ANALYSIS OF FAILURES OF DECODERS FOR LDPC CODES

by

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DEDICATION

To

Swapna, of course

And

Other family members

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ABSTRACT

Ever since the publication of Shannon's seminal work in 1948, the search for capacity achieving codes has led to many interesting discoveries in channel coding theory. Low-density parity-check (LDPC) codes originally proposed in 1963 were largely forgotten and rediscovered recently. The significance of LDPC codes lies in their capacity approaching performance even when decoded using low complexity sub-optimal decoding algorithms. Iterative decoders are one such class of decoders that work on a graphical representation of a code known as the Tanner graph. Their properties have been well understood in the asymptotic limit of the code length going to infinity. However, the behavior of various decoders for a given finite length code remains largely unknown.

An understanding of the failures of the decoders is vital for the error floor analysis of a given code. Broadly speaking, error floor is the abrupt degradation in the frame error rate (FER) performance of a code in the high signal-to-noise ratio domain. Since the error floor phenomenon manifests in the regions not reachable by Monte-Carlo simulations, analytical methods are necessary for characterizing the decoding failures. In this work, we consider hard decision decoders for transmission over the binary symmetric channel (BSC).

For column-weight-three codes, we provide tight upper and lower bounds on the guaranteed error correction capability of a code under the Gallager A algorithm as a function of the girth of the underlying Tanner graph. We accomplish this by

studying combinatorial objects known as trapping sets. For higher column weight codes, we establish bounds on the minimum number of variable nodes that achieve certain expansion as a function of the girth of the underlying Tanner graph, thereby obtaining lower bounds on the guaranteed error correction capability. We explore the relationship between a class of graphs known as cage graphs and trapping sets to establish upper bounds on the error correction capability.

We also propose an algorithm to identify the most probable noise configurations, also known as instantons, that lead to error floor for linear programming (LP) decoding over the BSC. With the insight gained from the above analysis techniques, we propose novel code construction techniques that result in codes with superior error floor performance.

CHAPTER 1

Introduction

If I have seen a little further it is by standing on the shoulders of Giants.

Sir Isaac Newton

1.1 Historical Background

As with any dissertation in information theory, we begin the historical background of coding and information theory with Shannon's work first published in 1948 [1]. Shannon proved that every communication channel has an associated *capacity*, and that reliable communication is possible at all rates up to the capacity and for communication at any rate above the capacity, the probability of error is bounded away from zero. Specifically, he showed that for all rates below the capacity, there exist a channel code and a decoding algorithm for which the probability of error can be made as small as possible. Shannon, however, did not provide explicit construction of such codes and since then the search for capacity achieving codes has led to many developments in the area of coding theory.

The initial developments in channel coding had a strong algebraic flavor, highlighted in the seminal work of Hamming [2] who was the first to propose linear block codes capable of correcting a single error. Hamming's work was followed by the discovery of multi-error correcting Bose-Chaudhuri-Hocquenghem (BCH) codes by Hocquenghem [3] and Bose and Ray-Chaudhuri [4]. Reed and Solomon [5] then

proposed the now ubiquitous Reed-Solomon (RS) codes, which are a special class of BCH codes. Efficient encoding and decoding algorithms for these codes that exploit the underlying algebraic structure have been studied in great detail [6]. Algebraic codes returned to the center of attention recently by the discovery of the so called *list decoding algorithms* by Sudan [7], which also allow soft-decision decoding of algebraic codes [8]. A more probabilistic flavor of channel coding can be found in the area of convolutional codes [6], which are generally decoded using the Viterbi algorithm [9]. However, both the above avenues did not lead to the construction of capacity approaching codes with low-complexity decoding algorithms.

The quest for capacity achieving codes bore fruit with the discovery of Turbo codes by Berrou, Glavieux, and Thitimajshima [10]. This was subsequently followed by the renaissance of low-density parity-check (LDPC) codes in the last decade. Gallager [11] proposed LDPC codes in his thesis which were largely forgotten for the next three decades, with Tanner's work [12] being the notable exception in which he laid the foundations for the study of codes based on graphs. MacKay [13] rediscovered LDPC codes and proved that there exist LDPC codes which achieve capacity under the sum-product algorithm. A wide variety of algorithms known in the areas of artificial intelligence, signal processing and digital communications can be derived as specific instances of the sum-product algorithm (the interested reader is referred to [14] for an excellent tutorial on this subject).

The significance of LDPC codes lies in their capacity approaching performance even when decoded by sub-optimal low complexity algorithms. One class of such decoders are the so-called iterative decoding algorithms, which include message-passing algorithms (variants of the belief propagation algorithm [15] and Gallager

type algorithms [11]), and bit-flipping algorithms [16] (serial and parallel).

By generalizing the seminal work of Gallager [11] to ensembles of codes, Richardson and Urbanke [17] analyzed LDPC codes under message passing decoders and showed that the bit error probability asymptotically tends to zero, whenever the noise is below a finite threshold. The authors of [17] also proposed the density evolution technique for computing the threshold and smaller signal-to-noise-ratio (SNR) behavior of LDPC codes performing over different decoders and channels. Richardson, Shokrollahi and Urbanke [18] used the density evolution approach for code optimization, specifically to derive capacity approaching degree distributions for irregular code ensembles. Bazzi, Richardson and Urbanke [19] derived exact thresholds for the binary symmetric channel (BSC) and the Gallager A algorithm.

Zyablov and Pinsker [16] were the first to analyze LDPC codes under bit flipping algorithms and demonstrated that regular LDPC codes with left degree greater than four are capable of correcting a fraction of errors. Sipser and Spielman [20] analyzed the bit flipping algorithms for LDPC codes using expansion based arguments. They also proposed a general class of asymptotically good codes known as expander codes, which were further analyzed and generalized by Barg and Zemor [21]. Burshtein and Miller [22] applied expander based arguments to message passing algorithms to show that these algorithms are also capable of correcting a fraction of errors. Recently, Burshtein [23] showed a vast improvement in the fraction of correctable errors for bit flipping algorithms.

The linear programming (LP) decoder is yet another sub-optimal decoder that has gained prominence in the recent years mainly due to its amenability to analysis. LP decoding was introduced by Feldman, Wainwright and Karger in [24] in which

they formulated the decoding problem as a linear program. Feldman and Stein [25] showed that LP decoding achieves capacity on any binary-input memoryless symmetric log-likelihood-ratio-bounded channel. Feldman *et al.* [26] used expander arguments to show that LP decoding corrects a fraction of errors. Daskalakis, Dimakis, Karp and Wainwright [27] considered probabilistic analysis of LP decoding to establish better bounds on the fraction of correctable errors.

1.2 Motivation and Problem Background

A common feature of all the analysis methods used in deriving the asymptotic results is that the underlying assumptions hold in the asymptotic limit of infinitely long code and/or are applicable to an ensemble of codes. Hence, they are of limited use for the analysis of a given finite length code. The performance of a code under a particular decoding algorithm is characterized by the bit-error-rate (BER) or the frame-error-rate (FER) curve plotted as a function of the SNR. A typical BER/FER vs SNR curve consists of two distinct regions. At small SNR, the error probability decreases rapidly with the SNR, with the curve looking like a *water fall*. The decrease slows down at moderate values turning into the *error-floor* asymptotic at very large SNR [28]. This transient behavior and the error-floor asymptotic originate from the sub-optimality of the decoder, i.e., the ideal maximum-likelihood (ML) curve would not show such a dramatic change in the BER/FER with the SNR increase. While the slope of the BER/FER curve in the waterfall region is the same for almost all the codes in the ensemble, there can be a huge variation in the slopes for different codes in the error floor region [29]. Since for sufficiently long codes, the error floor phenomenon manifests itself in the domain unreachable by brute force Monte-Carlo

simulations, analytical methods are necessary to characterize the FER performance.

Finite length analysis of LDPC codes is well understood for decoding over the binary erasure channel (BEC). The decoder failures in the error floor domain are governed by combinatorial structures known as stopping sets [30]. Stopping set distributions of various LDPC ensembles have been studied by Orlitsky *et al.* (see [31] and references therein for related works). Tian *et al.* [32] used this fact to construct irregular LDPC codes which avoid small stopping sets thus improving the guaranteed erasure recovery capability of codes under iterative decoding, and hence improving the error-floors. Unfortunately, such a level of understanding of the decoding failures has not been achieved for other important channels such as the BSC and the additive white Gaussian noise channel (AWGNC).

Failures of iterative decoders for graph based codes were first studied by Wiberg [33] who introduced the notions of *computation trees* and *pseudo-codewords*. Subsequent analysis of the computation trees was carried out by Frey *et al.* [34] and Forney *et al.* [35]. The failures of the LP decoder can be understood in terms of the vertices of the so-called *fundamental polytope* which are also known as pseudo-codewords. Vontobel and Koetter [36] introduced a theoretical tool known as the graph cover decoding and used it to establish connections between the LP decoder and the message passing decoders using the notion of the fundamental polytope. They showed that the pseudo-codewords arising from the Tanner graph covers are identical to the pseudo-codewords of the LP decoder. Vontobel and Koetter [37] also studied the relation between the LP and min-sum decoders. The performance of the min-sum decoder is governed by pseudo-codewords arising from computation-trees as well as graph covers.

For iterative decoding on the AWGNC, MacKay and Postol [38] were the first to discover that certain "near codewords" are to be blamed for the high error floor in the Margulis code [39]. Richardson [28] reproduced their results and developed a computation technique to predict the performance of a given LDPC code in the error floor domain. He characterized the troublesome noise configurations leading to error floor using combinatorial objects termed trapping sets and described a technique (of a Monte-Carlo importance sampling type) to evaluate the error rate associated with a particular class of trapping sets. The method from [28] was further refined for the AWGN channel by Stepanov *et al.* [40] who introduced the notion of *instantons*. In a nutshell, an instanton is a configuration of the noise which is positioned between a codeword (say zero codeword) and another pseudo-codeword (which is not necessarily a codeword). Incremental shift (allowed by the channel) from this configuration toward the zero codeword leads to correct decoding (into the zero-codeword) while incremental shift in an opposite direction leads to a failure. In principle, one can find this dangerous configuration of the noise exploring the domain of correct decoding surrounding the zero codeword, and finding borders of this domain – the so-called error-surface. If the channel is continuous, the error-surface consists of continuous patches while configuration of the noise maximizing the error-probability over a patch is called an instanton.

As stated above, the instantons that affect the decoder performance in the error floor region are extremely rare, and hence identifying and enumerating them is a challenging task. However once this difficulty is overcome, the knowledge of the trapping set/pseudo-codeword distribution can be used to evaluate the performance of the code. It can also be used to guide optimization of the code and design

improved decoding strategies.

Previous investigation of the problem includes the work by Kelley and Sridhara [41] who studied pseudo-codewords arising from graph covers and derived bounds on the minimum pseudo-codeword weight in terms of the girth and the minimum left-degree of the underlying Tanner graph. The bounds were further investigated by Xia and Fu [42]. Smarandache and Vontobel [43] found pseudo-codeword distributions for the special cases of codes from Euclidean and projective planes. Pseudo-codeword analysis has also been extended to the convolutional LDPC codes by Smarandache *et al.* [44]. Milenkovic *et al.* [45] studied the asymptotic distribution of trapping sets in regular and irregular LDPC code ensembles. Wang *et al.* [46] proposed an algorithm to exhaustively enumerate certain stopping and trapping sets.

Chernyak *et al.* [47] and Stepanov *et al.* [40] suggested to pose this problem of finding the instantons as a special optimization problem. This optimization method was built in the spirit of the general methodology, borrowed from statistical physics, guiding exploration of rare events which contribute the most to the BER/FER. The optimization method allowed to discover in [40], the set of most probable instantons for AWGN channel and iterative decoder. The operational utility of the method was illustrated on some number of moderate size examples and strong dependence of the instanton structure on the number of iterations was observed. The general optimization method was substantially improved and refined in [48] for the LP decoding over continuous channels (with main enabling example chosen to be the AWGNC). The pseudo-codeword-search algorithm of [48] was essentially exploring in an iterative way the Wiberg formula treating an instanton configuration as a median between a

pseudo-codeword and the zero-codeword. It was shown empirically that, initiated with a sufficiently noisy configuration, the algorithm converges to an instanton in sufficiently small number of steps, independent or weakly dependent on the code size. Repeated multiple times the method outputs the set of instanton configurations which can further be used to estimate the BER/FER performance in the transient and error-floor domain. (See also [49] for an exhaustive list of references for this and related subjects.)

An important outcome of all the work on error floors is the observation that the slope of the FER curve of a given code under hard decision decoding in the error floor region is related to the guaranteed error correction capability of the code. Hence, in this dissertation, we focus primarily on establishing lower and upper bounds on the number of errors that a decoding algorithm is guaranteed to correct. To this end, we primarily consider hard decision decoding algorithms but note that the general concepts are applicable to a wide variety of channels and decoding algorithms. Expander based arguments [20] provide the required lower bounds as it is known that a variety of decoding algorithms can correct a fixed fraction of errors if the underlying Tanner graph of a code is a good expander. It is well known that a random graph is a good expander with high probability [20], but the fraction of nodes having the required expansion is very small and hence the code length to guarantee correction of a fixed number of errors must be large. Moreover, determining the expansion of a given graph is known to be NP hard [50], and spectral gap methods cannot guarantee an expansion factor of more than 1/2 [20]. Also, expander based arguments cannot be applied to codes with column weight three, which are of interest in many practical scenarios owing to their low

decoding complexity. The approach in this dissertation is, therefore, to identify and analyze the failures of different hard decision decoding iterative algorithms for LDPC codes as a function of two important code parameters, namely column weight and girth of the underlying graph, which are easy to determine for any given code.

1.3 Contributions of this Dissertation

- We prove that a necessary condition for a column-weight-three code to correct all error patterns with up to $q \geq 5$ errors is to avoid all cycles up to length $2q$ in its Tanner graph representation. As a result, we prove that given $\alpha > 0$, at sufficiently large lengths, no code in the ensemble of regular codes with column weight three can correct α fraction of errors. These results have been reported in [51].

- We prove that a column-weight-three LDPC code with Tanner graph of girth $g \geq 10$ can correct all error patterns with up to $g/2 - 1$ errors under the Gallager A algorithm. This result, in conjunction with the necessary condition mentioned above, completely determines the slope of the FER curve in the error floor region. These results have been reported in [52].

- For higher column weight codes, we derive lower bounds on the number of variable nodes that achieve sufficient expansion as a function of the column weight and girth of the Tanner graph of the code. We also derive bounds on the size of the smallest possible trapping sets thereby establishing upper bounds on the guaranteed error correction capability. These results have been reported in [53].

- We illustrate how the knowledge of trapping sets can be used for (a) deriving trapping set/pseudo-codeword statistics for hard decision decoders and (b) construction of codes with superior error floor performance. These results have been reported in [54, 55, 56].

1.4 Organization of the Dissertation

In Chapter 2, we establish the notation and provide necessary background. We derive upper bounds and lower bounds on the error correction capability of column-weight-three codes under the Gallager A algorithm in Chapter 3 and Chapter 4 respectively. Expansion properties of Tanner graphs with higher column weight as a function of girth are derived in Chapter 5. We highlight the applications of the knowledge of trapping sets with illustrative numerical results in Chapter 6 and conclude with a few remarks in Chapter 7.

CHAPTER 2

Preliminaries

> We could, of course, use any notation we want; do not laugh at notations; invent them, they are powerful. In fact, mathematics is, to a large extent, invention of better notations.
>
> Richard P. Feynman (1963)

In this chapter, we provide a brief description of some fundamental concepts from coding theory. We start with a description of block codes and then proceed to define LDPC codes. We then discuss channel assumptions and decoding algorithms. We then proceed to describe decoding failures in general and highlight the importance of correction of a fixed number of errors and its relation to slope of the FER curve.

2.1 LDPC Codes

Definition 2.1. (Codes) We consider binary codes in this work.

- **(Block Code)** An (n, k) binary block code maps a message block of k information bits to a binary n-tuple. The rate r of the code is given by $r = k/n$.
- **(Linear Block Code)** An (n, k) binary linear block code, $\mathcal{C}$, is a subspace of $\mathrm{GF}(2)^n$ of dimension k.

- **(Parity-Check Matrix)** A parity check matrix $\mathbf{H}$ of $\mathcal{C}$ is a matrix whose columns generate the orthogonal complement of $\mathcal{C}$, i.e., an element $\mathbf{x} = (x_1, \ldots, x_n)$ of $\mathrm{GF}(2)^n$ is a codeword of $\mathcal{C}$ *iff* $\mathbf{x}\mathbf{H}^T = \mathbf{0}$. □

Definition 2.2. (Graphs) We adopt the standard notation in graph theory (see [57] for example).

- **(Graph)** A graph with set of nodes U and set of edges E is denoted by $G = (U, E)$. When there is no ambiguity, we simply denote the graph by G. The order of the graph is $|U|$ and the size of the graph is $|E|$.

- **(Neighborhood)** An edge e is an unordered pair $\{u_1, u_2\}$ of nodes and is said to be incident on u_1 and u_2. Two nodes u_1 and u_2 are said to be adjacent (neighbors) if there is an edge incident on them. The set of all neighbors of a node u is denoted by $\mathcal{N}(u)$.

- **(Degree of a node)** The degree of u, $d(u)$, is the number of its neighbors i.e., $d(u) = |\mathcal{N}(u)|$. A node with degree one is called a leaf or a pendant node. A graph is d-regular if all the nodes have degree d. The average degree $\overline{d}$ of a graph is defined as $\overline{d} = 2|E|/|U|$.

- **(Girth of a graph)** The girth $g(G)$ of a graph G, is the length of smallest cycle in G. When there is no ambiguity about the graph under consideration, we denote the girth simply by g.

- **(Bipartite graph)** $G = (V \cup C, E)$ denotes a bipartite graph with two sets of nodes; variable (left) nodes V and check (right) nodes C and edge set E. Nodes in V have neighbors only in C and vice versa. A bipartite graph is

said to be d_v-left regular if all variable nodes have degree d_v, d_c-right regular if all check nodes have degree d_c and (d_v, d_c) regular if all variable nodes have degree d_v and all check nodes have degree d_c. The girth of a bipartite graph is even. □

LDPC codes [11] are a class of linear block codes which can be defined by sparse bipartite graphs [12].

Definition 2.3. (LDPC Codes and Tanner Graphs) Let G be a bipartite graph with two sets of nodes: the set of variable nodes V with $|V| = n$ and the set of check nodes C with $|C| = m$. This graph defines a linear block code of length n and dimension at least $n - m$ in the following way: The n variable nodes are associated with the n coordinates of codewords. A binary vector $\mathbf{x} = (x_1, \ldots, x_n)$ is a codeword if and only if for each check node, the modulo two sum of its neighbors is zero. Such a graphical representation of an LDPC code is called the Tanner graph [12] of the code. The adjacency matrix of G gives $\mathbf{H}$, a parity check matrix of $\mathcal{C}$. □

It should be noted that the Tanner graph is not uniquely defined by the code and when we say the Tanner graph of an LDPC code, we only mean one possible graphical representation. In this work, • represents a variable node, □ represents an even degree check node and ■ represents an odd degree check node.

Definition 2.4. Code Ensembles

- **(Regular LDPC Codes)** A (d_v, d_c) regular LDPC code of length n denoted by (n, d_v, d_c) has a Tanner graph with n variable nodes each of degree d_v and nd_v/d_c check nodes each of degree d_c. This code has length n and rate $r \geq 1 - d_v/d_c$ [11].

- **(Ensemble)** **[17]** The ensemble of $\mathcal{C}^n(d_v, d_c)$of (d_v, d_c)-regular LDPC codes of length n is defined as follows. Assign to each node d_v or d_c "sockets" according to whether it is a variable node or a check node, respectively. Label the variable and check sockets separately with the set $[nd_v] := \{1, \ldots, nd_v\}$ in some arbitrary fashion. Pick a permutation π on nd_v letters at random with uniform probability from the set of all $(nd_v)!$ such permutations. The corresponding (labeled) bipartite graph is then defined by identifying edges with pairs of sockets and letting the set of such pairs be $\{(i, \pi(i)), i = 1, \ldots, nd_v\}$. □

2.2 Channel Assumptions

We assume that a binary codeword $\mathbf{x}$ is transmitted over a noisy channel and is received as $\mathbf{y}$. The support of a binary vector $\mathbf{x}$, denoted by $\text{supp}(\mathbf{x})$, is defined as the set of all variable nodes $v \in V$ such that $x_v \neq 0$.

Definition 2.5. Channels and Decoding

- **(Channel)** A channel is characterized by input alphabet $\mathcal{X}$, output alphabet $\mathcal{Y}$ and transition probability $\Pr(\mathbf{y}|\mathbf{x})$, the probability that $\mathbf{y} \in \mathcal{Y}$ is received given that $\mathbf{x} \in \mathcal{X}$ was sent.

- **(MAP Decoding)** The maximum a posteriori (MAP) decoding consists of finding the codeword $\mathbf{x} \in \mathcal{C}$ which maximizes $\Pr(\mathbf{x}|\mathbf{y})$, the probability that $\mathbf{x}$ was sent given that $\mathbf{y}$ is received.

- **(ML Decoding)** Under the assumption that all information words are equally likely, this is equivalent to ML decoding which finds a codeword $\mathbf{x}$ maximizing $\Pr(\mathbf{y}|\mathbf{x})$.

- **(Memoryless Channel)** A memoryless channel is a channel which satisfies

$$\Pr(\mathbf{y}|\mathbf{x}) = \prod_{v \in V} \Pr(y_v|x_v).$$

Hence it can be characterized by $\Pr(y_v|x_v), v \in V$, the probability that y_v is received given that x_v was sent.

- **(Log-Likelihood Ratio)** The negative log-likelihood ratio (LLR) corresponding to the variable node $v \in V$ is given by

$$\gamma_v = \log\left(\frac{\Pr(y_v|x_v = 0)}{\Pr(y_v|x_v = 1)}\right).$$

- **(BSC)** The BSC is a binary input memoryless channel with output alphabet $\{0, 1\}$. On the BSC with transition probability p, every transmitted bit $x_v \in \{0, 1\}$ is flipped [1] with probability p and is received as $y_v \in \{0, 1\}$. Hence,

$$\gamma_v = \begin{cases} \log\left(\frac{1-p}{p}\right) & \text{if } y_v = 0 \\ \\ \log\left(\frac{p}{1-p}\right) & \text{if } y_v = 1 \end{cases}$$

- **(AWGN Channel)** For the AWGN channel, we assume that each bit $x_v \in \{0, 1\}$ is modulated using binary phase shift keying (BPSK) and transmitted as $\overline{x}_v = 1 - 2x_v$ and is received as $y_v = \overline{x}_v + n_v$, where $\{n_v\}$ are i.i.d. $N(0, \sigma^2)$ random variables. Hence, we have

$$\gamma_v = \frac{2y_v}{\sigma^2}.$$

□

[1] The event of a bit changing from 0 to 1 and vice-versa is known as flipping.

2.3 Decoding Algorithms

In this section, we describe various decoding algorithms for decoding LDPC codes. We start with iterative decoders which can further be subdivided into message passing algorithms and bit flipping decoders. There is a wide variety of message passing algorithms; Gallager A/B, sum-product and min-sum to name a few. We then describe the LP decoder in the most general setting.

2.3.1 Message Passing Decoders

Message passing decoders operate by passing messages along the edges of the Tanner graph representation of the code. Gallager in [11] proposed two simple binary message passing algorithms for decoding over the BSC; Gallager A and Gallager B. There exist a large number of message passing algorithms (sum-product algorithm, min-sum algorithm, quantized decoding algorithms, decoders with erasures to name a few) [17] in which the messages belong to a larger alphabet.

Every round of message passing (iteration) starts with the variable nodes sending messages to their neighboring check nodes (first half of the iteration) and ends by the check nodes sending messages to their neighboring variable nodes (second half of the iteration). Let $\mathbf{y} = (y_1, \ldots, y_n)$, an n-tuple be the input to the decoder. Let $\omega_j(v, c)$ denote the message passed by a variable node $v \in V$ to its neighboring check node $c \in C$ in the j^{th} iteration and $\varpi_j(c, v)$ denote the message passed by a check node c to its neighboring variable node v. Additionally, let $\omega_j(v, :)$ denote the set of all messages from v, $\omega_j(v, : \backslash c)$ denote the set of messages from v to all its neighbors except to c and $\omega_j(: , c)$ denote the set of all messages to c. The terms $\omega_j(: \backslash v, c), \varpi_j(c, v), \varpi_j(c, : \backslash v), \varpi_j(: , v)$ and $\varpi_j(: \backslash c, v)$ are defined similarly.

At the end of each iteration, an estimate of each variable node is made based on the incoming messages and possibly the received value. The codeword estimate of the decoder at the end of j^{th} iteration is denoted as $\mathbf{y}^{(j)}$. To decode the message in complicated cases (when the message distortion is large) we may need a large number of iterations, although typically a few iterations would be sufficient. To speed up the decoding process one may check after each iteration whether the output of the decoder is a valid codeword, and if yes to terminate the decoding algorithm. In this work, we assume that the maximum number of iterations is M.

Gallager A/B Algorithm:

Gallager in [11] proposed two simple binary message passing algorithms for decoding over the BSC; Gallager A and Gallager B. With a slight abuse of the notation, let $|\varpi(:, v) = 0|$ and $|\varpi(:, v) = 1|$ denote the number of incoming messages to v which are equal to 0 and 1 respectively. Associated with every decoding round j and variable degree d_v is a threshold b_{j,d_v}. The Gallager B algorithm can then be defined as follows.

$$\omega_1(v, c) = y_v$$

$$\varpi_j(c, v) = \left(\sum_{u \in \mathcal{N}(c) \setminus v} \omega_j(u, c) \right) \bmod 2$$

$$\omega_j(v, c) = \begin{cases} 1, & \text{if } |\varpi_{j-1}(: \setminus c, v) = 1| \geq b_{j,d_v} \\ 0, & \text{if } |\varpi_{j-1}(: \setminus c, v) = 0| \geq b_{j,d_v} \\ y_v, & \text{otherwise} \end{cases}$$

The Gallager A algorithm is a special case of the Gallager B algorithm with $b_{j,d_v} = d_v - 1$ for all j.

Different rules to estimate a variable node after each iteration are possible and it is likely that changing the rule after certain iterations may be beneficial. However, the analysis of various scenarios is beyond the scope of this work. For column-weight-three codes only two rules are possible.

- Decision Rule A: if all incoming messages to a variable node from neighboring checks are equal, set the variable node to that value; else set it to its received value.

- Decision Rule B: set the value of a variable node to the majority of the incoming messages; majority always exists since the column weight is three.

We adopt Decision Rule A in Chapter 3 and Decision Rule B in Chapter 4 but note that the results in both the chapters hold for both the rules.

Sum-Product Algorithm

A decoding algorithm with a specific choice of how the messages are calculated from the channel output (the best possible one if messages are calculated locally in the Tanner's graph of the code) is called the sum-product algorithm. With a moderate abuse of notation, the messages passed in the sum-product algorithm are described below:

$$\begin{aligned} \omega_1(v,c) &= \gamma_v \\ \varpi_j(c,v) &= 2\tanh^{-1}\left(\prod_{u\in\mathcal{N}(c)\setminus v}\tanh\left(\frac{1}{2}\omega_j(u,c)\right)\right) \end{aligned}$$

$$\omega_j(v,c) \;=\; \gamma_v + \sum_{u \in \mathcal{N}(v)\setminus c} \varpi_{j-1}(u,v)$$

The codeword estimate at the end of j iterations is determined by the sign of $m_v^{(j)} = \gamma_v + \sum_{u \in \mathcal{N}(v)} \varpi_j(u,v)$. If $m_v^{(j)} > 0$ then $y_v^{(j)} = 0$, otherwise $y_v^{(j)} = 1$.

Min-Sum Algorithm

In the limit of high SNR, when the absolute value of the messages is large, the sum-product becomes the min-sum algorithm, where the message from the check c to the variable node v looks like:

$$\varpi_j(c,v) \;=\; \min_{u \in \mathcal{N}(c)\setminus v} \big|\omega_j(u,c)\big| \cdot \prod_{u \in \mathcal{N}(c)\setminus v} \operatorname{sign}\big(\omega_j(u,c)\big)$$

The min-sum algorithm has a property that the Gallager A/B and LP decoders also possess — if we multiply all the likelihoods γ_v by a factor, all the decoding would proceed as before and would produce the same result. Note that we don't have this "scaling" in the sum-product algorithm.

2.3.2 Bit Flipping Decoders

The bit flipping algorithms [16, 20] are iterative algorithms which do not belong to the class of message passing algorithms. We now describe the parallel and serial bit flipping algorithms. A constraint (check node) is said to be satisfied by a setting of variable nodes if the sum of the variable nodes in the constraint is even; otherwise the constraint is unsatisfied.

Parallel Bit Flipping Algorithm

- In parallel, flip each variable that is in more unsatisfied than satisfied constraints.
- Repeat until no such variable remains.

Serial Bit Flipping Algorithm

- If there is a variable that is in more unsatisfied than satisfied constraints, then flip the value of that variable.
- Repeat until no such variable remains.

2.3.3 Linear Programming Decoder

The ML decoding of the code $\mathcal{C}$ allows a convenient LP formulation in terms of the *codeword polytope* poly($\mathcal{C}$) whose vertices correspond to the codewords in $\mathcal{C}$. The ML-LP decoder finds $\mathbf{f} = (f_1, \ldots, f_n)$ minimizing the cost function $\sum_{v \in V} \gamma_v f_v$ subject to the $\mathbf{f} \in \text{poly}(\mathcal{C})$ constraint (see [24] for details). The formulation is compact but impractical because of the number of constraints exponential in the code length.

Hence a *relaxed* polytope is defined as the intersection of all the polytopes associated with the local codes introduced for all the checks of the original code. Associating $(f_1, \ldots, f_n)$ with bits of the code we require

$$0 \leq f_v \leq 1, \quad \forall v \in V \tag{2.1}$$

For every check node c, let $E_c = \{T \subseteq \mathcal{N}(c) : |T| \text{ is even}\}$. The polytope Q_c associated with the check node c is defined as the set of points $(\mathbf{f}, \mathbf{w})$ for which the

following constraints hold

$$0 \leq w_{c,T} \leq 1, \qquad \forall T \in E_c \tag{2.2}$$

$$\sum_{T \in E_c} w_{c,T} = 1 \tag{2.3}$$

$$f_v = \sum_{T \in E_c, T \ni v} w_{c,T}, \quad \forall v \in \mathcal{N}(c) \tag{2.4}$$

Now, let $Q = \cap_c Q_c$ be the set of points $(\mathbf{f}, \mathbf{w})$ such that (2.1)-(2.4) hold for all $c \in C$. (Note that Q, which is also referred to as the fundamental polytope [58, 36], is a function of the Tanner graph G and consequently the parity-check matrix H representing the code $\mathcal{C}$.) The linear code linear program (LCLP) can be stated as

$$\min_{(\mathbf{f},\mathbf{w})} \sum_{v \in V} \gamma_v f_v, \text{ s.t. } (\mathbf{f}, \mathbf{w}) \in Q.$$

For the sake of brevity, the decoder based on the LCLP is referred to in the following as the LP decoder. A solution $(\mathbf{f}, \mathbf{w})$ to the LCLP such that all f_vs and $w_{c,T}$s are integers is known as an integer solution. The integer solution represents a codeword [24]. It was also shown in [24] that the LP decoder has the ML certificate, i.e., if the output of the decoder is a codeword, then the ML decoder would decode into the same codeword. The LCLP can fail, generating an output which is not a codeword.

2.4 Decoder Failures

In this section, we define the various parameters needed to understand decoding failures. We deal with iterative decoders and LP decoders separately. To characterize the performance of a coding/decoding scheme over any output symmetric channel, one can assume, without loss of generality, the transmission of the all-zero-codeword, i.e. $\mathbf{x} = \mathbf{0}$. We make this assumption throughout the paper.

2.4.1 Trapping Sets for Iterative Decoders

Definition 2.6. (Trapping Sets [28]) Let $\mathbf{y}$ denote the input to the iterative decoder and $\mathbf{y}^{(j)}$ denote the codeword estimate of the decoder at the end of j^{th} iteration.

- A variable node v is said to be *eventually correct* if there exists a positive integer l_c such that for all $l_c \leq j \leq M$, $y_v^j = 0$.
- For an input $\mathbf{y}$, the *failure set* $\mathbf{T}(\mathbf{y})$ is defined as the set of variable nodes that are not eventually correct. The decoding on the input $\mathbf{y}$ is successful if and only if $\mathbf{T}(\mathbf{y}) = \emptyset$.
- If $\mathbf{T}(\mathbf{y}) \neq \emptyset$, then we say that $\mathbf{T}(\mathbf{y})$ is a *trapping set*. Since the failure sets of two different input vectors can be the same trapping set, we denote a trapping set simply by $\mathcal{T}$. A trapping set $\mathcal{T}$ is said to be an (a, b) trapping set if it has a variable nodes and b odd-degree check nodes in the sub-graph induced by $\mathcal{T}$. □

Remark 2.7. An (a, b) trapping set is not unique i.e., two trapping sets with same a and b can have different underlying topological structures (induced subgraphs). So, when we talk of a trapping set, we refer to a specific topological structure. In this paper, the induced subgraph is assumed to be known from the context.

Remark 2.8. In order to show that a given set of variable nodes $\mathcal{T}$ is a trapping set, we should exhibit a vector $\mathbf{y}$ for which $\mathbf{T}(\mathbf{y}) = \mathcal{T}$.

The definitions above do not provide the necessary and sufficient conditions for a set of variable nodes to be a trapping set. Hence, for hard decision decoding

algorithms, we define the notion of a *fixed set* for which such conditions can be derived and note that any fixed set is a trapping set.

Definition 2.9. (Inducing and Fixed Sets)

- Let $\mathbf{y}$, a binary n-tuple, be the input to a hard decision decoder. If $\mathbf{T}(\mathbf{y}) \neq \emptyset$, then we say that $\text{supp}(\mathbf{y})$ is an *inducing set*. The size of an inducing set is its cardinality.

- Let $\mathcal{F}$ be a set of variable nodes and let $\mathbf{y}$ with $\text{supp}(\mathbf{y}) = \mathcal{F}$ be the input to the Gallager A/B algorithm. If $\omega_j(v, c) = \mathbf{y}_v, \quad \forall j > 0$, then $\mathcal{F}$ is known as a fixed set and $\mathbf{y}$ is known as a fixed point.

- Let $\mathcal{F}$ be a set of variable nodes and let $\mathbf{y}$ with $\text{supp}(\mathbf{y}) = \mathcal{F}$ be the input to the bit flipping decoder (serial or parallel). $\mathcal{F}$ is a fixed set for the bit flipping algorithms if the set of corrupt variables after every iteration is $\mathcal{F}$. □

When a fixed point is the input to the decoder then the messages passed from variable nodes to check nodes along the edges are the same in every iteration. Since the outgoing messages from variable nodes are same in every iteration, it follows that the incoming messages from check nodes to variable nodes are also same in every iteration and so is the estimate of a variable after each iteration. In fact, the estimate after each iteration coincides with the received value. It is clear from the above definition that if the input to the decoder is a fixed point, then the output of the decoder is the same fixed point. It follows that for transmission over the BSC, if $\mathbf{y}$ is a fixed point and $\mathbf{T}(\mathbf{y}) \neq \emptyset$, then $\mathbf{T}(\mathbf{y}) = \text{supp}(\mathbf{y})$ is a trapping set as well as an inducing set.

Definition 2.10. (Critical Number) Let $\mathcal{T}$ be a trapping and let $\mathbf{y} \in \mathrm{GF}(2)^n$. Let $\mathbf{Y}(\mathcal{T}) = \{\mathbf{y}|\mathbf{T}(\mathbf{y}) = \mathcal{T}\}$. The critical number $\mathcal{T}_c$ of trapping set $\mathcal{T}$ for the Gallager algorithm is the minimum number of variable nodes that have to be initially in error for the decoder to end up in the trapping set $\mathcal{T}$, i.e.,

$$\mathcal{T}_c = \min_{\mathbf{Y}(\mathcal{T})} |\mathrm{supp}(\mathbf{y})|.$$

The most relevant trapping set in the error floor region is the trapping set with the least critical number. □

Definition 2.11. (Instanton) An instanton is a binary vector $\mathbf{i}$ such that $\mathbf{T}(\mathbf{i}) = \mathcal{T}$ for some trapping set $\mathcal{T}$ and for any binary vector $\mathbf{r}$ such that $\mathrm{supp}(\mathbf{r}) \subset \mathrm{supp}(\mathbf{i})$, $\mathbf{T}(\mathbf{i}) = \emptyset$. The size of an instanton is the cardinality of its support. □

2.4.2 Pseudo-codewords for LP decoders

In contrast to the iterative decoders, the output of the LP decoder is well defined in terms of pseudo-codewords.

Definition 2.12. (Pseudo-codewords [24]) *Integer pseudo-codeword* is a vector $\mathbf{p} = (p_1, \ldots, p_n)$ of non-negative integers such that, for every parity check $c \in C$, the neighborhood $\{p_v : v \in N(c)\}$ is a sum of local codewords. □

Alternatively, we can define a *re-scaled pseudo-codeword*, $\mathbf{p} = (p_1, \ldots, p_n)$ where $0 \leq p_v \leq 1$, $\forall v \in V$, simply equal to the output of the LCLP. In the following, we adopt the re-scaled definition.

We now define another important parameter, namely the cost associated with LP decoding of a given vector to a pseudo-codeword.

Definition 2.13. The cost associated with LP decoding of a vector $\mathbf{y}$ to a pseudo-codeword $\mathbf{p}$ is given by

$$C(\mathbf{y}, \mathbf{p}) = \sum_{v \in V} \gamma_v p_v.$$

□

In the case of the BSC, the likelihoods are scaled as

$$\gamma_v = \begin{cases} 1, & \text{if } y_v = 0; \\ -1, & \text{if } y_v = 1. \end{cases}$$

Hence, the cost associated with LP decoding of a binary vector $\mathbf{r}$ to a pseudo-codeword $\mathbf{p}$ is given by

$$C(\mathbf{r}, \mathbf{p}) = \sum_{i \notin \text{supp}(\mathbf{r})} p_i - \sum_{i \in \text{supp}(\mathbf{r})} p_i. \tag{2.5}$$

A given code $\mathcal{C}$ may have different Tanner graph representations and consequently potentially different fundamental polytopes. Hence, we refer to the pseudo-codewords as corresponding to a particular Tanner graph G of $\mathcal{C}$.

Definition 2.14. (Pseudo-codeword Weight [41, Definition 2.10]) Let $\mathbf{p} = (p_1, \ldots, p_n)$ be a pseudo-codeword distinct from the all-zero-codeword of the code $\mathcal{C}$ represented by Tanner graph G . Let q be the smallest number such that the sum of the q largest p_vs is at least $\left(\sum_{v \in V} p_v\right)/2$. Then, the *pseudo-codeword weight* of $\mathbf{p}$ is defined as follows:

- $w_{BSC}(\mathbf{p})$ for the BSC is

$$w_{BSC}(\mathbf{p}) = \begin{cases} 2q, & \text{if } \sum_{q} p_v = \left(\sum_{v \in V} p_v\right)/2; \\ 2q - 1, & \text{if } \sum_{q} p_v > \left(\sum_{v \in V} p_v\right)/2. \end{cases}$$

- $w_{AWGN}(\mathbf{p})$ for the AWGNC is

$$w_{AWGN}(\mathbf{p}) = \frac{(p_1 + \ldots + p_n)^2}{(p_1^2 + \ldots + p_n^2)}$$

The minimum pseudo-codeword weight of G denoted by $w_{min}^{BSC/AWGN}$ is the minimum over all the non-zero pseudo-codewords of G. □

Definition 2.15. (BSC Instanton) The BSC *instanton* $\mathbf{i}$ is a binary vector with the following properties: (1) There exists a pseudo-codeword $\mathbf{p}$ such that $C(\mathbf{i}, \mathbf{p}) \leq C(\mathbf{i}, \mathbf{0}) = 0$; (2) For any binary vector $\mathbf{r}$ such that $\text{supp}(\mathbf{r}) \subset \text{supp}(\mathbf{i})$, there exists no pseudo-codeword with $C(\mathbf{r}, \mathbf{p}) \leq 0$. The size of an instanton is the cardinality of its support. □

2.5 Error Correction Capability and Slope of the FER Curve

We establish the relation between the guaranteed error correction capability of LDPC codes and the slope of the FER curve in the error floor region under hard decision decoding algorithms.

Let p be the transition probability of the BSC and c_r be the number of configurations of received bits for which r channel errors lead to codeword (frame) error. The frame error rate (FER) is given by:

$$FER(p) = \sum_{r=i}^{n} c_r p^r (1-p)^{(n-r)}$$

where i is the minimal number of channel errors that can lead to a decoding error and n is length of the code.

On a semi-log scale the FER is given by

$$\begin{aligned}\log\left(FER(p)\right) &= \log\left(\sum_{r=i}^{n} c_r p^r (1-p)^{n-r}\right) \\ &= \log(c_i) + i\log(p) + \log\left((1-p)^{n-i}\right) \\ &\quad + \log\left(1 + \frac{c_{i+1}}{c_i} p(1-p)^{-1} + \ldots + \frac{c_n}{c_i} p^{n-i}(1-p)^{-i}\right)\end{aligned}$$

For small p, the expression above is dominated by the first two terms. That is,

$$\log\left(FER(p)\right) \approx \log(c_i) + i\log(p)$$

The $\log(FER)$ vs. $\log(p)$ graph is close to a straight line with slope equal to i, the minimal critical number. If two codes $\mathcal{C}^1$ and $\mathcal{C}^2$ have minimum critical numbers i_1 and i_2, such that $i_1 > i_2$, then the code C_1 will perform better than C_2 for small enough p, independent of the number of trapping sets. Hence, if a code can correct all error patterns with up to q errors, then the slope of the FER curve in the error floor region is at least $q + 1$.

CHAPTER 3

Error Correction Capability of Column-Weight-Three Codes: I

For example is not proof

Jewish proverb

In this chapter, we derive upper bounds on the error correction capability of column-weight-three LDPC codes when decoded using the Gallager A algorithm. We study the size of inducing sets of a code as a function of the girth of the underlying Tanner graph. The main consequence of the theorems proved in this chapter is the result that given $\alpha > 0$, at sufficiently large lengths, no code in the ensemble of regular codes with column weight three can correct α fraction of errors.

3.1 Conditions for a Fixed Set

We begin by deriving the necessary and sufficient conditions for a given set of variable nodes to be a fixed set.

Theorem 3.1. *Let $\mathcal{C}$ be a code with Tanner graph G in the ensemble of $(3, d_c)$ regular LDPC codes. Let $\mathcal{F}$ be a set of variable nodes with induced subgraph $\mathcal{I}$. Let the checks in $\mathcal{I}$ be partitioned into two disjoint subsets; $\mathcal{O}$ consisting of checks with odd degree and $\mathcal{E}$ consisting of checks with even degree. $\mathcal{F}$ is a fixed set iff : (a) Every variable node in $\mathcal{I}$ is connected to at least two checks in $\mathcal{E}$ and (b) No two checks of $\mathcal{O}$ are connected to a common variable node outside $\mathcal{I}$.*

Proof. We first show that the conditions of the theorem are sufficient. Let $\mathcal{F}$ be a set of variable nodes with induced subgraph $\mathcal{I}$ satisfying the conditions (a) and (b). Let $\mathbf{y}$ be the input to the decoder with $\text{supp}(\mathbf{y}) = \mathcal{F}$. Then,

$$\omega_1(v,:) \quad = \quad \begin{cases} 1, & v \in \mathcal{F} \\ 0, & \text{otherwise} \end{cases}$$

Let a check node $c_o \in \mathcal{O}$. Then,

$$\varpi_1(c_o, v) \quad = \quad \begin{cases} 0, & v \in \mathcal{F} \\ 1, & \text{otherwise} \end{cases}$$

Let a check node $c_e \in \mathcal{E}$. Then,

$$\varpi_1(c_e, v) \quad = \quad \begin{cases} 1, & v \in \mathcal{F} \\ 0, & \text{otherwise} \end{cases}$$

For any other check node c, $\varpi_1(c, v) = 0$. By the conditions of the theorem, at the end of first iteration, any $v \in \mathcal{F}$ receives at least two 1's and any $v \notin \mathcal{F}$ receives at most one 1. So, we have

$$\omega_2(v,:) \quad = \quad \begin{cases} 1, & v \in \mathcal{F} \\ 0, & \text{otherwise} \end{cases}$$

The messages passed in the subsequent iterations are same as the messages passed in the first iteration and by definition, $\mathcal{F}$ is a fixed set.

To see that the conditions stated are necessary, observe that for a variable node to send the same messages as in the first iteration, it should receive at least two messages which coincide with the received value. □

We note that Theorem 3.1 is a consequence of Fact 3 from [28].

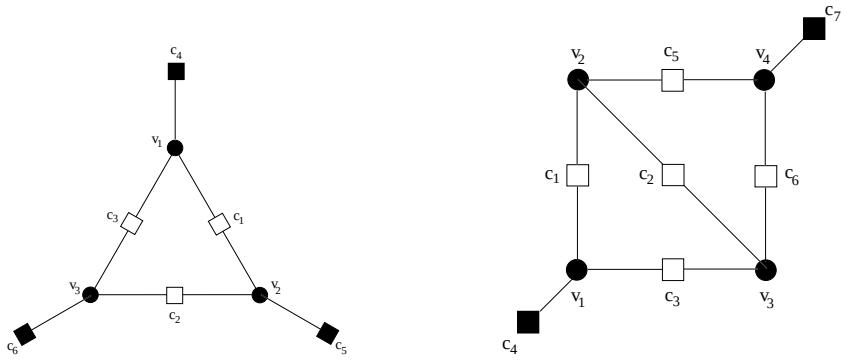


Figure 3.1: Subgraphs induced by (a) a $(3,3)$ trapping set (b) a $(4,2)$ trapping set

Since every fixed set is an inducing set, the conditions (a) and (b) of Theorem 3.1 are sufficient for a given set of variable nodes to be an inducing set and a trapping set. We illustrate the above definitions with an example.

Example 3.2. Fig.3.1(a) shows a subgraph induced by a set of three variable nodes $\{v_1, v_2, v_3\}$ in a Tanner graph of girth six. If no two odd degree check nodes from $\{c_4, c_5, c_6\}$ are connected to a variable outside the subgraph, then by Theorem 3.1, $\{v_1, v_2, v_3\}$ is a fixed set and therefore an inducing set of size three. If $\mathbf{y}$ with $\text{supp}(\mathbf{y}) = \{v_1, v_2, v_3\}$ is the input vector to the Gallager A decoder, then $\mathbf{T}(\mathbf{y}) = \{v_1, v_2, v_3\}$. This implies that $\{v_1, v_2, v_3\}$ is a trapping set and according to the terminology $\{v_1, v_2, v_3\}$ is a $(3,3)$ trapping set i.e., the induced subgraph has 3 variable nodes and 3 odd degree (here degree one) checks.

If two odd degree checks, say c_5 and c_6, are connected to another variable node, say v_4, then the subgraph induced by $\{v_1, v_2, v_3, v_4\}$ is depicted in Fig.3.1(b). Assuming that the check nodes c_4 and c_7 are not connected to a common variable node, $\{v_1, v_2, v_3, v_4\}$ is a fixed set and an inducing set of size four. If $\mathbf{y}$ with $\text{supp}(\mathbf{y}) = \{v_1, v_2, v_3, v_4\}$ is the input vector to the Gallager A decoder, then

$\mathbf{T}(\mathbf{y}) = \{v_1, v_2, v_3, v_4\}$. Consequently Fig. 3.1(b) depicts the subgraph induced by a $(4, 2)$ trapping set.

Now, assume that no two checks in $\{c_1, \ldots, c_7\}$ in Fig.3.1(b) are connected to a common variable node in $V \setminus \{v_1, v_2, v_3, v_4\}$. Let $\mathbf{y}$ with $\text{supp}(\mathbf{y}) = \{v_1, v_2, v_3\}$ be the input to the decoder. In this case, $\mathbf{T}(\mathbf{y}) = \{v_1, v_2, v_3\}$. On the other hand if $\mathbf{y}$ with $\text{supp}(\mathbf{y}) = \{v_1, v_2, v_4\}$ is the input to the decoder, then $\mathbf{T}(\mathbf{y}) = \{v_1, v_2, v_3, v_4\}$. This illustrates that an inducing set of size three can result in a trapping set of size four. We also note that a fixed set can contain inducing sets as its subsets. In this example, $\{v_1, v_2, v_3, v_4\}$ is a fixed set and its subsets $\{v_1, v_2, v_3\}$, $\{v_2, v_3, v_4\}$, $\{v_1, v_2, v_4\}$ and $\{v_1, v_3, v_4\}$ are inducing sets. □

3.2 Girth of Tanner Graph and Size of Inducing Sets

In this section, we prove a fundamental theorem concerning the guaranteed error correction capability of column-weight-three codes. We begin with a lemma in which we establish the relation between the size of inducing sets in a code as a function of underlying Tanner graph.

Lemma 3.3. *Let $\mathcal{C}$ with a Tanner graph G of girth g. If*

(i) [23] $g = 2$, then $\mathcal{C}$ has at least one inducing set of size one.

(ii) $g = 4$, then $\mathcal{C}$ has at least one inducing set of size two or three.

(iii) $g = 6$, then $\mathcal{C}$ has least one inducing set of size three or four.

(iv) $g = 8$, then $\mathcal{C}$ has least one inducing set of size four or five.

(v) $g \geq 10$, the set of variable nodes $\{v_1, v_2, \ldots, v_{g/2}\}$ involved in the shortest cycle is a fixed set (as well as an inducing set) of size $g/2$.

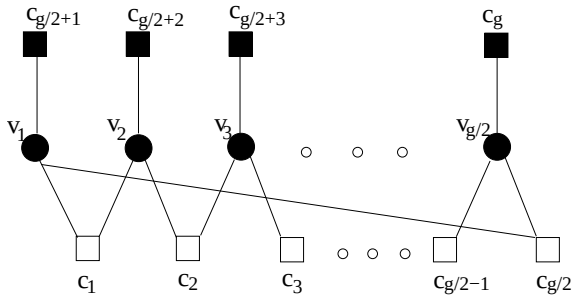


Figure 3.2: Illustration of a cycle of length g

Proof. See Section 3.3 for proofs of (i)-(iv). Since $\mathcal{C}$ has girth g, there is at least one cycle of length g. Without loss of generality, assume that $\{v_1, v_2, \ldots, v_{g/2}\}$ form a cycle of minimum length as shown in Fig.3.2. Let the even degree checks be $\mathcal{E} = \{c_1, c_2, \ldots, c_{g/2}\}$ and the odd degree checks be $\mathcal{O} = \{c_{g/2+1}, c_{g/2+2}, \ldots, c_g\}$. Note that each variable node is connected to two checks from $\mathcal{E}$ and one check from $\mathcal{O}$ and $c_{g/2+i}$ is connected to v_i. We claim that no two checks from $\mathcal{O}$ can be connected to a common variable node outside $\{v_1, v_2, \ldots, v_{g/2}\}$.

The proof is by contradiction. Assume c_i and c_j $(g/2 + 1 \leq i < j \leq g)$ are connected to a variable node v_{ij}. Then $\{v_i, \ldots, v_j, v_{ij}\}$ form a cycle of length $2(j - i + 2)$ and $\{v_j, \ldots, v_{g/2}, v_1, \ldots, v_i, v_{ij}\}$ form a cycle of length $2(g/2 - j + i + 2)$. Since $g \geq 10$,

$$\min(2(j - i + 2), 2(g/2 - j + i + 2)) < g.$$

This implies that there is a cycle of length less than g, which is a contradiction as the girth of the graph is g.

By Theorem 3.1, $\{v_1, v_2, \ldots, v_{g/2}\}$ is a fixed set and consequently an inducing set of size $g/2$. It is worth noting that $\{v_1, v_2, \ldots, v_{g/2}\}$ is a $(g/2, g/2)$ trapping set. □

It might be possible that in Lemma 3.3, the statements (ii)–(iv) can be made

stronger by further analysis, i.e., it might be possible to show that a code with Tanner graph of girth four always has an inducing set of size two, a code with Tanner graph of girth six has an inducing set of size three and a code with Tanner graph of girth eight has an inducing set of size four. However, these weaker lemmas are sufficient to establish the main theorem.

Corollary 3.4. *For a code in the standard $(3, d_c)$ regular LDPC code ensemble to correct all error patterns up to $q \geq 5$ errors, it is necessary to avoid all cycles up to length $2q$.* □

We now state and prove the main theorem.

Theorem 3.5. *Consider the standard $(3, d_c)$ regular LDPC code ensemble. Let $\alpha > 0$. Let N be the smallest integer satisfying*

$$\begin{aligned} \alpha N &> 2\left(\frac{\log N}{\log\left(2(d_c-1)\right)} + 1\right) \\ \alpha N &\geq 5. \end{aligned}$$

Then, for $n > N$, no code in the $\mathcal{C}^n(3, d_c)$ ensemble can correct all αn or fewer errors.

Proof. First observe that for any $n > N$, we have

$$\alpha n > 2\left(\frac{\log n}{\log\left(2(\rho-1)\right)} + 1\right). \tag{3.1}$$

From [Theorem C.1 [11]] and [Lemma C.1 [11]], we have the girth g of any code in $\mathcal{C}^n(3, \rho)$ is bounded by

$$g \leq 4\left(\frac{\log n}{\log\left(2(\rho-1)\right)} + 1\right) \tag{3.2}$$

For $n > N$, Equations (3.1) and (3.2) imply that for any code in the $\mathcal{C}^n(3, \rho)$ ensemble, the girth is bounded by

$$g < 2\alpha n.$$

The result now follows from Corollary 3.4. □

3.3 Proofs

We provide the missing proofs in this section.

3.3.1 Proof of Lemma 3.3.(i)

The ensemble of $(3, d_c)$ regular codes consists of codes whose Tanner graphs consist of a variable node connected to a check node via multiple edges. Such a Tanner graph is said to contain parallel edges. In [23], it was shown that such a code cannot correct a single worst case error. The proof is for the parallel bit flipping algorithm, but also applies to the Gallager A algorithm (see [23] for details). □

3.3.2 Proof of Lemma 3.3.(ii)

Let $\{v_1, v_2\}$ be the variable nodes that form a four cycle. The following two cases arise:

(1) The subgraph induced by v_1 and v_2 has three even degree checks c_1, c_2 and c_3. In this case, $\{v_1, v_2\}$ is a weight-two codeword and therefore an inducing set of size two.

(2) The subgraph induced by v_1 and v_2 has two even degree checks $\{c_1, c_2\}$ and two odd degree checks $\{c_3, c_4\}$. If c_3 and c_4 are not connected to a common variable node, then $\{v_1, v_2\}$ is a fixed set and hence an inducing set of size two. Now assume

that c_3 and c_4 are connected to a common variable node v_3. Then, $\{v_1, v_2, v_3\}$ is a fixed set and therefore an inducing set of size three. □

3.3.3 Proof of Lemma 3.3.(iii)

Since $g = 6$, there is at least one six cycle. Without loss of generality, we assume that $\{v_1, v_2, v_3\}$ together with the three even degree checks $\{c_1, c_2, c_3\}$ and the three odd degree checks $\{c_4, c_5, c_6\}$ form a six cycle as in Fig.3.1(a). The check nodes c_4, c_5 and c_6 are distinct, since otherwise the girth would be less than six. If no two checks from $\{c_4, c_5, c_6\}$ are connected to a common variable node, then $\{v_1, v_2, v_3\}$ is a fixed set and hence an inducing set of size three. On the contrary, assume that $\{v_1, v_2, v_3\}$ is not a fixed set. Then there exists a variable node v_4 which is connected to at least two checks from $\{c_4, c_5, c_6\}$. If v_4 is connected to all the three checks, then $\{v_1, v_2, v_3, v_4\}$ is the support of a codeword of weight four and it is easy to see that $\{v_1, v_2, v_3\}$ is an inducing set. Now assume that v_4 is connected to only two checks from $\{c_4, c_5, c_6\}$. Without loss of generality, let the two checks be c_5 and c_6. Let the third check connected to v_4 be c_7 as shown in Fig.3.1(b). If c_4 and c_7 are not connected to a common variable node then $\{v_1, v_2, v_3, v_4\}$ is a fixed set and hence an inducing set of size four. If c_4 and c_7 are connected to say v_5, we have two possibilities: (a) The third check is c_8 and (b) The third check of v_5 is c_2 (the third check cannot be c_1 or c_3 as this would introduce a four cycle). We claim that in both cases $\{v_1, v_2, v_3, v_4\}$ is an inducing set. The two cases are discussed below.

Case (a): Let $\text{supp}(\mathbf{y}) = \{v_1, v_2, v_3, v_4\}$.

$$\omega_1(v, :) \quad = \quad \begin{cases} 1, & v \in \{v_1, v_2, v_3, v_4\} \\ 0, & \text{otherwise} \end{cases}$$

The messages in the second half of the first iteration are,

$$\varpi_1(c_1, v) = \begin{cases} 1, & v \in \{v_1, v_2\} \\ 0, & \text{otherwise} \end{cases}$$

Similar equations hold for c_2, c_3, c_5, c_6. For c_4 we have

$$\varpi_1(c_4, v) = \begin{cases} 0, & v = v_1 \\ 1, & \text{otherwise} \end{cases}$$

Similar equations hold for c_7. At the end of the first iteration, we note that v_2 and v_3 receive all incorrect messages, v_1, v_4 and v_5 receive two incorrect messages and all other variable nodes receive at most one incorrect message. We therefore have $\mathbf{y}^{(1)} = \mathbf{y}$ and $\text{supp}(\mathbf{y}^{(1)}) = \{v_1, v_2, v_3, v_4\}$. The messages sent by variable nodes in the second iteration are,

$$\begin{aligned} \omega_2(v, :) &= 1, v \in \{v_1, v_2, v_3, v_4\} \\ \omega_2(v_5, c_8) &= 1, \\ \omega_2(v_5, \{c_4, c_7\}) &= 0, \\ \omega_2(v, :) &= 0, v \in V \setminus \{v_1, v_2, v_3, v_4, v_5\}. \end{aligned}$$

The messages passed in the second half of the second iteration are same as in the second half of first iteration, except that $\varpi(c_8, : \setminus v_5) = 1$. At the end of the second iteration, we note that v_2 and v_3 receive all incorrect messages, v_1, v_4 and v_5 receive two incorrect messages and all other variable nodes receive at most one incorrect message. The situation is same as at the end of first iteration. The algorithm runs for M iterations and the decoder outputs $\mathbf{y}^{(M)} = \mathbf{y}$ which implies that the set of variable nodes $\{v_1, v_2, v_3, v_4\}$ is an inducing set.

Case (b): The proof is along the same lines as for Case (a). The messages for the first iteration are the same. The messages in the first half of the second iteration are,

$$\begin{aligned}
\omega_2(v,:) &= 1, v \in \{v_1, v_2, v_3, v_4\} \\
\omega_2(v_5, c_2) &= 1, \\
\omega_2(v_5, \{c_4, c_7\}) &= 0, \\
\omega_2(v,:) &= 0, v \in V \setminus \{v_1, v_2, v_3, v_4, v_5\}.
\end{aligned}$$

The messages passed in the second half of the second iteration are same as in the second half of the first iteration, except that $\varpi(c_2, : \setminus \{v_2, v_3, v_5\}) = 1$ and $\varpi(c_2, \{v_2, v_3, v_5\}) = 0$. At the end of the second iteration, v_1, v_2, v_3, v_4 and v_5 receive two incorrect messages and all other variable nodes receive at most one incorrect message and hence $\mathbf{y}^{(2)} = \mathbf{y}$. The messages passed in the first half of the third iteration (and therefore subsequent iterations) are same as the messages passed in the first half of the second iteration. The algorithm runs for M iterations and the decoder outputs $\mathbf{y}^{(M)} = \mathbf{y}$ which implies that that the set of variable nodes $\{v_1, v_2, v_3, v_4\}$ is an inducing set. □

3.3.4 Proof of Lemma 3.3.(iv)

Let $\mathcal{T}_1 = \{v_1, v_2, v_3, v_4\}$ be the set of variable nodes that form an eight cycle (see Fig.3.3(a)). If no two checks from $\{c_5, c_6, c_7, c_8\}$ are connected to a common variable node, then $\mathcal{T}_1$ is a fixed set and also an inducing set of size four. On the other hand, if $\mathcal{T}_1$ is not a fixed set, then there must be at least one variable node which is connected to two checks from $\{c_5, c_6, c_7, c_8\}$. Assume that c_5 and c_7 are connected to v_5 and the third check of v_5 is c_9 (see Fig.3.3(b)). We claim that $\mathcal{T}_2 = \mathcal{T}_1 \cup \{v_5\}$ is

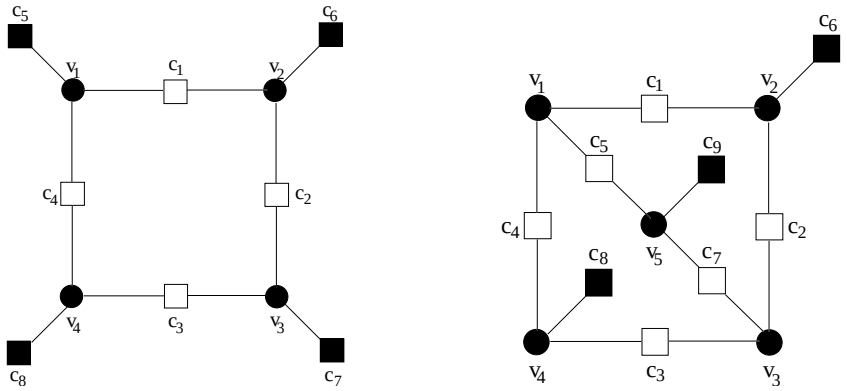


Figure 3.3: Subgraphs induced by (a) a $(4, 4)$ trapping set (b) a $(5, 3)$ trapping set

an inducing set. Let $\mathcal{I}_2$ be the subgraph induced by $\mathcal{T}_2$. Let $\mathcal{E}$ and $\mathcal{O}$ be as defined in Theorem 3.1.

Case 1: No two checks from $\mathcal{O} = \{c_6, c_8, c_9\}$ are connected to a common variable node. Then $\mathcal{T}_2$ is a fixed set and hence an inducing set of size five.

Case 2: All the three checks in $\mathcal{O}$ are connected to a common variable node, say v_6. Then $\mathcal{T}_2 \cup \{v_6\}$ is a codeword of weight six and it is easy to see that $\mathcal{T}_2$ is an inducing set.

Case 3: There are variable nodes connected to two checks from $\mathcal{O}$. There can be at most two such variable nodes (if there are three such variable nodes, they will form a cycle of length less than or equal to six violating the condition that the graph has girth eight). Note that if $\text{supp}(\mathbf{y}) = \mathcal{T}_2$, the decoder has a chance of correcting only if a check node in $\mathcal{E}$ receives an incorrect message from a variable node outside $\mathcal{T}_2$ in some j^{th} iteration. We now prove that this is not possible. Indeed in the first iteration,

$$\omega_1(v,:) \;=\; \begin{cases} 1, & v \in \mathcal{T}_2 \\ 0, & \text{otherwise} \end{cases}$$

By similar arguments as in the proof for Theorem 3.1, it can be seen that the only check nodes which send incorrect messages to variable nodes outside $\mathcal{T}_2$ are c_6, c_8 and c_9. There are now two sub cases.

Sub case 1: There is one variable node connected to two checks from $\mathcal{O}$. Let v_6 be connected to c_6 and c_8. It can be seen that the third check connected to v_6 cannot belong to $\mathcal{E}$ as this would violate the girth condition. So, let the third check be c_{10}. In the first half of the second iteration, we have

$$\omega_2(v,c) \;=\; \begin{cases} 1, & v \in \mathcal{T}_2 \text{ or } (v,c) = (v_6, c_{10}) \\ 0, & \text{otherwise} \end{cases}$$

The only check nodes which send incorrect messages to variable nodes outside $\mathcal{T}_2$, are c_6, c_8, c_9 and c_{10}. The variable node v_6 is connected to c_6 and c_8. If c_9 and c_{10} are not connected to any common variable node, we are done. On the other hand, let c_9 and c_{10} be connected to a variable node, say v_7. The third check of v_7 cannot be in $\mathcal{E}$. Proceeding as in the case of proof for Lemma 3.3(iii), we can prove that $\mathcal{T}_2$ is an inducing set by observing that there cannot be a variable node outside $\mathcal{T}_2$ which sends an incorrect message to a check in $\mathcal{E}$.

Sub case 2: There are two variable nodes connected to two checks from $\mathcal{O}$. Let c_6 and c_8 be connected to v_6 and c_6 and c_9 connected to v_7. Proceeding as above, we can conclude that $\mathcal{T}_2$ is an inducing set. □

CHAPTER 4

Error Correction Capability of Column-Weight-Three Codes: II

> Thus, be it understood, to demonstrate a theorem, it is neither necessary nor even advantageous to know what it means.
>
> Henri Poincare.

The primary goal in this chapter is to establish lower bounds on the guaranteed error correction capability of column-weight-three LDPC codes under Gallager A algorithm. While it is known empirically that codes with higher girth exhibit superior FER performance, the exact relation between girth and slope of FER curve in the error floor region has not been established. The central result of this chapter is that a column-weight-three LDPC code whose Tanner graph has girth $g \geq 10$, can correct all error patterns with up to $g/2 - 1$ errors. This result shows that the upper bounds derived in Chapter 3 are tight. The guaranteed error correction capability of codes with Tanner graphs of girth $4, 6$ and 8 are dealt in Appendix A. Hence, the problem of determining the slope of FER curves for column-weight-three codes in the error floor region is now settled. While this is an important result in itself, we note that the methodology of the proofs adopted in this chapter can be used to derive bounds for higher column weight codes. We begin by introducing the notation and then proceed to analyze the Gallager A algorithm for the independent iterations. We then establish our main theorem for codes with girth $g \geq 12$, such that $g/2$ is even. The cases in which $g/2$ is odd are dealt with later.

4.1 Notation

An edge e is an unordered pair $\{v, c\}$ of a variable node v and a check node c and is said to be incident on v and c. A directed edge $\vec{e}$ is an ordered pair (v, c) or (c, v) corresponding to the edge $e = \{v, c\}$. With a moderate abuse of notation, we denote directed edges by simple letters (without arrows) but specify the direction. The girth g is the length of the shortest cycle in G. For a given node u, the neighborhood of depth t, denoted by $\mathcal{N}_u^t$, is the induced subgraph consisting of all nodes reached and edges traversed by paths of length at most t starting from u (including u). The directed neighborhood of depth t of a directed edge $e = (v, c)$ denoted by $\mathcal{N}_e^t$, is defined as the induced subgraph containing all edges and nodes on all paths $e_1, \ldots, e_t$ starting from v such that $e_1 \neq e$ (see [17] for definitions and notation). In a Tanner graph with girth g, we note that $\mathcal{N}_u^t$ is a tree when $t \leq g/2 - 1$. Also, if $e_1 = (v, c)$ and $e_2 = (c, v)$, then $\mathcal{N}_{e_1}^{t_1} \cap \mathcal{N}_{e_2}^{t_2} = \phi$ for $t_1 + t_2 < g - 1$. Let l denote the number of independent iterations as defined in [11]. The original value of a variable node is its value in the transmitted codeword. We say a variable node is good if its received value is equal to its original value and bad otherwise. A message is said to be correct if it is equal to the original value of the corresponding variable node and incorrect otherwise. In this work, $\circ$ denotes a good variable node, $\bullet$ denotes a bad variable node and $\square$ denotes a check node. For output symmetric channels (see [17]), without loss of generality, we can assume that the all zero codeword is transmitted. We make this assumption throughout the chapter. Hence, a bad variable node has received value 1 and an incorrect message has a value of 1. A configuration of bad variable nodes is a subgraph in which the location of bad variables relative to each other is specified. A valid configuration $\mathcal{C}_g$ is a configuration of at most $g/2 - 1$ bad variable

nodes free of cycles of length less than g. The set of bad variable nodes in $\mathcal{N}_e^t$ is denoted by $\mathcal{B}(\mathcal{N}_e^t)$ and $|\mathcal{B}(\mathcal{N}_e^t)|$ is denoted by $B(\mathcal{N}_e^t)$. The number of bad variable nodes at depth t in $\mathcal{N}_e^t$ is denoted by b_e^t.

4.2 The first l iterations

We begin with a lemma describing the messages passed by the Gallager A algorithm in a column-weight-three code.

Lemma 4.1. *(i) If v is a bad variable node, then we have $\omega_1(v,:) = \{1\}$ and*

- *$\omega_j(v,:) = \{1\}$ if $|\varpi_{j-1}(:,v) = 1| \geq 2$, i.e., v sends incorrect messages to all its neighbors if it receives two or more incorrect messages from its neighboring checks in the previous iteration.*
- *$\omega_j(v,: \setminus c) = \{1\}$ and $\omega_j(v,c) = 0$ if $\varpi_{j-1}(: \setminus c, v) = \{0\}$ and $\varpi_{j-1}(c,v) = 1$, i.e., v sends one correct message and two incorrect messages if it receives one incorrect message from its neighboring checks in the previous iteration. The correct message is sent along the edge on which the incorrect message is received.*
- *$\omega_j(v,:) = \{0\}$ if $\varpi_{j-1}(:,v) = \{0\}$, i.e., v sends all correct messages if it receives all correct messages from its neighboring checks in the previous iteration.*

(ii) If v is a good variable node, then we have $\omega_1(v,:) = \{0\}$ and

- *$\omega_j(v,:) = \{0\}$ if $|\varpi_{j-1}(:,v) = 0| \geq 2$, i.e., v sends all correct messages if it receives two or more correct messages from its neighboring checks in the previous iteration.*

- $\omega_j(v, : \backslash c) = \{0\}$ *and* $\omega_j(v, c) = 1$ *if* $\varpi_{j-1}(: \backslash c, v) = \{1\}$ *and* $\varpi_{j-1}(c, v) = 0$, *i.e., v sends one incorrect message and two correct messages if it receives two incorrect messages from its neighboring checks in the previous iteration. The incorrect message is sent along the edge on which the correct message is received.*

- $\omega_j(v, :) = \{1\}$, *if* $\varpi_{j-1}(:, v) = \{1\}$, *i.e., v sends all incorrect messages if it receives all incorrect messages from its neighboring checks in the previous iteration.*

(iii) For a check node c, we have,

- $\varpi_j(c, v) = \omega_j(v, c) \oplus 1$, *if* $|\omega_j(:, c) = 1|$ *is odd, i.e., c sends incorrect messages along the edges on which it received correct messages and correct messages along the edges on which it received incorrect messages, if the total number of incoming incorrect messages from its neighboring variable nodes is odd.*

- $\varpi_j(c, v) = \omega_j(v, c)$, *if* $|\omega_j(:, c) = 1|$ *is even, i.e., c sends incorrect messages along the edges on which it received incorrect messages and correct messages along the edges on which it received correct messages, if the total number of incoming incorrect messages from its neighboring variable nodes is even.*

(iv) A variable node is estimated incorrectly at the end of an iteration if it receives at least two incorrect messages.

Proof. Follows from the description of the Gallager A algorithm. □

Now let v be a variable node which sends an incorrect message along the edge $e = (v, c)$ in the $(l+1)^{th}$ iteration. The message along e depends only on the variable

nodes and check nodes in $\mathcal{N}_e^{2l}$. Under the assumption that $\mathcal{N}_e^{2l}$ is a tree, the above observations provide a method to find all the possible configurations of bad variable nodes in $\mathcal{N}_e^{2l}$. We have the following two cases:

(a) v is a bad variable node: In this case, there must be at least one variable node in $\mathcal{N}_e^2$ which sends an incorrect message in the l^{th} iteration.

(b) v is a good variable node: In this case, there must be at least two variable nodes in $\mathcal{N}_e^2$ which send an incorrect message in the l^{th} iteration.

This is repeated l times until we reach the first iteration, at which point only the nodes and edges in $\mathcal{N}_e^{2l}$ would have been explored and all of these are guaranteed to be distinct as $\mathcal{N}_e^{2l}$ is a tree. Since only bad variables send incorrect messages in the first iteration, we can calculate the number of bad variables in each configuration. Specifically, let v be a variable node which sends an incorrect message along the $e = (v, c)$ in the second iteration. If v is a good variable node, then $\mathcal{N}_e^2$ must have at least two bad variable nodes. If v is bad, $\mathcal{N}_e^2$ must have at least one bad variable node. Following this approach, we have Fig. 4.1(a), Fig. 4.1(b) and Fig. 4.1(c) which show the possible configurations of bad variable nodes in $\mathcal{N}_e^{2l}$ so that v sends an incorrect message in the $(l+1)^{th}$ iteration, for $l = 1, l = 2$ and $l = 3$, respectively.

Remarks: (i) Fig. 4.1 shows configurations with at most $2l + 1$ bad variable nodes.

(ii) We do not illustrate configurations in which a bad variable node receives two incorrect messages in the l^{th} iteration, so that it sends an incorrect message in the $(l + 1)^{th}$ iteration. However, all such configurations can be found by considering configurations involving good variable nodes and converting a good variable node to a bad one. This increases the number of bad variable nodes in the configuration.

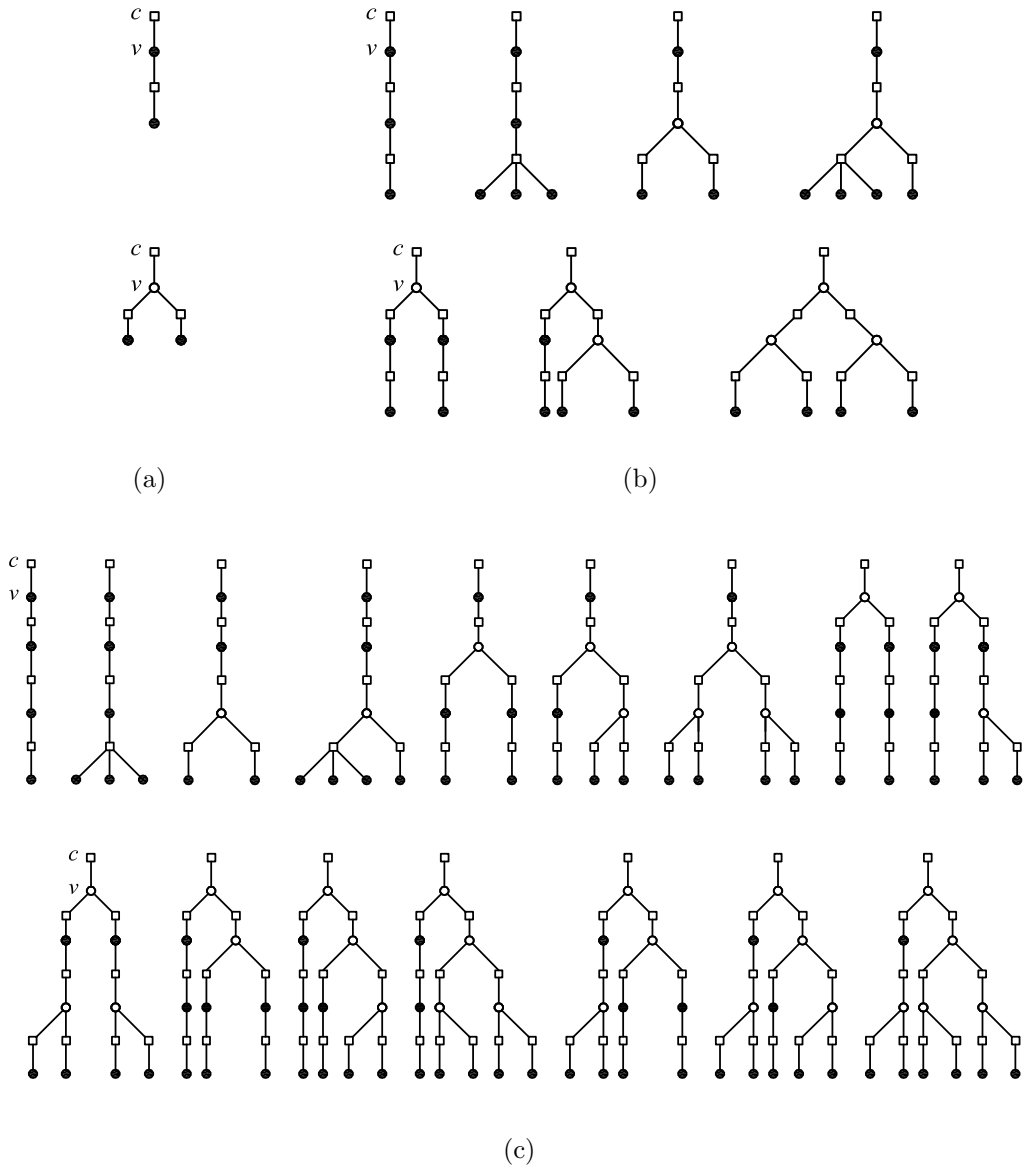


Figure 4.1: Possible configurations of at most $2l + 1$ bad variable nodes in the neighborhood of a variable node v sending an incorrect message to check node c in the $(l + 1)^{th}$ iteration for (a) $l = 1$, (b) $l = 2$ and (c) $l = 3$.

As will be seen later, such configurations are not relevant for establishing our main result.

The above observations help establish bounds on $B(\mathcal{N}_e^{2l})$, which we state in the following lemma.

Lemma 4.2. *(i) If v is a bad variable node sending an incorrect message on $e = (v, c)$ in the $(l+1)^{th}$ iteration and $\mathcal{N}_e^{2l}$ is a tree, then $B(\mathcal{N}_e^{2l}) \geq l+1$. If $B(\mathcal{N}_e^{2l-2}) = 1$, i.e., $b_e^t = 0$ for $t = 2, 4, \ldots, 2(l-1)$, then $B(\mathcal{N}_e^{2l}) \geq 2^{(l-1)} + 1$. If $B(\mathcal{N}_e^{2l-2}) = 2$, then $B(\mathcal{N}_e^{2l}) \geq 2^{(l-2)} + 2$.*

(ii) If v is a good variable node sending an incorrect message on $e = (v, c)$ in the $(l+1)^{th}$ iteration and $\mathcal{N}_e^{2l}$ is a tree, then $B(\mathcal{N}_e^{2l}) \geq 2l$. If $B(\mathcal{N}_e^{2l-2}) = 0$, then $B(\mathcal{N}_e^{2l}) \geq 2^l$. If $B(\mathcal{N}_e^{2l-2}) = 1$, then $B(\mathcal{N}_e^{2l}) \geq 2^{(l-1)} + 2^{(l-2)} + 1$. If $B(\mathcal{N}_e^{2l-2}) = 2$, then $B(\mathcal{N}_e^{2l}) \geq 2^{(l-1)} + 2$.

Proof. The proof is by induction on l. It is easy to verify the bounds for $l = 2$. Let the bounds be true for some $l \geq 2$. Let v_0 be a bad variable node sending an incorrect message on $e = (v_0, c)$ in the $(l+2)^{th}$ iteration. Further, assume that $\mathcal{N}_e^{2l+2}$ is a tree. Then, $\mathcal{N}_e^1$ has at least one check node c_1 which sends an incorrect message along the edge $e_1 = (c_1, v_0)$ in the $(l+1)^{th}$ iteration. This implies that $\mathcal{N}_e^2$ has at least one variable node v_2 sending an incorrect message in the $(l+1)^{th}$ iteration along the edge $e_2 = (v_2, c_1)$. Since a path of length 2 exists between v_0 and v_1, $\mathcal{N}_{e_2}^t \subset \mathcal{N}_e^{t+2}$.

If v_2 is a bad variable node, then $B(\mathcal{N}_{e_2}^{2l}) \geq l+1$ and consequently $B(\mathcal{N}_e^{2l+2}) \geq l+2$. If v_2 is a good variable node, then $B(\mathcal{N}_{e_2}^{2l}) \geq 2l$ and consequently $B(\mathcal{N}_e^{2l+2}) \geq 2l+1 > l+1$.

If $b_e^t = 0$ for $t = 2, 4, \ldots, 2l$, then v_2 is a good variable node such that $B(\mathcal{N}_{e_2}^{2l-2}) =$

0 which implies that $B(\mathcal{N}_{e_2}^{2l}) \geq 2^l$ by the induction hypothesis. Hence, $B(\mathcal{N}_e^{2l+2}) \geq 2^l + 1$.

If $B(\mathcal{N}_e^{2l}) = 2$ then either (a) v_2 is a bad variable node with $b_{e_2}^t = 0$ for $t = 2, 4, \ldots, 2(l-1)$ which implies that $B(\mathcal{N}_{e_2}^{2l}) \geq 2^{(l-1)} + 1$ by the induction hypothesis. Hence, $B(\mathcal{N}_e^{2l+2}) \geq 2^{(l-1)} + 2$, or (b) v_2 is a good variable node with $B(\mathcal{N}_{e_2}^{2l-2}) = 1$ which implies that $B(\mathcal{N}_{e_2}^{2l}) \geq 2^{(l-1)} + 2^{(l-2)} + 1$ by the induction hypothesis. Hence, $B(\mathcal{N}_e^{2l+2}) \geq 2^{(l-1)} + 2^{(l-2)} + 2 > 2^{(l-1)} + 2$.

By the principle of mathematical induction, the bounds are true for all l when v_0 is a bad variable node. The proofs are similar for the case when v_0 is a good variable node. □

4.3 Girth of Tanner Graph and Guaranteed Error Correction Capability

In this section, we prove that a column-weight-three code with Tanner graph of girth $g \geq 10$ can correct $g/2 - 1$ errors in $g/2$ iterations of the Gallager A algorithm. The proof proceeds by finding, for a particular choice of l, all configurations of $g/2 - 1$ or less bad variable nodes which do not converge in $l + 1$ iterations and then prove that these configurations also converge in subsequent iterations. When $g/2$ is even, we use $l = g/4 - 1$ (or $g/2 - 1 = 2l + 1$) and when $g/2$ is odd, we use $l = (g - 2)/4$ (or $g/2 - 1 = 2l$). We deal with these cases separately.

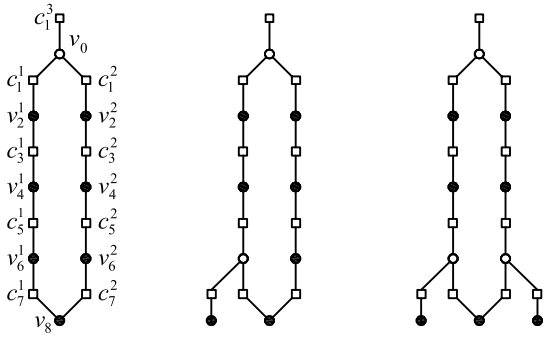
4.3.1 $g/2$ is even

Let v_0 be a variable node which receives two incorrect messages along the edges $e_1 = (c_1^1, v_0)$ and $e_2 = (c_1^2, v_0)$ at the end of $(l + 1)^{th}$ iteration. This implies that $N_{e_1}^1$ and $N_{e_2}^1$ each has a variable node, v_2^1 and v_2^2 respectively, that sends an incor-

rect message in the $(l+1)^{th}$ iteration to check nodes c_1^1 and c_1^2, respectively. Let $e_3 = (v_2^1, c_1^1)$, $e_4 = (v_2^2, c_1^2)$, $e_5 = (c_1^1, v_2^1)$, and $e_6 = (c_1^2, v_2^2)$ (see Fig. 4.2(a) for an illustration). All possible configurations of bad variable nodes in $\mathcal{N}_{e_3}^{2l}$ and $\mathcal{N}_{e_4}^{2l}$ can be determined using the method outlined in Section 4.2. Since there exists a path of length 3 between v_2^2 and c_1^1, we have $\mathcal{N}_{e_4}^{t} \subset \mathcal{N}_{e_5}^{t+3}$. Also, $\mathcal{N}_{e_3}^{t_1} \cap \mathcal{N}_{e_5}^{t_2} = \phi$ for $t_1 + t_2 < g - 1 = 4l + 3$. Therefore, $\mathcal{N}_{e_3}^{t_1} \cap \mathcal{N}_{e_4}^{t_2} = \phi$ for $t_1 + t_2 < 4l$. This implies that $\mathcal{N}_{e_3}^{2l}$ and $\mathcal{N}_{e_4}^{2l}$ can have a common node only at depth $2l$. The total number of bad variable nodes in $\mathcal{N}_{e_3}^{2l} \cup \mathcal{N}_{e_4}^{2l}$, $B(\mathcal{N}_{e_3}^{2l} \cup \mathcal{N}_{e_4}^{2l})$, in any configuration is therefore lower bounded by $B(\mathcal{N}_{e_3}^{2l-2}) + B(\mathcal{N}_{e_4}^{2l-2}) + \max(b_{e_3}^{2l}, b_{e_4}^{2l})$ or equivalently $\max\left(B(\mathcal{N}_{e_3}^{2l-2}) + B(\mathcal{N}_{e_4}^{2l}), B(\mathcal{N}_{e_3}^{2l}) + B(\mathcal{N}_{e_4}^{2l-2})\right)$. We are interested only in the valid configurations, i.e., at most $g/2 - 1$ bad variable nodes, free from cycles of length less than g. We divide the discussion into three parts: (1) we find all the possible valid configurations for the case when $g = 16$; (2) we then proceed iteratively for $g > 16$; (3) We consider the case $g = 12$ separately as the arguments for $g \geq 16$ do not hold for this case.

$g = 16$

Let v be a variable node which sends an incorrect message in the iteration $l + 1 = g/4 = 4$ along edge $e = (v, c)$, given that there are at most seven bad variables and $\mathcal{N}_e^7$ is a tree. Fig. 4.1(c) illustrates different configurations of bad variable nodes in $\mathcal{N}_e^6$. As remarked earlier, Fig. 4.1(c) does not show configurations in which a bad variable node has to receive two incorrect messages in an iteration to send an incorrect message along the third edge in the next iteration. It can be seen in the proof of Theorem 4.5 that these cases do not arise in valid configurations.



(a) (b) (c)

Figure 4.2: Configurations of at most 7 bad variable nodes, free of cycles of length less than 16, which do not converge in 4 iterations.

Let $v_0, v_2^1, v_2^2, e_1, e_2, e_3, e_4$ be defined as above with $l = 3$. Using the arguments outlined above (and the constraint that $g = 16$), all possible configurations such that $B(\mathcal{N}_{e_3}^{2l} \cup \mathcal{N}_{e_4}^{2l}) \leq 7$ can be found. Fig. 4.2 shows all such possible configurations.

$g \geq 20$

Let $\mathcal{C}_g$ be a valid configuration in which there exists a variable node v_0 which receives two incorrect messages along the edges $e_1 = (c_1^1, v_0)$ and $e_2 = (c_1^2, v_0)$ at the end of $(l+1)^{th}$ iteration. This implies that $\mathcal{N}_{e_1}^1$ and $\mathcal{N}_{e_2}^1$ each has a variable node, v_2^1 and v_2^2, respectively, that sends an incorrect message in the $(l+1)^{th}$ iteration. We have the following lemma.

Lemma 4.3. *v_2^1 and v_2^2 are bad variable nodes.*

Proof. The proof is by contradiction. We know that the total number of bad variables in any configuration is lower bounded by
$\max\left(B(\mathcal{N}_{e_3}^{2l-2}) + B(\mathcal{N}_{e_4}^{2l}), B(\mathcal{N}_{e_3}^{2l}) + B(\mathcal{N}_{e_4}^{2l-2})\right)$ and that $l > 3$. We have two cases.

(a) v_2^1 and v_2^2 are both good variable nodes: We first note that in any valid configuration, $B(\mathcal{N}_{e_3}^{2l-2}) \geq 2$. Otherwise, we have $B(\mathcal{N}_{e_3}^{2l-2}) = 1$, and from Lemma

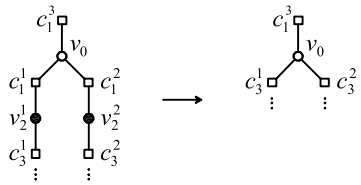


Figure 4.3: Construction of $\mathcal{C}_{g-4}$ from $\mathcal{C}_g$

2, $B(\mathcal{N}_{e_3}^{2l}) \geq 2^{(l-1)} + 2^{(l-2)} + 1 > 2l + 1$ or, we have $B(\mathcal{N}_{e_3}^{2l-2}) = 0$, and $B(\mathcal{N}_{e_3}^{2l}) \geq 2^l + 2 > 2l + 1$. Both cases are a contradiction as we have at most $2l + 1$ bad variable nodes. Hence, $B(\mathcal{N}_{e_3}^{2l-2}) \geq 2$.

Now, $B(\mathcal{N}_{e_4}^{2l}) \geq 2l$, and hence we have $B(\mathcal{N}_{e_3}^{2l} \cup \mathcal{N}_{e_4}^{2l}) \geq B(\mathcal{N}_{e_4}^{2l}) + B(\mathcal{N}_{e_3}^{2l-2}) \geq 2l + 2$, which is a contradiction.

(b) v_2^1 is a bad variable node and v_2^2 is a good variable node. The opposite case is identical.

First we claim that in any valid configuration, $B(\mathcal{N}_{e_3}^{2l-2}) \geq 2$. Since v_2^1 is a bad variable node, $B(\mathcal{N}_{e_3}^{2l-2}) \neq 0$. Assume that $B(\mathcal{N}_{e_3}^{2l-2}) = 1$. Then $B(\mathcal{N}_{e_3}^{2l}) \geq 2^{(l-1)}+1$. Again, $B(\mathcal{N}_{e_4}^{2l-2}) \geq 2$ implies that $B(\mathcal{N}_{e_3}^{2l} \cup \mathcal{N}_{e_4}^{2l}) \geq 2^{(l-1)} + 3 > 2l + 1$ (as $l > 3$), which is a contradiction. Hence, $B(\mathcal{N}_{e_3}^{2l-2}) \geq 2$.

Now, $B(\mathcal{N}_{e_3}^{2l-2}) \geq 2$ and $B(\mathcal{N}_{e_4}^{2l}) \geq 2l$, implies that $B(\mathcal{N}_{e_3}^{2l} \cup \mathcal{N}_{e_4}^{2l}) \geq 2l + 2$ which is a contradiction.

Hence, v_2^1 and v_2^2 are both bad variable nodes.

□

We now have the following theorem:

Theorem 4.4. *If $\mathcal{C}_g$ is a configuration which does not converge in $(l+1)$ iterations, then there exists a configuration $\mathcal{C}_{g-4}$ which does not converge in l iterations.*

Proof. In the configuration $\mathcal{C}_g$, v_2^1 and v_2^2 are bad variable nodes which send incorrect

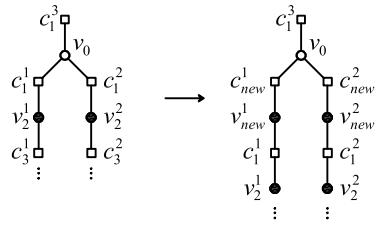
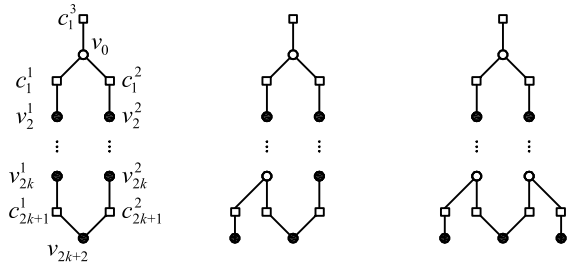


Figure 4.4: Construction of $\mathcal{C}_{g+4}$ from $\mathcal{C}_g$.

messages to check nodes c_1^1 and c_1^2, respectively, in the $(l+1)^{th}$ iteration. This implies that $\mathcal{N}_{e_3}^1$ and $\mathcal{N}_{e_4}^1$ each has a check node, c_3^1 and c_3^2, respectively, that sends an incorrect message in l^{th} iteration to v_2^1 and v_2^2, respectively. Now consider a configuration $\mathcal{C}_{g-4}$ constructed from $\mathcal{C}_g$ by removing the nodes $v_2^1, v_2^2, c_1^1, c_1^2$ and the edges connecting them to their neighbors and introducing the edges (v_0, c_3^1) and (v_0, c_3^2) (see Fig. 4.3). If $\mathcal{C}_g$ has at most $2l+1$ bad variable nodes and no cycles of length less than g, then $\mathcal{C}_{g-4}$ has at most $2(l-1)+1$ bad variable nodes and no cycles of length less than $g-4$. In $\mathcal{C}_{g-4}$ variable node v_0 receives two incorrect messages at the end of l iterations and hence $\mathcal{C}_{g-4}$ is a valid configuration which does not converge in l iterations. □

Theorem 4.4 gives a method to construct valid configurations of bad variable nodes for girth g from valid configurations for girth $g+4$. Also, if $\mathcal{C}_g^1$ and $\mathcal{C}_g^2$ are two distinct valid configurations, then the configurations $\mathcal{C}_{g-4}^1$ and $\mathcal{C}_{g-4}^2$ constructed from $\mathcal{C}_g^1$ and $\mathcal{C}_g^2$, respectively, are distinct. Hence, the number of valid configurations for girth g is greater than or equal to the number of valid configurations for girth $g+4$. Note that the converse of Theorem 4.4 is not true in general. However, for $g \geq 16$, we will show in Theorem 4.5 that any configuration for girth g can be extended to a configuration for girth $g+4$.



(a) (b) (c)

Figure 4.5: Configurations of at most $2l + 1$ bad variable nodes free of cycles of length less than $4l + 4$ which do not converge in $(l + 1)$ iterations.

Theorem 4.5. *For $g/2$ even and $l \geq 3$, there are only three valid configurations which do not converge in $(l + 1)$ iterations.*

Proof. For $l = 3$, we have $g = 16$ and there are only three valid configurations as given in Fig 4.2. So, for $g \geq 16$ and $g/2$ even, there can be at most three valid configurations. Each valid configuration for $g = 16$, can be extended to a configuration $\mathcal{C}_{20}$ for $g = 20$ by the addition of two bad variable nodes v_{new}^1 and v_{new}^2 in the following way. Remove the edges (v_0, c_1^1) and (v_0, c_1^2). Add bad variable nodes v_{new}^1 and v_{new}^2 and check nodes c_{new}^1 and c_{new}^2. Introduce the edges (v_0, c_{new}^1), (v_0, c_{new}^2), (v_{new}^1, c_{new}^1), (v_{new}^2, c_{new}^2), (v_{new}^1, c_1^1) and (v_{new}^2, c_1^2) (see Fig. 4.3 for an illustration). It can be seen that $\mathcal{C}_{20}$ is a valid configuration for girth $g = 20$. In general, the configurations constructed using the above method from the valid configurations for $g \geq 16$ are valid configurations for $g + 4$. Fig. 4.5 illustrates the three configurations for all $l \geq 3$. □

Remark: In all valid configurations $\mathcal{C}_g$ with $g \geq 16$, no bad variable node receives two incorrect messages at the end of the $(l + 1)^{th}$ iteration.

Theorem 4.6. *All valid configurations $\mathcal{C}_g$ converge to a codeword in $g/2$ iterations.*

Proof. We prove the theorem for one configuration for $g = 16$ only. The proof is similar for other configurations. At the end of fourth iteration, let v_0 receive two incorrect messages (see Fig 4.2(a)). It can be seen that there cannot exist another variable node (either good or bad) which receives two incorrect messages without violating the $g = 16$ constraint. Also, v_8 receives all correct messages and $v_2^1, v_2^2, v_4^1, v_4^2, v_6^1, v_6^2$ receive one incorrect message each from $c_3^1, c_3^2, c_5^1, c_5^2, c_7^1, c_7^2$, respectively. In the fifth iteration, we have

$$\begin{aligned}
\omega_5(v_0, c_1^3) &= 1, \\
\omega_5(v, : \backslash c) &= \{1\}, \quad (v, c) \in \{(v_2^1, c_3^1), (v_2^2, c_3^2), \\
&\qquad (v_4^1, c_5^1), (v_4^2, c_5^2), (v_6^1, c_7^1), (v_6^2, c_7^2)\}, \\
\omega_5(v, c) &= 0, \quad \text{otherwise.}
\end{aligned}$$

In the sixth iteration, we have

$$\begin{aligned}
\omega_6(v_0, c_1^3) &= 1, \\
\omega_6(v, : \backslash c) &= \{1\}, \quad (v, c) \in \{(v_2^1, c_3^1), (v_2^3, c_3^2), (v_4^1, c_5^1), \\
&\qquad (v_4^2, c_5^2)\}, \\
\omega_6(v, c) &= 0, \quad \text{otherwise.}
\end{aligned}$$

In the seventh iteration, we have,

$$\begin{aligned}
\omega_7(v_0, c_1^3) &= 1, \\
\omega_7(v, : \backslash c) &= \{1\}, \quad (v, c) \in \{(v_2^1, c_3^1), (v_2^2, c_3^2)\} \\
\omega_7(v, c) &= 0, \quad \text{otherwise.}
\end{aligned}$$

Finally in the eighth iteration, we have,

$$\begin{aligned} \omega_8(v_0, c_1^3) &= 1, \\ \omega_8(v, c) &= 0, \quad \text{otherwise} \end{aligned}$$

At the end of eighth iteration, no variable node receives two incorrect messages and hence the decoder converges to a valid codeword.

□

$g = 12$

In this case, $l = 2$ and the incoming messages at the end of the second iteration are independent. We need to prove that any code with Tanner graph with $g = 12$, can correct all error patterns of weight less than six. Let v be a variable node which sends an incorrect message in the third iteration along edge $e = (v, c)$ given that there are at most 5 bad variables and $\mathcal{N}_e^5$ is a tree. Fig. 4.1(c) illustrates different configurations of bad variable nodes in $\mathcal{N}_e^4$.

Fig. 4.6 shows all possible configurations of five or less bad variable nodes which do not converge to a codeword at the end of three iterations. However, all the configurations converge to a codeword in six iterations. The proofs for configurations in Fig. 4.6(a)-(h) are similar to the proof for configuration in Fig. 4.2(a) and are omitted. Since, configuration (i) has only four bad variable nodes, a complete proof for convergence requires considering all possible locations of the fifth bad variable node, but other than that the structure of the proof is identical to that of the proof for the configuration in Fig. 4.2(a). It is worth noting that in this case, there exist configurations in which a bad variable receives two incorrect messages at the

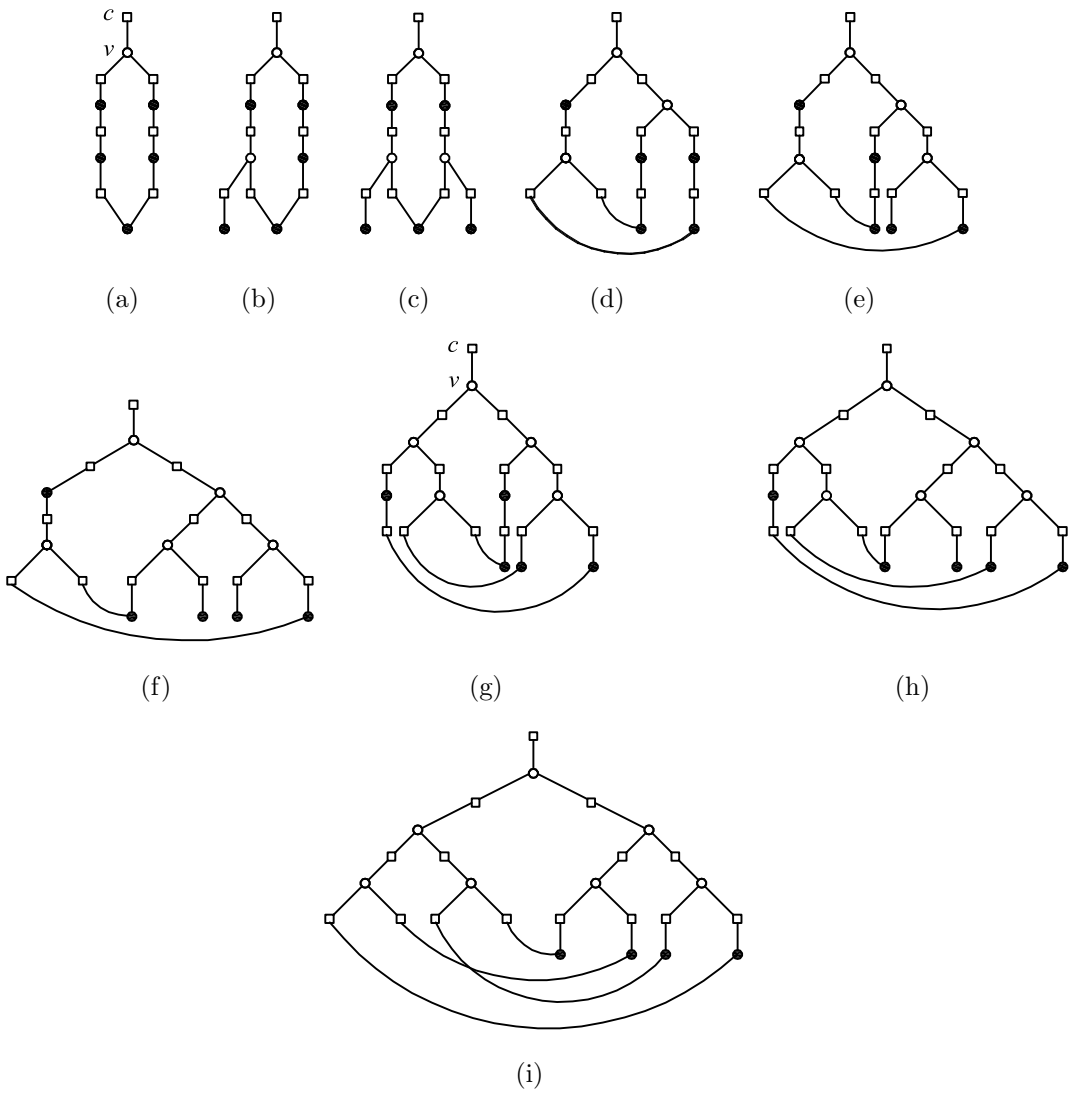


Figure 4.6: Configurations of at most 5 variable nodes free of cycles of length less than 12 which do not converge in 3 iterations.

end of the third iteration. However, all the configurations eventually converge to a codeword.

4.3.2 $g/2$ is odd

In this case, we have $l = (g-2)/4$ and we need to prove that the code is capable of correcting all error patterns of weight $g/2 - 1 = 2l$ or less. The methodology of the proof is similar to the proof in the case when $g/2$ is even. In this case, we have $\mathcal{N}_{e_3}^{i} \cap \mathcal{N}_{e_4}^{j} = \phi$ for $i + j < 4l - 2$. This implies that $\mathcal{N}_{e_3}^{2l}$ and $\mathcal{N}_{e_4}^{2l}$ can have a common node at depth $2l-1$. Therefore, in any configuration, $B(\mathcal{N}_{e_3}^{2l} \cup \mathcal{N}_{e_4}^{2l})$ is lower bounded by $\max\left(B(\mathcal{N}_{e_3}^{2l-4}) + B(\mathcal{N}_{e_4}^{2l}), B(\mathcal{N}_{e_3}^{2l}) + B(\mathcal{N}_{e_4}^{2l-4})\right)$. The valid configurations in this case are the ones which satisfy $B(\mathcal{N}_{e_3}^{2l} \cup \mathcal{N}_{e_4}^{2l}) \leq 2l$. We again deal with $g \geq 14$ and $g = 10$ separately.

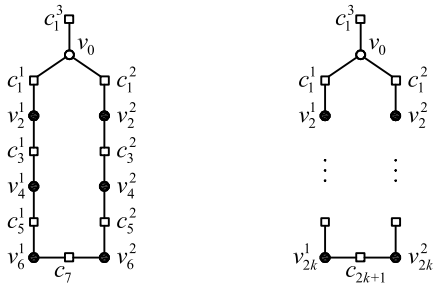
$g \geq 14$

Lemma 4.7. *For $g = 14$, there is only one configuration of six bad variable nodes which does not converge in four iterations.*

Proof. Using arguments outlined above and the configurations in Fig. 4.1(b) along with the constraint that $g \geq 14$, we conclude that there is only one configuration which does not converge in four iterations, which is shown in Fig. 4.7(a). □

Lemma 4.8. *If $\mathcal{C}_g$ with $g \geq 14$ is a valid configuration which does not converge in $l + 1$ iterations, then v_1^1 and v_1^2 are bad variable nodes*

Proof. Similar to the proof of Lemma 4.8. □



(a) (b)

Figure 4.7: (a) Configuration of at most 6 bad variable nodes free of cycles of length less than 14 which does not converge in 4 iterations (b) Configuration of at most $2l$ bad variable nodes free of cycles of length less than $4l + 2$ which does not converge in $l + 1$ iterations.

Theorem 4.9. *If $\mathcal{C}_g$ is a valid configuration which does not converge in $l + 1$ iterations, then there exists a valid configuration $\mathcal{C}_{g-4}$ which does not converge in l iterations.*

Proof. Similar to the proof of Theorem 4.4. □

Theorem 4.10. *For $l \geq 3$, there is only one valid configuration which does not converge in $l + 1$ iterations.*

Proof. For $l = 3$, we have $g = 14$ and there is only one configuration. For $l = 4$, the number of valid configurations cannot be more than one. The valid configuration for $g = 14$, can be extended to a configuration for $g = 18$ (in the same manner as in Theorem 4.5). In general, the valid configuration for girth g can be extended to a valid configuration for girth $g + 4$. Fig. 4.7(b) shows $\mathcal{C}_g$ for all $g \geq 14$. □

Theorem 4.11. *The configuration $\mathcal{C}_g$ converges to a codeword in $g/2$ iterations.*

Proof. Similar to the proof of Theorem 4.6. □

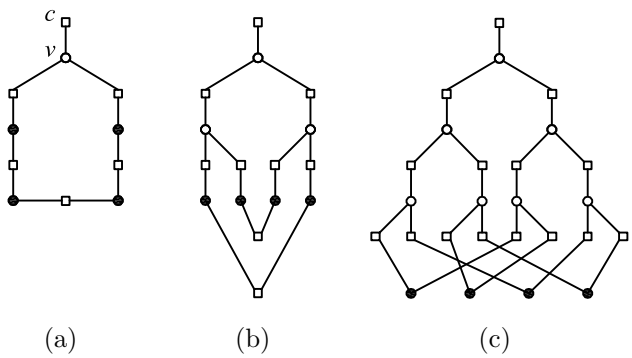


Figure 4.8: Configurations of at most 4 variable nodes free of cycles of length less than 10 which do not converge in 3 iterations.

$g = 10$

In this case, $l = 2$ and there are three configurations which do not converge at the end of the third iteration. Fig. 4.8 shows the three configurations. It can be shown that these configurations converge in five iterations.

CHAPTER 5

Error Correction Capability of Higher Column Weight Codes

All generalizations are false, including this one.

Mark Twain

In Chapters 3 and 4, we have focused on column-weight-three LDPC codes decoded using the Gallager A algorithm. In this chapter, we turn our attention toward higher column weight codes. The analysis of such codes under Gallager A/B is quite complicated and hence we consider the simpler algorithm namely the parallel bit flipping algorithm. Recall that the expander based arguments can be applied to these codes. The approach in this chapter is to find the size of variable node sets which expand sufficiently, as a function of the girth of the Tanner graph. We also establish upper bounds on the guaranteed error correction capability by studying the size of smallest possible fixed sets in a Tanner graph of given column weight and girth. By saying that a code with column weight d_v and girth g is not guaranteed to correct q errors, we mean that there exists a code with column weight d_v and girth g that fails to correct q errors or in other words, there exists at least one code which has an inducing set of size q. We start with an overview of expander based arguments. We then proceed to establish relation between girth and expansion of left-regular graphs. We then study cage graphs and their relation to fixed sets.

5.1 Expansion and Error Correction Capability

Sipser and Spielman [20] analyzed the performance of the bit flipping algorithms using the expansion properties of the underlying Tanner graph of the code. We summarize the results from [20] below for the sake of completeness. We start with the following definitions from [20].

Definition 5.1. Let $G = (U, E)$ with $|U| = n_1$. Then *every set of at most* m_1 *nodes expands by a factor of* δ if, for all sets $S \subset U$

$$|S| \leq m_1 \Rightarrow |\{u_2 : \exists u_1 \in S \text{ such that } (u_1, u_2) \in E\}| > \delta|S|.$$

□

We consider bipartite graphs and expansion of variable nodes only.

Definition 5.2. A graph is a $(d_v, d_c, \alpha, \delta)$ expander if it is a (d_v, d_c) regular bipartite graph in which every subset of at most α fraction of the variable nodes expands by a factor of at least δ. □

The following theorem from [20] relates the expansion and error correction capability of an (n, d_v, d_c) LDPC code with Tanner graph G when decoded using the parallel bit flipping decoding algorithm.

Theorem 5.3. *[20, Theorem 11] Let* G *be a* $(d_v, d_c, \alpha, (3/4 + \epsilon)d_v)$ *expander over* n *variable nodes, for any* $\epsilon > 0$*. Then, the parallel bit flipping algorithm will correct any* $\alpha_0 < \alpha(1 + 4\epsilon)/2$ *fraction of errors after* $\log_{1-4\epsilon}(\alpha_0 n)$ *decoding rounds.* □

Notes:

1. The serial bit flipping algorithm can also correct $\alpha_0 < \alpha/2$ fraction of errors if G is a $(d_v, d_c, \alpha, (3/4)d_v)$ expander.

2. The results hold for any left regular code as expansion is needed for variable nodes only.

From the above discussion, it is observed that finding the number of variable nodes which are guaranteed to expand by a factor of at least $3d_v/4$, gives a lower bound on the guaranteed error correction capability of LDPC codes.

5.2 Column Weight, Girth and Expansion

In this section, we prove our main theorem which relates the column weight and girth of a code to its error correction capability. We show that the size of variable node sets which have the required expansion is related to the well known Moore bound [59, p.180]. We start with a few definitions required to establish the main theorem.

5.2.1 Definitions

Definition 5.4. The *reduced graph* $G_r = (V \cup C_r, E'_r)$ of $G = (V \cup C, E')$ is a graph with vertex set $V \cup C_r$ and edge set E'_r given by

$$\begin{aligned} C_r &= C \setminus C_p,\ C_p = \{c \in C : \text{c is a pendant node}\} \\ E'_r &= E' \setminus E'_p,\ E'_p = \{(v_i, c_j) \in E : c_j \in C_p\}. \end{aligned}$$

□

Definition 5.5. Let $G = (V \cup C, E')$ be such that $\forall u \in V, d(u) \leq d_v$. The d_v *augmented graph* $G_{d_v} = (V \cup C_{d_v}, E'_{d_v})$ is a graph with vertex set $V \cup C_{d_v}$ and edge

set E'_{d_v} given by

$$\begin{aligned} C_{d_v} &= C \cup C_a, \text{ where } C_a = \bigcup_{i=1}^{|V|} C_a^i \text{ and} \\ C_a^i &= \{c_1^i, \ldots, c_{d_v - d(v_i)}^i\}; \\ E'_{d_v} &= E' \cup E'_a, \text{ where } E'_a = \bigcup_{i=1}^{|V|} E_a'^i \text{ and} \\ E_a'^i &= \{(v_i, c_j) \in V \times C_a : c_j \in C_a^i\}. \end{aligned}$$

□

Definition 5.6. [20, Definition 4] The *edge-vertex incidence graph* $G_{ev} = (U \cup E, E_{ev})$ of $G = (U, E)$ is the bipartite graph with vertex set $U \cup E$ and edge set

$$E_{ev} = \{(e, u) \in E \times U : u \text{ is an endpoint of e}\}.$$

□

Notes:

1. The edge-vertex incidence graph is right regular with degree two.

2. $|E_{ev}| = 2|E|$ and $g(G_{ev}) = 2g(G)$.

Definition 5.7. An *inverse edge-vertex incidence graph* $G_{iev} = (V, E'_{iev})$ of $G = (V \cup C, E')$ is a graph with vertex set V and edge set E'_{iev} which is obtained as follows. For $c \in C_r$, let $N(c)$ denote the set of neighbors of c. Label one node $v_i \in N(c)$ as a root node. Then

$$\begin{aligned} E'_{iev} &= \{(v_i, v_j) \in V \times V : v_i \in N(c), v_j \in N(c), \\ &\qquad i \neq j, \ v_i \text{ is a root node, for some } c \in C_r\}. \end{aligned}$$

□

Notes:

1. Given a graph, the inverse edge-vertex incidence graph is not unique.

2. $g(G_{iev}) \geq g(G)/2$, $|E'_{iev}| = |E'_r| - |C_r|$ and $|C_r| \leq |E'_r|/2$.

3. $|E'_{iev}| \geq |E'_r|/2$ with equality only if all checks in C_r have degree two.

4. The term inverse edge-vertex incidence is used for the following reason. Suppose all checks in G have degree two. Then the edge-vertex incidence graph of G_{iev} is G.

The *Moore bound* [59, p.180] denoted by $n_0(d, g)$ is a lower bound on the least number of vertices in a d-regular graph with girth g. It is given by

$$\begin{aligned} n_0(d, g) = n_0(d, 2r+1) &= 1 + d\sum_{i=0}^{r-1}(d-1)^i, \ g \text{ odd} \\ n_0(d, g) = n_0(d, 2r) &= 2\sum_{i=0}^{r-1}(d-1)^i, \ g \text{ even.} \end{aligned}$$

In [60], it was shown that a similar bound holds for irregular graphs.

Theorem 5.8. *[60] The number of nodes $n(\overline{d}, g)$ in a graph of girth g and average degree at least $\overline{d} \geq 2$ satisfies*

$$n(\overline{d}, g) \geq n_0(\overline{d}, g).$$

□

Note that $\overline{d}$ need not be an integer in the above theorem.

5.2.2 The Main Theorem

We now state and prove the main theorem.

Theorem 5.9. *Let G be a $d_v \geq 4$-left regular Tanner graph G with $g(G) = g$. Then for all $k_1 < n_0(d_v/2, g/2)$, any set of k_1 variable nodes in G expands by a factor of at least $3d_v/4$.*

Proof. Let $G^{k_1} = (V_1^k \cup C_1^k, E_1^k)$ denote the subgraph induced by a set of k_1 variable nodes V^{k_1}. Since G is d_v-left regular, $|E^{k_1}| = d_v k_1$. Let $G_r^{k_1} = (V^{k_1} \cup C_r^{k_1}, E_r^{k_1})$ be the reduced graph. We have

$$\begin{aligned} |C^{k_1}| &= |C_r^{k_1}| + |C_p^{k_1}| \\ |E_1^k| &= |E_p^{k_1}| + |E_r^{k_1}| \\ |E_p^{k_1}| &= |C_p^{k_1}| \\ |C_p^{k_1}| &= d_v k_1 - |E_r^{k_1}|. \end{aligned}$$

We need to prove that $|C_1^k| > 3d_v k_1/4$.

Let $f(k_1, g/2)$ denote the maximum number of edges in an arbitrary graph of order k_1 and girth $g/2$. By Theorem 2, for all $k_1 < n_0(d_v/2, g/2)$, the average degree of a graph with k_1 nodes and girth $g/2$ is less than $d_v/2$. Hence, $f(k_1, g/2) < d_v k_1/4$. We now have the following lemma.

Lemma 5.10. *The number of edges in $G_r^{k_1}$ cannot exceed $2f(k_1, g/2)$ i.e.,*

$$|E_r^{k_1}| \leq 2f(k_1, g/2).$$

Proof. The proof is by contradiction. Assume that $|E_r^{k_1}| > 2f(k_1, g/2)$. Consider $G_{iev}^{k_1} = (V^{k_1}, E_{iev}^{k_1})$, an inverse edge vertex incidence graph of G^{k_1}. We have

$$|E_{iev}^{k_1}| > f(k_1, g/2).$$

This is a contradiction as $G^{k_1}_{eiv}$ is a graph of order k_1 and girth at least $g/2$. □

We now find a lower bound on $|C^k_1|$ in terms of $f(k_1, g/2)$. We have the following lemma.

Lemma 5.11. $|C^{k_1}| \geq d_v k_1 - f(k_1, g/2)$.

Proof. Let $|E^{k_1}_r| = 2f(k_1, g/2) - j$ for some integer $j \geq 0$. Then $|E^{k_1}_p| = d_v k_1 - 2f(k_1, g/2) + j$. We claim that $|C^{k_1}_r| \geq f(k_1, g/2) + j$. To see this, we note that

$$\begin{aligned} |E^{k_1}_{iev}| &= |E^{k_1}_r| - |C^{k_1}_r|, \text{ or} \\ |C^{k_1}_r| &= |E^{k_1}_r| - |E^{k_1}_{iev}|. \end{aligned}$$

But

$$\begin{aligned} |E^{k_1}_{iev}| &\leq f(k_1, g/2) \\ \Rightarrow |C^{k_1}_r| &\geq 2f(k_1, g/2) - j - f(k_1, g/2) \\ \Rightarrow |C^{k_1}_r| &\geq f(k_1, g/2) - j. \end{aligned}$$

Hence we have,

$$\begin{aligned} |C^{k_1}| &= |C^{k_1}_r| + |C^{k_1}_p| \\ \Rightarrow |C^{k_1}| &\geq f(k_1, g/2) - j + d_v k_1 - 2f(k_1, g/2) + j \\ \Rightarrow |C^{k_1}| &\geq d_v k_1 - f(k_1, g/2). \end{aligned}$$

□

The theorem now follows as

$$f(k_1, g/2) < d_v k_1/4$$

and therefore

$$|C^{k_1}| > 3d_v k_1/4.$$

□

Corollary 5.12. *Let $\mathcal{C}$ be an LDPC code with column weight $d_v \geq 4$ and girth g. Then the bit flipping algorithm can correct any error pattern of weight less than $n_0(d_v/2, g/2)/2$.* □

5.3 Cage Graphs and Trapping Sets

In this section, we first give necessary and sufficient conditions for a given set of variables to be a fixed set. We then proceed to define a class of interesting graphs known as cage graphs [61] and establish a relation between cage graphs and fixed sets. We then give an upper bound on the error correction capability based on the sizes of cage graphs. The proofs in this section are along the same lines as in Section 5.2. Hence, we only give a sketch of the proofs.

Theorem 5.13. *Let $\mathcal{C}$ be an LDPC code with d_v-left regular Tanner graph G. Let $\mathcal{T}$ be a set of variable nodes with induced subgraph $\mathcal{I}$. Let the checks in $\mathcal{I}$ be partitioned into two disjoint subsets; $\mathcal{O}$ consisting of checks with odd degree and $\mathcal{E}$ consisting of checks with even degree. Then $\mathcal{T}$ is a fixed set for bit flipping algorithm iff : (a) Every variable node in $\mathcal{I}$ has at least $\lceil d_v/2 \rceil$ neighbors in $\mathcal{E}$, and (b) No $\lfloor d_v/2 \rfloor + 1$ checks of $\mathcal{O}$ share a neighbor outside $\mathcal{I}$.*

Proof. We first show that the conditions stated are sufficient. Let $\mathbf{x}_{\mathcal{T}}$ be the input to the bit flipping algorithm, with support $\mathcal{T}$. The only unsatisfied constraints are in $\mathcal{O}$. By the conditions of the theorem, we observe that no variable node is involved

in more unsatisfied constraints than satisfied constraints. Hence, no variable node is flipped and by definition $\mathbf{x}_{\mathcal{T}}$ is a fixed point implying that $\mathcal{T}$ is a fixed set.

To see that the conditions are necessary, observe that for $\mathbf{x}_{\mathcal{T}}$ to be a fixed set, no variable node should be involved in more unsatisfied constraints than satisfied constraints. □

Remark: Theorem 5.13 is a consequence of Fact 3 from [28].

To determine whether a given set of variables is a fixed set, it is necessary to not only know the induced subgraph but also the neighbors of the odd degree checks. However, in order to establish general bounds on the sizes of fixed sets given only the column weight and the girth, we consider only condition (a) of Theorem 5.13 which is a necessary condition. A set of variable nodes satisfying condition (a) is known as a *potential fixed set*. A fixed set is a potential fixed set that satisfies condition (b). Hence, a lower bound on the size of the potential fixed set is a lower bound on the size of a fixed set. It is worth noting that a potential fixed set can always be extended to a fixed set by successively adding a variable node till condition (b) is satisfied.

Definition 5.14. [61] A (d, g)-*cage graph*, $G(d, g)$, is a d-regular graph with girth g having the minimum possible number of nodes. □

A lower bound, $n_l(d, g)$, on the number of nodes $n_c(d, g)$ in a (d, g)-cage graph is given by the Moore bound. An upper bound $n_u(d, g)$ on $n_c(d, g)$ (see [61] and

references therein) is given by

$$n_u(3,g) = \begin{cases} \frac{4}{3} + \frac{29}{12}\, 2^{g-2} & \text{for g odd} \\ \frac{2}{3} + \frac{29}{12}\, 2^{g-2} & \text{for g even} \end{cases}$$

$$n_u(d,g) = \begin{cases} 2(d-1)^{g-2} & \text{for g odd} \\ 4(d-1)^{g-3} & \text{for g even} \end{cases}.$$

Theorem 5.15. *Let $\mathcal{C}$ be an LDPC code with d_v-left regular Tanner graph G and girth g. Let $|\mathcal{T}(d_v, g)|$ denote the size of smallest possible potential fixed set of $\mathcal{C}$ for the bit flipping algorithm. Then,*

$$|\mathcal{T}(d_v, g)| = n_c(\lceil d_v/2 \rceil, g/2).$$

Proof. We first prove the following lemma and then exhibit a potential fixed set of size $n_c(\lceil d_v/2 \rceil, g/2)$.

Lemma 5.16. $|\mathcal{T}(d_v, g)| \geq n_c(\lceil d_v/2 \rceil, g/2)$.

Proof. Let $\mathcal{T}_1$ be a fixed set with $|\mathcal{T}_1| < n_c(\lceil d_v/2 \rceil, g/2)$ and let G_1 denote the induced subgraph of $\mathcal{T}_1$. We can construct a $(\lceil d_v/2 \rceil, g/2')$- cage graph $(g/2' \geq g/2)$ with $|\mathcal{T}_1| < n_c(\lceil d_v/2 \rceil, g/2)$ nodes by removing edges (if necessary) from the inverse edge-vertex of G_1 which is a contradiction. □

We now exhibit a potential fixed set of size $n_c(\lceil d_v/2 \rceil, g/2)$. Let $G_{ev}(\lceil d_v/2 \rceil, g/2)$ be the edge-vertex incidence graph of a $G(\lceil d_v/2 \rceil, g/2)$. Note that $G_{ev}(\lceil d_v/2 \rceil, g/2)$ is a left regular bipartite graph with $n_c(\lceil d_v/2 \rceil, g/2)$ variable nodes of degree $\lceil d_v/2 \rceil$ and all checks have degree two. Now consider $G_{ev,d_v}(\lceil d_v/2 \rceil, g/2)$, the d_v augmented graph of $G_{ev}(\lceil d_v/2 \rceil, g/2)$. It can be seen that $G_{ev,d_v}(\lceil d_v/2 \rceil, g/2)$ is a potential fixed set. □

Theorem 5.17. *There exists a code $\mathcal{C}$ with d_v-left regular Tanner graph of girth g which fails to correct $n_c(\lceil d_v/2 \rceil, g/2)$ errors.*

Proof. Let $G_{ev,d_v}(\lceil d_v/2 \rceil, g/2)$ be as defined in Theorem 5.15. Now construct a code $\mathcal{C}$ with column weight d_v and girth g starting from $G_{ev,d_v}(\lceil d_v/2 \rceil, g/2)$ such that the set of variable nodes in $G_{ev,d_v}(\lceil d_v/2 \rceil, g/2)$ also satisfies condition (b) of Theorem 5.13. Then, by Theorem 5.13 and Theorem 5.15, the set of variable nodes in $G_{ev,d_v}(\lceil d_v/2 \rceil, g/2)$ with cardinality $n_c(\lceil d_v/2 \rceil, g/2)$ is a fixed set and hence $\mathcal{C}$ fails to decode an error pattern of weight $n_c(\lceil d_v/2 \rceil, g/2)$. □

Remark: We note that for $d_v = 3$ and $d_v = 4$, the above bound is tight. Observe that for $d = 2$, the Moore bound is $n_0(d, g) = g$ and that a cycle of length g with g variable nodes is always a potential trapping set. In fact, for a code with $d_v = 3$ or 4, and Tanner graph of girth greater than eight, a cycle of the smallest length is always a trapping set (from Chapter 3).

CHAPTER 6

Applications and Numerical Results

Prediction is difficult, especially of the future.

Neils Bohr

The focus of the previous chapters was to establish theoretical bounds on the guaranteed error correction capability of LDPC codes. Short to moderate length codes generally have Tanner graphs with girth six or eight. In this chapter, we focus on such codes and show the statistics of trapping sets and instantons for different codes. We also illustrate how to construct codes with superior error floor performance.

6.1 Trapping Set Statistics for Different Codes

In this section, we present statistics of trapping sets for the following three different column-weight-three codes (i) A (816,3,6) regular random LDPC code taken from MacKay's webpage [63] (ii) The (2640,3,6) Margulis code [39] (iii) The (155,3,5) group structured code also known as Tanner code [64]. By making use of Theorem 3.1, we identify potential fixed sets and trapping sets. We then enumerate different trapping sets by graph search techniques and by exploiting the structure of the codes. The $(3, 3)$ trapping set depicted in Fig. 3.1(a) is isomorphic to a six cycle and the $(4, 4)$ trapping set illustrated in Fig. 3.3(a) is isomorphic to an eight cycle. The $(4, 2)$ trapping sets (Fig. 3.1(b)) are one node extensions of $(3, 3)$ trapping sets and the $(5, 3)$ trapping sets (Fig. 3.3(b)) are one node extensions of $(4, 4)$

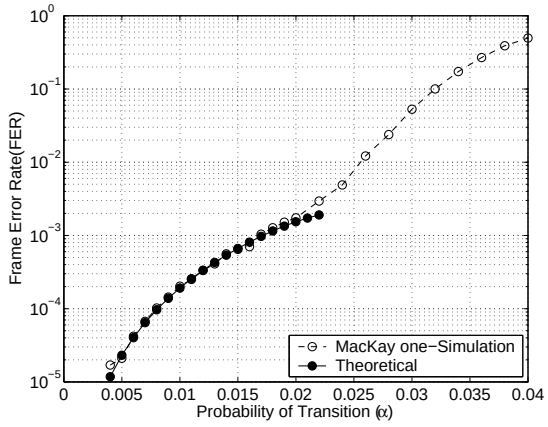
Table 6.1: Different Codes and Their Parameters

Code	$\|V\|$	$\|C\|$	**g**	**Relevant Trapping Sets**	**No. of Trapping Sets**
MacKay's code one	1008	504	6	(3,3);(4,4);(5,3)	165; 1215; 14
Margulis code	2640	1320	8	(4,4);(5,5)	1320; 11088
Tanner code	155	93	8	(5,3)	155

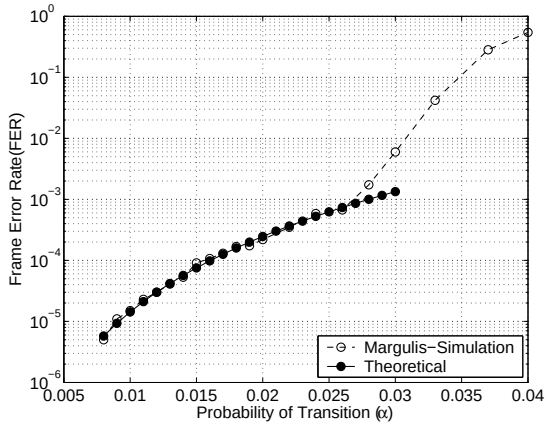
trapping sets. The number of eight cycles in the Margulis code can be found using its automorphism group which has size 1320. Similarly the group structure of the Tanner code can be used to count the number of $(5, 3)$ trapping sets. For random codes, brute force search techniques are used to enumerate different trapping sets. Table 6.1 shows the trapping set statistics for the above three codes. The knowledge of the trapping sets can be used to estimate the FER performance in the error floor region. The details of the method will not be discussed in this work. The interested reader is referred to [54] for more details. We, however, illustrate the results in Fig. 6.1.

6.2 Instanton Statistics for LP Decoding over the BSC

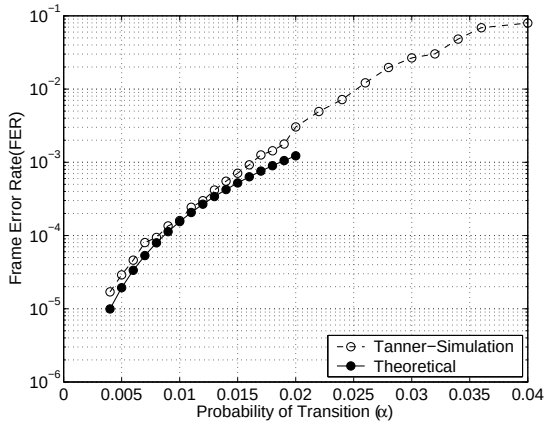
The instantons and pseudo-codewords for LP decoding over the BSC cannot purely be determined by combinatorial and topological considerations as trapping sets and fixed sets for iterative decoders. Instead we use the instanton search algorithm (ISA) to enumerate all instantons of a given code. The ISA was first described in [65]. The reader is referred to Appendix B for details of the ISA. It is however worth noting



(a)



(b)



(c)

Figure 6.1: FER plots for different codes (a) MacKay's random code one (b) The Margulis code (c) The Tanner code

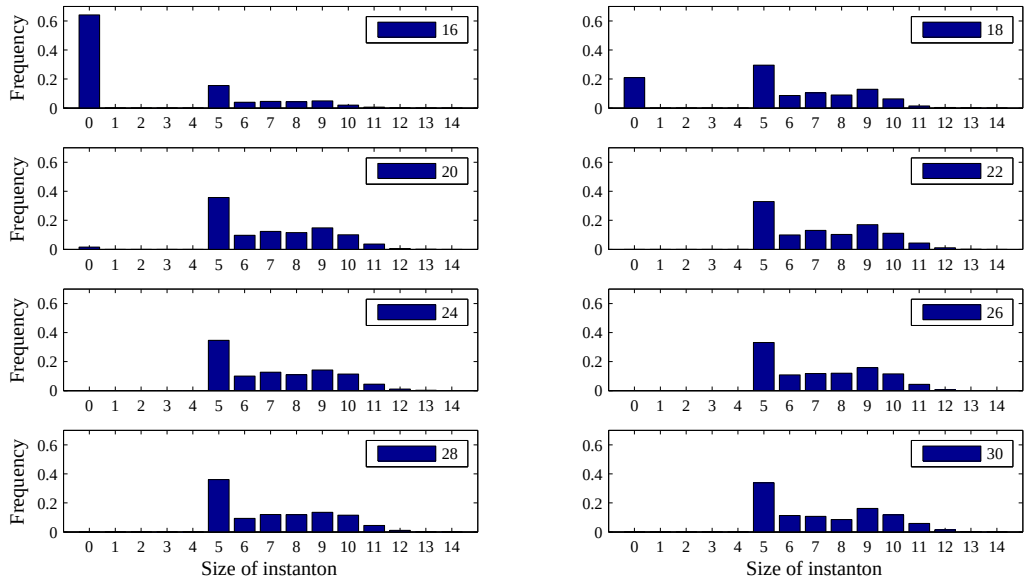


Figure 6.2: Frequency of instanton (ISA outputs) sizes for different weights of inputs. (Total length of the bars (for any of the sub-plots) should sum to one.) The bar-graphs were obtained by running the ISA for 2000 different random initiations with the fixed number of flips (ranging from 16 to 30). Numbers at zero (if any) show the frequency of patterns decoded into the all-zero-codeword.

that once a few instantons have been found by the ISA, the topological structure of these instantons can be exploited to find other instantons.

For illustration purposes, we choose the Tanner code. We ran 2000 ISA trials using random inputs with fixed number of initiation flips. Fig. 6.2 shows the frequency of the instanton sizes for the number of initiation flips ranging from 16 to 30. The value at zero should be interpreted as the number of patterns that decode to the all-zero-codeword. It can be seen that if the initial noise vector consists of 22 or more flips, then it converges to a pseudo-codeword different from the all-zero-codeword in all of 2000 cases.

Note that Fig. 6.2 shows a count of the total number of instantons of the same size, so that multiple trials of ISA may correspond to the same instanton. To correct

for this multiplicity in counting, one can also find it useful (see discussions below) to study the total number of unique instantons observed in the ISA trials, coined the Instanton-Bar-Graph. Fig. 6.3 shows the number of distinct instantons of a given size for 2000 and 5000 random initiations with 20 flips. One finds that the total number of ISA outputs of size 5 after 2000 trails is 720, however representing only 155 distinct instantons. In this case (of the Tanner code), we can independently verify [1] that the total number of instantons of size 5 is indeed 155, thus confirming that our algorithm has found all the instantons of length 5 detecting each of them roughly 4 times. Obviously, the total number of distinct instantons of size 5 does not change with further increase in the number of trails. This observation emphasizes utility of the sub-plots, with different number of initiations, as the comparison allows to judge the sufficiency (or insufficiency) of the number of trials for finding all the given size instantons. Extending the comparison to larger size (> 5) instantons, one observes that the numbers change in transition from 2000 to 5000 trials, thus indicating that the statistics is insufficient (at least after 2000 trials) as some of the instantons have not been found yet. The smallest weight instanton found by the ISA is 5 thereby indicating that the slope of the FER curve in the error floor region is five.

6.3 Code Construction Avoiding Certain Topological Structures

The minimum pseudo-codeword weight (as well as the trapping set size) increase with increase of the Tanner graph girth (see [41, 52]). While girth optimized codes

[1] We have observed that all the instantons of size 5 are in fact the $(5, 3)$ trapping sets described in [54]. Further investigation of the topological structure of instantons will be dealt in future work.

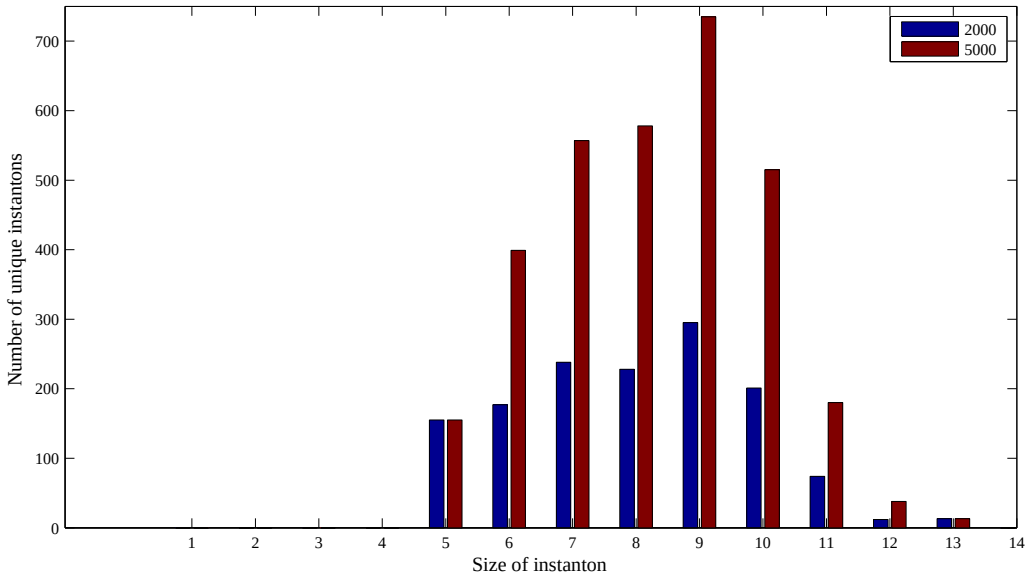


Figure 6.3: Instanton-Bar-Graph showing the number of unique instantons of a given weight found by running the ISA with 20 random flips for 2000 and 5000 initiations respectively.

are known to perform better in general, the code length and degree distribution place a fundamental restriction on the best achievable girth. Observing that the instantons for different decoding algorithms performing over different channels have a common underlying topological structure (e.g. the $(5, 3)$ trapping set in the case of the Tanner code), it is natural to discuss design of a similar but new code which excludes these troublesome structures. In fact, this suggests a natural code optimization technique with an improved instanton distribution. Starting with a reasonably good code (constructed either algebraically or by the progressive edge growth (PEG) method [66]), we find the most damaging instantons and their underlying topological structure. We then construct a new code avoiding such subgraphs (either by swapping edges, by increasing code length, or utilizing a combination of both). We iterate this procedure till the code can no longer be optimized or we reach a

computationally unbearable complexity.

For Gallager A decoding, it has been proved in [56] that codes with Tanner graphs of girth 8 which avoid the $(5, 3)$ trapping set and weight 8 codewords can correct all the error patterns of weight 3 or less. While proving a similar result might be difficult for the iterative decoder over the AWGN channel and LP decoder, such considerations nonetheless play a role in our code design strategy. An algorithm, suggesting construction of a code meeting the Gallager A-related conditions, was provided in [56]. This algorithm can be seen as a generalization of the PEG algorithm [66]. Given a list of forbidden subgraphs, at every step of the algorithm, an edge is established such that the resulting graph at that stage does not consist of any of the forbidden subgraphs. (The PEG algorithm is a special case forbidding cycles shorter than a given threshold.)

Using the algorithm proposed in [56], we constructed a new code of length 155 with uniform left degree 3 and with most check nodes with degree 5. By construction, this code avoids $(5, 3)$ trapping sets. This results in a steeper FER slope of 4 in the error floor domain under the Gallager A decoder, as shown in Fig. 6.4. The minimum weight instanton for LP decoder over the BSC found by the ISA is 6. Fig. 6.4 shows FER performance of the Tanner code and the new code under Gallager A and LP decoders. The instanton distribution for LP decoding over the BSC for the Tanner code and the new code found by running the ISA for 2000 times is shown in Fig. 6.5. All the above statistics illustrate the superiority of the new code.

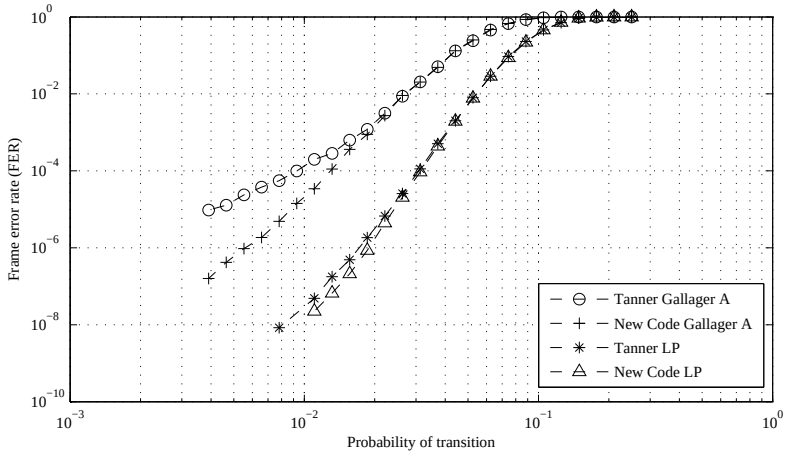


Figure 6.4: FER performance of the Tanner code and the new code.

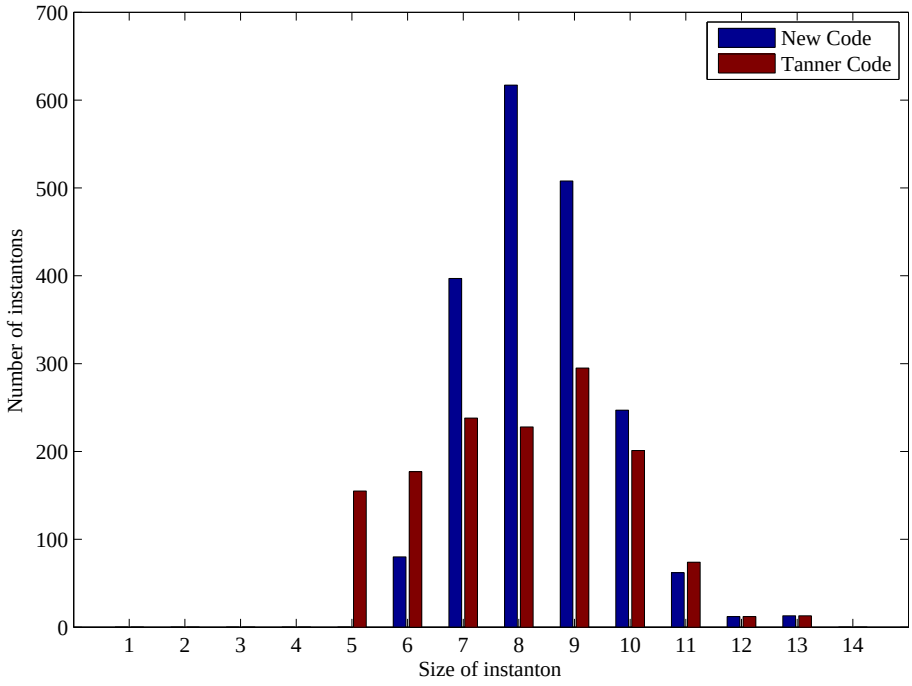


Figure 6.5: Instanton weight distribution for the Tanner code and the new code for LP decoding over the BSC as found by running the ISA 2000 times.

CHAPTER 7

Concluding Remarks

> I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.
>
> Sir Isaac Newton

We have investigated the problem of guaranteed error correction capability of LDPC codes under various hard decision decoding algorithms. The results presented in this work are only a first step toward understanding the decoding failures of various decoders for finite length LDPC codes. One readily identifies that the discussion in the preceding chapters was limited to regular binary LDPC codes and specific hard decision decoding algorithms. We can identify a large class of problems that arise as direct or indirect consequences of the results reported in this dissertation. We list below a few ongoing investigations and also indicate possible future directions for research.

- **Extension to irregular codes:** Trapping set/ pseudo-codeword analysis for ensembles of irregular codes is an important theoretical problem as it is generally known that irregular degree distributions achieve capacity on a variety of channels [18].

- **Extension to soft decision decoding algorithms:** Hard decision algorithms are convenient from an analysis point of view but most of the practical systems employ soft decision algorithms. Analysis of multi-bit decoders is a first step in this process.

- **Analysis of classes of codes:** A wide variety of structured codes have been investigated for practical implementations. An important class of such codes is the protograph-based codes [67]. Trapping set/pseudo-codeword analysis for such codes allows construction of codes with low-complexity encoders/decoders.

- **Extension to non-binary LDPC codes:** Many properties of non-binary LDPC codes [11] are largely unknown owing to the complexity of the decoding algorithms. Study of simple hard decision algorithms and their asymptotic analysis are one of the many interesting problems in this area.

- **Practical LP Decoders:** While very attractive from a theoretical point of view, LP decoders suffer from huge computational complexity. Methods that successively improve LP decoding by adding constraints hold lot of promise for implementation purposes. The interested reader is referred to [68, 69, 70, 71, 72] for introduction to these problems.

- **Extension to GLDPC codes:** Generalized LDPC (GLDPC) codes [12] have many interesting properties and it would be interesting to see how their error correction capability depends on column weight and error correction capability of the sub-code (see [53] for initial results).

While the above mentioned problems are important from a theoretical stand-

point, it should be kept in mind that the ultimate purpose of the analysis methods developed and presented in this work is to aid in the construction of codes with good error floor performance. To this end, it is of utmost importance to formulate better and tighter bounds for higher column weight codes and derive necessary and sufficient conditions to guarantee correction of certain number of errors. Answers to the above questions will help in system architectures that guarantee superior error floor performance.

APPENDIX A

Column-Weight-Three Codes with Tanner Graphs of Girth Less Than 10

In Chapter 3, we established a relation between the size of inducing set and the girth of Tanner graph for a column-weight-three LDPC code. The results of Chapter 3 and Chapter 4, provide unambiguous bounds on minimum number of errors correctable by the Gallager A algorithm for codes with Tanner graphs of girth greater than eight. However, there still remains some ambiguity about codes with girth less than or equal to eight.

We have shown in Lemma 3.3, that a code with girth $g = 4$ always has an inducing set of size two or three. It is unlikely for a code with girth $g = 4$ not to contain an inducing set of size two and even if such a code exists it might be very difficult to construct such a code. So, it is safe to assume that a code with Tanner graph of girth $g = 4$ cannot correct two errors. We are now tempted to ask under what conditions can it correct a single error. It can easily be seen that this is possible if the Tanner graph does not contain codewords of weight two, a result that is hardly surprising. Similarly, it can be said that a code with Tanner graph of girth $g = 6$ cannot correct three errors and can correct two errors if the Tanner graph avoids codewords of weight four. We will not elaborate on these results as it is easy to consider all possible configurations and derive the conditions.

The problem of correction of three errors is relatively more complicated. As we have observed that codes with Tanner graph of girth $g = 6$ cannot correct three

errors (with high probability), we conclude that the girth should be at least eight. However, girth of eight alone cannot guarantee correction of three errors as there can be trapping sets that have critical number three. In this chapter, we prove that apart from the $(3,3)$ trapping sets, there exist two more trapping sets, namely $(5,3)$ trapping set and $(8,0)$ trapping set that can have critical number three. We also show that avoiding structures isomorphic to these trapping sets guarantees correction of three errors.

A.1 The Case of Codes with Tanner Graphs of Girth Eight

Lemma A.1. *There exist (5,3) and (8,0) trapping sets with critical number three and no (5,3) or (8,0) trapping sets with critical number less than three.*

Proof. We prove the lemma for the case of $(5,3)$ trapping sets and omit the proof for $(8,0)$ trapping sets. Consider the $(5,3)$ trapping set shown in Fig. A.1. Let $V^1 := \{v_1^1, v_2^1, v_3^1\}$ be the set of variables which are initially in error. Let $C^1 := \{c_1^1, c_2^1, \ldots, c_9^1\}$ and $V^2 := \{v_1^2, v_2^2\}$. Also, assume that no variable node in $V \backslash (V^1 \cup V^2)$, has two or more neighbors in C^1. In the first iteration, we have:

$$\omega_1(v,c) = \begin{cases} 1 & \text{if } v \in V^1 \\ 0 & \text{otherwise} \end{cases} \tag{A.1}$$

$$\overline{\omega}_1(c,v) = \begin{cases} 1 & \text{if } c \in C^1,\ v \notin V^1 \\ 0 & \text{otherwise} \end{cases} \tag{A.2}$$

Consequently, all variable nodes in V^2 are decoded incorrectly at the end of the first

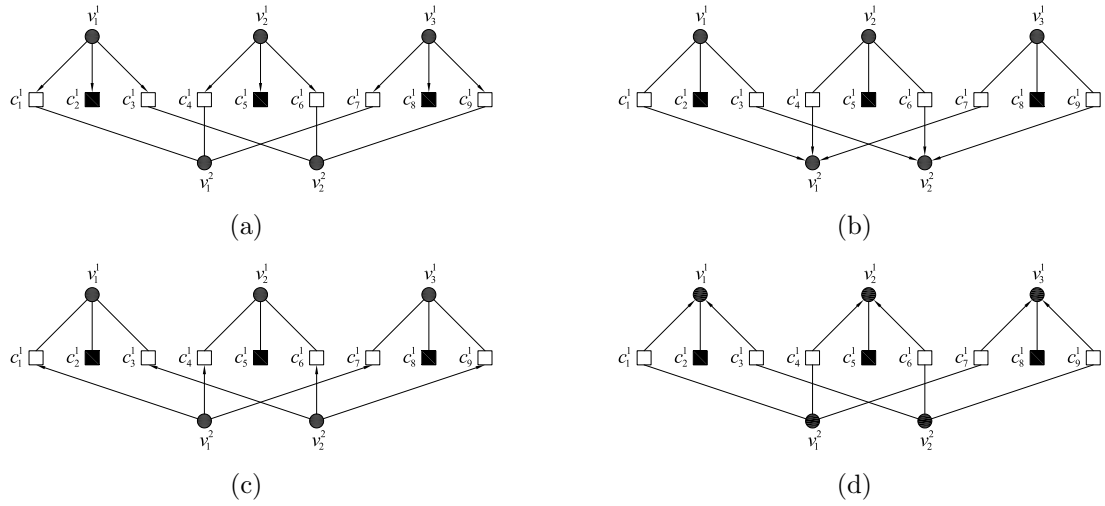


Figure A.1: Illustration of message passing for a $(5, 3)$ trapping set: (a) variable to check messages in round one; (b) check to variable messages in round one; (c) variable to check messages in round two; and (d) check to variable messages in round two. Arrow-heads indicate the messages with value 1.

iteration. In the second iteration:

$$\omega_2(v, c) = \begin{cases} 1 & \text{if } v \in V^2 \\ 0 & \text{otherwise} \end{cases} \tag{A.3}$$

$$\overline{\omega}_2(c, v) = \begin{cases} 1 & \text{if } c \in C^1 \backslash \{c_2^1, c_5^1, c_8^1\},\ v \notin V^2 \\ 0 & \text{otherwise} \end{cases} \tag{A.4}$$

and all variable nodes in V^1 are decoded incorrectly. Continuing in this fashion, $\omega_3(v, c) = \omega_1(v, c)$ and $\overline{\omega}_3(c, v) = \overline{\omega}_1(c, v)$. That is, the messages being passed in the Tanner graph would repeat after every two iterations. Hence, three variable nodes in error initially can lead to a decoder failure and therefore, this $(5, 3)$ trapping set has critical number equal to three. □

Theorem A.2. *To guarantee that three errors in a column-weight-three LDPC code can be corrected by the Gallager-A algorithm, it is necessary to avoid* $(3, 3)$*,* $(5, 3)$ *and* $(8, 0)$ *trapping sets in its Tanner graph.*

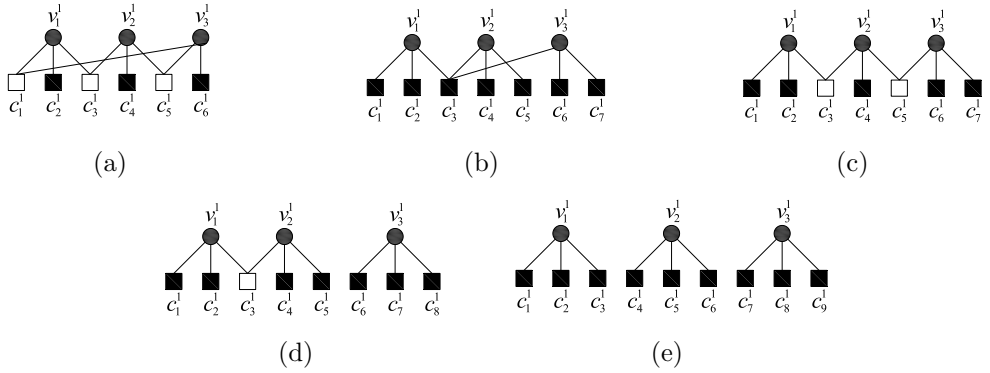


Figure A.2: All the possible subgraphs that can be induced by three variable nodes in a column-weight-three code

Proof. Follows from the discussion above. □

We now state and prove the main theorem.

Theorem A.3. *If the Tanner graph of a column-weight-three LDPC codes has girth eight and does not contain a subgraph isomorphic to a* $(5, 3)$ *trapping set or a subgraph isomorphic to an* $(8, 0)$ *trapping set, then any three errors can be corrected using the Gallager-A algorithm.*

Proof. Let $V^1 := \{v_1^1, v_2^1, v_3^1\}$ be the three erroneous variables and C^1 be the set of the checks connected to the variables in V^1. In a column-weight-three code (free of cycles of length four) the variables in V^1 can induce only one of the five subgraphs given in Fig. A.2. In each case, $\omega_1(v, :) = \{1\}$ if $v \in V^1$ and is 0 otherwise. The proof proceeds by examining these subgraphs one at a time and proving the correction of the three erroneous variables in each case.

Subgraph 1: Since the girth of the code is eight, it has no six cycles. Hence, the configuration in Fig. A.2(a) is not possible.

Subgraph 2: The variables in V^1 induce the subgraph shown in Fig. A.2(b).

At the end of the first iteration:

$$\overline{\omega}_1(c,v) = \begin{cases} 1 & \text{if } c \in C^1,\ v \notin V^1 \\ 0 & \text{otherwise} \end{cases} \tag{A.5}$$

There cannot exist a variable node which is connected to two or more checks in the set C^1 without introducing either a six-cycle or a subgraph isomorphic to $(5,3)$ trapping set. At the end of first iteration, $\overline{\omega}_1(:,v) = \{0\}$ for all $v \in V^1$. Furthermore, there exists no $v \notin V^1$ for which $\overline{\omega}_1(:,v) = \{1\}$. Hence, if a decision is made after the first iteration, a valid codeword is found and the decoder is successful.

Subgraph 3: The variables in V^1 induce the subgraph shown in Fig. A.2(c). At the end of the first iteration:

$$\overline{\omega}_1(c,v) = \begin{cases} 1 & \text{if } c \in C^1 \backslash \{c_3^1, c_5^1\},\ v \notin V^1 \\ 1 & \text{if } c \in \{c_3^1, c_5^1\},\ v \in V^1 \\ 0 & \text{otherwise} \end{cases} \tag{A.6}$$

For no $v \notin V^1$, $\overline{\omega}_1(:,v) = \{1\}$ as this would introduce a four-cycle or a six-cycle in the graph. For any $v \notin V^1$, $\omega_2(v,c) = 1$ only if $\overline{\omega}_1(: \backslash c, v) = \{1\}$. This implies that v has two checks in $C^1 \backslash \{c_3^1, c_5^1\}$. Let V^2 be the set of such variables. We have the following lemma:

Lemma A.4. *There can be at most one variable in V^2.*

Proof. Suppose $|V^2| = 2$. Specifically, assume $V^2 = \{v_1^2, v_2^2\}$. The proof is similar for $|V^2| > 2$. First note that for any $v \in V^2$, v cannot be connected to c_4^1 as it would create a six-cycle. Next, let $C_1^1 := \{c_1^1, c_2^1\}$ and $C_2^1 := \{c_6^1, c_7^1\}$. Then, v cannot have both checks in either C_1^1 or C_2^1 as this would cause a four-cycle. Hence, v has one check in C_1^1 and one check in C_2^1. Assume without loss of generality that v_1^2 is

connected to c_1^1 and c_6^1. Then, v_2^2 cannot be connected to c_1^1 and c_7^1 as this would form a six-cycle. v_2^2 cannot be connected to c_2^1 and c_7^1 as it would create a $(5,3)$ trapping set. Hence, $|V^2| < 2$. □

Let $v_1^2 \in V^2$ be connected to c_1^1, c_6^1 and an additional check c_1^2. In the second iteration:

$$\omega_2(v,c) = \begin{cases} 1 & \text{if } v \in \{v_1^1, v_3^1\},\ c \notin \{c_3^1, c_5^1\} \\ 1 & \text{if } v = v_2^1 \\ 1 & \text{if } v = v_1^2,\ c = c_1^2 \\ 0 & \text{otherwise} \end{cases} \tag{A.7}$$

$$\overline{\omega}_2(c,v) = \begin{cases} 1 & \text{if } c \in C^1 \backslash \{c_4^1\},\ v \notin V^1 \\ 1 & \text{if } c \in \{c_3^1, c_4^1, c_5^1\},\ v \neq v_2^1 \\ 1 & \text{if } c = c_1^2,\ v \neq v_1^2 \\ 0 & \text{otherwise} \end{cases} \tag{A.8}$$

We have the following lemma:

Lemma A.5. *There cannot exist any variable $v \notin V^1 \bigcup V^2$ such that it receives two or more incorrect messages at the end of the second iteration.*

Proof. Suppose there existed a variable v such that it received two incorrect messages in the second iteration. Then, it would be connected to two checks in the set $C^1 \bigcup \{c_1^2\}$. This is not possible as it would introduce a four-cycle, six-cycle or a $(5,3)$ trapping set (*e.g.* if v is connected to c_4^1 and c_1^2, it would form a $(5,3)$ trapping set). □

Thus, in the third iteration:

$$\omega_3(v,c) = \begin{cases} 1 & \text{if } v \in \{v_1^1, v_3^1\},\ c \notin \{c_3^1, c_5^1\} \\ 1 & \text{if } v = v_1^2,\ c = c_1^2 \\ 0 & \text{otherwise} \end{cases} \tag{A.9}$$

$$\overline{\omega}_3(c,v) = \begin{cases} 1 & \text{if } c \in \{c_1^1, c_2^1, c_6^1, c_7^1\},\ v \notin \{v_1^1, v_3^1\} \\ 1 & \text{if } c = c_1^2,\ v \neq v_1^2 \\ 0 & \text{otherwise} \end{cases} \tag{A.10}$$

At the end of the third iteration, $\overline{\omega}_3(:,v) = \{0\}$ for all $v \in V^1$. Also, we have the following lemma:

Lemma A.6. *There exists no $v \notin V^1$ such that $\overline{\omega}_3(:,v) = \{1\}$.*

Proof. Suppose there exists v such that $\overline{\omega}_3(:,v) = \{1\}$. Then, v is connected to three checks in the set $\{c_1^1, c_2^1, c_6^1, c_7^1, c_1^2\}$. This implies that $\overline{\omega}_2(:,v) = \{1\}$. However, from Lemma A.5 it is evident that no such v exists. □

Hence, if a decision is made after the third iteration, a valid codeword is found and the decoder is successful.

Subgraph 4: The variables in V^1 induce the subgraph shown in Fig. A.2(d). At the end of the first iteration:

$$\overline{\omega}_1(c,v) = \begin{cases} 1 & \text{if } c \in C^1 \backslash \{c_3^1\},\ v \notin V^1 \\ 1 & \text{if } c = c_3^1,\ v \in \{v_1^1, v_2^1\} \\ 0 & \text{otherwise} \end{cases} \tag{A.11}$$

For no $v \in V \backslash V^1$, $\overline{\omega}_2(:,v) = \{1\}$. For any $v \in V \backslash V^1$, $\omega_2(v,c) = 1$ only if $\overline{\omega}_1(: \backslash c, v) = \{1\}$. Let V^2 be the set of all such variables. We have the following lemma:

Lemma A.7. *(i) V^2 has at most four variables, and (ii) No two variables in V^2 can share a check in $C \backslash C^1$.*

Sketch of the Proof: There exists no variable which is connected to two checks from the set $\{c_1^1, c_2^1, c_3^1, c_4^1, c_5^1\}$ as it would introduce a four-cycle or a six-cycle. However, a variable node can be connected to one check from $\{c_1^1, c_2^1, c_3^1, c_4^1, c_5^1\}$ and to one check from $\{c_6^1, c_7^1, c_8^1\}$. There can be at most four such variable nodes. When four such variable nodes exist, none are connected to c_3^1. Also, these four variable nodes cannot share checks outside the set $C^1 \backslash \{c_3^1\}$.

Let these four variable nodes be labeled v_1^2, v_2^2, v_3^2 and v_4^2 and their third checks c_1^2, c_2^2, c_3^2 and c_4^2, respectively. Let $C^2 := \{c_1^2, c_2^2, c_3^2, c_4^2\}$. Hence, in the second iteration:

$$\omega_2(v,c) = \begin{cases} 1 & \text{if } v \in \{v_1^1, v_2^1\},\ c \neq c_3^1 \\ 1 & \text{if } v \in V^2,\ c \in C^2 \\ 0 & \text{otherwise} \end{cases} \tag{A.12}$$

$$\overline{\omega}_2(c,v) = \begin{cases} 1 & \text{if } c \in \{c_1^1, c_2^1, c_4^1, c_5^1\},\ v \notin \{v_1^1, v_2^1\} \\ 1 & \text{if } c \in C^2,\ v \notin V^2 \\ 0 & \text{otherwise} \end{cases} \tag{A.13}$$

At the end of the second iteration $\overline{\omega}_2(:,v) = \{0\}$ for all $v \in V^1$. Moreover, for no $v \notin V^1$, $\overline{\omega}_2(:,v) = \{1\}$. So, if a decision is made after the second iteration, a valid codeword is reached and the decoder is successful.

Subgraph 5: The variables in V^1 induce the subgraph shown in Fig. A.2(e). At the end of the first iteration:

$$\overline{\omega}_1(c,v) = \begin{cases} 1 & \text{if } c \in C^1,\ v \notin V^1 \\ 0 & \text{otherwise} \end{cases} \tag{A.14}$$

If there exists no variable $v \in V \backslash V^1$ such that $\overline{\omega}_1(:, v) = \{1\}$, a valid codeword is reached after the first iteration. Suppose this is not the case. Let V^2 be the set of variables which receive two or more incorrect messages. Then, we have the following lemma:

Lemma A.8. *(i) There exists one variable $v_1^2 \in V^2$ such that $\overline{\omega}_1(:, v_1^2) = \{1\}$, and (ii) V^2 has at most three variables which receive two incorrect messages at the end of the first iteration. Furthermore, they cannot share a check in $C \backslash C^1$.*

Proof. We omit the proof of Part (ii) as it is straightforward. Part (i) is proved as follows: If there existed no variable, v_1^2, such that $\overline{\omega}_1(:, v_1^2) = \{1\}$, then the decoder would converge in one iteration. Next, suppose $v_1^2, v_2^2 \in V^2$ such that $\overline{\omega}_1(:, v_1^2) = \overline{\omega}_1(:, v_2^2) = \{1\}$. Without loss of generality, let v_1^2 be connected to c_1^1, c_4^1 and c_7^1. Then, v_1^2 would share two checks in the set C^1. It is then not possible to connect v_2^2 without introducing a six-cycle or a $(5, 3)$ trapping set (*e.g.*, if v_1^2 is connected to c_2^1, c_5^1 and c_8^1, then it would introduce a $(5, 3)$ trapping set). □

Let the third checks connected to v_2^2, v_3^2 and v_4^2 be c_1^2, c_2^2 and c_3^2, respectively and let $C^2 := \{c_1^2, c_2^2, c_3^2\}$ In the second iteration:

$$\omega_2(v, c) = \begin{cases} 1 & \text{if } v = v_1^2 \\ 1 & \text{if } v \in V^2 \backslash \{v_1^2\},\ c \in C^2 \\ 0 & \text{otherwise} \end{cases} \tag{A.15}$$

$$\overline{\omega}_2(c, v) = \begin{cases} 1 & \text{if } c \in \{c_1^1, c_3^1, c_7^1\},\ v \neq v_1^2 \\ 1 & \text{if } c \in C^2,\ v \notin V^2 \\ 0 & \text{otherwise} \end{cases} \tag{A.16}$$

There cannot exist a variable node which is connected to one check from C^2 and to

one check from $\{v_1^1, v_4^1, v_7^1\}$. Also, there cannot be a variable node which is connected to all three checks in the set C^2 as this would introduce a graph isomorphic to the $(8, 0)$ trapping set. However, there can be at most two variable nodes which receive two incorrect messages from the checks in C^2, say v_1^3 and v_2^3. Let the third checks connected to them be c_1^3 and c_2^3, respectively. Let $V^3 := \{v_1^3, v_2^3\}$ and $C^3 := \{c_1^3, c_2^3\}$. At the end of the second iteration, variables v_1^1, v_2^1 and v_3^1 receive one incorrect message each. Variables in the set V^3 receive two incorrect messages each. Therefore, in the third iteration, we have:

$$\omega_3(v, c) = \begin{cases} 1 & \text{if } v \in V^1,\ c \notin \{c_1^1, c_4^1, c_7^1\} \\ 1 & \text{if } v \in V^3,\ c \in C^3 \\ 0 & \text{otherwise} \end{cases} \tag{A.17}$$

$$\overline{\omega}_3(c, v) = \begin{cases} 1 & \text{if } c \in C^1 \backslash \{c_1^1, c_4^1, c_7^1\},\ v \notin V^1 \\ 1 & \text{if } c \in C^3,\ v \notin V^3 \end{cases} \tag{A.18}$$

At the end of the third iteration, $\overline{\omega}_3(:, v) = \{1\}$ for all $v \in V^1$. Furthermore, for no $v \notin V^1$, $\overline{\omega}_3(:, v) = 1$. So, if a decision is made after the third iteration, a valid codeword is reached and the decoder is successful. $\square$

APPENDIX B

Finding Instantons for LP Decoding Over the BSC

The pseudo-codewords and instantons for LP decoding cannot entirely be determined from combinatorial considerations. Hence, we adopt an algorithmic approach to identify and enumerate instantons for LP decoding over the BSC.

Definition B.1. The median noise vector (or simply the median) $M(\mathbf{p})$ of a pseudo-codeword $\mathbf{p}$ distinct from the all-zero-codeword is a binary vector with support $S = \{v_1, v_2, \ldots, v_q\}$, such that $p_{v_1}, \ldots, p_{v_q}$ are the $q(= \lceil (w_{BSC}(\mathbf{p}) + 1)/2 \rceil)$ largest components of $\mathbf{p}$. □

One observers that, $C(M(\mathbf{p}), \mathbf{p}) \leq 0$. From the definition of $w_{BSC}(\mathbf{p})$, it follows that at least one median exists for every $\mathbf{p}$. Also, all medians of $\mathbf{p}$ have $\lceil (w_{BSC}(\mathbf{p}) + 1)/2 \rceil$ flips. The proofs of the following two lemmas are now apparent.

Lemma B.2. *The LP decoder decodes a binary vector with q flips into a pseudo-codeword $\mathbf{p}$ distinct from the all-zero-codeword iff $w_{BSC}(\mathbf{p}) \leq 2q$.* □

Lemma B.3. *Let $\mathbf{p}$ be a pseudo-codeword with median $M(\mathbf{p})$ whose support has cardinality q. Then $w_{BSC}(\mathbf{p}) \in \{2q-1, 2q\}$.* □

Lemma B.4. *Let $M(\mathbf{p})$ be a median of $\mathbf{p}$ with support S. Then the result of LP decoding of any binary vector with support $S' \subset S$ and $|S'| < |S|$ is distinct from $\mathbf{p}$.*

Proof. Let $|S| = q$. Then by Lemma B.3, $w_{BSC}(\mathbf{p}) \in \{2q-1, 2q\}$. Now, if $\mathbf{r}$ is any binary vector with support $S' \subset S$, then $\mathbf{r}$ has at most $q-1$ flips and therefore by Lemma B.2, $w_{BSC}(\mathbf{p}) \leq 2(q-1)$, which is a contradiction. □

Lemma B.5. *If $M(\mathbf{p})$ converges to a pseudo-codeword $\mathbf{p}_M \neq \mathbf{p}$, then $w_{BSC}(\mathbf{p}_M) \leq w_{BSC}(\mathbf{p})$. Also, $C(M(\mathbf{p}), \mathbf{p}_M) \leq C(M(\mathbf{p}), \mathbf{p})$.*

Proof. According to the definition of the LP decoder, $C(M(\mathbf{p}), \mathbf{p}_M) \leq C(M(\mathbf{p}), \mathbf{p})$.

If $w_{BSC}(\mathbf{p}) = 2q$, then $M(\mathbf{p})$ has q flips and by Lemma B.2, $w_{BSC}(\mathbf{p}_M) \leq 2q = w_{BSC}(\mathbf{p})$.

If $w_{BSC}(\mathbf{p}) = 2q - 1$, then $M(\mathbf{p})$ has q flips and $C(M(\mathbf{p}), \mathbf{p}) < 0$. Hence, $w_{BSC}(\mathbf{p}_M) \leq 2q$ by Lemma B.2. However, if $w_{BSC}(\mathbf{p}_M) = 2q$, then $C(M(\mathbf{p}), \mathbf{p}_M) = 0$, which is a contradiction. Hence, $w_{BSC}(\mathbf{p}_M) \leq 2q - 1 = w_{BSC}(\mathbf{p})$. □

The following lemma follows from the definition of the cost of decoding (the pseudo-codeword cost):

Lemma B.6. *Let $\mathbf{i}$ be an instanton. Then for any binary vector $\mathbf{r}$ such that supp($\mathbf{i}$) $\subset$ supp($\mathbf{r}$), there exists a pseudo-codeword $\mathbf{p}$ satisfying $C(\mathbf{r}, \mathbf{p}) \leq 0$.*

Proof. Since $\mathbf{i}$ is an instanton, there exists a pseudo-codeword $\mathbf{p}$ such that $C(\mathbf{i}, \mathbf{p}) \leq 0$. From Definition 2.13 we have,

$$\sum_{v \notin \text{supp}(\mathbf{i})} p_v - \sum_{v \in \text{supp}(\mathbf{i})} p_v \leq 0.$$

Since, supp($\mathbf{i}$) $\subset$ supp($\mathbf{r}$) and $p_v \geq 0, \forall v$, we have

$$\sum_{v \notin \text{supp}(\mathbf{r})} p_v - \sum_{v \in \text{supp}(\mathbf{r})} p_v \leq 0,$$

thus yielding

$$C(\mathbf{r}, \mathbf{p}) \leq 0.$$

□

The above lemma implies that the LP decoder fails to decode every vector $\mathbf{r}$ whose support is a superset of an instanton to the all-zero- codeword. We now have the following corollary:

Corollary B.7. *Let* $\mathbf{r}$ *be a binary vector with support* S*. Let* $\mathbf{p}$ *be a pseudo-codeword such that* $C(\mathbf{r}, \mathbf{p}) \leq 0$*. If all binary vectors with support* $S' \subset S$ *such that* $|S'| = |S| - 1$*, converge to* $\mathbf{0}$*, then* $\mathbf{r}$ *is an instanton.* □

The above lemmas lead us to the following lemma which characterizes all the failures of the LP decoder over the BSC:

Lemma B.8. *A binary vector* $\mathbf{r}$ *converges to a pseudo-codeword different from the all-zero-codeword iff the support of the vector* $\mathbf{r}$ *contains the support of an instanton as a subset.* □

The most general form of the above lemma can be stated as following: if $\mathbf{x}$ is the transmitted codeword and $\mathbf{y}$ is the received vector, then $\mathbf{y}$ converges to a pseudo-codeword different from $\mathbf{x}$ *iff* the $\text{supp}(\mathbf{y} + \mathbf{x})$, where the addition is modulo two, contains the support of an instanton as a subset.

From the above discussion, we see that the BSC instantons are analogous to the minimal stopping sets for the case of iterative/LP decoding over the BEC. In fact, Lemma B.8 characterizes all the decoding failures of the LP decoder over the BSC in terms of the instantons and can be used to derive analytical estimates of the code performance given the weight distribution of the instantons. In this sense, the instantons are more fundamental than the minimal pseudo-codewords [43, 41] for the BSC (note, that this statement does not hold in the case of the AWGN channel). Two minimal pseudo-codewords of the same weight can give rise to different number

of instantons. This issue was first pointed out by Forney *et al.* in [35]. (See Examples 1, 2, 3 for the BSC case in [35].) It is also worth noting that an instanton converges to a minimal pseudo-codeword.

It should be noted that finding pseudo-codewords with fractional weight d_{frac} is not equivalent to finding minimum weight pseudo-codewords. The pseudo-codewords with fractional weight d_{frac} can be used to derive some instantons, but not necessarily the ones with the least number of flips. However, as d_{frac} provides a lower bound on the minimum pseudo-codeword weight, it can be used as a test if the ISA actually finds an instanton with the least number of flips. In other words, if the number of flips in the lowest weight instanton found by the ISA is equal to $\lceil d_{frac}/2 \rceil$, then the ISA has indeed found the smallest size instanton.

B.1 Instanton Search Algorithm and its Analysis

In this Section, we describe the Instanton Search Algorithm. The algorithm starts with a random binary vector with some number of flips and outputs an instanton.

Instanton Search Algorithm

Initialization (j=0) step: Initialize to a binary input vector $\mathbf{r}$ containing sufficient number of flips so that the LP decoder decodes it into a pseudo-codeword different from the all-zero-codeword. Apply the LP decoder to $\mathbf{r}$ and denote the pseudo-codeword output of LP by $\mathbf{p}^1$.

$j \geq 1$ step: Take the pseudo-codeword $\mathbf{p}^j$ (output of the $(j-1)$ step) and calculate its median $M(\mathbf{p}^j)$. Apply the LP decoder to $M(\mathbf{p}^j)$ and denote the output by $\mathbf{p}_{M_j}$. By Lemma B.5, only two cases arise:

- $w_{BSC}(\mathbf{p}_{M_j}) < w_{BSC}(\mathbf{p}^j)$. Then $\mathbf{p}^{j+1} = \mathbf{p}_{M_j}$ becomes the j-th step output/$(j+$

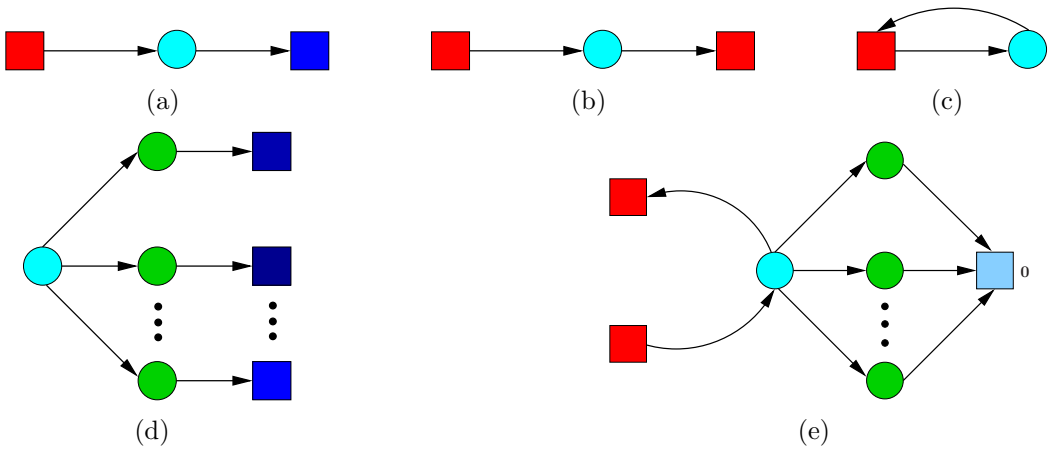


Figure B.1: Squares represent pseudo-codewords and circles represent medians or related noise configurations (a) LP decodes median of a pseudo-codeword into another pseudo-codeword of smaller weight (b) LP decodes median of a pseudo-codeword into another pseudo-codeword of the same weight (c) LP decodes median of a pseudo-codeword into the same pseudo-codeword (d) Reduced subset (three different green circles) of a noise configuration (e.g. of a median from the previous step of the ISA) is decoded by the LP decoder into three different pseudo-codewords (e) LP decodes the median (blue circle) of a pseudo-codeword (low red square) into another pseudo-codeword of the same weight (upper red square). Reduced subset of the median (three configurations depicted as green circles are all decoded by LP into all-zero-codeword. Thus, the median is an instanton.

1) step input.

- $w_{BSC}(\mathbf{p}_{M_j}) = w_{BSC}(\mathbf{p}^j)$. Let the support of $M(\mathbf{p}^j)$ be $S = \{i_1, \ldots, i_{k_j}\}$. Let $S_{i_t} = S \backslash \{i_t\}$ for some $i_t \in S$. Let $\mathbf{r}_{i_t}$ be a binary vector with support S_{i_t}. Apply the LP decoder to all $\mathbf{r}_{i_t}$ and denote the i_t-output by $\mathbf{p}_{i_t}$. If $\mathbf{p}_{i_t} = \mathbf{0}, \forall i_t$, then $M(\mathbf{p}^j)$ is the desired instanton and the algorithm halts. Else, $\mathbf{p}_{i_t} \neq \mathbf{0}$ becomes the j-th step output/$(j+1)$ step input. (Notice, that Lemma B.4 guarantees that any $\mathbf{p}_{i_t} \neq \mathbf{p}^j$, thus preventing the ISA from entering into an infinite loop.)

Fig. B.1 illustrates different scenarios arising in the execution of the ISA. Here, the squares represent pseudo-codewords and the circles represent binary vec-

tors (noise configurations). Two squares of the same color have identical pseudo-codeword weight and two circles of the same color consist of same number of flips. Fig. B.1(a) shows the case where a median, $M(\mathbf{p}^l)$, of a pseudo-codeword $\mathbf{p}^j$ converges to a pseudo-codeword $\mathbf{p}_{M_j}$ of a smaller weight. In this case, $\mathbf{p}^{j+1} = \mathbf{p}_{M_j}$. Fig. B.1(b) illustrates the case where a median, $M(\mathbf{p}^j)$, of a pseudo-codeword $\mathbf{p}^j$ converges to a pseudo-codeword $\mathbf{p}_{M_j}$ of the same weight. Fig. B.1(c) illustrates the case where a median, $M(\mathbf{p}^j)$, of a pseudo-codeword $\mathbf{p}^j$ converges to the pseudo-codeword $\mathbf{p}^j$ itself. In the two latter cases, we consider all the binary vectors whose support sets are subsets of the support set of $M(\mathbf{p}^j)$ and the vectors contain one flip less. We run the LP decoder with the vectors as inputs and find their corresponding pseudo-codewords. One of the non-zero pseudo-codewords found is chosen at random as $\mathbf{p}^{j+1}$. This is illustrated in Fig. B.1(d). Fig. B.1(e) shows the case when all the subsets of $M(\mathbf{p}^j)$ (reduced by one flip) converge to the all-zero-codeword. $M(\mathbf{p}^j)$ itself could converge to $\mathbf{p}^j$ or some other pseudo-codeword of the same weight. In this case, $M(\mathbf{p}^j)$ is an instanton constituting the output of the algorithm.

We now prove that the ISA terminates (i.e., outputs an instanton) in the number of steps of the order the number of flips in the initial noise configuration.

Theorem B.9. *$w_{BSC}(\mathbf{p}^j)$ and $|supp(M(\mathbf{p}^j))|$ are monotonically decreasing. Also, the ISA terminates in at most $2k_0$ steps, where k_0 is the number of flips in the input.*

Proof. If $\mathbf{p}^{j+1} = \mathbf{p}_{M_j}$, then $w_{BSC}(\mathbf{p}^{j+1}) < w_{BSC}(\mathbf{p}^j)$. Consequently, $|\text{supp}(M(\mathbf{p}^{j+1}))| \leq |\text{supp}(M(\mathbf{p}^j))|$.

If $\mathbf{p}^{j+1} = \mathbf{p}_{i_t}$, then $w_{BSC}(\mathbf{p}_{i_t}) \leq 2(|\text{supp}(M(\mathbf{p}^j))| - 1) < w_{BSC}(\mathbf{p}^j)$. Consequently, $|\text{supp}(M(\mathbf{p}^{j+1}))| \leq |\text{supp}(M(\mathbf{p}^j))|$.

Since $w_{BSC}(\mathbf{p}^j)$ is strictly decreasing, the weight of pseudo-codeword at step j

decreases by at least one compared to the weight of the pseudo-codeword at step $j-1$. Since by Lemma B.2, $w_{BSC}(\mathbf{p}^1) \leq 2k_0$, the algorithm can run for at most $2k_0$ steps. □

Remarks: (1) By "sufficient number of flips", we mean that the initial binary vector should be noisy enough to converge to a pseudo-codeword other than the all-zero-codeword. While any binary vector with a large number of flips is almost guaranteed to converge to a pseudo-codeword different from the all-zero-codeword, such a choice might also lead to a longer running time of the ISA (from Theorem B.9). On the other hand, choosing a binary vector with a few number of flips might lead to convergence to the all-zero-codeword very often, thereby necessitating the need to run the ISA for a large number of times.

(2) Theorem B.9 does not claim that the algorithm finds the minimum weight pseudo-codeword or the instanton with the smallest number of flips. However, it is sometimes possible to verify if the algorithm has found the minimum weight pseudo-codeword. Let w_{min}^{ISA} denote the weight of the minimum weight pseudo-codeword found by the ISA. If $w_{min}^{ISA} = 2\lceil d_{frac}/2 \rceil - 1$, then $w_{min}^{ISA} = w_{min}^{BSC}$.

(3) At some step j, it is possible to have $w_{BSC}(\mathbf{p}_{M_j}) = w_{BSC}(\mathbf{p}^j)$ and incorporating such pseudo-codewords into the algorithm could lead to lower weight pseudo-codewords in the next few steps. However, this inessential modification was not included in the ISA to streamline the analysis of the algorithm.

(4) While we have shown that $w_{BSC}(\mathbf{p}^j)$ decreases by at least unity at every step, we have observed that in most cases, it decreases by at least two. This is due to the fact that the pseudo-codewords with odd weights outnumber pseudo-codewords with even weights. As a result, in most cases, the algorithm converges in less than

k_0 steps. (For illustration of this point see example discussed in the next Section.)

(5) At any step, there can be more than one median, and the ISA does not specify which one to pick. Our current implementation suggests to pick a median at random. Also, the algorithm does not provide clarification on the choice of the pseudo-codeword for the case when more than one noise configurations from the subset $\mathbf{r}_{i_t}$ converge to pseudo-codewords distinct from the all-zero-codeword. In this degenerate case, we again choose a pseudo-codeword for the next iteration at random. Note that one natural deterministic generalization of the randomized algorithm consists of exploring all the possibilities at once. In such a scenario, a tree of solutions can be built, where the root is associated with one set of initiation flips, any branch of the tree relates to a given set of randomized choices (of medians and pseudo-codewords), and any leaf corresponds to an instanton.

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