site y in $\mathcal{L}'$, the corresponding block y^o of $\mathcal{L}$ is explicitly given by

$$y^o = \{x : by_i - \frac{b-1}{2} \leq x_i \leq by_i + \frac{b-1}{2}, 1 \leq i \leq d\} \tag{3.1}$$

for odd blocking factor b; and

$$y^o = \{x : by_i - \frac{b-2}{2} \leq x_i \leq by_i + \frac{b}{2}, 1 \leq i \leq d\} \tag{3.2}$$

for even blocking factor b. We also restrict our attention to a special kind of deterministic probability kernel: There is a function $\phi_y(\sigma)$ that depends only on σ through y^o, and $T_y(\sigma, \sigma'_y) = 2\delta(\phi_y(\sigma), \sigma'_y)$.

Our basic assumption is that the original interaction J lies in a Banach space $\mathcal{B}_r$, with norm

$$||J||_r = \sup_{x \in \mathcal{L}} \sum_{X : x \in X} |J(X)| e^{rl(x,X)}, \tag{3.3}$$

where the constant $r \geq 0$. Correspondingly, there is a paired Banach space $\mathcal{B}_r^*$. As

$$
\begin{aligned}
|\sum_X J_1(X)J_2(X)| &\leq \sum_X |J_1(X)| \sum_{x\in X} \frac{1}{|X|}|J_2(X)| \\
&= \sum_{x\in\mathcal{L}} \sum_{X:x\in X} \frac{1}{|X|}|J_1(X)||J_2(X)| \\
&\leq \sum_{x\in\mathcal{L}} \sup_{x\in X} \frac{1}{|X|}|J_2(X)|e^{-rl(x,X)} \sum_{X:x\in X} |J_1(X)|e^{rl(x,X)} \\
&\leq \sup_{x\in\mathcal{L}} \sum_{X:x\in X} |J_1(X)|e^{rl(x,X)} \cdot \sum_{x\in\mathcal{L}} \sup_{x\in X} \frac{1}{|X|}|J_2(X)|e^{-rl(x,X)},
\end{aligned}
$$

a suitable $\mathcal{B}_r^*$ norm is defined by

$$
||J||_r^* = \sum_{x\in\mathcal{L}} \sup_{x\in X} \frac{1}{|X|}|J(X)|e^{-rl(x,X)}. \tag{3.4}
$$

Notice that here $\mathcal{B}_r$ is technically not the dual space of $\mathcal{B}_r^*$, and $\mathcal{B}_r^*$ is technically not the dual space of $\mathcal{B}_r$. The spaces $\mathcal{B}_r$ and $\mathcal{B}_r^*$ are paired in the sense that each one is part of the dual space of the other, or in other words, each one consists of continuous linear functions defined on the other. We study the situation when $||J||_r = 0$ (indication of infinite temperature). We consider the spectrum of the linearization $\mathrm{L}(J)$ of two commonly used RG transformations, decimation and deterministic majority rule with odd blocking factor. We show that this spectrum is of an unusual kind: dense point spectrum for which the adjoint operators $\mathrm{L}^*(J)$ have no point spectrum at all, but only residual spectrum.

Remark. Here spectrum is crudely divided into 3 types [4]: For a bounded linear operator A acting on a Banach space $\mathrm{A} : B \to B$,

- λ is in the point spectrum $\iff$ there exists $B \ni u \neq 0$, such that $(\mathrm{A}-\lambda)u = 0$, i.e., Kernel$(\mathrm{A} - \lambda\mathrm{I})$ is nontrivial.

- λ is in the residual spectrum $\iff$ λ is not in the point spectrum, and $\overline{\text{Range}(\text{A} - \lambda\text{I})} \neq B$.

- λ is in the continuous spectrum $\iff$ λ is not in the point spectrum or the residual spectrum, $\text{Range}(\text{A} - \lambda\text{I}) \neq B$, and $\overline{\text{Range}(\text{A} - \lambda\text{I})} = B$.

This definition is too simple to fully capture the notion of continuous spectrum, but it will be adequate for our purposes.

Israel [10] found the operator bound of $\text{L}(J)$ for decimation in a Banach algebra setting, but did not go into detail about the spectral type of this transformation. He also briefly mentioned the operator bound of $\text{L}(J)$ for majority rule on the triangular lattice. These results are extended by the present investigation, which includes the spectral type of $\text{L}(J)$ and $\text{L}^*(J)$ for decimation (Theorems 3.1.1 and 3.1.2) and majority rule (Theorems 3.2.1 and 3.2.2). Even though this investigation is focused on the RG transformation acting on a system very close to a trivial interaction, it serves as a test case—after all, if it is reasonably difficult to compute the spectrum of the RG map, then one can get an idea of what to expect by computing in a simple case. If even this case has bizarre spectral properties, then it may serve as a lesson in what to expect in more general situations.

Remark. The same spectral results will hold if we use the weaker Banach space norm (4.1), to which expansion and combinatorial techniques are applicable, as in later chapters.

Proposition 3.0.3. *Suppose the original interaction J is at infinite temperature. Then for every subset W of the original lattice and every subset Z of the image lattice,* *the partial derivative* $\frac{\partial J'(Z)}{\partial J(W)}$ *of the RG transformation is given by the expression*

$$\frac{\partial J'(Z)}{\partial J(W)} = \sum_{\sigma} \sum_{\sigma'} \prod_{y \in \mathcal{L}'} T_y(\sigma, \sigma'_y) \sigma_W \sigma'_Z. \tag{3.5}$$

Proof. When J is at infinite temperature, i.e., $||J||_r = 0$, $J(X) = 0$ for every subset X of the original lattice. Equation (2.10) is thus simplified. □

Definition 3.0.1. *For every subset Z of the image lattice, the linearization* $\mathrm{L}(J)$ *of the RG transformation for J at infinite temperature is given by a linear function of K (which indicates variation from infinite temperature),*

$$\mathrm{L}(J)K(Z) = \sum_{W} \frac{\partial J'(Z)}{\partial J(W)} K(W), \tag{3.6}$$

where W ranges over all finite subsets of the original lattice.

Remark. The same linearization definition applies for J at high temperature, as in Chapter 5.

Definition 3.0.2. *The adjoint of the linearization* $\mathrm{L}^*(J)$ *of the RG transformation for J at infinite temperature is characterized by the usual correspondence between adjoint operators,*

$$\sum_{X} K_1(X)\mathrm{L}(J)K_2(X) = \sum_{Y} K_2(Y)\mathrm{L}^*(J)K_1(Y), \tag{3.7}$$

where X ranges over all finite subsets of the image lattice, and Y ranges over all finite subsets of the original lattice.

Definition 3.0.3. *A constant pure magnetic field is one such that* $K(X) = 0$ *except for one-point sets* $\{x\}$, *where* $K(\{x\}) = m$, *a constant.*

3.1 Spectrum of the linearization of decimation transformation and its adjoint at infinite temperature

Proposition 3.1.1. *Consider decimation transformation with blocking factor b and a probability kernel defined by*

$$\phi_y(\sigma) = \sigma_{by}, \tag{3.8}$$

where $by = b(y_1, ..., y_d) = (by_1, ..., by_d)$. *Suppose the original interaction* J *is at infinite temperature. Then for every subset* Z *of the image lattice, the linearization* $\mathrm{L}(J)$ *of this transformation is given by the expression*

$$\mathrm{L}(J)K(Z) = K(bZ), \tag{3.9}$$

where $bZ = \cup_{z \in Z}\{bz\}$.

Proof. We evaluate (3.5) explicitly:

$$\frac{\partial J'(Z)}{\partial J(W)} = \sum_{\sigma} \delta(W, bZ) = \delta(W, bZ), \tag{3.10}$$

where δ is the Kronecker delta function. □

Proposition 3.1.2. *Consider the adjoint of decimation transformation with blocking factor b and a probability kernel defined by*

$$\phi_y(\sigma) = \sigma_{by}. \tag{3.11}$$

Suppose the original interaction J *is at infinite temperature. Then for every subset* Z *of the original lattice, the adjoint of the linearization* $\mathrm{L}^*(J)$ *of this transformation is given by the expression*

$$\mathrm{L}^*(J)K(Z) = \begin{cases} K(Y) & \textit{if } Z = bY; \\ 0 & \textit{otherwise.} \end{cases} \tag{3.12}$$

Proof. We notice that in this case, (3.7) becomes

$$\sum_X K_1(X)\mathrm{L}(J)K_2(X) \;=\; \sum_X K_1(X)K_2(bX). \tag{3.13}$$

□

Remark. Equations (3.9) and (3.12) are derived from finite lattices, but will hold for infinite lattices in this special case, which is the setting for the following spectral discussion.

Theorem 3.1.1 (Israel)**.** *Suppose the original interaction J is at infinite temperature. Then in the Banach Space $\mathcal{B}_r$, the spectrum of the linearization of the decimation transformation* $\mathrm{L}(J)$ *is all point spectrum,* $|\lambda| \leq 1$.

Proof. The proof of this theorem follows from several propositions. □

Proposition 3.1.3. $||\mathrm{L}(J)|| = 1$.

Proof. We check that for each fixed $x \in \mathcal{L}$, $\sum_{X:x\in X} |\mathrm{L}(J)K(X)|e^{rl(x,X)} \leq ||K||_r$, which would imply $||\mathrm{L}(J)|| \leq 1$. By (3.9),

$$\sum_{X:x\in X} |\mathrm{L}(J)K(X)|e^{rl(x,X)} = \sum_{X:x\in X} |K(bX)|e^{rl(x,X)}$$

$$\leq \sum_{X:bx\in bX} |K(bX)|e^{rl(bx,bX)} \leq \sum_{X:bx\in X} |K(X)|e^{rl(bx,X)} \leq ||K||_r. \tag{3.14}$$

The claim is verified when we realize that a constant pure magnetic field is an eigenvector with eigenvalue 1. □

Corollary 3.1.1. *Every eigenvalue λ of* $\mathrm{L}(J)$ *satisfies* $|\lambda| \leq 1$.

Proposition 3.1.4. *Every* $|\lambda| \leq 1$ *is an eigenvalue.*

Proof. For a generic λ, we display one eigenvector here. In fact, with some further thought, it is not hard to show that there are infinitely many eigenvectors for each λ. The eigenvector K is defined by,

$$K(\{(b^n, 0, ..., 0)\}) = \lambda^n K(\{(1, 0, ..., 0)\}) = \lambda^n \tag{3.15}$$

for $n \geq 0$, and for all the other subsets X, $K(X)$ is set to zero. □

Moreover, we have stricter restrictions on the eigenvector K that lies in a Banach space $\mathcal{B}_r : r > 0$.

Proposition 3.1.5. *For $\lambda \neq 0$ and for every finite subset $|X| > 1$, we must have $K(X) = 0$ for the eigenvector K.*

Proof. This follows from the observation that we can always pick a site, say x, in X, such that $l(x, X) > 0$. As a result, $b^n x$ is a site in $b^n X$, and $l(b^n x, b^n X) = b^n l(x, X) > 0$. Since $\mathrm{L}(J)K(X) = K(bX)$, we must have $K(b^n X) = \lambda^n K(X)$. Then due to the fact that K is an eigenvector, we need to ensure that

$$|\lambda|^n |K(X)| e^{r b^n l(x,X)} < \infty. \tag{3.16}$$

□

Theorem 3.1.2. *Suppose the original interaction J is at infinite temperature. Then in the Banach Space $\mathcal{B}_r^*$, the spectrum of the adjoint of the linearization of the decimation transformation $\mathrm{L}^*(J)$ is all residual spectrum, $|\lambda| \leq 1$.*

Proof. The proof of this theorem follows from several propositions. □

Proposition 3.1.6. $||\mathrm{L}^*(J)|| \leq 1$.

Proof. By (3.12),

$$\sum_{x\in\mathcal{L}} \sup_{x\in X} \frac{1}{|X|} |\mathrm{L}^*(J)K(X)| e^{-rl(x,X)} \leq \sum_{bx\in\mathcal{L}} \sup_{bx\in bX} \frac{1}{|X|} |K(X)| e^{-rl(bx,bX)} \leq \sum_{x\in\mathcal{L}} \sup_{x\in X} \frac{1}{|X|} |K(X)| e^{-rl(x,X)}. \tag{3.17}$$

□

Proposition 3.1.7. *For every $\lambda \neq 0$, there is no nontrivial eigenvector.*

Proof. Fix an arbitrary finite subset X. For $\lambda \neq 0$, $K(X)$ is either zero or a nonzero constant multiple of $K(\{0\})$ as a result of the action of $\mathrm{L}^*(J)$. In particular, $\lambda K(\{0\}) = \mathrm{L}^*(J)K(\{0\}) = 0$, which implies $K(X) = 0$. □

Proposition 3.1.8. *For $\lambda = 0$, there is no nontrivial eigenvector.*

Proof. Suppose the nontrivial eigenvector $K(X) = m \neq 0$ for some finite subset X, then the crucial fact that we can always find Y, with $\mathrm{L}^*(J)K(Y) = K(X)$ will do the job. As $\mathrm{L}^*(J)K(Y) = \lambda K(Y) = 0$, we reach a contradiction. □

Corollary 3.1.2. *In the Banach Space $\mathcal{B}_r^*$, the point spectrum of* $\mathrm{L}^*(J)$ *is empty.*

Proof of Theorem 3.1.2 continued. In order to verify our theorem, the only thing left to show now is that for $|\lambda| \leq 1$, $\overline{\mathrm{Range}(\lambda I - \mathrm{L}^*(J))} \neq \mathcal{B}_r^*$. We divide into two cases: $|\lambda| < 1$ and $|\lambda| = 1$.

Case 1: $|\lambda| < 1$.

Define $K(\{(b^n, 0, ..., 0)\}) = \bar{\lambda}^n$, and $K(X) = 0$ for all other subsets X. We have

$$||K||_r^* = \sum_{n=0}^{\infty} |\bar{\lambda}|^n = \frac{1}{1 - |\bar{\lambda}|} < \infty, \tag{3.18}$$

which says that K lies in $\mathcal{B}_r^*$. However, K can not be approximated by any K' in $\text{Range}(\lambda I - \text{L}^*(J))$. To see this, note that

$$K'(X) = \begin{cases} \lambda S(X) - S(Y) & \text{if } X = bY; \\ \lambda S(X) & \text{otherwise} \end{cases} \tag{3.19}$$

for some S that lies in $\mathcal{B}_r^*$. Also

$$\begin{aligned} ||K - K'||_r^* &= \sum_{x\in\mathcal{L}} \sup_{x\in X} \frac{1}{|X|} |K(X) - K'(X)| e^{-rl(x,X)} \\ &\geq \sum_{x=(b^n,0,...,0)} \sup_{x\in X} \frac{1}{|X|} |K(X) - K'(X)| e^{-rl(x,X)} \\ &\geq \sum_{n=0}^{\infty} |K(\{(b^n,0,...,0)\}) - K'(\{(b^n,0,...,0)\})|. \end{aligned} \tag{3.20}$$

However,

$$\sum_{n=0}^{\infty} \overline{K(\{(b^n,0,...,0)\})} K'(\{(b^n,0,...,0)\}) = \sum_{n=0}^{\infty} \lambda^n K'(\{(b^n,0,...,0)\})$$

$$= \lambda S(\{(1,0,...,0)\}) + \lambda(\lambda S(\{(b,0,...,0)\}) - S(\{(1,0,...,0)\})) + \cdots = 0. \tag{3.21}$$

It follows that

$$\begin{aligned} &\sum_{n=0}^{\infty} |K(\{(b^n,0,...,0)\}) - K'(\{(b^n,0,...,0)\})| \\ \geq &\sqrt{\sum_{n=0}^{\infty} |K(\{(b^n,0,...,0)\}) - K'(\{(b^n,0,...,0)\})|^2} \\ &\geq \sqrt{\sum_{n=0}^{\infty} |\bar{\lambda}^n|^2} = \sqrt{\frac{1}{1-|\bar{\lambda}|^2}} > 0. \end{aligned} \tag{3.22}$$

Case 2: $|\lambda| = 1$.

Define $K(\{(1,0,...,0)\}) = 1$, and $K(X) = 0$ for all other subsets X. Suppose there exists a K' approximating K such that

$$\begin{aligned}\frac{1}{2} &\geq ||K-K'||_r^* = \sum_{x\in\mathcal{L}} \sup_{x\in X} \frac{1}{|X|}|K(X)-K'(X)|e^{-rl(x,X)} \\ &\geq \sum_{x=(b^n,0,...,0)} \sup_{x\in X} \frac{1}{|X|}|K(X)-K'(X)|e^{-rl(x,X)} \\ &\geq \sum_{n=0}^{\infty} |K(\{(b^n,0,...,0)\})-K'(\{(b^n,0,...,0)\})|\end{aligned}$$

$$= |\lambda K(\{(1,0,...,0)\})-1| + |\lambda K(\{(b,0,...,0)\})-K(\{(1,0,...,0)\})| + \cdots \quad (3.23)$$

Then, as $|\lambda|=1$, we would have, for any $n \geq 0$,

$$|K(\{(b^n,0,...,0)\})| \geq \frac{1}{2}. \quad (3.24)$$

But then,

$$||K||_r^* = \sum_{x\in\mathcal{L}} \sup_{x\in X} \frac{1}{|X|}|K(X)|e^{-rl(x,X)} \geq \sum_{n=0}^{\infty} |K(\{(b^n,0,...,0)\})| = \infty, \quad (3.25)$$

which is a contradiction. □

Remark. Notice the similarity between the adjoint operators $\mathrm{L}(J)/\mathrm{L}^*(J)$ in our Banach spaces and left/right translation in l^∞/l^1. $\mathrm{L}(J)$ acts like left translation and $\mathrm{L}^*(J)$ acts like right translation on sequences $(X, bX, ...)$ for all possible subsets X. Moreover, ignoring multiplicity of the eigenvalues, the spectrum of $\mathrm{L}(J)$ is the same as that of left translation in l^∞, and the spectrum of $\mathrm{L}^*(J)$ is the same as that of right translation in l^1. This might be related to the fact that the norms in our Banach spaces are something like combinations of l^∞ and l^1 norms.

3.2 Spectrum of the linearization of majority rule transformation and its adjoint at infinite temperature

Proposition 3.2.1. *Consider majority rule transformation with odd blocking factor b and a probability kernel defined by*

$$\phi_y(\sigma) = \mathrm{sign}\left(\sum_{x\in y^o} \sigma_x\right). \tag{3.26}$$

Suppose the original interaction J is at infinite temperature. Then for every subset Z of the image lattice, the linearization $\mathrm{L}(J)$ *of this transformation is given by the expression*

$$\mathrm{L}(J)K(Z) = \sum_{W:W\subset Z^o} \prod_{z\in Z} \chi(W\cap z^o)K(W), \tag{3.27}$$

where $\chi(W\cap z^o) = \sum_\sigma \sum_{\sigma'} T_z(\sigma, \sigma'_z)\sigma_{W\cap z^o}\sigma'_z$.

Proof. We evaluate (3.5) explicitly:

$$\frac{\partial J'(Z)}{\partial J(W)} = \sum_\sigma \sigma_{W\setminus Z^o} \prod_{z\notin Z} \sum_{\sigma'} T_z(\sigma, \sigma'_z) \prod_{z\in Z} \sum_\sigma \sum_{\sigma'} T_z(\sigma, \sigma'_z)\sigma_{W\cap z^o}\sigma'_z. \tag{3.28}$$

Since $\sum_\sigma \sigma_{W\setminus Z^o} = 0$ for W not completely contained inside Z^o, it follows that $W \subset Z^o$. □

Proposition 3.2.2. *Consider majority rule transformation with odd blocking factor b and a probability kernel defined by*

$$\phi_y(\sigma) = \mathrm{sign}\left(\sum_{x\in y^o} \sigma_x\right). \tag{3.29}$$

Suppose the original interaction J is at infinite temperature. Then for every subset Z of the original lattice, the adjoint of the linearization $\mathrm{L}^*(J)$ *of this transformation is given by the expression*

$$\mathrm{L}^*(J)K(Z) = \begin{cases} \prod_{W_n} \chi(W_n)K(\cup\{n\}) & \text{if } Z = \cup W_n; \\ 0 & \text{otherwise,} \end{cases} \tag{3.30}$$

where $W_n \subset n^o$*.*

Proof. We notice that in this case, (3.7) becomes

$$\begin{aligned}\sum_X K_1(X)\mathrm{L}(J)K_2(X) &= \sum_X K_1(X) \sum_{Y:Y\subset X^o} \prod_{x\in X} \chi(Y\cap x^o)K_2(Y) \\ &= \sum_{Y=\cup W_n} K_2(Y)\prod_{W_n}\chi(W_n)K_1(\cup\{n\}). \end{aligned} \tag{3.31}$$

□

Remark. Equations (3.27) and (3.30) are derived from finite lattices, but will hold for infinite lattices in this special case, which is the setting for the following spectral discussion.

Theorem 3.2.1. *Suppose the original interaction* J *is at infinite temperature. Then in the Banach Space* $\mathcal{B}_r$*, the spectrum of the linearization of the majority rule transformation* $\mathrm{L}(J)$ *is all point spectrum,* $|\lambda| \leq s\binom{s-1}{\frac{s-1}{2}}/2^{s-1}$*.*

Proof. The proof of this theorem follows from several propositions. □

Proposition 3.2.3. *Consider Ising-type spin system on an odd polygon* A *with cardinality* $|A|$*, fix a certain vertex* V *and a certain subset* W *of the vertices. If* $\sigma'_a \in \{+1,-1\}$ *satisfies* $\sigma_A\sigma'_a > 0$*, then*

$$|\sum_\sigma \sigma_W\sigma'_a| \leq \sum_\sigma \sigma_V\sigma'_a = \binom{|A|-1}{\frac{|A|-1}{2}}/2^{|A|-1}, \tag{3.32}$$

where $\binom{n}{k}$ *is the binomial coefficient.*

Proof. We first show that $\sum_\sigma \sigma_W\sigma'_a = 0$ for any W with even cardinality. This is due to a symmetry argument. If there is a spin configuration with $\sigma_W\sigma'_a = 1$, then flipping the spins at every vertex, we will have a configuration with $\sigma_W\sigma'_a = (-1)^{|W|}(-1) = (-1)^{|W|+1} = -1$. Vice versa. Thus the total sum will be zero.

Next we investigate into the special case $\sum_\sigma \sigma_V \sigma'_a$ where V is any fixed vertex. The explicit calculation is easy to carry out. Due to symmetry, we only consider $\sigma_V = 1$ in the following, and there are $|A|-1$ vertices for which the spins are yet to be assigned.

- $\sigma'_a = 1$, if there are more 1's than -1's in the overall spin configuration, i.e., as long as the number of -1's does not exceed $\frac{|A|-1}{2}$. It is not hard to see that there are $\binom{|A|-1}{0} + \binom{|A|-1}{1} + \cdots + \binom{|A|-1}{\frac{|A|-1}{2}}$ of them.
- $\sigma'_a = -1$, if there are more -1's than 1's in the overall spin configuration, i.e., as long as the number of -1's exceeds $\frac{|A|-1}{2}$. Again, it is not hard to see that there are $\binom{|A|-1}{\frac{|A|+1}{2}} + \binom{|A|-1}{\frac{|A|+3}{2}} + \cdots + \binom{|A|-1}{|A|-1} = \binom{|A|-1}{\frac{|A|-3}{2}} + \binom{|A|-1}{\frac{|A|-5}{2}} + \cdots + \binom{|A|-1}{0}$ of them.

In conclusion, when $\sigma_V = 1$, there are $\binom{|A|-1}{\frac{|A|-1}{2}}$ more spin configurations for σ'_a to be 1 rather than to be -1. Similar result holds for $\sigma_V = -1$. Thus considering all possible spin configurations, there are $2\binom{|A|-1}{\frac{|A|-1}{2}}$ more spin configurations for $\sigma_V \sigma'_a$ to be 1 rather than to be -1. It follows that $\sum_\sigma \sigma_V \sigma'_a = \binom{|A|-1}{\frac{|A|-1}{2}}/2^{|A|-1}$.

Finally we consider $\sum_\sigma \sigma_W \sigma'_a$ for any W with odd cardinality. Without loss of generality, suppose $V \subset W$. For a fixed spin configuration, $\sigma_V \sigma'_a \neq \sigma_W \sigma'_a$ can only occur when there is an odd number of -1's and an odd number of 1's in the spin configuration for vertices in $W \backslash V$. For such a configuration, we notice the following important fact: Suppose it has the extra property that unequal numbers of -1's and 1's are assigned for the remaining $|A|-1$ vertices of $A \backslash V$, then if we flip the spins at every vertex other than V, $\sigma_V \sigma'_a$ will change sign. Moreover, at the same time, the sign of $\sigma_W \sigma'_a$ also changes, so the total sum does not change. Therefore, we see that the difference in $\sum_\sigma \sigma_W \sigma'_a$ and $\sum_\sigma \sigma_V \sigma'_a$ can only be caused by the following scenario: Equal numbers of -1's and 1's are assigned for the remaining $|A|-1$

vertices of $A\backslash V$, and there is an odd number of -1's and an odd number of 1's in the spin configuration for vertices in $W\backslash V$. It is not hard to see that there are at most $2\binom{|A|-1}{\frac{|A|-1}{2}}$ of them. Thus $\sum_\sigma \sigma_W \sigma'_a$ varies between $-\binom{|A|-1}{\frac{|A|-1}{2}}/2^{|A|-1}$ and $\binom{|A|-1}{\frac{|A|-1}{2}}/2^{|A|-1}$, and our claim follows. □

Proposition 3.2.4. $||\mathrm{L}(J)|| = s\binom{s-1}{\frac{s-1}{2}}/2^{s-1}$.

Proof. We check that for each fixed $x \in \mathcal{L}$,

$$\sum_{X:x\in X} |\mathrm{L}(J)K(X)|e^{rl(x,X)} \le s||K||_r \binom{s-1}{\frac{s-1}{2}}/2^{s-1},$$

which would imply $||\mathrm{L}(J)|| \le s\binom{s-1}{\frac{s-1}{2}}/2^{s-1}$. As $x \in X$, $\mathrm{L}(J)K(X)$ is a linear combination of $K(Y)$'s, each one with coefficient bounded above by $\binom{s-1}{\frac{s-1}{2}}/2^{s-1}$ by (3.32). Ignoring the coefficients of $K(Y)$'s, we can then collect terms according to which one of the sites in x^o belongs to Y. (When $|Y \cap x^o| > 1$, $K(Y)$ can be classified into either one of the s groups.) Moreover, each Y has size no smaller than X, the exponential factor changes to a larger quantity after the action of $\mathrm{L}(J)$. We see that each collection is bounded above by $||K||_r$ by definition. The claim is verified when we realize that a constant pure magnetic field is an eigenvector with eigenvalue $s\binom{s-1}{\frac{s-1}{2}}/2^{s-1}$. □

Corollary 3.2.1. *Every eigenvalue* $|\lambda| \le s\binom{s-1}{\frac{s-1}{2}}/2^{s-1}$.

Proposition 3.2.5. *Every* $|\lambda| \le s\binom{s-1}{\frac{s-1}{2}}/2^{s-1}$ *is an eigenvalue.*

Proof. For a generic λ, we display one eigenvector here. In fact, with some further thought, it is not hard to show that there are infinitely many eigenvectors for each λ. The eigenvector K is defined by,

$$K(\{(\frac{b-1}{2}, ..., \frac{b-1}{2})\}) = 2^{s-1}\lambda/\binom{s-1}{\frac{s-1}{2}} - (s-1), \tag{3.33}$$

and

$$K(\{x\}) = 1 \tag{3.34}$$

for $(\frac{b-1}{2}, ..., \frac{b-1}{2}) \neq x \in 0^o$. In general, for $n \neq 0$, K is defined by $sK(\{m\}) = 2^{s-1}\lambda / \binom{s-1}{\frac{s-1}{2}} K(\{n\})$ for $m \in n^o$. For all the other subsets X, $K(X)$ is set to zero. □

Corollary 3.2.2. *The spectrum of* $\mathrm{L}(J)$ *diverges as* $\sqrt{\frac{2s}{\pi}}$ *as the blocking factor* b *gets large.*

Proof. This follows from an easy application of Stirling's formula:

$$s\binom{s-1}{\frac{s-1}{2}}/2^{s-1} \sim \frac{s\sqrt{2\pi \cdot (s-1)}(s-1)^{s-1}e^{-(s-1)}}{2\pi\frac{s-1}{2}\left(\frac{s-1}{2}\right)^{s-1}e^{-(s-1)}2^{s-1}} \sim \sqrt{\frac{2s}{\pi}}. \tag{3.35}$$

□

Theorem 3.2.2. *Suppose the original interaction* J *is at infinite temperature. Then in the Banach Space* $\mathcal{B}_r^*$*, the point spectrum of the adjoint of the linearization of the majority rule transformation* $\mathrm{L}^*(J)$ *is empty.*

Proof. The proof of this theorem follows from several propositions. □

Proposition 3.2.6. *For every* $\lambda \neq 0$ *and* $\lambda \neq \binom{s-1}{\frac{s-1}{2}}/2^{s-1}$*, there is no nontrivial eigenvector.*

Proof. Fix an arbitrary finite subset X. For $\lambda \neq 0$, $K(X)$ is either zero or a nonzero constant multiple of $K(\{0\})$ as a result of the action of $\mathrm{L}^*(J)$. In particular, $\lambda K(\{0\}) = \mathrm{L}^*(J)K(\{0\}) = \binom{s-1}{\frac{s-1}{2}}/2^{s-1}K(\{0\})$, which implies that $K(\{0\}) = 0$. □

Proposition 3.2.7. *For* $\lambda = 0$*, there is no nontrivial eigenvector.*

Proof. Suppose the nontrivial eigenvector $K(X) = m \neq 0$ for some finite subset X, then the crucial fact that we can always find Y, with $\mathrm{L}^*(J)K(Y)$ a nonzero constant multiple of $K(X)$ will do the job. As $\mathrm{L}^*(J)K(Y) = \lambda K(Y) = 0$, we reach a contradiction. □

Proposition 3.2.8. *For* $\lambda = \binom{s-1}{\frac{s-1}{2}}/2^{s-1}$, *every nontrivial eigenvector has norm infinity.*

Proof. We must have $K(\{0\}) = m \neq 0$ in order for K to be nontrivial. As

$$\binom{s-1}{\frac{s-1}{2}}/2^{s-1}K(\{x\}) = \mathrm{L}^*(J)K(\{x\}) = \binom{s-1}{\frac{s-1}{2}}/2^{s-1}K(\{0\}) \tag{3.36}$$

for $x \in 0^o$, we see that $K(\{x\}) = m$ also. Following similar fashion, $K(\{n\}) = m$ for arbitrary n. But then, $||K||_r^* = \sum_{x\in\mathcal{L}} \sup_{x\in X} \frac{1}{|X|}|K(X)| = \infty$. □

Chapter 4
A CLUSTER EXPANSION APPROACH TO RENORMALIZATION GROUP TRANSFORMATIONS

This chapter treats the rigorous definition of the RG map for classical Ising-type lattice systems. A cluster expansion is used to justify this definition in the infinite volume limit at high temperature.

Our basic assumption is that the original interaction J lies in a Banach space $\mathcal{B}_r$, with norm

$$||J||_r = \sup_{x\in\mathcal{L}} \sum_{X:x\in X} |J(X)|e^{r|X|}, \tag{4.1}$$

where the constant $r > 0$ and $|X|$ denotes the cardinality of the set X. We will show that when $||J||_r$ is small (indication of high temperature), the renormalized interaction J' lies in a Banach space $\mathcal{B}_{r'}$ with $r' < r$. We will also verify the existence of the partial derivatives $\frac{\partial J'(Z)}{\partial J(W)}$ of the RG transformation. The idea is that one starts with a J and finds a complicated but explicit expansion that defines a J' (and $\frac{\partial J'(Z)}{\partial J(W)}$). This expansion is derived from the formal expressions, but it is itself well-defined and convergent.

Other papers have derived similar results using different methods. Israel [10] used a Banach algebra framework. Kashapov [11] worked with cumulants (semi-invariants), with estimates that relied on methods of Malyshev [14]. The work we are presenting here is a reasonably straightforward application of the cluster expansion machinery, and so may be valuable to people who want to see how this approach may be of use in RG applications. In particular, it may help serve as an introduction to the important work of Haller and Kennedy [9] and Olivieri and Picco [19].

4.1 Cluster expansion for the renormalized interaction

To understand the following Proposition 4.1.1, we need to introduce some combinatorial concepts. A hypergraph is a set of sites together with a collection Γ of nonempty subsets. Such a nonempty set is referred to as a hyper-edge or link. Two links are block-connected if they both intersect some block. The support of a hypergraph is the set $\cup\Gamma$ of sites that belong to some set in Γ. A hypergraph Γ is block-connected if the support of Γ is nonempty and cannot be partitioned into nonempty sets with no block-connected links. In our current setting, a subset X of $\mathcal{L}$ defines a subset X' of $\mathcal{L}'$, corresponding to the set of blocks that have non-empty intersection with X. Thus a hypergraph Γ on $\mathcal{L}$ defines a hypergraph Γ' on $\mathcal{L}'$. We use Γ_c to indicate block connectivity of the hypergraph Γ_c, and write $\Gamma_c^* = \cup\Gamma_c'$ for the support of Γ_c' in the image lattice.

Proposition 4.1.1. *For fixed values of the renormalized spins σ', the partition function $W(\sigma')$ has the cluster representation*

$$W(\sigma') = \sum_{\Delta} \prod_{N \in \Delta} w_N, \tag{4.2}$$

where Δ is a set of disjoint subsets N's of $\mathcal{L}'$, and

$$w_N = \sum_{\Gamma_c^* = N} \alpha(N, \Gamma_c, \{\sigma'\}_N), \tag{4.3}$$

and the sum here is over block-connected hypergraphs Γ_c on $\mathcal{L}$ whose images in $\mathcal{L}'$ have support N. The contribution of each block-connected hypergraph is given by

$$\alpha(N, \Gamma_c, \{\sigma'\}_N) = \sum_{\sigma} \prod_{y \in N} T_y(\sigma, \sigma'_y) \prod_{X \in \Gamma_c} \left(e^{J(X)\sigma_X} - 1\right). \tag{4.4}$$

Proof. When the original Hamiltonian H is at high temperature ($||J||_r$ small), we can rewrite $e^{\sum_X J(X)\sigma_X}$ as a perturbation around zero interaction (infinite temperature),

$$\begin{aligned} W(\sigma') &= \sum_{\sigma} \prod_{y\in\mathcal{L}'} T_y(\sigma, \sigma'_y) \prod_{X} \left(1 + e^{J(X)\sigma_X} - 1\right) \\ &= \sum_{\sigma} \prod_{y\in\mathcal{L}'} T_y(\sigma, \sigma'_y) \sum_{\Gamma} \prod_{X\in\Gamma} \left(e^{J(X)\sigma_X} - 1\right), \end{aligned} \tag{4.5}$$

where Γ is a set of subsets X's of $\mathcal{L}$.

We are going to organize the sum over hypergraphs in (4.5) in the following way. Each hypergraph Γ on $\mathcal{L}$ has a support in $\mathcal{L}'$, which breaks up into block-connected parts. Let Δ be the parts, and for $N \in \Delta$, let $S(N)$ be the corresponding block-connected hypergraph on this part, i.e., $S(N)^* = N$. Then summing over hypergraphs Γ is equivalent to summing over Δ and functions S with the appropriate property. Furthermore, the product over N in Δ and the links in $S(N)$ is equivalent to the product over the corresponding Γ. We have

$$W(\sigma') = \sum_{\sigma} \prod_{y\in\mathcal{L}'} T_y(\sigma, \sigma'_y) \sum_{\Delta} \sum_{S} \prod_{N\in\Delta} \prod_{X\in S(N)} \left(e^{J(X)\sigma_X} - 1\right). \tag{4.6}$$

By independence, the sum over σ can be factored over Δ, and this gives

$$W(\sigma') = \sum_{\Delta} \prod_{y\notin\cup\Delta} \sum_{\sigma} T_y(\sigma, \sigma'_y) \sum_{S} \prod_{N\in\Delta} \sum_{\sigma} \prod_{y\in N} T_y(\sigma, \sigma'_y) \prod_{X\in S(N)} \left(e^{J(X)\sigma_X} - 1\right). \tag{4.7}$$

Notice that because of (2.6), many of the T_y factors sum to 1, (4.7) can be simplified,

$$W(\sigma') = \sum_{\Delta} \sum_{S} \prod_{N\in\Delta} \alpha(N, S(N), \{\sigma'\}_N). \tag{4.8}$$

And by the distributive law,

$$\sum_{S} \prod_{N\in\Delta} \alpha(N, S(N), \{\sigma'\}_N) = \prod_{N\in\Delta} \sum_{\Gamma_c^*=N} \alpha(N, \Gamma_c, \{\sigma'\}_N). \tag{4.9}$$

Therefore

$$W(\sigma') \;=\; \sum_{\Delta}\prod_{N\in\Delta}\sum_{\Gamma_c^*=N}\alpha(N,\Gamma_c,\{\sigma'\}_N). \tag{4.10}$$

Our claim thus follows. □

We rewrite (4.2) in the following way to apply standard results on cluster expansion,

$$\begin{aligned}\sum_{\Delta}\prod_{N\in\Delta} w_N \;&=\; \sum_{p=0}^{\infty}\frac{1}{p!}\sum_{N_1,\ldots,N_p}\prod_{\{i,j\}}(1-c(N_i,N_j))\,w_{N_1}\cdots w_{N_p}\\ &=\; \sum_{p=0}^{\infty}\frac{1}{p!}\sum_{N_1,\ldots,N_p}\sum_{G}\prod_{\{i,j\}\in G}(-c(N_i,N_j))\,w_{N_1}\cdots w_{N_p},\end{aligned} \tag{4.11}$$

where G is a graph with vertex set $\{1,...,p\}$ and

$$c(N_i,N_j)=\begin{cases}1 & \text{if } N_i \text{ and } N_j \text{ overlap;}\\ 0 & \text{otherwise.}\end{cases} \tag{4.12}$$

Proposition 4.1.2. *The frozen block spin free energy* $\log(W(\sigma'))$ *is given by the cluster expansion*

$$\log(W(\sigma'))=\sum_{p=1}^{\infty}\frac{1}{p!}\sum_{N_1,\ldots,N_p}C\left(N_1,...,N_p\right)w_{N_1}\cdots w_{N_p}, \tag{4.13}$$

where

$$C\left(N_1,...,N_p\right)=\sum_{G_c}\prod_{\{i,j\}\in G_c}\left(-c(N_i,N_j)\right), \tag{4.14}$$

and G_c *is a connected graph with vertex set* $\{1,...,p\}$.

Proof. The effect of taking the logarithm is that the sum over graphs is replaced by the sum over connected graphs:

$$\log(W(\sigma'))=\sum_{p=1}^{\infty}\frac{1}{p!}\sum_{N_1,\ldots,N_p}\sum_{G_c}\prod_{\{i,j\}\in G_c}(-c(N_i,N_j))\,w_{N_1}\cdots w_{N_p}. \tag{4.15}$$

Our claim thus follows. □

We have seen in (4.3) and (4.4) that w_N only depends on image sites in N, thus a natural conclusion here is that $J'(Z)$ vanishes unless Z is a subset of $\cup_1^p N_i$. Also, for every fixed σ', it is possible to bound (4.3) by

$$|w_N| \leq v_N = \sum_{\Gamma_c^* = N} \prod_{X \in \Gamma_c} \left(e^{|J(X)|} - 1 \right). \tag{4.16}$$

We have the following theorem:

Theorem 4.1.1. *The renormalized coupling constants J' (2.7) are given by the cluster expansion*

$$J'(Z) = \sum_{p=1}^{\infty} \sum_{Z \subset \cup_1^p N_i} \frac{1}{p!} C\left(N_1, ..., N_p\right) \sum_{\sigma'} \sigma'_Z w_{N_1} \cdots w_{N_p}, \tag{4.17}$$

and may be bounded above by

$$|J'(Z)| \leq \sum_{p=1}^{\infty} \sum_{Z \subset \cup_1^p N_i} \frac{1}{p!} \left| C\left(N_1, ..., N_p\right) \right| |v_{N_1}| \cdots |v_{N_p}|. \tag{4.18}$$

Remark. Equation (4.17) is derived from finite lattices, but will be taken as the definition of the renormalized interaction $J'(Z)$ in the infinite volume limit, following standard interpretation of statistical mechanics.

4.2 Convergence of the cluster expansion for the pinned frozen block spin free energy

Theorem 4.2.1 (Koteckÿ-Preiss)**.** *Recall (4.12) and (4.16). Take $1 < M < e^r$. Suppose that*

$$\sum_{N'} c(N, N') v_{N'} M^{|N'|} \leq |N| \log(M). \tag{4.19}$$

Then we have convergence of the cluster expansion for the pinned frozen block spin free energy $\log(W(\sigma'))$,

$$\sum_{p=1}^{\infty} \sum_{N_1,...,N_p : \exists i N_i = N} \frac{1}{p!} \left| C\left(N_1, ..., N_p\right) \right| |v_{N_1}| \cdots |v_{N_p}| \leq v_N M^{|N|}. \tag{4.20}$$

Moreover, the avoidance probability for every $Y \subset \mathcal{L}'$ *also has a convergent power series expansion,*

$$\left| \sum_{\Delta'} \prod_{N \in \Delta'} w_N / \sum_{\Delta} \prod_{N \in \Delta} w_N \right|$$

$$\leq \exp\left(\sum_{p=1}^{\infty} \frac{1}{p!} \sum_{N_1,...,N_p} \left| C\left(N_1, ..., N_p\right) \right| c(Y, \cup_1^p N_i) |w_{N_1}| \cdots |w_{N_p}| \right) \leq M^{|Y|}, \tag{4.21}$$

where Δ' *is a set of disjoint subsets of* $\mathcal{L}' \backslash Y$*, and* Δ *is a set of disjoint subsets of* $\mathcal{L}'$*. By (4.12) and (4.14), here we are only counting contributions of block-connected* N_i*'s that are also block-connected to* Y*.*

Proposition 4.2.1. *Consider the original coupling constants* J *with the Banach space norm* $||J||_r$*. Suppose this norm is sufficiently small. Suppose that* $1 < M < e^r$*. Then for each block spin site* y*, we have*

$$\sum_{y \in N} v_N M^{|N|} \leq \log(M). \tag{4.22}$$

Remark. It is sufficient that the norm satisfy

$$||J||_r \leq \frac{\log(M) c^2}{2s\left(c + \log(M)\right)}, \tag{4.23}$$

where $c = \frac{1}{\sqrt{\epsilon}} - 1$ and $\epsilon = M e^{-r}$.

Remark. The inequality (4.22) is a standard sufficient condition for (4.19). It will be applied in the following Theorems 4.3.1 and 4.4.1.

Proof. We notice that when $||J||_r$ is small (say $||J||_r \leq \frac{1}{2}$), $e^{|J(X)|} - 1 \leq 2|J(X)|$ by the mean value theorem. Also, it easily follows from (4.1) that for all X with cardinality m and containing a fixed x, $\sum |J(X)| \leq ||J||_r e^{-rm}$. More importantly, for $\Gamma_c^* = N$, $|N| \leq \sum |X|$ with X in Γ_c. We have

$$\begin{aligned}\sum_{y\in N} v_N M^{|N|} &\leq \sum_{y\in N}\sum_{\Gamma_c^*=N} M^{|N|} \prod_{X\in\Gamma_c} 2|J(X)| \\ &\leq \sum_{y\in N}\sum_{\Gamma_c^*=N} \prod_{X\in\Gamma_c} 2|J(X)|M^{|X|} \\ &= \sum_{y\in\Gamma_c^*} \prod_{X\in\Gamma_c} 2|J(X)|M^{|X|}. \end{aligned} \tag{4.24}$$

We say that a hypergraph Γ_c is block-rooted at y if its support intersects a fixed block y^o. Let $a_n(y)$ be the contribution of all block-connected hypergraphs with n links that are block-rooted at y,

$$a_n(y) = \sum_{y\in\Gamma_c^*:|\Gamma_c|=n} \prod_{X\in\Gamma_c} 2|J(X)|M^{|X|}. \tag{4.25}$$

Then

$$\sum_{y\in N} v_N M^{|N|} \leq \sum_{n=1}^{\infty} \sup_{y\in\mathcal{L}'} a_n(y). \tag{4.26}$$

Let a_n be the supremum over y of the contribution of block-connected hypergraphs with n links that are block-rooted at y, i.e., $a_n = \sup_{y\in\mathcal{L}'} a_n(y)$. It seems that once we show that a_n is exponentially small, the geometric series above will converge, and our claim might follow. To estimate a_n, we relate to some standard combinatorial facts [15]. The rest of the proof follows from a series of lemmas. □

Lemma 4.2.1. *Let a_n be the supremum over y of the contribution of block-connected hypergraphs with n links that are block-rooted at y. Then a_n satisfies the recursive bound*

$$a_n \leq 2s||J||_r \sum_{m=1}^{\infty} \epsilon^m \sum_{k=0}^{m} \binom{m}{k} \sum_{a_{n_1},\ldots,a_{n_k}:n_1+\cdots+n_k+1=n} a_{n_1}\cdots a_{n_k} \tag{4.27}$$

for $n \geq 1$, *where* $\binom{m}{k}$ *is the binomial coefficient.*

Proof. We first linearly order the points x in $\mathcal{L}$ and also linearly order the subsets X of $\mathcal{L}$. This naturally induces a linear ordering of the points y in $\mathcal{L}'$. For a fixed but arbitrarily chosen y in $\mathcal{L}'$, we examine (4.25). Write $\Gamma_c = \{X_1\} \cup \Gamma_c^1$, where X_1 is the least X in Γ_c with $y^o \cap X_1 \neq \emptyset$. There must be such a set, since $y \in \Gamma_c^*$. Moreover, there must be some $x \in y^o$ such that $x \in X_1$, of which there are s possibilities, as the block cardinality is s. Then

$$a_n(y) \leq s \sum_{m=1}^{\infty} \sup_{x \in \mathcal{L}} \sum_{X_1 : x \in X_1, |X_1| = m} 2|J(X_1)| M^m \sum_{\Gamma_c^1} \prod_{X \in \Gamma_c^1} 2|J(X)| M^{|X|}. \tag{4.28}$$

As a consequence,

$$a_n(y) \leq \sum_{m=1}^{\infty} 2s||J||_r \epsilon^m \sum_{\Gamma_c^1} \prod_{X \in \Gamma_c^1} 2|J(X)| M^{|X|}. \tag{4.29}$$

The remaining hypergraph Γ_c^1 has $n-1$ subsets and breaks into $k : k \leq m$ block-connected components $\Gamma_1, ..., \Gamma_k$ of sizes $n_1, ..., n_k$, with $n_1 + \cdots + n_k = n-1$. The set of such components may be empty, or it could just be the original block-connected set. For each component Γ_i, there is a least block y_i^o through which it is block-connected to X_1. The image $\{y_i\}$ of these blocks is a subset of X_1' in $\mathcal{L}'$, thus has no more than $k : k \leq m$ points, and the components are block-rooted at these image sites. Furthermore, different Γ_i's correspond to disjoint Γ_i''s, as $y_i \in \Gamma_i'$, the map from the components to this image is injective. So we have

$$a_n(y) \leq \sum_{m=1}^{\infty} 2s||J||_r \epsilon^m \sum_{k=0}^{m} \binom{m}{k} \sum_{a_{n_1}, \ldots, a_{n_k} : n_1 + \cdots + n_k + 1 = n} a_{n_1} \cdots a_{n_k}. \tag{4.30}$$

Our inductive claim follows by taking the supremum over all y in $\mathcal{L}'$. Finally, we look at the base step: $n = 1$. In this simple case, as reasoned above, we have

$$\begin{aligned} a_1 &= \sup_{y\in\mathcal{L}'} \sum_{y\in\Gamma_c^*:|\Gamma_c|=1} \prod_{X\in\Gamma_c} 2|J(X)|M^{|X|} \\ &\le s\sum_{m=1}^{\infty} \sup_{x\in\mathcal{L}} \sum_{X:x\in X,|X|=m} 2|J(X)|M^m \\ &= \sum_{m=1}^{\infty} 2s||J||_r\epsilon^m, \end{aligned} \tag{4.31}$$

and this verifies our claim. □

Clearly, $\sum_{y\in N} v_N M^{|N|}$ will be bounded above by $\sum_{n=1}^{\infty} \bar{a}_n$, if

$$\bar{a}_n = 2s||J||_r \sum_{m=1}^{\infty} \epsilon^m \sum_{k=0}^{m} \binom{m}{k} \sum_{\bar{a}_{n_1},\ldots,\bar{a}_{n_k}:n_1+\cdots+n_k+1=n} \bar{a}_{n_1}\cdots\bar{a}_{n_k} \tag{4.32}$$

for $n \ge 1$, i.e., equality is obtained in the above lemma.

Lemma 4.2.2. *Consider the coefficients $\bar{a}_n$ that bound the contributions of block-connected and block-rooted hypergraphs with n links. Let $w = \sum_{n=1}^{\infty} \bar{a}_n z^n$ be the generating function of these coefficients. Then the recursion relation (6.27) for the coefficients is equivalent to the formal power series generating function identity*

$$w = 2s||J||_r z \sum_{m=1}^{\infty} \epsilon^m (1+w)^m = 2s||J||_r z \frac{\epsilon(1+w)}{1-\epsilon(1+w)}. \tag{4.33}$$

Proof. Notice that $(1+w)^m = \sum_{k=0}^{m} \binom{m}{k} w^k$, thus

$$w = 2s||J||_r z \sum_{m=1}^{\infty} \epsilon^m \sum_{k=0}^{m} \binom{m}{k} w^k. \tag{4.34}$$

Writing completely in terms of z, we have

$$\sum_{n=1}^{\infty} \bar{a}_n z^n = 2s||J||_r \sum_{m=1}^{\infty} \epsilon^m \sum_{k=0}^{m} \binom{m}{k} \sum_{\bar{a}_{n_1},\ldots,\bar{a}_{n_k}:n_1+\cdots+n_k+1=n} \bar{a}_{n_1}\cdots\bar{a}_{n_k} z^n. \tag{4.35}$$

Our claim follows from term-by-term comparison. □

Lemma 4.2.3. *If w is given as a function of z as a formal power series by the generating function identity (6.28), then this power series has a nonzero radius of convergence $|z| \leq \frac{1}{2s||J||_r}c^2$. For big enough c, this radius of convergence is arbitrarily large, and in particular, the series will converge for $z = 1$, i.e., the sum of the bounds on the contributions of block-connected and block-rooted hypergraphs converges.*

Proof. Without loss of generality, assume $z \geq 0$. Set $z_1 = 2s||J||_r z$. Solving (6.28) for z_1 gives

$$z_1 = \frac{w\left(1 - \epsilon(1+w)\right)}{\epsilon(1+w)}. \tag{4.36}$$

By elementary calculus, this increases as w goes from 0 to c to have values z_1 from 0 to c^2. It follows that as z_1 goes from 0 to c^2, the w values range from 0 to c. □

Proof of Proposition 4.2.1 continued. We notice that in the above lemma, $w = \sum_{n=1}^{\infty} \bar{a}_n z^n = c$ corresponds to $z_1 = 2s||J||_r z = c^2$, which implies that for each n,

$$\bar{a}_n \leq c(2s||J||_r)^n c^{-2n}. \tag{4.37}$$

Gathering all the information we have obtained so far,

$$\begin{aligned} \sum_{y \in N} v_N M^{|N|} &\leq \sum_{n=1}^{\infty} c(2s||J||_r)^n c^{-2n} \\ &= c\frac{\frac{2s||J||_r}{c^2}}{1 - \frac{2s||J||_r}{c^2}} \leq \log(M) \end{aligned} \tag{4.38}$$

by (4.23). □

4.3 Estimation of the size of the renormalized interaction

Proposition 4.3.1. *Let r' be the parameter in the norm for the renormalized coupling constants J'. Then this norm may be bounded in terms of the cluster expansion* *by*

$$||J'||_{r'} \leq \sum_{z\in Z}\sum_{p=1}^{\infty}\sum_{N_1,...,N_p:\exists i N_i = Z}\frac{1}{p!}\left|C\left(N_1,...,N_p\right)\right|\left|v'_{N_1}\right|\cdots\left|v'_{N_p}\right|, \tag{4.39}$$

where $v'_{N_i} = (2e^{r'})^{|N_i|}v_{N_i}$.

Proof. For each subset Z of the image lattice, we have seen that we can estimate $J'(Z)$ by (4.18). Then we have for each z in the image lattice $\mathcal{L}'$,

$$\sum_{z\in Z}|J'(Z)|e^{r'|Z|} \leq \sum_{z\in Z}\sum_{p=1}^{\infty}\sum_{Z\subset\cup_1^p N_i}\frac{1}{p!}\left|C\left(N_1,...,N_p\right)\right|\left|v_{N_1}\right|\cdots\left|v_{N_p}\right|e^{r'(|N_1|+\cdots+|N_p|)}. \tag{4.40}$$

Notice that $z\in Z\subset\cup_1^p N_i$ implies that for some i, we have $z\in N_i$ and $Z\subset\cup_1^p N_i$. We get a larger bound by interchanging the order of summation:

$$\sum_{z\in Z}|J'(Z)|e^{r'|Z|} \leq \sum_{p=1}^{\infty}\sum_{N_1,...,N_p:\exists i z\in N_i}\sum_{Z\subset\cup_1^p N_i}\frac{1}{p!}\left|C\left(N_1,...,N_p\right)\right|\left|v_{N_1}\right|\cdots\left|v_{N_p}\right|e^{r'(|N_1|+\cdots+|N_p|)}. \tag{4.41}$$

But then we can bound the Z sum by $2^{|N_1|+\cdots+|N_p|}$, and we get the bound

$$\sum_{z\in Z}|J'(Z)|e^{r'|Z|} \leq \sum_{p=1}^{\infty}\sum_{N_1,...,N_p:\exists i z\in N_i}\frac{1}{p!}\left|C\left(N_1,...,N_p\right)\right|\left|v_{N_1}\right|\cdots\left|v_{N_p}\right|(2e^{r'})^{|N_1|+\cdots+|N_p|}. \tag{4.42}$$

Now bound this by

$$\sum_{z\in Z}|J'(Z)|e^{r'|Z|} \leq \sum_{z\in Z}\sum_{p=1}^{\infty}\sum_{N_1,...,N_p:\exists i N_i = Z}\frac{1}{p!}\left|C\left(N_1,...,N_p\right)\right|\left|v'_{N_1}\right|\cdots\left|v'_{N_p}\right|. \tag{4.43}$$

Our claim thus follows. □

Theorem 4.3.1. *Consider the original coupling constants J with the Banach space norm $||J||_r$. Take $r' < r$. Suppose that $1 < M < \frac{e^{r-r'}}{2}$. Then for small enough $||J||_r$ norm, the norm of the renormalized coupling constants J' satisfies the bound*

$$||J'||_{r'} \leq \log(M). \tag{4.44}$$

Remark. It is sufficient that the norm satisfy

$$||J||_r \leq \frac{\log(M)c'^2}{2s\,(c' + \log(M))}, \tag{4.45}$$

where $c' = \frac{1}{\sqrt{\epsilon'}} - 1$ and $\epsilon' = 2Me^{r'-r}$.

Proof. The proof of this theorem follows from Theorem 4.2.1, Propositions 4.2.1 and 4.3.1 applied in the following way. (4.45) implies

$$\sum_{z \in Z} v_Z (2e^{r'} M)^{|Z|} \leq \log(M), \tag{4.46}$$

by Proposition 4.2.1, which further implies

$$||J'||_{r'} \leq \sum_{z \in Z} v_Z (2e^{r'} M)^{|Z|}. \tag{4.47}$$

by Theorem 4.2.1 and Proposition 4.3.1. Our claim then follows by combining (4.46) and (4.47). □

4.4 Existence of the partial derivatives of the renormalization group transformation

We have shown in (4.2) that the denominator of (2.10) has a cluster representation. We now examine the effect of multiplying σ_W to this cluster representation as in the numerator of (2.10). There will be two kinds of terms. In some of these, none of the block-connected components intersect W, so for these terms one gets a product of σ_W

with a product of independent w_N's. For the other terms one decomposes Δ into one block-connected component that is connected to W and remaining block-connected components that are not. The result is the representation

$$\sigma_W W(\sigma') = \sum_{R,\Delta'} \tilde{w}_R \prod_{N\in\Delta'} w_N, \tag{4.48}$$

where $R = \emptyset$ or $R \cap W' \neq \emptyset$, and $\tilde{w}_R$ is a sum over hypergraphs Δ_R with $\cup\Delta_R = R$ such that W, Δ_R is block-connected. Therefore

$$\frac{\partial J'(Z)}{\partial J(W)} = \sum_{\sigma'} \sigma'_Z \frac{\sum_{R,\Delta'} \tilde{w}_R \prod_{N\in\Delta'} w_N}{\sum_{\Delta} \prod_{N\in\Delta} w_N}. \tag{4.49}$$

Remark. Equation (4.49) is derived from finite lattices, but will be taken as the definition of the partial derivative $\frac{\partial J'(Z)}{\partial J(W)}$ in the infinite volume limit, following standard interpretation of statistical mechanics.

Theorem 4.4.1. *Consider the original coupling constants J with the Banach space norm $||J||_r$. Suppose this norm is sufficiently small. Then for every subset W of the original lattice and every subset Z of the image lattice, the partial derivative $\frac{\partial J'(Z)}{\partial J(W)}$ of the RG transformation (4.49) is well-defined.*

Proof. The proof of this theorem is a more involved application of the Kotecký-Preiss result [13]. By (4.21),

$$\left| \sum_{\Delta'} \prod_{N\in\Delta'} w_N / \sum_{\Delta} \prod_{N\in\Delta} w_N \right| \leq M^{|R\cup W'|}. \tag{4.50}$$

Also notice that $N \in \Delta'$ implies $N \cap (R \cup W') = \emptyset$. We have

$$|\tilde{w}_R| \leq \sum_{\Delta_R} \prod_{Y\in\Delta_R} |w_Y|. \tag{4.51}$$

To verify our claim, we need to estimate

$$\sum_{R} |\tilde{w}_R| M^{|R \cup W'|} \leq \sum_{\Delta_R} M^{|W|} \prod_{Y \in \Delta_R} |w_Y| M^{|Y|}. \tag{4.52}$$

But this is easy, remove W, the remaining hypergraph breaks up into $k : 0 \leq k \leq |W|$ block-connected components. So this last quantity is bounded by

$$M^{|W|} \sum_{k=0}^{|W|} \binom{|W|}{k} (\log(M))^k = M^{|W|} (1 + \log(M))^{|W|}. \tag{4.53}$$

□

Chapter 5

APPROXIMATE BAND PROPERTY OF THE MATRIX OF PARTIAL DERIVATIVES AT HIGH TEMPERATURE

This chapter continues the theme of Chapter 4, exploring the spectral properties of the RG transformation at high temperature. We concentrate our attention on finite-range (i.e., there is a constant S such that $J(X) = 0$ for $\mathrm{diam}(X) > S$) and translation-invariant Hamiltonians. We will show that the matrix of partial derivatives in this case displays an approximate band property. This justifies the well-definedness of the RG linearization.

Remark. *For later purposes, we point out that the finite-range assumption on the Hamiltonian implies a weaker assumption, finite-body, i.e., there is a constant D such that $J(X) = 0$ for $|X| > D$, where D only depends on the maximum possible range S and the number of dimensions d.*

5.1 Some combinatorial results

Proposition 5.1.1. *Consider the original coupling constants J with the Banach space norm $||J||_r$. Suppose this norm is sufficiently small. Suppose that $1 < M < e^r$. Then for each block spin site y, we have*

$$\sum_{y \in N : |N| > P} v_N M^{|N|} \leq \epsilon(P), \tag{5.1}$$

where

$$\epsilon(P) = c\frac{\left(\frac{2s||J||_r}{c^2}\right)^{\frac{P}{D}}}{1 - \frac{2s||J||_r}{c^2}}, \tag{5.2}$$

and for a fixed $||J||_r$ that satisfies (4.23), $\epsilon(P) \to 0$ as $P \to \infty$.

Proof. Due to the finite-body assumption on the Hamiltonian, any block-connected hypergraph that is block-rooted at y and with cardinality greater than P will have at least P/D links. By (4.37), this implies

$$\sum_{y \in N : |N| > P} v_N M^{|N|} \leq \sum_{n=P/D}^{\infty} c(2s||J||_r)^n c^{-2n} = \epsilon(P) \tag{5.3}$$

□

Proposition 5.1.2. *Consider the original coupling constants J with the Banach space norm $||J||_r$. Suppose this norm is sufficiently small. Suppose that $1 < M < e^r$. Then for every $Y \subset \mathcal{L}'$, we have*

$$\sum_{p=1}^{\infty} \frac{1}{p!} \sum_{N_1,...,N_p : |\cup_1^p N_i| > P} |C\left(N_1, ..., N_p\right)| c(Y, \cup_1^p N_i) |w_{N_1}| \cdots |w_{N_p}| \leq |Y| \epsilon(P). \tag{5.4}$$

Proof. This follows from Proposition 5.1.1. Remove Y, the remaining hypergraph is still block-connected by (4.14). Moreover, there can be at most $|Y|$ choices for where it is pinned down. □

5.2 Band structure of the partial derivatives of the renormalization group transformation

Theorem 5.2.1. *Suppose the original interaction J is at high temperature. Suppose also that it is finite-range and translation-invariant. Then there is an approximate band property for the matrix of partial derivatives: For subset W of the original* *lattice and subset Z of the image lattice that are sufficiently far apart, the partial derivative $\frac{\partial J'(Z)}{\partial J(W)}$ of the RG transformation (4.49) is arbitrarily small.*

Remark. Recall (5.2). Let

$$l(W,Z) = \inf\{||w-z|| : w \in W', z \in Z\} \tag{5.5}$$

be the distance between W and Z measured in the image lattice. If

$$l(W,Z) > S(|W|P + QK), \tag{5.6}$$

then

$$|\frac{\partial J'(Z)}{\partial J(W)}| \leq M^D \left(1+\log(M)\right)^D \left(\frac{\epsilon(P)}{\log(M)} + \left(\epsilon(Q)+\epsilon(K)\right)(1+P)DM^{(1+P)D}\right). \tag{5.7}$$

Proof. Fix a P that is large enough. We rewrite (4.49) as

$$\frac{\partial J'(Z)}{\partial J(W)} = \sum_{\sigma'} \sigma'_Z \frac{\sum_{|R|>|W|P,\Delta'} \tilde{w}_R \prod_{N\in\Delta'} w_N}{\sum_{\Delta} \prod_{N\in\Delta} w_N} + \sum_{\sigma'} \sigma'_Z \frac{\sum_{|R|\leq|W|P,\Delta'} \tilde{w}_R \prod_{N\in\Delta'} w_N}{\sum_{\Delta} \prod_{N\in\Delta} w_N}. \tag{5.8}$$

Following, we will verify the smallness of (5.8) by examining the two terms on the right-hand side separately.

Case 1: $|R| > |W|P$. Similarly as in the proof of Theorem 4.4.1, we estimate (4.52). Remove W, the remaining hypergraph (with cardinality greater than $|W|P$) breaks up into $k : 0 \leq k \leq |W|$ block-connected components, so at least one of them has cardinality greater than P. By (5.2), the contribution of this hypergraph is bounded by

$$M^{|W|}\epsilon(P)\sum_{k=0}^{|W|}\binom{|W|}{k}\left(\log(M)\right)^{k-1} = \frac{\epsilon(P)}{\log(M)} M^{|W|}\left(1+\log(M)\right)^{|W|}. \tag{5.9}$$

Case 2: $|R| \leq |W|P$. We need to do a more careful analysis for this case. By the Kotecký-Preiss theorem [13], (4.22) not only implies the inequality (4.21), but also an equality,

$$\sum_{\Delta'} \prod_{N \in \Delta'} w_N / \sum_{\Delta} \prod_{N \in \Delta} w_N$$
$$= \exp\left(- \sum_{p=1}^{\infty} \frac{1}{p!} \sum_{N_1,...,N_p} C\left(N_1, ..., N_p\right) c(R \cup W', \cup_1^p N_i) w_{N_1} \cdots w_{N_p} \right). \quad (5.10)$$

For notational convenience, we will denote the right-hand side of (5.10) by $F(\infty, \infty)$, where the first parameter of F indicates the cardinality restriction over the subsets N_i's under consideration, whereas the second parameter of F indicates the maximum number of N_i's allowed in the expansion. It is straightforward that for fixed Q and K,

$$F(\infty, \infty) = F(\infty, \infty) - F(Q, \infty) + F(Q, \infty) - F(Q, K) + F(Q, K). \quad (5.11)$$

We first examine $F(\infty, \infty) - F(Q, \infty)$. For every subset N of $\mathcal{L}'$, define

$$u_N = \begin{cases} w_N & \text{if } |N| \leq Q; \\ 0 & \text{otherwise.} \end{cases} \quad (5.12)$$

The difference in F can then be interpreted as induced by evaluating (5.10) using two sets of parameters w_N and u_N. By (4.21), these two parameter sets both lie in the region of analyticity of (5.10), thus intuitively, the difference can be as small as desired when Q is large enough. In fact, it is bounded by $|R \cup W'| M^{|R \cup W'|} \epsilon(Q)$ by the mean value theorem, applied to (4.21) and (5.4). Fix such a Q. We next examine $F(Q, \infty) - F(Q, K)$. This difference can be regarded as the tail of the convergent series (5.10), thus should also be small when K is large enough. We again refer to (4.21) and (5.4), and conclude that it is bounded by $|R \cup W'| M^{|R \cup W'|} \epsilon(K)$. Fix such a K. For these two situations, the only thing left to show now is that

$$\sum_{|R|\leq|W|P} |\tilde{w}_R| \tag{5.13}$$

is finite, but this naturally follows from (4.53).

Finally, we examine $F(Q,K)$. Recall that by (4.3) and (4.4), w_N only depends on image sites in N. As $R \cup W'$ and $\cup_1^p N_i$ is block-connected, $F(Q,K)$ will only depend on image sites in a finite region (roughly a ball with radius $S(|W|P+QK)$). If Z is outside this region, then

$$\sum_{|R|\leq|W|P} \tilde{w}_R F(Q,K) \tag{5.14}$$

is a constant with respect to σ'_Z, thus, when summing over all possible image configurations σ' as in (5.8), it vanishes. □

Proposition 5.2.1. *Suppose the original interaction J is at high temperature. Suppose also that it is finite-range and translation-invariant. Then for subset W of the original lattice and subset Z of the image lattice, as the distance $l(W,Z)$ between W and Z gets large, the partial derivative $|\frac{\partial J'(Z)}{\partial J(W)}|$ decays sub-exponentially, a little slower than* $\exp(-l(W,Z)^{1/2})$.

Proof. For notational convenience, we denote $l(W,Z)$ simply by l. Take

$$P = \frac{1}{|W|}\left(\frac{l}{2S}\right)^{\alpha}, \tag{5.15}$$

and

$$Q = K = \left(\frac{l}{2S}\right)^{\beta}, \tag{5.16}$$

where $0 < \alpha < \beta \leq 1/2$. We examine (5.7) closely. The first factor,

$$M^D \left(1+\log(M)\right)^D,$$

is just a constant. The second factor is more complicated and merits more attention. The first term, $\epsilon(P)/\log(M)$, decays like $\exp(-l^\alpha)$, whereas the second term,

$$(\epsilon(Q)+\epsilon(K))\,(1+P)DM^{(1+P)D},$$

decays like $\exp\left(-l^\beta + l^\alpha\right) \sim \exp\left(-l^\beta\right)$. Piecing it all together, $|\frac{\partial J'(Z)}{\partial J(W)}|$ decays sub-exponentially, like $\exp(-l^\alpha)$. □

5.3 Existence of the renormalization group linearization

Proposition 5.3.1. *Consider finite-range and translation-invariant Hamiltonians. Fix a subset Z of the image lattice. Let $n(E)$ be the number of subsets W of the original lattice that are at most E-distance away from Z,*

$$n(E) = \#\{W : l(W,Z) \leq E\}. \tag{5.17}$$

Then $n(E)$ grows polynomially in E.

Proof. Due to our finite-range and translation-invariant assumptions on the Hamiltonian,

$$n = \sup_{y\in\mathcal{L}'} \#\{W : y \in W'\} < \infty. \tag{5.18}$$

Thus $n(E)$ grows at the same rate as the volume of a d-dimensional ball with radius E, i.e., polynomial growth E^d. □

Theorem 5.3.1. *Suppose the original interaction J is at high temperature. Suppose also that it is finite-range and translation-invariant. Then the linearization* $\mathrm{L}(J)$ *of the RG transformation (3.6) is well-defined.*

Proof. This is mainly due to the fact that sub-exponential decay dominates polynomial growth. Take K with $||K||_r$ small. As $||K||_\infty \leq ||K||_r$, $||K||_\infty$ is small also. By Propositions 5.2.1 and 5.3.1,

$$|\mathrm{L}(J)K(Z)| \leq \sum_{n=0}^{\infty} \sum_{n \leq l(W,Z) < n+1} \left| \frac{\partial J'(Z)}{\partial J(W)} \right| |K(W)| \tag{5.19}$$

$$\sim ||K||_\infty \sum_{n=0}^{\infty} \exp(-n^\alpha)(n+1)^d. \tag{5.20}$$

Our claim then follows from the integral test. □

Remark. Notice that here we have shown the existence of the RG linearization $\mathrm{L}(J)$, but have not been able to show that $\mathrm{L}(J)K$ also lies in a Banach space (except for one with uniform norm). In the future we hope to overcome this difficulty. And our more ambitious goal is to prove actual Gâteaux differentiability (or even Fréchet differentiability) of the RG transformation, which would require examining higher order terms.

Chapter 6
SPECTRAL PROPERTIES OF THE RENORMALIZATION GROUP AT CRITICAL TEMPERATURE

In this chapter, we employ cluster expansion techniques to study spectral properties of the RG transformation of finite-range and translation-invariant Hamiltonians at critical temperature. We will extend the theoretical result of Haller and Kennedy, showing that the partial derivatives of the RG transformation exist under a certain condition. Roughly speaking, the condition is that the collection of measures $\mu_{\sigma'}$ is in the high-temperature phase uniformly in the block spin configuration σ', which was verified in two special cases [9]: Decimation with spacing $b = 2$ on the square lattice with $\beta < 1.36\beta_c$, and the Kadanoff transformation with parameter p on the triangular lattice in a subset of the β, p plane that includes values of β greater than β_c. The main difficulty here is that the original interaction J is not small at the critical temperature, thus a direct cluster expansion is not applicable as in previous chapters. However, there is a marvellous estimate on long range energies that will provide us with the smallness needed. We first review the relevant results in [9].

For notational convenience, we will denote $\prod_{y\in\mathcal{L}'} T_y(\sigma, \sigma'_y) e^{\sum_X J(X)\sigma_X}$ by e^{-H} in the following. As shown in [9], this modified Hamiltonian H will also be finite-range.

6.1 Review of relevant results

The key idea in [9] is to divide $\mathcal{L}$ into blocks that are L sites long on each side. They refer to them as L-blocks, and choose L large enough, so that these L-blocks are commensurate with the blocks in the RG transformation, i.e., each RG block is a subset of an L-block. They then divide these L-blocks into 2^d types. For illustration purposes, they restrict their attention to two dimensions, so there would be 4 types of L-blocks, labelled by $i = 1, 2, 3, 4$. Let $\sum_i$ denote the summation over the spins which are in a type-i L-block. Then trivially,

$$\sum_\sigma e^{-H} = \sum_4 \sum_3 \sum_2 \sum_1 e^{-H}. \tag{6.1}$$

They start by considering $\sum_1 e^{-H}$ and define F^1 by

$$\exp(-F^1) = \sum_1 e^{-H}. \tag{6.2}$$

So F^1 is a function of the spins in blocks of types 2, 3, 4 and the boundary spins. And the above sum factors into a product over type 1 blocks of the sum over the spins in that block. However, when they try to compute $\sum_2 \exp(-F^1)$ in a similar fashion, they run into difficulty. F^1 can contain terms which involve spins in more than one type 2 block. Thus this sum does not factor into a product of independent sums over the type 2 blocks. To proceed, they distinguish long-range terms $F^1_{\text{LR}} = \sum_{B:\text{LR}} F^1_B$ supported on sets of sites with diameter greater than L that prevent the factorization from short-range terms $F^1_{\text{SR}} = \sum_{B:\text{SR}} F^1_B$ that do not. Then $\sum_2 \exp(-F^1_{\text{SR}})$ will factor into a product over the type 2 blocks.

They continue the above constructions iteratively, always throwing out the long-range terms that prevent the factorization. Eventually, after performing all the summation, they get F^4. They then define a modified expectation E, given by

$$Ef = \exp(F^4) \sum_4 \sum_3 \sum_2 \sum_1 \exp(-H + F^1_{\text{LR}} + F^2_{\text{LR}} + F^3_{\text{LR}}) f. \tag{6.3}$$

It follows that

$$\sum_\sigma e^{-H} = e^{-F^4} V(\sigma'), \tag{6.4}$$

where

$$V(\sigma') = E\exp(-F^1_{\mathrm{LR}} - F^2_{\mathrm{LR}} - F^3_{\mathrm{LR}}). \tag{6.5}$$

For each allowable long-range B (small enough to fit inside some $3L$ by $3L$ square), they define

$$K(B) = \exp(-F^1_B - F^2_B - F^3_B) - 1. \tag{6.6}$$

If $||F^i_B||_\infty$ is small, then $||K(B)||_\infty$ will be small also, and a cluster expansion may be applicable as desired. The smallness of $||F^i_B||_\infty$ is established in [9] as follows.

Definition 6.1.1. *Let V be a finite set of sites in the original lattice, τ a boundary condition for V, and σ' a block spin configuration. Define a probability measure which depends on V, τ and σ' by*

$$\mu_{\sigma',V,\tau}(F) = \frac{\sum_\sigma e^{-H}F(\sigma)}{\sum_\sigma e^{-H}}. \tag{6.7}$$

Hypothesis: Suppose there exist constants $a < \infty$ and $m > 0$ such that for every finite subset V of the lattice, every two sites $i, j \in V$, every boundary condition τ, and every block spin configuration σ',

$$|\mu_{\sigma',V,\tau}(\sigma_i\sigma_j) - \mu_{\sigma',V,\tau}(\sigma_i)\mu_{\sigma',V,\tau}(\sigma_j)| \leq ae^{-m|i-j|}. \tag{6.8}$$

Lemma 6.1.1. *Suppose that **Hypothesis** holds. For a finite volume V, a boundary condition τ outside of V, and a block spin configuration σ', let $F_{\sigma',V,\tau}$ be the free energy, i.e., minus the logarithm of the partition function. Then there is a constant n such that for every finite volume V, every boundary condition τ, every block spin configuration σ', and every two sites $i, j \notin V$ (with space in-between large enough), we have **Condition**:*

$$|\sum_{\tau_i,\tau_j}\tau_i\tau_j F_{\sigma',V,\tau}| \leq ne^{-m|i-j|}. \tag{6.9}$$

*Here m is the same constant that appears in **Hypothesis**.*

Lemma 6.1.2. *Suppose that* ***Condition*** *holds. Then there are constants c', p, L_0 such that*

$$||F_B^i||_\infty \leq c' L^p e^{-mL} \tag{6.10}$$

if $L \geq L_0$ and B is long-range. Here m is the same constant that appears in ***Condition****.*

The upshot is that when **Hypothesis** holds, there is a function $\epsilon(L)$ such that

$$||K(B)||_\infty \leq \epsilon(L) \tag{6.11}$$

for every allowable B, with $\epsilon(L) \to 0$ as $L \to \infty$. This justifies an expansion for $V(\sigma')$:

$$V(\sigma') = \sum_\Delta \prod_{N \in \Delta} w_N, \tag{6.12}$$

where Δ is a set of disjoint subsets N's of $\mathcal{L}''$, and

$$w_N = \sum_{\Gamma_C^* = N} E\left(\prod_{B \in \Gamma_C} K(B)\right). \tag{6.13}$$

To understand the above expression, we need to introduce some more combinatorial concepts. Two links of a hypergraph are L-connected if they are within a distance cL apart, where c is a constant that only depends on the number of dimensions d as shown in [9]. A hypergraph Γ is L-connected if the support of Γ is nonempty and cannot be partitioned into nonempty sets with no L-connected links. In our current setting, a subset X of $\mathcal{L}$ defines a subset X'' of $\mathcal{L}''$, corresponding to the set of L-blocks that have non-empty intersection with X. Conversely, for each site y in $\mathcal{L}''$, there is a corresponding L-block y^* that is a subset of $\mathcal{L}$. Thus a hypergraph Γ on $\mathcal{L}$ defines a hypergraph Γ'' on $\mathcal{L}''$. We use Γ_C to indicate L-connectivity of the hypergraph Γ_C, and write $\Gamma_C^* = \cup \Gamma_C''$ for the support of Γ_C'' in $\mathcal{L}''$.

6.2 Convergence of the cluster expansion for the pinned frozen block spin free energy

Proposition 6.2.1. *Take $M > 1$. Suppose L is sufficiently large so that $\epsilon(L)$ is sufficiently small. Then for each L-block site y, we have*

$$\sum_{y \in N} |w_N| M^{|N|} \leq \log(M). \tag{6.14}$$

Remark. It is sufficient that $\epsilon(L)$ satisfy

$$\epsilon(L) \leq \frac{\log(M)(p-1)^p}{c(Mp)^p\,(1 + (p-1)\log(M))}. \tag{6.15}$$

Here every union B of L-blocks under consideration is allowable, i.e., consisting of at most $p = 3^d$ L-blocks, and

$$c = \sum_{m=1}^{pL^d} \sup_{x \in \mathcal{L}} \#\{B : x \in B, |B| = m\} < \infty, \tag{6.16}$$

due to our finite-range and translation-invariant assumptions on the Hamiltonian.

Remark. The inequality (6.14) is a standard sufficient condition for convergence of the cluster expansion (Kotecký-Preiss). It will be applied in the following Theorem 6.3.1.

Proof.

$$\begin{aligned}
\sum_{y \in N} |w_N| M^{|N|} &\leq \sum_{y \in N} \sum_{\Gamma_C^* = N} M^{|N|} \prod_{B \in \Gamma_C} ||K(B)||_\infty && (6.17)\\
&\leq \sum_{y \in N} \sum_{\Gamma_C^* = N} M^{p|\Gamma_C|} \left(\epsilon(L)\right)^{|\Gamma_C|} && (6.18)\\
&= \sum_{y \in \Gamma_C^*} \left(M^p \epsilon(L)\right)^{|\Gamma_C|}. && (6.19)
\end{aligned}$$

We say that a hypergraph Γ_C is L-rooted at y if its support intersects a fixed L-block y^*. Let $a_n(y)$ be the number of all L-connected hypergraphs with n links that are L-rooted at y,

$$a_n(y) = \#\{y \in \Gamma_C^* : |\Gamma_C| = n\}. \tag{6.20}$$

Let a_n be the supremum over y of the number of L-connected hypergraphs with n links that are L-rooted at y, i.e., $a_n = \sup_{y \in \mathcal{L}''} a_n(y)$. Then

$$\sum_{y \in N} |w_N| M^{|N|} \leq \sum_{n=1}^{\infty} a_n \left(M^p \epsilon(L)\right)^n. \tag{6.21}$$

It seems that once we show that a_n grows at most exponentially with n, the geometric series above will converge for small enough $\epsilon(L)$, and our claim might follow. To estimate a_n, we relate to some standard combinatorial facts [15]. The rest of the proof follows from a series of lemmas. □

Lemma 6.2.1. *Let a_n be the supremum over y of the number of L-connected hypergraphs with n links that are L-rooted at y. Then a_n satisfies the recursive bound*

$$a_n \leq c \sum_{k=0}^{p} \binom{p}{k} \sum_{a_{n_1}, \ldots, a_{n_k} : n_1 + \cdots + n_k + 1 = n} a_{n_1} \cdots a_{n_k} \tag{6.22}$$

for $n \geq 1$, where $\binom{p}{k}$ is the binomial coefficient.

Proof. We first linearly order the points x in $\mathcal{L}$ and also linearly order the union of L-blocks B of $\mathcal{L}$. This naturally induces a linear ordering of the points y in $\mathcal{L}''$. For a fixed but arbitrarily chosen y in $\mathcal{L}''$, we examine (6.20). Write $\Gamma_C = \{B_1\} \cup \Gamma_C^1$, where B_1 is the least B in Γ_C with $y^* \subset B_1$. There must be such a union of L-blocks, since $y \in \Gamma_C^*$. Moreover, every $x \in y^*$ will satisfy $x \in B_1$. Then

$$a_n(y) \leq \sum_{m=1}^{pL^d} \sup_{x \in \mathcal{L}} \sum_{B_1 : x \in B_1, |B_1| = m} \#\{\Gamma_C^1\}. \tag{6.23}$$

As a consequence,

$$a_n(y) \leq c\#\{\Gamma_C^1\}. \tag{6.24}$$

The remaining hypergraph Γ_C^1 has $n-1$ subsets and breaks into $k : k \leq p$ L-connected components $\Gamma_1, ..., \Gamma_k$ of sizes $n_1, ..., n_k$, with $n_1 + \cdots + n_k = n - 1$. The set of such components may be empty, or it could just be the original L-connected set. For each component Γ_i, there is a least L-block y_i^* through which it is L-connected to B_1. The image $\{y_i\}$ of these L-blocks is a subset of B_1'' in $\mathcal{L}''$, thus has no more than $k : k \leq p$ points, and the components are L-rooted at these image sites. Furthermore, different Γ_i's correspond to disjoint Γ_i'''s, as $y_i \in \Gamma_i''$, the map from the components to this image is injective. So we have

$$a_n(y) \leq c \sum_{k=0}^{p} \binom{p}{k} \sum_{a_{n_1},...,a_{n_k}: n_1+\cdots+n_k+1=n} a_{n_1} \cdots a_{n_k}. \tag{6.25}$$

Our inductive claim follows by taking the supremum over all y in $\mathcal{L}''$. Finally, we look at the base step: $n = 1$. In this simple case, as reasoned above, we have

$$\begin{aligned} a_1 &= \sup_{y \in \mathcal{L}''} \#\{y \in \Gamma_C^* : |\Gamma_C| = 1\} \\ &\leq \sum_{m=1}^{pL^d} \sup_{x \in \mathcal{L}} \#\{B : x \in B, |B| = m\} \\ &= c, \end{aligned} \tag{6.26}$$

and this verifies our claim. □

Clearly, $\sum_{y \in N} |w_N| M^{|N|}$ will be bounded above by $\sum_{n=1}^{\infty} \bar{a}_n \left(M^p \epsilon(L)\right)^n$, if

$$\bar{a}_n = c \sum_{k=0}^{p} \binom{p}{k} \sum_{\bar{a}_{n_1},...,\bar{a}_{n_k}: n_1+\cdots+n_k+1=n} \bar{a}_{n_1} \cdots \bar{a}_{n_k} \tag{6.27}$$

for $n \geq 1$, i.e., equality is obtained in the above lemma.

Lemma 6.2.2. *Consider the coefficients $\bar{a}_n$ that bound the number of L-connected and L-rooted hypergraphs with n links. Let $w = \sum_{n=1}^{\infty} \bar{a}_n z^n$ be the generating function of these coefficients. Then the recursion relation (6.27) for the coefficients is equivalent to the formal power series generating function identity*

$$w = cz(1+w)^p. \tag{6.28}$$

Proof. Notice that $(1+w)^p = \sum_{k=0}^{p} \binom{p}{k} w^k$, thus

$$w = cz \sum_{k=0}^{p} \binom{p}{k} w^k. \tag{6.29}$$

Writing completely in terms of z, we have

$$\sum_{n=1}^{\infty} \bar{a}_n z^n = c \sum_{k=0}^{p} \binom{p}{k} \sum_{\bar{a}_{n_1},\ldots,\bar{a}_{n_k}: n_1+\cdots+n_k+1=n} \bar{a}_{n_1} \cdots \bar{a}_{n_k} z^n. \tag{6.30}$$

Our claim follows from term-by-term comparison. □

Lemma 6.2.3. *If w is given as a function of z as a formal power series by the generating function identity (6.28), then this power series has a nonzero radius of convergence $|z| \leq \frac{(p-1)^{p-1}}{cp^p}$.*

Proof. Without loss of generality, assume $z \geq 0$. Set $z_1 = cz$. Solving (6.28) for z_1 gives

$$z_1 = \frac{w}{(1+w)^p}. \tag{6.31}$$

By elementary calculus, this increases as w goes from 0 to $1/(p-1)$ to have values z_1 from 0 to $(p-1)^{p-1}/p^p$. It follows that as z_1 goes from 0 to $(p-1)^{p-1}/p^p$, the w values range from 0 to $1/(p-1)$. □

Proof of Proposition 6.2.1 continued. We notice that in the above lemma, $w = \sum_{n=1}^{\infty} \bar{a}_n z^n = 1/(p-1)$ corresponds to $z_1 = cz = (p-1)^{p-1}/p^p$, which implies that for each n,

$$\bar{a}_n \leq (cp^p)^n (p-1)^{-(1+(p-1)n)}. \tag{6.32}$$

Gathering all the information we have obtained so far,

$$\begin{aligned} \sum_{y\in N} |w_N| M^{|N|} &\leq \sum_{n=1}^{\infty} (c(Mp)^p \epsilon(L))^n (p-1)^{-(1+(p-1)n)} &(6.33)\\ &= \frac{\frac{c(Mp)^p\epsilon(L)}{(p-1)^p}}{1-\frac{c(Mp)^p\epsilon(L)}{(p-1)^{p-1}}} \leq \log(M) &(6.34)\end{aligned}$$

by (6.15). □

6.3 Existence of the partial derivatives of the renormalization group transformation

We have shown in (6.12) that the denominator of (2.10) has a cluster representation. We now examine the effect of multiplying σ_W to this cluster representation as in the numerator of (2.10). There will be two kinds of terms. In some of these, none of the L-connected components intersect W, so for these terms one gets a product of σ_W with a product of independent w_N's. For the other terms one decomposes Δ into one L-connected component that is connected to W and remaining L-connected components that are not. The result is the representation

$$\sigma_W V(\sigma') = \sum_{R,\Delta''} \tilde{w}_R \prod_{N\in\Delta''} w_N, \tag{6.35}$$

where $R = \emptyset$ or $R \cap W'' \neq \emptyset$, and $\tilde{w}_R$ is a sum over hypergraphs Δ_R with $\cup\Delta_R = R$ such that W, Δ_R is L-connected. Therefore

$$\frac{\partial J'(Z)}{\partial J(W)} = \sum_{\sigma'} \sigma'_Z \frac{\sum_{R,\Delta''} \tilde{w}_R \prod_{N\in\Delta''} w_N}{\sum_{\Delta} \prod_{N\in\Delta} w_N}. \tag{6.36}$$

Remark. Equation (6.36) is derived from finite lattices, but will be taken as the definition of the partial derivative $\frac{\partial J'(Z)}{\partial J(W)}$ in the infinite volume limit, following standard interpretation of statistical mechanics.

Theorem 6.3.1. *Suppose that **Hypothesis** holds. Then for every subset W of the original lattice $\mathcal{L}$ and every subset Z of the image lattice $\mathcal{L}'$, the partial derivative $\frac{\partial J'(Z)}{\partial J(W)}$ of the RG transformation (6.36) is well-defined.*

Proof. By the Kotecký-Preiss theorem [13], (6.14) implies

$$\left|\sum_{\Delta''}\prod_{N\in\Delta''} w_N \Big/ \sum_{\Delta}\prod_{N\in\Delta} w_N\right| \le M^{|R\cup W''|}. \tag{6.37}$$

Also notice that $N \in \Delta''$ implies $N \cap (R \cup W'') = \emptyset$. We have

$$|\tilde{w}_R| \le \sum_{\Delta_R}\prod_{Y\in\Delta_R} |w_Y|. \tag{6.38}$$

To verify our claim, we need to estimate

$$\sum_{R} |\tilde{w}_R| M^{|R\cup W''|} \le \sum_{\Delta_R} M^{|W|} \prod_{Y\in\Delta_R} |w_Y| M^{|Y|}. \tag{6.39}$$

But this is easy, remove W, the remaining hypergraph breaks up into $k : 0 \le k \le |W|$ L-connected components. So this last quantity is bounded by

$$M^{|W|}\sum_{k=0}^{|W|}\binom{|W|}{k}(\log(M))^k = M^{|W|}\left(1+\log(M)\right)^{|W|}. \tag{6.40}$$

□

Remark. It may be shown using similar techniques as in Chapter 5 that the matrix of partial derivatives $\frac{\partial J'(Z)}{\partial J(W)}$ in this case also displays an approximate band property, which in turn justifies the well-definedness of the RG linearization at critical temperature. But still, we would meet the same difficulty as in high temperature when it comes to proving actual Gâteaux differentiability.

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