INTERACTING WITH ALGEBRA:
MATHEMATICIANS, MATHEMATICS EDUCATORS, AND TEACHERS
MAKING SENSE OF ALGEBRA CONTENT

by
Joshua D. Chesler

A Dissertation Submitted to the Faculty of the
DEPARTMENT OF MATHEMATICS
In Partial Fulfillment of the Requirements
For the Degree of
DOCTOR OF PHILOSOPHY
In the Graduate College
THE UNIVERSITY OF ARIZONA

2 0 0 9
As members of the Dissertation Committee, we certify that we have read the dissertation prepared by Joshua D. Chesler entitled Interacting with Algebra: Mathematicians, Mathematics Educators, and Teachers Making Sense of Algebra Content and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

Rebecca McGraw
Date: May 14, 2009

Marta Civil
Date: May 14, 2009

Deborah Hughes Hallett
Date: May 14, 2009

Walter Doyle
Date: May 14, 2009

Final approval and acceptance of this dissertation is contingent upon the candidate’s submission of the final copies of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

Rebecca McGraw
Date: May 14, 2009

Dissertation Director: Rebecca McGraw
STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfillment of requirements for an advanced degree at the University of Arizona and is deposited in the University Library to be made available to borrowers under rules of the Library.

Brief quotations from this dissertation are allowable without special permission, provided that accurate acknowledgment of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the head of the major department or the Dean of the Graduate College when in his or her judgment the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.

SIGNED: Joshua D. Chesler
Acknowledgements

Rebecca McGraw has been an essential part of my success in graduate school. About five years ago, she was the professor for my first mathematics education course. Through this course, she helped me discover and appreciate the wonderful complexity and diversity of mathematics education. Since then, it is through her guidance that I have become a “mathematics educator” and completed this dissertation. As my advisor, she has generously shared her knowledge, her time, and her advice. To Rebecca, I extend my deepest gratitude.

I am also very thankful for my other committee members. Marta Civil, Deborah Hughes Hallett, and Walter Doyle have each contributed to this dissertation. During my oral defense and my dissertation proposal they treated me as a peer and expressed genuine interest in my work. I was able to make sense of my dissertation data largely because of their advice and feedback.

The participants in the 2006 and 2008 Making Connections workshops were very kind to allow me to record and study their interactions. I thank them for their openness and for being themselves. I also thank William McCallum and the Institute for Mathematics & Education for facilitating and encouraging this study.

Graduate school has been an amazing experience which I will always value. The best part has been the people. To my professors and to my fellow graduate students, I extend my gratitude for your friendship and your support. I would especially like to acknowledge Daniel Bartlett; he has been and will continue to be an inspiration.

To my friends and family, thank you for believing in me and for providing me with a life outside of graduate school. It has been a treat to live in the same state as my parents and sister for the first time since 1990. Finally, I would like to thank my wife, Aisha, whose support and patience has been everything.
DEDICATION

for Aisha
# Table of Contents

**List of Tables** ................................................................. 9

**List of Figures** ............................................................... 10

**Abstract** ........................................................................... 11

**Chapter 1. Introduction** .................................................. 12

**Chapter 2. Review of Selected Literature** ......................... 17
  2.1. Algebra ........................................................................ 18
    2.1.1. Algebra in The Schools ........................................ 18
    2.1.2. Sources of Meaning in Algebra ............................ 21
    2.1.3. Algebraic Activity ............................................. 24
    2.1.4. Research on the Learning of Algebra .................... 25
  2.2. Curriculum .................................................................... 28
    2.2.1. Alternative Stances on Curriculum Use ................ 30
    2.2.2. Temporal Phases of Curriculum ............................ 31
    2.2.3. Teachers Interacting with and Learning from Curriculum 32
  2.3. Professional Groups’ Beliefs about Algebra and Mathematics 35
    2.3.1. Teachers’ Beliefs about Algebra and Mathematics .... 35
    2.3.2. Teacher Learning and Knowledge ......................... 38
    2.3.3. Mathematicians and Mathematics Educators .......... 41
  2.4. Conclusion .................................................................. 44

**Chapter 3. Methods** .......................................................... 45
  3.1. Context of the Study .................................................. 46
    3.1.1. The Workshop Participants ................................ 48
    3.1.2. The 12 Algebra Problems ................................... 49
    3.1.3. The Categorization Task .................................... 51
  3.2. Data Collection ......................................................... 54
  3.3. Data Analysis ............................................................ 56
    3.3.1. Creating the Data Set ........................................ 57
    3.3.2. Pilot Study ...................................................... 58
    3.3.3. Analysis of the 2006 and 2008 Data .................... 61
  3.4. Trustworthiness ......................................................... 65
Table of Contents—Continued

Chapter 4. Results and Discussion ........................................... 67
  4.1. The Workshop Organizer’s Views on Algebra and on the 12 Problems . . . 68
    4.1.1. Diminished Roles for Functions and Graphs .......................... 69
    4.1.2. Summary - Chris’ View of Algebra ................................... 72
  4.2. Interacting with Algebra ............................................... 72
    4.2.1. Number 1 .................................................................. 72
    4.2.2. The Role of the Student in Framing the Problem Analysis ......... 81
    4.2.3. Focus on Structure ..................................................... 86
    4.2.4. The Wording of Questions .......................................... 91
    4.2.5. Summary - Interacting with Algebra ................................ 96
  4.3. Views of Algebra ......................................................... 97
    4.3.1. Number 11 ................................................................ 97
    4.3.2. Algebraic Form vs. Function ........................................ 105
    4.3.3. Mathematical Habits .................................................. 117
    4.3.4. Translating between English and Mathematics ................... 122
    4.3.5. Summary - Views of Algebra ....................................... 125
  4.4. Conclusion ................................................................. 126

Chapter 5. Implications ......................................................... 128
  5.1. Wording and Translation ............................................... 128
  5.2. Understanding Interactions with Algebra ............................ 133
  5.3. The Objects of Algebra ................................................. 139
  5.4. Cultivating the Flash ..................................................... 141
  5.5. Limitations ............................................................... 142
  5.6. Final Remarks ............................................................ 144

Appendix A. The 12 Problems .............................................. 146

Appendix B. 2008 Recording Guide ........................................ 151

Appendix C. Interview Protocols ............................................. 154
  C.1. Interview Protocol for All Participants .............................. 154
    C.1.1. Algebra ................................................................. 154
    C.1.2. Curriculum ........................................................... 155
    C.1.3. Professional Culture/Beliefs .................................... 155
  C.2. Supplemental Questions for Chris Norris ......................... 156

Appendix D. Surveys ............................................................ 158
  D.1. Mathematics Background Survey for Teachers ..................... 158
  D.2. Mathematics Background Survey for Mathematicians ............ 158
Table of Contents—Continued

Appendix E. TexNicCenter ......................................................... 160

Appendix F. Results of the Categorization Task ............................ 161

References ................................................................................. 162
LIST OF TABLES

Table 3.1. Making Connections Participants. (M - Middle School Teacher, H - High School Teacher) ................................................. 48
Table 3.2. 2006 Groupings. The numbers in the parentheses are the question numbers included in each group. ............................... 53
Table 3.3. 2008 Groupings. The numbers in the parentheses are the question numbers included in each group. ............................... 53
Table 3.4. Full Transcripts from the Making Connections Workshops ............................... 58
Table 3.5. Codes developed during pilot study .................................. 60
Table 3.6. Start codes for 2008 analysis. ........................................ 62
Table 3.7. Keyword Groups. Keywords are separated by commas. ............... 63
Table 4.1. Categorization of Number 1 ........................................... 73
Table 4.2. Word counts from the categorization tasks (90 minutes in 2006, 60 minutes in 2008). “Usage per hour” is calculated for each professional group using totals from both workshops. ............................................. 82
Table 4.3. Categorization of Number 11 .......................................... 98
Table B.1. 2008 Making Connections Audio Recordings - Part 1 ................ 151
Table B.2. 2008 Making Connections Audio Recordings - Part 2 ................ 152
Table B.3. 2008 Making Connections Video Recordings .......................... 153
Table F.1. 2006 Groupings. The numbers in the parentheses are the question numbers included in each group. ............................... 161
Table F.2. 2008 Groupings. The numbers in the parentheses are the question numbers included in each group. ............................... 161
LIST OF FIGURES

FIGURE 1.1. Professional Groups - Math Educators (ME), Mathematicians (M), and Teachers (T) - interacting with each other as framed by the Making Connections algebra content. .................................................. 15
FIGURE 1.2. Professional Groups - Math Educators (ME), Mathematicians (M), and Teachers (T) - interacting with Making Connections algebra problems. ........ 15

FIGURE 2.1. Remillard’s “framework of component’s of teacher-curriculum relationship” (2005). .......................................................... 29
FIGURE 2.2. Hill, Ball, and Schilling’s (2008) domain map for MKT. .......... 40

FIGURE 4.1. Problem Number 9. .................................................. 69
FIGURE 4.2. Problem Number 1. ................................................. 73
FIGURE 4.3. Problem Number 2. .................................................. 83
FIGURE 4.4. Problem Number 4. .................................................. 84
FIGURE 4.5. Problem Number 7. .................................................. 87
FIGURE 4.6. Problem Number 10. ................................................. 90
FIGURE 4.7. Problem Number 2. .................................................. 93
FIGURE 4.8. Problem Number 5. .................................................. 94
FIGURE 4.9. Problem Number 11 ................................................... 97
FIGURE 4.10. Problem Number 10. .............................................. 106
FIGURE 4.11. Problem Number 4. ............................................... 110
FIGURE 4.12. Problem Number 6. ............................................... 113
FIGURE 4.13. Problem Number 8. ............................................... 118
FIGURE 4.14. Problem Number 5. ............................................... 123

FIGURE E.1. Partial screen shot of TexNicCenter. .............................. 160
This dissertation examines the interactions of mathematicians, mathematics educators, and teachers with a set of 12 algebra problems. The data are from two Making Connections workshops (2006 and 2008) which included members of these three professional groups; each workshop was comprised of 13 participants. The primary analytical focus was a task in which each professional group separately discussed and categorized the 12 algebra problems; interviews were conducted with the participants from the 2008 workshop. The methods and analysis of this study were framed by research on the teaching and learning of algebra, on curriculum use as a participatory relationship, and on the dispositions and beliefs of members of the three professional groups.

The study indicated that teachers, as compared to the other two groups, drew more heavily upon their knowledge of students as they made sense of the algebra problems. Teachers were at times concerned with the wording of questions as an obstacle for students. The mathematicians and mathematics educators put a greater focus on algebraic structure as they discussed the problems whereas the teachers put a greater focus on graphing. I connect these findings to the workshop participants’ views of algebra.
Chapter 1

INTRODUCTION

A few years ago I went back to Middle School. During the 2005-2006 school year, as an NSF GK-12 fellow, I spent 10 to 15 hours each week in Middle School math and science classrooms. At the time I was a graduate student in mathematics who had studied anthropology as an undergraduate. I returned to middle school both as a representative of the Tucson mathematics community and as someone expecting to be a part of some interesting cultural interactions. And that is what happened.

For a year I collaborated with the teachers at this middle school. We planned lessons together, looked at student work and curricular materials together, interacted with students together, and discussed mathematics together. Something interesting happened though; when we looked at mathematics, in the form of curricular materials or student work, we often noticed and valued very different aspects. But it was rarely the case that one of us was right and the others were wrong. We were looking at the same mathematics but interpreting it differently.

This dissertation is, to some extent, my effort to understand this phenomena. At a broad level I examine how members of different professional groups make sense of and are influenced by mathematical content. More specifically, I examine how some mathematicians, mathematics educators, and teachers interacted with algebra content. The sites for the research were two workshops in which members of these professional groups were tasked with discussing and categorizing a set of twelve algebra problems compiled by the workshops’ organizer, a mathematician. These Making Connections workshops were not designed for the purposes of this research project. But they were inspired by an appreciation for these groups’ differing perspectives on and important roles in K-12 mathematics education.
For the purposes of this dissertation, I use rather narrow definitions of these three professional groups. A mathematician is someone with a Ph.D. in mathematics who is a professor in a university mathematics department and has research interests and activities primarily in mathematics. A mathematics educator is someone with a Ph.D. in mathematics or in education who is involved with the preparation of K-12 teachers and has research interests in mathematics education. For the purposes of the present study, a teacher is a middle school or high school algebra teacher. Herein, I borrow Remillard’s (2005) definition of curriculum or curricular materials as “printed, often published resources designed for use by teachers and students during instruction” (p. 213).

Mathematicians, mathematics educators, and teachers play essential roles in the development and implementation of curricular materials. Yet the differing foci of their professions may frame school mathematical content and the associated pedagogies in different and not necessarily complementary ways. Some research has focused on how teachers’ individual characteristics influence their interpretations and uses of curricular materials (e.g., Remillard, 2005; Stein, Remillard, & Smith, 2007; Lambdin & Preston, 1995; Remillard & Bryans, 2004); this includes such characteristics as content knowledge, pedagogical content knowledge, beliefs, goals, and identity. Certainly this can be said of mathematicians and mathematics educators as well; their personal identities and knowledge influence their interactions with mathematics. In the coming chapters, I look across and within each of these three professional groups and examine their patterns of interaction with the set of 12 Making Connections algebra problems.

One justification for this study is that it may help illuminate certain aspects of curriculum development and enactment. Remillard (1999) has noted that we “have limited knowledge of how teachers interact with and use curriculum materials” (p. 316). In this dissertation, I examine how Making Connections workshop participants (teachers, mathematicians, and mathematics educators) interacted with a set of algebra problems and I connect this analysis to their views of algebra in general, to their professional identities, and to features of the problems themselves.
It is critical that curriculum developers pay careful attention to the multiple ways that their materials communicate with the teacher. They must consider how they are addressing the teacher through the design of their materials, how they expect the teacher to respond to their suggestions, and how they represent what it means to use their resource. (Remillard, 2005, p. 240)

A modest extension of the preceding quote would suggest that successful curricula must communicate across the professional groups which develop and use curricula. Designing curricula which can effectively communicate across disciplines must be informed by an understanding of the relationships between members of each of these disciplines and mathematics curricula. The inclusion of mathematicians and mathematics educators in my analysis will help to put in relief certain characteristics that are prominent within the group of teachers; it also will contribute to an understanding of the multiple ways that curricular materials communicate with mathematicians, mathematics educators, and teachers.

This cross-disciplinary perspective will support commentary on the dispositions and beliefs of members of these three professional groups in regard to algebra and to mathematics. In contrast to research on teachers’ beliefs and knowledge, there has been limited research about these cultural aspects for mathematicians and mathematics educators. However, such research could provide insight into the variation and similarities in the perspectives and values of mathematicians, mathematics educators, and teachers. It could inform curriculum development, teacher training, and professional development activities, all of which may involve collaboration or interaction across the three professional groups.

The Making Connections workshops included mathematicians, mathematics educators, and teachers interacting with each other (both within and across professional groups) and interacting with algebra. The algebra content was comprised of 12 problems selected and/or written by the conference organizer, a mathematician. The content included topics which would traditionally be included in Pre-Algebra, Algebra I, and Algebra II classes. One way to think about the Making Connections workshop would be as members of these three professions interacting with each other as framed by the 12 Making Connections problems (Figure 1.1). However, for the purposes of the present study I have instead focused on how
each of these groups interacts with the algebra content (Figure 1.2).

![Diagram](image)

Figure 1.1. Professional Groups - Math Educators (ME), Mathematicians (M), and Teachers (T) - interacting with each other as framed by the Making Connections algebra content.

![Diagram](image)

Figure 1.2. Professional Groups - Math Educators (ME), Mathematicians (M), and Teachers (T) - interacting with Making Connections algebra problems.

To support this perspective, I will, in part, frame my study by Remillard’s (2005) framework for the teacher-curriculum relationship which conceptualizes curriculum use as teacher and curriculum interacting with each other. This framework acknowledges that teachers construct their understanding of curricular materials yet are also influenced by these materials. For this reason, Figure 1.2 represents the relationship between each professional group and the algebra content as a double-headed arrow.

This way of thinking about individuals’ relationships to curriculum provides the language, the backdrop, and the justification for addressing research questions about mathematicians, mathematics educators, and teachers interacting with the Making Connections algebra problems. The present study is also framed by research on the teaching and learning of algebra and on the beliefs and dispositions of members of these three professions. Understanding how these stakeholders in mathematics education interact with algebra is of particular interest because of algebra’s prominence in school curricula, its role as a
gatekeeper to educational opportunities, and because there are competing views about the appropriate content of school algebra (Kieran, 2007).

My data is primarily drawn from a particular task in which the Making Connections participants, within their professional groups, analyzed and categorized the 12 algebra problems. I have data from the 2006 and 2008 workshops, both of which used the same set of problems. During the 2008 workshop, I interviewed and surveyed participants in order to better understand their backgrounds and their views of algebra. These data have allowed me to address the following questions:

1. What knowledge did the participants draw from as they analyzed the Making Connections algebra problems? How did they justify their claims related to what a problem is about? What made a question difficult/easy?

2. What views of algebra and of mathematics informed the participants’ interactions with the content? What mathematical habits did the participants value and notice in student work?

The following chapters document the existing research and theory which have framed my study (Chapter 2), my research methods (Chapter 3), and the results of my research (Chapter 4). The final chapter (Chapter 5) is a discussion of the results and their implications for mathematics education.
Chapter 2

Review of Selected Literature

The research questions of the present study focus on how mathematicians, mathematics educators, and mathematics teachers interacted with the mathematics embodied in the Making Connections algebra problems. There are three overlapping categories of research which make up the theoretical framework:

1. Research on the learning and teaching of school algebra,

2. Research on interactions with curricula, and


The Making Connections algebra problems were compiled and/or authored by the conference organizer Chris Norris. He had a point of view and some purpose in choosing to use these particular problems. The inclusion of research on school algebra in the theoretical framework provides structure in which to situate not just Chris’ algebra problems but also the interpretations of and interactions with these problems by all participants in the workshop. Indeed, a view of school algebra is influencing the Making Connections participants’ interactions with these problems.

Analysis of these interactions with the algebra problems is also framed by research on interactions with curricular materials in general. There are many factors which influence these interactions. These factors include the presentation and content of the Making Connections problem set. They also include less tangible characteristics of the workshop participants who are interpreting these problems. For this reason, the theoretical framework also includes research on these disciplinary groups’ beliefs about and dispositions toward mathematics in general and algebra in particular.
2.1 Algebra

What is algebra? What is school algebra? These two interconnected questions are at the heart of this section. Drouhard (2004) notes “two main ways to address the question of the nature of algebra: one is based on the nature of the problems you can solve with algebra... and the other is based on the specific ‘algebraic’ way to solve the problems” (p. 41). He describes this question as difficult to answer yet important because “the way you teach algebra depends dramatically on what you believe algebra is; therefore this question is worth addressing” (p. 44). Kieran (2007) reports on Lee’s (1997) dissertation which reported on responses to the question “What is Algebra?” from mathematicians, teachers, students, and mathematics education researchers. Among the diversity she discovered there was a prevailing theme to think of algebra as an activity, it is something you do.

In this section I offer some frameworks to help make sense of this diversity of ideas about the nature of algebra. First, I describe prominent ways in which algebra is manifested in classrooms. Next, I present frameworks for categorizing and thinking about the sources of meaning in algebra and about algebraic activity. Lastly, I report on some research on student learning of algebra which is relevant to the present study. Discussion of research about how mathematicians, mathematics educators, and teachers view algebra will, for the most part, be reserved for Section 2.3.

2.1.1 Algebra in The Schools

Kieran (2007) describes the content of school algebra as ranging between reform-oriented and traditional. Reform-minded algebra often is characterized by an emphasis on functions, real-world problems, and a value on multiplicity of techniques (such as the use of technology). Included in this the view of algebra are the multiple ways of representing functions including graphical, tabular, and symbolic representations. A traditional view conceptualizes algebra as generalized arithmetic and often focuses on symbolic manipulations, recognition of forms, simplification of expressions, solving equations, and fac-
toring polynomials. The content of a traditional algebra classroom is generally within a polynomial-equation framework whereas the content of reformed classrooms is generally within a functions framework. The former puts an emphasis on letter-symbolic aspects of algebra, the latter on a combination of mathematical representations. Within the context of this manuscript, my use of the terms “traditional” and “reformed” will refer to the type of algebraic content included rather than the pedagogical practices which have, at times, been associated with these labels.

Chazan (2000) connected the issue of student engagement to the nature of the content of algebra classes. He described his experience teaching algebra in two different secondary classroom settings, the second of which differed from the first in that it was comprised mostly of non-college-bound students. Working in the second setting, he found that a functions-based approach to algebra provided explanatory and exploratory opportunities and increased student engagement.

A functions-based approach helped make it possible to explain to students the characteristics of solutions to traditional Algebra One tasks, thus enabling students to explore such problems before learning an algorithm, and it helped [my co-teacher and I] engage our students in the task of generating relevance for school algebra. (p. 97)

Reformed algebra is often connected with the use of computing technology such as graphing calculators (Kieran & Yerushalmy, 2004) and to “real world” situations and modeling (e.g., Zbiek & Conner, 2006). Within a traditional framework, however, functions are not the organizing idea of an algebra class nor are the representations, associated tools, or applications of functions. Rather, the traditional version of algebra, as Pimm (1995) described it, “is about form and about transformation” (p. 88).

This is not to say that functions and graphs, for example, never play roles in a traditional classroom. Nor does it imply that functions-oriented classrooms never address structure or the objects valued in traditional classroom. Kaput (2000b) described the idea of function as “an extremely powerful organizer of mathematical activity across topics and grade levels”
(p. 4) but noted that other “forms of algebra” are also powerful. He noted four other “forms” of algebraic reasoning: generalizing, manipulating symbols, studying structure and abstraction, and modeling. However, Kieran (2007) warned that “hybrid” curricula which reflect both traditional and reform-oriented ideas may lead to confusion amongst students. For example, Chazan and Yerushalmy (2003) note that the distinction between equations and functions may become a source of confusion.

Thus, there is some ambiguity and diversity in what one means by “school algebra”. I now present a brief look at how the National Council of Teachers of Mathematics (NCTM) and the American Diploma Project (ADP), two prominent forces in determining algebra curricula, envision school algebra. Both endorse emphases on functions and multiple representations in the algebra classroom.

NCTM The NCTM Standards (NCTM, 2000) include algebra as a strand that extends from prekindergarten to 12th grade. The goal of the algebra strand is to enable students to:

- understand patterns, relations, and functions;
- represent and analyze mathematical situations and structures using algebraic symbols;
- use mathematical models to represent and understand quantitative relationships;
- analyze change in various contexts (p. 37).

The first and third of these goals are clearly more aligned with an emphasis on functions and multiple representations, ideas which Kieran (2007) labeled as characteristic of reformed algebra. The second and fourth goals, perhaps less conspicuously, are also aligned with these ideas as well. Throughout the standards, the NCTM endorses achieving these goals through exploring graphical representations. Ideas associated with traditional mathematics such as structure and symbolic manipulation are certainly included in the NCTM Standards as well, however there is a consistent emphasis on functions and multiple representations.
The American Diploma Project network includes 34 states and almost 85% of all U.S. public school students. ADP states have agreed to align their standards with mathematics benchmarks established by the ADP. These benchmarks “describe the specific content and skills that graduates must have mastered by the time they leave high school if they expect to succeed in postsecondary education or in high-growth jobs” (Achieve, 2009). The algebra benchmarks are:

- Perform basic operations on algebraic expressions fluently and accurately
- Understand functions, their representations and their properties
- Apply basic algebraic operations to solve equations and inequalities
- Graph a variety of equations and inequalities in two variables, demonstrate understanding of the relationships between the algebraic properties of an equation and the geometric properties of its graph, and interpret a graph
- Solve problems by converting the verbal information given into an appropriate mathematical model involving equations or systems of equations; apply appropriate mathematical techniques to analyze these mathematical models; and interpret the solution obtained in written form using appropriate units of measurement

These ADP benchmarks emphasize the ideas of functions (and their representations), graphing, and mathematical modeling. They call for reform-oriented, function-oriented version of school algebra.

2.1.2 Sources of Meaning in Algebra

Kieran connects the diversity in the foci of school algebra to the diverse sources of meaning in algebra.

The diversity of perspectives underlying the content of school algebra suggests that the question of where algebraic meaning comes from is intimately related
to the nature of the algebra courses experienced by students. Courses with a traditional focus on literal expressions, equations, equivalent forms, properties and structure will offer quite different meaning-building experiences from those oriented toward functions and the interplay of tabular, graphical, and symbolic representations. (Kieran, 2007, p. 712)

She identifies four main sources of meaning in algebra. The first source of meaning comes from the algebraic structure itself. The focus is on structure and on interpretation of the letters and symbols. Linchevski and Livneh (1999) describe structure sense as the ability “to use equivalent structures of an expression flexibly and creatively” (p. 191). They endorse algebra instruction that “promotes the decomposition and recomposition of expressions” and that helps students make sense of “the mental gymnastics needed in manipulating expressions” (p. 191) in order to promote structure sense. They found that obstacles to structure sense in algebraic contexts have analogs in purely numerical contexts. Livneh and Linchevski (2007) described teaching arithmetic for algebraic purposes as a potential means through which to encourage algebraic structure sense and tested this with at-risk students. For example, do problems like “Is 75 - 25 + 25 equal or not equal to 75 - 50?” help prepare students for problems like “Is 16 - 4x + 3x equal or not equal to 16 - 7x?”

They found that teaching arithmetic for algebraic purposes was successful “to some extent” with at-risk students.

Hoch and Dreyfus (2004) list some of the abilities which comprise structure sense.

Structure sense, as it applies to high school algebra, can be described as a collection of abilities. These abilities include the ability to: see an algebraic expression or sentence as an entity, recognise an algebraic expression or sentence as a previously met structure, divide an entity into sub-structures, recognise mutual connections between structures, recognise which manipulations it is possible to perform, and recognise which manipulations it is useful to perform. (p. 50-51)

They cite as an example a student’s work with the following equation:

$$1 - \frac{1}{n + 2} - \left(1 - \frac{1}{n + 2}\right) = \frac{1}{132}.$$
The student could have had the structure sense to recognize that \((1 - \frac{1}{n+2})\) is being subtracted from itself so the left hand side is equivalent to zero and therefore the equation has no solution. Instead, the student began by multiplying both sides by \(132(n + 2)\) to clear the denominators, following a procedure instead of using structure sense.

Others have emphasized *symbol sense* which often is afforded a similarly amorphous definition and considerable overlap with the concept of structure sense. Arcavi (1994) offered a list of skills and examples to describe symbol sense rather than offering a definition. For example, a student would exhibit symbol sense if she solved the equation \(3x + 5 = 4x\) non-algorithmically by realizing that, in order to obtain four \(x\)’s on the right from three \(x\)’s on the left, another \(x\) would need to be added to \(3x\) thus \(x\) must be 5. This example would also be considered a demonstration of structure sense. Bergsten (1999) defines symbol sense as “an appreciation for the power of symbolic thinking, an understanding of when and why to apply it, and a feel for mathematical structure” (p. 123).

The second source of meaning in algebra comes from other mathematical representations. That is, the meaning which students build is very much dependent on the orientation of the choices of algebraic representations (symbolic, graphical, numeric) and the linkages between these. An emphasis on meaning-building from translations between representations is associated with a reform-oriented curriculum and is considered to be crucial by many (e.g., Kaput, 2000a). In the context of this source of meaning, some have focused on students’ *representational fluency* (e.g., Heid & Blume, 2008; Nathan & Kim, 2007) which includes “the ability to construct various external representations and the ability to interpret features of one representation in the context of another representation of the same mathematical entity or in the context of the real-world situation being represented” (Heid & Blume, 2008, p. 68).

The final two sources of meaning come from sources external to the symbols and representations of algebra. The third source of meaning making is from the problem context; this includes the “real” aspects of “real-world” and modeling problems. In other words, meaning-building may take place in problem-solving activities relevant to real-world con-
texts and word problems.

Lastly, meaning may be built from sources external to the mathematics and the problem context. This includes metaphors, lived experience, gestures, bodily movements, and artifacts that are interwoven into a mathematical activity. This is a broad category which includes activities such as meaning-building about functions and their representations through student use of motion sensors for graphing calculators to graph their motion real-time (as described by Arzarello, Pezzi, & Robutti, 2003).

### 2.1.3 Algebraic Activity

Kieran (2007) also offers a framework for algebraic activity. The GTG model is a conceptualization of the activities of school algebra. It categorizes the activities of school algebra into three types: Generational, Transformational, and Global/meta-level.

Generational activities include forming expressions and equations and work with variables, unknowns, equality, parameters. Broadly, it is the use of algebra as a language to express meaning. For example, generational activities could include modeling problem situations or describing patterns algebraically. Transformational activities, also known as rule-based activities) are rule-based and include operations deriving from the field axioms, factoring, expanding, and changing expressions while maintaining equivalence. These activities include more than just skill-based work and includes meaning building around algebraic objects and rules. Generational activities draw upon multiple representations for developing meaning of algebraic symbols and objects. With transformational activities, there is more of a letter-symbolic focus.

Global/meta-level activities are involve general mathematical processes and habits in which algebra is used as a tool. They often provide the motivation and context for using and learning algebra and include activities such as problem solving, working with patterns, justifying, proving, generalizing, and modeling. These are mathematical activities, like problem solving, that could be emphasized outside the context of algebra.
It should be noted that algebraic activities can be any combination of generational, transformational, and global/meta-level simultaneously. Meaning-building happens in all of these categories. Furthermore, none of these types of activities is exclusively identified with any particular emphasis in algebra (reformed or traditional). However, the nature of the meaning which is built through these three types of algebraic activities is dependent upon the version of algebra that is emphasized in the classroom.

2.1.4 Research on the Learning of Algebra

The previous sections documented frameworks for four main sources of algebraic meaning and for three types of algebraic activity. Kieran (2007) surveyed studies which documented student difficulty with every possible combination of meaning-source and type of algebraic activity. These studies dealt with the essential objects of algebra including variables, expressions, equations, functions, and graphs. In this section I will review some research literature about student learning of algebra that is relevant to the Making Connections problem set or to themes that emerged in the present research.

One relevant focus of study is the transition from arithmetic to algebra. Herscovics and Linchevski (1994) describe "a cognitive gap between arithmetic and algebra, a cognitive gap that can be characterized as the students’ inability to operate spontaneously with or on the unknown" (p. 59). Linchevski and Livneh (1999) found that student difficulties with structure sense in algebra often mirror difficulties they have in purely numerical settings.

Others have found that student difficulty with the objects and symbols of arithmetic may carry over to algebra. For example, Knuth (2006) described student misconceptions of the equal sign; many students demonstrate an operational rather than relational understanding. Vlassis (2004) noted student difficulties with the minus sign. As with the equal sign there are multiple ways in which the minus sign can be interpreted, for example it can be viewed as a property of a number (negativity) or as an operation (subtraction) or it may have a different sort of operational meaning (take the opposite of). She describes the minus sign
as a conceptual obstacle for many beginning algebra students.

Difficulties have also been documented which lie completely within the domain of algebra. In a multi-national study which included 2nd year university students in the United States, Vaiyavutjamai, Ellerton, and Clements (2005) reported on difficulties understanding the key theoretical concepts that underly solving quadratic equations. For example, in the equation \((x - 3)(x - 5) = 0\), eleven out of the 18 students who correctly solved this equation and were subsequently interviewed thought that the \(x\) in the first set of parentheses had a different value than the \(x\) in the second set.

Some research has drawn attention to students’ lack of knowledge of the underlying algebraic properties which justify transformational activity (e.g., Demby, 1997). Many of these studies provide ample examples of students applying rules incorrectly while others have focused instead on students’ engagement with the visual form of the algebraic expressions. Kirshner and Awtry (2004) present evidence that “students respond spontaneously to the visual patterns of the notational display apart from engagement with the declarative content of the rules” (p. 225). For example, it is true that \((a + b)c = ac + bc\), however it is not generally true that \((a + b)c = a^c + b^c\). Kirshner and Awtry would attribute this to a “misperception of the forms” rather than “misunderstanding of the meaning” of the rules (p. 229, emphasis in original). They would characterize the rule \((a + b)c = ac + bc\) as having visual salience since both sides of the equation appear to be naturally related. In their study, 114 seventh grade students were, for the first time, exposed to rules which were both visually salient and not visually salient and given pre- and post-tests. The researchers found that the visually salient rules were retained at a higher rate but students were more likely to overgeneralize them by applying them where inappropriate.

Hoch and Dreyfus (2004) also examined student recognition of form and structure. In their study of 11th graders, they found that few used structure sense to factor quadratic expressions like \((2x + 3)^2 - 12(2x + 3) + 36\) (22 out of 88 students used structure sense with this expression). They noted that students in “advanced” classes were more likely to use structure sense and that those who used structure sense (rather than just manipulation
skills) were more likely to get the correct answer.

Other research on the learning of algebra has focused on functions and graphs. Lobato et al. (2003) examined student understanding of graphical ideas. They describe how some students understand slope as a difference rather than as a ratio and connected this partly to the language, specifically the phrase “goes up by”, used in the classroom. Yerushalmy (2000) conducted a longitudinal study of two students learning algebra over the course of three years using a functional approach. He noted that a focus on “bridging between representations” may not be sufficient for helping students appreciate algebraic symbols (p. 145).

Nathan and Koedinger (2000b) investigated student fluency with verbal representations. In a study of 76 high-school students who had completed a year of algebra and in a subsequent follow-up study with 171 students, they found that students had more difficulty solving symbolic-equation problems than verbally presented problems (more on this study in Section 2.3). They cite the variety of entry points and solution approaches as possible factors for this. In a 2004 follow-up study (Koedinger & Nathan, 2004) they liken learning formal mathematical representations to learning a foreign language:

Students acquire some explicit knowledge of the grammar of algebra, such as what a term or a factor is. However, it is likely that much of the knowledge of parsing algebraic expressions is perceptual learning of “chunks” that implicitly characterize the syntactic structure of expressions and equations. (p. 159, emphasis in original)

Their use of the word “chunks” is borrowed from Servan-Schreiber and Anderson’s (1990) work on “a theory of immediate perception and automatic learning” (p. 607), an unintentional learning phenomena, in which strings of symbols are encoded into hierarchical and grammatical chunks by the learner.

Knuth (2000) examined 178 high school students’ understanding of the connections between graphical and symbolic representations. He found an overwhelming reliance on symbolic representations. Students were proficient with the connections between represen-
tations on routine problems but they may not have developed flexible and robust connections in general. Students showed particular difficulty with a particular type of problem which presented a graph and asked students to comment on the related equation. Kieran (2007) has suggested that older students (high school and college) may exhibit a symbolic preference whereas younger students have difficulties with symbolic approaches.

2.2 Curriculum

The present study is, in part, framed by Remillard’s (2005) description of curriculum use as a participatory relationship between teacher and curriculum. Her framework acknowledges that teachers construct their understanding of curricular materials yet are also influenced by these materials. Figure 2.1 represents this relationship and many of the factors which influence both the teacher-curriculum relationship and it’s manifestation in planned and enacted curricula.

Of particular interest are the characteristics of curriculum which affect this participation. The shaded outer ring of the right-hand circle represents the “subjective scheme”, the subjective filter through which the teacher (and society, in general) encounter curricular materials. In the curriculum circle, Remillard (2005) lists aspects of curriculum that go beyond examination of mathematical content and pedagogy. She includes more “subtle and often unintended” components which are also influential in the teacher-curriculum relationship (p. 237).

Among these influential components are the voice and look of a curriculum which provide a frame for examining the author-reader relationship constructed by the text. Herbel-Eisenmann (2007) operationalized the notion of voice by looking at linguistic forms, representation of agency, and the “cohesiveness” of the text. She looked at the use of imperatives, personal pronouns, and modality1. For example, an imperative or command can

---

1 Modality here refers to the likelihood or authority attached to an utterance. This draws attentions to “hedges” such as “You may think that” or “You probably discovered.”
be inclusive (such as “consider”) and include the reader as a thinker or it can be exclusive (such as “build”) which assumes that meaning is already shared between author and reader.

She analyzed a unit of the Connected Mathematics (CMP) curriculum. Her focus of analysis was on the text as an objectively given structure; that is, her concern was the structure and the discourse of the text rather than teachers’ interactions with it. Among her discoveries was the absence of first-person (singular and plural) pronouns and the use of the word “you” 263 times. This sort of language use could distance the author from the reader. It may also serve to remove human agency as phrases like “the equation shows you” animate an inanimate object (the equation) and depict an absolutist view of mathematics “as a system that can act independent of humans” (Herbel-Eisenmann, 2007, p. 358). In her analysis, even the standards-based CMP curriculum was found to be susceptible to “the hegemony of the traditional forms of discourse” (p. 361).
Remillard’s (2005) framework is grounded in the view that curriculum use is an act of interpretation. Collopy’s (2003) study of how two teachers interpreted and used scripted conversations in the teacher’s guide for the *Investigations* curriculum is exemplary of research framed by this view of curriculum. One of the teachers who they researched used the scripted conversations as scripts to be read with the students, the other teacher used them to anticipate ideas which may surface in class discussions.

Though Remillard’s (2005) framework addresses *teachers* interacting with curricula, in the present study I extend certain essential aspects of this framework to mathematicians and mathematics educators. I use as a framing concept the idea that an individual *interacts with* curriculum. Furthermore, I acknowledge that these interactions are influenced by aspects of the curriculum and characteristics of the individual regardless of professional affiliation.

### 2.2.1 Alternative Stances on Curriculum Use

There are, of course, alternatives to Remillard’s view of “curriculum use as interpretation”. Stein et al. (2007) note two other prominent stances on curriculum use amongst researchers. The first of these is “curriculum use as following or subverting the text”. Studies within this framework investigate teachers’ fidelity to the intended curriculum. For example, Manouchehri and Goodman (1998), in their study of 66 middle school mathematics teachers over 2 years, found variation in how teachers implemented standards-based curricular materials. Teachers differed in the time they spent on materials, in their expectations of students, and in the extent of their efforts to build classroom cultures which supported use of the curriculum. Within the framework of “curriculum use as interpretation”, the notion of fidelity is not particularly useful as a research frame:

… fidelity as a descriptor of curriculum use may be a misleading construct. While examining how teachers understand and use particular features of curriculum materials is certainly valuable, the evidence suggests that a written curriculum cannot fully capture or represent teaching… teachers play a fundamental role in reading and interpreting the offerings in curricular resources
The second alternative stance is “curriculum use as participating with”; that is, the teacher collaborates with the curricular materials. This category of research does not represent a major philosophical shift from the “curriculum use as interpretation” perspective; rather, it is characterized by a focus on the teacher-text relationship. Studies which adopt this stance emphasize and illustrate the complexity of the transformation from written to intended curriculum. An example is Remillard’s (2000) study of two fourth-grade teachers’ learning during the first year in which they used a textbook that was a commercial publisher’s response to the NCTM standards. She found that they interacted with the same text in different ways yet each encountered opportunities to learn about the nature of mathematics, the mathematics content, and student learning.

2.2.2 Temporal Phases of Curriculum

Stein et al. (2007) describe the various ways in which curriculum manifests as “unfolding in a series of temporal phases from the printed page (the written curriculum), to the teachers’ plans for instruction (the intended curriculum), to the actual implementation of curricular-based tasks in the classroom (the enacted curriculum)” (p. 321, emphasis in original). They draw attention to the interpretation and interactive processes that happen within and between these phases. This framework fits well with the view of the curriculum process embodied by Figure 2.1 above.

Stein et al (2007) summarize research which seeks to characterize and explain the ways in which curriculum is transformed within and between the phases of curriculum use. The literature attributes a prominent role to the teacher yet, in general, the factors which influence these transformations are portrayed as complex and diverse. The most commonly studied teacher characteristics which influence curriculum use are listed in the “Teacher” circle in Figure 2.1. Heaton (1992) gave an example of how a teacher’s lack of content knowledge led to ineffective enacted curriculum. Manouchehri and Goodman (1998) doc-
umented teachers with limited content knowledge avoiding units they did not understand, failing to answer students’ why questions, and failing to see connections between activities within a unit. These studies provide examples of how teachers’ beliefs influence the intended and enacted curriculum.

2.2.3 Teachers Interacting with and Learning from Curriculum

Ball and Cohen (1996) note that, “despite their central role in the instructional system, . . . curriculum materials have played an uneven role in practice” (p. 6). They present three potential reasons for this. First, curriculum developers do not adequately account for the role of the teacher and her/his professional development needs; they cite “New Math” as one prominent example of this. Next, there is great diversity in the ways in which teachers interact with curriculum. This is often due to lack of curricular guidance; thus the implementation of curriculum is strongly shaped by teachers’ beliefs and interpretations of the materials. Material is actively adapted by teachers to suit the needs of their particular students. Third, there is often a negative perception of textbooks, especially among reform-oriented teachers.

Despite these obstacles to curriculum implementation, Ball and Cohen (1996) observed that “the relationship between textbooks and teachers has rarely been taken up with much care or imagination” (1996, p. 7). They call for closer attention to curricular enactment in the creation of curricular materials. They adopt a view in alignment with Remillard’s that enacted curriculum is constructed in the interaction between teachers, students, and materials in context. They describe five intersecting domains which are relevant to the enactment of curriculum and across which teachers much work:

- Teachers are influenced by their perceptions of their students and of student knowledge and learning.

- Teachers are influenced by their own understanding of the mathematics. This shapes their interpretations of the curriculum, how they interact with students, and how they
focus and frame the content for the students.

- Teachers modify and choose tasks for students and navigate instructional resources in order to design instruction.
- Teachers must work within the intellectual and social environment of the class.
- Teachers are influenced by their perceptions of the community and policy contexts in which they work.

These domains mesh nicely with the view of curriculum embodied by Remillard’s framework in Figure 2.1. Ball and Cohen (1996) call for curriculum materials to “make central the work of enacting curriculum” (p. 7).

Remillard and Bryans (2004) studied the ways in which the reform-oriented *Investigations* curriculum supported teacher learning among eight teachers at an urban elementary school. Over the course of two years, the researchers did several classroom observations and interviews of each of the teachers. The interviews focused on teachers’ views of mathematics, student learning, teaching, teacher learning, and the *Investigations* curriculum. Though there was substantial variation in the teachers’ views of the curriculum, they described 3 broad categories of teachers’ use of the curriculum: intermittent and narrow, adopting and adapting, and thorough piloting. They introduced the construct of *orientation toward curriculum* as a means of describing the teacher-curriculum relationship:

We define this orientation as a set of perspectives and dispositions about mathematics, teaching, learning, and curriculum that together influence how a teacher engages and interacts with a particular set of curriculum materials and consequently the curriculum enacted in the classroom and the subsequent opportunities for student and teacher learning (p. 364).

They noted that those who thoroughly piloted the curriculum were, with one exception, the youngest teachers. They concluded that without support for teachers using reform-oriented, standards-based curricula like *Investigations*, “the impact of these curriculum materials is likely to be unpredictable and varied” (p. 386).
The researchers documented opportunities for the teachers to learn about mathematics, teaching, and student thinking created by their interactions with the curriculum. They identified four categories of opportunities to learn: “(a) expanding one’s repertoire of activities; (b) insights into student thinking; (c) explorations of mathematics; and (d) constructing the teacher’s role in orchestrating student learning” (Remillard & Bryans, 2004, p. 380). The learning opportunities created by each teacher varied and were influenced by her/his orientation toward curriculum (i.e., intermittent and narrow, adopting and adapting, or thorough piloting).

Similar to the construct of orientation toward curriculum though broader in scope, Collopy (2003) describes a teacher’s identity as “the constellation of interconnected beliefs and knowledge about subject matter, teaching, and learning as well as personal self-efficacy and orientation toward work and change” (p. 289). Like Remillard and Bryans (2004), Collopy notes that if a teacher’s beliefs about mathematics or about the role of curricula in the classroom are not consistent with the curriculum then that teacher may not use the curriculum as intended. Her study describes two elementary teachers’ divergent responses to the curricula as a function of their identities.

Lambdin and Preston (1995) documented use of the Connected Mathematics (CMP) curriculum with a focus on changes in classroom and professional culture. Observational, interview, and questionnaire data were collected during the course of a year for 44 teachers (though only 34 provided complete data). They created three “caricatures” to describe patterns and themes they observed amongst these teachers and how the teachers related to the underlying philosophy of CMP. The caricature of the frustrated methodologists was teachers with relatively strong mathematics backgrounds whose established classroom practices were in conflict with the goals of CMP and were resistant to change. The teachers on the grow were teachers with weak mathematics backgrounds but who were open to change and gradually grew in comfort and effectiveness as CMP teachers. The standards bearers had strong backgrounds in pedagogy and mathematics and their established educational routines and philosophies aligned with the CMP curriculum.
Stein et al (2007) note that the role of the curriculum itself has received “minimal attention in the literature” (p. 356). They raise the question of how particular features of the curriculum influence the teacher’s use of it. Some studies have focused on the ways in which different categories of curricula influence teacher use. For example, Chavez (2003) found that teachers using standards-based (vs. conventional) curricula felt less comfortable in adapting the curriculum and in determining their course objectives. Others have focused on more specific curriculum features such as the sequencing of the curriculum (e.g., Tarr, Chavez, Reys, & Reys, 2006) or the features of curricula which encourage teacher education (e.g., Collopy, 2003).

2.3 Professional Groups’ Beliefs about Algebra and Mathematics

There is, without doubt, great diversity within each of the three professional groups involved in Making Connections. However, I am addressing questions about how workshop participants, as representatives of their disciplines, interacted with the Making Connections curricular materials. Therefore I have conducted my analysis with an eye toward studies and theory about the beliefs and dispositions of mathematics teachers, mathematics educators, and mathematicians. I include here frameworks for describing teachers’ beliefs and knowledge about mathematics in general and algebra in particular.

There is less theorizing and research which directly addresses the beliefs of mathematicians and mathematics educators (as compared to the teachers). The structure of this section reflects this disproportion. However, ways of conceptualizing mathematical beliefs and knowledge of teachers can provide useful descriptors for the nature mathematical beliefs regardless of disciplinary affiliation.

2.3.1 Teachers’ Beliefs about Algebra and Mathematics

Thompson (1992) defined a teacher’s conception of the nature of mathematics as “that teacher’s conscious or subconscious beliefs, concepts, meanings, rules, mental images, and
preferences concerning the discipline of mathematics” (p. 132). Descriptions of the nature of mathematics have often drawn upon the distinction between absolutist and a fallibilist views of mathematics (e.g., Lerman, 1990). Ernest (1992) describes conceptions about mathematics in terms of two dichotomies, the prescriptive-descriptive distinction and the process-product distinction. A prescriptive view of mathematics would be one that emphasizes the rules over descriptions of the nature of mathematics. Ernest’s second distinction is between mathematics as a “process of inquiry and coming to know, a continually expanding field of human creation and invention” as opposed to mathematics as a finished product (p. 99).

Doerr (2004) notes that “there has been little research about teachers’ knowledge and practice and its development with respect to the teaching and learning of algebra” (p. 285). I will, however, describe some of the existing research in this area which are relevant to the present study. Doerr’s survey of such research on teachers’ algebraic knowledge led to the following statement:

We found no research evidence in the USA that would suggest that teachers see the concept of function as an integrating theme for algebra instruction across the curriculum despite this being envisioned in the curriculum standards of the National Council of Teachers of Mathematics. This is not to say that individual teachers do not see functions as an integrating theme for the algebra curriculum and instruction. (p. 278)

Such a statement about the absence of evidence perhaps leads to more questions than answers.

Nathan and Koedinger (2000b) investigated how 67 secondary school mathematics teachers and 35 mathematics education researchers predicted student difficulties with a set of arithmetic and algebra problems. Most of the researchers and teachers predicted greater difficulty with word-problems and algebra than with symbolic problems and arithmetic thus endorsing a view which Nathan and Koedinger referred to as the Symbolic-Precedence Model; students’ symbolic reasoning strictly precedes verbal problem solving and arith-
metic skills precede algebraic skills. This view is similar to that commonly portrayed in textbooks. However, analysis of student problem-solving strategies on these problems indicated that a *Verbal-Precedence Model* prevailed; that is, “contrary to teachers’ expectations, students experienced greater difficulties when solving symbolic-equation problems than when solving verbally presented problems” (p. 179).

In a follow-up study (Nathan & Koedinger, 2000a), 107 K-12 teachers were asked to rank-order (by difficulty) the mathematics problems from the previous study (Nathan & Koedinger, 2000b) and to complete a Likert-scale assessment about the teachers’ beliefs. Teachers’ views of mathematics, math instruction, and student learning were assessed on each of the following six constructs:

1. Algebraic procedures are the best method for mathematical problem solving.

2. Students can learn and invent effective problem-solving methods different from those which are taught.

3. Arithmetic problems are easier and should be presented before algebra. Word problems are difficult and should appear later in the curriculum.

4. Teachers should encourage invented problem-solving strategies.

5. Correct answers should be emphasized over the students’ reasoning processes (product over process).

6. Alternative or invented solution methods indicate knowledge gaps.

They found that, across grade levels, teachers tended to reflect reform-oriented views of math. Elementary teachers were the most likely to agree with reform-oriented views and high-school teachers were the least likely. Middle school teachers did best at ranking the difficulty of the problems (as measured by comparison to student responses) whereas teachers from other levels displayed a *Symbolic-Precedence Model*. High school teachers valued invented strategies the least. Accordingly, Nathan and Koedinger noted that high
school teachers are more susceptible to a what they called an “expert blind spot” since their greater content-area expertise makes them “personally more distant from the difficulties of their novice students” (Nathan & Koedinger, 2000a, p. 229).

Nathan and Petrosino (2003) tested the “expert blind spot” hypothesis in a study of 48 pre-service secondary teachers with varying levels of expertise. Their results support the hypothesis that the more advanced a participant’s mathematics education, the more likely she/he is “to view symbolic reasoning and mastery of equations as a necessary prerequisite for word equations and story problem solving” (p. 905). This is in contrast with students’ actual performance patterns and indicates that those with more subject-matter knowledge tend to “view student development through a domain-centric lens” (p. 918).

Hadjidemetriou and Williams (2001) also investigated teacher perceptions of task difficulty related to graphing amongst 14 and 15 year old students in the UK. They expressed doubt about whether teachers were aware of student misconceptions of graphing. In instances where teachers mis-estimated the difficulty of a question they cited two main causes: (1) the teachers misunderstood the question itself, and (2) teachers sometimes overestimated the difficulty of a question because they assumed the necessity of advanced knowledge and did not recognize that alternative solution methods existed.

### 2.3.2 Teacher Learning and Knowledge

Shulman (1986) offered a framework for categorizing teacher knowledge. He described three categories of content knowledge: subject matter content knowledge, pedagogical knowledge, and curricular knowledge. Of particular interest is Shulman’s conception of pedagogical content knowledge (PCK). Pedagogical Content Knowledge includes knowledge of student thinking and student misconceptions. He describes it as

... subject matter knowledge *for teaching*. [It’s a] particular form of content knowledge that embodies the aspects of content most germane to its teachability... I include, for the most regularly taught topics in one’s subject area, the most useful forms of representation of those ideas, the most powerful analogies,
illustrations, examples, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others... Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. (p. 9, emphasis in original)

Hill, Ball, and Schilling (2008) “unpacked” Shulman’s conceptualization of content knowledge and offered a finer-grained view of pedagogical content knowledge (PCK) and subject-matter knowledge. They define knowledge of content and students (KCS) as:

...content knowledge intertwined with knowledge of how students think about, know, or learn this particular content. KCS is used in tasks of teaching that involve attending to both the specific content and something particular about learners, for instance, how students typically learn to add fractions and the mistakes or misconceptions that commonly arise during the process. (p. 375)

KCS includes common student errors, students’ understanding of content (interpreting student work), student developmental sequences, and common student computational strategies. It is a primary component of PCK; through the construct of KCS they are separating teacher knowledge of student thinking about mathematical content from other components of PCK. Namely, KCS is distinct from “knowledge of content and teaching” (KCT) which, for example, includes knowledge of how to remedy student errors. KCS is also distinct from any curricular knowledge which may be considered part of PCK.

Hill, Ball, and Schilling (2008) situate these types of teacher knowledge within a larger model of mathematical knowledge for teaching (MKT) (see Figure 2.2). Included in this model is an “unpacking” of subject-matter knowledge as well. It includes knowledge which is used in mathematics but is also common to other professions, this is “common content knowledge” (CCK). Also included is specialized content knowledge (SCK) which “allows teachers to engage in particular teaching tasks, including how to represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems” (p. 378, emphasis in original).
Lastly, subject matter knowledge is partly comprised of knowledge at the mathematical horizon which includes awareness of connections to other content which students will encounter in the future (Ball, 1993; Ball & Bass, 2002).

![Figure 2.2. Hill, Ball, and Schilling’s (2008) domain map for MKT.](image)

The constructs of PCK and KCS are outcomes of the realization that expert content knowledge may not be adequate for teaching effectively (Ball & Bass, 2002). Cochran-Smith and Lytle (1999) also provide a framework for characterizing different types of teacher knowledge; it is perhaps more nuanced than Shulman’s as it more explicitly includes a category for practical knowledge. They distinguish between knowledge-for-practice, knowledge-in-practice, and knowledge-of-practice. They describe knowledge-for-practice as the formal knowledge and theory generated by education researchers for teachers to use in practice. Knowledge-in-practice is the practical knowledge, “what very competent teachers know as it is embedded in practice and in teachers’ reflections on practice” (p. 250). Knowledge-of-practice is “constructed collectively within local and broader communities” (p. 274) as teachers practice inquiry about teaching, learning, and knowledge. It is the knowledge which teachers learn from practice in their own classrooms and schools.
Artique et al. (2001) proposed another derivative of Shulman’s framework specifically for analyzing the teaching of algebra. Their framework is called the Multidimensional Grid for Professional Competence in elementary Algebra (MGPCA). It consists of three interrelated dimensions: epistemological, cognitive, and didactic. The epistemological dimension includes the content and structure of algebra as well as its role and connections to mathematics and the world in general. The cognitive dimension includes knowledge about the learning of algebra, student thinking, and common student misconceptions. The didactic dimension includes knowledge of the curriculum, educational resources, and different instructional methods.

### 2.3.3 Mathematicians and Mathematics Educators

As noted above, there are substantially fewer studies related to the beliefs and knowledge of mathematicians and mathematics educators. However, many of the constructs for categorizing knowledge of teachers may be applied to these lesser-studied disciplinary groups; for example, a mathematician may draw upon pedagogical content knowledge. Mura (1995) did one of the few studies on views about mathematics held by mathematics educators and compared this to data from a similar study about mathematicians (Mura, 1993). She received completed questionnaires from 63 tenured or tenure-track mathematics educators at Canadian universities. The respondents were asked to define mathematics and to name up to ten influential books in the discipline of mathematics. She compared the results to a study of 173 Canadian mathematicians which utilized the same survey and methodology. In neither study did all respondents answer both questions.

Amongst the two studies (Mura, 1995, 1993), Mura identified 14 themes which were used to define mathematics and found diversity within each group in their descriptions of the nature of mathematics. A statistical analysis revealed that the mathematics educators made “proportionally more frequent references to the ideas of mathematics as the study of patterns and mathematics making use of inductive thinking processes” (p. 391). Patterns
were referenced by 37.3% of the mathematics educators, inductive thinking by 17.6% (versus 4.9% and 2.9%, respectively, for mathematicians). She noted that “the ideas expressed by mathematics educators and by mathematicians show more similarities than differences” (p. 395).

Burton (1999b, 1999a, 2001) also reported on the diversity of mathematicians’ views and practices. She conducted life-history interviews with 70 mathematicians (35 female, 35 male) in the British Isles and proposed an epistemological model for describing knowing in mathematics. She included 5 components in this model: person- and cultural/social-relatedness, the aesthetics of mathematical thinking invoked, intuition, styles of thinking, and connectivities. Most of the mathematicians in her study embraced the benefits of joint work (only 4 out of 70 claimed never to work with others) and had an appreciation for the power of mathematics. But, in general, there was a lack of homogeneity amongst the mathematicians. Within the group there was a diverse range of understandings of what mathematics is and of ways of understanding mathematics. There were differing views about the nature of mathematics, specifically in response to the question of whether mathematics is invented or discovered.

Burton (1999b) focused on intuition as something that has not been emphasized in the mathematics education literature but was seen by most of the 70 mathematicians as necessary for developing mathematical knowledge. Burton cites Hersh’s (1998) list of meanings and uses of the word “intuition” which includes intuition as “the opposite of rigor”, as “visual”, as “convincing in the absence of proof”, as “incomplete”, as “based on a physical model or on some special examples”, or as “holistic” as opposed to “detailed or analytic” (pp. 61-62). Burton and Hersh both characterize the notion of “intuition” as vague and as the object of diverse opinions (positive and negative) by mathematicians.

Sierpinska and Kilpatrick (1998) posed the following question: “If a mathematics educator studies mathematics, is it the same object for him or her as it is for a mathematician who studies mathematics?” (p. 5). The research of Burton (1999b, 1999a, 2001) and Mura (1995, 1993) seems to indicate that views of mathematics vary greatly even within
disciplinary groups. Their work indicated only a few differences between these disciplinary
groups.

Bass (2006), writing as a research mathematician who is involved with school mathe-
matics education, expressed his preferences in school mathematics curricula as follows:

In my view, the focus within the New Math [curriculum] on mathematical
structure remains an appropriate theme for school mathematics, and its loss has weakened the curriculum. New Math’s critical failure was to naively im-
plement this via abrupt axiomatic formalism rather than through a process of
organic generalization from intuitive beginnings. (pp. 421-422, emphasis in
original)

Bass also described the value of collaborations between mathematicians and mathematics
educators. For Bass, the multiple disciplinary-based perspectives and sensibilities that re-
sult from such collaborations make “visible important things” that would be missed by a
singular disciplinary perspective (p. 423).

Bass (1997) writes a bit about his view of the dispositions of mathematicians to educa-
tion. He describes a very domain-centric approach:

The disposition of many mathematicians toward the problems of education
well reflects their professional culture, which implicitly demeans the impor-
tance and substance of pedagogy. Mathematical scientists typically address
educational issues exclusively in terms of subject matter content and technical
skills. (p. 20)

For him, content knowledge is separate from the knowledge needed to effectively commu-
nicate mathematics. This attitude is reflected in his work on the mathematical knowledge
for teaching (MKT) model (Hill et al., 2008).

Al Cuoco was trained as a research mathematician and “for the past three decades has
spent most of [his] time thinking about high school mathematics - teaching it, working
with people who teach it or plan to teach it, and writing materials for it” (Cuoco, 2001, p.
168). Cuoco (2001) expressed his view that most teachers do not see connections between
the mathematics they are exposed to as undergraduates and the mathematics they teach.
He claims this is especially true within the field of algebra. Secondary teachers do not see connections between abstract algebra and school algebra; these connections include an emphasis on structure. He sees this as unfortunate: “As a result, high school algebra has evolved into a subject that is almost indistinguishable from the precalculus study of functions” (p. 169). Cuoco also expresses a broader concern about the state of teacher preparation. Teachers need more than “the facts” of mathematics, they need to develop a “mathematical taste” (p. 170) and to become familiar with “the spirit of doing mathematics” (p. 174). That is, they need to be able to determine what to emphasize and to have a sense for important concepts and connections. Thus, Bass and Cuoco share the view that content knowledge alone is not sufficient for good teaching and they both endorse a view of mathematics in the schools with a greater focus on mathematical structure.

### 2.4 Conclusion

The present study is an examination of how teachers, mathematics educators, and mathematicians interacted with a set of algebra problems. I investigate how the statements and decisions made about the nature of these problems reflect the Making Connections participants’ views of algebra and of mathematics. The analysis which follows is framed by existing research and theory about school algebra, interactions with curricular materials, and the beliefs/knowledge of the participating professional groups.
Chapter 3

Methods

The focus of data collection and analysis was the Making Connections participants’ interactions with the algebra content which was present at the Making Connections workshops in 2006 and 2008. I wanted to identify and interrogate patterns and themes which may be characteristic of the three professional groups involved in the workshop. Broadly, my methodology was designed to address the question: How do mathematicians, mathematics educators, and teachers interact with algebra content? Specifically, my focus was on the following questions:

1. What knowledge did the participants draw from as they analyzed the Making Connections algebra problems? How did they justify their claims related to what a problem is about? What made a question difficult/easy?

2. What views of algebra and of mathematics informed the participants’ interactions with the content? What mathematical habits did the participants value and notice in student work?

The Making Connections workshop was not something that I designed in order to research these questions. It was conceived of by Chris Norris, a mathematician, who described his motivation for the workshop as follows:

Because interesting conversations come out of it. I think it’s educational for everybody involved [mathematicians, teachers, and mathematics educators] to see the perspective of the other group in a useful way. It’s educational, you know, it gives them useful insights. I think working with real student work and talking about it with real teachers is very different from reading a textbook and writing your opinion about the way it’s doing things. I think it’s more likely to lead to people thinking, you know, mathematicians thinking about the
mathematics of school in a way that, to keep their feet on the ground so to speak. They have some awareness of what actually goes on in classrooms and how people think. I think teachers might sometimes be inclined to not think about the mathematical issues and rather than yelling at them that they should, exposing them to an experience where they’re working with mathematicians probably has a better effect. (Interview, 9/12/2008)

I chose to research the Making Connections workshops because I had an interest in the ways in which members of different professional groups interacted with mathematics curricular materials\(^1\). Making Connections presented an opportunity to observe members of three professional groups, each a stake-holder in K-12 mathematics education, interacting with each other and with algebra content. The research questions enumerated above are a fine-tuning of my broader interest in these interactions; in their design, I was conscious of the data which would be available through the Making Connections workshops. In this chapter, I describe my methodology for exploring these questions. In the sections which follow I will describe the context of the study and my data collection and analysis processes.

3.1 **Context of the Study**

Making Connections was a 3-day workshop held annually in the southwestern United States from 2006 to 2008. Participants were mathematicians, mathematics educators, and mathematics teachers from geographically diverse areas. These regional groups were each comprised of multiple members of two or three of these professional groups (a complete list of participants will be provided in Table 3.1). The workshops were framed by analyses of a common set of algebra problems and student work on these problems; the student work was produced in the classrooms of the participating teachers each of whom had administered the problems and brought their students’ work to the workshop. The 2006 and the 2008 workshops used the same set of 12 algebra problems (see Appendix A) which

\(^1\)I am using Remillard’s (2005) definition of curriculum or curricular materials as “printed, often published resources designed for use by teachers and students during instruction” (p. 213).
covered topics traditionally found in pre-Algebra, Algebra I, and Algebra II classes. These two workshops are the sources of data for the present study.

The workshop was motivated by a recognition of the crucial roles that mathematicians, mathematics educators, and teachers play in K-12 mathematics education and by the belief that each group has something unique to contribute. The structure of the workshop encouraged mathematics-related discourse, framed by the Making Connections problem set, both within and across professional groups. Both the 2006 and 2008 workshops were structured as follows:

1. The Making Connections problems were discussed and categorized within the professional groups; this activity will be referred to as the “categorization task”. Each group then reported on its findings.

2. Each teacher described her/his classroom and students. Each discussed the implementation of the Making Connections problem set in her/his classroom. The teachers brought their students’ work on the set of problems and presented some work which they found interesting.

3. Regional groups met to discuss development and implementation of professional development and/or curricular material inspired by the set of problems. Each group then reported on its product.

The primary analytical focus of the present study is the first of these activities, the categorization task. Broadly, I examined how the participants interacted with the algebra content as representatives of their professions. The workshop was framed by Chris’ encouragement for participants to act as representatives of their professional groups; this is exemplified by the following introductory remark from 2006:

If everyone could be unabashedly who they are that would be helpful. The mathematicians should be trying to bring to the discussion the mathematician’s sensibility. Don’t correct yourself because you say, “Well, I know that’s probably not the way the educators or teachers would think about it.” You really
are here in some sense representing your profession. The same goes for the teachers, and the educators. Although we are here for collaboration, we are also here to figure out how the intellectual and scholarly input from these three professions mingle. So don’t go too far in the direction of trying to sort of accommodate the intellectual impulse from the other profession. (Chris Norris)

### 3.1.1 The Workshop Participants

Both the 2006 and 2008 workshops were comprised of cross-disciplinary groups from three different geographic regions. The 2006 workshop included groups from the southeast, the mountain west, and from the southwest United States each of which included at least one representative from each of the three professional groups. The 2008 workshop included groups from the west, mid-Atlantic, and southwest United States. There were no mathematics educators present at the 2008 workshop. Table 3.1 details the compositions of the two workshops. Only the organizer, Chris Norris, attended both workshops.

<table>
<thead>
<tr>
<th>Year</th>
<th>Region</th>
<th>Teachers</th>
<th>Mathematicians</th>
<th>Math Educators</th>
</tr>
</thead>
<tbody>
<tr>
<td>2006</td>
<td>Southwest</td>
<td>Tom Luft (M)</td>
<td>Chris Norris</td>
<td>Susan Oster</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Ursa Harper (M)</td>
<td>Craig Miller</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Nick Porter</td>
<td></td>
</tr>
<tr>
<td>2006</td>
<td>Mtn West</td>
<td>Sheila Eastman (H)</td>
<td>Kara Nolte</td>
<td>Kevin Martin</td>
</tr>
<tr>
<td>2006</td>
<td>Southeast</td>
<td>Tammy Vonce (M)</td>
<td>Tracy Coleman</td>
<td>Doug Garrett</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Tanya Iverson</td>
</tr>
<tr>
<td>2008</td>
<td>Southwest</td>
<td>Beth Larson (H)</td>
<td>Chris Norris</td>
<td>Larry Donahue</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Nadya King (H)</td>
<td></td>
<td>Shane Tyson</td>
</tr>
<tr>
<td>2008</td>
<td>Mid-Atlantic</td>
<td>Greg Davis (H)</td>
<td>Ian Snyder</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Keith Young (H)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Kevin Lewis (H)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2008</td>
<td>West</td>
<td>Carrie Dixon (M)</td>
<td>Frasier Grant</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Judith Hardy (M)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Michael Chow (M)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 3.1. Making Connections Participants.** (M - Middle School Teacher, H - High School Teacher)

At the time of the workshops, all of the teachers listed in Table 3.1 were or had previ-
ously been teachers of algebra at the middle or high school level or both. Sheila Eastman (2006) and Keith Young (2008) were the only two who were not practicing teachers at the time; both were employed in administrative/advisory mathematics positions within their school districts. Each of the practicing teachers administered some or all of the Making Connections problems in their classrooms (Sheila administered the problems in the classroom of a cooperating teacher).

The instructions from Chris Norris, the conference organizer, for administering the problems in the classrooms were not prescriptive and allowed the teachers to choose how to use the Making Connections problems. Each teacher implemented the problems differently in her or his classroom. Some teachers used all the problems, others were selective. Some had students work collaboratively, others required individual work on the problems. Some teachers administered the problems in class while others assigned them for homework. Although the decision making processes which underly this diversity are surely researchable and interesting they are out of scope for the present study.

The teachers from the 2008 workshop were interviewed and each completed a survey (these data sources will be discussed in more depth below in Section 3.2). However, this data was not collected from the 2006 participants and less is known of their backgrounds. The group of teachers in 2008 was quite diverse in their teaching experience ranging from recent college graduates (Carrie Dixon and Michael Chow) to Keith Young who had 20 years of algebra teaching-related experience. All worked in public middle or high schools with the exception of Judith Hardy who taught at a charter middle school which was affiliated with a university.

3.1.2 The 12 Algebra Problems

The two Making Connections workshops (2006 and 2008) were each framed by the same set of 12 algebra problems (see Appendix A). The problems encompassed a range of content which includes topics traditionally included in pre-algebra, Algebra I, and Algebra II
classes. The problems were compiled by Chris Norris, the conference organizer, and were
not intended to be a comprehensive set of algebra topics. He explained:

I wouldn’t say [the problem set] was representative [of the entirety of school
algebra], no. Well, okay, I did want to have some questions about expressions
and some questions about equations so in that sense I was trying to cover both
of those. I wanted a range of levels of difficulty from pre-Algebra, the first
questions are really, the first question is just about numbers, numerical expres-
sions. All the way to, you know, the last ones are reasonably challenging.
(Interview, 9/12/2008)

The preceding quote is representative of Norris’ view of expressions and equations
as the central objects in algebra. Likewise, his exclusion of functions and graphs from
his description is consistent with his view of algebra. Norris’ views of algebra will be
discussed in more depth in Section 4.1.

But, as documented in Chapter 2, there are many views of what is or should be the con-
tent of school algebra. Likewise, there are different types of algebraic activity and sources
of meaning that could be emphasized in the classroom. Indeed, the analytical focus of the
present study is the way in which the workshop participants interacted with the Making
Connections problems and the views of algebra which influenced these interactions. A
corollary to this would be an examination of how the interactions with the Making Con-
nections problems influenced the participants’ views of algebra however this is not within
the scope of the present study. For the purpose of establishing context, the following are
features of the problem set which do not explicitly draw upon a particular view of algebra;
this list is essentially comprised of tallies about the appearance of the algebraic objects
described in Chapter 2 and about the structures of the problems.

- Eleven of the 12 problems involve the use of a variable or unknown (all except prob-
  lem number 1).

- Three of the problems (10, 11, 12) involve quadratic expressions, functions, or equa-
tions.
• Each of the 12 problems prompts the student to write some sort of prose, that is, something more than a number, expression, equation, or function.
  
  – “Explain how you could have predicted your answer without doing the calculation.” or some variation (1, 6, 12)
  
  – “Explain why you made the choice you did.” (2, 3, 4)
  
  – “Explain what this tells you.” or some variation (5, 10)
  
  – “Explain how you chose your answer.” (7, 9, 11)
  
  – “Decide if the two methods give the same answer.” (8)

• Six of the problems (1, 2, 3, 4, 6, 12) presented a table in which each row contained an expression or equation. For each row, the student is prompted to chose which one of the mutually exclusive and exhaustive column headings applied. For example, a row in the table for number 12 presents the equation $3(x - 3)(x + 2) = 0$ and the student must choose whether the equation has two, one, or no solutions.

• Three of the problems were multiple choice (7, 9, 11).

• The only explicit mention of a graphing idea came as an incorrect multiple choice answer (B) that mentions “slope” (number 9).

• The only explicit reference to functions was the use of function notation in number 10. The relationship between height ($h$) and time ($t$) was given by $h(t) = -16t^2 + 1024$.

• The equal sign appears in four problems (6, 7, 10, 12).

### 3.1.3 The Categorization Task

The categorization tasks from the 2006 and 2008 workshops comprise a major source of data for the analysis and results reported in Chapter 4. This task involved workshop participants, separated by professional group, analyzing and categorizing the Making Connections
problems. The instructions for the categorization activity, as communicated by Chris Norris to the whole group, emphasized mathematical content. The following are Chris Norris’ instructions for this task from the 2006 workshop.

What I’d like really in broad terms is a report on what is the mathematical content of these problems, what are they about, your take on that coming from your profession. So there are a lot of questions one could think about asking. The focus is the mathematical content. There are three things we are giving you as instructions to organize the work... the first thing we’d like to try to do is organize the problems into groups according to the mathematics that students would need or wrestle with or have to have as they attempt to solve these problems. So what do you think is the sort of mathematical content of the problems and do they fall naturally into different groups?... The second is... what are the concepts, the ideas, and skills that students need to know to solve these problems and the third is are there any problems within these groups you’ve arranged that seem particularly difficult and why do they seem difficult.

(Chris Norris, 2006)

The instructions for the 2008 categorization task were similar. At both workshops, this task was not framed with any more specificity than is described in the preceding quote. For example, the participants were not provided with lists of categories from which to choose. The categorization task and its intended outcomes were open to interpretation beyond Norris’ guidelines. It is possible that some of the variation between groups which is noted in Chapter 4 may be due to this openness. Regardless of the participants’ interpretations, it is worth noting that the categorization task may not have corresponded exactly to the practices that members of any of these professional groups do within the context of their professions. However, the task provided insight into the present research questions.

Tables 3.2 and 3.3 show the finished products of the categorization tasks from 2006 and 2008, respectively. These categories were conceived of and negotiated by the workshop participants; they were not selected from any sort of list of potential categories. It should be noted that not every group in each year spoke about each of the 12 problems. This was in part due to time constraints (90 minutes in 2006 and 60 minutes in 2008). Although
the final categorizations were negotiated and agreed upon within each professional group, there were instances of discord which ranged from an exchange of ideas to one instance of a seemingly heated debate. In Chapter 4, I will draw attention to some of these disagreements which help to address the research foci enumerated above. The final categories by no means capture the breadth or the depth of the discussions related to the Making Connections problems. All of these caveats have been accommodated in the analysis.
3.2 Data Collection

My goal was to investigate the interactions between members of the professional groups and the 12 Making Connections problems. The following data were collected at the two workshops:

- Audio and video recordings from the 2006 and 2008 workshops,
- Student work on the Making Connections problem set,
- Field notes from the 2008 workshop,
- Audio recordings from interviews with the 2008 Making Connections participants, and
- Surveys of the 2008 Making Connections participants.

I was not present at the 2006 workshop; the data collection was administered by workshop participants and a videographer. A total of three audio recorders and one video camera were used. During the workshop, there were up to three simultaneously working groups since the workshop was structured so that, at certain times, participants were separated by professional group or by geographic region. All of the professional, regional, and whole group sessions were audio-recorded. Only the whole group discussions were video recorded in entirety, each of the other sessions was only partially video recorded because of limited video resources.

I was present during the 2008 workshop and had the use of three video cameras and three audio recorders. There was also an integrated microphone system in the room in which all of the whole group sessions and some of the regional and professional sessions took place. All professional, regional, and whole group sessions were both audio and video recorded. All interviews were audio recorded. The tables in Appendix B list all the recordings from the 2008 workshop. My role during the 2008 workshop was as an
observer and as an interviewer. I took field notes throughout the workshop and managed
the audio/video recording process. I did not participate in any of the categorization or
planning tasks.

Student Work At both workshops, teachers brought work which their students had done
on the Making Connections problem set. However, the student work is tangential and not
essential to address the research questions which I’ve posed. For instance, student work
is important insofar as it may prompt workshop participants to reveal details about their
views of algebra. However, in neither workshop was the student work drawn upon heavily
during the categorization task. Especially during the 2008 workshop, this was at least in
part due to the overwhelmingly large quantity of student work which was present (there
were more than 2000 pages of student work in 2008). Additionally, only a small portion of
the students went through the IRB consenting process and the majority of their work was,
therefore, not available as the object of research. For these reasons, student work was not
directly included my analysis.

Interviews The interviews were an opportunity to probe participants’ beliefs about algebra,
their opinions of the Making Connections problem set, and their general impressions of the
workshop. The interviews took place in a private room with just myself and individual
participants. They were conducted on each day of the workshop, either in the morning
before the workshop began, after it ended in the afternoon, or during the lunch break. My
interviews with Shane Tyson and Chris Norris were delayed until after the workshop due
to schedule conflicts.

I conducted what Brenner (2006) described as “open-ended” or “qualitative” inter-
views; the intent of the interviews was to give “an informant the space to express meaning
in his or her own words and to give direction to the interview process” (p. 357). The in-
terviews ranged between 24 minutes and one hour. The interview protocol was organized
around the themes of algebra, curriculum, and interactions between the disciplinary groups.
The protocol was loosely followed; the course of each interview was largely guided by the
informants’ responses. However, each interview began with the grand tour question “What is algebra?”.

The interview protocol is in Appendix C; it is divided into sections with questions relevant to algebra, curriculum, and professional culture/beliefs. In general, teachers were asked more about curriculum use than were the mathematicians. Supplemental questions were asked of Chris Norris during his interview; the intent was to understand his motivations for the workshop and for choosing the Making Connections problems. These supplemental questions may also be found in Appendix C.

Surveys During the 2008 workshop, I administered surveys to the participants in order to get information about their professional and educational backgrounds. There were separate surveys for teachers and mathematicians. The survey questions are reproduced in Appendix D.

Multiple Data Sources My data is comprised of surveys and interviews from 2008, recordings from 2006 and 2008, and observational data/field notes from 2008. This diversity of data sources supports the recognition and exploration of the complexity related to my research questions. This perspective is in accord with that of Coffey and Atkinson (1996) who endorse “a sensitive appreciation of complexity and variety” and reject “vulgar triangulation” (p. 14). In other words, data from multiple sources cannot simply be summed to form an increasingly accurate aggregate representation of the interactions which took place during the Making Connections workshops. It was through my analysis of these data that I attempted to reveal and to construct their complexity and addressed questions of how mathematicians, mathematics educators, and teachers interact with algebra content.

3.3 Data Analysis

I approached the data analysis process as part of a larger cyclical research process. I embraced the following advice of Coffey and Atkinson.
The process of analysis should not be seen as a distinct stage of research; rather, it is a reflexive activity that should inform data collection, writing, further data collection, and so forth. Analysis... should be seen as part of the research design and of the data collection. (Coffey & Atkinson, 1996, p. 6)

In this section, I begin by describing the data set on which the analysis was based; this essentially amounts to an accounting of the relevant transcripts. The section is then structured more or less as a chronological account of my research and analysis processes.

3.3.1 Creating the Data Set

The core of the data set was comprised of full transcripts from the categorization tasks during the 2006 and 2008 workshops. This included five sessions totaling approximately 6½ hours. In addition, during the 2008 workshop I conducted interviews with 12 of the 13 participants; Carrie Dixon (teacher) chose not to participate in an interview. Each of these interviews lasted between 24 and 60 minutes and each was transcribed in its entirety. Table 3.4 displays this information.

I also created partial transcripts of other specific parts of the workshop such as the introductory remarks by Chris Norris and his instructions for the categorization task. I personally transcribed all of these recordings with the exception of the transcript for the 2006 categorization task by mathematicians. That session was transcribed by Rebecca McGraw, my dissertation advisor, as part of a pilot study which will be reported on below in Section 3.3.2.

I considered the transcription process to be a part of the larger analytic process. By transcribing most of the data myself I was able to become acquainted or re-acquainted with the data. As I transcribed, I recorded my analytical ideas and made note of interesting dialog in a Microsoft Word document called “Ideas from Transcribing”. During this early phase of analysis, the dialog which captured my interest was that which corresponded to the work on algebra which was included in my literature review (Chapter 2). I cautiously revisited the ideas contained in this file throughout the analytical process. The caution was
<table>
<thead>
<tr>
<th>2006</th>
<th>2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>Categorization Task</td>
<td>Categorization Task</td>
</tr>
<tr>
<td>- Mathematicians (90min)</td>
<td>- Mathematicians (60min)</td>
</tr>
<tr>
<td>- Teachers (90min)</td>
<td>- Teachers (60min)</td>
</tr>
<tr>
<td>- Math Educators (60min)</td>
<td>Interviews with Teachers</td>
</tr>
<tr>
<td></td>
<td>- Greg Davis (33min)</td>
</tr>
<tr>
<td></td>
<td>- Keith Young (35min)</td>
</tr>
<tr>
<td></td>
<td>- Kevin Lewis (30min)</td>
</tr>
<tr>
<td></td>
<td>- Judith Hardy (35min)</td>
</tr>
<tr>
<td></td>
<td>- Michael Chow (24min)</td>
</tr>
<tr>
<td></td>
<td>- Beth Larson (32min)</td>
</tr>
<tr>
<td></td>
<td>- Nadya King (28min)</td>
</tr>
<tr>
<td>Interviews with Mathematicians</td>
<td></td>
</tr>
<tr>
<td>- Larry Donahue (46min)</td>
<td></td>
</tr>
<tr>
<td>- Chris Norris (60min)</td>
<td></td>
</tr>
<tr>
<td>- Ian Snyder (24min)</td>
<td></td>
</tr>
<tr>
<td>- Frasier Grant (29min)</td>
<td></td>
</tr>
<tr>
<td>- Shane Tyson (45min)</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.4. Full Transcripts from the Making Connections Workshops

necessitated by the process through which the file was created; some of the ideas were recorded early on in the transcription process when I had a less-complete view of the data.

Thus, it is these full and partial transcripts which are the primary data for the present study. The data set was also comprised of completed surveys. I did not, however, directly include student work as part of the data set largely due to incomplete human subjects consent, limited information about the students, and lack of necessity to address my research questions which focused on the professional groups rather than the students.

3.3.2 Pilot Study

I became involved in researching the Making Connections workshop during the Fall of 2007. It was suggested to me by my dissertation advisor, Rebecca McGraw, as a potentially fruitful site of research which matched my interests in interpretation and enactment
of curricular materials and in the dispositions of the professional groups which are often stakeholders in mathematics education. At that time, I had access to the audio and video recordings from the 2006 workshop (see Section 3.2). I familiarized myself with the workshop by watching the videos and taking notes on the contents of these videos. I also began to record ideas for gathering data at and researching the 2008 workshop. At this point, I recognized that Making Connections presented a potentially interesting opportunity for researching the ways in which mathematicians, teachers, and mathematics educators made sense of algebraic content.

Since my interest was in researching the professional groups’ interactions with algebra content, the categorization task quickly became the focus of analysis. I transcribed the categorization tasks in which the teachers and the mathematics educators participated. Dr. McGraw transcribed the mathematician’s categorization session. At this point, my approach to the data was inductive as I tried to describe the categories that emerged. I free coded the data by marking up printed transcripts with notes and codes. I recorded the codes which emerged in a spreadsheet; this initial set of codes is list in Table 3.5.

Within and through this set of codes, I became aware of salient themes related to the participants’ understandings of and about algebra. It raised the following questions:

- Did the conference participants have different views of algebra? Did they emphasize different mathematical objects (equations, expressions, functions, graphs)?

- What did the participants value in the Making Connections problems and in student work?

- How did the workshop participants make sense of the problem set? How did they justify their claims about these problems?

Furthermore, I wanted to link each of these questions to the professional affiliations of the participants. That is, within the professional groups were there distinguishing patterns or dispositions related to these research questions?
<table>
<thead>
<tr>
<th>Code</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>assessment of difficulty</td>
<td>making judgments about the difficulty of a problem</td>
</tr>
<tr>
<td>encapsulation</td>
<td>adding a mathematical label(descriptor) to a problem</td>
</tr>
<tr>
<td>expression of goal</td>
<td>mentioning educative goal for class or student</td>
</tr>
<tr>
<td>intuition vs. procedural knowledge</td>
<td>discussing ideas related to rules, sense-making, etc.</td>
</tr>
<tr>
<td>justification</td>
<td>justifying a claim about a problem</td>
</tr>
<tr>
<td>mathematical representations &amp; objects</td>
<td>mentioning equations, expressions, functions, graphs, or tables</td>
</tr>
<tr>
<td>problem novelty</td>
<td>discussing how novel a problem is</td>
</tr>
<tr>
<td>skills</td>
<td>discussing skills needed to solve problem</td>
</tr>
<tr>
<td>solution</td>
<td>speaking about how student or self would solve this problem</td>
</tr>
<tr>
<td>student errors</td>
<td>discussing student errors</td>
</tr>
<tr>
<td>student habits</td>
<td>speaking about student habits (positively or negatively)</td>
</tr>
<tr>
<td>student understanding vs. written work</td>
<td>expressing idea that student may understand something but not be able to express it in writing</td>
</tr>
<tr>
<td>student work</td>
<td>speaking about student work</td>
</tr>
<tr>
<td>symbols</td>
<td>focusing on symbol use</td>
</tr>
<tr>
<td>wording</td>
<td>discussing of wording of question (positively or negatively)</td>
</tr>
</tbody>
</table>

**Table 3.5. Codes developed during pilot study**

*Building on the Pilot Study* The questions raised from the pilot study guided my literature review. I included research and theory about school algebra and about the dispositions and beliefs of mathematicians, mathematics educators, and teachers. I also needed some framework through which to examine and describe the act of interpreting and interacting with the problem set. This framework came from research and theory about interactions with curricular materials. These three components of my literature review (algebra, professional cultures, and curricular interactions) informed both the design of my study and the analysis and coding which followed.
3.3.3 Analysis of the 2006 and 2008 Data

My analytical approach to the pilot study was inductive; I looked for emergent themes and categories in the data. The codes which were produced during the pilot study were useful insofar as they provided direction for conducting a literature review and designing data collection for 2008. After completing data collection during the 2008 workshop, I coded (and re-coded for the 2006 workshop) the entire set of data which is represented by the transcripts listed above in Table 3.4. These data were from both the 2006 and 2008 workshops.

My approach to this round of coding was more deductive than was my approach in the pilot study. The theoretical constructs discussed in Chapter 2 and the research questions enumerated at the beginning of this chapter largely framed my interactions with the data set. For example, I was conscious of the differing emphases in school algebra which Kieran (2007) described. That is, my codes reflected the question: Did the Making Connections participants emphasize functions or generalized arithmetic? Likewise, Kieran’s categorization of algebraic activities prompted me to focus on the global/meta level themes that Making Connections participants discussed. However, throughout the process I was also open to emergent themes and I was always mindful of the analytic work I had done in the pilot study.

I used three main tools for data management and for generating concepts from and with the data. Each tool had limitations but they were each useful for the tasks of data reduction and analysis. These tools were (1) Transana (Wisconsin Center for Education Research, 2006), (2) manual coding/memoing, and (3) TeXnicCenter (TeXnicCenter.org, 2008). The overall process of coding and breaking the textual data into useful parts (i.e., unitizing) was iterative and cyclic. What follows is a chronological account of this process and a description of the tools which I used.

I began with a very broad start list of codes which are detailed in Table 3.6. These categories had considerable overlap and were applied to both the interviews and to the cat-
egorization tasks, though some codes (e.g., Justification of Categories) were less applicable to the interviews. In addition to these analytic codes, I used some housekeeping codes for organizational purposes; these included the code “Categorization” for noting when a category was suggested and the set of codes “Problem 1, 2, . . . , 12” to mark which problems were being discussed.

<table>
<thead>
<tr>
<th>Code</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algebra</td>
<td>Discussing one or many of the objects of algebra (equations, expressions, etc.)</td>
</tr>
<tr>
<td>Justification of Categories</td>
<td>Providing a reason for categorizing a particular problem in a certain way</td>
</tr>
<tr>
<td>Students</td>
<td>Discussing student(s) - real or imagined</td>
</tr>
<tr>
<td>Mathematical Habits</td>
<td>Discussing mathematical habits (positively or negatively)</td>
</tr>
<tr>
<td>Meta Themes</td>
<td>Talking about mathematics</td>
</tr>
</tbody>
</table>

Table 3.6. Start codes for 2008 analysis.

For this coding, I began by using the qualitative analysis software Transana. This software allows codes, or “keywords”, to be assigned to dialog of any length, allows multiple codes to be assigned to a single segment of transcript, and facilitates the generation of reports sorted by individual codes. Transana also supports the grouping of keywords and, in my subsequent iterations of coding, my start codes transformed into a related set of “keyword groups” containing finer-grained keywords/codes; these are listed in Table 3.7.

But there were limitations to Transana. I did not do my transcription in Transana since it was incompatible with my transcription foot-pedal; instead I used the software Express Scribe (Software, 2008). Because of this I needed to manually insert Transana-style time codes into my transcripts in order to precisely associate notes or memos with particular “clips” or segments of the transcripts. This task proved to be quite tedious. Furthermore, though Transana had support for generating some reports, they were not customizable or easily editable. I worked around this by making notes on printed transcripts and by using TexNicCenter as an analytic tool. These methods together provided a way to organize
Keyword Group | Keywords/Codes
--- | ---
Ideas about Algebra | Defining Algebra, Equation, Expression, Function, Generalized Arithmetic, Graph, Multiple Representations
Justification of Categories | “How I would solve it”, “How a student would/should solve it”, Appeal to Mathematical Properties/Rules
Student Activity | Student Errors, Referencing a student (real or imagined)
Mathematical Habits | Intuition, Predicting, Procedure, Flash of Insight, Rigor/Precision, Translating from English to Math
Meta Themes | Using the phrase “Doing the Math”, Discussing Problem Context, Discussing Educational Goals, Focus on Structure
The Problem Set | The Wording of Problems, Question Difficulty

Table 3.7. Keyword Groups. Keywords are separated by commas.

exemplar clips and interesting units of data and to build a robust set of notes around them. My utilization of Transana evolved into its use as a way to manage analytic codes.

TexNicCenter is an environment for creating documents in \LaTeX, a typesetting package that separates content from layout. The TexNicCenter user types the content of a document while explicitly specifying the hierarchical structure (e.g., \texttt{\{section[Analysis]\}” marks the beginning of a section called “Analysis”) and the formatting (e.g., \texttt{\{textbf hello\}” will put the word “hello” in bold). In the header of the document the user specifies the document’s style (e.g., APA format). The document, filled with commands dictating style and format, is saved with a “.tex” extension and then must be compiled as if it were a computer program; the output is a presentable “.pdf” file which embodies all the specified formatting.

This piece of software was not intended as a tool for qualitative analysis but it has many advantages over the way in which I was able to use Transana. Advantages of TexNicCenter included:

- Different sections of a document can be stored in different files which allows for superior organization.
TexNicCenter will display the hierarchical structure (Chapters, Sections, Subsections) of a document so that it is easily navigable just by clicking section headings.

LaTeX supports the addition of comments to text which will be displayed in the “.tex” file but not in the “.pdf” file.

These features accommodated what was lacking in Transana. I created section headings in TexNicCenter which matched my analytic categories and pasted exemplar segments of dialog, related information, and my notes into the appropriate sections. There were, of course, instances where a single segment was pasted into multiple sections.

I see several advantages to using TexNicCenter as an analytical tool. First, it is not subject to the limitations of Transana (as I used it) which were enumerated above. Second, the ability to easily navigate the document’s structure and add hidden comments allows for a nice integration of analysis and writing. Finally, by “commenting out” (i.e., hidden from the final compiled version but still present in the “.tex” file) selected sections, there is a record of my thought process and I will always have access to that extra information even though it does not appear in the final document.

Through multiple iterations, I added comments to this document, renamed and moved section headings, and was able to recognize which ideas were justifiable as “results” and which were perhaps topics for future studies. I created a file called “analysis.tex” which organized analytically important units of data within an evolving analytical structure; a partial screen shot of this file as viewed in TexNicCenter is in Appendix E. Through this process, analysis became merged with the writing of the results section. Segments and notes were copied over to the “.tex” file which became my Results chapter. Segments which were not of use were “commented out”. Some shorthand notes were converted into prose while others were hidden. The structure and content of “Chapter 4: Results and Discussion” is the outcome of this process.

Throughout this iterative process I searched for disconfirming and/or complicating evidence. One of my goals while I was engaging in this activity of data reduction was to
simultaneously engage in what Coffey and Atkinson (1996) referred to as “data complication”. I attempted to “expand, transform, and reconceptualize” the data. The main way I accomplished this was built into the cyclical analytic process which I engaged in and which was supported by the tools I used. Later in this process, I revisited some of the data which was less of a direct focus of analysis by re-watching and taking notes on videos of whole group discussions about the Making Connections problems from both workshops.

The analytically relevant categories which evolved from this process allowed me to address the following questions:

**Algebra**

What view of algebra did participants emphasize? What roles do graphs and functions play?

**Justification of Categorizations**

How did participants justify their claims about the Making Connections problems? To what extent did participants draw upon student activity or mathematical properties?

**The Problems**

What made a problem difficult? What role did the wording of the problem play?

**Meta Themes**

What aspects about mathematics and algebra did the participants value? To what extent was recognizing structure valued?

The treatment of these questions is the content of “Chapter 4: Results” and the outcome of the methods which have been described in this chapter.

### 3.4 Trustworthiness

The trustworthiness of this study relies upon the breadth and depth of the methods of inquiry and of the analytic process. I have collected multiple data sources and subjected them to methodical, iterative analyses while simultaneously documenting my reflections
and decisions. These characteristics are among the general guidelines to valid research which Coffey and Atkinson (1996) have outlined. Throughout the process I have sought and received expert feedback and advice. I received feedback from my dissertation committee when I presented my dissertation proposal which included reports on the pilot study and on the methodology for data collection during the 2008 workshop. Rebecca McGraw, my dissertation adviser provided me with feedback throughout the analysis and writing process.

Despite the geographic diversity of the Making Connections participants, I was able to conduct member checks with one member of each of the professional groups. This group included Susan Oster (mathematics educator, 2006), Beth Larson (teacher, 2008), and Chris Norris (mathematician and workshop organizer, 2006 and 2008). To each of these participants, I showed portions of a near-final version of “Chapter 4: Results” which include discussions specifically related to their participation in the workshop. Neither Oster nor Larson expressed any concerns with the way they were depicted. Chris Norris expressed accordance with the depiction of his view of algebra and offered further clarification related to particular quotes; I updated this document to reflect his comments. In general, my analytic process was cyclic and extended from the data collection phase through the final writing of results; I have striven to think about the data, through the data, and with the data.
Chapter 4

RESULTS AND DISCUSSION

In this chapter I address the following research questions:

1. What knowledge did the participants draw from as they analyzed the Making Connections algebra problems? How did they justify their claims related to what a problem is about? What made a question difficult/easy?

2. What views of algebra and of mathematics informed the participants’ interactions with the content? What mathematical habits did the participants value and notice in student work?

For each of these questions, I examine differences and similarities across the three professional groups. The results reported herein do not constitute comprehensive answers to these questions. They do, however, offer insight into the mechanisms by which mathematicians, teachers, and mathematics educator may interact with and make sense of algebra content.

This chapter is comprised of two main sections:

1. Interacting with Algebra

2. Views of Algebra

Each of these sections begins with an analysis of how each of the three professional groups interacted with one demonstrative question from the set of Making Connections problems. These analyses are intended to introduce and frame the themes which will be addressed in this chapter. Dividing this chapter into the two primary sections listed above is useful but, in a sense, artificial. A reporting and analysis of the interactions which took place with and around the Making Connection problems cannot neatly be separated from the participants’ views of algebra.
Before the two main sections are presented I present an overview of how the conference organizer, Chris Norris, thought about and chose the Making Connections problems. I also briefly discuss his views of algebra in general and how that is reflected in these problems.

4.1 The Workshop Organizer’s Views on Algebra and on the 12 Problems

Chris Norris, the workshop organizer and a mathematician, compiled the set of 12 Making Connections algebra problems. The problems came from a number of sources but largely from the College Algebra textbook which he is helping to author. Additional input and authorship of some of the pre-Algebra questions were provided by a retired high school algebra teacher. I intend to use Chris’ intentions and interpretations of these problems as a point of comparison in order to highlight certain aspects of the interactions which conference participants had with and around these problems. To be clear, I will not use his viewpoint as the developer of this curricular material as a benchmark for determining “correct” interpretation of the problems; the research questions and theoretical framework do not support such a perspective which embraces ideas of “curriculum fidelity”. However, I do ultimately want to make a statement about the nature and variety of these interpretations. To support this, what follows is a description of Chris’ ideas about algebra and his intentions for the set of problems.

For the most part, Chris’ view of algebra is as generalized arithmetic. This is a departure from a view of algebra characterized by a greater emphasis on functions and on multiple representations such as graphs and tables which Kieran (2007) describes as a prominent alternative viewpoint. The following quote describes Chris’ view of algebra and school algebra.

I’d say algebra is the idea of using symbols to stand for numbers, forming expressions that describe calculations with numbers using symbols... [School algebra] seems to be solving equations and studying expressions is what a lot of school algebra is. (Chris Norris, interview, 9/12/2008)
Indeed, Chris focused on expressions and equations although he acknowledged the alternative viewpoint which emphasizes objects such as functions and graphs. This was evident when he spoke about algebra in general and when he described his motivation for compiling this particular set of algebra problems.

I guess I think there’s a progression of ideas in algebra and it starts with expressions and equations and variables and using symbols. Then as you get more advanced you think about functions and the functions that are defined by expressions, the equations you get out of functions, and so on. There are people who do it the other way around which I think is fine but I was interested in these exercises of getting at the, at what I was saying was the basic level of expressions and equations mostly. (Chris Norris, interview, 9/12/2008)

4.1.1 Diminished Roles for Functions and Graphs

If the tickets for a concert cost $p$ each, the number of people who will attend is $2500 - 80p$. Which of the following best describes the meaning of the 80 in this expression?

A. The price of an individual ticket.
B. The slope of the graph of attendance against ticket price.
C. The price at which no-one will go to the concert.
D. The number of people who will decide not to go if the price is raised by one dollar.

Explain how you chose your answer.

Figure 4.1. Problem Number 9.

Chris’ professed openness to the idea of functions as a part of algebra became more nuanced and complicated later in the same interview as will be expressed in quotes presented below. It is important to note that many of the Making Connections problems could be approached by thinking about functional ideas such as relationships between inputs and outputs (or independent and dependent variables). However, with the exception of number
10, these ideas were packaged as thinking about how some expression was changing as a variable changed. For example, in problem number nine (Figure 4.1) no dependent variable (i.e., the number of people attending) is explicitly defined. The expression \(2500 - 80p\) is embedded within a sentence rather than having been presented as something of the form \(f(p) = 2500 - 80p\) which explicitly calls attention to the functional relationship between ticket cost and attendance. The following interchange about number nine between myself and Chris indicates an intentional decentralizing of functions within this set of problems.

Chris: Yeah, I mean, I guess it was intentional to focus on the expression itself. I mean the fact of the matter is you can pose these questions without ever naming a dependent variable. I mean the dependent variable is kind of extraneous.

Josh: Well it’s implicitly defined, right?

Chris: Yeah, right, yeah, absolutely it’s there in some sense but it, I guess it was a way of writing the question to force people to focus on the structure of the expression itself because there was nothing else to look at. (Interview, 9/12/2008)

Chris also expressed his view that introducing functions at the high school or middle school level is unnecessary and potentially confusing.

The whole emphasis on relations and functions is some residue of the new math or something which has just become a funny little ossified relic in the curriculum… That’s my big point about expressions versus functions. Functions are objects which have no inherent expression associated with them at all. Most of the time we use expressions to define functions. Different expressions can define the same function. I mean unless you keep these two notions separate in your mind you’re going to run into all sorts of trouble… Making a big deal about [functions] is bizarre. It’s just like making a big deal about something before anybody was worried about it. (Chris Norris, interview, 9/12/2008)

In the following bit of dialog which directly followed the preceding quote, Chris makes it very clear that he is skeptical of making functions a central theme in algebra classes.
Josh: But this is often the central theme of algebra classes. It’s the central theme of the college algebra class [is taught at your university].

Chris: Right, I know, it’s nuts.

Josh: You think it’s nuts?

Chris: Yeah, I think it’s nuts, even in college algebra. (Interview, 9/12/2008)

Thus, Chris expressed that including functions in an algebra class is not, in and of itself, problematic but making them a centerpiece could detract from algebraic ideas like expressions, equations, and structure.

Neither did Chris view graphs as having a central place in algebra. Kieran (2007) described functions and graphs as being among the main themes of reform-minded algebra. Chris acknowledged these objects’ popularity in many algebra classrooms but professed that he did not agree with this emphasis. In the following quote, Chris talks about the role of graphs in algebra as being a tool through which to explore the central themes of symbols and structure.

I think that graphs can be useful and I don’t think of them as an essential object in algebra which is probably a point where other people would disagree. I think of them as a tool that you would use in doing algebra. I didn’t have any graphs here [in the set of 12 problems] because I was more interested in what I think of as the central object which is exploring the symbolic representations and exploring the structure of expressions and equations. (Chris Norris, interview, 9/12/2008)

In fact, the only place where a graphing concept is explicitly used in the set of 12 questions is in (incorrect) answer choice B in number 9 (Figure 4.1). The correct answer for this problem is D.
4.1.2 Summary - Chris’ View of Algebra

Chris Norris has a very specific view of what algebra is and what school algebra should be. The following statements describe his perspective:

- Structure, symbol use, expressions, and equations are important components of algebra and school algebra.
- Functions and graphs are not essential parts of algebra.

Furthermore, he mentioned several student habits and practices which he values. Chris wants “kids to be able to read an equation and say ‘What is this equation telling me?’” He wants them “to see structure in expressions and recognize what it might be good for by fooling around with it, by thinking about the values you plug in to it, by seeing, you know, by seeing broad structure” (Interview, 2008). These themes and Chris’ views of algebra will be addressed in more depth throughout this chapter.

4.2 Interacting with Algebra

This section begins with a careful examination of how each of the three professional groups (mathematicians, teachers, and mathematics educators) interacted with and categorized problem number one. The themes which are developed will frame the rest of this section.

4.2.1 Number 1

Teachers In 2006, the teachers decided that number one (Figure 4.2) was about signed numbers and properties of real numbers. There was little discussion about and instant agreement with this categorization. By “properties of real numbers”, the teachers were referring to “properties. . . which would include your order of operations, your like terms, and . . .”, referring to “other things such as . . .” (Interview, 2006).

1The 2006 group of teachers categorized every question as “mathematical reasoning”. Since this omnibus category does not allow for discrimination between problems it will not be addressed until Section 4.3.3.

2A complete listing of all categorizations for both 2006 and 2008 is available in Appendix F.
Is the expression positive, negative, or zero?

<table>
<thead>
<tr>
<th></th>
<th>Positive</th>
<th>Negative</th>
<th>Zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 - 10 + (-8)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$10 - (-8) - 2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$8 - 10 - 2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-8 - 2 - (-10)$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For each expression, explain how you could have predicted your answer without doing the calculation, if possible.

- $2 - 10 + (-8)$
- $10 - (-8) - 2$
- $8 - 10 - 2$
- $-8 - 2 - (-10)$

**Figure 4.2. Problem Number 1.**

<table>
<thead>
<tr>
<th>Teachers</th>
<th>Math Educators</th>
<th>Mathematicians</th>
</tr>
</thead>
<tbody>
<tr>
<td>2006</td>
<td>2006</td>
<td>2006</td>
</tr>
<tr>
<td>Signed Numbers</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Properties of Real #s</td>
<td>Reasoning about quantities</td>
<td>Number &amp; Operations</td>
</tr>
<tr>
<td>Mathematical Reasoning</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2008</td>
<td>2008</td>
<td>2008</td>
</tr>
<tr>
<td>Predicting</td>
<td></td>
<td>Static Expressions</td>
</tr>
</tbody>
</table>

**Table 4.1. Categorization of Number 1**

your distributive property, all of those” (Sheila Eastman, HS³ 2006). There was no discussion of why these categories were chosen, but their focus was more on knowing the properties or rules necessary to answer questions about signed numbers.

This focus differs markedly from what the teachers noticed in 2008 when they described this as a question about “predicting”. It appears as though the two groups of teachers drew upon cues from the different parts of the problem in deciding on the categorization. The 2008 teachers were concerned that, as a question about prediction, it is easier to just do the arithmetic.

³Throughout the rest of this dissertation “MS” will indicate middle school and “HS” will indicate high school.
Yeah, I found the same thing. [My students] were, in their explanations, they were working the problem and kind of explaining the problem until that very last step and just saying their answer as their prediction. (Kevin Lewis, HS)

They also expressed skepticism about the utility of asking a student to make a prediction about a question for which it may be easier to do the computations.

Judith Hardy (MS): . . . I don’t know how you can do it without doing the math.

Greg Davis (HS): Well just to kind of throw this out, I’m putting my headset into the mind of a 15 year old. Why would I want to predict this? Regardless of the size of the numbers why would I predict that if, you know, I have access to a calculator. I know I’m trying to make it into a mathematical question because what is the purpose of predicting?

In the above bit of dialog, Greg Davis relates Judith’s comment to how a student may think about predicting in this context. This focus on student activity was extended to these teachers’ classroom practices as they tried to make sense of prediction in the context of problem number one.

Kevin Lewis (HS): So can we think of kind of maybe the type of problems that we see in our classes that would parallel something like this or would require that kind of thinking [about prediction]? Can you think of kind of an analogous problem or piece of your courses?

Greg Davis (HS): I was thinking about polynomials, when you’re talking about quadratics, like in Algebra II, something that just popped in my head because I just finished it up, is predict for me where would the ball hit the ground, talking about parabolas. How long, how many seconds predictably would it be? Something like that prediction is extremely useful.
Nadya King (HS): And also it helps because a low level student maybe will see why we want our kids to check their work. They don’t want to go through and do it again. If they had originally gone through the problem, made a prediction and then done the work to then check that they match their prediction that’s a way for them to know whether they need to go through it, fully rework the problem or whether they have an answer so a low level student can use a prediction, if used that way, can be helpful.

Kevin Lewis: And maybe also as a kind of test-taking skill, something like this could be used in eliminating, at least predicting these values can be used for eliminating answer choices on a test question or something like that.

Carrie Dixon (MS): Yeah I mean because kids, when we’re doing the testing, are doing this all the time, those things are tricks.

In the preceding dialog, Greg speaks about using quadratic models to describe the flight of a ball. The utility of using such a mathematical model is to predict things such as where the ball will land. This is perhaps a different type of predicting than is communicated by Nadya, Kevin, and Carrie who talk about prediction as a way to get a sense for the would-be outcome of a calculation and as a skill for test-taking. As such, the teachers’ label of “predicting” may have had different meanings even within the group.

The dialog, however, does illustrate a focus on classroom practices and student activity as a framework for approaching the categorization task. They discussed the utility of this sort of problem as a means to develop or reinforce test-taking skills or to highlight the importance of students “checking their work”. There was never an explicit discussion of why this question should be categorized as “predicting” but consensus was reached subsequent to a discussion that focused on the sort of classroom-oriented concerns described above and on the ways which students would approach this problem. Furthermore, this categorization as “predicting” was chosen despite the fact that there was widespread agreement within the group that to predict would be unusual and perhaps superfluous in a problems such as this which requires relatively simple computations.
Mathematics Educators For the mathematics educators in 2006, number one was about “reasoning about quantities”. They discussed the strategies that they would like students to use in order to determine (or predict) if the given expressions were positive, negative, or zero without doing the arithmetic.

Susan Oster: So given kind of the ideal of what we would like students to do, this number, problem number one, then does it become a question about, I don’t know, a sense of the number line, a sense of a big positive and slightly smaller negative ends up being on the positive side of the spectrum? I mean is that the kind of math thinking and content that you’d like to have?

Doug Gar-rett: Right, so you’re imagining sort of actually doing these things on the number line whereas I’m imaging sort of rewriting everything as addition and then sort of putting my positives here and negatives here. Not necessarily doing it in the order that’s there but sort of breaking the expression into to two parts and then sort of weighing each part in some sense. You know? Both of these perspectives I think are -

Susan Oster: ones you want kids to end up having.

Thus, they discussed ways to reason about the quantities in this problem without doing the arithmetic and suggest two different approaches for doing so. For the mathematics educators, this problem was not about arithmetic; rather it was about a type of reasoning which draws upon number sense and operation sense. In negotiating this, they focused on two non-procedural strategies that students could use to approach the task of “predicting”. In fact, three of the five categories which mathematics educators used to describe these problems were labeled as “reasoning about” some mathematical object.

The mathematics educators, as did other the groups, spent some time talking about how typical this and other Making Connections problems were or were not. They spoke of the types of approaches that a student may actually take for these problems. This pedagogical content knowledge which was drawn upon is evident in the following exchange.
Doug Garrett: Well, just deciding whether the thing is positive, negative, or zero; that’s a new spin on it that students might not typically see.

Kevin Martin: It’s interesting because I suspect... that the way students are gonna do it is through computation.

Doug Garrett: Do the calculation and then -

Kevin Martin: and then say that’s what this is as opposed to sitting back looking at it and just going okay, this has to be.

Tanya Iversen: Uh huh. [in agreement]

The mathematics educators were also inspired to talk about the roles and practices of teachers. Doug Garrett, acknowledging that he could be making a “gross generalization”, provoked the following dialog about the novelty of the prompt to “predict” in problem number one.

Doug Garrett: What I’m talking about is that as part of the instruction, whatever it is, developing this sensibility of, well being able to say something about the answer without actually getting the answer. That’s hardly ever emphasized. Most problems are taught with the goal of getting the answer and, you know, once the students have gotten the answer to the problem they’re done in some large sense. Whereas when you’re doing these things if you, you know, part of the discussion is about this meta-question. Could you have predicted something about your answer without actually doing the calculation? That’s the thing I’m claiming is not taught very much.

Susan Oster: Well then it seems to me that there would be quite a bit of potential then for problems like this to be part of, you know, professional development activities.

The mathematics educators discussed both the types of reasoning that they hoped students would do and the classroom activities that they hoped teachers would encourage with problem number one. The way in which they categorized this problem reflected their interest in the student reasoning which they hoped would be inspired by this problem.
Mathematicians  The mathematicians in 2006 decided that number one was about “Number and Operations”; the group from 2008 said it was about “Static Expressions”. “Number and operations” refers to the skills needed to interpret the given mathematical expressions correctly. The category “static expressions” is a statement of the context of the expression insofar as the student is not required to think about a variable or unknown which changes or about the effect of changing a variable or unknown; this is in contrast to their category “dynamic expressions”.

In 2006, some of the mathematicians listed the skills to solve this problem citing: integer arithmetic and order of operations (Tracey Coleman), absolute value (Craig Miller), and positive/negative numbers and magnitudes (Kara Nolte). Other mathematicians in the 2006 group, however, had a different approach to analyzing this problem:

Nick Porter: I guess I sort of thought this was about picturing the integers as a big line of numbers and thinking of adding a positive number as going to the right and subtracting a positive number as going to the left and that sort of thing.

Chris Norris: Yes. And there is certainly that. I also thought that in some sense it was about algebra in the sense that it was about reading expressions and interpreting them correctly.

In the preceding exchange there is more of a focus on global/meta level activities than had been communicated by the mathematicians who had listed skills. Porter presents a way of thinking about the problem rather than a list of skills or rules necessary to solve it. Norris, the author of the problems, links it to the more explicitly algebraic problems in the Making Connections problem set.

Throughout the mathematicians’ discussion of these problems, they, at times, produced lists of the mathematical properties necessary to solve a problem; for example, Kara Nolte described number two as being about “distributive law, order of operations, and all the things we said for the first one.” However, the final categories which the mathematicians produced focused more on global/meta level themes than on lists of skills. The following exchange is exemplary of how this group dealt with both skills and meta-themes during the
categorization task; in this case, Tracy and Craig discuss the need to have mastery of skills (e.g., order of operations) in order to have access to the meta-activity of making predictions:

| Tracy Coleman: | Yeah, I liked that it asked “can you predict”, or “explain how you could have predicted your answer without doing the calculation” and that does fit in with what you were saying about picturing the number line. |
| Craig Miller: | That’s related to what you were saying about order of operations though, because you can’t predict it if you don’t know how to, if you don’t know how you would go about doing it in the first place, and not be able to have an idea. |

It was often Chris Norris who directed the group away from a focus on skills needed to solve problems. As the compiler/author of the problem set and the organizer of the workshop, Norris’ comments may have carried disproportionate authority. Larry Donahue, from the 2008 group, “couldn’t come up with a way of doing [number one] other than calculating it.” He asked Norris directly about the purpose of the problem.

| Larry Donahue: | What are you trying to get at? Are you trying to get at sort of approximation skills? Are you trying to get at sort of recognizing the negative of a negative is, you’re adding, that kind of thing. I can see it being, again I’m envisioning other problems instead of 10, 8 and 2 it’s 16, 10, and 100 or something where the size is so overwhelming it becomes more an approximation problem than worrying about, okay, I’ve got a negative negative eight, what does that mean? That sort of thing. |

Seemingly aware that his response would undermine the activity of negotiating a categorization, Norris redirected Donahue’s question to the group.

| Chris Norris: | What do other people think? What might be the purpose? |
| Frasier Grant: | I feel like that problem would actually be very different if you did have like 1, 10, and 100 where you could, where that is more like estimating with the dominant term. Whereas here it seems like it’s kind of mental math but not exactly estimating, but looking at features. What you’re doing instead of just doing it all out and here it doesn’t save you a whole lot but in some cases it does...
Chris Norris: Ian?

Ian Snyder: There’s some problems where you want students not to just dive right in and maybe this is an example of the kind of problem where you want the students to look at maybe all four questions at one time and see if there’s some uniform approach that would enable them to do each of them without getting, without doing the arithmetic.

Chris Norris: So I mean I can tell you what one of the purposes I have with these sorts of problems, and you can tell me if this one doesn’t, was simply to encourage students to make, have any reflectivity, do any reflection at all on what they’re doing. Students find this sort of question really bizarre because mostly they just want to dive right in.

In this exchange, the 2008 mathematicians spoke about some student habits which they value; this included “looking at features”, looking for a “uniform approach”, having “reflectivity”, and not just “diving in”. Despite this discussion of the mathematical habits which they believe students should embrace, the 2008 mathematicians categorized this question as being about “static expressions”. This categorization emphasized the mathematical object (the expression) and drew a distinction with problems where the student was required to think about the expression changing as a variable changes.

Summary - Number One I have just presented a view of the categorization activity for problem number one. This will be a launching point for the discussion which follows about how the members of these professional groups interacted with the Making Connections algebra content. Just in the analysis of problem number one, it begins to become evident that members of each professional group draw from multiple types of knowledge such as knowledge of students and of mathematics. Likewise, certain student activities were valued within the preceding bits of dialog; for example, the teachers spoke of students checking their work, the mathematics educators expressed interest in student reasoning, and the mathematicians spoke about various activities related to “not diving in”.

There were diverse perspectives expressed within the professional groups both looking across and within workshops (2006 vs. 2008). For example, the way in which the 2008 group of teachers spoke about the word “predict” may indicate a difference in perspective on just that single word/concept. From group to group, there was variation in the amount of time spent discussing number one, from less than a minute (teachers 2006) to more than 10 minutes (teachers 2008). Furthermore, the final versions of the categorizations presented by the groups often did not directly correspond to the discussions which took place about the problems themselves.

But it is certainly no surprise that monolithic professional cultures do not exist or that a conversation about a problem cannot be encapsulated by a single word or phrase. However there are patterns and themes which emerged in the ways that members of these professional groups made sense of and interacted with these problems. These patterns are the focus of much of the rest of the chapter.

4.2.2 The Role of the Student in Framing the Problem Analysis

When a Making Connection participant made a claim or observation about a problem, how did she/he justify this claim (if at all)? The claims which are of primary interest are those which contributed to the categorization of the problems. In this section, I argue that teachers, moreso than the other two professional groups, used ideas about real students or their views of typical students to frame their discussion of the Making Connections problems and to justify their claims about the problems.

As was documented in the discussion of problem number one, all three groups drew from diverse types of knowledge. However, in both 2006 and 2008, the teachers discussed students to a greater extent than did the other two professional groups. During both workshops, the teachers referenced students during their discussions of every problem except for problem number one in 2006 (they spoke about this problem for less than 30 seconds total). The teachers referred to their own students in their discussions of the Making Con-
nections problems; for example, Nadya King (HS 2008), in discussing number four, said “So my kids didn’t know what it meant ‘for increasing values of $a$’”. They also drew upon pedagogical content knowledge to talk about students in general or about their own conceptions of typical students. Tom Luft (MS 2006), for example, said “I also think that kids don’t understand that multiplying by one fifth is dividing by 5 very well” when discussing number five, a problem which was categorized as being about “rational number operations” in 2006.

The mathematicians and mathematics educators also engaged in these sorts of discourse related to students and student work. They, of course, were less familiar with the work which students did in particular teachers’ classrooms; because of this, they referenced these students or classroom experiences less frequently. In general, though, there was a difference in the frequency with which teachers referenced students and drew upon classroom experiences in their discussions.

<table>
<thead>
<tr>
<th></th>
<th>they</th>
<th>them</th>
<th>student(s)</th>
<th>kid(s)</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teachers 2006</td>
<td>474</td>
<td>130</td>
<td>26</td>
<td>129</td>
<td>759</td>
</tr>
<tr>
<td>Teachers 2008</td>
<td>308</td>
<td>75</td>
<td>50</td>
<td>72</td>
<td>505</td>
</tr>
<tr>
<td>usage per hour</td>
<td>312.8</td>
<td>82</td>
<td>30.4</td>
<td>80.4</td>
<td>505.6</td>
</tr>
<tr>
<td>Math Educators 2006</td>
<td>119</td>
<td>79</td>
<td>73</td>
<td>37</td>
<td>308</td>
</tr>
<tr>
<td>usage per hour</td>
<td>79.3</td>
<td>52.7</td>
<td>48.7</td>
<td>24.7</td>
<td>205.3</td>
</tr>
<tr>
<td>Mathematicians 2006</td>
<td>103</td>
<td>51</td>
<td>58</td>
<td>10</td>
<td>222</td>
</tr>
<tr>
<td>Mathematicians 2008</td>
<td>79</td>
<td>46</td>
<td>51</td>
<td>0</td>
<td>176</td>
</tr>
<tr>
<td>usage per hour</td>
<td>72.8</td>
<td>38.8</td>
<td>43.6</td>
<td>4</td>
<td>159.2</td>
</tr>
</tbody>
</table>

Table 4.2. Word counts from the categorization tasks (90 minutes in 2006, 60 minutes in 2008). “Usage per hour” is calculated for each professional group using totals from both workshops.

The teachers’ greater focus on students was particularly notable in pronoun usage across groups; the teachers would talk about what “they”, the students, would or should or did do for a particular problem. Table 4.2 shows the frequency with which members of each professional group used certain words which reference the students. A potential caveat of this word-count pertains to the use of the words “they” and “them”; the word count was not
sufficiently sophisticated to detect the noun which each instance of these words modified however the vast majority of the usages of these words referred to “the students”. Likewise, there were instances where students were referred to but none of the four words included in the word-count were used; for instance, phrases like “mine did” or “a couple did” were not detected. The difference in rates of usage of words which refer to students is quite telling; teachers used these words more than 500 times per hour. In contrast, mathematics educators used these words 205 times per hour and mathematicians used them 159 times per hour.

<table>
<thead>
<tr>
<th>You are simplifying $7 - 2(3 - 8x)$. Which of the expressions are possible results after the first step?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5(3 - 8x)$</td>
</tr>
<tr>
<td>$7 - 2(-5x)$</td>
</tr>
<tr>
<td>$7 - 6 - 16x$</td>
</tr>
<tr>
<td>$7 - 6 + 16x$</td>
</tr>
</tbody>
</table>

For each expression, explain why you made the choice you did.

**Figure 4.3. Problem Number 2.**

There were many reasons why students were discussed during the categorization tasks. All professional groups discussed perceived student errors or misconceptions sometimes as a means to critique the Making Connections problems. Indeed, while discussing algebra problems intended for use with students it does not seem unusual to at times to discuss students interactions with these problems. However, the difference in how professional groups referenced students was not limited to a difference in quantity. In contrast to the two other professional groups, there were instances amongst the teachers where students were used as a means to justify claims about particular problems.

An illustrative example is the teachers’ discussion of problem number two (Figure 4.3) from 2006; Ursa Harper (MS) justifies her interpretation of the question by referring to the mistakes her students made. She is a bit dismissive of Sheila Eastman’s (HS) claim that the problem is about the distributive law because that is not what Ursa perceived her students to
have struggled with. It is worth noting that Sheila was the only person in the 2006 teachers’
group who did not administer the problems to her own class.

Sheila Eastman: You know, with what [the question writers] did here truly it’s just the distributive property because they wanted them to just come up with one [possible first step].

Ursa Harper: But what the kids made mistakes with were order of operations. They also made mistakes on like terms was the big one for my kids.

Sheila Eastman: Okay, order of operations, like terms, distributive property.

Ursa Harper: Bigger than distributive property was order of op, was the like terms.

Sheila Eastman: Oh, okay.

Ursa Harper: I was shocked when I looked at it [the student work]. I was like “excuse me”.

This discussion lead directly to the categorization of number two as a problem about “properties of real numbers” which, for the 2006 teachers, included concepts such as order of operations and like terms.

| In the expressions below, \(a\) and \(x\) are positive numbers. For each expression explain the effect of increasing \(a\): does the value of the expression increase, decrease, or remain unchanged? |
|-----------------|-----------------|-----------------|
| \(ax + 1\)       | Increases       | Decreases       | Remains unchanged |
| \(x + a\)        |                 |                 |                  |
| \(x - a\)        |                 |                 |                  |
| \(a + x - (2 + a)\)|                 |                 |                  |

For each expression, explain why you made the choice you did.

**Figure 4.4. Problem Number 4.**

A similar discussion took place about problem number four (Figure 4.11). Ursa (MS) cites student errors with the distributive law and number sense as the justification that the
problem is about these properties. Tom Luft (MS) and Tammy Vonce (MS) contributed to the discussion by suggesting other ways in which students could approach the problem, namely ideas about graphing, functions, and signed numbers.

Tom Luft: I would, um, extend that to, um, the idea of a function because I think the way my kids approach it is more of an input-output style... 

Tammy Vonce: Put in different inputs, look at the outputs.

Tom Luft: Input-output, they looked at in slope-intercept form. You know, they saw $ax + 1$ and they said, well, this slope’s positive, it’s going up, this slope’s negative it’s going down, that kind of thing. So I mean I think that’s an underlying principle. Whether or not all kids would see it that way, depending on their experience and stuff, I would think that most kids would just plug in and just see what happens.

Tammy Vonce: Well, you know, my kids, now that you’re saying that it reminds me, they looked at it as signed numbers. (2006)

This problem was ultimately categorized as being about “signed numbers”. The categorization of this problem was negotiated largely through an examination of what students did with the problem.

Generally, the teachers discussed multiple ways in which students could and did solve each of the problems. This was, in part, what lead the teachers, in both 2006 and 2008, to place individual problems in multiple categories. Their interactions with the Making Connections problem set were framed to a large extent by their experiences with the problems in their classrooms. This was not the sole factor influencing this relationship but it was prominent in their discussions as they engaged in the categorization task.

Mathematicians and mathematics educators also referenced (real or imagined) students as they interacted with the Making Connections problem set. However, this happened less frequently as indicated by Table 4.2 and was rarely a sustained focus of their discourse. Perhaps it would have been a larger focus if they had administered the problem set in
classrooms of students but this is unclear from the present data. The teachers based their analysis of the problems, in a large part, on what their students did with the problems.

4.2.3 Focus on Structure

What did the teachers, mathematics educators, and mathematicians notice in or about the Making Connections problems? As discussed previously (Section 4.2.1), there were diverse ways of talking about these problems both between and within groups. However, patterns did emerge in the extent to which each of the professional groups noticed or discussed the structure of algebraic objects. To the mathematicians, and to a lesser extent to the mathematics educators, the idea of structure was central to these problems. This was reflected in their categorizations, in their discussions during the categorization tasks, and during the interviews with the mathematicians from the 2008 workshop.

Structure is one of the global/meta themes in mathematics which Kieran (2007) wrote about. Many consider it to be one of or the sole central idea in algebra. During our interview, Ian Snyder (mathematician 2008) described algebra as follows:

Well, one description that might work to get us started is generalized arithmetic. You want to be able to look at the structure of a collection of arithmetic problems or a big arithmetic problem and it takes some algebraic structure to accomplish that.

Chris Norris spoke of structure as the concept which connects the ideas in school algebra with the abstract algebra of higher mathematics (Interview, 2008). These ideas about algebra will be discussed in more depth in Section 4.3.2.

In algebra there are an infinite number of equivalent ways to represent any expression. For example, $3x + 6$, $3(x + 2)$, $\frac{6x+12}{2}$ and $2x + x + 9 - 3$ are all equivalent expressions with different structures. From one perspective, their equivalence is ultimately derived from the axiomatic properties (associativity, commutativity, etc.) which justify algebraic manipulations in general. So when I say that mathematicians “noticed structure” I mean that they
placed value on and derived meaning from the particular choice of structure for the expression or algebraic object. This is related to what Linchevski and Livneh (1999) described as **structure sense**, the ability “to use equivalent structures of an expression flexibly and creatively” (p. 191).

You plan to drive 300 miles at 55 miles per hour, stopping for a two-hour rest. You want to know \( t \), the number of hours the journey is going to take. Which of the following equations would you use?

\[
\begin{align*}
(A) \quad 55t &= 190 \\
(B) \quad 55 + 2t &= 300 \\
(C) \quad 55(t + 2) &= 300 \\
(D) \quad 55(t - 2) &= 300 
\end{align*}
\]

Explain how you chose your answer.

**Figure 4.5. Problem Number 7.**

The following discussion about problem seven (Figure 4.5) is illustrative of the ways in which structure may not be noticed. The teachers in 2008 spoke about how they and their students had to solve this by “working backwards”. Working backwards, in this case, refers to finding an expression for the unknown, \( t \), of the form \( t = \ldots \) and then, through algebraic manipulation, matching it to one of the four multiple choice responses (which are not presented as \( t = \ldots \)).

Nadya King (HS): So how many of you went, how many of you started by evaluating the four [multiple choice] answers rather than starting your own?

Judith Hardy (MS): No.

Greg Davis (HS): I didn’t have the time.

Nadya King: So we all did our own way and then went back and checked the answers?
Nadya King: How many of your students did the same thing, did it their own way and then went back?

Beth Larson: Almost all of my students said “I didn’t do it that way, I set it up”. I don’t know maybe it’s just . . .

Nadya King: All of your students did what?

Beth Larson: They did the working backwards, what I consider working backwards, but they even called it working backwards where you find the time and you work, you know, you don’t set it up the way they did here but . . .

Nadya King: So they worked independently of the four answers.

Beth Larson: Uh huh. And then they compared it.

Nadya King: Mine went through the four answers and then got themselves totally in a befuddled state. They were trying to figure out which of the four answers worked instead working it.

This way of approaching and discussing problem seven is contrary to what Chris Norris had intended for the question as the following indicates.

I mean, so another way of doing [number seven] which is probably the way I had in mind when I wrote this question is “Which one of these equations is saying to me that I drove 55 miles per hour and I want to get a total of 300 and I took a 2 hour rest?” So I want people to say well, of course, C is a sort of a distractor because I’ll take that 2 hours I’ll add it to the time that, you know, then you say, no that’s not right, really I should subtract the 2 hours from the total time then that’s the time I was driving so the answer is D. And then A and B are just like way off base I guess, so maybe it’s not such a great question . . .

I mean the intent of the question was to sort of, that little talking through the equations that I just did, that’s a skill you want kids to have. You want kids to be able to read an equation and say “What is this equation telling me?” (Chris Norris, interview, 2008).

His intention was for the structure of the given equations to be meaningful. That is, if $t$ represents total time, then $(t - 2)$ represents time spent driving and, since “rate $\times$ time $=$
distance”, D is the correct answer. The teachers, and their students, who interacted with this algebra problem did not discuss or work with the structure presented in the answer choices. There was no indication that they did “that little talking through the equations”; instead they solved for $t$ and “worked backwards”. Unfortunately, there was very little discussion about this problem amongst the mathematicians or the mathematics educators. However, the mathematics educators did categorize number seven as “reasoning about symbolic forms” in 2006, a category which acknowledges the role of structure.

At the end of their discussion of number seven, one of the 2006 middle school teachers, Tom Luft, did speak about about “equivalent forms” of equations.

I think [this problem is also about] recognizing equivalent forms of the equation $[D = RT]$ because it, like, I think a lot of kids probably could figure it out but then which of those equations is the thing you actually did.

This comment, however, does not seem to reflect the idea of sense-making by noticing or examining the structure of an algebraic object nor did other instances in which teachers spoke about “equivalencies”. It is more a statement about students, after “figuring it out”, being able to find something equivalent to their answer among the given choices. Moreover, in my examination of the teachers’ categorization tasks from both workshops I found very few instances where the teachers may have discussed meaning as coming from the algebraic structure. In 2006, during discussion of problem number 10 (Figure 4.10) Ursa Harper (MS) mentioned that “what they [the students] had to do is think about what that formula knew but they didn’t”; she was referring to the fact that the factored form of a quadratic expression is convenient for finding the zeros of that expression. Sheila Eastman (HS 2006) spoke about “understanding the factored form” in the context of this problem as well. Teachers from both workshops similarly spoke about understanding the vertex form in regard to number 11. However, in general, throughout both workshops there was little or no explicit discussion of structure as a source of meaning amongst the teachers. Likewise, structural ideas were not reflected in their categorizations despite the inclusion
of “comparing equivalencies” in the 2008 categories.

A peanut, dropped at time $t = 0$ from an upper floor of the Empire State Building, is at a height, $h$, in feet above the ground $t$ seconds later given by

$$h(t) = -16t^2 + 1024$$

What does the factored form

$$h(t) = -16(t - 8)(t + 8)$$

tell us about when the peanut hits the ground?

**Figure 4.6. Problem Number 10.**

In contrast, Chris was not the only mathematician or mathematics educator to make strong statements which valued the structure of algebraic objects. For example, Susan Oster (mathematics educator 2006) described numbers 10 and 11 as asking “what does the structure of a particular symbolic form, what kind of information is easily obtained from a certain symbolic form?” There were explicit statements such as this from other mathematicians and mathematics educators. There were also more subtle expressions of this focus on form such as the following:

I think it is curious that this particular function [number 10: $h(t) = -16t^2 + 1024$] is something you can set equal to zero and find the roots of very easily without factoring it, right? Because there is no linear term. (Nick Porter, mathematician, 2006)

That is, Nick looked at the structure of the given function and noticed that it contained no term of the form $kt$ where $k$ is a non-zero coefficient. He had the structure sense to realize that he could solve for the roots without making use of the factored form (as the problem had suggested).

In 2006, both the mathematicians and the mathematics educators highlighted structural ideas in their categorizations of the problems. The mathematicians in 2006 categorized four of the 12 problems as either “Reading Expressions and Equations: Purpose of Different
Forms” (numbers 10 and 11) or “Reading Expressions and Equations: What Form of an Equation Is Helpful” (numbers 6 and 12). Tracy Coleman (mathematician 2006) referenced number 12, a problem about the solutions of quadratic equations, as she spoke about “what form is well-suited to the given purpose. To some extent 12 is about that too because sometimes you can tell easily how many solutions and paths and sometimes it’s much more tricky.” Drawing on similar ideas, the mathematics educators in 2006 categorized four problems (numbers 7, 9, 10, and 11) as “Reasoning about Symbolic Forms”.

In their interactions with the Making Connections algebra content, the teachers stood apart from the mathematicians and the mathematics educators in regard to the extent to which they discussed structure. Perhaps this is connected to the prominence which was given to students’ explanations in their discussion of the problems. For the mathematicians, it is possible that the presence of Chris Norris during their categorization tasks played a role in establishing structure as a central theme within that group. The interviews with the 2008 mathematicians, however, affirm that structure is, for them, a core idea in algebra; this will be explored in Section 4.3.2.

### 4.2.4 The Wording of Questions

In Remillard’s (2005) conceptualization of the participatory relationship between teacher and curricular materials she highlights the roles of both the teacher and the curricular materials. In the Making Connections workshops, the interactions between the participants and the set of 12 algebra problems were, of course, influenced by the problems themselves. In this section, I will direct some attention to features of the problem set.

I will (perhaps tangentially) address the question which Chris Norris raised when he introduced the categorization task to the Making Connections participants: What makes a problem difficult? Specifically, I will address participants’ comments about the wording of the Making Connections problems since the wording was often cited by the teachers as causing difficulty for their students. By “wording”, I am referring to word choice, notation,
During the 2006 problem categorization task, the teachers described the wording as problematic for their students on seven out of the 11 problems which they discussed for more than one minute (numbers 4-7, 9, 10, 12). The group of teachers in 2008 discussed the wording on 10 of the 12 problems as a source of difficulty for students (all except numbers 6 and 7 were discussed in this way). I will present some examples below to help characterize these concerns about wording.

In fact, members of each of the three professional groups, at various times, expressed displeasure with the wording of the problems however the most vocal group on this topic was the teachers. This, no doubt, is related to the fact the teachers had administered these problems in their classrooms and had encountered student difficulties first hand. The mathematics educators in 2006 were, in general, critical of the wording of the problems but, unlike the groups of teachers, they did not provide many details relevant to the wording of specific problems. Instead there were general comments such as the following:

Doug Garrett: One interesting thing to think about is it’s harder I think for us [the mathematics educators], with what we know, not to suggest how these problems could be improved, you know?

Susan Oster: [jokingly] We could group them that way [referring to categorizing the problems]. These problems could be improved by, in this way.

Tanya Iverson: These require the least amount of change. These require the most change.

In fact, Doug Garrett, one of the mathematics educators, had made his own unsolicited revisions of many of the Making Connections problems prior to attending the workshop (his revisions are not part of the present data set). However, the mathematicians made only a couple of comments regarding wording as an obstacle; this may have been due to the presence of Chris Norris, who compiled/authored the problems, during the mathematicians’ categorization tasks.
You are simplifying $7 - 2(3 - 8x)$. Which of the expressions are possible results after the first step?

<table>
<thead>
<tr>
<th></th>
<th>Possible</th>
<th>Not Possible</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5(3 - 8x)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$7 - 2(-5x)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$7 - 6 - 16x$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$7 - 6 + 16x$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For each expression, explain why you made the choice you did.

**Figure 4.7. Problem Number 2.**

For the teachers, the wording of problems was often perceived to be unfamiliar or generally not intelligible for the students. For example, Judith Hardy (MS teacher 2008) did not like the wording of problem number two (Figure 4.7). She expressed that a particular phrase may be confusing to the students.

> [Problem number two says] “possible correct results”. Someone might interpret, the students interpreted that as, maybe it’s like one of the steps in solving the problem. I mean they just didn’t know how to I guess interpret it. Maybe the wording on this one’s bad.

A slight variation on Judith’s complaint occurred when the wording was criticized, not just for being unfamiliar, but for straying from the typical language used in the classroom. Ursa Harper (MS teacher) expressed the following in 2006 about problem number nine (Figure 4.1).

> And I didn’t like exactly the way D [the correct answer] was worded. “The number people of people who decide not to go if the price is raised by a dollar.” And I teach them it’s the “change per” and the word “per” wasn’t in there.

The wording of problems with regard to symbol use was also seen as an obstacle at times. Problem number nine provides an example that was mentioned within the group of mathematics educators (2006), the group of mathematicians (2006), and within the group of teachers (2006 and 2008). They cited, as a source of confusion, the notation “$p$” to represent “$p$ dollars” where $p$ is the price of a concert ticket. Similarly, problem number
five (Figure 4.8) required students to perform some numerical operations “on a number $b$”. Michael Chow (MS teacher 2008) described the issue that arose from this choice of wording:

> That was the problem in my class because they’re used to saying variables are letters, numbers are numbers, now they’re saying “a number $b$”, and like, that’s a letter, it can’t be both.

Both “$p$” and “a number $b$” have something in common in that they are both embracing the fact that variables or unknowns represent numbers. Thus, for example, since $p$ is a number that represents the price of something it is valid, but maybe not typical, to write a dollar sign in front of it.

<table>
<thead>
<tr>
<th>In (a)-(c),</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) Write an algebraic expression representing each of the given operations on a number $b$.</td>
</tr>
<tr>
<td>(ii) Are the expression equivalent? Explain what this tells you.</td>
</tr>
<tr>
<td>(a) “Multiply by one fifth”</td>
</tr>
<tr>
<td>“Divide by one third”</td>
</tr>
<tr>
<td>(b) “Multiply by one fifth”</td>
</tr>
<tr>
<td>“Divide by five”</td>
</tr>
<tr>
<td>(c) “Multiply by 0.4”</td>
</tr>
<tr>
<td>“Divide by five halves”</td>
</tr>
</tbody>
</table>

**Figure 4.8. Problem Number 5.**

Problem five (Figure 4.8) presented difficulties for students because of its structure. Michael Chow (MS teacher 2008) and Tammy Vonce (MS teacher 2006) both reported that students interpreted the hyphen in the phrase “In (a)-(c)” as a minus sign and the “a” and “c” as variables. Carrie Dixon (MS teacher 2008) reported on student confusion stemming from the quotation marks and called the question “terribly written”. The fact that the wording and structure of this (and other) problems caused confusion for students was perceived
negatively by the teachers. The following exchange from the teachers’ categorization task in 2008 reflects this:

Carrie Dixon (MS): So I don’t think we got anything out of this one just because the wording was so -

Greg Davis (HS): Before we even get to the mathematics the language has got to be comprehensible.

Michael Chow (MS): Yeah, it was just above [the students’] level.

During the teachers’ categorization task in 2006, Tom Luft (MS) summed up these concerns:

I think that could be a general comment overall. I thought the directions on several problems the kids are like, “What am I supposed to do on this thing?” And, you know, and I would clarify the directions because we wanted to get, we wanted to get data on the problems not on their ability to interpret the directions on the problems.

But to what extent does “clarifying” the problem change the problem? Luft’s quote seems to imply that changing the wording of a problem does not change problem. Certain aspects of the Making Connections participants’ views of the connection between language and mathematics will be explored in Section 4.3.4 and in Chapter 5.

This question about the effect of changing the wording of a problem has a corollary which concerns the structure(s) of the algebraic object(s) in the problem. For example, problem nine presents the linear expression “2500 – 80p”. In 2006, mathematician Nick Porter raised the concern that students may have a hard time interpreting the expression because it is not in what Tracy Coleman (mathematician 2006) referred to as the “canonical form” $y = mx+b$; that is, the expression wasn’t written as $-80p+2500$. In fact, Beth Larson (HS teacher 2008) cited this as a difficulty for her students: “If you’re thinking linear and you think $mx+b$, that’s the formula students think of, $-80p+2500$, maybe they would have gotten it more, but my students just saw 80, slope of that.” But I am hesitant to classify
a concern such as this being about “wording” because, as was explored in Section 4.2.3, the structure of algebraic objects is of central focus to some. Here, perhaps in contrast to Luft’s statement, Larson’s quote may imply that changing the structure of an algebraic object does, in some sense change the problem.

4.2.5 Summary - Interacting with Algebra

Prior to conducting this research, I was tempted to think that members of professional groups would draw upon the specialized (perhaps stereotypical) knowledge of their fields when analyzing the Making Connections problem set. In general, this did not prove to be true in the Making Connections workshops. Members of each of the three groups drew upon mathematical content knowledge and pedagogical content knowledge in their analyses. For example, within each professional group, student strategies for solving problem number one were discussed and so were mathematical ideas and skills associated with the problem (e.g., order of operations, distributive law). Moreover, there was diversity within each of the groups in their discussions of these algebra problems. However, there were some patterns that emerged:

1. As compared to the other two professional groups, the teachers’ analyses of the problems was more heavily guided by drawing upon knowledge of students and classrooms.

2. The teachers did not focus on structure as much as the mathematicians or the mathematics educators.

3. The teachers negatively viewed the impact of the unfamiliar wording of the Making Connections problems on students’ ability to solve the problems.
4.3 Views of Algebra

The previous section addressed the ways in which mathematicians, mathematics educators, and teachers interacted with Making Connections algebra content. These interactions were influenced by the views that they have of algebra and of mathematics. In this section, I draw upon data related to the problem analysis/categorization task and to the 2008 interviews in order to investigate the participants’ views of algebra and how these views played out in the Making Connections workshop. I begin with an examination of the categorization task for problem number 11 as a means to introduce many of the themes from this section and to reinforce themes from the preceding section.

4.3.1 Number 11

A street vendor of t-shirts finds that if the price of a t-shirt is set at $p$, the revenue from a day’s sales is $p(900 - 60p)$. She wants to choose the price that will yield the greatest revenue. The best form of this expression for figuring what price to set is

A. $p(900 - 60p)$
B. $-60(p - 7.5)^2 + 3375$
C. $-60p(p - 15)$
D. $900p - 60p^2$

Explain how you chose your answer.

The four answer choices for number 11 are equivalent to each other. Chris intended for the correct answer choice to be choice B. Indeed, this is what I chose when I first saw this problem. My rationale was that this expression is in a standard or canonical form called the “vertex form”. In this form, I was able to identify immediately that the vertex occurs where $p = 7.5$ and to combine this with the knowledge that the vertex of the graph of this particular quadratic expression (which is a downward opening parabola) is the highest
point on the graph. This highest point in turn corresponds to the highest revenue. One way to find the vertex form is by using a process called “completing the square”.

However, as will be discussed below, there are ways to arrive at answer choice B which do not invoke ideas of graphing. There are also justifications for choosing other answers. For example, Larry Donahue (mathematician 2008) mentioned a reason why C could be chosen:

I know when I taught pre-calc I had students who would have looked at this problem and said, you know looked at part C and said, “Okay, it’s zero when \( p \) equals zero and it’s zero when \( p \) equals 15 and I know the parabola has to be at its max or min halfway between the zeros so it must be 7.5”.

Donahue’s strategy requires thinking about the expression graphically and it draws upon his knowledge of students. Answer choice D could also be justified; Kevin Lewis (HS teacher 2008) mentioned that he had given this question to some calculus students and one of them chose D because it is the easiest to differentiate and this provides a means for finding the maximum.

*Teachers* The group of teachers from 2006 categorized this problem as being about “translating words into expressions/equations”, “quadratics”, and “reasoning from graphs”. The
2008 group of teachers categorized it as about “translating” and about “comparing equivalencies”. By “translating”, both of these groups were referring to the act of converting verbal representations to symbolic or graphical representations. There was little or no explanation amongst the teachers in either year as to why this problem was considered to involve “translation”. Although, during the 2006 workshop, Tom Luft (MS) briefly expressed some dissent about the categorization of number 11 as “translating” but never explained why. This idea of translating will be discussed in more detail in Section 4.3.4.

As mentioned in Section 4.2.1, the categorizations were not always reflective of what was discussed for specific problems. In 2006, Sheila Eastman (HS) received general agreement when she described number 11 as being about “equivalent expressions, meaning the understanding that two expressions are equivalent when they’re very different forms, understanding how to manipulate them algebraically” although this idea was not reflected in the categorization. Among the 2006 group of teachers, there was immediate agreement (and little discussion) that this problem was about “quadratics”. The idea that this question was about graphing was discussed as follows:

Ursa Harper (MS): Eleven demonstrates, I think kids that have a lot more experience with the graphing calculators, 11 is real straight-forward. And those kids that don’t, because my kids obviously didn’t have nearly enough experience with graphing calculators for quadratics. We didn’t do nearly enough. Because I thought, I went “wow, this one’s really easy”. But then almost nobody got it.

Tammy Vonce (MS): Okay, mine didn’t get it either.

Sheila Eastman (HS): They have to understand the idea of parabolics here. They have to understand there’s a max and a min and a vertex.

Ursa Harper: And if it’s in that form [vertex form] you can pick out the vertex and we didn’t do that form in my class.

Sheila Eastman: So maybe we should just say understanding quadratics.
Tom Luft (MS): I’m saying the vertex form of a parabolic function, that’s what this problem’s about and we don’t do that in that class. Even though I teach them how to find the vertex, negative \( \frac{b}{2a} \), and they know it’s a min/max but this form is not taught in my class.

Ursa Harper: It’s a second year algebra concept as far as I know.

Tom Luft: It’s an Algebra II, that’s an Algebra II topic.

Ursa Harper: It’s an intermediate algebra concept, it’s not a -

Tammy Vonce: My kids didn’t know what to do with it.

The “vertex form” is mentioned in this discussion but, to some extent, it is dismissed because it does not fit the context of their Algebra I classrooms. It is ultimately not reflected in their groupings. Moreover they express that, in order to answer problem 11, a student would need to know the essential features of the graph of a quadratic equation/function. The group of teachers in 2006 seemed to put a high value on graphing activities in algebra (more in Section 4.3.2); Sheila Eastman (HS) described it as “one thing NCTM has really pushed. The idea of visually looking at things.” For number 11, it is unclear to me how using a graphing calculator, as Ursa Harper (MS) suggested, would help determine which of these expressions would be “best” since they are all equivalent and would all have the same graph.

The group of teachers from 2008 focused most of their discussion about number 11 on the vertex form. There was no discussion about their categorization of this problem as “translating”. Some spoke of completing the square as a technique for obtaining the vertex form and as an approach for this problem. They considered this within the contexts of their classrooms and their students; this is evident in the following exchange between two high school teachers.
Nadya King: Our regular Algebra II kids have a hard time going from one form to vertex form. I think vertex is the hardest form to learn so we don’t, it’s not an emphasis in regular Algebra II. Our honors Algebra II do it, higher levels do it, but our regular Algebra II does not get to vertex.

Beth Larson: And if I had something rather than standard form, they’re thinking $x$-intercepts. But they could use the $x$-intercepts and go halfway between them.

Thus converting to vertex form is perceived to be a difficult task for their students. However, this task is not actually required in number 11 since choice B is the only one which has the *structure* of vertex form and which corresponds to the process of completing the square. Throughout the discussions by teachers in both 2006 and 2008 there were frequent references to student activity and the contexts of their classrooms as exemplified in the preceding bits of dialog. The 2006 group of teachers expected students to solve number 11 by drawing upon graphical ideas and ultimately categorized it as being about graphing. They also categorized it as “quadratics” drawing upon a feature of the focal algebraic object in the problem. The 2008 teachers’ categorization of this problem as “comparing equivalences” drew upon the structure of the problem (not of the algebraic object) which presented four equivalent expressions for the student to examine. The teachers from both workshops also said this problem was about “translating” from English to mathematics, a topic which will be examined with some detail in Section 4.3.4.

*Mathematics Educators* Mathematics educators placed an emphasis on the structure of the expressions in problem number 11. Within their group, as with the teachers in 2006, it was acknowledged that students may have difficulty with completing the square but this did not seem to influence their choices while categorizing as it did for the teachers. The mathematics educators categorized this problem as “reasoning about symbolic forms”. The following exchange captures the mathematics educators’ focus on structure alongside an awareness of “the student”; all people in the group agreed with this categorization as indicated by the following exchange from 2006.
Kevin Martin: But I mean you’re looking for a maximum in number 11 so whether or not the kids would reason this way at all or not I sincerely doubt it because I don’t know any kid that would ever complete the square to get that expression in B which is maybe the easiest way to get the answer for me.

Susan Oster: It’s also like what does the structure, what does the structure of a particular symbolic form, what kind of information is easily obtained from a certain symbolic form?

Tanya Iver son: From number 11.

Kevin Martin: Yes.

Doug Garrett: Well, 10 and 11.

The mathematics educators’ discussion of problem 11 did not extend much beyond what is represented by the dialog above. However, their discussion evokes the ideas from Section 4.2.3: Focus on Structure. They did not mention ideas related to graphing in their short discussion of number 11.

Mathematicians The mathematicians also noticed structure in number 11. In 2006, they categorized this as “reading expressions and equations: purpose of different forms”. Tracy Coleman (2006) described the importance of noticing structure but acknowledged the difficulty of this for students. In the following quote, Coleman discusses the importance of structure and connects students’ difficulties to classroom practices which emphasize the mechanics of completing the square.

[There is] a theme that comes out of some of these [like 10 & 11] that a different form for an expression might be useful for a different purpose, that’s a really important thing, but I’m sure that most of the kids were kind of lost, because it is fairly sophisticated, I think, to be able to realize that. . . I doubt that they [the students] would even come up with [choice B] because completing the square is something that someone told them to do that and then they carry it out.
Chris Norris (2006) picked up on this theme but lamented that knowing the standard forms does not always mean understanding them.

Certainly standard forms that have names and they spend a lot of their lives being told to put them from one form into another, but how much do they think about why it’s important to put them into forms, why do you want to factor, why do you want to put things into completed square form. (emphasis in original)

During the 2008 workshop, the group of mathematicians discussed these same themes but Chris suggested an alternative approach which is revealed in the following exchange.

Larry Donahue: And you have to know that completing the square is the right way to do it, you have to know that when you complete the square that tells you the vertex of the parabola.

Chris Norris: Okay, but I don’t agree. You don’t have to know anything about completing the square. I’m not proposing what I’m about to say as reasoning that would occur in students but you don’t need to know anything about completing the square to know that [choice] B answers the question for you, right? You could look at the structure of that expression and you could say to yourself, okay, I’ve got 33,375 subtract, and I’m subtracting something from it and what I’m subtracting is always positive or zero because it’s 600 times a square and squares are always positive so clearly I want to subtract zero to get the largest value for this expression so I’ll make $p$ be 7.5. I mean that’s what I would like as an answer to why you chose that. I’m not making a claim about that being a realistic answer, on the other hand that’s what you want as a mathematician when you look at that completed the square, that’s what you see. You see all that story in a flash.

When I asked Chris about this approach in our interview, he expressed a different opinion about how realistic this approach is. He said that it was the approach he “was looking for”. This approach is remarkable because it requires no knowledge about graphs or about completing the square. He talks about noticing structure and thinking about the numbers and operations which comprise that structure. But what does it mean to do this thinking “in a flash”? I will address this question in Section 4.3.3.
During the 2008 workshop, the mathematicians also discussed the idea that, although the algebraic objects presented in number 11 are expressions, they can be thought of as defining a functional relationship (revenue as a function of price). In response to Frasier Grant’s comment about changing the value of price to examine changes in revenue, Chris said the following:

Right. One way of saying this is that some of these are really about functions. So 11... is you’ve got four different expressions for the same function and the question is which expression is useful for which purpose, for the purpose of just discovering some property of that function. Where is its maximum? Where are the zeros? You know, whatever.

It is possible to identify functional relationships in several of the Making Connections problems but it is also possible to interpret these problems without any focus on functions. In Section 4.3.2, I will examine how Making Connections participants’ interactions with these problems did or did not reflect functional ideas.

Summary - Number 11 In this discussion of number 11, the themes of “Section 4.2: Interacting with Algebra” are visible. All three groups drew from content knowledge and pedagogical content knowledge in their analyses. However, the teachers framed their discussion heavily with their knowledge of (real or imagined) students. The mathematicians and mathematics educators put a heavy focus on structure in discussing number 11.

The interactions with number 11 were also framed by the participants’ beliefs about algebra. In the following sections, I will explore the participants’ beliefs about algebra. As reported in Section 4.1, Chris Norris did not believe that graphs play a large role in algebra. Indeed, the approach he “was looking for” for number 11 did not rely on any ideas about graphs. However, for the teachers from 2006, this was a question about graphing. The participants’ views of algebra influenced their interactions with the Making Connections problems. Is algebra about generalized arithmetic or is it about functions? How important are graphs and structure in algebra? I will examine the relevance of these questions to the
interactions in the problem categorization tasks. I will also report on the student habits and algebraic activity which were valued by the Making Connections participants.

4.3.2 Algebraic Form vs. Function

In this section I address the questions: How did the Making Connections participants think about algebra? What types of algebraic activities did they value? Kieran (2007) described two competing views of school algebra. One of these views emphasizes algebra as generalized arithmetic in which form, structure, and symbolic manipulation are emphasized. This view is often called a traditional view of algebra; the objects of interest are expressions and equations whereas functions have a minor role. This was the view which Chris Norris embraced and which inspired the set of 12 Making Connections problems (see Section 4.1). The second view is often referred to as reformed or reform-oriented algebra. It embraces functions as a central object of algebra and values multiplicity of representations including graphs and tables. I will focus on the ways in which participants spoke about functions and graphs, two reform-oriented objects, in their interactions with the Making Connections problem set.

Functions A focus on functions puts an emphasis on input/output relations (or independent and dependent variables). Functions can often be represented using algebraic expressions or equations but this is not always possible. Chris Norris did not see functions as playing a large role in algebra and expressed concern about the way they are treated in algebra classes.

... there are lots of different ways of representing the same number but the distinction between the representation and the object itself is like systematically confused and mucked up and messed around and a huge cause of trouble. The same things happen in algebra. That's my big point about expressions versus functions. Functions are objects which have no inherent expression associated with them at all. Most of the time we use expressions to define functions. Different expressions can define the same function. I mean unless you keep these
two notions separate in your mind you’re going to run into all sorts of trouble. (interview, 2008)

This is similar to Kieran’s (2007) warning that “hybrid” curricula which reflect both traditional and reform-oriented ideas may lead to confusion amongst students particularly with the distinction between equations and functions.

A peanut, dropped at time $t = 0$ from an upper floor of the Empire State Building, is at a height, $h$, in feet above the ground $t$ seconds later given by

$$h(t) = -16t^2 + 1024$$

What does the factored form

$$h(t) = -16(t - 8)(t + 8)$$
tell us about when the peanut hits the ground?

**Figure 4.10. Problem Number 10.**

Problems like number 11, which was discussed in Section 4.3.1, implicitly define a functional relationship. That is, the phrase “the revenue from a day’s sales is $p(900 - 60p)$” can be interpreted to mean that revenue is a function of price. It is possible to characterize problem 11 as asking the student to think about how the expression $p(900 - 60p)$ changes as $p$ changes. This problem (along with numbers four and nine) was therefore categorized as “dynamic expressions” by the mathematicians in 2008. Number 10 (Figure 4.10) was the only problem in the set of 12 which explicitly drew attention to a functional relation through the use of notation; the notation $h(t)$ signals that height, $h$, is a function of time, $t$.

I asked Chris about these choices during our 2008 interview.

**Josh:** Sometimes you write, like in number nine, you say the number of people who will attend is some expression. As opposed to defining a variable for the number of people who attend or something. Or expressing it as a function.

**Chris:** Or calling it $f(p)$ or something, yep.
Josh: In number 10 you actually do express it as, I guess it’s called functional notation.

Chris: Mmm Hmm. [agreeing]

Josh: But in others you kind of just say “here’s an expression” and it does represent something. From what I’ve observed with how the teachers reacted to these problems I feel like they felt those to be kind of unfamiliar to them.

Chris: Because there was no independent variable. I’m sorry, no dependent variable.

Josh: Yeah, because it wasn’t, because it was just like, “Hey, the amount of money you’re going to make, or the number of people who attend is $2500 - 80p$”.

Chris: Yeah, I mean, I guess it was intentional to focus on the expression itself. I mean the fact of the matter is you can pose these questions without ever naming a dependent variable. I mean the dependent variable is kind of extraneous.

Josh: Well it’s implicitly defined, right?

Chris: Yeah, right, yeah, absolutely it’s there in some sense but it, I guess it was a way of writing the question to force people to focus on the structure of the expression itself because there was nothing else to look at.

As was discussed above (Section 4.2.3), Chris was not always successful at forcing a focus on structure. Instead, function-related ideas like graphs were often the focus of participants’ discussions about the problems; this was especially true amongst the teachers, as will be argued in this section. The following discussion amongst mathematicians during the 2006 categorization task exemplifies how functions may be present in a discussion but not ultimately the focus.

Chris Norris: Uh, let’s look at some of these ones about equations. Uh, 12, 6, 7, now I think 10 is not really [about equations], it’s more about functions. It would be interesting to know which one we’d group 10 with actually. Uh, we have these two groups, 1 and 2 about expression, and 4 and 9 are about functions or values of expressions, 10 is explicitly about connecting the algebraic form of the expression to interpreting its values, of course these later ones are more difficult, or at least they’re higher level-
Tracy Coleman: Definitely higher level.

Chris Norris: All these ones about quadratics. So let’s talk a little bit about 10. What do people have to say about the skills involved in that one, before we go on to the equations?

Tracy Coleman: Well, I said they need to be able to interpret the meaning of an expression in context. But I was really focusing more on the expression than the function aspect of it. You know, you have to understand that -

Chris Norris: Well none of these are explicitly about functions, although this one does use function notation -

Tracy Coleman: And I think the reason I was grouping it right away with solving equations is that you have to know the point of factoring, what’s the point of factoring polynomials? Well you’re trying to, you do that so as to find the roots!

Kara Nolte: So even before that they have to be able to interpret that when it hits the ground that \( h \) equals zero.

Craig Miller: So that is kind of understanding what the function values mean.

Tracy Coleman: And you also have to understand that a product is zero when at least one of the factors is zero.

Chris began the preceding discussion by acknowledging that numbers 4 and 9 could be thought about either as functions or expressions; ultimately they were grouped as expressions in both 2006 and 2008. He then focuses on the algebraic form of 10 more than the function, a sentiment that Tracy shares. Number 10 does, in fact, present a function in function notation but the focus was ultimately on the structure of the expression that defines the output of the function, \(-16(t - 8)(t + 8)\), and why this gives information about when the peanut hits the ground. Problem number 10 was grouped by the mathematicians as “reading expressions and equations: purpose of different forms” in 2006.

In the 2008 workshop, the mathematicians again shifted focus from the function notation in problem 10 and categorized this problem as being about equations.
Larry Donahue: Yeah, I mean, is 10, is there any way, sensible way to do 10 other than to just recognize that [you must solve for \( t \) when \( h = 0 \)]?

Chris Norris: That’s an interesting observation to me that we’re doing this classification into the problems and we’re not classifying this as a function one because it reduces to an expression... and so is 10 about expressions or equations.

Frasier Grant: So I think it’s equations because what it’s asking you to do is to solve.

Chris Norris: But it could also be about expressions because it’s just asking you to stare at that expression and notice that it’s zero when \( t \) is 8.

The preceding dialog acknowledges that, despite their categorization as “equation”, this problem could be thought to be about functions, expressions, or equations. Chris’ reference to “staring at” and “noticing” things about an expression is a reference to a sort of expert gaze which will be discussed in Section 4.3.3.

The mathematics educators in 2006 discussed number 10 in relation to number 11. The group agreed that both problems required recognizing that “there are different, yeah, forms, there are different quadratic representations that that give” different information (Doug Garrett, 2006). Their focus was not on the functional relationship; the source of meaning was the structure of the algebraic object.

Number 10 was the only problem which used function notation. Most of the teachers from 2006 mentioned that the function notation caused problems for their students because their (middle school) students hadn’t been exposed to it yet. The teachers from both workshop who administered this problem reported very limited student success and attributed it to the students’ lack of familiarity with functions (though Tom Luft, MS 2006, attributed it to students needing more time). For example, Tammy Vonce (MS 2006) reported that her students learned how to factor “but we don’t really ever get into what a quadratic means. I mean I might talk about a parabola for like a day and that’s it.” Ursa Harper (MS 2006) did draw some attention to the structure by noting that, since the factored form was presented
in the problem, the students had to “think about what that formula knew but they didn’t”. In 2008, Beth Larson (HS) mentioned that her students would have had more success if the problem “would have given them more cues or [instructed them to] graph or even had a graph on the side”. Indeed, in 2006, the teachers categorized this problem as about graphing (among other things). Teachers from both workshops regarded number 10 as a problem about “translating” (this will be explored in Section 4.3.4.

In the expressions below, \( a \) and \( x \) are positive numbers. For each expression explain the effect of increasing \( a \): does the value of the expression increase, decrease, or remain unchanged?

<table>
<thead>
<tr>
<th>Expression</th>
<th>Increases</th>
<th>Decreases</th>
<th>Remains unchanged</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ax + 1 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x + a )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x - a )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( a + x - (2 + a) )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For each expression, explain why you made the choice you did.

**Figure 4.11. Problem Number 4.**

Despite the teachers’ concerns about the unfamiliarity of function notation, ideas about functions were present during their categorization tasks. For example, problem number four (Figure 4.11) is presented without any explicit reference to functions but students were required to think about how expressions change as the “positive number” \( a \) changes.

In the following bit of dialog, teachers from the 2006 workshop discuss this problem:

**Tom Luft (MS):** I would extend [the idea of thinking about the expressions changing] to the idea of a function because I think the way my kids approach it is more of an input-output style, you know, and like what you said plugging in and see what happens. But really that comes down to the idea of range and domain and input-output and I think most of the kids, probably because of the way I taught it to them, organized t-charts and they looked at as...

**Tammy Vonce (MS):** Put in different inputs, look at the outputs.
Tom Luft: Input-output, they looked at it in slope-intercept form. You know, they saw $ax + 1$ and they said, well, this slope’s positive, it’s going up, this slope’s negative it’s going down, that kind of thing. So I mean I think that’s an underlying principle.

Thus, Tom and Tammy relate this problem to functional ideas and then to related ideas about graphing. (We also, once again, see the invocation of student activity to frame the problem analysis.) Indeed, graphing is another idea which is characteristic of reformed algebra. It may serve to explicitly draw attention to an input-output (functional) relationship.

Within the Making Connections workshops, the mathematicians recognized the existence of functional ideas but focused instead on the structure of expressions or equations. There was not a significant amount of discussion amongst the mathematics educators explicitly relating to functions (possibly because they were running short on time by the time they discussed number 10). The teachers’ discussed functional ideas throughout the two workshops although this focus most strikingly manifested in emphasizing graphs and graphing. The teachers’ discussions of functions, in and of themselves, did not not extend much beyond what has been documented in this section.

Graphs Graphs often play a large role in reform-oriented algebra. They provide opportunities for links between multiple representations such as graphs, tables, and symbols and have been endorsed by some as a means of meaning-building in algebra (e.g., Kaput, 1987). Furthermore, they inherently invoke the functional relationship between independent and dependent variable as represented on the horizontal and vertical axes, respectively. Chris Norris described graphs as potentially useful but not essential to algebra. As noted in Section 4.1.1, graphs were not explicitly present in the Making Connections problem set.

I didn’t have any graphs here [in the set of 12 problems] because I was more interested in what I think of as the central object which is exploring the symbolic representations and exploring the structure of expressions and equations.

(Chris Norris, interview, 2008)

---

4I am considering graphs to be objects inherently related to functions as is consistent with the literature reviewed in Section 2.1. Although, it is possible that some Making Connections participants did not consider these concepts to be connected. This will be discussed further in Chapter 5.
Discussion of graphs was most widespread amongst the teachers in the 2006 workshop as is evidenced by their categorization of five problems as about graphs. Graphs also played a large role for the group of teachers in the 2008 workshop; during the categorization task they discussed graphing as a potential strategy for five of the Making Connections problems (numbers 4, 7, 9, 10, 11). Graphical ideas came up less frequently within the groups of mathematicians and mathematics educators. Each of these groups discussed slope in relation to number nine (Figure 4.1), a problem which made explicit mention of slope. However, it is possible to think about slope as a rate of change in a purely non-graphical context; the group of mathematicians in 2008 did not speak about slope in a necessarily graphical context. Other than number nine, graphical ideas were mentioned in relation to only one other problem by the 2006 mathematicians (# 4), by the 2008 mathematicians (# 11), and by the 2006 mathematics educators (# 12).

As discussed above, the teachers in 2006 related problem number four to graphs. This was also true amongst the teachers in 2008 who discussed thinking about slope as a means of approaching this problem. Number four is a problem which presents expressions but perhaps implicitly evokes a functional relationship. The functional-graphical interpretation of number four was not limited to teachers. In 2006, Tracy Coleman (mathematician) suggested that understanding slope could be important to solving number four.

Graphs were also discussed in the context of equations. The group of teachers from 2006 categorized number six (Figure 4.12) and number 12 as problems about “graph theory”. The name of this category was changed to “reasoning from graphs” when it was pointed out by mathematicians that the label “graph theory”, amongst mathematicians, refers to vertex-edge graphs rather than Cartesian graphs.

Problem number six presented nine linear equations and asked students to comment on the signs of their solutions (if any). Within the group of teachers in 2006, it was Tom Luft (MS) who initially endorsed thinking about this question in terms of graphs by suggesting that students could graph both sides of the equation using a calculator and look for the point of intersection. He saw it as a way to motivate students, as higher-level thinking, and as an
Say whether each equation has a positive solution, a negative solution, a zero solution, or no solution.

<table>
<thead>
<tr>
<th></th>
<th>Solution is positive</th>
<th>Solution is negative</th>
<th>Solution is zero</th>
<th>No solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( 7x = 5 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>( 3x + 5 = 7 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>( 5x + 3 = 7 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>( 5 - 3x = 7 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>( 3 - 5x = 7 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>( 8x + 11 = 2x + 3 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>G</td>
<td>( 11 - 2x = 8 - 4x )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>H</td>
<td>( 8x + 3 = 8x + 11 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>( 8x + 3x = 2x + 11 )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Could you have predicted the answers for any of the equations without solving it? Which ones, and how?

Figure 4.12. Problem Number 6.

age-appropriate tool for meaning-building. This is communicated in the following trio of quotes from Tom Luft (2006):

- [The Students] love it. They love using the intersect command [on a graphing calculator]. I mean a calculator, it’s like boom. They love it... I mean it’s motivating for them to understand what it does because they can use the technology.
- A mathematician might look at [an equation with linear expressions on each side] and say oh there’s no, those are parallel lines, there’s no solution.
- They have to have the visual connection, I mean they just, they’re not done developmentally. They have to see what it means visually.

Much as Tom Luft cited the supposed behaviors of mathematicians to justify a graphical approach to number six, Sheila Eastman (HS) cited the National Science Foundation (NSF) as justification for approaching number nine graphically.

I think we should add graph theory to [number nine] because I do think that if you’re in, you know, one of the systemic initiative projects, and stuff, out of NSF I think that that is what you’d call that problem. I think that you would have them graph it or start with a table and get some things down and realize,
"oh it’s linear, oh that’s right, it’s linear, it’s in slope intercept form”. You know and then kind of go from there. (Sheila Eastman, 2006)

In the 2006 group of teachers, five out of 12 of the Making Connections problems were categorized as “reasoning from graphs” though four of these five problems (all except number nine) were placed in multiple categories. Chris Norris’ perspective on number nine was discussed in Section 4.1.1. This is the problem in which the notion of graphing appears most explicitly, as part of an incorrect multiple choice answer which mentions the “slope of the graph”. This problem caused confusion amongst teachers and students in both 2006 and 2008 and many were inclined to choose the incorrect choice which referenced graphs.

Indeed, within the groups of teachers, there were other instances related to graphing around which confusion arose. For example, the teachers in 2008 deliberated over applying graphical ideas to number four. In the expression $ax + 1$, when explaining the effect of increasing $a$, there was confusion over whether $a$ or $x$ would be the slope. It is possible that, since Chris intended these problems to be about structural aspects of the expression, the application of a graphical perspective resulted in unproductive or confusing interactions between the Making Connections teachers and the problem set.

**Defining Algebra** The set of Making Connections problems was compiled by someone with a very specific point of view about what algebra is. Chris Norris considers algebra to be about structure and generalized arithmetic; functions and graphs do not play large roles in Chris’ view of algebra. However, the groups teachers from both workshops often focused on graphs during the problem categorization task. What, if anything, does this say about their views of algebra?

In 2008, I interviewed the 12 of the 13 Making Connections participants (there were only mathematicians and teachers present in 2008). I started each interview by asking the very broad question “What is algebra?”. There was a range of reactions within and across the groups (mathematicians and teachers); some answered immediately while others needed some time to think. In their responses to this question and follow-up questions
most of the participants embraced the traditional viewpoint of algebra which emphasizes symbolic manipulation and generalized arithmetic.

Of the seven teachers interviewed only Beth Larson (HS) diverged from this view. She described algebra as “the development in students of the abstraction, the ability to describe a situation in several different ways... graphically, numerically, as a sequence, connecting all the different parts of it. But really it’s an abstraction of a situation or a problem or a context or a pattern.” Of the mathematicians, Shane Tyson (who was not present for the categorization task) strayed the furthest from the traditional viewpoint by also describing algebra as “a formal means of modeling the real world”.

Not every participant directly addressed the role that graphs and functions play in algebra but among those who commented on these objects, the responses were varied within each professional group. Amongst the teachers, graphs were described as not necessary (Judith Hardy, MS) and also as a “main focus” (Kevin Lewis, HS). Greg Davis (HS) described graphing as an “important tool” that, though it may not necessarily need to be part of an algebra class, if graphs were to not be included then “there would be a whole section of individuals who’d be lost without it”. Amongst the mathematicians (excluding Chris Norris), those who commented on graphs, described them to be a part of algebra (Larry Donahue) and also to be over-emphasized but potentially powerful (Frasier Grant). There was less commentary on functions but, again, the comments represented a wide range of opinions. Greg Davis described functions as being “essential” in algebra. Ian Snyder (mathematician) thought that functions are not a crucial part of an algebra class but should be taught in preparation for calculus.

Summary - Algebraic Form vs. Function Overall, the 2008 participants gave definitions of and spoke about algebra from a traditional perspective during the interviews. However, during the categorization task, amongst the teachers, ideas relating to graphs were often the focus. This is despite the fact that Chris Norris did not intend these Making Connections problems to provoke ideas about graphing and functions. Unfortunately, I do not have
interview data from the 2006 group; the group of teachers from 2006 seemed to embrace the idea of graphing more than any other group.

The relevance of Remillard’s (2005) framework is worth noting here. I have framed the categorization task as people (teachers, mathematics educators, and mathematicians) interacting with the Making Connections problem set. The outcomes of these interactions often revealed aspects of algebra (functions, graphs) which were not intended by the author/compiler of these curricular materials (Chris Norris) and which highlighted aspects of reformed algebra which were not always present when the participants spoke about algebra in the interviews.

Returning to the ideas of Section 4.2: Interacting with Algebra, I argued that the teachers drew heavily upon student activity in their analyses of the problems. It often was pedagogical content knowledge, thinking about the way a student would or should solve a problem, which justified that a problem was about graphing for the teachers. This shaped their analyses more than their more formal definitions and discussions of algebra during their interviews.

This tension between competing views of algebra was present in the preceding analysis. It was also present throughout the categorization tasks. It was often hard for the participants to sort out the three potentially central objects of algebra: expressions, equations, and functions (which subsume graphs). For example, the mathematicians had discussions about whether particular problems were about equations or expressions:

If you look ahead at 8 and 9 it’s kind of interesting because they actually ask about expressions but there’s an implied equation. (Frasier Grant, 2008)

Beth Larson (high school teacher 2008) cited this as a difficulty for her students on number 11:

I’ll tell you what messed [my students] up and I would have never guessed this, the fact that [the answer choices] were expressions and not equations. I don’t know why that messed mine up.
Even Chris Norris, when discussing number 10, avoided discussing the interrelatedness of expressions, equations, and functions:

Ten seems to be, to almost be some sort of nexus between all three of these categories [expressions, equations, and functions]. Anyway, who knows?\(^5\) (2008)

Overall, the teachers put more of an emphasis on graphs than the mathematicians and mathematics educators. Functions, in and of themselves, were discussed less often (as compared to graphs) within the groups of teachers. In mathematicians’ discussions, functional ideas (when mentioned) often yielded to ideas about expressions, equations, or structure. The teachers’ focus on graphs was often justified by reference to student actions or pedagogical benefits even though it did not correspond to their own definitions of algebra or Chris’ intention for the problems. Although, some of the Making Connections problems were not easy to classify as about expression, equation, or function.

4.3.3 Mathematical Habits

In this section, I examine the student habits and activities which Making Connections participants valued as they interacted with the problem set and with the student work that was present at the workshops. One habit which all groups valued was student reasoning and sense-making as opposed to the application of algorithms. The following bit of dialog between two teachers from the 2006 workshop addresses their frustration that students are often resistant to the type of algebraic reasoning that they value.

\(^5\)Chris expressed, during the member check, that saying “Who knows?” was a means to close the discussion without going into excessive detail about the relationship between these three objects and that it was not an expression of confusion.
Sheila Eastman (HS): You know, that is what I noticed so much in looking through the papers from our school is if [the students] knew the algorithm they got the problem right. If they didn’t they’d, and they didn’t have a sense of reasoning out the problem if they didn’t know the algorithm. As much as we try, as much we work with kids to break them away from the algorithms, to make them, you know, just get a sense for the mathematics and they still, as much as we may think that we want to give them other options, I don’t know if they want them. They’re really into this. And yet they hate them you know because it’s more memorization that’s the thing.

Tammy Vonce (MS): But they want you to show them [the procedure]. I’ve had students say to me after explaining and really giving an in depth explanation, “Okay, but just show me how to do it”.

To convert from miles to kilometers, Abby takes the number of miles, \( m \), doubles it, then subtracts 20% of the result. Renato first divides the number of miles by 5, and then multiplies the result by 8.

\hspace{1cm} a. Write an algebraic expression for each method.

\hspace{1cm} b. Use your answer from part (a) to decide if the two methods give the same answer.

**Figure 4.13. Problem Number 8.**

Within the context of number eight (Figure 4.13), some of the mathematics educators spoke about a type of reasoning which did not depend on symbol use but should either be a part of algebra or precede algebra. They were the only group to specifically talk about algebraic reasoning without the use of letters standing for numbers.

Doug Garrett: Number eight, yeah. I mean, you know, for me, well let’s see, Abby doubles it and subtracts 20% so let’s see you’re at 200% then you’re da da da so you’re at 160%. Here divide by 5 then multiply by 8 well that’s the same as multiplying by 8 and dividing by 5 that’s 8/5 so 1.6. These are equivalent and I didn’t use any letters.

Susan Oster: Right, so the way the question’s worded it’s like forcing the students to write -
Doug Garrett: To write letters...

Susan Oster: An algebraic expression. Why? Why force?

Doug Garrett: My point is when we’re teaching algebra we, then again I’m generalizing, we typically don’t encourage... what some teachers might call non-algebraic thinking. To me it’s algebraic thinking, you know what I just described. But because it didn’t use letters some people might claim it’s not algebraic, okay? But I think that kind of thinking should be encouraged, you know, along the way as part of learning algebra, before learning algebra, as part of learning algebra, you know, whatever.

Susan Oster: It’s part of becoming mathematically proficient.

Doug Garrett: Right.

The word “reasoning” was used frequently throughout the workshops in multiple contexts. The 2006 teachers categorized all 12 problems as being about “mathematical reasoning” and three of the five categories created by the mathematics educators in 2006 were of the form “reasoning about [some mathematical object]”. Perhaps there is some vagueness which results from the breadth of usage of this word. However, for all Making Connections participants the term “reasoning” involved being able to solve a problem without totally relying on an algorithm, rule, or procedure.

But why were the Making Connections problems so much about reasoning? Why is “reasoning” invoked by problem number one which asks students whether arithmetic expressions such as “2 − 10 + (−8)” are positive, negative, or zero? The reason typically cited amongst the Making Connections participants was that the problems explicitly asked for explanations such as “explain why you made the choice you did”. Problem number one (and numbers 6 and 12) explicitly asked students to explain how they could have “predicted” the answer without doing calculations or solving equations.
As referenced in the dialog between Sheila Eastman (HS) and Tammy Vonce (MS) above, asking a student to explain something does not always result in the student engaging in reasoning. In fact, during the Making Connections categorization task, within every professional group in both workshops, there was skepticism about why a student would bother to predict (as they were asked to do) the result of an expression as simple as “2 – 10 + (−8)”. In fact, there was criticism of all three problems which asked the student to predict. Chris explained his rationale for the “prediction” questions:

But it’s not even so much that I want [the students] to be able to predict without doing the math, I want them to have the habit of at least looking at the expression before they do anything, sort of contemplate before doing, and see if you notice anything. Sometimes you don’t notice anything and just do the math, other times you say “oh, obviously”. (categorization task, 2006)

Chris also described why he structured the problems in such a way that students “did the math” before they predicted:

I thought it would be irritating for students to tell them they had to do this [problem] without doing [the calculations], so that’s why they’re all structured where you get to just do the math first, but then you answer a question about how you might have been able to do it. (categorization task, 2006)

Some participants were critical of this strategy and saw no point in predicting something that had already been solved, this attitude was also evident in student answers.

Chris spoke about the moment at which a student would say “oh, obviously” in a variety of ways. In 2008, he referred to “the flash” of insight which he spoke about in regard to number 11 (Section 4.3.1). During the 2006 categorization task, the group of mathematicians adopted the term “leaping out” to describe noticing something about structure instantly and at an unconscious level. During our 2008 interview, Chris and I discussed the origins of this “flash”.
Chris Norris: [When I solve a problem] I’ve sort of got this whole narrative going, you know, of an action, of a calculation. I’m performing this calculation. And you perform the calculation you sort of force yourself to interpret everything. But I think some people look at algebraic expressions like pictures and you’re meant to extract something from that picture, right? You know, [to them] it’s not a dynamic object that’s describing something happening, . . .

Josh: So you talk about this internal narrative when you read these [problems] but you also talk about this flash, like you look at [the problem] and you notice something.

Chris Norris: I think it’s tied up together. I think you develop that ability to notice from repeating the internal narrative lots of times. If you look at a factored form and if you think about it as multiplying a whole bunch of numbers together and, oh gee, it would be nice if one of them was zero because that’s what I’m looking for. That’s the sort of narrative. And after a while you don’t need to tell the whole story anymore. You’re seeing that characteristic of it, it’s still a dynamic characteristic of the factor. It’s its property of possibly being zero and then making the whole expression zero.

Josh: So the flash is something that needs to be cultivated?

Chris Norris: Right.

Josh: It’s not something that, as a mathematician, you have special access to?

Chris Norris: No, and my guess, and this is a research question, but my guess is that it’s cultivated by dynamic activity with that expression repeatedly.

Chris was not the only one who spoke about this expert-level skill of unconsciously noticing salient characteristics of algebraic expressions. Beth Larson (HS teacher 2008) spoke about how she naturally analyzes equations and finds patterns. Judith Hardy (MS teacher 2008) similarly spoke about immediately spotting equivalent expressions. They both mentioned the challenges of encouraging these habits in their students. Tracy Coleman
(mathematician 2006) warned that the encouragement of this sort of “noticing” should not be done to the detriment of developing computational proficiency.

The mathematics educators spoke of a different way for students to solve problems non-procedurally. Kevin Martin (2006) spoke about “laziness” as a virtue in students. By this he meant inventing a strategy specific to the problem which would be easier than the standard algorithmic procedure. This theme was picked up upon by the entire group of mathematics educators in 2006.

Across all groups there was an emphasis on student reasoning as a desirable approach to the Making Connections problems as opposed to using standard procedures. The Making Connections problems were, for the most part, perceived as intended to promote student reasoning largely because the problems explicitly asked for explanations. The mathematics educators spoke about some specific forms of reasoning, they valued algebraic reasoning in the absence of symbols and they valued invented strategies. A more nebulous habit which Chris and others spoke of was the expert’s gaze through which important structural aspects could be noticed unconsciously and instantly. (This “flash” of insight is surely related to structure sense as described in Section 2.1.) In general, there were not clear divisions between the disciplinary groups in their discussions of what sort of student habits and behaviors were valued.

4.3.4 Translating between English and Mathematics

In both 2006 and 2008, the groups of teachers (which shared no common members) categorized a large portion of the Making Connections problems as being about translating from English to mathematics. In 2006, five out of the twelve problems were categorized as such and in 2008 six problems were categorized at such. For example, number eight (Figure 4.13), which asked students to write expressions to convert from miles to kilometers, was categorized as being about “translating” by the teachers in both workshops.

On one level, “translating” refers to converting from a verbal representation to a sym-
bolic representation. However, I was intrigued by the communication of this idea through the phrase “translating between English and math”. Is there an idea of algebra or mathematics which might be underlying the use of this phrase or which might be inspired in a student who hears this phrase? Is the act of translating, itself, a part of mathematics?

<table>
<thead>
<tr>
<th>(i) Write an algebraic expression representing each of the given operations on a number ( b ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>(ii) Are the expression equivalent? Explain what this tells you.</td>
</tr>
<tr>
<td>(a) “Multiply by one fifth”</td>
</tr>
<tr>
<td>“Divide by one third”</td>
</tr>
<tr>
<td>(b) “Multiply by one fifth”</td>
</tr>
<tr>
<td>“Divide by five”</td>
</tr>
<tr>
<td>(c) “Multiply by 0.4”</td>
</tr>
<tr>
<td>“Divide by five halves”</td>
</tr>
</tbody>
</table>

**Figure 4.14. Problem Number 5.**

The following dialog from the teachers’ 2008 grouping activity is exemplary of the way in which they spoke about “translating”. They are referencing problem number five (Figure 4.14) which both the 2006 and the 2008 groups of teachers categorized as “translating”.

| Judith Hardy (MS): | And I think it goes back to what you were saying about converting the language of English to the language of math too. That you have to have, I think, a pretty decent grasp of English and then be able to convert that. |
| Nadya King (HS): | Yeah, because we were portraying equivalent expressions again. But before we can even get to that comparison we have to dig through all those words. |

In the group discussions in both 2006 and 2008, the phrase “translating from English to math” was picked up upon and used by members of all disciplinary groups. During the problem categorization tasks in 2006, the mathematicians brought up translating twice (numbers 5 and 8) and the mathematics educators mentioned this sort of “translating” once
(number 7). The phrase itself was never explicitly discussed during the workshop. However, in 2008, I asked about “translating from English to mathematics” in the interviews. The mathematicians all agreed that the phrase is potentially misleading since they believe that this sort of translation is *in itself* a mathematical activity. The responses amongst the teachers were a bit more mixed (recall that no mathematics educators were interviewed).

Keith Young (HS teacher 2008) spoke about translation in a way that perhaps evoked an image of a boundary between mathematics and English where the mathematical activity happens on just one side of the boundary.

Josh: But you could argue that the math *is* doing that translation. It’s not like it becomes [math] as a process of it, but the math is the process.

Keith Young: Mmmm, you see I understand what you’re saying there but the main reason for doing that translation is so you can easily do the mathematical manipulations that you need to do. But then once you’ve manipulated it, whatever form you’ve manipulated it in, you’ve got to be able to take that and put it back… into English terms.

Greg Davis (HS teacher 2008), after a similar prompt, more explicitly expressed a sentiment that the mathematics begins after the verbal representation has been translated:

I see what you’re saying but I don’t think necessarily mathematics is being able to translate, I think a skill you need in order to do mathematics well is to be able to translate.

So when people use the phrase “translating between English and math” there can be a range of views of mathematics which are underlying it. They may think that the mathematical activities begin after the translating has been completed or may feel that the translation *is* a mathematical activity. The mathematicians in 2008 all agreed with the latter analysis though the 2008 group of teachers expressed more varied opinions.
4.3.5 Summary - Views of Algebra

Chris Norris emphasized the ideas of structure and expressions and equations in his view of algebra. These were his foci as he created the Making Connections problem set and he consciously de-emphasized graphs and functions. The interactions between workshop participants and the problem set did not always reflect these views of algebra.

Especially amongst the teachers in both the 2006 and 2008 workshops, ideas about graphs often dominated their analyses and their discussions. However, all but one teacher interviewed during the 2008 workshop defined algebra in a traditional way which emphasized generalized arithmetic. This indicates that, in their interactions with the Making Connections problems, it was aspects related to students’ algebraic activity that drove their decision making about the problems more than the way they formally defined algebra in the interviews. There were features of the problems and traits of the teachers which, in their interactions, often expressed this “hybrid” view of algebra.

The teachers, in the interviews and in the workshop, often described graphs as pedagogically productive objects for students. Their views of algebra were more mediated by student activity and by pedagogical content knowledge than were the views of mathematicians (mathematics educators were not interviewed). Differences in views of algebra may also be underlying the use and interpretation of the phrase “translating between English and math”.

All Making Connections participants favored student reasoning over reliance on algorithms and procedures. A mathematical habit which was frequently endorsed by Chris and expressed in different language at various times by other participants is the expert “flash” of structural insight into an algebraic object. However, at times it was difficult for Making Connections participants to merely choose which algebraic object was the focus of a particular Making Connections problem.
4.4 Conclusion

In this chapter, I examined the following questions by focusing on Making Connections participants’ interactions with the set of 12 algebra problems.

1. What knowledge did the participants draw from as they analyzed the Making Connections algebra problems? How did they justify their claims related to what a problem is about? What made a question difficult/easy?

2. What views of algebra and of mathematics informed the participants’ interactions with the content? What mathematical habits did the participants value and notice in student work?

The results reported on in the preceding sections do not constitute comprehensive answers to these questions. They do, however, offer insight into the mechanisms by which mathematicians, teachers, and mathematics educator may interact with and make sense of algebra content. For each of the above questions, I offer the following partial answers:

*What knowledge did the participants draw from as they analyzed the Making Connections algebra problems? How did they justify their claims related to what a problem is about?*

Each of the three professional groups drew from both subject matter and pedagogical content knowledge. However, as compared to the other two professional groups, the teachers’ analyses of the problems was more heavily guided by drawing upon knowledge of students and classrooms. Furthermore, unlike the other two groups, teachers referenced student activity as a means to justify claims about the categorization of problems.

*What made a question difficult/easy?* The teachers negatively viewed the impact of the unfamiliar wording of the Making Connections problems on students’ ability to solve the problems.

*What views of algebra and of mathematics informed the participants’ interactions with the content?* The teachers did not focus on structure as much as the mathematicians or the mathematics educators. Teachers focused on graphing more than the other two groups but
they, for the most part, expressed traditional views of algebra during the 2008 interviews as did the mathematicians. Inquiry into the phrase “translating from English to math” revealed different perspectives on “when or where the mathematics begins”. Specific ways in which these views of algebra may have informed the participants’ interactions with the Making Connections problems will be discussed in Chapter 5.

What mathematical habits did the participants value and notice in student work? All three professional groups valued reasoning; that is, approaches to problems which did not totally rely upon memorized procedures. Participants spoke, at times, about an expert gaze which includes the unconscious and often instantaneous recognition of structure in algebraic objects.

In general, I discovered both diversity within professional groups and similarities across groups. Looking across disciplinary groups, there were differences in the extent to which interactions with the Making Connections problem set were framed by students. There were also differences in what was noticed, whether it be structure in the algebra or the wording of the problems. In the next chapter I more explicitly connect these observations to differences in the views of algebra expressed in the workshop.
Chapter 5

IMPLICATIONS

The preceding chapter was organized according to two main themes: (1) Interacting with Algebra and (2) Views of Algebra. This division was organizationally convenient but was not entirely organic. Making Connections participants’ interactions with the algebra content were framed by their views of algebra. Conversely, their views of algebra were potentially influenced by the interactions with the Making Connections content. In this section I will explore some of these ideas and connect them to the literature which was reviewed in Chapter 2.

5.1 Wording and Translation

Throughout the Making Connections workshops I observed interactions which served to complexify the relationship between the written and the understood. It was the wording of the problems which was often cited, especially amongst the teachers, as a source of confusion and difficulty for the students. In parallel to this, the teachers from both workshops categorized about half of the problems as being about “translating from English to Math”. At times, the remaining computational tasks (after the “translation”) were described as “doing the math”.

The teachers’ greater focus on issues related to wording and translation is perhaps associated with a greater knowledge of content and students (KCS). This may contrast with what Bass (1997) described as a domain-centric approach by mathematicians to the problems of education. Indeed, this domain-centricity may have manifested in the greater focus on structure and the lesser focus on students amongst mathematicians and mathematics educators as they interacted with the Making Connections problems. Bass warned that this approach “implicitly demeans the importance and substance of pedagogy” (p. 20). It should
be noted that the present study has presented no evidence that the teachers’ greater emphasis on KCS is in any way associated with deficient mathematical subject matter knowledge. On the contrary, in the view of Hill, Ball, and Schilling (2008) knowledge of content and students is an essential component of teacher knowledge. Additionally, the teachers were drawing upon their direct knowledge of how their students interacted with the Making Connections problems. Their access to this knowledge was unique among the three professional groups and surely was a prominent factor in the observations made throughout this dissertation.

However, in this section I focus more on beliefs about, rather than knowledge about, algebra which may have contributed to the focus on “translating from English to math”. As discovered in the 2008 interviews, there were two prominent ways to conceptualize the act of “translating”: (1) it is what a student needs to do in order to get to the mathematics (i.e., the translation is not mathematics), or (2) it is, in itself, a mathematical activity. The second of these perspectives was endorsed by all of the mathematicians yet there was variation within the group of teachers. Cutting across these two perspectives is the student activity of converting from a verbal representation to a symbolic representation; the difference is in how this activity relates to and fits in with ideas about mathematics. As Kieran (2007) reported, a focus on multiple mathematical representations is associated with a reformed view of algebra and the representation itself can serve as a source of meaning for an algebra problem. Within this context, the construct of representational fluency (Heid & Blume, 2008; Nathan & Kim, 2007) is relevant.

The wording of the Making Connections problems was, at times, described as an obstacle to students’ success. Indeed, this perception is consistent with Nathan and Koedinger’s (2000b) finding that (7th through 12th grade) teachers and mathematics educators perceived verbally presented problems to be more difficult than symbolically presented problems. Perhaps their description of this “symbolic-precedence” is related to a particular view of mathematics or of algebra; this possibility could be the focus of future study. Within the Making Connections workshop, one manifestation of the criticism of the problems’ word-
ing was Tom Luft’s (MS teacher 2006) clarification of the problems for his students. That is, because of the problematic wording/directions, he “would clarify the directions because we wanted to get... data on the problems not on [the students’] ability to interpret the directions on the problems”. Here, Luft was perhaps communicating an opinion that the wording of a problem is largely separate from the mathematics of the problem. This may correspond to the view that mathematics begins after the translation from English.

Certainly, a student who solves the “clarified” problem has engaged in a different task than a student who solves the originally-worded problem. The question, however, is whether or not the student has engaged in a different mathematical task. The answer is perhaps, among other things, dependent on an individual’s view of mathematics. If the wording of a problem is an obstacle then, for some, the problem may be about the wording; that is, the problem may be about “translating from English to math”.

It is possible that the teachers’ concerns about the wording of the Making Connections problems were, at least in part, byproducts of the unusual circumstances through which the Making Connections problems were introduced to their classrooms. First, the problem set was distributed to teachers with no specific instructions for implementation; they were free to do what they pleased with the problems. Furthermore, there was not much specificity about the how this student work would be used in the workshop; the resulting student work was described by Chris Norris as a means to help frame a cross-disciplinary content analysis. Also, the problem set was not a part of their regular curricular materials and may have had significant style or content differences from their usual classroom resources. In particular, there was no framing or explanation of the problems which indicated the intended mathematical purposes. In this sense, there may have been something novel about these problem and the problems may have been quite far removed from what the students typically interacted with. So, for example, Tom Luft’s “clarification” of the problems may not have reflected a particular view of algebra. Rather, it may have been nothing more than a practical means through which to produce materials for the workshop; he may or may not have believed that he was changing the mathematics of the problems by clarifying them.
Nonetheless, the interviews with teachers from 2008 (Luft was in the 2006 group which was not interviewed) did reveal varying opinions about the relationship of “translating” to mathematics.

But if we accept the view that “translating” is a part of mathematics, then any sort of clarification (regardless of the motivation) needs to be done with care. Hiebert and Grouws (2007), in their survey of literature on teaching for conceptual understanding, cited as a key feature “the engagement of students in struggling or wrestling with important mathematical concepts” (p. 387). Is “translating from English to math” an “important mathematical concept” with which students should be allowed to struggle? It is my opinion that it is. In its communication standard, the NCTM (2000) notes that “because mathematics is so often conveyed in symbols, oral and written communication about mathematical ideas is not always recognized as an important part of mathematics education” and endorses “speaking, writing, reading, and listening in mathematics classes” (p. 59). This viewpoint is compatible with the idea that “translating” is a mathematical activity. In fact, the NCTM makes this same word-choice (“translation”) when addressing the conversion between representations; for example, in their discussion of the Representation strand, they write about a word problem “translated into a mathematical representation” (p. 69). Although, the status of translating as a mathematical activity is not directly discussed.

The phrase “translating from English to math” may, in itself, communicate a viewpoint that mathematics begins after the translation is complete. During the Making Connections teachers’ presentations about their students’ work on the problems, there were instances where Making Connections participants would look at a student’s work and say something like, “This student understood the mathematics but just couldn’t put it in writing.” For example, during the teachers’ presentations in 2006, Tom Luft (MS teacher) commented:

I have those kids that know how to do stuff but they can’t explain it. They’re not verbal and they’re not linguistic but they are so mathematical, they can just do stuff.
Is this a description of students’ difficulties “translating from math to English”? Regardless of the view of mathematics held by a teacher, mathematician, or mathematics educator who says phrases such as “translating from English to math” or “the student understood the mathematics but couldn’t write it”, these phrases may imply a divide between the written and the understood in mathematics, between the wording of a problem (as not being mathematics) and the computations required by the problem (as mathematics).

But what is the appropriate way to discuss “translating”? Is it a skill or ability which is subsumed by the concept of transformational fluency? Is it a component of mathematical knowledge that needs to be labeled and made explicit? If we accept the view that “translation” is a part of mathematics, then what is an appropriate way to describe a students’ ability to take a mathematical approach to a verbally described situation? If converting back-and-forth between verbal and symbolic representations is a mathematical activity, then I fear that phrases like “translating between English and math” and “you understand the math but just can’t write it” may communicate something to the contrary to students. It may communicate that the mathematics portion of a word problem consists of the rule-based computations and manipulation. If a teacher holds a view that mathematics does not include “translating” then how does that affect her/his pedagogical decisions? For instance, how do teachers working with English language learners make use of English word problems? And do the decisions they make reflect or endorse the particular views of mathematics just described? These are all potential topics for future study.

At the very least, the present research has shown that the use of the phrase “translating between English and math” can correspond to varying views about which parts of a word problem are mathematical. Furthermore, the wording of a problem may be perceived as an obstacle to student engagement with mathematics rather than as a part of the mathematics. However, it would be useful to unpack and to characterize the ways in which wording may present obstacles. For instance, if a problem is difficult because the wording is unfamiliar then that has different consequences than if the wording is plainly inaccurate. Indeed, interpreting problems which use unfamiliar wording or symbols is an activity which many
students are often required to do on standardized assessments. The ability to flexibly translate from verbal to symbolic representations may be undermined if students are not given opportunity to constructively struggle with unfamiliar wording or if they have a view of mathematics which does not include “translation” as mathematical skill or knowledge.

In this section, I have related particular views of algebra to interactions with the algebra content. I described how an individual’s perception of “translating” as a mathematical activity may be related to how she or he perceives unfamiliar wording in an algebra problem. This perception may ultimately lead to decisions about curricular planning and enactment such as “clarifying problems”. The following section continues this examination of the relationship between interactions with and views of algebra; it is a discussion of how the results of Chapter 4 may contribute to an understanding of mathematicians’, mathematics educators’, and especially teachers’ interactions with algebra.

5.2 Understanding Interactions with Algebra

The present study is largely framed by Remillard’s (2005) idea that the teachers interact with curricular materials. I have extended this idea to view the Making Connections categorization tasks as mathematicians, mathematics educators, and teachers interacting with the Making Connections problem set. By framing these activities as interactions, my intention has been to highlight that the nature of the Making Connections problems was determined by the participatory relationship (as Remillard calls it) between the workshop participants and the problem set itself. This relationship is influenced by characteristics of the individual (beliefs, pedagogical content knowledge, etc.) and by features of the curriculum (voice, representation of concepts, etc.).

Remillard’s (2005) framework for teacher-curriculum relationship (Figure 2.1) is intended as a framework for studies on curriculum use. The Making Connections workshops were not exactly about curriculum use. Although the participating teachers had implemented the problem set in their classrooms, the decisions they made regarding implemen-
tation were neither the focus of the workshops nor of the research and analysis described herein. However, the results described in Chapter 4 are relevant to curriculum use; though the categorization task was removed from the typical environment and context in which teachers use curriculum, the Making Connections workshops provided insight into the ways in which teachers interact with curricular materials. For example, teachers made sense of the problem set with frequent reference to and guidance by their perceptions of students (as discussed in Section 4.2.2).

Remillard (2005) calls for more research into several aspects of her framework. How do teacher characteristics “contribute to, shape, or constrain” the teacher-curriculum relationship (p. 237)? How do curriculum features affect these interactions? How can we characterize variations in interpretation? In what ways are some interpretations more reasonable than others? Indeed, these broad questions may be partly informed by the specific research results which have been addressed in this dissertation. For example, the previous section explored how particular beliefs about algebra may influence interactions with novelly worded problems. Moreover, the present study has broadened the scope of the teacher-curriculum framework to include interactions of mathematicians and mathematics educators with the Making Connections problems. This view across professional groups allows for comment about the dispositions of members of all three groups and puts in relief some characteristics which were more pronounced within the groups of teachers in the context of the Making Connections workshops. Furthermore, this broad scope may inform collaborations and communication across professional groups.

Though diversity was observed within and between all groups during the Making Connections workshops, the present study provides opportunities to comment on teachers’ interactions with the Making Connections algebra problems as compared to the other two professional groups and to make connections to their views of algebra. For example, in the preceding section, issues with the wording of problems were discussed in relation to the views of algebra and mathematics which may be associated with the phrase “translating between English and math”. But how did the teachers’ views of algebra relate to their
greater emphasis on students during the categorization tasks? I now address this question.

Chris Norris, the compiler/author of the Making Connections problem set, did not consider graphing to be a prominent theme in the 12 problems or an important piece of algebra in general; though he did acknowledge that graphs can be useful tools for doing algebra. However, both the 2006 and 2008 groups of teachers spoke about graphing in relation to five of the problems. For Chris, a major theme within the Making Connections problem set was the structure of expressions and equations. This theme was discussed amongst the mathematicians and mathematics educators but was generally not discussed amongst the teacher. It is perhaps necessary to clarify here that I do not present the opinions of Chris Norris in contrast to those of the teachers in order to make a statement about the correctness of the teachers’ interpretations of and interactions with the problem set. On the contrary, I wish to highlight the inevitability that interactions with curricular materials are interpretative acts; an individual encounters curricular materials through her or his own subjective filter. In light of this, questions about fidelity or correctness of interpretation are not relevant to my analysis and are potentially inconsistent with my perspective that Making Connections participants interacted with the 12 problems.

The teachers’ discussions of the 12 problems differed from Chris’ intentions (and from the discussions by mathematicians and mathematics educators) as they pertained to graphing and to structure. Is it possible that this is connected to the prominence of the roles of students in their analyses of the problems? For instance, structure was a focus for both the mathematicians and the mathematics educators in 2006 as they discussed and categorized problems 10 and 11. Teachers from both workshops discussed graphing and “translating” in relation to these problems which they perceived as difficult for the students. The Making Connections teachers often spoke about graphing in terms of its pedagogical efficacy and as a strategy for students to approach a problem. Tom Luft (middle school teacher 2006) discussed how students “have to have a visual connection”. Ursa Harper (middle school teacher 2006) described number 11 as “straight-forward” for students who have experience with graphing calculators. Thus, it is possible that the prominence of teachers’ percep-
tions of students in their interactions with the Making Connections problems was, in part, responsible for these differences with the mathematicians and mathematics educators. That is, their greater emphasis on graphing and their lesser emphasis on structure may have been related to their focus on how students would approach the problems.

In the interviews of the 2008 group of teachers, six out of seven described algebra in a traditional way (e.g., generalized arithmetic) as did Chris Norris and all of the interviewed mathematicians. However, graphs are generally associated with reformed approaches to algebra. By emphasizing graphs, were the teachers infusing the Making Connections problems with a view of algebra different from that intended by the problems’ author? The answer to this is unclear especially since the interviews did not reveal particularly reformed views of algebra. There is little evidence that the Making Connections teachers’ emphasis on graphing was a sign of a reformed, function-oriented view of algebra. This is certainly consistent with Doerr’s (2004) report that there is “no research evidence in the USA that would suggest that teachers see the concept of function as an integrating theme for algebra instruction across the curriculum” (p. 278). However, the Making Connections teachers’ perceptions of students certainly were factors in shaping their discussions of the nature and content of these problems especially as they pertained to graphing.

Chris’ intentions to focus on structure and to deemphasize graphs were not acknowledged within the teachers’ categorization tasks at both workshops. This, once again, highlights the interpretive nature of interactions with curricular materials. Furthermore, there are associated implications for developers of curricular materials. Chris’ development of the Making Connections problem set was guided by his opinions about the nature of algebra. The problems were distributed to the participating teachers with no explanation of his intentions or viewpoints and with no instructions for implementation. For the most part, his intended foci for the problem set were not implicitly communicated to the teachers through his presentation of the problems. In order to have successfully communicated his intentions through these curricular materials to the teachers it would have been advisable to have considered the role that student activity plays in teachers’ appraisals of algebra problems. At
least within the context of the Making Connections workshop, a domain-centric lens was insufficient for consistently communicating ideas to teachers through curricular materials.

Of the three professional groups at the Making Connections workshops, the teachers were the only group to have administered the problem set to students and were the most familiar with the student work. This surely was influential in their greater emphasis on student activity during the categorization task. Perhaps the results of this study would have been different had the teachers not implemented the problem set. However, this potential caveat is not particularly relevant to the discussion in this chapter; by the nature of their craft, teachers have access to information about student activity and are concerned with implementation of curricular materials. It is important for developers of curricular materials and for instructors of pre-service and in-service teachers to acknowledge that teachers’ interactions with algebra content may be largely framed by their relationships with and perceptions of students. It would be of use to study this idea further within the context of classroom curriculum use.

Thus far, in this chapter, I have connected aspects of the Making Connections participants’ interactions with the 12 algebra problems to their views of algebra. My focus has primarily been upon the teachers. During the categorization tasks, perceptions of students played a larger role for teachers than for the other two professional groups; I hypothesized that teachers’ greater emphasis on graphing may also have resulted from a perceived appreciation for the pedagogical efficacy of graphs perhaps more than from views of algebra which embrace functions as a central theme. Similarly, mathematicians’ and mathematics educators’ greater focus on structure in their interactions with the Making Connections problems may correspond to ideas about structure as a central concept in algebra. This was certainly true for Chris Norris.

I also noted that the unfamiliar wording of the problems may have affected the participants’ interactions with the problems; this focus on wording may overlap with ideas about the nature of the relationship between mathematics and English. There is much opportunity for further research on the relationship between an individual’s global/meta level ideas
about mathematics and how this affects her or his interactions with curricular materials. Conversely, there is a need for further understanding of how specific features of curricular materials may influence an individual’s ideas about mathematics and algebra. Similarly, Stein et al. (2007) have called for investigation of how particular features of curriculum affect how teachers use it. The present study perhaps offers some direction through the observation that successful communication with teachers through curricular materials may need to be conscious of (1) the influence of teachers’ perceptions of students and (2) ideas about the relationship between the wording and the mathematics.

Looking across professional groups, there were also similarities in the ways in which mathematicians, mathematics educators, and teachers interacted with the Making Connections algebra problems. All three groups drew from both subject matter knowledge and from pedagogical content knowledge (PCK). Hill, Ball, and Schilling (2008) offered an “unpacking” of these types of knowledge as part of their mathematical knowledge for teaching (MKT) framework; this could offer a finer-grained view of the knowledge which informed the interactions at the Making Connections workshop. For instance, within pedagogical content knowledge (PCK), all three groups drew upon knowledge of content and students (KCS) more so than any of the other components of PCK - knowledge of content for teaching (KCT) and knowledge of curriculum. But what sort of teacher knowledge is drawn upon when teachers present unfamiliar problems to their students? Specifically, what knowledge is drawn from if a teacher chooses to “clarify” the wording versus if she or he chooses to help a student to constructively struggle with the wording? The knowledge which would guide the subsequent implementation would surely be varied - specialized content knowledge, KCS, KCT, and knowledge of curriculum could all play roles. I have approached these issues from a different perspective by asking: What are the beliefs about (rather than knowledge of) algebra which inspire the teachers’ decision about whether to clarify or to allow students to struggle? The components of MKT are all essential types of teacher knowledge. However, mathematical knowledge for teaching can either be supported or undermined by the teachers’ beliefs about mathematics.
Constructs such as PCK and MKT for describing teacher knowledge have also been useful for discussing the knowledge of mathematicians and mathematics educators as they interacted with the Making Connections problem set. However, the primary goal of this section has been to discuss and problematize features of the participatory relationship between teacher and curricular materials. I have discussed perceptions of students, beliefs about algebra, and the wording/presentation of the problems as factors which help to define this relationship. In the next section, I return to an examination of views of algebra as a means to further comment on the interactions between individuals and curricular materials.

5.3 The Objects of Algebra

The landscape of school algebra is often depicted as divided between reformed and traditional versions of algebra (Kieran, 2007). Reformed algebra is typically associated with a focus on functions and graphs whereas traditional algebra focuses more on equations and expressions. Hybrid approaches which reflect both traditional and reform-oriented ideas may lead to confusion amongst students about such things as the distinction between equations and functions (Chazan & Yerushalmy, 2003). In Chapter 4, I discussed instances of confusion in determining which algebra-related object (expression, equation, or function) was the focus of certain Making Connections problems.

Some of these problems were structured in such a way that an expression was embedded in a sentence so that it defined a functional relationship (e.g., numbers 10 and 11 - Sections 4.3.1 and 4.3.2). There were no additional cues such as chapter/section headings or curricular guides to frame the problems for the teachers, mathematicians, and mathematics educators. The interpretations were varied and there was, at times, uncertainty in identifying whether a problem was about equations, expressions, or functions. At the very least, this indicates that sorting out these algebra-related objects is not always an easy task, even for the experts present at the Making Connections workshops. It is less clear whether or not this is a necessary task. For example, every member of the 2008 group of mathematicians
could answer problem number 10. Yet this group of experts deliberated over whether it was about expressions, equations, or functions. What is added or lost by explicitly labeling the focal algebraic object in problems like numbers 10 and 11?

The Making Connections problem set was encountered by students and workshop participants outside of the contexts of textbooks, pacing calendars, and curricula. This atypical presentation of algebra content provokes some questions. First, is the presentation of the Making Connections problems analogous to standardized testing in that neither may supply cues such as chapter headings to signal the focus of problems? If, for example, problem number 11 were to appear in an algebra textbook in a chapter called “Functions” does it constitute a different mathematical task than if it were to appear in chapters titled “Equations” or “Expressions” or “Word Problems”? Would the teachers and students have perceived or approached this problem differently?

The preceding questions are all potential inspiration for future study. With these questions in mind, however, the distinction between reformed and traditional algebra is perhaps easier to respect philosophically than in the practices of teaching or of interacting with algebra content. When I asked Chris “What is algebra?” during our interview, the content of his answer was consistent with his descriptions of the Making Connections problems. But the question of whether or not something qualifies as algebra is different from the question of whether or not it belongs in an algebra classroom. For example, the teachers cited graphs as part of an algebra class because they are pedagogically useful. Ian Stewart (mathematician 2008) described functions as part of an algebra class because they are on the “mathematical horizon” (Hill et al., 2008) as a part of calculus.

A theme throughout this chapter has been that mathematicians’, mathematics educators’, and teachers’ views of algebra shape their interactions with algebra content. Conversely, their interactions with content help to shape their views of algebra. These observations are not particular to algebra but, within the context of secondary mathematics, competing views of algebra are contested and “hybridized”. Kieran (2007) speculates that the majority of algebra programs are hybrid and calls for teaching experiments to help
students distinguish function approaches from equation approaches (i.e., reformed from traditional) and to meaningfully integrate the two approaches. Indeed, student flexibility and proficiency with both of these approaches is desirable. But the act of distinguishing between them was at times a difficult task even for the experts present at the Making Connections workshops. I caution that, as with number 10, it is possible that someone may understand and solve a problem but may not be able to distinguish whether it embodies an equation or function approach. A focus on making such distinctions could possibly distract from global/meta-level themes concerning the reading, interpretation, and understanding of algebra.

5.4 Cultivating the Flash

This section is a departure from the preceding discussion of the Making Connections participants' interactions with algebra content. Rather, I examine an idea which was raised by workshop participants regarding mathematical habits. Specifically, I consider some implications for the expert gaze which Chris and others spoke about as a means through which they approach algebra problems. Among other things, this was referred to as a “flash” of insight and was connected to instantaneously recognizing structure and patterns within the Making Connections problems.

At least for Chris, this was connected to his view that structure is a central theme in algebra. This emphasis on structure was explicitly expressed by other Making Connections participants during the 2008 interviews (mathematicians Ian Snyder and Frasier Grant). This perspective is also endorsed by Cuoco (2001) and Bass (2006) who, like Chris, are mathematicians with interests in mathematics education.

But noticing important structural aspects of algebraic objects “in a flash” may not be a pedagogically empowering concept. Yet it is surely the case that teachers do notice things “in a flash” which their students do not. During the 2008 workshop, Judith Hardy (MS teacher) and Beth Larson (HS teacher) acknowledged the difficulty of encouraging this skill
in their students. In Chapter 4 I related this to the idea of structure sense. It is also possibly connected to the “perceptual learning of ‘chunks’ that implicitly characterize the syntactic structure of expressions and equations” (p. 607) as described by Nathan and Koedinger (2004). It also could be related to Burton’s (1999b) description of intuition which she found to be unemphasized in the mathematics education literature but considered by most of the 70 mathematicians in her study as necessary for developing mathematical knowledge.

But how can this “flash” be cultivated? In the context of structure sense, Linchevski and Livneh (1999) have endorsed algebra instruction that “promotes the decomposition and recomposition of expressions” and that helps students make sense of “the mental gymnastics needed in manipulating expressions” (p. 191). Chris suggested that students may develop an expert gaze through repeating internal narratives in which students “interpret everything”. He described this as a dynamic activity which, through repetition, fosters what he describes as “flashes” of insight into algebraic structure. He described his inclusion of prompts like “Explain what this tells you.” and “Could you have predicted your answer?” as attempts to elicit internal narratives in students.

At some level, “repeating the internal narrative lots of times” may consist of students consistently engaging in sense-making activity as they do the “mental gymnastics” of manipulating equations and expressions. My literature review revealed few studies which examine the impact of specific teaching strategies on developing students’ structure sense (see Section 2.1.2). Exploring techniques for encouraging structure sense or for fostering the expert gaze which Chris and others described would certainly merit further study. However, such studies should be cautious that the expert practices of a research mathematician (Chris Norris) may not necessarily translate well to general pedagogical practices.

5.5 Limitations

In various parts of this dissertation I have presented the research results as relevant to curricular use, professional development, teacher training, and curricular development. Indeed,
the Making Connections workshops involved mathematicians, mathematics educators, and teachers interacting with curricular materials and such interactions may inform all of the activities listed in the previous sentence. However, the context of the workshops was artificial; the categorization task and other components of the workshop are not activities which are typical of curricular use/development or teacher training. I have mitigated this caveat by incorporating interview data, by connecting the findings to existing research, and by generally trying to embrace the complexity of the Making Connections data. In this section I explicitly describe limitations which arose from the context and the structure of the Making Connections workshops.

There were 25 total workshops participants from 2006 and 2008 combined. It is unclear to what degree these participants represented cross-sections of their professional groups. The size and compositions of these groups limit the generalizability of the study. Perhaps less tangible are concerns that arose from the specific way in which tasks were arranged in the workshops. As previously noted, the fact that only the teachers had administered the problems before the workshop certainly contributed to their greater focus on student activity. Moreover, the scheduling of the categorization task before the teachers’ reports on student work may have similarly influenced this finding. In fact, this schedule choice may have implicitly communicated the idea that there could be meaningful discourse about the Making Connections problems without much consideration for student work. That is, it may have endorsed a domain-centric approach particularly to the participants who had not used these problems in a classroom context (i.e., the mathematicians and mathematics educators). Thus, the structure of the workshop could have been partly responsible for the findings of this study.

There are also potential limitations related to the power differentials present at the workshops. Specifically, the workshops took place on a university campus; from the teachers’ perspectives it is possible that the workshop location and the status of the other participants as university professors may have influenced their interactions during the workshops. This is particularly true during the whole group and regional sessions when participants were
in cross-disciplinary groups (the categorization task was intra-disciplinary). Within all the professional groups, the presence of Chris Norris, as the developer of the Making Connections problem set, may have limited some criticism of the problems or may have resulted in deference to his opinions. Likewise, my association with the university as a graduate student and as a researcher may have had similar effects during the interviews.

The atypical context of the Making Connections workshops created limitations for the present study. The tasks which comprised the workshops did not exactly correspond to any of the practices that are typically associated with mathematicians, mathematics educators, or teachers. However, it was this unusual context which allowed for comparisons across professional groups and promoted sustained discourse about a novel set of algebra problems. Indeed, a usual context for curricular interactions may not have directly informed the research questions which I have addressed.

5.6 Final Remarks

In this dissertation, I have discussed perceptions of students, beliefs about algebra and mathematics, and the wording/presentation of problems as factors which partly determine teachers’, mathematicians’, and mathematics educators’ interactions with the algebra content at two Making Connections workshops. I have looked across and within professional groups in order to make some observations about how each group makes sense of algebra content and about their views of algebra and student interactions with algebra. I have discussed the impact in terms of communication across the professional groups; this includes curriculum developers communicating with teachers through curricular materials and communication across professions in the context of professional development and teacher training.

As mathematical scientists, as mathematics education researchers, and as teachers in universities, colleges, community colleges and schools, we must begin to see our concerns for graduate, undergraduate, and K12 education as parts of an integrated educational enterprise in which we have to learn to communicate
and collaborate across cultural, disciplinary, and institutional borders, just as we are called upon to do in mathematical sciences research. (Bass, 1997, p. 21)

The results and discussions which I have presented will, I hope, contribute to the successful communication and collaboration across professional groups. I am optimistic that positive impacts to teacher practice and knowledge can result from teachers’ interactions with thoughtful curricular materials. I am equally optimistic that improvements in algebra education can result from collaborations across professional groups which respect differing perspectives and acknowledge common interests and knowledge.


Appendix A

THE 12 PROBLEMS

1. Is the expression positive, negative, or zero?

<table>
<thead>
<tr>
<th>Expression</th>
<th>Positive</th>
<th>Negative</th>
<th>Zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 - 10 + (-8)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$10 - (-8) - 2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$8 - 10 - 2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-8 - 2 - (-10)$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For each expression, explain how you could have predicted your answer without doing the calculation, if possible.

- $2 - 10 + (-8)$
- $10 - (-8) - 2$
- $8 - 10 - 2$
- $-8 - 2 - (-10)$

2. You are simplifying $7 - 2(3 - 8x)$. Which of the expressions are possible results after the first step?

<table>
<thead>
<tr>
<th>Expression</th>
<th>Possible</th>
<th>Not Possible</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5(3 - 8x)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$7 - 2(-5x)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$7 - 6 - 16x$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$7 - 6 + 16x$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For each expression, explain why you made the choice you did.

- $5(3 - 8x)$
- $7 - 2(-5x)$
- $7 - 6 - 16x$
- $7 - 6 + 16x$
3. Which expressions could represent the perimeter of the polygon below?

\[ a + b^2 + c^4 + d \]

\[ abbcddd \]

\[ a + 2b + 4c + d \]

\[ ab^2c^4d \]

<table>
<thead>
<tr>
<th>Could represent the perimeter</th>
<th>Could not represent the perimeter</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ a + b^2 + c^4 + d ]</td>
<td></td>
</tr>
<tr>
<td>[ abbcddd ]</td>
<td></td>
</tr>
<tr>
<td>[ a + 2b + 4c + d ]</td>
<td></td>
</tr>
<tr>
<td>[ ab^2c^4d ]</td>
<td></td>
</tr>
</tbody>
</table>

For each expression, explain why you made the choice you did.

- \[ a + b^2 + c^4 + d \]
- \[ abbcddd \]
- \[ a + 2b + 4c + d \]
- \[ ab^2c^4d \]

4. In the expressions below, \( a \) and \( x \) are positive numbers. For each expression explain the effect of increasing \( a \): does the value of the expression increase, decrease, or remain unchanged?

<table>
<thead>
<tr>
<th>( ax + 1 )</th>
<th>Increases</th>
<th>Decreases</th>
<th>Remains unchanged</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x + a )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x - a )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( a + x - (2 + a) )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For each expression, explain why you made the choice you did.

- \( ax + 1 \)
- \( x + a \)
- \( x - a \)
- \( a + x - (2 + a) \)
5. In (a)-(c),

(i) Write an algebraic expression representing each of the given operations on a number $b$.

(ii) Are the expressions equivalent? Explain what this tells you.

(a) “Multiply by one fifth”
“Divide by one third”
(b) “Multiply by one fifth”
“Divide by five”
(c) “Multiply by 0.4”
“Divide by five halves”

6. Say whether each equation has a positive solution, a negative solution, a zero solution, or no solution.

<table>
<thead>
<tr>
<th></th>
<th>Solution is positive</th>
<th>Solution is negative</th>
<th>Solution is zero</th>
<th>No solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$7x = 5$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>$3x + 5 = 7$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>$5x + 3 = 7$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>$5 - 3x = 7$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>$3 - 5x = 7$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>$8x + 11 = 2x + 3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>G</td>
<td>$11 - 2x = 8 - 4x$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>H</td>
<td>$8x + 3 = 8x + 11$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>$8x + 3x = 2x + 11x$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Could you have predicted the answers for any of the equations without solving it? Which ones, and how?

7. You plan to drive 300 miles at 55 miles per hour, stopping for a two-hour rest. You want to know $t$, the number of hours the journey is going to take. Which of the following equations would you use?

(A) $55t = 190$   
(B) $55 + 2t = 300$  
(C) $55(t + 2) = 300$ 
(D) $55(t - 2) = 300$

Explain how you chose your answer.
8. To convert from miles to kilometers, Abby takes the number of miles, $m$, doubles it, then subtracts 20% of the result. Renato first divides the number of miles by 5, and then multiplies the result by 8.

a. Write an algebraic expression for each method.

b. Use your answer from part (a) to decide if the two methods give the same answer.

9. If the tickets for a concert cost $p$ each, the number of people who will attend is $2500 - 80p$. Which of the following best describes the meaning of the 80 in this expression?

A. The price of an individual ticket.
B. The slope of the graph of attendance against ticket price.
C. The price at which no-one will go to the concert.
D. The number of people who will decide not to go if the price is raised by one dollar.

Explain how you chose your answer.

10. A peanut, dropped at time $t = 0$ from an upper floor of the Empire State Building, is at a height, $h$, in feet above the ground $t$ seconds later given by

$$h(t) = -16t^2 + 1024$$

What does the factored form

$$h(t) = -16(t - 8)(t + 8)$$

tell us about when the peanut hits the ground?
11. A street vendor of t-shirts finds that if the price of a t-shirt is set at $p$, the revenue from a day's sales is \( p(900 - 60p) \). She wants to choose the price that will yield the greatest revenue. The best form of this expression for figuring what price to set is

A. \( p(900 - 60p) \)
B. \(-60(p - 7.5)^2 + 3375\)
C. \(-60p(p - 15)\)
D. \(900p - 60p^2\)

Explain how you chose your answer.

12. Say whether the equations have two solutions, one solution, or no solutions.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Two solutions</th>
<th>One solution</th>
<th>No solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3(x - 3)(x + 2) = 0)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((x - 2)(x - 2) = 0)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((x + 5)(x + 5) = -10)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((x + 2)^2 = 17)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((x - 3)^2 = 0)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3(x + 2)^2 + 5 = 1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-2(x - 1)^2 + 7 = 5)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-2(x - 1)^2 + 7 = 5)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Could you have predicted the answers for any of the equations without solving it? Which ones, and how?
# Appendix B

## 2008 Recording Guide

<table>
<thead>
<tr>
<th>Date</th>
<th>Title</th>
<th>Start</th>
<th>End</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>6/12/08</td>
<td>Mathematicians Content Analysis</td>
<td>09:30</td>
<td>10:30</td>
<td>handheld</td>
</tr>
<tr>
<td>6/12/08</td>
<td>Morning Intro &amp; Teachers Content Analysis</td>
<td>09:00</td>
<td>10:30</td>
<td>handheld</td>
</tr>
<tr>
<td>6/12/08</td>
<td>Morning Intro</td>
<td>09:00</td>
<td>09:30</td>
<td>mic system</td>
</tr>
<tr>
<td>6/12/08</td>
<td>Teachers Content Analysis</td>
<td>09:30</td>
<td>10:30</td>
<td>mic system</td>
</tr>
<tr>
<td>6/12/08</td>
<td>Whole Group Content Analysis</td>
<td>11:00</td>
<td>12:30</td>
<td>handheld</td>
</tr>
<tr>
<td>6/12/08</td>
<td>Whole Group Content Analysis</td>
<td>11:00</td>
<td>12:30</td>
<td>mic system</td>
</tr>
<tr>
<td>6/12/08</td>
<td>Whole Group - Teacher Presentations</td>
<td>01:30</td>
<td>03:30</td>
<td>handheld</td>
</tr>
<tr>
<td>6/12/08</td>
<td>Whole Group - Teacher Presentations</td>
<td>01:30</td>
<td>03:30</td>
<td>mic system</td>
</tr>
<tr>
<td>6/12/08</td>
<td>Whole Group - Teacher Presentations</td>
<td>00:00</td>
<td>00:00</td>
<td>handheld</td>
</tr>
<tr>
<td>6/12/08</td>
<td>Whole Group - Teacher Presentations</td>
<td>00:00</td>
<td>00:00</td>
<td>mic system</td>
</tr>
<tr>
<td>6/12/08</td>
<td>Larry Donahue Interview</td>
<td>05:15</td>
<td>05:45</td>
<td>handheld</td>
</tr>
<tr>
<td>6/13/08</td>
<td>Frasier Grant Interview</td>
<td>08:40</td>
<td>09:10</td>
<td>handheld</td>
</tr>
<tr>
<td>6/13/08</td>
<td>RegionalPlanningSession1 - West</td>
<td>09:15</td>
<td>10:30</td>
<td>handheld</td>
</tr>
<tr>
<td>6/13/08</td>
<td>RegionalPlanningSession1 - West</td>
<td>09:15</td>
<td>10:30</td>
<td>mic system</td>
</tr>
<tr>
<td>6/13/08</td>
<td>RegionalPlanningSession1 - MidAtlantic</td>
<td>09:15</td>
<td>10:30</td>
<td>handheld</td>
</tr>
<tr>
<td>6/13/08</td>
<td>RegionalPlanningSession1 - Southwest</td>
<td>09:15</td>
<td>10:30</td>
<td>handheld</td>
</tr>
<tr>
<td>6/13/08</td>
<td>RegionalPlanningSession2 - MidAtlantic</td>
<td>01:00</td>
<td>00:00</td>
<td>handheld</td>
</tr>
</tbody>
</table>

*Table B.1. 2008 Making Connections Audio Recordings - Part 1*
<table>
<thead>
<tr>
<th>Date</th>
<th>Title</th>
<th>Start</th>
<th>End</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>6/13/08</td>
<td>WholeGroup planning session morning</td>
<td>11:00</td>
<td>00:00</td>
<td>handheld</td>
</tr>
<tr>
<td>6/13/08</td>
<td>WholeGroup planning session morning</td>
<td>11:00</td>
<td>00:00</td>
<td>mic system</td>
</tr>
<tr>
<td>6/13/08</td>
<td>RegionalPlanningSession2 - West</td>
<td>01:00</td>
<td>00:00</td>
<td>handheld</td>
</tr>
<tr>
<td>6/13/08</td>
<td>RegionalPlanningSession2 - West</td>
<td>01:00</td>
<td>00:00</td>
<td>mic system</td>
</tr>
<tr>
<td>6/13/08</td>
<td>Judith Hardy Interview</td>
<td>12:00</td>
<td>12:30</td>
<td>handheld</td>
</tr>
<tr>
<td>6/13/08</td>
<td>Michael Chow Interview</td>
<td>12:30</td>
<td>00:00</td>
<td>handheld</td>
</tr>
<tr>
<td>6/13/08</td>
<td>Afternoon Introduction</td>
<td>00:00</td>
<td>00:00</td>
<td>mic system</td>
</tr>
<tr>
<td>6/13/08</td>
<td>RegionalPlanningSession2 - South-west</td>
<td>01:00</td>
<td>00:00</td>
<td>handheld</td>
</tr>
<tr>
<td>6/13/08</td>
<td>WholeGroup planning session 2 afternoon</td>
<td>03:30</td>
<td>05:00</td>
<td>handheld</td>
</tr>
<tr>
<td>6/13/08</td>
<td>WholeGroup planning session 2 afternoon</td>
<td>03:30</td>
<td>05:00</td>
<td>mic system</td>
</tr>
<tr>
<td>6/13/08</td>
<td>Greg Davis Interview</td>
<td>05:00</td>
<td>05:30</td>
<td>handheld</td>
</tr>
<tr>
<td>6/14/08</td>
<td>Kevin Lewis Interview</td>
<td>00:00</td>
<td>00:00</td>
<td>handheld</td>
</tr>
<tr>
<td>6/14/08</td>
<td>Nadya King Interview</td>
<td>00:00</td>
<td>00:00</td>
<td>handheld</td>
</tr>
<tr>
<td>6/14/08</td>
<td>Ian Snyder</td>
<td>00:00</td>
<td>00:00</td>
<td>handheld</td>
</tr>
<tr>
<td>6/14/08</td>
<td>Morning Intro - Whole Group</td>
<td>00:00</td>
<td>00:00</td>
<td>handheld</td>
</tr>
<tr>
<td>6/14/08</td>
<td>Morning Intro - Whole Group</td>
<td>00:00</td>
<td>00:00</td>
<td>mic system</td>
</tr>
<tr>
<td>6/14/08</td>
<td>RegionalPlanningSession3 - West</td>
<td>09:00</td>
<td>10:30</td>
<td>handheld</td>
</tr>
<tr>
<td>6/14/08</td>
<td>RegionalPlanningSession3 - West</td>
<td>09:00</td>
<td>10:30</td>
<td>mic system</td>
</tr>
<tr>
<td>6/14/08</td>
<td>RegionalPlanningSession3 - South-west</td>
<td>09:00</td>
<td>10:30</td>
<td>handheld</td>
</tr>
<tr>
<td>6/14/08</td>
<td>RegionalPlanningSession3 - MidAtlantic</td>
<td>09:00</td>
<td>10:30</td>
<td>handheld</td>
</tr>
<tr>
<td>6/14/08</td>
<td>WholeGroupPresentations</td>
<td>00:00</td>
<td>00:00</td>
<td>handheld</td>
</tr>
<tr>
<td>6/14/08</td>
<td>WholeGroupPresentations</td>
<td>00:00</td>
<td>00:00</td>
<td>mic system</td>
</tr>
<tr>
<td>6/14/08</td>
<td>Beth Larson</td>
<td>00:00</td>
<td>00:00</td>
<td>handheld</td>
</tr>
<tr>
<td>6/14/08</td>
<td>Keith Young</td>
<td>00:00</td>
<td>00:00</td>
<td>handheld</td>
</tr>
<tr>
<td>6/16/08</td>
<td>Shane Tyson</td>
<td>00:00</td>
<td>00:00</td>
<td>handheld</td>
</tr>
<tr>
<td>9/12/08</td>
<td>Chris Norris</td>
<td>00:00</td>
<td>00:00</td>
<td>handheld</td>
</tr>
</tbody>
</table>

Table B.2. 2008 Making Connections Audio Recordings - Part 2
<table>
<thead>
<tr>
<th>Date</th>
<th>Title</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>6/12/08</td>
<td>Morning Intro</td>
<td>Sony</td>
</tr>
<tr>
<td>6/12/08</td>
<td>Whole Group Content Analysis</td>
<td>Sony</td>
</tr>
<tr>
<td>6/12/08</td>
<td>Teachers Content Analysis</td>
<td>Sony</td>
</tr>
<tr>
<td>6/12/08</td>
<td>Mathematicians Content Analysis</td>
<td>Panasonic</td>
</tr>
<tr>
<td>6/12/08</td>
<td>Whole Group - Teacher Presentations</td>
<td>Sony</td>
</tr>
<tr>
<td>6/13/08</td>
<td>RegionalPlanningSession1 - West</td>
<td>Sony</td>
</tr>
<tr>
<td>6/13/08</td>
<td>RegionalPlanningSession1 - Southwest</td>
<td>Panasonic</td>
</tr>
<tr>
<td>6/13/08</td>
<td>RegionalPlanningSession1 - MidAtlantic</td>
<td>JVC</td>
</tr>
<tr>
<td>6/13/08</td>
<td>WholeGroup planning session 1 morning</td>
<td>Sony</td>
</tr>
<tr>
<td>6/13/08</td>
<td>Afternoon Intro</td>
<td>Sony</td>
</tr>
<tr>
<td>6/13/08</td>
<td>RegionalPlanningSession2 - West</td>
<td>Sony</td>
</tr>
<tr>
<td>6/13/08</td>
<td>RegionalPlanningSession2 - Southwest</td>
<td>Panasonic</td>
</tr>
<tr>
<td>6/13/08</td>
<td>RegionalPlanningSession2 - MidAtlantic</td>
<td>JVC</td>
</tr>
<tr>
<td>6/14/08</td>
<td>RegionalPlanningSession3 - West</td>
<td>Sony</td>
</tr>
<tr>
<td>6/14/08</td>
<td>RegionalPlanningSession3 - Southwest</td>
<td>Panasonic</td>
</tr>
<tr>
<td>6/14/08</td>
<td>RegionalPlanningSession3 - MidAtlantic</td>
<td>JVC</td>
</tr>
<tr>
<td>6/14/08</td>
<td>morning Intro</td>
<td>Sony</td>
</tr>
<tr>
<td>6/14/08</td>
<td>WholeGroupPresentations</td>
<td>Sony</td>
</tr>
</tbody>
</table>

*Table B.3. 2008 Making Connections Video Recordings*
Appendix C

INTERVIEW PROTOCOLS

C.1 Interview Protocol for All Participants

C.1.1 Algebra

- What is Algebra?
- What do you think of Chris’s problems? What did your students think?
- How would you characterize Chris’s problems? Was there a central theme that linked them? How do they fit in with what should be learned in school algebra? Do you think they represent a particular attitude about algebra?
- Some ideas about algebra came up when talking about these problems yesterday? How big a role do you think each of them plays in Algebra? (this will hopefully get at the transformational/generational distinction)
  - Graphs - Not present in Chris’s problems
  - Numerical representations - not present in Chris’ problems.
  - Expressions
  - Equations
  - Functions
  - Symbolic manipulation
  - Multiple representations
  - Real-world problems, “in context”
- What knowledge is necessary for a student to complete these questions?
• How did the questions fit into what you’ve done in your class throughout the year? Would you have used these questions otherwise? Why or why not?

C.1.2 Curriculum

Most of these questions pertain to teachers only (not mathematicians).

• Were these sorts of problems familiar to your students? How did you deal with problems that may have been unfamiliar to your students? How can you characterize the source of the unfamiliarity? (This may get at the “voice” of the questions.)

• Did you learn anything from these problems?

• Would you have otherwise used any of these questions in your class? Why or why not?

• Where would these questions fit into the curriculum/pacing calendar that you use?

• What role does curriculum (textbook, standards, pacing calendar) play in your classroom?

C.1.3 Professional Culture/ Beliefs

• What expertise do you feel you’ve brought to the interdisciplinary sessions?

• In general, what is your role relative to the other two groups here? How do you see that playing out in the interdisciplinary sessions?

• What is beneficial/challenging about working with these other disciplinary groups?

• How would you describe your philosophy on teaching math? How do you describe your role as mathematics teacher?

• What is mathematics? How do you think students learn it? How does this play out in your teaching?
C.2 Supplemental Questions for Chris Norris

• Why did many of these questions prompt the student to predict? Could you comment on the importance of predicting in the questions.

• How do you feel about criticisms of the wording of the questions? Is it sometimes not justified?

• Is it bad for students to be confused?

• Let’s go through the questions you wrote, tell me your motivation for each

  – In number 4, why do you use an $x$ as a constant?
  – The teachers hated the wording in number 5. Why do you think that is?
  – #7 - You write “You want to know $t$” but none of the equations are of the form “$t=$”. Why?
  – People mentioned using graphs or thinking about graphs to solve questions like #4 and #12. You’ve made no mention of graphs in your problem set. What do you see as the role of graphing in Algebra?
  – The teachers’ discussion of number 9 displayed a lot of confusion. Why do you think that is?
  – In numbers 10 and 11, you seem to be emphasizing the different “forms” of quadratics. In number 10 and parts of number 6 (part I) you’ve talked about “just staring” at it and seeing things “in a flash”. What did you mean by that?
  – In number 11, you have a list of expressions rather than something of the form “Revenue=...”. Also in #9 this is the case. Why?

• The teachers talked a lot about the language of the problems converting English to math and vice versa. Did this come up with the mathematicians? Do you see a dichotomy between language and mathematics like this?
• Themes to explore: Equivalent forms may be useful in different circumstances. Importance of standard forms and notation. Recognizing structure.

• You use expressions in many questions that represent the output of a functional relationship which is only implicitly defined. Why?

• Respond to this quote from Lockhart’s Lament (Lockhart, 2008): “No mathematician in the world would bother making these senseless distinctions: 2 1/2 is a ‘mixed number’, while 5/2 is an ‘improper fraction’ ” (p. 59).
Appendix D

SURVEYS

D.1 Mathematics Background Survey for Teachers

1. What is your educational background?

2. Where did you receive your K-12 education?

3. What is your current job?

4. How long have you held this position?

5. What is your employment history relevant to mathematics?

6. What type of training or professional development have you participated in as preparation for K-12 teaching?

7. What mathematics courses and grade levels do you currently teach?

8. What additional mathematics courses and grade levels have you taught in the past?

9. What was your primary motivation in your decision to become a mathematics teacher?

D.2 Mathematics Background Survey for Mathematicians

1. What is your educational background?

2. Where did you receive your K-12 education?

3. What is your current job?

4. How long have you held this position?
5. What is your employment history relevant to mathematics?

6. What is your current field of study/research interest?

7. What former research focuses have you had?

8. Do you regularly interact with K-12 teachers? If so, how?

9. In what other capacities have you had professional interaction with K-12 education?
Appendix E

\textbf{TexNicCenter}

The following figure is a screen shot of the upper left corner of TexNicCenter displaying the file “analysis.tex”. The panel on the left displays the hierarchical structure of the document and may be used for navigation purposes.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{TexNicCenter_screen_shot.jpg}
\caption{Partial screen shot of TexNicCenter.}
\end{figure}

\subsection{T06}
Did not discuss - only mention.

\subsection{2}
Yes, "the kids"!

\subsection{3}
TD: Okay. Next one I thougt.

*Spoke about their own students
ST: I had a few that identif.

\subsection{4}
TD: Yeah, yeah, I agree. Of you know, taking an algebra class...
know it makes them think the problems but as I look through

Appendix F

RESULTS OF THE CATEGORIZATION TASK

<table>
<thead>
<tr>
<th>Teachers</th>
<th>Math Educators</th>
<th>Mathematicians</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Signed Numbers (1,2,4,6)</td>
<td>• Reasoning about quantities (1,4)</td>
<td>1. Number &amp; Operations (1,2)</td>
</tr>
<tr>
<td>• Properties of Real #s (1,2,3,4,6)</td>
<td>• Standard meaning of algebraic notation (2,3)</td>
<td>2. Writing Expressions and Equations (3,5,7,8)</td>
</tr>
<tr>
<td>• Rational Number Operations (5,6,7,8)</td>
<td>• Equivalence of expressions (5,8)</td>
<td>3. Reading Expressions and Equations (4,6,9,10,11,12)</td>
</tr>
<tr>
<td>• Translating words into expressions/equations (5,7,8,10,11)</td>
<td>• Reasoning about equations (6,12)</td>
<td>3.1 Change (4,9)</td>
</tr>
<tr>
<td>• Quadratics (10,11,12)</td>
<td>• Reasoning about symbolic form(s) (7,9,10,11)</td>
<td>3.2 Purpose of Different Forms (10,11)</td>
</tr>
<tr>
<td>• Graph Theory changed to Reasoning from Graphs (6,9,10,11,12)</td>
<td></td>
<td>3.3 What form of an equation is helpful (6,12)</td>
</tr>
<tr>
<td>• Mathematical Reasoning (all)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table F.1. 2006 Groupings. The numbers in the parentheses are the question numbers included in each group.

<table>
<thead>
<tr>
<th>Teachers</th>
<th>Mathematicians</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Translating (4,5,7,9,10,11)</td>
<td>• Equations (6,7,10,12)</td>
</tr>
<tr>
<td>• Comparing Equivalencies (2,3,5,8,11)</td>
<td>• Static Expressions (1,2,3,5,8)</td>
</tr>
<tr>
<td>• Predicting (1,4,6,12)</td>
<td>• Dynamic Expressions (4,9,11)</td>
</tr>
</tbody>
</table>

Table F.2. 2008 Groupings. The numbers in the parentheses are the question numbers included in each group.
References


