THE CHORDAL LOEWNER EQUATION DRIVEN BY BROWNIAN MOTION WITH LINEAR DRIFT

by
Benjamin N. Dyhr

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As members of the Dissertation Committee, we certify that we have read the dissertation prepared by Benjamin N. Dyhr entitled The Chordal Loewner Equation Driven by Brownian Motion with Linear Drift and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

Thomas G. Kennedy
Date: 2 July 2009

William G. Faris
Date: 2 July 2009

Douglas M. Pickrell
Date: 2 July 2009

Mikhail Stepanov
Date: 2 July 2009

Final approval and acceptance of this dissertation is contingent upon the candidate’s submission of the final copies of the dissertation to the Graduate College. I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

Dissertation Director: Thomas G. Kennedy
Date: 2 July 2009
STATEMENT BY AUTHOR

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SIGNED: Benjamin N. Dyhr
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Schramm-Loewner evolution (SLE$_\kappa$) is an important contemporary tool for identifying critical scaling limits of two-dimensional statistical systems. The SLE$_\kappa$ one-parameter family of processes can be viewed as a special case of a more general, two-parameter family of processes we denote SLE$_{\mu \kappa}$. The SLE$_{\mu \kappa}$ process is defined for $\kappa > 0$ and $\mu \in \mathbb{R}$; it represents the solution of the chordal Loewner equations under special conditions on the driving function parameter which require that it is a Brownian motion with drift $\mu$ and variance $\kappa$. We derive properties of this process by use of methods applied to SLE$_\kappa$ and application of Girsanov’s Theorem. In contrast to SLE$_\kappa$, we identify stationary asymptotic behavior of SLE$_{\mu \kappa}$. For $\kappa \in (0, 4]$ and $\mu \neq 0$, we present a pathwise construction of a process, $\hat{\gamma}$, with stationary temporal increments and stationary imaginary component and relate $\hat{\gamma}$ to the limiting behavior of the SLE$_{\mu \kappa}$ generating curve. Our main result is a spatial invariance property of $\hat{\gamma}$ achieved by defining a top-crossing probability for points $z \in \mathbb{C}$ with respect to the generating curve.
INTRODUCTION

In the last decade, analytic methods originally developed by Greg Lawler, Wendelin Werner and the late Oded Schramm have dramatically advanced our understanding of statistical mechanics on two-dimensional lattices. The family of stochastic processes now called Schramm-Loewner evolution constitutes an original approach for identifying fine-mesh scaling limits of certain discrete systems and has proven to be a rich object of study from a purely mathematical perspective.

Physicists have long been interested in a variety of discrete, two-dimensional systems on a lattice thought to have meaningful scaling limits as the mesh width of the lattice goes to zero. Some of these systems are parameterized paths in the plane defined explicitly by a discrete random process, for example, self avoiding walks and loop-erased random walks. Other systems, like site percolation on the triangular lattice, uniform spanning trees and the Ising model, can be used to implicitly define interfaces with nontrivial scaling limits. Numerical analysis and techniques developed by physicists predicted the existence of a class of these models, referred to as critical systems, with scaling limits given by probability measures (on sets of curves) that are invariant with respect to conformal transformations in the plane in a certain well-defined sense or conformally invariant. The amazing fact Schramm discovered was that every probability distribution of random curves supported in the closed upper half-plane, \( \gamma : [0, \infty) \rightarrow \mathbb{H} \), exhibiting conformal invariance and a stochastic requirement called the domain Markov property is described by a certain distribution of solutions of a differential equation with complex-analytic origins called the Loewner equation. The Loewner equation has an input parameter, called the driving function, that can be taken to be any continuous function \( \psi : [0, \infty) \rightarrow \mathbb{R} \). Schramm-Loewner evolution, or SLE\( _{\kappa} \), is the solution of the Loewner equation when the driving function is taken to be \( \psi(t) = \sqrt{\kappa} W_t \), where \( W_t \) is a standard Brownian motion. This
choice of driving function initiated a breakthrough in research on 2-d critical systems by providing a candidate scaling limit for the critical systems mentioned above and several other lattice systems thought to possess conformally invariant scaling limits.

A society of mathematicians has now successfully utilized the SLE$_\kappa$ process to rigorously describe several conformally invariant scaling limits in 2-d statistical physics, verify predictions about these systems originally made by physicists and derive laws of random variables associated with these systems. The SLE$_\kappa$ process has also motivated the construction of other models with conformally invariant scaling limits, and generalizations of SLE$_\kappa$ abound.

One particular generalization of SLE$_\kappa$ that has been considered is the introduction of a drift term into the driving function of SLE$_\kappa$, say $\psi(t) = \sqrt{\kappa} B_t + \xi(t)$ ([1], [2], [23]). These forms of Loewner equation solutions have been considered as candidate scaling limits for off-critical statistical systems. Off-critical models are achieved by taking scaling limits of the aforementioned systems near, but not at, the critical parameter. These off-critical models rely on taking the limit of a parameter towards the critical value as one takes the fine-mesh scaling in manners that achieve nondegenerate scaling limits that are not equivalent to SLE$_\kappa$.

From an analytic perspective, perhaps the most basic form of $\xi(t)$ is linear, that is, $\xi(t) := \mu t$ for some $\mu \neq 0$ (addition of a constant only results in a spatial translation of the system). This work is the first in-depth examination of the Loewner equation with driving function given by $U_t = \sqrt{\kappa} B_t + \mu t$, $\kappa > 0$, $\mu \neq 0$. We denote this process SLE$_\kappa^\mu$. We find that several methods used in the past to analyze SLE$_\kappa$ can be used in a similar manner to prove analogous theorems about SLE$_\kappa^\mu$. Beyond these basic results comparing and contrasting SLE$_\kappa$ and SLE$_\kappa^\mu$, we also describe asymptotic stationarity properties of SLE$_\kappa^\mu$ that have no correspondence in the theory of SLE$_\kappa$. For $\kappa \in (0, 4)$, SLE$_\kappa^\mu$ generates a self-avoiding, complex-valued, random path $\gamma(t)$, supported in the upper half plane for all $t > 0$. We prove that as $t \to \infty$ this path behaves like a certain temporally stationary 2-d process that is an interesting mathematical object.
in its own right.

In Chapter 1, we first review two discrete statistical systems that have been associated with SLE$_\kappa$; namely, critical site percolation on the triangular lattice and the loop-erased random walk (LERW). Rigorous proofs that critical percolation interfaces and LERW converge to SLE$_6$ and SLE$_2$ are established in [7] and [20], respectively. These are not the only models that have been either proven or conjectured to converge to an SLE$_\kappa$ process, but they are instructive models with respect to a natural derivation of the Schramm-Loewner equation that we provide in Section 1.2.2. It is in our interest to review this derivation because it also provides justification for our interest in SLE$_\mu$.

Chapter 1 also includes an introduction to the Loewner equation and collects definitions of object associated with the Loewner equation that we will need. After reviewing the Loewner equation and defining SLE$_\kappa$, we collect properties of SLE$_\kappa$ that are used in the sequel.

Next, in the second chapter, we establish basic properties of SLE$_\mu$ that do not require complicated proof. Some of these properties are acquired from methods borrowed from existing SLE$_\kappa$ literature; several other results carry over directly from properties of SLE$_\kappa$ by way of Girsanov’s theorem. Many of the results established in Chapter 2 will be integral in the development of our main results.

Our main results concern the asymptotic behavior of the SLE$_\mu$ process. In Chapter 3, we use the reverse Loewner flow to define a process that exhibits certain temporal stationarity properties and represents the asymptotic behavior of SLE$_\mu$. These results are established only for $\kappa \in (0, 4]$, the range of $\kappa$ for which the path generated by SLE$_\kappa$ is self-avoiding. In the final section, we obtain a crossing probability with respect to the limiting process defined in the previous section; we then prove this crossing probability is invariant with respect to horizontal spatial shifts. This spatial invariance is the main result of the dissertation; its proof requires a logarithmic correction factor embedded in the definition of the object, which is interesting in itself.
The existence of the stationarity limit of $\text{SLE}_\kappa^\mu$ we describe also gives rise to a number of conjectures suitable for future research investment.

Finally, please observe the following notational conventions we will use throughout the paper. As is customary in mathematical literature, for any function $f$ and set $S$, we use $f(S)$ to denote the set $\{f(x) : x \in S\}$. Furthermore, for paths $\gamma : [0, \infty) \to \mathbb{C}$ and intervals $I \subset \mathbb{R}$, we write $\gamma I := \{\gamma(t) : t \in I\}$; that is, parentheses around the parentheses defining the interval itself will be omitted. We will avoid the index $i$ for sequence elements, but use $i$ to denote the imaginary number $\sqrt{-1}$ nonetheless. Throughout the paper we reserve the symbol $\Omega$ for the probability space underlying the stochastic processes we discuss and use w.p.1 to abbreviate ‘with probability one’ in context of the probability measure being used.
1. BACKGROUND

The Riemann mapping theorem, a foundational result in the theory of conformal maps, describes three-parameter families of conformal maps between any two simply-connected (strict) subsets of the plane. By prescribing certain restrictions on these conformal maps we can construct a one-to-one relationship between conformal maps satisfying these restrictions and simply-connected subsets of the plane. For example, by designating the image of three distinct boundary points on the simply connected set, we completely determine the conformal map, so the space of simply connected sets in the plane, each with three boundary points \( z_1, z_2, z_3 \in \mathbb{C} \) identified, is in one-to-one correspondence with the space of conformal maps.

Given the highly geometric nature of conformal maps, a statistical physicist is lead to question whether there are random sets in the plane whose distribution is somehow invariant under a certain class of conformal maps. The related concept of a space of random continuous curves, defined on simply connected subsets of the plane, whose distribution is invariant under conformal transformations between these subsets dates back to the earliest rigorous development of the theory of stochastic processes. Indeed, the scaling limit of a two-dimensional Brownian motion conditioned to connect two points on the boundary and remain in the interior of a simply connected domain, or Brownian excursion, was shown to be conformally invariant in the 1940’s, less than twenty years after Norbert Wiener’s construction of Brownian motion. The result follows from the coincidence between the harmonic measure associated with Brownian motion and harmonic functions on sets in the plane; in turn, the property of a function being harmonic is invariant under conformal transformation.

Using the Donsker invariance principle, we can also describe Brownian motion as the scaling limit of a simple random walk, and the Brownian excursion can be thought of as a scaling limit of discrete excursions on lattice subsets in a similar manner. Thus,
by our remarks above, we can say the weak limit of a random excursion on a lattice is conformally invariant. Many other discrete statistical systems of interest to mathematicians and physicists are thought to have scaling limits that can be described by distributions of continuous, fractal paths in the plane. Some of these scaling limits are thought to be conformally invariant; however, it is often difficult even to conclude that a scaling limit exists. Examples of such two-dimensional models are the self avoiding walk (SAW), the loop erased random walk (LERW), critical percolation interfaces, critical Ising model interfaces and uniform spanning tree (UST). All of these models have scaling limits (by conjecture for SAW) that have been identified with solutions of a stochastic differential equation called the chordal Schramm-Loewner equation. These solutions are abbreviated SLE_κ; κ is a positive real parameter that has a fixed, distinct value for each of the aforementioned models. As we will see, the predicted conformal invariance property of scaling limits of the models listed above leads one naturally to the Schramm-Loewner equation.

The background material we review here provides a brief survey of 2-d lattice systems associated with SLE_κ including a description of two models which best motivate Schramm’s derivation of SLE_κ (and SLE_κ^κ). Since our definition of chordal Schramm Loewner evolution will be supported on the upper half plane, we will describe the discrete systems in the context of the upper half plane. These discrete models can be described in arbitrary simply connected domains, but we will avoid a comprehensive introduction to each model. A much more detailed survey of the models that have scaling limits corresponding to SLE_κ has been written by Gregory Lawler ([16]).

After surveying these discrete systems, we establish our notational conventions while stating and motivating the precise definition of chordal Schramm-Loewner equation. We will see that the solution of the equation, SLE_κ, is a natural candidate for the scaling limit of certain processes. In fact, although the definition of SLE_κ customarily uses a Brownian motion with nonzero drift, we will see in the sequel that inclusion of a drift term in the Brownian motion also produces a process with similar
1.1. Lattice Systems and Conformal Invariance

1.1.1. Critical percolation

The critical percolation model in the upper half plane, $\mathbb{H}$, can be visualized as a tiling of $\mathbb{H}$ with equilateral hexagons, each of which is colored black or white with probability $1/2$ independent of the color of any of the other hexagons. This is equivalent to designating a black or white assignment to every vertex on a regular triangular lattice in $\mathbb{H}$ in the same manner. This model can be defined for any probability $p \in [0, 1]$, but $p = 1/2$ is the unique critical value for which there is no infinite cluster of either color w.p.1 ([10]). This suggests the existence of a nondegenerate scaling limit if one takes the width of the lattice spacing to converge to zero. Recently, a scaling limit for interfaces of these random clusters has been shown to exist and to coincide with a certain solution of the Schramm-Loewner equation defined in the sequel ([6], [7]).

We can describe one particular random path as follows. Arrange the half plane hexagonal tiling so that the centers of the bottom row of hexagons are located on $\mathbb{R} \subset \mathbb{H}$ with $0 \in \mathbb{H}$ located at the boundary of two hexagons. Next, fix all of the hexagons
in $\mathbb{H}$ on the positive real line to be black and fix all of the hexagons located on the negative real line to be white, as pictured in Figure 1. By construction, the hexagons centered on $\mathbb{R}^+$ are part of an infinite cluster of black hexagons, and the hexagons centered on $\mathbb{R}^-$ are part of an infinite cluster of white hexagons. We can now define a random path, $\gamma : [0, \infty) \to \mathbb{H}$, initiated at 0 and following the boundary between the infinite cluster of black hexagons and the infinite cluster of white hexagons, as seen in Figure 1. It is not particularly important how we parametrize this path because the conformal invariance properties we desire of the scaling limit will only hold up to reparametrization; for clarity we let $\gamma(n), n \geq 1$, be located at the $n$th vertex of the hexagonal lattice that $\gamma$ visits, and linearly interpolate $\gamma$ along the edges of the hexagons for $t \notin \mathbb{N}$. This construction defines a probability measure on continuous paths $\gamma : [0, \infty) \to \mathbb{H}$.

We will not go into details here, but any for other simply connected domain $D \subset \mathbb{C}$ ($D \neq \mathbb{C}$) we can construct a similar percolation exploration process by allowing the hexagonal mesh to become sufficiently fine and designating an initial point $z \in \partial D$ ($0$ in the case of $\mathbb{H}$) distinct from a terminal point $w \in \partial D$ ($\infty$ in the case of $\mathbb{H}$). Here we are ignoring some technicalities related to the exact meaning of ‘a lattice approximation of $D$’ but emphasize that the mesh spacing of this lattice approximation of $D$ is at least fine enough that the filled hexagons form a simply connected approximation of $D$.

Consider the support of the process $\gamma$ up to a fixed time step $n$. Let $\Gamma_n$ denote the set of all hexagons that share a hexagonal edge with $\gamma[0, n]$. This set will consist of a sequence of white hexagons that border the right side of the (directed) path $\gamma : [0, n] \to \mathbb{H}$ and a sequence of black hexagons that border the left side of $\gamma$. Further, there is a well-defined simply connected set, $H_n$, given by the connected component of $\mathbb{H} \setminus (\Gamma_n \cup \mathbb{R})$ that contains $\infty$ (Figure 2). The important thing to notice is that $\gamma([0, n])$ only depends on the colors of the hexagons in $\Gamma_n$ and is independent of the colors of the hexagons in $H_n$. It is possible that $\gamma(n)$ is not on the the boundary between $\Gamma_n$
and $H_n$; if so, $\gamma(n)$ can be evolved according to the realizations of hexagons in $\mathbb{H} \setminus H_n$ until $\gamma(n + k)$ is on the boundary of $H_n$ for some $k \in \mathbb{N}$ with $H_{n+k} = H_n$. Therefore, without loss of generality, we can assume that $\gamma(n) \in \partial H_n$.

Each of the two clusters of hexagons connected to those hexagons in $\Gamma_n \cup \mathbb{R}$ bordering $H_n$ are infinite, one black and one white. For each possible realization of the set $H_n$, we can define a similar percolation exploration process in this new domain that is initiated at $\gamma(n)$. The independence of the black/white probability at each individual site in the original exploration process yields the following important fact: The measure on infinite curves given by the infinite percolation exploration process, $\tilde{\gamma}[0, \infty)$, defined on $H_n$ and initiated at $\gamma(n)$, is the same as the measure on infinite curves, $\gamma[n, \infty)$, defined by original percolation exploration process and conditioned on $\gamma[0, n]$. This feature of percolation is called the domain Markov property. One expects a continuum version of this property to be preserved by the fine-mesh scaling limit.

Unlike the domain Markov property, conformal invariance only will hold in the fine-mesh scaling limit of the percolation exploration process, described as follows. Let $\delta$ denote the edge length of our regular hexagonal lattice. If we center our reference frame at $0 \in \mathbb{H}$ and take $\delta \to 0$, then, in accordance with the criticality of $p = 1/2$,
it is reasonable to conjecture that the percolation exploration process will converge to a probability measure on curves that are nondegenerate w.p.1. This conjecture is supported by other, more acute observations. For models where $p$ is set equal to a value in $[0, 1]/\{1/2\}$, the system has a finite correlation length, an indicator of the strength of correlations between the sizes of distant finite-clusters of sites. Experimental evidence reveals that the correlation length goes to $\infty$ as $p \to 1/2$ and suggests that the model has no characteristic length at $p = 1/2$. This suggests a scale-invariant, fractal structure in the limit as $\delta \to 0$; a detailed exposition of these ideas is given in [5].

As already indicated, the percolation exploration process can be defined on any simply connected domain by carefully specifying how one approximates the domain with increasingly fine hexagonal lattices. Assume the fine-mesh scaling limit of the percolation exploration process exists as a measure on curves. Given a simply connected domain $D \subset \mathbb{C}$ and boundary points $x, y \in \partial D$, let $\mu_{D,x,y}$ denote the measure on (infinite length) curves, $\gamma$, given by the scaling limit of the percolation exploration process in $D$ with boundary conditions defined so that $\gamma$ joins $x$ with $y$. Conformal invariance is defined in terms of all such domains $D \subset \mathbb{C}$; the property requires that for all triplets $(D, x, y)$ and $(D', x', y')$ and conformal maps $f : D \to D'$, the image measure of $\mu_{D,x,y}$ under $f$ is equivalent to $\mu_{D',x',y'}$. Consider the special case that $(D, x, y) = (\mathbb{H}, 0, \infty)$ and $(D', x', y') = (\mathbb{H}/\gamma(0, t], \gamma(t), \infty)$. Let $f : \mathbb{H} \to \mathbb{H}/\gamma(0, t]$ be a conformal map that takes $0$ to $\gamma(t)$ and fixes $\infty$. One implication of conformal invariance is the equivalence (up to time reparametrization) of the measure $\mu_{\mathbb{H}/\gamma(0, t]}.\gamma(t), \infty}$ and the image of $\mu_{\mathbb{H}, 0, \infty}$ under $f$. We will see that this conjectured equivalence combined with the domain Markov property motivate the definition of the Schramm-Loewner equation.

Using Schramm-Loewner evolution, the existence of the full scaling limit of the percolation process has been rigorously established ([6]). We will not give any good intuitive reason for one to expect conformal invariance of the scaling limit of the
percolation exploration process, but the result had been conjectured by physicists more than 30 years ago ([24]). A proof that assumes the scaling limit exists and then proves conformal invariance was first accomplished in [29] by Smirnov. In Section 1.2.2 we explain how the domain Markov property and conformal invariance are used to derive the Schramm-Loewner equation.

1.1.2. Loop-erased random walk

In order to define the loop-erased random walk, we first need the random walk half-plane excursion. Consider all two-dimensional $n$-step walks, $S_n : \mathbb{N} \to \mathbb{C}$ with $S_0 = 0$ and $X_j := S_{j+1} - S_j \in \{i, 1, -i, -1\}$ for $j \geq 0$. We can extend the domain of $S$ to the positive real numbers by linear interpolation to get a subset of the space of continuous curves $S : [0, \infty) \to \mathbb{C}$ indexed by sequences $(X_0, X_1, \cdots, X_n)$. By letting $n \to \infty$, we get a 2-dimensional (infinite) simple walk indexed by infinite sequences of the form $(X_0, X_1, \cdots)$ with $X_j \in \{i, 1, -i, -1\}$.

One can induce a probability measure on infinite simple walks by prescribing transition probabilities to the increments $X_j$, $j \geq 0$. For the simple random walk and upper half plane excursion, these transition probabilities only depend on the position of $S_j$ at time $j$. Thus we can completely describe the transition probabilities by the function $p : \mathbb{Z} + i\mathbb{Z} \to \mathbb{Z} + i\mathbb{Z}$, given by $p(z, w) := \mathbb{P}[\hat{S}_{j+1} = w | \hat{S}_j = z]$ and satisfying $\sum_{w \in \mathbb{Z} + i\mathbb{Z}} p(z, w) = 1$ for all $j \in \mathbb{N}$. The simple random walk is the distribution of infinite simple walks, denoted $S^{(0)}_t$, achieved by letting

$$p(z, z + 1) = p(z, z - 1) = p(z, z + i) = p(z, z - i) = \frac{1}{4},$$

and $p(z, w) = 0$ otherwise.

By Donsker’s invariance principle, the scaled process given by

$$S^{(N)}_{t} := N^{-1/2} S^{(0)}_{2Nt}$$

where
converges weakly to a two-dimensional Brownian process, \( B_t = B^1_t + iB^2_t \), as \( N \to \infty \).

We want to define a similar process, \( \hat{S} \), with \( \hat{S}_0 := 0, \hat{S}_1 := i \) and \( \hat{S}_t \in \mathbb{H} \) for all \( t > 1 \). More constructive descriptions of the process we define are found in the literature, but, for brevity, we define the process by explicitly stating the transition probabilities \( p(z, w) \) for times \( t \geq 1 \), as follows:

\[
\hat{p}(z, z+1) = \hat{p}(z, z-1) = \frac{1}{4} \\
\hat{p}(z, z+i) = \frac{\Im z + 1}{4\Im z} \\
\hat{p}(z, z-i) = \frac{\Im z - 1}{4\Im z}.
\]

These properties may seem somewhat arbitrary to the reader, a consequence of our avoidance of lengthy, constructive definition of the model. In fact, the random walk \( \hat{S} \) coincides exactly with what is justly described as a simple random walk in \( \mathbb{C} \) conditioned to stay in \( \mathbb{H} \) or \textit{random walk \( \mathbb{H} \)-excursion}. This is not an obvious correspondence; a theorem is required to establish the coincidence between the two processes ([17]). Note that the continuation, \( \{\hat{S}_t\}_{t>n} \), of a path \( \hat{S} \) conditioned to have \( \hat{S}_n = z \in \mathbb{H} \) is completely independent of \( \{\hat{S}_t\}_{t\leq n} \); that is, it is a Markov process. It can be shown that the scaled random walk \( \mathbb{H} \)-excursion,

\[
\hat{S}^{(N)}_t := N^{-1/2}\hat{S}_{2Nt},
\]

converges weakly to the so-called \( \mathbb{H} \)-\textit{excursion} given by, \( \hat{B}_t = B^1_t + i|B^*_t| \), where \( B^1_t \) is a 1-dimensional Brownian motion and \( B^*_t \) is an independent 3-dimensional Brownian motion [17]. It is known that \( |B^*_t| \), hence \( \hat{B}_t \), diverges to \( \infty \) (in fact, this a consequence of Theorem 2.1.1 of Section 1.2.4, below). This coincides with the fact that \( \hat{S}_t \) is not a recurrent process, which is important for the following construction of the loop-erased random walk.

More generally, consider any 2-d simply connected domain, \( D \neq \mathbb{C} \), with designated boundary points \( z, w \in \partial D \). Roughly speaking, the \textit{Brownian excursion on} \( D \)
from $z$ to $w$ is a 2-dimensional Brownian motion $B_t$ with $B_0 = z$ that is conditioned to stay in $D$ until it leaves $D$ through $w$. A precise definition of this process requires a lengthy development we wish to avoid here; however, it ultimately follows that the Brownian excursion on $D$ from $z$ to $w$, as defined in the literature, is equivalent to the image of the $\mathbb{H}$-excursion under a conformal map, $F_D : \mathbb{H} \to D$, mapping $0 \mapsto z$ and $\infty \mapsto w$, modulo reparametrization. It can also be shown that the Brownian excursion on $D$ from $z$ to $w$ coincides with the fine-mesh scaling limit of random walk excursion measures defined on lattice approximations of the domain $D$. In other words, the random walk $\mathbb{H}$-excursion is a member of a family of measures on curves defined in simply connected domains $D \subset \mathbb{C}$ ($D \neq \mathbb{C}$) and possessing conformally invariant scaling limits. Precise definitions of Brownian excursions and rigorous conformal invariance proofs are found in [17].

Now, if $\hat{S} = (\hat{S}_1, \hat{S}_2, \ldots)$ is a $\mathbb{H}$-excursion, transience of $\hat{S}$ allows us to construct a self-avoiding path $(0, \hat{S}_{j_1}, \hat{S}_{j_2}, \ldots)$ achieved by chronologically removing loops of $\hat{S}$, as they occur. Implicitly, this means that each subsequence $(S_{j_k}, S_{(j_k+1)}, \ldots, S_{(j_{(k+1)}-1)})$ is a finite random walk in $\mathbb{H}$ that terminates at its initial point. The process $(L_k)_{k \in \mathbb{N}} := (S_{j_k})_{k \in \mathbb{N}}$ is called the $\mathbb{H}$-loop-erased random walk. The loop erasure procedure used to define the $\mathbb{H}$-loop-erased random walk can also be applied to random walk excursions in arbitrary simply connected domains, $D$, with designated initial and terminal points, $z$ and $w$, to get a $D$-loop-erased random walk from $z$ to $w$. It was conjectured that the fine-mesh scaling limits of such objects retains the conformal invariance property of the Brownian excursions on simply connected domains $D$ since the loop erasure procedure is not explicitly dependent on the parametrization for the $\mathbb{H}$-excursion and only depended on the order in which the loops are erased.

With some work, one can also prove that the domain Markov property also holds for LERW. In the context of the $\mathbb{H}$, this means that if we condition the $\mathbb{H}$-LERW on its first $n$-steps (note that this is a condition on the entire history of the underlying
$\mathbb{H}$-excursion), then the resulting measure on paths is the same measure one gets by defining the $\mathbb{H} \setminus \gamma(0,N]$-LERW directly.

LERW provides another example of a 2-d statistical system where the domain Markov property and expected conformal invariance of the scaling limit provide motivations for the derivation of $\text{SLE}_\kappa$ outlined in the sequel. In fact, Schramm specifically had the scaling limit of LERW in mind when he discovered the $\text{SLE}_\kappa$ process. Later the convergence of LERW to $\text{SLE}_2$ was established rigorously by Lawler, Schramm and Werner in [20].

1.1.3. Other models

Several other 2-d lattice systems have been associated with $\text{SLE}_\kappa$. It will become clear in Section 1.2.2 that any system for which $\text{SLE}_\kappa$ is a candidate scaling limit are self-avoiding systems (with the possibility of self-intersections that are not self-crossings). Currently, the following associations between model and scaling limit have been established rigorously:

- LERW $\leftrightarrow$ $\text{SLE}_2$
- Ising model $\leftrightarrow$ $\text{SLE}_3$
- Harmonic explorer $\leftrightarrow$ $\text{SLE}_4$
- Critical percolation $\leftrightarrow$ $\text{SLE}_6$
- Uniform spanning tree $\leftrightarrow$ $\text{SLE}_8$

It is also conjectured that the 2-d upper half plane self-avoiding walk has a scaling limit given by chordal $\text{SLE}_{8/3}$ ([21]), and there is strong numerical evidence that this conjecture holds [12].

Thematic of each of these models, there is a critical parameter. That is, each of these models can be viewed as a sub-model of a larger, one-parameter family of
models. As an example, the black/white probability $p = 1/2$ for critical percolation is a critical parameter: if we define the model for $p \neq 1/2$ then the scaling limit is trivial; that is, it is dominated by infinitesimal black sites or dominated by infinitesimal by white sites. In essence, at these critical values, the statistical models do not have a finite correlation length, allowing for the existence of a nontrivial scaling limit.

1.1.4. Near-critical models

In order to achieve a nondegenerate fine-mesh scaling limit of the critical site percolation model and LERW defined above, it is essential that the model is at criticality when one takes the scaling limit. That is, for critical site percolation in the upper half plane, the fixed probability of each individual site being open (or closed) must be equal to $1/2$, and the fixed horizontal transition probabilities for LERW, $\hat{p}(z, z + 1) = \hat{p}(z, z - 1) = 1/4$, cannot be skewed with $\hat{p}(z, z + 1) \neq \hat{p}(z, z - 1)$. In such cases the scaling limit of the interface/path can be shown to degenerate to either the positive or negative real line, depending which horizontal direction the skew in the fixed probabilities favors.

However, it is possible to obtain non-trivial scaling limits of the associated measures if one adjusts the transition probabilities closer and closer to the critical transition probabilities as one takes the scaling limits. These off-critical or near-critical models have been considered in mathematical physics literature long before SLE$_{\kappa}$ was introduced, at least in the case of the site percolation model ([13]). The success of SLE$_{\kappa}$ as a viable scaling limit for the several critical 2-d systems listed above has motivated an effort to consider variations of SLE$_{\kappa}$ as possible scaling limits of off-critical models.

Work has been done on the possibility of SLE$_{\kappa}$ variations as a viable scaling limit for off-critical systems. Recent results on off-critical percolation have given a strong indication that this is not a realistic endeavor, at least in the case of percolation ([23]).
In the case of LERW, there have been slightly more successful efforts in utilizing SLE\(_\kappa\) and its variations as a tool for approaching off-critical systems([2], [1]). These approaches are more physically based, and are not necessarily in line with the specific variation of SLE\(_\kappa\) in this paper; however, it should be noted that possible descriptions of the off-critical regime was one of the original motivations of this work.

1.2. Chordal Schramm Loewner Evolution

1.2.1. The Loewner equation

The chordal Loewner evolution is a non-autonomous vector field in the upper half plane, \(\mathbb{H}\). Soon we will reveal the deep connection between the differential equation that defines chordal Loewner evolution and families of conformal maps from subsets of \(\mathbb{H}\) to \(\mathbb{H}\), but these connections are not immediate from the definition. Given a vector field that evolves in time, we can always define the reverse flow of the vector field by letting time evolve backwards and reversing the direction of the vector field. The reverse flow of the Loewner evolution is often used in our methods, so we define a slightly more general notation for the chordal Loewner equation than typically found in the literature. This notation will accompany both the forward and reverse flow of the chordal Loewner equation.

For any continuous function \(U : \mathbb{R} \to \mathbb{R}\) and any fixed \(s \in \mathbb{R}\), we use \(g_{t}^{(s)}(z)\) to denote the complex-valued solution to the initial value problem,

\[
\frac{\partial}{\partial t}g_{t}^{(s)}(z) = \frac{2}{g_{t}^{(s)}(z) - U_{s+t}}; \quad g_{0}^{(s)}(z) = z \in \mathbb{H}, \quad (1.1)
\]

We call \(U\) the driving function of the process \(g_{t}^{(s)}\).

If \(U_{0} = 0\), then \(g_{t}^{(0)}(z)\), \(t \geq 0\), is the conventional definition of the chordal Loewner equation typically found in the literature. For all other values of \(s\) and \(U_{s}\) in \(\mathbb{R}\), the
function $\tilde{g}_t(z) := g_t(s)(z + U_s) - U_s$ satisfies,

$$
\frac{\partial}{\partial t} \tilde{g}_t(z) = 2 \frac{g_t(s)(z + U_s) - U_{s+t} + U_s - U_s}{\tilde{g}_t(z) - (U_{s+t} - U_s)};
$$

(1.2)

$$
\tilde{g}_0(z) = z + U_s - U_s = z.
$$

Hence, any solution $\{g_t(s)\}_{t>0}$ of (1.1) is simply a translation of a solution of the chordal Loewner equation that adheres to the conventional definition of the chordal Loewner equation with driving function, $\tilde{U}_t := U_{s+t} - U_s$.

Note that the flow vectors defined by the chordal Loewner equation always have a negative vertical component and a horizontal component that points away from the position of the driving function with respect to the position of the real part of the vector.

For the unfamiliar, it is helpful to consider the solution of the equation when the driving function is identically zero, $U \equiv 0$. In this case, one can explicitly solve (1.1) by separation of variables to get,

$$
g_t^{(0)}(z) = \sqrt{4t + z^2} := \sqrt{|4t + z^2|} \exp\left(\frac{i}{2} \arg(4t + z^2)\right),
$$

(1.3)

where the branch of the argument is $\arg(\cdot) \in (0, 2\pi)$. In particular, for $y > 0$,

$$
g_t^{(0)}(iy) = \sqrt{4t - y^2}.
$$

The solution of (1.1) is undefined at times $t$ for which $g_t^{(s)}(z) - U_{s+t} = 0$, so $g_t^{(0)}(iy) = \sqrt{4t - y^2}$ is only a valid solution for $t \in [0, y^2/4]$. This illustrates the existence of an increasing sequence of subsets, $K_t \subset \mathbb{H}$, for which the solution, $g_t(z)$, of (1.1) no longer exists. For $U \equiv 0$, $K_t$ is given by the imaginary interval $i(0, 2\sqrt{t}]$. An additional example is detailed in the Appendix, section 5.2, where separation of variables is used to derive an implicit equation for $K_t$ when $U_t := \mu t$ for $\mu > 0$. Our primary interest is in the case $U_t = \sqrt{\kappa} W_t + \mu t$, where $W_t$ is a standard Brownian motion, so the solution for the case $U_t = \mu t$ is enlightening. In particular, when $\mu > 0$, the driving function and the growth of the real part of the set $K_t$ tend towards positive infinity,
and when $\mu > 0$, the driving function and the growth of the real part of the set $K_t$ tend towards negative infinity.

The previous example motivates a precise characterization of the domain of the complex function $g_t^{(s)}$ as a subset of $\mathbb{H}$. We extend the parenthetical superscript notation to the Loewner evolution domains by writing $g_t^{(s)} : H_t^{(s)} \to \mathbb{H}$. For each $z \in \mathbb{H}$, let

$$T^{(s)}(z) := \sup\{t \geq 0 : \min_{r \in [0,t]} |g_r^{(s)}(z) - U_{r-s}| > 0\},$$

and $K_t^{(s)} := \{z \in \mathbb{H} : T^{(s)}(z) \leq t\}$. The families $\{K_t^{(s)}\}_{t>0}$ and $\{g_t^{(s)}\}_{t>0}$ are each referred to as the Loewner chain associated with the driving function $\{U_{t-s}\}_{t>0}$. Since solutions of (1.1) are well-defined and continuous for $t \in [0,T^{(s)}(z)]$, the domain of $g_t^{(s)}$ can and will be taken to be $H_t^{(s)} := \{z \in \mathbb{H} : z \not\in K_t^{(s)}\}$. The sets $\{K_t^{(s)}\}_{t>0}$ and $\{H_t^{(s)}\}_{t>0}$ play a central role in the theory and its relationship to conformal maps. A few more definitions, that follow, will allow us to make the connection transparent.

We have that $K_t^{(s)}$ is one-parameter set of increasing subsets of $\mathbb{H}$, and it is easily seen that, for each $t > 0$, $K_t^{(s)}$ is a compact $\mathbb{H}$-hull, as defined in the Appendix, Section 5.1. Some work reveals that $K_t^{(s)}$ is continuously increasing with respect to $t$; that is, for all $T > 0$, and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $t \leq T$, there exists a bounded and connected set $S \subseteq H_t^{(s)}$ with diameter not larger than $\epsilon$ such that $S$ disconnects $K_{t+\delta}^{(s)} \setminus K_t^{(s)}$ from infinity in $\mathbb{H} \setminus K_t^{(s)}$ [30]. The set of all continuously increasing compact $\mathbb{H}$-hulls has a direct correspondence with solutions of a more general version of the chordal Loewner evolution that will not be introduced here; details of the more general theory Loewner equation can be found in [17].

It can be shown that $g_t^{(s)}(H_t^{(s)}) = \mathbb{H}$; we do this by first introducing the reverse flow of the Loewner equation. The reverse Loewner flow, $\{f_t^{(s)}(z)\}_{t>0}$, is the unique solution of

$$\frac{\partial}{\partial t} f_t^{(s)}(z) = \frac{-2}{f_t^{(s)}(z) - U_{s-t}} ; \quad f_0(z) = z \in \mathbb{H}.$$  

(1.4)
Note that if \( g_t(z) = w \), then \( g_t(z) \bigg|_{r=0} = w \), and, by (1.1), \( \{g_t(z)\}_{r>0} \) satisfies,

\[
\frac{\partial}{\partial r} g_t(z) = \frac{-2}{g_t(z) - U_{s+t-r}},
\]

By definition, \( \{f_r^{(s+t)}(w)\}_{r>0} \) also solves this initial value problem, and the solution is unique. Therefore

\[
f_t^{(s+t)}(w) = g_t(z) = z,
\]

and we have shown \( f_t^{(s+t)} \) is the inverse of \( g_t(z) \).

Note that, for all \( r > 0 \) and \( z \in \mathbb{H} \),

\[
\frac{\partial}{\partial r} 3f_r^{(s+t)}(z) > 0,
\]

so \( |f_r^{(s+t)} - U_{s+t-r}| > 0 \) and the domain restrictions encountered for \( g_t(z) \) do not apply to the inverse maps, \( f_r^{(s+t)} \). Thus the domain of \( f_t^{(s)} \) is \( \mathbb{H} \), and the inverse relationship between \( g_t(z) \) and \( f_t^{(s+t)} \) allows us to conclude that \( g_t(z) \) is a bijection from \( H_t^{(s)} \) to \( \mathbb{H} \).

Using this inverse relationship, it is easily shown that for \( s \in \mathbb{R} \), \( t > 0 \) and \( r > 0 \),

\[
\begin{align*}
  g_t^{(s)} &= f_r^{(s+t+r)} \circ g_{t+r}^{(s)} \quad \text{and} \\
  g_t^{(s)} &= g_{t+r}^{(s-r)} \circ f_r^{(s)}. 
\end{align*}
\]  

(1.5) \hspace{1cm} (1.6)

We will often use \( f_t^{(s+t)} \) to avoid using the cumbersome inverse function notation, \( [g_t^{(s)}]^{-1} \).

While working on the Bieberbach conjecture, Charles Loewner identified the Loewner equation as a differential equation satisfied by certain analytic functions. To discover the chordal Loewner equation the way he might have, one considers conformal maps, \( g_A \), from \( \mathbb{H} \setminus A \) to \( \mathbb{H} \) with \( A \) given by a compact \( \mathbb{H} \)-hull and \( g_A(\infty) = \infty \).

The set \( \mathbb{H} \setminus A \) necessarily contains \( \infty \) so the expansion of \( g_A \) about \( z = \infty \) has the form

\[
g_A(z) = b_{-1}z + b_0 + b_1z^{-1} + ... \]
As detailed in Appendix Section 5.1, \( g_A(z) \) is uniquely determined if we require that it satisfy

\[
\lim_{z \to \infty} [ g_A(z) - z ] = 0, \tag{1.7}
\]

or, equivalently, \( b_{-1} = 1 \) and \( b_0 = 0 \). Now consider a continuously increasing family of compact \( \mathbb{H} \)-hulls \( \{ A_t \}_{t>0} \) and a corresponding family of conformal maps, \( g_{A_t}(z) \), satisfying (1.7). These maps will have an expansion about \( z = \infty \) of the form

\[
g_{A_t}(z) = z + a(t)z^{-1} + \ldots,
\]

and it can be shown that \( a(t) \), called the half-plane capacity of \( A_t \), is an increasing continuous function; therefore, we can reparametrize the family \( \{ A_t \}_{t>0} \) by a continuous function \( \rho \) such that \( a(\rho(t)) = 2t \). It can be shown that continuously increasing families of compact \( \mathbb{H} \)-hulls of this form comprise the Loewner chain associated with (1.1) for some continuous driving function \( U_t : [0, \infty) \to \mathbb{R} \). One proves this by utilizing classical methods of complex analysis like the Beurling estimate; rigorous development of these estimates can be found in [17].

Conversely, it can be shown that solutions of (1.1) are analytic functions with an expansion about \( z = \infty \) given by

\[
g_{A_t}(z) = z + 2tz^{-1} + \ldots,
\]

where \( A_t \) is given by the Loewner chain \( K_t^{(s)} \). This can be shown directly, and a proof is detailed in Appendix section 5.1. The primary observation is that the solutions of (1.1) are families of conformal maps \( \{ g_t^{(s)} : H_t^{(s)} \to \mathbb{H} \}_{t>0} \) that correspond to continuously increasing Loewner chains. In applications, the Loewner chain itself is often the (candidate) scaling limit of a discrete process being considered. We will want to make qualitative descriptions of \( K_t^{(s)} \) by studying the boundary behavior of the maps \( g_t^{(s)} \) on the boundary points of \( H_t^{(s)} \) in \( K_t^{(s)} \).

In many important cases, the Loewner chain \( K_t^{(s)} \) is generated by a curve, \( \gamma^{(s)} \), with \( \gamma^{(s)}(0) = U_s \) and \( \gamma^{(s)}([0, \infty)) \subset \overline{\mathbb{H}} \), meaning that for each time \( t \geq 0 \), the set
$H_t^{(s)} = \mathbb{H} \setminus K_t^{(s)}$ is the unbounded component of $\mathbb{H} \setminus \gamma([0,t])$. Certain conditions on the driving function, like Lipschitz continuity, can be used to ensure the corresponding Loewner chain is of this form, but there exists Loewner chains that are not generated by curves. Curiously, Brownian motion is a “borderline” driving function with respect to a Hölder $\alpha$-continuity exponent (jargon defined in the following subsection) that dictates the existence of a generating curve. It is difficult, but possible, to prove that a generating function exists w.p.1 in this case. More details on this matter are reviewed in Subsection 1.2.3, below. The driving functions considered in this study will be shown to correspond to Loewner chains that are generated by a curve, so we limit our scope to Loewner chains generated by a curve for the remainder.

Suppose the Loewner chain $\{g_t^{(s)}\}_{t>0}$ is generated by $\gamma^{(s)}(t)$. Since $f_t^{(s+t)}$ is an analytic map with domain $\mathbb{H}$, it has a unique continuous extension to $\mathbb{R}$ [25]. The continuous extension of $f_t^{(s)}$ then satisfies

$$f_t^{(s+t)}(U_{s+t}) = \lim_{z \to U_{s+t}} f_t^{(s+t)}(z) = \lim_{r \to 0} f_t^{(s+t)}(\gamma^{(s+t)}(r)),$$

since, by construction, $\gamma^{(s+t)}(r)$ grows continuously with $\gamma^{(s+t)}(0) := U_{s+t}$. The function $f_t^{(s+t)}$ can be viewed as a correspondence between the location, on $\mathbb{R}$, of the infinitesimal growth of the Loewner chain $K_t^{(s+t)}$ and the location, on $\partial H_t^{(s)}$, of the infinitesimal growth of the Loewner chain $K_t^{(s)}$. For $r > 0$. Since the growth of either Loewner chain given is dictated by the growth of the generating curve, we must have

$$\lim_{r \to 0} f_t^{(s+t)}(\gamma^{(s+t)}(r)) = \gamma^{(s)}(t),$$

so that for all $s \in \mathbb{R}$ and $t > 0$,

$$f_t^{(s+t)}(U_{s+t}) = \gamma^{(s)}(t) \quad (1.8)$$

In terms of the (forward) Loewner chain $g_t^{(s)}$, we have

$$g_t^{(s)}(\gamma^{(s)}(t)) = U_{s+t}. \quad (1.9)$$
We now have a clear picture of the connection between families of conformal maps and the chordal Loewner equation. We would like to identify generating curves that are good candidates for scaling limits of the discrete processes described in Section 1.1. This means that the driving function for the corresponding Loewner chain will be a random process, and the Loewner equation will be a stochastic differential equation. In the next section, we identify a two parameter family of candidate driving processes and then discuss the technicalities surrounding the existence of the generating curve.

1.2.2. Defining properties of Schramm-Loewner evolution

Schramm-Loewner evolution, or SLE\(_\kappa\), can be precisely defined as the Loewner chain, \(K_t^{(0)}\) (or \(g_t^{(0)}\)), corresponding to a driving function given by a 1-dimensional Brownian motion; however, a more pedagogic approach is to define SLE\(_\kappa\) as a Loewner chain that satisfies three specific properties we will review now. Mathematicians and physicists have reason to expect that scaling limits of the discrete processes discussed in Section 1.1 should satisfy these properties, and Oded Schramm applied these conditions to Loewner chains in order to deduce a driving function whose Loewner chain represents a scaling limit of LERW.

The scaling limits of the discrete processes above are conjectured to satisfy the Markovian property and be invariant under conformal transformations. That is, if \(\gamma\) is a process supported in \(\mathbb{H}\) and the process, as defined on the simply connected set \(\mathbb{H} \setminus \gamma([0,t])\), is given by \(\tilde{\gamma}\), then the image of \(\tilde{\gamma}\) under a conformal transformation from \(\mathbb{H} \setminus \gamma([0,t])\) to \(\mathbb{H}\) is distributionally equivalent to \(\gamma\) and independent of \(\gamma([0,t])\).

The relationship between \(K_t^{(s)}\) and \(g_t^{(s)}\) allows us to translate these properties into the specific requirements on \(g_t^{(s)}\); namely, for \(s, t > 0\), (i) the distribution of \(g_t^{(s)} - U_s\) only depends on \(t\) and not on \(s\), and (ii) \(g_t^{(s)}\) is independent of \(\{g_r^{(0)}\}_{r \in [0, s]}\). The relationship, (1.9), between \(\gamma\) and \(U\) allows us to convert these requirements into the following conditions on the driving function,
• $t \mapsto U_t$ is continuous w.p.1;
• for all $r \in [0, s)$ and $t > s$, $U_t - U_s$ is independent of $U_r$;
• the distribution of $U_t - U_s$ only depends on $t - s$.

By algebra, one easily deduces that these assumptions imply that $E[U_t]$ and $E[(U_t - E[U_t])^2]$ are linear functions, say $E[U_t] = \mu t$ and $E[(U_t - E[U_t])^2] = \kappa t$. It follows that $(\sqrt{\kappa})^{-1}(U_t - \mu t)$ is a continuous martingale with variation,

$$E[((\sqrt{\kappa})^{-1}(U_t - \mu t))^2] = t,$$

so, by Levy’s martingale characterization of Brownian motion, $W_t := (\sqrt{\kappa})^{-1}(U_t - \mu t)$ is a standard Brownian motion or standard Wiener process [11]. Thus $U_t = \sqrt{\kappa}W_t + \mu t$ is the only driving process that will produce Loewner chains satisfying the criteria above. When considering the corresponding Loewner flow, our intuition is benefited from the Brownian scaling relationship,

$$\sqrt{\kappa}W_t \overset{D}{=} W_{\kappa t}.$$

Early efforts to describe a scaling limit of a discrete system with Loewner chains had been focused on discrete interfaces described in sections 1.1.1-1.1.3, where one expects to have either scale invariance or invariance with respect to reflections about the imaginary axis. Either of these conditions imply $\mu = 0$, and SLE$_\kappa$ is defined to be the Loewner chain corresponding to the driving function $U_t = \sqrt{\kappa}W_t$, with $\kappa > 0$ and $\mu \in \mathbb{R}$. We are interested in the case of nonzero drift, $\mu > 0$, and use the notation SLE$_\kappa^\mu$ to denote Loewner chain corresponding to the driving function $U_t = \sqrt{\kappa}W_t + \mu t$, $\mu > 0$.

The *Schramm-Loewner equation* is given by,

$$\frac{\partial}{\partial t} g_t^{(s)}(z) = \frac{2}{g_t^{(s)}(z) - \sqrt{\kappa}W_{s+t}}; \quad g_0(z) = z \in \mathbb{H},$$

(1.10)
where \( s \) is fixed and, for each choice of \( s \), \( W_{s+t} - W_s \) is standard Brownian motion. In the literature, \( \text{SLE}_\kappa \) is defined to be the special case \( s = 0 \). Inclusion of the superscript \((s)\) only results in a random horizontal translation of the Loewner chain by \( U_s \) per equation (1.2), and is useful for notational reasons. Scale invariance allows us to conclude that the function

\[
h_t(z) := \frac{g_t(\sqrt{\kappa} z)}{\sqrt{\kappa}}
\]

is distributionally equivalent (up to reparametrization) to \( g_t(z) \), and (1.10) implies that

\[
\frac{\partial}{\partial t} h_t^{(s)}(z) = \frac{a}{h_t^{(s)}(z) - W_{s+t}} ; \quad h_0^{(s)}(z) = z, \quad (1.11)
\]

where \( a := 2/\kappa \). Below, we will identify ranges or \textit{phases} of the value \( \kappa \) that correspond to different qualitative behavior of \( \text{SLE}_\kappa \). The parameter \( a \) is often used as an alternative to \( \kappa \) for indexing the qualitative behavior of \( \text{SLE}_\kappa \), and we reserve the notation \( a = 2/\kappa \) throughout.

Our next task is to establish the existence of a generating curve for these Loewner chains. The existence of a generating curve is not guaranteed, so we need to identify which values of \( \kappa \), if any, correspond to a Loewner chain generated by a curve w.p.1.

1.2.3. Existence of the generating curve

We have shown that the solution \( g_t^{(s)} : H_t^{(s)} \to \mathbb{H} \) of (1.1) exists and is unique for any continuous driving function \( U : \mathbb{R} \to \mathbb{R} \), but it is not true in general that the Loewner chain \( K_t^{(s)} \) is generated by a curve. If \( K_t^{(s)} \) is generated by \( \gamma^{(s)} \), then \( f_t^{(s+t)}(U_t) = \gamma(t) \) in the extension of \( f \) to \( \mathbb{R} \); conversely, if the limit

\[
\gamma^{(s)}(t) := \lim_{y \to 0} f_t^{(s+t)}(U_{t+s} + iy) \quad (1.12)
\]

exists and defines a continuous \( \gamma^{(s)} : [0, \infty) \to \mathbb{H} \), then \( K_t^{(s)} \) is generated by \( \gamma^{(s)} \).

Therefore, establishing the existence of a generating curve for \( K_t^{(s)} \) requires careful
analysis of boundary behavior of the conformal maps $f_t^{(s+t)}$ near $U_t$; in particular, bounds on the complex derivative near $U_t$.

Schramm-Loewner evolution is in fact generated by a curve, but the proof is somewhat technical and will not be reviewed here. In [16], a corollary of the main results in the article describes a probabilistic bound for $\frac{\partial}{\partial z} f_t^{(s+t)}(z)$ near $U_t$ that allows for a short proof of the existence of the generating curve.

It is worth commenting that Brownian motion represents a ‘borderline’ case of driving functions for Loewner chain generated by curve according to the following theorem due to Marshall and Rohde [22],

**Theorem 1.2.1.** There exists $C' > 0$ such that if $U_t$ satisfies,

$$|U_t - U_s| \leq C' \sqrt{t - s}$$

then $K_t^{(s)}$ is generated by a simple curve $\gamma$; moreover, there exists $C > C'$ and a continuous function $U : \mathbb{R} \to \mathbb{R}$ such that, for all $s, t > 0$,

$$|U_t - U_s| \leq C \sqrt{t - s},$$

but the Loewner chain $K_t^{(s)}$ corresponding to $\{U_{t-s}\}_{t>0}$ is not generated by a curve.

For finite $T$, a function $U : [0, T] \to \mathbb{R}$ is said to be Hölder $\alpha$-continuous if there exists $C > 0$ such that for all $s, t \in [0, T]$, $|U_t - U_s| \leq C|t - s|^\alpha$. Then, by the preceding theorem, if $U$ is Hölder $\alpha$-continuous on finite intervals for $\alpha < 1/2$, then $K_t^{(s)}$ is generated by a simple curve. Curiously, on finite intervals the process $U_t := \sqrt{\kappa} W_t + \mu t$ is, w.p.1, Hölder $\alpha$-continuous for all $\alpha < 1/2$ but is not Hölder $1/2$-continuous [17].

For the remainder, we are interested in Loewner chains corresponding to $U_t = \sqrt{\kappa} W_t + \mu t$ for $\kappa > 0$, denoted SLE$_\kappa$ if $\mu = 0$ and SLE$_\kappa^\mu$ if $\mu \neq 0$. In the next section, the last of the introduction, we collect more results on SLE$_\kappa$ before proceeding to establish basic properties of SLE$_\kappa^\mu$ in Chapter 2.
1.2.4. Real-valued SLE$_\kappa$ and phases

An important property of SLE$_\kappa$ is the relationship it has to the Bessel-$d$ stochastic differential equation when one solves the Schramm-Loewner equation for initial conditions in $\mathbb{R}$. The connection between these equations allows us to use classical results on solutions of the Bessel stochastic differential equation to identify which values of $\kappa$ correspond to certain topological properties of the generating curve.

For any continuous driving function, $U_t$, define $g_t$ as the solution to
\[
\frac{\partial}{\partial t}g_t(x) = \frac{2}{g_t(x) - U_t}; \quad g_0(x) = x \in \mathbb{R} \setminus \{0\},
\] defined for all times
\[
t \leq T(x) := \sup\{t \geq 0 : \min_{r \in [0,t]} |g_r(x) - U_r| > 0\}.\]

This defines a nonautonomous vector field on $\mathbb{R}$ and a family of increasing sets $R_t := \{x : t \leq T(x)\}$ called the real-valued Loewner chain.

Now let $U_t = \sqrt{\kappa}W_t$, where $W_t$ is a standard Brownian motion, then $X_t(x) := (g_t(\sqrt{\kappa}x) - U_t)/\sqrt{\kappa}$ satisfies the stochastic differential equation given by
\[
dX_t(x) = \frac{a}{X_t(x)}dt + dW_t, \quad X_0(x) = x > 0,
\]
with $a = 2/\kappa$, as usual. Here we are using the distributional equivalence $-W_t \overset{D}{=} W_t$ and letting $dW_t$ be the stochastic differential associated with the Brownian motion $-W_t$. The process $-W_t(x)$ is another standard Brownian motion, so we need only consider $x > 0$ since $-X_t(-x) \overset{D}{=} X_t(x)$. The solution to (1.16) is called the Bessel-$d$ process for $d := 2a + 1$ since it is distributionally equivalent to the magnitude, $|B_t|$, of a $d$-dimensional Brownian motion when $d \in \mathbb{N}$.

Let $K_t$ denote the Loewner chain corresponding to $U_t$. It is easily shown that $R_t = \mathbb{R} \cap K_t$, and $X_t(\cdot)$ is a continuous function on $\mathbb{R}^+ \setminus R_t$, so $x < y$ implies that $T(x) \leq T(y)$ w.p.1. Using existence of the generating curve, $\gamma$, we can conclude that,
for finite $T(x)$,
\[ \lim_{t \to T(x)} \gamma(t) \geq x; \]
in particular, $\gamma(T(x)) \in \mathbb{R}$. This allows us to use classical analysis of (1.16) to infer qualitative properties of the SLE$_\kappa$ curve.

The following distributional properties of the family $\{T(x)\}_{x > 0}$, described in terms of the parameter $a$, are results of standard techniques in stochastic calculus; for a full exposition of the proof of the following theorem, we refer the reader to Proposition 1.2 in [17]. The essential step of the proof in [17] is an application of Ito’s formula to the one-dimensional exiting distribution of $X_t(x)$, with $x \in [x_1, x_2]$, given by
\[ \phi(x; x_1, x_2) := P[X_\sigma(x) = x_2] \text{ where } \sigma := \inf\{t : X_t(x) \in \{x_1, x_2\}\}. \]
This produces a deterministic differential equation for $\phi$ that can be solved explicitly in terms of $x$, $x_1$ and $x_2$. One can then take $x_1 \to 0$ and $x_2 \to \infty$ to determine if $T(x)$ is finite. We will apply the same technique for our analysis of the real-valued SLE$_\kappa$ in Section 2.1, below. Here, we summarize the phases of real-valued SLE$_\kappa$ in terms of $\kappa$ without proof:

**Theorem 1.2.2.** Take $T(x)$ as in (1.15).

- If $\kappa < 4$, then w.p.1 $\inf_{t \geq 0} X_t^x > 0$ for all $x > 0$.
- If $\kappa \leq 4$, then w.p.1 $T_x = \infty$ for all $x > 0$.
- If $\kappa > 4$, then w.p.1 $T_x < \infty$ for all $x > 0$.
- If $\kappa \geq 4$, then w.p.1 $T_x < T_y$ for all $0 < x < y$.
- If $4 < \kappa < 8$ and $0 < x < y$, then $P[T_x = T_y] > 0$.

The properties of the generating curve, $\gamma^{(s)}$, we prove in the next corollary will depend on $\kappa$, but not on $s$. This is because $\gamma^{(s)}(t)$ is just a horizontal translation
of the generating curve $\gamma(t)$ that corresponds to the driving function $U_{t+s} - U_s$, and for all values of $s$, $W_t^{(s)} := W_{s+t} - W_s$ is just another Brownian motion, as will be proved in the next paragraph. Also, our main results will make extensive use of the reverse Loewner flow, and we will need to evaluate our Brownian driving function for negative times. This requires a clear definition of the bi-infinite Brownian motion. Let $W_{t,-}$ and $W_{t,+}$ be independent, standard Brownian motions. A bi-infinite Brownian motion, $W_t$, is defined by letting $W_t := W_{t,+}$ for $t \geq 0$ and letting $W_t := W_{-t,-}$ for $t < 0$.

Now we prove that, for any $r \in \mathbb{R}$, $W_t^{(r)} := W_{r+t} - W_t$ is another bi-infinite Brownian motion. First consider $r < 0$. For $t > 0$, it follows that

$$W_t^{(r)} = \begin{cases} W_{-r-t,-} - W_{-r,-} & 0 < t < |r| \\ W_{t+r,+} - W_{-r,-} & t \geq |r| \end{cases}.$$ 

Recall that the covariance function for Brownian motion is given by $C(s, t) = \min\{s, t\}$. It follows that the covariance of the process $\{W_t^{(r)}\}_{t > 0}$ is given by

$$C(s, t) = \begin{cases} E[(W_{r-s,-} - W_{-r,-})(W_{r-t,-} - W_{-r,-})] & s < t < |r| \\ E[(W_{r-s,-} - W_{-r,-})(W_{t+r,+} - W_{-r,-})] & s > |r| < t \\ E[(W_{s+r,+} - W_{-r,-})(W_{t+r,+} - W_{-r,-})] & |r| < s < t \\ \min\{-r - s, -r - t\} - \min\{-r - s, -r\} - \min\{-r, -r - t\} + (-r) & s < t < |r| \\ \min\{-r - s, t + r\} - \min\{-r - s, -r\} - \min\{-r, t + r\} + (-r) & s < |r| < t \\ \min\{s + r, t + r\} - \min\{s + r, -r\} - \min\{-r, t + r\} + (-r) & |r| < s < t \\ \min\{s, t\}. \end{cases}$$

Therefore, $\{W_t^{(r)}\}_{t > 0}$ is a standard Brownian motion. A similar calculation shows $\{W_t^{(r)}\}_{t < 0}$ is also a Gaussian process with covariance $C(s, t) = \min\{s, t\}$, as is needed to conclude $\{W_t^{(r)}\}_{t < 0}$ is a Brownian motion. Lastly, since $\{W_t\}_{t < 0}$ is a Brownian motion independent of $\{W_t\}_{t > 0}$, $\{W_t^{(r)}\}_{t < 0}$ is necessarily independent of the latter process; further, $\{W_t\}_{t \in (-r, 0)}$ is independent of $\{W_{-t}\}_{t \leq -r}$, which implies that $\{W_t^{(r)}\}_{t < 0}$ is independent of $\{W_t^{(r)}\}_{t > 0}$ per requirement of the definition. This is enough to conclude $W_t^{(r)}$ is a bi-infinite Brownian motion for $r < 0$. The case $r > 0$ follows from a similar proof.
Since $W_t^{(s)} \overset{d}{=} W_t$ and $U_0 := \sqrt{\kappa}W_0 = 0$, it follows from (1.2) that
\begin{align*}
\{\gamma^{(s)}(t) - U_s\}_{t > 0} \overset{d}{=} \{\gamma^{(0)}(t)\}_{t > 0}.
\end{align*}
(1.17)

This property will be used frequently when we establish our main results. In the proof of the following theorem, we use the property to quickly generalize the qualitative description of the standard generating curve $\gamma^{(0)}$ to the randomly shifted generating curve, $\gamma^{(s)}$, for arbitrary $s \in \mathbb{R}$.

**Corollary 1.2.1.** Fix $s \in \mathbb{R}$ and let $\gamma^{(s)}(t)$ be the generating curve corresponding to SLE$_\kappa$. Then

- If $0 < \kappa \leq 4$, then $\gamma^{(s)}(t)$ is a simple curve with $\gamma^{(s)}((0, \infty)) \subset \mathbb{H}$ and
  \[ \lim_{t \to \infty} |\gamma^{(s)}(t)| = \infty \text{ w.p.} 1 \]
- If $\kappa > 4$, then $\gamma^{(s)}(t)$ is self-intersecting and
  \[ \bigcup_{t > 0}(K^{(s)}_t \cup R^{(s)}_t) = \mathbb{H} \text{ w.p.} 1 \]

**Proof.** ($\kappa < 4$) First take $\kappa \in (0, 4]$. By (1.17), we can take $s = 0$ without loss of generality. By Theorem 2.1.1, $T(x) := T^{(0)}(x) = \infty$; therefore, $\gamma(t) := \gamma^{(0)}(t) \in \mathbb{H}$ for all $t > 0$.

To prove that $\gamma$ is simple, first apply (1.17) to get that, for each $r > 0$, $\gamma^{(r)}(t) - U_r \in \mathbb{H}$ for all $t > 0$. We will assume that $\gamma(t_1) = \gamma(t_2)$ for some $t_1 \neq t_2$ and pursue a contradiction. If $r \in (t_1, t_2)$, then $\gamma^{(r)}(t_2 - r) \in \mathbb{H}$ w.p.1 so
\[ f_r^{(r)}(\gamma^{(r)}(t_2 - r)) = \gamma(t_2) = \gamma(t_1) \in \mathbb{H}. \]

This implies that $g_r^{(0)}(\gamma(t_1)) \in \mathbb{H}$, but this contradicts the necessity $T(\gamma(t_1)) = t_1 < r$. Thus $\gamma(t)$ is a simple curve.
Now we show $|\gamma(t)| \to \infty$ as $t \to \infty$ for $\kappa \in (0, 4]$. Using the notation $g_t := g_t^{(0)}$, note that for $c > 0$,

$$\frac{\partial}{\partial t}[cg_t(z)] = \frac{2}{cg_t(z) - c\sqrt{\kappa}W_t} \frac{\partial}{\partial t} \frac{2}{cg_t(z) - \sqrt{\kappa}W_t}$$

so, up to reparametrization, $\gamma$ is distributionally invariant with respect to scaling. Therefore, $\gamma(t) \to \infty$ if and only if there exists a $\delta > 0$ such that w.p.1 there exists $T > 0$ such that $|\gamma(t)| > \delta$ for all $t > T$ or, equivalently,

$$\mathbb{P}\left[\liminf_{t \to \infty} |\gamma(t)| > 0\right] = 1.$$  

To establish this, we will use facts about $\gamma^{(1)}$ to prove the statement for $\gamma$. We have that $\gamma^{(1)} - \sqrt{\kappa}W_1 \overset{\mathcal{D}}{=} \gamma$ so $\gamma^{(1)}$ is a simple curve w.p.1. Consider the unique continuous extension of $f_1^{(1)}$ to $\mathbb{R}$. This is equal to the conformal inverse of $g_t$ on $\mathbb{R}$. Since $g_t(\mathbb{R}) = R_t \cap \mathbb{R}$, $f_1^{(1)}(R_t \cap \mathbb{R}) = \mathbb{R}$. Also, $\partial R_t = \{r^-, r^+\}$ for some $r^- \in \mathbb{R}^-$ and $r^+ \in \mathbb{R}^+$, and it follows that

$$f_1^{(1)}(r^-) = 0;$$
$$f_1^{(1)}(r^+) = 0;$$
$$f_1^{(1)}(U_1) = \gamma(1).$$

Further, $r^-, r^+$ are the only pre-images of 0 under $f_1^{(1)}$. Note that $\partial H_1$ is double-valued on $\gamma((0, 1))$; precisely, the ‘right side’ of $\gamma((0, 1))$ is the image of $(U_1, r^+)$ under $f_1^{(1)}$, and the ‘left side’ of $\gamma((0, 1))$ is the image of $(r^-, U_1)$ under $f_1^{(1)}$.

The generating curve divides $\mathbb{H}$ into a connected component of $\mathbb{H} \setminus \gamma((0, \infty))$ connected to $-1$, say $H^-$, and the connected component of $\mathbb{H} \setminus \gamma((0, \infty))$ connected to 1, say $H^+$. Let $N_\epsilon(w) := \{z \in \mathbb{H} : |z - w| < \epsilon\}$. If we can show that w.p.1, there exists $\delta > 0$ such that

$$\gamma^{(1)}((0, \infty)) \cap (N_\delta(r^-) \cup N_\delta(r^+)) = \emptyset, $$  

(1.19)
then it follows that there exists $\epsilon > 0$ such that

$$N_\epsilon(0) \cap H^- \subset f_1^{(1)}(N_\delta(r^-)),$$

and

$$N_\epsilon(0) \cap H^+ \subset f_1^{(1)}(N_\delta(r^+))$$

so $\gamma((1, \infty)) \cap N_\epsilon(0) = \emptyset$, from which (1.18) immediately follows.

To prove that (1.19) holds for some $\delta > 0$, first note that scale invariance and distributional symmetry of $\gamma^{(1)}((0, \infty))$ reduces the problem to showing w.p.1 there exists $\delta > 0$ such that

$$\gamma^{(1)}((0, \infty)) \cap N_\delta(1) = \emptyset.$$  \hspace{1cm} (1.20)

For any $r > 0$, consider the stopping time $\sigma_r := \inf\{t : \gamma^{(1)}(t) \cap N_\delta(1) \neq \emptyset\}$. Note that there exists an $r > 0$ such that $\sigma_r = \infty$ if and only if (1.20) holds for some $\delta > 0$.

We choose to pursue a contradiction based on the assumption that, with positive probability, $\sigma_r < \infty$ for all $r > 0$. Since $g_{\sigma_r}$ is a conformal map with $g_{\sigma_r}(\infty) = \infty$, the harmonic measure of $[U_{\sigma_r}, g_{\sigma_r}(1)] \subset \mathbb{R} = \partial \mathbb{H}$ with respect to $\infty$ is the same as that of $f^{(\sigma_r)}_{\sigma_r}([U_{\sigma_r}, g_{\sigma_r}(1)]) \subset \partial H_{\sigma_r}$ with respect to $\infty$. The latter is composed by taking the disjoint union of $[0, 1]$ with the right side of $\gamma(0, \sigma_r]$. The harmonic measure of this set with respect to infinity is given by the probability that a complex-valued Brownian motion initiated at infinity (defined using a limit) exits $H_{\sigma_r}$ somewhere in this set. But geometric consideration show us that the Brownian motion must pass through the gap between $\gamma(\sigma_r)$ and 1 which is no larger than $|\gamma(\sigma_r) - 1| < r$. Basic properties of the harmonic measure (see [17]) then imply that for some constant $k > 0$,

$$g_{\sigma_r}(1) - U_{\sigma_r} < k|\gamma(\sigma_r) - 1| < kr.$$  \hspace{1cm} (1.21)

By assumption, this holds for arbitrarily small $r$, contradicting Theorem 2.1.1, which states that $X_t(x) := (g_t(\sqrt{k}x) - U_t)$ satisfies

$$\inf_{t \geq 0} X_t^x > 0 \text{ w.p.1.}$$

Thus, $\gamma(t) \to \infty$ as $t \to \infty$ w.p.1. \hfill $\Box$
The proof of the theorem for $\kappa = 4$ is slightly more technical because it can be shown that $\inf_t X_t^x = 0$, so the contradiction achieved at the end of the previous section requires more work. The result in the second item, for the case $\kappa > 4$, requires topological considerations combined with the Bessel stochastic equation result, $T(x) < \infty$ w.p.1. Details of each proof can be found in [17].

It follows from the theorem, for example, that the scaling limit of the percolation exploration process ($\kappa = 6$) introduced in Section 1.1.1 is self-intersecting and intersects $\mathbb{R}$ w.p.1. We can also conclude that the LERW ($\kappa = 2$) scaling limit is a simple curve w.p.1. The theorem also can be used to prove $\gamma$ is a space filling curve if $\kappa \geq 8$. We will show that the existence of self-intersections of the SLE$^\mu_\kappa$ generating curve exhibits the same dependence on $\kappa$ and is independent of $\mu$. The construction of the primary object of our study, in Section 3.2, will only use that $\gamma$ is a simple curve for $\kappa \leq 4$. 
As shown in Section 1.2.3, if one requires a family of solutions, \( \{ g_t^{(s)} \}_{t > 0} \), of (1.1) to satisfy the conditions,

- The distribution of \( \{ g_t^{(s)} - U_s \}_{t > 0} \) only depends on \( t \) and not on \( s \)
- \( \{ g_t^{(s)} \}_{t > 0} \) is independent of \( \{ g_r^{(0)} \}_{r \in [0,s]} \),

then it follows that the driving function of (1.1) must have the form

\[
U_t = \sqrt{\kappa} W_t + \mu t
\]

where \( W_t \) is standard Brownian motion, \( \mu \in \mathbb{R} \) and \( \kappa > 0 \). The case \( \mu = 0 \) is the Loewner chain \( \text{SLE}_\kappa \) described in the introduction. We use \( \text{SLE}_\kappa^\mu \) to denote the Loewner chain \( \{ g_t^{(s)} \}_{t > 0} \) in the case that \( \mu \neq 0 \). Clearly, if \( \mu > 0 \) and \( K_t \) is the \( \text{SLE}_\kappa^\mu \) Loewner chain then \( \{ x + i y : -x + i y \in K_t \} \) is distributionally equivalent to \( \text{SLE}_\kappa^{-\mu} \) so we need only consider \( \mu > 0 \) when deriving properties of \( \text{SLE}_\kappa^\mu \).

Many properties of \( \text{SLE}_\kappa^\mu \) follow from analogous properties of \( \text{SLE}_\kappa \) immediately or by mimicking specific methods that have been used to study \( \text{SLE}_\kappa \); for example, in the first section of this chapter we analyze a process similar to the Bessel-\( d \) process to derive properties of the real-valued \( \text{SLE}_\kappa^\mu \) process. In the second section it is shown that a \( \text{SLE}_\kappa^\mu \) generating curve exists w.p.1 by utilizing a classical result in stochastic analysis called Girsanov’s theorem. This theorem tells us that \( \text{SLE}_\kappa^\mu \) and \( \text{SLE}_\kappa \) are mutually absolutely continuous measures when restricted to events that depend only on the natural filtrations, \( \{ \mathcal{F}_t : t \in [0,T] \} \), of the Brownian motion in the driving function. We use Girsanov’s theorem to carry over other properties of \( \text{SLE}_\kappa \) to \( \text{SLE}_\kappa^\mu \) that will be essential for our main results in Chapter 3.1. Using properties of the real-valued \( \text{SLE}_\kappa^\mu \) process and the existence of the generating curve, we are able to
identify phases of the generating curve analogous to the phases of SLE$\kappa$ described in Corollary 1.2.1.

Finally, by taking advantage of the fact that Brownian motion with positive drift, $U_t = \sqrt{\kappa}W_t + \mu t$, has finite last hitting times $T(x) := \sup\{t : U_t = x\}$ w.p.1, we can use inequalities derived directly from the Loewner differential equation to establish strict (random) bounds that crudely characterize the generating curve, but are useful in the development of our main results.

For the bulk of this chapter we state our results in terms of the notation $\{g_t\}_{t>0} := \{g_t^{(0)}\}_{t>0}$, and it will be clear from the context whether we are referring to the real-valued Loewner chain, $g_t : \mathbb{R} \to \mathbb{R}$, or the Loewner chain on $\mathbb{H}$. Clearly $U_0 := \sqrt{\kappa}W_0 + \mu \cdot 0 = 0$, so by (1.2), $\{g_t\}_{t>0} \overset{d}{=} \{g_t^{(s)} - U_s\}_{t>0}$. Therefore, for each $s \in \mathbb{R}$ the results in this chapter also apply to the process $\{g_t^{(s)} - U_s\}_{t>0}$. We will also use the abbreviated notations $\{f_t\}_{t>0} := \{f_t^{(0)}\}_{t>0}$, $\{H_t\}_{t>0} := \{H_t^{(0)}\}_{t>0}$, $\{K_t\}_{t>0} := \{K_t^{(0)}\}_{t>0}$ and $T(z) := T^{(0)}(z)$.

### 2.1. Real-valued SLE$\mu$$_\kappa$

The intent of this section is to identify real-valued SLE$\mu$$_\kappa$ with a process closely related to the Bessel-$d$ discussed in Section 1.2.4. We will use this analysis to prove a result, analogous to Theorem 1.2.2 for SLE$\kappa$, describing ranges of $\kappa$ that correspond to the probability that the flow of real-valued SLE$\mu$$_\kappa$ exists for all positive times. In the next section, we combine the results from this section with Girsanov’s theorem to identify various qualitative characteristics, or phases, of the generating curve that depend on $\kappa$ and sgn($\mu$).

Real-valued SLE$\mu$$_\kappa$ is given by the family of solutions, $g_t(x)$, to the initial value problem

$$
\frac{\partial}{\partial t} g_t(x) = \frac{2}{g_t(x) - U_t}; \quad g_0(x) = x \in \mathbb{R} \setminus \{0\},
$$

(2.1)
where \( U_t = \sqrt{\kappa} W_t + \mu t \) for \( \kappa > 0, \mu \neq 0 \) and \( W_t \) is a standard Brownian motion. The solutions are defined up to time

\[
T(x) := \sup \{ t \geq 0 : \min_{r \in [0, t]} |g_r(x) - U_r| > 0 \},
\]

and we use \( R_t := \{ x : t \leq T(x) \} \) to denote the real-valued Loewner chain corresponding to \( U_t \).

**Theorem 2.1.1.** Take \( T(x) \) as in (2.2) and \( \mu > 0 \).

- If \( \kappa \leq 4 \), then w.p.1 \( T(x) = \infty \) for all \( x \neq 0 \).
- If \( \kappa \leq 4 \) and \( x > 0 \) then \( \inf_{t > 0} (g_t(x) - U_t) = 0 \).
- If \( \kappa > 4 \) and \( x > 0 \), then w.p.1 \( T(x) < \infty \).
- If \( \kappa > 4 \) and \( x < 0 \), then \( q(x) := P[T(x) < \infty] \in (0, 1) \).

**Proof.** Take \( g_t(x) \) and \( T(x) \) as in (2.1) and (2.2), respectively, and let \( \mu > 0 \). Define \( X_t^x := (g_t(\sqrt{\kappa} x) - U_t)/\sqrt{\kappa} \). With this notation, (2.1) can be written as

\[
dX_t^x = \left( \frac{a}{X_t^x} - \eta \right) dt + dW_t, \quad X_0^x = x \neq 0, \quad (2.3)
\]

where \( a = 2/\kappa \) and \( \eta = \mu/\sqrt{\kappa} \). Here, we also used the distributional equivalence \( W_{-t} \overset{D}{=} W_t \) to change the sign of \( dW_t \) for notational convenience.

First consider initial values \( X_0^x = x > 0 \). Suppose \( 0 < x_1 < x < x_2 < \infty \). Let \( \sigma := \inf \{ t : X_t^x \in \{x_1, x_2\} \} \) and \( \phi(x) := P[X_\sigma^x = x_2] \). It follows that

\[
E[\phi(X_\sigma^x)|\mathcal{F}_t] := \begin{cases} 
\phi(X_t^x) & \text{if } t < \sigma \\
\phi(X_\sigma^x) & \text{if } t \geq \sigma \end{cases} = \phi(X_{t \wedge \sigma}^x).
\]

Next, if \( s < t \), it then follows from the previous equality that

\[
E[E[\phi(X_\sigma^x)|\mathcal{F}_t]|\mathcal{F}_s] = E[\phi(X_\sigma^x)|\mathcal{F}_s] = \phi(X_{s \wedge \sigma}^x);
\]
therefore, the process \( \phi(X_{t \wedge \sigma}^x) \) is a martingale. Also, if we assume for now that \( \phi \in C^2 \) (this will be justified later) and apply Ito's formula, we get

\[
d\phi(X_{t \wedge \sigma}^x) = \phi'(X_{t \wedge \sigma}^x)dW_t + \left( \phi'(X_{t \wedge \sigma}^x) \left( \frac{a}{X_{t \wedge \sigma}^x} - \eta \right) + \frac{1}{2} \phi''(X_{t \wedge \sigma}^x) \right) dt,
\]

so, because \( \phi(X_{t \wedge \sigma}^x) \) is a martingale, we must have that \( \phi \) satisfies the boundary value problem given by

\[
\phi'(x) \left( \frac{a}{x} - \eta \right) + \frac{1}{2} \phi''(x) = 0; \quad \phi(x_1) = 0, \quad \phi(x_2) = 1. \tag{2.4}
\]

Let \( \psi(x) = \phi'(x) \). We can use separation of variables to get

\[
\psi(x) = ke^{2\eta x}x^{-2a},
\]

for some constant \( k > 0 \); therefore, for some constant \( c \in \mathbb{R} \),

\[
\phi(x) = k \int_{x_1}^x e^{2\eta \xi} \xi^{-2a} d\xi + c.
\]

The boundary conditions in (2.4) give

\[
k = \left( \int_{x_1}^{x_2} e^{2\eta \xi}\xi^{-2a} d\xi \right)^{-1}, \quad c = 0;
\]

that is,

\[
\phi(x) = \tilde{\phi}(x) := \frac{1}{k} \int_{x_1}^x e^{2\eta \xi}\xi^{-2a} d\xi. \tag{2.5}
\]

Here, \( \phi \) is still defined by \( \phi(x) := \mathbb{P}[X_{t \wedge \sigma}^x = x_2] \), but \( \tilde{\phi} \) is defined by the right hand side of (2.5), which is the unique solution of (2.4). We make this distinction so that we
can justify our presumption that \( \phi \) is a \( C^2 \) function, as follows. Ito's formula gives us

\[
\tilde{\phi}(X_{t\wedge \sigma}^x) = x + \int_0^t \tilde{\phi}'(X_{s\wedge \sigma}^x) dX_{s\wedge \sigma}^x + \frac{1}{2} \int_0^t \tilde{\phi}''(X_{s\wedge \sigma}^x) ds
\]

\[
= k \int_0^t \exp (2 \eta X_{s\wedge \sigma}^x) (X_{s\wedge \sigma}^x)^{-2a} dX_{s\wedge \sigma} + k \int_0^t \frac{(2 \eta)}{2} \exp (2 \eta X_{s\wedge \sigma}^x) (X_{s\wedge \sigma}^x)^{-2a} \exp (2 \eta X_{s\wedge \sigma}^x) (X_{s\wedge \sigma}^x)^{-2a-1} ds
\]

\[
= k \int_0^t \exp (2 \eta X_{s\wedge \sigma}^x) (X_{s\wedge \sigma}^x)^{-2a} \left( \frac{a}{X_{s\wedge \sigma}^x} - \eta \right) ds + k \int_0^t \exp (2 \eta X_{s\wedge \sigma}^x) (X_{s\wedge \sigma}^x)^{-2a} \left( \frac{a}{X_{s\wedge \sigma}^x} - \eta \right) ds
\]

This allows us to conclude \( \tilde{\phi}(X_{t\wedge \sigma}^x) \) is a bounded martingale. Therefore, we can apply the optional sampling theorem to get

\[
\phi(x) := P[X_{t\wedge \sigma}^x = x] = E[\tilde{\phi}(X_{\infty\wedge \sigma}^x)|\mathcal{F}_0] = \tilde{\phi}(x).
\]

That is, without assuming \( \phi \in C^2 \), we can show that \( \phi \) is identical to \( \tilde{\phi} \), so \( \phi \in C^2 \) since \( \tilde{\phi} \in C^2 \).

Now, \( \kappa \leq 4 \) implies \( a \geq 1/2 \). In this case we have that for each \( x_2 > 0 \), \( \phi(x) = \phi(x; x_1, x_2) \) satisfies

\[
\lim_{x_1 \to 0} \phi(x; x_1, x_2) = \lim_{x_1 \to 0} \int_{x_1}^x e^{2 \eta \xi \xi^{-2a} d\xi} = 1,
\]

so the first item in the theorem is satisfied for initial conditions \( x > 0 \). If we instead fix \( x_1 \in (0, x) \) and let \( x_2 \) vary, it follows that

\[
\lim_{x_2 \to \infty} \phi(x; x_1, x_2) = \lim_{x_2 \to \infty} \int_{x_1}^x e^{2 \eta \xi \xi^{-2a} d\xi} = 0,
\]

which implies the second item in the theorem.

In the case \( \kappa > 4 \), the integral in the numerator and denominator of (2.5) converge as \( x_1 \to 0 \), so we can conclude that for each \( x_2 > x \),

\[
q(x, x_2) := \lim_{x_1 \to 0} \phi(x; x_1, x_2) \in (0, 1).
\]
Note that $1 - q(x, x_2)$ is a lower bound on the probability that $T(x) < \infty$, so this event holds with positive probability. Further, $q(x, x_2) \to 0$ as $x_2 \to \infty$, and the fourth item in the theorem follows.

Now consider initial values $X_0^x = x < 0$ and let $x_2 > x > x_1 > 0$. Let $y := -x$, $y_2 = -x_2, y_1 = -x_1$ and $Y^y_t = -X^x_t$, so $0 < y_1 < y < y_2 < \infty$. In this case,

$$dY^y_t = \left( \frac{a}{Y^y_t} + \eta \right) dt + dW_t, \quad Y^y_0 = y > 0.$$  

As before, let $\sigma := \inf \{ t : Y^y_t \in \{ y_1, y_2 \} \}$ and $\phi(y) := P[Y^y_{\sigma} = y_2]$, so that

$$E[\phi(Y^y_{\sigma}) | F_t] := \begin{cases} \phi(Y^y_t) & \text{if } t < \sigma \\ \phi(Y^y_{\sigma}) & \text{if } t \geq \sigma = \phi(Y^y_{t \wedge \sigma}). \end{cases}$$  

Proceeding exactly as in the previous case, we can conclude $\phi(Y^y_{t \wedge \sigma})$ is a bounded martingale, and $\phi$ is a $C^2$ function that satisfies the boundary value problem given by

$$\phi'(y) \left( \frac{a}{y} + \eta \right) + \frac{1}{2} \phi''(y) = 0 \ ; \ \phi(y_1) = 0, \ \phi(y_2) = 1. \quad (2.6)$$  

The unique solution of this boundary value problem is

$$\phi(y) = \phi(y; y_1, y_2) = \int_{y_1}^{y_2} e^{-2n \xi} \xi^{-2a} d\xi.$$

(2.7)

First consider $\kappa \leq 4$, or, equivalently, $a \geq 1/2$. If we fix $y_2 > 0$, then

$$\lim_{y_1 \to 0} \phi(y; y_1, y_2) = 1,$$

and the first item of the theorem follows for $x < 0$.

If $\kappa > 4$, then $\phi(y; y_1, y_2)$, then for each $y_2 > y$

$$q(y, y_2) := \lim_{y_1 \to 0} \phi(y; y_1, y_2) \in (0, 1).$$

As in the case $x > 0$, $1 - q(y, y_2)$ is an lower bound on the probability that $T(-x) < \infty$, so this event holds with positive probability. However, in contrast to the case $x > 0$,

$$\lim_{y_2 \to \infty} q(y, y_2) > 0$$

for each $y > 0$. This proves the last item of the theorem and concludes the proof. \qed
Corollary 2.1.1. Take $T(x)$ as in (2.2) and $\mu > 0$. If $\kappa > 4$, then $q(x) := P[T(x) < \infty]$ is the cumulative density function for the random variable given by $\inf\{x: T(x) < \infty\}$.

Proof. Let the antecedent hold. The previous theorem gives us $q(x) = 1$ for $x \geq 0$ and $q(x) \in (0, 1)$ for $x < 0$. Suppose $x < x_0 < 0$. The processes $g_t(x), g_t(x_0)$ and $U_t$ are continuous processes, so $T(x) \geq T(x_0)$ always holds. Thus $q(x) := P[T(x) < \infty] = P[\forall x_0 \geq x, T(x_0) < \infty]$. This is enough.

2.2. Existence of the Generating Curve, Phases and Properties

The intent of this thesis is to understand the behavior of the SLE$^{\mu}_\kappa$ generating curve, but proving the existence of the SLE$^{\mu}_\kappa$ generating curve directly from the Loewner differential equation that defines the process is no easy task. Fortunately, a classical result of stochastic analysis, the Girsanov theorem, can be used to show that the existence of the SLE$^\kappa$ generating curve [26] implies the existence of the SLE$^{\mu}_\kappa$ generating curve.

More generally, if $\{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration associated with the Brownian motion $\{W_t\}_{t \geq 0}$, Girsanov’s theorem implies that any event in the filtration $\mathcal{F}_T$, $T < \infty$, that occurs w.p.1 with respect to the probability measure, $P$, associated with SLE$^\kappa$ occurs w.p.1 with respect to the probability measure $\tilde{P}$ associated with SLE$^{\mu}_\kappa$ [11]; in other words, the measures $P$ and $\tilde{P}$ are mutually absolutely continuous with respect to events that occur in finite time.

In this section, we collect several properties of SLE$^{\mu}_\kappa$ that can be derived directly from known properties of SLE$^\kappa$ with the aid of Girsanov’s theorem. In addition to establishing the existence of the generating curve, we include results that will be essential in our description and proof of a spatial invariance property of the asymptotic limit of the SLE$^{\mu}_\kappa$ generating curve.
2.2.1. Girsanov’s theorem, existence of the generating curve

Use $W_t = \{W_t, \mathcal{F}_t\}_{t \geq 0}$ to denote a standard, 1-dimensional Brownian motion with filtration $\mathcal{F}_t$ and probability measure $\mathbb{P}$. Define

$$V_t := \exp \left( \nu W_t - \frac{1}{2} \nu^2 t \right). \quad (2.8)$$

Ito’s rule gives us that for the function $f(x) := \exp(x)$ and semimartingale process $X_t := \nu W_t - \frac{1}{2} \nu^2 t$,

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dW_s + \int_0^t f'(X_s) d(-\frac{1}{2} \nu^2 s) + \frac{1}{2} \int_0^t f''(X_t) d\langle W \rangle_s$$

$$= f(X_0) + \int_0^t \exp(X_s) dW_s - \frac{1}{2} \nu^2 \int_0^t \exp(X_s) ds + \frac{1}{2} \int_0^t \exp(X_t) \nu^2 ds$$

$$= 1 + \int_0^t \exp(X_s) dW_s;$$

that is, $V_t$ satisfies the stochastic differential equation,

$$dV_t = V_t X_t dW_t; \quad V_0 = 1.$$ 

Therefore, $V_t$ is a martingale, and for each $T \geq 0$ we can define the family of probability measures,

$$\tilde{\mathbb{P}}_T[A] = \mathbb{E}[1_A V_T] \quad A \in \mathcal{F}_T, \quad (2.9)$$

which satisfy the consistency condition $\tilde{\mathbb{P}}_T[A] = \tilde{\mathbb{P}}_t[A]$ for $t \in [0, T]$ and $A \in \mathcal{F}_t$.

We hardly need Girsanov’s theorem in its full generality and will not state it here (refer to [11]); rather, we review the implications of the theorem in the context of the process $V_t$, which represents the Girsanov transformation of the constant function $t \mapsto \nu$. In this context, Girsanov’s theorem states that the process $\tilde{W}_t = W_t - \nu t, t \geq 0$, is a standard Brownian motion with respect to probability measures $\{\tilde{\mathbb{P}}_t\}_{t \geq 0}$ and filtration $\mathcal{F}_t$. Equivalently, $W_t = \tilde{W}_t + \nu t$ is a Brownian motion with linear drift $\nu$ with respect to the measures $\{\tilde{\mathbb{P}}_t\}_{t \geq 0}$. In particular, the measures $\tilde{\mathbb{P}}_t$ and $\mathbb{P}_t$ are mutually absolutely continuous, so for $A \in \mathcal{F}_t$ and $T \geq t$, $\mathbb{P}_T(A) = 1$ if and only if $\tilde{\mathbb{P}}_T(A) = 1$. 


For an example of an application of Girsanov’s theorem in the context of SLE$_{\kappa}^{\mu}$, consider the event

$$A_t := \left\{ \bar{K}_s \mid s \in [0,t] \text{ is generated by a simple curve} \right\},$$

where $\bar{K}_s$ is the Loewner chain corresponding to the $\bar{P}_t$-measurable Brownian motion, $\bar{W}_t$. As shown in Section 1.2.4, the event $A_t$ occurs $\bar{P}_t$ almost surely. Girsanov’s theorem allows to conclude that $\bar{W}_t = W_t + \mu t$ is a Brownian motion with linear drift under $P_t$, and the measures $P_t$ and $\bar{P}_t$ are mutually absolutely continuous. Therefore the family of compact $\mathbb{H}$-hulls, $K_t$, associated with the $P_t$-measurable $\bar{W}_t = W_t + \mu t$ are also generated by a simple curve w.p.1. This is a difficult result to prove if one does not use Girsanov’s theorem to extend the known result for SLE$_1$.

This application of Girsanov’s theorem is certainly not limited to the case $\kappa = 1$, hence the following corollary.

**Corollary 2.2.1.** Let $\mu \in \mathbb{R}$. W.p.1 for each $t > 0$, the SLE$_{\kappa}^{\mu}$ Loewner chain, $K_t$, is generated by a curve $\gamma : [0,t] \to \mathbb{H}$. For $\kappa \leq 4$ and $t > 0$, $\gamma[0,t]$ is a simple curve with $\gamma(0,t] \subset \mathbb{H}$.

**Proof.** As shown in [16], we have that for each $t > 0$, the SLE$_{\kappa}$ Loewner chain, $\{K_s\}_{s \in [0,t]}$ is generated by a curve $\{\gamma(s)\}_{s \in [0,t]}$ w.p.1, and the generating curve is simple for $\kappa \leq 4$. In other words, the $\mathcal{F}_t$-measurable event that $\{K_s\}_{s \in [0,t]}$ is not generated by a (simple) curve for $\kappa > 0$ (and $\kappa \leq 4$) has measure 0. Girsanov’s theorem implies that for $\nu := \frac{\mu}{\sqrt{\kappa}}$ and $V_t$ defined as in (2.8), the process $\bar{W}_t = W_t + \nu t$ is a Brownian motion with respect to the measures $\bar{P}_t$ defined by (2.9). Thus, we have that $\sqrt{\kappa}\bar{W}_t$ is a Brownian motion with variance $\kappa$ with respect to $\bar{P}_t$. We also have that $\sqrt{\kappa}W_t$ is Brownian motion with variance $\kappa$ with respect to $P_t$ and $\sqrt{\kappa}\bar{W}_t = \sqrt{\kappa}W_t + \mu t$. For finite $t$, we have that $P_t$ and $\bar{P}_t$ are mutually absolutely continuous and the events described in the statement of the theorem occur $\bar{P}_t$ almost surely so the corollary follows. \qed
The Girsanov transformation, $V_t$, is also commonly used to extend known probability densities associated with Brownian motion to more general processes; the sum of a Brownian motion (with any variance) and a deterministic drift component is a straightforward example of one such generalization. The case of a linear drift term is among the most elementary to analyze, and different types of estimates can be derived. Some of these will be useful in describing the SLE$_\kappa^\mu$ generating curve.

For notational brevity, we now omit the $t$ subscripts from the measures $P$ and $\tilde{P}$. Take $V_t$ as in (2.8) with $\nu = \mu$ and $\tilde{P} = \tilde{P}_t$ as in (2.9). As mentioned above, $W_t$ can be viewed as a Brownian motion with linear drift $\mu t$ with respect to $\tilde{P}$. Brownian motion is known to have passage times, $T_b := \inf\{t \geq 0 : W_t = b\}$, with density

$$P[T_b \in dt] = \frac{|b|}{\sqrt{2\pi t^3}} \exp(-\frac{b^2}{2t})dt. \quad (2.10)$$

The optional sampling theorem for bounded stopping times of a submartingale can be applied to yield the equation

$$\tilde{P}[T_b \in dt] = \frac{|b|}{\sqrt{2\pi t^3}} \exp(-\frac{(b - \mu t)^2}{2t})dt. \quad (2.12)$$

Also, taking the limit of (2.11) as $t \to \infty$ and the Laplace transformation given by

$$E[\exp(-\mu^2/2T_b)] = \exp(-|b|\sqrt{2\mu^2}),$$

we get

$$\tilde{P}[T_b < \infty] = \exp(\mu b - |\mu b|). \quad (2.13)$$

These observations yield short proofs of basic results on passage times and random bounds for Brownian motion with drift. Many of these basic properties will be necessary when we prove our main results, so we collect some of these results in the
following lemma. These results can be proved by other means but we choose to view them as consequences of Girsanov’s transformation. We omit the details of the proofs of equations (2.10) through (2.13) and the following lemma; the reader should refer to [11] for a thorough treatment.

**Lemma 2.2.1.** Suppose \( U_t = \sqrt{\kappa}W_t + \mu t \) where \( W_t \) is a standard Brownian motion and \( \kappa, \mu > 0 \). Then

\[
P[T_b < \infty] = \begin{cases} 
\exp(2\mu b/\sqrt{\kappa}) & b < 0 \\
1 & b \geq 0 
\end{cases}
\]

Further,

\[
P[-\inf_{t>0} U_t \in db] = (2\mu/\sqrt{\kappa}) \exp(-2\mu b/\sqrt{\kappa})db, \quad b > 0
\]

So, w.p.1 there exists \( C < 0 \) such that \( U_t > C \) for all \( t > 0 \).

2.2.2. Phases

Before introducing the phases of \( \text{SLE}^\mu_\kappa \), one should note the limitations to which properties of \( \text{SLE}_\kappa \) can be carried over to \( \text{SLE}^\mu_\kappa \) with Girsanov’s theorem. In particular, the theorem does not apply to events dependent upon the entire history of the \( \text{SLE}_\kappa \) process. For example, we cannot conclude that the \( \text{SLE}^\mu_\kappa \) generating curve diverges to \( \infty \) w.p.1 even though this is a known property of the \( \text{SLE}_\kappa \) generating curve (1.2.1).

Corollary 2.2.1, above, indicates that the generating curves for \( \text{SLE}_\kappa \) and \( \text{SLE}^\mu_\kappa \) have some common properties. On the other hand, some of the items in Theorem 2.1.1 indicate a contrast between the real-valued \( \text{SLE}_\kappa \) process and the real-valued \( \text{SLE}^\mu_\kappa \) process with respect to certain events that require knowledge of the entire history of the process. Our next corollary combines these results to describe phases of the generating curve of the \( \text{SLE}^\mu_\kappa \) process analogous, but not equivalent, to the phases of the \( \text{SLE}_\kappa \) process that were identified in Section 1.2.4.
Note that in the following proof we use $\Re \gamma(t) \to \infty$ as $t \to \infty$ for $\kappa \in (0, 4]$. We will not prove this divergence until Chapter 3, Theorem 3.1.1, but the proof of Theorem 3.1.1 will not depend on the corollary we prove now.

**Corollary 2.2.2.** Let $\gamma$ be the generating curve corresponding to $\text{SLE}_k^\mu$. Then

- If $0 < \kappa \leq 4$, then $\gamma(t)$ is a simple curve with $\gamma(0, \infty) \subset \mathbb{H}$ and
  \[
  \lim_{t \to \infty} |\gamma(t)| = \infty \text{ w.p.1}. 
  \]

- If $\kappa > 4$, then $\gamma(t)$ is a self-intersecting curve and w.p.1
  \[
  [0, \infty) \subset \bigcup_{t>0} K_t, \text{ but } \bigcup_{t>0} \overline{K}_t \neq \mathbb{H}. 
  \]

**Proof.** First let $\gamma$ be the generating curve corresponding to $\text{SLE}_k^\mu$ with $\kappa \in (0, 4]$, $\mu > 0$. Corollary 2.2.1 gives us that $\gamma(0, t]$ is a simple curve and is contained in $\mathbb{H}$ w.p.1 for all $t > 0$. Thus $\gamma(0, \infty) \subset \mathbb{H}$. Using the fact that $\Re \gamma(t) \to \infty$ as $t \to \infty$ (Theorem 3.1.1), $|\gamma(t)| \to \infty$ as $t \to \infty$ for $\kappa \in (0, 4]$.

Now assume $\kappa > 4$. Self-intersections of $\gamma$ can in fact be obtained by inheriting the same property of $\text{SLE}_k^\mu$ via Girsanov’s theorem. However, the proof of this result is omitted from Subsection 1.2.4, so we argue directly in the interest of self-containment. For each $s > 0$, the bi-infinite Brownian motion $W_t^{(s)} := W_{s+t} - W_s$ has the property that the Brownian motions $\{W_{t+}^{(s)}\}_{t>0} := \{W_{s+t} - W_s\}_{t>0}$ and $\{W_{t-}^{(s)}\}_{t>0} := \{W_{s-t} - W_s\}_{t>0}$ are independent. Therefore $\{U_{s+t} - U_s\}_{t>0} = \{W_{t+}^{(s)} + \mu t\}_{t>0}$ and $\{U_{s-t} - U_s\}_{t>0} = \{W_{t-}^{(s)} - \mu t\}_{t>0}$ are independent Brownian motions with drifts $\mu t$ and $-\mu t$, respectively. In turn, the reverse and forward Loewner chains, $\{f_t^{(s)}\}_{t>0}$ and $\{g_t^{(s)}\}_{t>0}$, are independent processes. Using Corollary 2.1.1 and $\{g_t\}_{t>0} \overset{D}{=} \{g_t^{(s)} - U_s\}_{t>0}$, one can conclude that there is a nonzero probability that for some $t > 0$ and $x \in [U_s - 1, U_s]$, $\gamma^{(s)}(t) = x$. This probability does not depend on $s$, and since $U_s \to \infty$ as $s \to \infty$ w.p.1, it is easily shown there exists $s > 0$ such that $\gamma^{(s)}(0, \infty) \cap [0, U_s] \neq \emptyset$. Choose
such an $s$, and suppose $t_0$ is such that $x = \gamma^{(s)}(t_0) \in \gamma^{(s)}(0, \infty) \cap [0, U_s]$. Take $T$
defined by
\[
T := \inf \{ t \geq 0 : f_t^{(s)}(x) - U_{s-t} = 0 \}.
\]
Clearly,
\[
\frac{\partial}{\partial t} f_t^{(s)}(x) = \frac{-2}{f_t(x) - U_{s-t}} > 0
\]
for all $t \in [0, T]$, so $T < \infty$ w.p.1; in fact, since $U_{s-t} > x$ holds for $t \in [0, T]$, we must have $T < s$. This shows that $\gamma$
has a self-intersection since,
\[
\gamma(s + t_0) = f_s^{(s)}(\gamma^{(s)}(t_0)) = f_{s-t_0}^{(s-T)}(U_{s-T}) \in \partial K_{s-T} \subset \gamma(0, s - T).
\]
So we conclude $\gamma$ is not a simple curve w.p.1 for $\kappa > 4$.

By the third item of Theorem 2.1.1 and the fact that $T(x) < \infty$ implies $x \in \overline{K_t}$
immediately yield,
\[
[0, \infty) \subset \bigcup_{t>0} \overline{K_t} \text{ w.p.1.}
\]

Using Lemma 2.2.1, we get that there exists a random $x < 0$ such that $U_t > x$ for all $t > 0$. It follows that if $\Re z < x$ then,
\[
\frac{\partial}{\partial t} g_t(z) = \frac{2(\Re g_t(z) - U_t)}{(\Re g_t(z) - U_t)^2 + (\Im g_t(z))^2} < 0
\]
for all $t > 0$. Consequently, $U_t - g_t(z) > U_t - x > 0$ for all $t > 0$. In summary,
$T(z) = \infty$ for all $z$ such that $\Re z < x$, so
\[
\bigcup_{t>0} \overline{K_t} \subset \{ z \in \mathbb{H} : \Re z > x \} \neq \mathbb{H}.
\]
This completes the proof.

\[\square\]\n
2.2.3. Continuity of $\text{SLE}_\kappa^\mu$ left-crossing probability

One example of a useful computational formula associated with $\text{SLE}_\kappa$ is Schramm’s
left crossing probability, $p^\ast(z)$, of a point $z \in \mathbb{H}$ with respect to the $\text{SLE}_\kappa$ generating
curves. This function on $H$ is defined for $\kappa \in (0, 8)$, and its definition is given in terms of winding numbers. For $\kappa \in (0, 4]$, an equivalent, more easily stated definition is given by,

$$P[\gamma \text{ crosses left of } z] = p^*(z) = P[z \in H^+_\infty]$$  \hspace{1cm} (2.14)

where $H^+_\infty$ is defined to be the connected component of $\bar{H} \setminus \gamma[0, \infty)$ that contains $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$. If $\kappa \in (0, 4]$, then $\gamma$ is simple and $\gamma(t) \to \infty$ w.p.1, so $\bar{H} \setminus \gamma[0, \infty)$ has exactly two simply connected components; thus, $p^*$ is well-defined.

It is important to point out that any definition of $p^*$ depends on the conjecture that $\gamma(t)$ diverges to $\infty$ as $t \to \infty$. As noted above, this is a known property of the SLE$_\kappa$ generating curve that cannot be carried over to SLE$_\mu^\kappa$ via Girsanov’s theorem, but Corollary 2.2.2 states that the result carries over to SLE$_\mu^\kappa$ for $\kappa \in (0, 4]$ (although Corollary 2.2.2 depends on Theorem 3.1.1, results in Section 3.1 will not depend on the existence of the left crossing probability, so our logic will not be circular.)

Before we proceed to Schramm’s definition of the SLE$_\kappa$ left crossing probability for all $\kappa \in (0, 8)$, it should also be pointed out that we would like to have,

$$P[\gamma \text{ crosses left of } z] = 1 - P[\gamma \text{ crosses right of } z].$$  \hspace{1cm} (2.15)

Here, the event ‘$\gamma$ crosses right of $z$’ is described by ‘$\gamma$ does not cross left of $z$ and $z \notin \gamma(0, \infty)$;’ therefore, (2.15) holds if and only if

$$P[z \in \gamma(0, \infty)] = 0.$$  \hspace{1cm} (2.16)

This property of the SLE$_\kappa$ generating curve is proven for all $\kappa \in (0, 8)$ in [26], and it can be extended to the SLE$_\mu^\kappa$ generating curve via Girsanov’s theorem, as follows.

**Theorem 2.2.1.** If $\gamma$ is the SLE$_\mu^\kappa$ generating curve for $\mu \in \mathbb{R}$ and $0 < \kappa < 8$, then for all $z \in \bar{H} \setminus \{0\},$

$$P[z \in \gamma(0, \infty)] = 0.$$


**Proof.** Let the antecedent hold. For \( \mu = 0 \) and \( \kappa \in (0, 8) \), the result is known [26], and it follows that for all \( t > 0 \), \( P[z \in \gamma(0, t)] \leq P[z \in \gamma(0, \infty)] = 0 \). For finite \( t > 0 \), Girsanov’s theorem can be applied to get that for all \( \kappa \in (0, 8) \), \( \mu > 0 \) and \( t > 0 \), the generating curve satisfies \( P[z \in \gamma(0, t)] = 0 \). With \( \Re \gamma(t, \infty) := \{ \Re \gamma(s) : s \in (t, \infty) \} \), we then have that

\[
P[z \in \gamma(0, \infty)] \leq P[z \in \gamma(0, t)] + P[z \in \gamma(t, \infty)] \\
\leq P[z \in \gamma(0, t)] + (1 - P[\Re \gamma(t, \infty) \subset (\Re z, \infty)]) \\
= 1 - P[\Re \gamma(t, \infty) \subset (\Re z, \infty)].
\]

Since we also have that \( \Re \gamma(t) \to \infty \) as \( t \to \infty \) w.p.1, taking the limit of the previous inequality as \( t \to \infty \) yields the desired result.

Schramm defines the SLE\(_\kappa\) left crossing probability, \( p^* \), for \( \kappa \in (0, 8) \) as follows. Fix \( z \in \mathbb{H} \). For each \( t > 0 \), let \( \sigma_t(s) \) be the closed path given by

\[
\sigma_t(s) := \begin{cases} 
\gamma(t) & 0 \leq s \leq t \\
|\gamma(t)| \exp\{i(t + 1 - s) \arg(\gamma(t))\} & t < s \leq t + 1 \\
|\gamma(t)|(t + 2 - s) & t + 1 < s \leq t + 2
\end{cases}
\]

Let \( W(z; t) \) be the winding number of the closed path \( \sigma_t : [0, t + 2] \to \mathbb{H} \) with respect to the point \( z \) (for a definition of the winding number, see [14]). We have that \( \gamma \) does not cross itself, \( |\gamma(t)| \to \infty \) and (2.16) holds. Therefore,

\[
W(z) := \lim_{t \to \infty} W(z; t) \in \{0, 1\} \text{ w.p.1.}
\]

We say \( \gamma \) passes left of \( z \) if \( W(z) = 1 \); otherwise, if \( W(z) = 0 \), we say \( \gamma \) passes right of \( z \). Of course, the value of \( W(z) \) is random, and we define the left crossing probability at \( z \) by

\[
p^*(z) := P[\gamma \text{ passes left of } z].
\]

In the case of SLE\(_\kappa\), one can define a crossing probability at \( z \) with respect to the SLE\(_\kappa\) exactly as Schramm defined \( p^* \) provided that the SLE\(_\kappa\) generating curve diverges.
to $\infty$ as $t \to \infty$. As already noted above, this result is established for $\kappa \in (0, 4]$ in Chapter 3, and we denote the SLE$_{\kappa}^\mu$ left crossing probability by $p : \mathbb{H} \to [0, 1]$.

Schramm derived an explicit formula for $p^*$ by following a strategy similar to our proof of Theorem 2.1.1 in that he applies Ito’s formula to a known martingale and sets the drift component to 0 to derive a deterministic Fokker-Plank equation for a two-point hitting density. In the proof of Theorem 2.1.1, the two-point hitting density is related to the probability of the event $T(x) < \infty$; Schramm’s two-point hitting density is related to the left crossing probability, $p^*(z)$. The partial differential equation attained for $p^*(z) = p^*(x + iy)$ is

$$
\left( \frac{2x}{|z|^2} \right) \frac{\partial p^*}{\partial x} - \frac{2y}{|z|^2} \frac{\partial p^*}{\partial y} + \frac{\kappa}{2} \frac{\partial^2 p^*}{\partial x^2} = 0,
$$

with boundary conditions $p^*(z) \uparrow 1$ as $\arg z \downarrow 0$ and $p^*(z) \downarrow 0$ as $\arg z \uparrow \pi$. The scale invariance of SLE$_\kappa$ suggests a substitution $w = x/y$ that allows one to reduce (2.17) to an equation in one variable that can be solved explicitly in terms of hypergeometric functions [27].

A formal derivation of a partial differential equation for the SLE$_\kappa^\mu$ crossing probability $p(z) = p(x + iy)$ that parallels Schramm’s derivation yields

$$
\left( \frac{2x}{|z|^2} - \mu \right) \frac{\partial p}{\partial x} - \frac{2y}{|z|^2} \frac{\partial p}{\partial y} + \frac{\kappa}{2} \frac{\partial^2 p}{\partial x^2} = 0.
$$

(2.18)

There is no trick that allows one to reduce this to an ordinary differential equation in one variable, as was done by Schramm for SLE$_\kappa$, and it does not appear that an explicit solution to (2.18) can be attained. In fact the derivation of (2.18) is not even technically justified because, as in the proof of Theorem 2.1.1, the only way to prove that $p$ is in fact a $C^2$ function is to explicitly solve (2.18) and argue this property of $p$ a posteriori by using Ito’s formula. Therefore, (2.18) is not really useful for us, so we omit the details of its formal derivation.

Our main result will require that $p$ is at least a continuous function, and a proof of this statement is the primary intent of this subsection. Consistent with the theme
of the section, this is established by applying Girsanov’s theorem to carry over known results for $\text{SLE}_\kappa$, as follows.

**Theorem 2.2.2.** The $\text{SLE}_\kappa$ left crossing probability, $p$, is continuous on $\mathbb{H}$.

**Proof.** First note that, for $z, w \in \mathbb{H}$ with $\delta := \frac{|z - w|}{2}$,

$$|p(z) - p(w)| = |\mathbb{P}[\gamma \text{ crosses left of } z] - \mathbb{P}[\gamma \text{ crosses left of } w]|$$

$$\leq \mathbb{P}[\gamma \text{ crosses left of } z \text{ and crosses right of } w]$$

$$+ \mathbb{P}[\gamma \text{ crosses left of } w \text{ and crosses right of } z]$$

$$< \mathbb{P} \left[ \gamma(0, \infty) \cap N_\delta \left( \frac{z + w}{2} \right) \neq \emptyset \right]$$

Therefore, it is enough to show that for all $z \in \mathbb{H}$,

$$\mathbb{P}[\gamma(0, \infty) \cap N_\delta(z) \neq \emptyset] \to 0 \text{ as } \delta \to 0. \quad (2.19)$$

For this proof only, let $\gamma^*$ denote the $\text{SLE}_\kappa$ generating curve. It is shown in [4] that

$$\mathbb{P}[\gamma^*(0, \infty) \cap N_\delta(z) \neq \emptyset] \to 0 \text{ as } \delta \to 0.$$

Clearly, this implies that for finite $t$,

$$\mathbb{P}[\gamma^*(0, t) \cap N_\delta(z) \neq \emptyset] \to 0 \text{ as } \delta \to 0. \quad (2.20)$$

Note that for any $t > 0$ we have

$$\mathbb{P}[\gamma(0, \infty) \cap N_\delta(z) \neq \emptyset] \leq \mathbb{P}[\gamma(0, t) \cap N_\delta(z) \neq \emptyset] + \mathbb{P}[\gamma(t, \infty) \cap N_\delta(z) \neq \emptyset]$$

$$\leq \mathbb{P}[\gamma(0, t) \cap N_\delta(z) \neq \emptyset] + \mathbb{P}[\gamma(t, \infty) \cap \{w : |w| < |z| + 1 \} \neq \emptyset].$$

Let $\epsilon > 0$. We can use Theorem 3.1.1 to choose $T > 0$ large enough that

$$\mathbb{P}[\gamma(T, \infty) \cap \{w : |w| < |z| + 1 \} \neq \emptyset] < \frac{\epsilon}{2}$$

and, given such a $T$, the Girsanov transformation and (2.20) allow us to choose $\delta < 1$ small enough that,

$$\mathbb{P}[\gamma(0, T) \cap N_\delta(z) \neq \emptyset] < \frac{\epsilon}{2},$$
That is, for all $\epsilon > 0$ there exists a $\delta > 0$ so that
\[
P[\gamma(0, \infty) \cap N_\delta(z) \neq \emptyset] \leq P[\gamma(0, T) \cap N_\delta(z) \neq \emptyset] + P[\gamma(T, \infty) \cap \{w : |w| > |z|\} \neq \emptyset]
\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Thus (2.19) has been established, and this completes the proof.

2.3. SLE$^\mu_\kappa$ Corridor

By considering the geometry of the Loewner equation vector field and using visual intuition to understand the non-autonomous flow of the system when the driving function is taken to be $U_t = \sqrt{\kappa} W_t + \mu t$, for $\kappa, \mu > 0$, one can predict and easily derive certain random bounds on the support of the generating curve, $\gamma$. Specifically, we can show that for fixed $x \in \mathbb{R}$, $p(x + iy) \to 1$ as $y \to \infty$.

The result below should be viewed as a crude bound that restricts the region in which one expects the support of $\gamma(0, \infty) \subset \overline{\mathbb{H}}$ to reside. It will be helpful for streamlining the proofs of our main results in the sequel, and provides intuition for visualizing the generating curve.

**Theorem 2.3.1.** Let $\gamma$ be the SLE$^\mu_\kappa$ generating curve for any $\kappa, \mu > 0$. For all $r > 0$, w.p.1 there exist $x < 0$ and $y > 0$ such that
\[
\gamma(0, \infty) \subset \mathcal{Y}^r_{x,y} := ([x, r] \times [0, y]) \cup ([r, \infty) \times [0, \infty)).
\]

**Proof.** Let $\kappa, \mu > 0$ and $r > 0$. For $z \in \mathbb{H}$ we have that $T(z) < \infty$ only if $g_t(z) - U_t = 0$ for some $t < \infty$. If $T_z = \infty$, then $z \notin \gamma(0, \infty)$. Thus, our strategy will be to prove that w.p.1 there exist $x < 0$ and $y > 0$ such that for all $z \in \mathbb{H} \setminus \mathcal{Y}^r_{x,y}$, $g_t(z) - U_t$ is bounded away from 0 for all $t > 0$. 
By Lemma 2.2.1, w.p.1 there exists \( x < 0 \) such that \( x < \sqrt{\kappa W_t + \mu t} \) for all \( t > 0 \).

Now, for all \( z \in \{ w \in \mathbb{H} : \Re w \leq x \} \),
\[
\frac{\partial}{\partial t} \Re g_t(z) = \frac{2(\Re g_t(z) - U_t)}{|g_t(z) - U_t|^2} > 0
\]
holds only if \( \Re g_t(z) > U_t \). But \( \Re g_0(z) = \Re z < U_0 \), so \( \frac{\partial}{\partial t} \Re g_s(z) \big|_{s=0} \leq 0 \). It follows that for each \( t > 0 \), \( \Re g_t(z) \leq U_t \) and \( \frac{\partial}{\partial s} \Re g_s(z) \big|_{s=t} \leq 0 \). Thus, for all \( z \in \{ w \in \mathbb{H} : \Re w \leq x \} \), \( g_t(z) - U_t \) is bounded away from 0 for all \( t > 0 \).

Next, again by Lemma 2.2.1, w.p.1 there exist a last hitting time \( T > 0 \) satisfying \( U_T = r + 1 \) and
\[
\inf_{t \geq T} U_t(z) = r + 1.
\]

Let \( z \in \{ w \in \mathbb{H} : x \leq \Re w \leq r \} \). For \( t \in [0, T(z)] \), \( \Im g_t(z) \) satisfies the differential inequality given by
\[
0 > \frac{\partial}{\partial t} \Im g_t(z) = \frac{-2\Im g_t(z)}{|g_t(z) - U_t|^2} > \frac{-2}{\Im g_t(z)} =: F(\Im g_t(z)). \tag{2.21}
\]

Let \( y = \Im z \) and let \( M_t(y) \) denote the solution of the initial value problem given by
\[
\frac{\partial}{\partial t} M_t(y) = -\frac{2}{M_t(y)} \quad ; \quad M_0(y) = y,
\]
then, provided \( y^2 > 4T \), it follows that \( M_T = \sqrt{y^2 - 4T} > 0 \). Then, using (2.21) along with the fact that \( F(\xi) \) increases as \( \xi \) increases for \( \xi > 0 \), we may conclude that for \( \Im z > 2\sqrt{T} \),
\[
\Im g_T(z) > M_T(\Im z) = \sqrt{(\Im z)^2 - 4T}.
\]

In turn, this allows for the crude bound,
\[
\frac{\partial}{\partial t} |g_t(z)| = \frac{2}{|g_t(z) - U_t|} < \frac{2}{\Im g_t(z)} < \frac{2}{\sqrt{(\Im z)^2 - 4T}} \tag{2.22}
\]

Now set \( y = 2\sqrt{(T)^2 + T} \) and let \( z \in T^r_{x,y} \). If \( \Re z < x \), we already have that
$T(z) = \infty$ from above; otherwise, $\Im z \geq y$ and it follows from (2.22) that

$$
\Re g_T(z) < \Re z + |\Re g_T(z) - \Re z|
$$

$$
< \Re z + |g_T(z) - z|
$$

$$
< \Re z + \frac{2T}{\sqrt{(\Im z)^2 - 4T}}
$$

$$
< \Re z + \frac{2T}{\sqrt{y^2 - 4T}}
$$

$$
< \Re z + 1
$$

Further, by definition of $T$, $U_t > \Re z + 1$ for all $t > T$, so we can repeat our argument used for $\Re z < x$, above, to conclude that $T(z) = \infty$ for all $z \in \Upsilon^r_{x,y}$. This completes the proof. \qed
3. A STATIONARY PROCESS IN THE LIMIT

In the previous chapter we defined $\text{SLE}_\kappa^\mu$ and used corollaries of established results on $\text{SLE}_\kappa$ to describe similarities of the two processes. We also used lower bounds on Brownian motion with positive linear drift to describe confidence regions for the long term behavior of the generating curve. These estimates suggest that the generating curve exhibits asymptotic stability in a distributional sense.

Our main contribution is the characterization of a process, $\hat{\gamma} : (-\infty, \infty) \to \mathbb{H}$, intended to represent the asymptotic behavior of the $\text{SLE}_\kappa^\mu$ generating curve for $\kappa \in (0, 4]$ and $\mu > 0$. The $\hat{\gamma}$ path is almost surely continuous with $\Re \hat{\gamma}(t) \to \pm\infty$ as $t \to \pm\infty$, respectively. Time increments of the process, say $\hat{\gamma}(s + \Delta s) - \hat{\gamma}(s)$, are related to limits of the $\text{SLE}_\kappa^\mu$ increments, $\gamma(t + \Delta s) - \gamma(t)$, as $t \to \infty$, and, for any fixed $s \in \mathbb{R}$, the distribution of the random variable $\Im \hat{\gamma}(s)$ are related to the limit of $\Im \gamma(t)$ as $t \to \infty$.

The construction of $\hat{\gamma}$ utilizes infinite limits of the reverse Loewner flow, and the inclusion of a logarithmic correction term is an essential part of the construction. Specifically, the logarithmic correction allows for finiteness of $\hat{\gamma}$; that is, for all $t \in \mathbb{R}$, we find that, $|\hat{\gamma}(t)| < \infty$ w.p.1. In summary, the $\hat{\gamma}$ path is acquired by taking logarithmic corrections of infinite limits of the reverse Loewner flow associated with $\text{SLE}_\kappa^\mu$ and then utilizing differential inequalities derived directly from the Loewner equation to prove that it is nondegenerate.

In the final subsection, we define a crossing probability for points $z \in \mathbb{H}$ with respect to $\hat{\gamma}$ analogous to the left crossing probabilities defined in Section 2.2.3 for $\text{SLE}_\kappa$ and $\text{SLE}_\kappa^\mu$. Recall that a formula for crossing probabilities of the $\text{SLE}_\kappa^\mu$ process could not be attained in Section 2.2.3 because methods used for $\text{SLE}_\kappa$ to derive the crossing probability by explicitly solving a Fokker-Planck type equation could not be applied to $\text{SLE}_\kappa^\mu$; however, we were able to establish continuity of the left crossing
probability $p : \mathbb{H} \to [0, 1]$ with Theorem 2.2.2, an essential result we build upon in the latter part of this section. Using the $\hat{\gamma}$ process, we prove that for fixed $y > 0$, $p(x + iy)$ converges as $x \to \pm \infty$. This is a consequence of our main result: the image of $(-\infty, \infty)$ under $\hat{\gamma}$ is distributionally invariant with respect to spatial shifts in the real component of $\mathbb{H}$.

3.1. Existence of a Limit with Stationary Increments with Respect to Time

Our basic strategy is to utilize infinite limits of the reverse Loewner flow to identify stationarity properties for increments of and imaginary components of the SLE$^\mu$ generating curve, $\gamma(t)$, in the limit as $t \to \infty$. We briefly outline the mechanisms that make this work before proceeding to our first lemma.

For $\mu > 0$, $\kappa > \in (0, 4]$ and $t > 0$, the random point $\gamma(t)$ on the SLE$^\mu_\kappa$ generating curve is in $\mathbb{H}$ w.p.1. This allows us to use the reverse Loewner flow to analyze $\gamma(t)$ as $t \to \infty$ by utilizing the relationship $\gamma(t) = f_r^{(r)}(\gamma(t - r))$. Let $W_t$ be a bi-infinite Brownian motion. For $r \in \mathbb{R}$, we have from Section 1.2.4 that $W_{t(r)} := W_{r+t} - W_r$ is another bi-infinite Brownian motion; by algebra, it follows that if $U_t := \sqrt{\kappa}W_t + \mu t$ for $\kappa, \mu > 0$, then we also have that $U_{r-t} - U_r \overset{D}{=} U_{-t}$. This allows one to write

$$\gamma(r) = f_r^{(r)}(U_r) \overset{D}{=} f_r^{(0)}(\gamma(0)) - U_{-r} = f_r(0) - U_{-r},$$

$$\gamma(r + t) = f_r^{(r)}(\gamma(t)) \overset{D}{=} f_r^{(0)}(\gamma(t)) - U_{-r} = f_r(\gamma(t)) - U_{-r}. $$

Therefore,

$$\lim_{r \to \infty} \Im \gamma(r) \overset{D}{=} \lim_{r \to \infty} \Im f_r(0),$$

$$\lim_{r \to \infty} [\gamma(r + t) - \gamma(r)] \overset{D}{=} \lim_{r \to \infty} [f_r(\gamma(t)) - f_r(0)]$$

provided the limits in each equation exist. All four of these limits can be shown to exist in a distributional sense, but pointwise functional limits exist w.p.1 only for the right hand sides of (3.1) and (3.2). The functions in the limits on the left hand side
of these equations have multiple accumulation points w.p.1, but they converge in the sense that the real indexed sequences
\[ \{\Im \gamma(r)\}_{r>0} \text{ and } \{\gamma(r+t) - \gamma(r)\}_{r>0} \]
converge in law as \( r \to \infty \). It will not be clear that these random variables converge in law until we establish the existence of the limits in the right hand sides of (3.1) and (3.2) and prove that these equations hold.

In Theorem 3.1.1, below, we show that \( \Re \gamma(t) \to \infty \). Also, the difference \( U_t - \gamma(t) \to \infty \) w.p.1. It turns out that the latter divergence is problematic when using the reverse Loewner flow to get a handle on the asymptotic behavior of \( \gamma \); however, we prove that the addition of a logarithmic correction term in the limit allows us to control this growth (Theorem 3.1.3). Precisely, we are able to show that
\[
\lim_{r \to \infty} [U_r - \gamma(r) - \frac{2}{\mu} \log r] \overset{D}{=} \lim_{r \to \infty} [0 - f_r(0) - \frac{2}{\mu} \log r] = \lim_{r \to \infty} [-f_r(0) - \frac{2}{\mu} \log r] < \infty \text{ w.p.1}
\]
In words, the difference \( U_t - \gamma(t) \) converges to \( \frac{2}{\mu} \log r + C \) for some (random) constant \( C \) w.p.1. This description of the asymptotic behavior of the difference \( U_t - \gamma(t) \) constitutes a pivotal step in the proof of invariance of \( \hat{\gamma} \) with respect to horizontal translations.

As shown above, \( U_{s+t} - U_s \overset{D}{=} U_t \), and it immediately follows that
\[
\{g_t^{(s)} - U_s\}_{t \geq 0} \overset{D}{=} \{g_t\}_{t \geq 0} \text{ and } \{f_t^{(s)} - U_s\}_{t \geq 0} \overset{D}{=} \{f_t\}_{t \geq 0}
\]
for all \( s \in \mathbb{R} \). These distributional equivalences combined with the inverse relationship, \( f_t^{(s+t)} = [g_t^{(s)}]^{-1} \), allow us to relate the reverse Loewner flow to the forward evolution of the generating curve and provide an essential tool for the proofs of the following lemmas.

Our first lemma tells us that for all \( s \in \mathbb{R} \) and \( z \in \mathbb{H} \), \( \Re f_t^{(s)}(z) - U_{s-t} \to \infty \) as \( t \to \infty \). By establishing that \( f_t^{(s)}(z) \) grows distant of \( U_{s-t} \), we will be able to show that derivatives given in the Loewner equation grow small.
Lemma 3.1.1. Let $A \subset \mathbb{H}$ be compact, $s \in \mathbb{R}$ and $f_t^{(s)}$ be the reverse flow (initiated at time $s$) of SLE$_{\kappa}^{\mu}$ for any $\mu, \kappa > 0$. For all $l > 0$ there exists a time $T > 0$ w.p.1 such that for all $z \in A$, and $t > T$, $\Re f_t^{(s)}(z) - U_{s-t} > l$.

Proof. Let the antecedent hold. By (3.3), we may take $s = 0$ without loss of generality.

First consider a single point $z \in A$. We first prove that $|f_t(z) - U_{-t}|$ is unbounded over $t \in [0, \infty)$. This is done by arriving at a contradiction from the presumption that there exists a bound, $L > 0$, for which, for all $t > 0$, $|\Re f_t(z) - U_{-t}| < L$ and $\Im f_t(z) < L$. If such a bound exists then we have that

$$\frac{\partial}{\partial t} \Im f_t(z) = \frac{2 \Im f_t(z)}{(\Re f_t(z) - U_{-t})^2 + (\Im f_t(z))^2} > \min \left\{ \frac{2 \Im z}{L^2 + (\Im z)^2}, \frac{2}{L} \right\} > 0,$$

for all $t > 0$. This implies $\Im f_t(z) \to \infty$, a contradiction.

Hence one or both of $|\Re f_t(z) - U_{-t}|$ and $\Im f_t(z)$ must be unbounded over $t \in [0, \infty)$. Suppose that only $\Im f_t(z)$ is unbounded and $|\Re f_t(z) - U_{-t}|$ is bounded. Since $\Im f_t(z)$ is strictly increasing, $\Im f_t(z) \uparrow \infty$, and for all $L > 0$ there exists a $t_L$ such that for all $s > t_L$, $\Im f_s(z) > L$. If we choose $L > 4/\mu$, then we also get that, for all $s > t_L$,

$$\left| \frac{\partial}{\partial r} \Re f_r(z) \right|_{r=s} < \frac{\partial}{\partial r} \left| f_r(z) \right|_{r=s} < \frac{2}{L} < \frac{\mu}{2}.$$ (3.4)

We will use (3.4) below, but first recall that the Brownian motion with drift $U_t = \sqrt{\kappa} W_t + \mu t$ satisfies $U_t \to \pm \infty$ as $t \to \pm \infty$ w.p.1 for $\mu > 0$. This can be proved directly, or, alternatively, can be viewed as an immediate consequence of the law of the iterated logarithm [31]. The same statement obviously holds for the Brownian motion with drift given by $U_t - (\mu/2)t$; therefore, w.p.1 there exists $t^* > t_L$ such that, for all $s < -t^*$,

$$\sqrt{\kappa} W_s < -\frac{\mu}{2} s + (\sqrt{\kappa} W_{t^*} + \mu t^* + l).$$ (3.5)
Equivalently, for all \( s < -t^* \),

\[
U_s - U_{-t^*} = (\sqrt{\kappa} W_s + \mu s) - U_{-t^*} < \left(-\frac{\mu}{2} s + U_{-t^*} + l + \mu s \right) - U_{-t^*} = \frac{\mu}{2} s + l,
\]
or,

\[
-U_s > -\frac{\mu}{2} s - l - U_{-t^*}, \tag{3.6}
\]

Since we also have that by differential inequality (3.4), for all \( s < -t^* \),

\[
\Re f_{-s}(z) - \Re f_{t^*}(z) > \frac{\mu}{2} s, \quad \text{or}
\]

\[
\Re f_{-s}(z) > \Re f_{t^*}(z) + \frac{\mu}{2} s,
\]

we get that as \( s \to -\infty \),

\[
\Re f_{-s}(z) - U_s > \Re f_{t^*}(z) + \frac{\mu}{2} s - \frac{2}{L} s - l - U_{-t^*} \to \infty,
\]
a contradiction. Thus \( |\Re f_t(z) - U_{-t}| \) is unbounded. Finally, since \( U_{-t} \to -\infty \) as \( t \to \infty \) and \( \left( \frac{\partial}{\partial s} \Re f_s(z) \right)_{s=t} > 0 \) whenever \( \Re f_t(z) - U_{-t} < 0 \), we must have that \( \Re f_t(z) - U_{-t} \to \infty \). Thus we have established the theorem for singleton sets \( A = z \).

Now suppose \( A \subset \mathbb{H} \) is an arbitrary compact set. Let \( l > \max\{1, \frac{4}{\mu}\} \). For each \( z \in A \) we can find a (random) time \( T^* = T^*(z) \) such that \( \Re f_t(z) - U_{-t} > 2l \) for all \( t > T^* \). For the same reason we had existence of a time \( t^* \) satisfying (3.5) in the first part of the proof, there will also eventually be a time \( T = T(z) > T^* \) such that for all \( t < -T \),

\[
U_t < U_{-T} + l - \frac{\mu}{2} (t + T). \tag{3.7}
\]

Following the same line of reasoning that follows (3.5) then allows us to conclude that if \( w \) is such that \( |f_T(w) - f_T(z)| < 1 < l \), then for all \( t > T \),

\[
\frac{\partial}{\partial r} f_r(w) \bigg|_{r=t} > -\frac{\mu}{2}
\]

so that

\[
f_t(w) - U_{-t} > L \text{ for all } t > T. \tag{3.8}
\]
Now, continuity of $f_T$ ensures that there exists $\delta = \delta(z) < \frac{1}{2} \min_{z \in K} \Im(z)$ such that $|w - z| < \delta$ implies $|f_T(w) - f_T(z)| < 1 < l$. Let $N_z = \{w \in \mathbb{C} : |w - z| < \delta(z)\}$. The set $\bigcup_{z \in A} N_z$ is an open cover of $A$; since $A$ is compact, we can choose a finite subcover, say $N_{z_1} \cup N_{z_2} \cup \cdots \cup N_{z_k}$. Let $T^{\max} := \max_k T(z_k)$. It follows from (3.8) that if $w \in N_{z_j}$ for any $j \in \{1, \ldots, k\}$, then $f_t(w) - U_t > l$ for all $t > T^{\max}$. Since $N_z$ covers $A$, we have that for all $w \in A$, $f_t(w) - U_t > l$ for all $t > T^{\max}$, as needed. Since $A$ and $l$ were chosen arbitrarily, this completes the proof.

In the previous chapter, we could not use Girsanov’s theorem to carry over the property that the SLE$_\kappa$ generating curve diverges to infinity w.p.1 to SLE$_\mu^\kappa$ since this event depends on the entire future of the process. With the previous lemma, we can now use an alternative proof to establish this property for SLE$_\mu^\kappa$, $0 < \kappa \leq 4$.

**Theorem 3.1.1.** For $\mu > 0$ and $0 < \kappa \leq 4$, the SLE$_\mu^\kappa$ generating curve satisfies

$$\Re \gamma(t) \to \infty \text{ as } t \to \infty \text{ w.p.1.} \quad (3.9)$$

**Proof.** By Lemma 2.2.1, 1-dimensional Brownian motion with positive drift diverges to $-\infty$ as $t \to -\infty$ w.p.1. Then w.p.1 there exists $R > 0$ such that for all $t < 0$,

$$U_t < R + \frac{3\mu}{4} t. \quad (3.10)$$

Obviously, this also implies that,

$$U_t < R + \frac{\mu}{2} t. \quad (3.11)$$

Thus, for $z$ such that $\Re z > R + 8/\mu$ and $\Im z \geq 0$, we have that

$$\left| \frac{\partial}{\partial s} \Re f_s(z) \right|_{s=0} = \frac{2|\Re f_0(z)|}{\Re f_0(z)^2 + \Im f_0(z)^2} \leq \frac{2}{|\Re f_0(z)|} = \frac{2}{\Re z} < \frac{\mu}{4}$$
It follows that the continuous function \( \xi(t) := \frac{\partial}{\partial t} \Re f_t(z) \) satisfies \( \xi(0) > -\mu/4 \). We now pursue a contradiction under the assumption there exists a time \( s > 0 \) for which \( \xi(s) < -\mu/2 \). Since \( \Re f_s(z) \) is continuous, \( s_0 := \inf\{s > 0 : \xi(s) \leq -\mu/2\} \) exists and is unique. It follows that,

\[
\frac{-2}{\Re f_{s_0}(z) - U_{-s_0}} < \frac{-2(\Re f_{s_0}(z) - U_{-s_0})}{(\Re f_{s_0}(z) - U_{-s_0})^2 + (\Im f_{s_0}(z))^2} = \xi(s_0) \leq -\frac{\mu}{2};
\]

consequently, \( 4/\mu > \Re f_{s_0}(z) - U_{-s_0} \). Also, by (3.11), we have

\[
-U_{-s_0} > -R + \frac{\mu}{2} s_0;
\]

therefore,

\[
\frac{4}{\mu} > \Re f_{s_0}(z) - U_{-s_0} > \Re f_{s_0}(z) - R + \frac{\mu}{2} s_0.
\]

In other words, the differentiable function \( \phi(s) := \Re f_s(z) - 2/\mu + \mu s/2 \) satisfies \( \phi(s_0) < 4/\mu \) and \( \phi(0) > 6/\mu \) (since \( \Re f_0(z) > 8/\mu \)); hence, by the intermediate value theorem, there exists \( s \in (0, s_0) \) with

\[
\frac{\partial}{\partial s} \phi(s) < 0;
\]

that is, \( \xi(s) + \frac{\mu}{2} < 0 \) or \( \xi(s) < -\mu/2 \). This contradicts our assumption that \( s_0 = \inf\{s > 0 : \xi(s) \leq -\mu/2\} \). Thus, for all \( t > 0 \),

\[
\frac{\partial}{\partial t} \Re f_t(z) > -\frac{\mu}{2}. \tag{3.12}
\]

It immediately follows from (3.12) and (3.10) that for all \( z \in \overline{\mathbb{H}} \) with \( \Re z > R' := R + 8/\mu \),

\[
\lim_{t \to \infty} [\Re f_t(z) - U_{-t}] \to \infty. \tag{3.13}
\]

Note that \( \{\gamma(t)\}_{t>0} \) is independent of the events (3.11) and (3.12) since the former only depends on the forward Loewner flow and the latter events depend on the reverse Loewner flow. Using Corollary 2.3.1, it follows w.p.1 there exists \( x < 0 \) and \( y > 0 \) such that

\[
\gamma(0, \infty) \subset ([x, R'] \times [0, y]) \cup ([R', \infty] \times [0, \infty]), \tag{3.14}
\]
where the random bounds $x$ and $y$ are independent of the random bound $R'$. Since $0 < \kappa \leq 4$, $\gamma$ is a simple curve w.p.1 (Theorem 2.2.2); therefore, $\gamma[1, \infty) \cap ([x, R'] \times [0, y])$ is a compact subset of $\mathbb{H}$. We can then apply Lemma 3.1.1 to get that for all $M > 0$, there exists $s > 0$ such that for any $t \geq 1$ with $\gamma(t) \in ([x, R'] \times [0, y])$,

$$\inf_{r \geq s+1} (\Re \gamma(r)(t+r) - U_{-r}) > M.$$  

Similarly, we can apply (3.13) to get that for all $M > 0$, there exists $S > 0$ such that for any $t \geq 1$ with $\Re \gamma(t) > R'$,

$$\inf_{r \geq S+1} (\Re \gamma(r)(t+r) - U_{-r}) > M.$$  

Letting $t = 1$, the previous results imply that $\inf_{r \geq s} (\Re \gamma(r)(1+r) - U_{-r}) \to \infty$ at $s \to \infty$ w.p.1. Since $\{\gamma(r)(r+1) - U_{-r}\}_{r>0} \overset{D}{=} \{\gamma(r+1)\}_{r>0}$, it then follows that

$$\inf_{r \geq s} (\Re \gamma(r)) > \inf_{r \geq s} (\Re \gamma(r+1)) \to \infty \text{ as } s \to \infty \text{ w.p.1.}$$

This completes the proof.

A remark is in order here on the different ways in which the SLE$_{\kappa}$ and SLE$_{\mu}$ generating curves diverge to $\infty$. In the case of SLE$_{\kappa}$, it can be shown as a consequence of Schramm’s formula for the left crossing probability that $\Re \gamma(t)$ is a recurrent process, so $\Im \gamma(t) \to \infty$ w.p.1 immediately follows. On the other hand, for SLE$_{\mu}$, the previous result can be used to show $\Im \gamma(t) \not\to \infty$ as $t \to \infty$.

Our next lemma details the rate at which $\Re f_t(s)(z) - U_{s-t} \to \infty$ as $t \to \infty$, establishing that it is at least linear. We will need this result to establish convergence of the logarithmically-corrected reverse Loewner flow.

**Lemma 3.1.2.** Let $A \subset \mathbb{H}$ be compact, $s \in \mathbb{R}$ and $f_t^{(s)}$ be the reverse flow (initiated at time $s$) of SLE$_{\mu}^\kappa$ for any $\mu, \kappa > 0$. W.p.1 there exists $C > 0$ and $T > 0$ such that $t > T$ implies that for all $z \in A$,

$$\Re f_t(s)(z) - U_{s-t} > Ct.$$  \hspace{1cm} (3.15)
**Proof.** Let the antecedent hold. Proceeding as in the beginning of the proof of Lemma 3.1.1, since $A$ is compact, if we can show (3.15) holds for arbitrary $z \in A$ for some positive constant, $C > 0$, then we may conclude that

$$\inf_{z \in A} \{\Re f_t(z) - U_{-t}\} > C' t$$

for some positive constant $C' > 0$, as required.

Let $z \in A$. We have that \( \{f^{(s)}_t - U_s\}_{t>0} \overset{D}{=} \{f_t\}_{t>0} \) for all $s \in \mathbb{R}$, so it is enough to show that w.p.1 there exists $C > 0$ and $T > 0$ such that $t > T$ implies

$$\Re f_t(z) - U_{-t} > Ct.$$ 

By Lemma 3.1.1, for each $l > 0$, there exists a random $T_l \in (0, \infty)$ such that for all $t > T_l$,

$$\Re f_t(z) - U_{-t} > l.$$ 

(3.16)

Also, we can use the same logic used to establish (3.11) in the proof of Theorem 3.1.1 to get that for any fixed $R > 0$ and any $s \in \mathbb{R}$,

$$q(R; s) := \mathbb{P} \left[ \forall t > 0, U_{s-t} - U_s < R - \frac{\mu t}{2} \right] > 0.$$ 

The distributional equivalence, \( \{U_{s_1-t} - U_{s_1}\}_{t>0} \overset{D}{=} \{U_{s_2-t} - U_{s_2}\}_{t>0}, s_1, s_2 \in \mathbb{R} \), allows us to conclude that for distinct $s_1, s_2 > T_l$, $q(l; s_1) = q(l; s_2)$. It is easily shown that the correlation of the events in the definition of $q(l; s_1)$ and $q(l; s_2)$ decays to zero as $s_2 - s_1$ increases; therefore, w.p.1 there exists $S > T_l$ such that, for all $t > S$,

$$U_{-S-t} - U_{-S} < \frac{l}{2} - \frac{\mu t}{2}.$$ 

(3.17)

Letting $l = 16/\mu$, we have that there exists $S > T_{16/\mu}$ such that,

$$U_{-S-t} - U_{-S} < \frac{8}{\mu} - \frac{\mu t}{2},$$

and the definition of $T_{16/\mu}$ implies that for all $t > 0$,

$$\Re f_{S+t}(z) - U_{-S-t} > \frac{16}{\mu}.$$
Then, proceeding as in the section of the proof of the previous theorem that starts at (3.11) and ends at (3.12) (with $R$ in the argument now given by $8/\mu$), we get that for all $t > S$,
\[
\frac{\partial}{\partial t} \Re f_t(z) > -\frac{\mu}{2}.
\]
In fact, it is clear that the argument from the previous proof can be strengthened slightly, to get that there exists $K < \mu/2$ such that for all $t > S$,
\[
\frac{\partial}{\partial t} \Re f_t(z) > -K.
\]
Combining this with (3.1), (3.16) follows immediately for
\[
C := \mu/2 - K,
\]
\[
T := S.
\]
This completes the proof.

We now introduce a logarithmic correction term of the reverse flow of $\text{SLE}_\kappa^\mu$ that will allow us to identify the asymptotic behavior of $\gamma(t)$ conjectured in the introduction to this chapter. For notational convenience we will use the function $\log^+ : \mathbb{R} \to \{ x : x \geq 0 \}$ defined by
\[
\log^+(t) := \begin{cases} 
0 & t < 1 \\
\log t & t \geq 1
\end{cases}.
\]

**Theorem 3.1.2.** For $\mu, \kappa > 0$, $s \in \mathbb{R}$ and $z \in \mathbb{H}$, the limit,
\[
\lim_{t \to \infty} \left[ f_t^{(s)}(z) + \frac{2}{\mu} \log^+(t-s) \right],
\]
(3.18)
of reverse flow of $\text{SLE}_\kappa^\mu$ exists and is finite w.p.1. Further, if $A \subset \mathbb{H}$ is compact, then (3.18) converges uniformly over $z \in A$.

**Proof.** We have that $\{ f_t^{(s)} - U_s \}_{t>0} \overset{d}{=} \{ f_t \}_{t>0}$, and it will be implicit from our calculation that temporal shifts of $\log^+$ will not affect the existence of the finite limit (3.18), so we can take $s = 0$ without loss of generality.
For $t > 1$ and $0 < r < t - 1$, we can write
\[
 f_t(z) = \frac{2}{\mu} \log^+(t) = f_r(z) + \frac{2}{\mu} \log^+(\mu t) - \frac{2}{\mu} \log^+(\mu) \]
\[
 = f_r(z) - \frac{2}{\mu} \log^+(\mu) + \int_0^{t-r} - \frac{2}{f_r(\xi)} d\xi + \frac{2}{\mu} \left( \log(\mu r) + \int_{\mu r}^1 \frac{1}{\xi} d\xi \right) \]
\[
 = f_r(z) - \frac{2}{\mu} \log^+(\mu) + \int_0^{t-r} - \frac{1}{f_r(\xi)} d\xi + \frac{2}{\mu} \int_{\mu r}^1 \frac{1}{\xi} d\xi \]
\[
 = f_r(z) + \frac{2}{\mu} \log(r) + \frac{2}{\mu} \int_0^{t-r} \frac{1}{f_r(\xi)} d\xi + \frac{2}{\mu} \int_{\mu r}^1 \frac{1}{\xi + \mu r} d\xi. \]

Note that the logarithmic correction term was designed so that the linear term in the driving function got canceled out in the numerator of the previous integrand.

Take $A$ as in the antecedent. To prove that (3.18) holds, it is enough to show that this integral converges uniformly over $z \in A$ as $t \to \infty$, so we need to argue that
\[
 \left| \int \frac{f^{(-r)}_{\xi/\mu}(f_r(z)) - \frac{\sqrt{\mu}}{\mu} W_{r-\xi/\mu} + (\mu - 1)r}{(f^{(-r)}_{\xi/\mu}(f_r(z)) - U_{r-\xi/\mu})(\xi + r)} d\xi \right| \quad (3.19) \]
decays sufficiently fast as $\xi \to \infty$. Omitting constant terms that do not effect the asymptotic decay, we need only prove the quantity given by
\[
 \left| \frac{f^{(-r)}_{\xi/\mu}(f_r(z)) - \frac{\sqrt{\mu}}{\mu} W_{r-\xi/\mu}}{(f^{(-r)}_{\xi/\mu}(f_r(z)) - U_{r-\xi/\mu})(\xi + r)} \right| \quad (3.20) \]
decays like $\xi^{-(1+q)}$ for some $q > 0$ to ensure convergence of the integral. Further, since the $\xi + r$ term in the denominator makes a linear contribution to the decay, it is enough to show that for some $r > 0$, w.p.1 there exist constants $M, q > 0$ such that for all $z \in A$ and $\xi > 0$,
\[
 \left| \frac{f^{(-r)}_{\xi/\mu}(f_r(z)) - \frac{\sqrt{\mu}}{\mu} W_{r-\xi/\mu}}{f^{(-r)}_{\xi/\mu}(f_r(z)) - U_{r-\xi/\mu}} \right| \leq M\xi^{-q}. \quad (3.21) \]

Since $A$ is compact, Lemma 3.1.2 gives the existence of constants $r, C > 0$ such that for all $z \in A$ and $\xi > 0$,
\[
 f^{(-r)}_{\xi/\mu}(f_r(z)) - U_{r-\xi/\mu} = f_{r+\xi/\mu}(z) - U_{r-\xi/\mu} > C(r + \xi/\mu). \quad (3.22) \]
To control the growth of the numerator we first apply Lemma 3.1.2 again to get that for all \( z \in A \) and \( \xi > 0 \)

\[
\left| f^{(-r)}_{\xi/\mu}(f_r(z)) \right| \leq \left| f^{(-r)}_0(f_r(z)) \right| + \int_0^{\xi/\mu} \left| \left( \frac{\partial}{\partial t} f^{(-r)}_{t}(f_r(z)) \right)_{t=s} \right| ds
\]

\[
= \left| f^{(-r)}_0(f_r(z)) \right| + \int_0^{\xi/\mu} \left| \frac{-2}{f^{(-r)}_s(f_r(z)) - U_{-r-s}} \right| ds
\]

\[
< \left| f^{(-r)}_0(f_r(z)) \right| + \int_0^{\xi/\mu} \left| \frac{2}{C(r + s)} \right| ds
\]

\[
< \left| f^{(-r)}_0(f_r(z)) \right| + \frac{2}{C} \int_0^{\xi/\mu} \left| \frac{1}{r + s} \right| ds
\]

\[
< \left| f^{(-r)}_0(f_r(z)) \right| + \frac{2}{C} \left[ \frac{1}{r} + \int_0^{\xi/\mu} \frac{1}{s} ds \right]
\]

\[
= \text{constant} + \frac{2}{C} \log \left( \frac{\xi}{\mu} \right).
\]

We can also conclude that the Brownian motion comprising the rest of numerator has, w.p.1, constants \( C' \in \mathbb{R} \) and \( M' > 0 \) such that

\[
|W_{-r-\xi/\mu}| < C' + M' \xi^{3/4}
\]

for all \( \xi > 0 \). This may be viewed as a consequence of the law of the iterated logarithm, but can be proved more easily. Since the asymptotic growth of \( f^{(-r)}_{\xi/\mu}(f_r(z)) \) in the numerator of the left hand side of (3.22) is logarithmic, it is also dominated by \( C' + M' \xi^{3/4} \) for sufficiently large \( \xi \). Thus, for all \( z \in A \) and \( \xi > 0 \)

\[
\left| f^{(-r)}_{\xi/\mu}(f_r(z)) - \frac{\sqrt{K}}{\mu} W_{-r-\xi/\mu} \right| \leq \text{constant} + M' \xi^{3/4}.
\]

This proves that (3.21) holds for \( q \in (0,1/4) \) uniformly over \( z \in A \), and we are done.

We are now prepared to define the process, \( \hat{\gamma} \), and show that it represents the asymptotic behavior of the SLE\(_\kappa^\mu \) generating curve. Our next theorem shows that \( \hat{\gamma} \) is well-defined, finite and continuous.
Theorem 3.1.3. Let $f_t^{(s)}$ be the reverse Loewner flow of $SLE^\kappa_\mu$ for any $\mu > 0$ and $0 < \kappa \leq 4$. W.p.1, the sequence of functions given by

$$\gamma_n(t) := \gamma^{(t-n)}(n) + \frac{2}{\mu} \log^+(n-t)$$

(3.23)

converges pointwise to a finite continuous function $\hat{\gamma} : \mathbb{R} \to \mathbb{H}$ as $n \to \infty$.

Proof. Let the antecedent hold. First note that for $\delta \in [0, 2]$, we have that

$$f_n^{(r)}(\gamma^{(r)}(\delta)) + \frac{2}{\mu} \log^+(n-r) = f_n^{(r-1)}(\gamma^{(r-1)}(1+\delta)) + \frac{2}{\mu} \log^+(n-r)$$

$$= f_n^{(r-1)}(\gamma^{(r-1)}(1+\delta)) + \frac{2}{\mu} \log^+(n-(r-1)) + \frac{2}{\mu} \log^+ \frac{n-r}{n-(r-1)}.$$

Now,

$$\lim_{n \to \infty} \frac{2}{\mu} \log^+ \frac{n-r}{n-(r-1)} = 0,$$

so, if the limit exist,

$$\lim_{n \to \infty} \left[ f_n^{(r)}(\gamma^{(r)}(\delta)) + \frac{2}{\mu} \log^+(n-r) \right] = \lim_{n \to \infty} \left[ f_n^{(r-1)}(\gamma^{(r-1)}(1+\delta)) + \frac{2}{\mu} \log^+(n-(r-1)) \right].$$

For $\kappa \in (0, 4]$, the compact set $\{\gamma^{(r-1)}(1+\delta)\}_{\delta \in [0, 2]}$ is contained in $\mathbb{H}$. Thus we can apply Theorem 3.1.2 and the previous equation to get that $\{\Gamma_n : [0, 2] \to \mathbb{H}\}_{n>1}$ given by $\Gamma_n(\delta; r) = \Gamma_n(\delta) = f^{(r)}_n(\gamma(\delta)) + \frac{2}{\mu} \log^+(n-r)$ is a sequence of continuous functions that converges uniformly. It follows that $\lim_{n \to \infty} \Gamma_n(\delta)$ is a finite, continuous function of $\delta$; further,

$$\{\lim_{n \to \infty} \Gamma_n(\delta)\}_{\delta \in [0, 2]} \subset \mathbb{H}$$

since $\Im f^{(r-1)}_n(z)$ strictly increases as $n$ increases for any $z \in \mathbb{H}$.
Now take $t \in \mathbb{R}$. Writing
\[
\lim_{n \to \infty} \gamma_n(t - 1 + \delta) = \left[ f_n^{(t-1+\delta)}(U_{t-1+\delta}) + \frac{2}{\mu} \log^+(n - (t - 1 + \delta)) \right] \\
= \lim_{n \to \infty} \left[ f_n^{(t-1)}(\gamma^{(t-1)}(\delta)) + \frac{2}{\mu} \log^+(n + \delta - (t - 1)) \right] \\
+ \frac{2}{\mu} \lim_{n \to \infty} \log \frac{n - (t - 1 + \delta)}{n + \delta - (t - 1)} \\
= \lim_{n \to \infty} \left[ f_n^{(t-1)}(\gamma^{(t-1)}(\delta)) + \frac{2}{\mu} \log^+(n - t - 1) \right] \\
= \lim_{n \to \infty} \Gamma_n(\delta; t - 1),
\]
reveals that \( \hat{\gamma}(t - 1 + \delta) = \lim_{n \to \infty} \gamma_n(t - 1 + \delta) \in \mathbb{H} \) for \( \delta \in [0, 2] \) and is continuous at \( \delta = 1 \). Since \( t \) was taken arbitrarily, \( \gamma(t) \) is continuous for all \( t \in \mathbb{R} \), as needed. \( \square \)

The imaginary component and temporal increments of the process \( \hat{\gamma} : \mathbb{R} \to \mathbb{H} \) exhibit stationarity properties with respect to time that are stated precisely in the following theorem. These are properties expected in the long behavior of the SLE\( _\kappa \) generating curve, \( \gamma \), but are more easily proved directly from the construction of \( \hat{\gamma} \), as follows. In the following theorem, we use the notation \( \mathcal{B}(\mathbb{R}^n) \) to denote the Borel sets in \( \mathbb{R}^n \).

**Theorem 3.1.4.** Let \( \hat{\gamma} : \mathbb{R} \to \mathbb{H} \) be defined as in Theorem 3.1.3 and \( s \in \mathbb{R} \).

If \( (s_1, \ldots, s_n) \in \mathbb{R}^n \) with \( s_1 < \cdots < s_n \) and \( B_j \in \mathcal{B}(\mathbb{R}) \), \( j = 1, \ldots, n \), then
\[
\mathbb{P}[\mathfrak{I}\hat{\gamma}(s_1) \in B_1, \ldots, \mathfrak{I}\hat{\gamma}(s_n) \in B_n] = \mathbb{P}[\mathfrak{I}\hat{\gamma}(s_1 + s) \in B_1, \ldots, \mathfrak{I}\hat{\gamma}(s_n + s) \in B_n].
\]

If \( (s_1, \ldots, s_n) \in \mathbb{R}^n \), \( (\Delta_1, \ldots, \Delta_n) \in \mathbb{R}^n \) and \( B_j \in \mathcal{B}(\mathbb{R}^2) \), \( j = 1, \ldots, n \), then
\[
\mathbb{P}[\hat{\gamma}(s_1 + \Delta_1) - \hat{\gamma}(s_1) \in B_1, \ldots, \hat{\gamma}(s_n + \Delta_n) - \hat{\gamma}(s_n) \in B_n] = \\
\mathbb{P}[\hat{\gamma}(s_1 + s + \Delta_1) - \hat{\gamma}(s_1 + s) \in B_1, \ldots, \hat{\gamma}(s_n + s + \Delta_n) - \hat{\gamma}(s_n + s) \in B_n].
\]

**Proof.** By definition, for \( j = 1, \ldots, n \) and \( z \in \mathbb{H} \), \( f^{(s_j)}(z) \) satisfies the Loewner equation,
\[
\frac{\partial}{\partial t} f^{(s_j)}(z) = \frac{-2}{f^{(s_j)}(z) - U_{s_j - t}}; \quad f^{(s_j)}_0(z) = z, \quad (3.24)
\]
and \( f_t^{(s_j+s)}(z) \) satisfies the Loewner equation,
\[
\frac{\partial}{\partial t} f_t^{(s_j+s)}(z) = \frac{-2}{f_t^{(s_j+s)}(z) - U_{s_j+s-t}}.; \quad f_0^{(s_j+s)}(z) = z, \tag{3.25}
\]
and we have that \( f_t^{(s_j)} : \mathbb{H} \to H_t^{(s_j)} \) and \( f_t^{(s_j+s)} : \mathbb{H} \to H_t^{(s_j+s)} \) have unique extensions to \( \partial \mathbb{H} = \mathbb{R} \). Since \( (U_t - U_{s_n})_{t \leq s_n} \) is equivalent in distribution to \( (U_t - U_{s_n+s})_{t \leq s_n+s} \), it follows that the processes \( (f_t^{(s_n)} - U_{s_n}) \big|_{t > 0} \) and \( (f_t^{(s_n+s)} - U_{s_n+s}) \big|_{t > 0} \) are also equivalent in distribution as functions of \( z \in \mathbb{H} \). We then have the distributional equivalence of \( n \)-tuples,
\[
(f_t^{(s_1)}(U_{s_1}), \ldots, f_t^{(s_n)}(U_{s_n})) \overset{D}{=} (f_t^{(s_1+s)}(U_{s_1+s}), \ldots, f_t^{(s_n+s)}(U_{s_n+s}) - U_{s_n+s}).
\]
Note that the previous equality holds since the joint distribution of the two \( n \)-tuples depend entirely on \( (U_t - U_{s_n})_{t \leq s_n} \) and \( (U_t - U_{s_n+s})_{t \leq s_n+s} \), respectively.

In particular,
\[
(\Im f_t^{(s_1)}(U_{s_1}), \ldots, \Im f_t^{(s_n)}(U_{s_n})) \overset{D}{=} (\Im f_t^{(s_1+s)}(U_{s_1+s}), \ldots, \Im f_t^{(s_n+s)}(U_{s_n+s})).
\]
Therefore,
\[
(\Im \hat{\gamma}(s_1), \ldots, \Im \hat{\gamma}(s_n)) = \lim_{t \to \infty} (\Im f_t^{(s_1)}(U_{s_1}), \ldots, \Im f_t^{(s_n)}(U_{s_n})) \overset{D}{=} \lim_{t \to \infty} (\Im f_t^{(s_1+s)}(U_{s_1+s}), \ldots, \Im f_t^{(s_n+s)}(U_{s_n+s})) = (\Im \hat{\gamma}(s_1 + s), \ldots, \Im \hat{\gamma}(s_n + s)),
\]
and the first claim follows immediately.

Proof of the latter item uses a similar strategy. There is no loss of generality in supposing that \( \Delta_j > 0 \) and \( s_n + \Delta_n > s_j + \Delta_j \) for \( j = 1, \ldots, n \). Proceeding as above, we find that the processes \( (f_t^{(s_n+\Delta_n)} - U_{s_n+\Delta_n}) \big|_{t > 0} \) and \( (f_t^{(s_n+s+\Delta_n)} - U_{s_n+s+\Delta_n}) \big|_{t > 0} \) are equivalent in distribution as functions of \( z \in \mathbb{H} \). Further, the joint distribution
\[
(f_t^{(s_1+\Delta_1)}(U_{s_1+\Delta_1}) - f_t^{(s_1)}(U_{s_1}), \ldots, f_t^{(s_n+\Delta_n)}(U_{s_n+\Delta_n}) - f_t^{(s_n)}(U_{s_n})) \tag{3.26}
\]
depends entirely on \( (f_t^{(s_n+\Delta_n)} - U_{s_n+\Delta_n}) \big|_{t > 0} \) and the joint distribution
\[
(f_t^{(s_1+s+\Delta_1)}(U_{s_1+s+\Delta_1}) - f_t^{(s_1+s)}(U_{s_1+s}), \ldots, f_t^{(s_n+s+\Delta_n)}(U_{s_n+s+\Delta_n}) - f_t^{(s_n+s)}(U_{s_n+s})) \tag{3.27}
\]
depends entirely on \((f_t^{(s_n+s+\Delta_n)} - U_{s_n+s+\Delta_n})\) \(t>0\). By also noting that the differences in each component of (3.26) and (3.27) nullify the random horizontal translations that arise from driving function terms, we can conclude that the \(n\)-tuples in (3.26) and (3.27) are also equivalent in distribution. In order to relate these \(n\)-tuples to \(\hat{\gamma}\), note that for each \(j = 1, \cdots, n\), we have

\[
\hat{\gamma}(s_j + \Delta_j) - \hat{\gamma}(s_j) = \lim_{t \to \infty} \left[ f_t^{(s_j+\Delta_j)}(U_{s_j+\Delta_j}) + \frac{2}{\mu} \log^+(t - (s_j + \Delta_j)) \right]
- \lim_{t \to \infty} \left[ f_t^{(s_j)}(U_{s_j}) + \frac{2}{\mu} \log^+(t - s_j) \right]
= \lim_{t \to \infty} \left[ f_{t+\Delta_j}^{(s_j)}(U_{s_j+\Delta_j}) + \frac{2}{\mu} \log^+(t + \Delta_j - (s_j + \Delta_j)) \right]
- \lim_{t \to \infty} \left[ f_t^{(s_j)}(U_{s_j}) + \frac{2}{\mu} \log^+(t - s_j) \right]
= \lim_{t \to \infty} \left[ f_t^{(s_j+\Delta_j)}(U_{s_j+\Delta_j}) - f_t^{(s_j)}(U_{s_j}) \right],
\]

and a similar derivation yields

\[
\hat{\gamma}(s_j + s + \Delta_j) - \hat{\gamma}(s_j + s) = \lim_{t \to \infty} \left[ f_t^{(s_j+s+\Delta_j)}(U_{s_j+s+\Delta_j}) - f_t^{(s_j+s)}(U_{s_j+s}) \right].
\]

Equivalence in distribution of (3.26) and (3.27) then implies that

\[
(\hat{\gamma}(s_1 + \Delta_1) - \hat{\gamma}(s_1), \cdots, \hat{\gamma}(s_n + \Delta_n) - \hat{\gamma}(s_n)) \overset{D}{=} (\hat{\gamma}(s_1 + s + \Delta_1) - \hat{\gamma}(s_1 + s), \cdots, \hat{\gamma}(s_n + s + \Delta_n) - \hat{\gamma}(s_n + s)),
\]

from which the second item in the theorem follows immediately.

Now that we have defined the \(\hat{\gamma}\) process and identified the previous stationarity properties, we want to verify that this process characterizes the long term behavior of \(\gamma\) as suggested at the beginning of this chapter. The next theorem is a precise statement of the sense in which \(\hat{\gamma}\) encapsulates the asymptotic behavior of \(\gamma(t)\) as \(t \to \infty\). The theorem is stated in terms of convergence in law of probability measures, defined as follows. A (real-indexed) sequence of probability measures \(P_t\) on \(\mathbb{R}^k\) is said to converge in law to a probability measure \(P\) on \(\mathbb{R}^k\), written \(P_t \to P\), if for every
bounded, continuous, real-valued function on $\mathbb{R}^k$,
\[
\int f dP_n \to \int f dP \text{ as } n \to \infty.
\]

**Theorem 3.1.5.** Take $\hat{\gamma}: \mathbb{R} \to \mathbb{H}$ as above, and let $\gamma$ be the SLE$_κ^\mu$ generating curve with $\mu > 0$ and $0 < \kappa \leq 4$.

- For every $\Delta > 0$, there exists a probability measure $P^{(\Delta)}$ on $\mathbb{R}^2$ that gives the law of the random variable $\hat{\gamma}(s + \Delta) - \hat{\gamma}(s)$ for any $s \in \mathbb{R}$; furthermore, if $P_t^{(\Delta)}$ denotes the probability measure on $\mathbb{R}^2$ induced by the random variable $\gamma(t + \Delta) - \gamma(t)$, then $P_t^{(\Delta)} \to P^{(\Delta)}$ as $t \to \infty$.

- There exists a probability measure $Q$ on $\mathbb{R}$ that gives the law of the random variable $\Im \hat{\gamma}(s)$ for any $s \in \mathbb{R}$; furthermore, if $Q_t$ denotes the probability measure on $\mathbb{R}$ induced by the random variable $\Im \gamma(t)$, then $Q_t \to Q$ as $t \to \infty$. Moreover, for all intervals $I = (a, b] \subset \mathbb{R}$, $a < b$,
\[
\lim_{t \to \infty} P[\Im \gamma(t) \in I] = P[\Im \hat{\gamma}(s) \in I] \tag{3.28}
\]

**Proof.** We start with the first item. By Theorem 3.1.4, the law of the random variable $\hat{\gamma}(s + \Delta) - \hat{\gamma}(s)$ does not depend on $s$, so for any $s \in \mathbb{R}$ we denote the probability measure associated with $\hat{\gamma}(s + \Delta) - \hat{\gamma}(s)$ by $P^{(\Delta)}$. By definition, $\hat{\gamma}(s + \Delta) - \hat{\gamma}(s)$ is given by
\[
\hat{\gamma}(s + \Delta) - \hat{\gamma}(s) = \lim_{n \to \infty} \left[ f^{(s+\Delta)}_{n+\Delta}(U_{s+\Delta}) + \frac{2}{\mu} \log^+(n + \Delta - (s + \Delta)) - (f^{(s)}_n(U_s) + \frac{2}{\mu} \log^+(n - s)) \right]
\]
\[
= \lim_{n \to \infty} \left[ f^{(s+\Delta)}_{n+\Delta}(U_{s+\Delta}) - f^{(s)}_n(U_s) \right]
\]
\[
= \lim_{n \to \infty} \left[ f^{(s)}_n(f^{(\Delta)}_{s+\Delta}(U_{s+\Delta})) - f^{(s)}_n(U_s) \right]
\]
\[
= \lim_{n \to \infty} \left[ f^{(s)}_n(\gamma(\Delta)) - f^{(s)}_n(U_s) \right]
\]
\[
= \lim_{n \to \infty} \left[ \gamma^{(s-n)}(n + \Delta) - \gamma^{(s-n)}(n) \right].
\]
Let $P_n^{(\Delta)}$ denote the law of the random variable inside the limit. We have that the limit converges w.p.1, so for any bounded, continuous, real-valued function $f$ on $\mathbb{R}^2$, it follows that $f(\gamma^{(s-n)}(n + \Delta) - \gamma^{(s-n)}(n)) \rightarrow f(\hat{\gamma}(s + \Delta) - \hat{\gamma}(s))$ as $n \rightarrow \infty$ w.p.1. Thus,

$$\int f(\xi)dP_n^{(\Delta)}(\xi) \rightarrow \int f(\xi)dP^{(\Delta)}(\xi).$$

for all bounded continuous functions $f$; that is, $P_n^{(\Delta)} \rightarrow P^{(\Delta)}$ as $n \rightarrow \infty$.

Clearly, we can replace the natural number index $n$ in the sequence $\gamma^{(s-n)}(n + \Delta) - \gamma^{(s-n)}(n)$ with a real-valued index $t > 0$ and arrive at the same result, $P_t^{(\Delta)} \rightarrow P^{(\Delta)}$ as $t \rightarrow \infty$ w.p.1. Further,

$$\gamma^{(s-t)}(t + \Delta) - \gamma^{(s-t)}(t) \overset{D}{=} \gamma(t + \Delta) - \gamma(t),$$

so $P_t^{(\Delta)}$ is in fact the (real-valued) sequence of probability measures described in the theorem statement. This is enough for the first item.

We proceed in a similar manner to get the latter item. Again by Theorem 3.1.4, the law of the random variable $\Im \hat{\gamma}(s)$ does not depend on $s$, so for any $s \in \mathbb{R}$ we denote the probability measure associated with $\Im \hat{\gamma}(s)$ by $Q$. By definition, $\hat{\gamma}(s + \Delta) - \hat{\gamma}(s)$ is given by

$$\Im \hat{\gamma}(s) = \lim_{n \rightarrow \infty} \Im[f_n^{(s)}(U_s) + \frac{2}{\mu} \log^+(n - s)]$$

$$= \lim_{n \rightarrow \infty} \Im f_n^{(s)}(U_s)$$

$$= \lim_{n \rightarrow \infty} \Im \gamma^{(s-n)}(n).$$

Let $Q_n$ denote the law of the random variable $\Im \gamma^{(s-n)}(n)$. We have that the limit converges w.p.1, so for any bounded, continuous, real-valued function $f$ on $\mathbb{R}$, it follows that $f(\Im \gamma^{(s-n)}(n)) \rightarrow f(\hat{\gamma}(s))$ as $n \rightarrow \infty$. Thus,

$$\int f(\xi)dQ_n(\xi) \rightarrow \int f(\xi)dQ(\xi).$$

for all bounded continuous functions $f$; that is, $Q_n \rightarrow Q$ as $n \rightarrow \infty$. 

Again, we may replace the natural number index $n$ in the sequence $\Im^\gamma(s-n)(n)$ with a real-valued index $t > 0$ and arrive at the same result, $Q_t \to Q$ as $t \to \infty$. Since we also have that $\Im^\gamma(s-t)(t) \overset{P}{=} \Im^\gamma(t)$, it follows that $P_t$ is in fact the (real-valued) sequence of probability measures described in the theorem statement, as needed.

The one thing left to prove is the last statement in the second item of the theorem. We have that $\Im^\gamma(t) \uparrow \Im^{\hat{\gamma}}(s)$ as $t \to \infty$; therefore, if $\Im^{\hat{\gamma}}(s) = \lim_{t \to \infty} \Im^\gamma(t) = a_0 > a$, then there exists a $T$ such that for all $t > T$, $\Im^\gamma(t) \in (a_0, a)$. Therefore, for intervals of the form $(a, \infty)$, we can write

$$\int_{\Omega} \mathbb{1}_{\{\Im^{\hat{\gamma}}(s) \in (a, \infty)\}} dQ = \int_{\Omega} \lim_{t \to \infty} \mathbb{1}_{\{\Im^\gamma(t) \in (a, \infty)\}} dQ.$$

We also have that $f_t(s)(U_s) - U_{s-t} \overset{D}{=} f_t^{(t)}(U_t) - U_{t-t} = f_t^{(t)}(U_t)$, so

$$\Im^\gamma(s-t)(t) = \Im(f_t(s)(U_s) - U_{s-t}) \overset{D}{=} \Im f_t^{(t)}(U_t) = \Im^\gamma(t).$$

Putting these equations together and using the bounded convergence theorem, we get that for $I := (b, \infty)$ with $b \in \mathbb{R},$

$$P[\Im^{\hat{\gamma}}(s) \in I] = \int_{\Omega} \mathbb{1}_{\{\Im^{\hat{\gamma}}(s) \in I\}} dQ$$

$$= \int_{\Omega} \lim_{t \to \infty} \mathbb{1}_{\{\Im f_t^{(s)}(U_s) \in I\}} dQ$$

$$= \lim_{t \to \infty} \int_{\Omega} \mathbb{1}_{\{\Im f_t^{(s)}(U_s) \in I\}} dQ$$

$$= \lim_{t \to \infty} P[\Im f_t^{(s)}(U_s) \in I]$$

$$= \lim_{t \to \infty} P[\Im^\gamma(s-t)(t) \in I]$$

$$= \lim_{t \to \infty} P[\Im^\gamma(t) \in I].$$
We then get that for $b > a > 0$,

$$
P[\Im \hat{\gamma}(s) \in (a, b)] = P[\Im \hat{\gamma}(s) \in (0, b)] - P[\Im \hat{\gamma}(s) \in (0, a)]$$

$$= (1 - P[\Im \hat{\gamma}(s) \in (b, \infty)]) - (1 - P[\Im \hat{\gamma}(s) \in (a, \infty)])$$

$$= (1 - \lim_{t \to \infty} P[\Im \gamma(t) \in (b, \infty)]) - (1 - \lim_{t \to \infty} P[\Im \gamma(t) \in (a, \infty)])$$

$$= \lim_{t \to \infty} ((1 - P[\Im \gamma(t) \in (b, \infty)]) - (1 - P[\Im \gamma(t) \in (a, \infty)]))$$

$$= \lim_{t \to \infty} (P[\Im \gamma(t) \in (0, b)] - P[\Im \gamma(t) \in (0, a)])$$

$$= \lim_{t \to \infty} P[\Im \gamma(t) \in (a, b)],$$

as needed.

3.2. Spatial Invariance with Respect to Horizontal Translation

In Section 2.2.3, Girsanov’s theorem was used to show that for all $\kappa \in (0, 8)$, $\mu \in \mathbb{R}$ and $z \in \mathbb{H}$, the SLE$_\kappa$ generating curve $\gamma := \gamma^{(0)}$ satisfies $P[z \in \gamma(0, \infty)] = 0$ (Theorem 2.2.1), and the left crossing probability, $p(z)$, is well-defined and continuous (Theorem 2.2.2). A crossing probability can also be defined for the limiting curve $\hat{\gamma}$ defined in the previous section. We do this below and use $\hat{p}$ to denote what is more aptly named the top crossing probability in the case of $\hat{\gamma}$. Once this definition is in hand, we describe a spatial invariance property of $\hat{\gamma}$ in terms of $\hat{p}$; namely, we prove that if $z \in \mathbb{H}$ and $x \in \mathbb{R}$, then $\hat{p}(z) = \hat{p}(z + x)$. Unlike the temporal stationarity of the imaginary part and time increments of $\hat{\gamma}$, shown in Theorem 3.1.4, spatial stationarity of $\hat{\gamma}$ does not follow directly from the construction of the process.

As with the SLE$_\kappa$ process, the SLE$_\mu$ process has the potential to represent the scaling limit of certain 2-dimensional discrete processes of mathematical physics. Invariance of $\hat{\gamma}$ with respect to horizontal spatial shifts is a property one would always expect of discrete processes that deserve SLE$_\mu$ as a candidate for their scaling limit.

Our first task is to define the top crossing probability, $\hat{p}$. In Section 2.2.3, the definition of the left crossing of $z \in \mathbb{H}$ by the curve $\gamma$ was given in terms of a (random)
winding number, \( W(z) \in \{0, 1\} \), related to the SLE\( ^{\kappa} \) generating curve \( \gamma \). Roughly speaking, we want to pass this definition through the limit,

\[
z = \lim_{t \to \infty} \left[ f_t(w) + \frac{2}{\mu} \log^+(s) \right],
\]

and define \( z \) to be crossed over the top by \( \hat{\gamma} \) if and only if \( w \) is crossed to the left by \( \gamma \). In order for this definition to be consistent, we will need that the map,

\[
F_{\infty}(w) := \lim_{t \to \infty} \left[ f_t(w) + \frac{2}{\mu} \log^+(t) \right], \tag{3.29}
\]

is a bijection, a special case of the following lemma.

**Lemma 3.2.1.** Let \( \kappa \in (0, 4] \) and \( s \in \mathbb{R} \). The map \( F_{\infty}^{(s)} : \mathbb{H} \to \mathbb{H} \setminus \gamma(-\infty, s] \) given by

\[
F_{\infty}^{(s)}(w) := \lim_{t \to \infty} \left[ f_t^{(s)}(w - \frac{2}{\mu} \log^+(s)) + \frac{2}{\mu} \log^+(s + t) \right], \tag{3.30}
\]

is a bijection.

**Proof.** Let \( s > 0 \) and take \( F_{\infty}^{(s)} \) as in (3.30). First we establish injectivity of \( F_{\infty}^{(s)} \).

Suppose \( w_1, w_2 \in \mathbb{H} \) and \( w_1 \neq w_2 \). Note that

\[
F_{\infty}^{(s)}(w_2) - F_{\infty}^{(s)}(w_1) = \lim_{t \to \infty} \left[ f_t^{(s)}(w_2 - \frac{2}{\mu} \log^+(s)) - f_t^{(s)}(w_1 - \frac{2}{\mu} \log^+(s)) \right].
\]

Also note that \( f_t^{(s)} \overset{D}{=} f_t + U_s \), so

\[
f_t^{(s)}(w_2 - \frac{2}{\mu} \log^+(s)) - f_t^{(s)}(w_1 - \frac{2}{\mu} \log^+(s)) \overset{D}{=} f_t(\psi_s(w_2)) - f_t(\psi_s(w_1)),
\]

where \( \psi_s \) is the injective map given by \( z \mapsto z - \frac{2}{\mu} \log^+(s) \); therefore, w.p.1 \( F_{\infty}^{(s)} \) is an injection if and only if w.p.1 \( F_{\infty} \) is an injection. We now prove the latter statement.

For \( w_1, w_2 \) as above, we have that for all \( r > 0 \), \( f_r \) is a conformal map, so \( f_r(w_1) \neq f_r(w_2) \). Also, for \( j = 1, 2 \), we have

\[
F_{\infty}(w_j) := \lim_{t \to \infty} f_t(w_j) + \frac{2}{\mu} \log^+(t)
\]

\[
= \lim_{t \to \infty} f_t^{(-r)}(f_r(w_j)) + \frac{2}{\mu} \log^+(r + t);
\]
hence, it is enough to show that for some \( r > 0 \),

\[
\lim_{t \to \infty} \left[ f_t^{(-r)}(f_r(w_2)) - f_t^{(-r)}(f_r(w_1)) \right] \neq 0
\]

which holds if and only if either

\[
\Re f_t^{(-r)}(f_r(w_2)) - \Re f_t^{(-r)}(f_r(w_1)) \to 0 \quad \text{as} \quad t \to \infty,
\]

fails to hold, or

\[
\Im f_t^{(-r)}(f_r(w_2)) - \Im f_t^{(-r)}(f_r(w_1)) \to 0 \quad \text{as} \quad t \to \infty
\]

fails to hold.

Now, as a consequence of Theorem 3.1.2, we have that for some \( y > 0 \), for each of \( j = 1, 2 \) and all \( t > 0 \),

\[
\Im f_t^{(-r)}(f_r(w_j)) < \Im F_\infty(w_j) < y.
\]

Also, using Lemma 3.1.2, we can conclude that there exists \( r > 0 \) such that for all \( t > 0 \),

\[
\Re f_t^{(-r)}(f_r(z_j)) - U_{-r-t} > 2y \tag{3.31}
\]

for each of \( j = 1, 2 \).

At this point, our strategy is to divide the proof into different geometric cases. Let \( z_j = f_r(w_j), \ j = 1, 2 \). For each \( t > 0 \), define the set

\[
R_t^\leq := \left\{ z \in \mathbb{H} : \frac{d}{ds} \Re f_s^{(-r-t)}(z) \bigg|_{s=0} < \frac{d}{dt} \Re f_s^{(-r)}(z_2) \bigg|_{s=t} < 0 \right\}. \tag{3.32}
\]

This set is given by the inside of the semicircle,

\[
\left\{ z \in \mathbb{H} : \Re \frac{2}{z - U_t} = \frac{d}{ds} \Re f_s^{(-r)}(z_2) \bigg|_{s=t} \right\} ;
\]

explicitly, if \( C_t := \frac{d}{ds} \Re f_s^{(-r)}(z_2) \bigg|_{s=t} \), then

\[
R_t^\leq = \left\{ z \in \mathbb{H} : \left( \Re z - U_t \right)^2 + (\Im z)^2 < \left( \frac{1}{C_t} \right)^2 \right\}. \tag{3.33}
\]
Now, if $z_1$ is in $R_0^<$, then either (i) $f_t^{(-r)}(z_1) \in R_t^<$ for all $t > 0$ and we are done or (ii) the stopping time $\tau := \inf\{t > 0 : f_t^{(-r)}(z_1) \notin R_t^<\}$ is finite. In the latter case, we still have $f_t^{(-r)}(z_1) < y$ for all $t > 0$, and the definition of $\tau$ implies that

$$\frac{\partial}{\partial t} \left( \Re f_t^{(-r)}(z_2) - \Re f_t^{(-r)}(z_1) \right) > 0$$

for $t < \tau$.

Geometric considerations then reveal that at the time $\tau$ when $f_t^{(-r)}(z_1)$ leaves $R_t^<$, it must follow that

$$\Im f_{\tau}^{(-r)}(z_1) - \Im f_{\tau}^{(-r)}(z_2) > \Re f_0^{(-r)}(z_2) - \Re f_0^{(-r)}(z_1).$$

For clarity, we pause here and explain how the previous step represents a central, repetitive step of our argument for injectivity. The repeated assumption is that a real or imaginary component of $f_t^{(r)}(z_2) - f_t^{(r)}(z_1)$ is assumed to have a derivative that is the same sign as itself, hence the absolute value of this component difference is growing. In time, the difference can evolve to a position (in the vector field) where the sign of that component of $f_t^{(r)}(z_2) - f_t^{(r)}(z_1)$ is opposite the sign of its derivative, but in each case the other, imaginary or real, component of the absolute value of the difference will be larger than the absolute value of the initial difference of the first component. That is, for all $t > 0$, we claim

$$\max \left\{ \left| \Re f_t^{(r)}(z_2) - \Re f_t^{(r)}(z_1) \right|, \left| \Im f_t^{(r)}(z_2) - \Im f_t^{(r)}(z_1) \right| \right\} > 0,$$

which is enough to conclude that $F_\infty(w_1) \neq F_\infty(w_2)$.

Continuing with our explicit procedure, let $r' \in \{r, r+\tau\}$; here, the consideration of $r' = r + \tau$ represents the continuation of the case $z_1 \in R_0^<$ and the consideration of $r' = r$ represents the beginning of the case $z_1 \in I_0^>$ where

$$I_t^> := \left\{ z \in \mathbb{H} : \frac{d}{ds} \Im f_s^{(-r-t)}(z) \bigg|_{s=0} > \frac{d}{dt} \Im f_s^{(-r)}(z_2) \bigg|_{s=t} > 0 \right\}.$$
This set is given by the inside of the circle,
\[
\left\{ z : \Re \frac{2}{z-U_t} = \frac{d}{dt} \Re f_s^{(-r')}(z) \right\}_{s=t}^\infty.
\]
Explicitly, if \( D_t := \frac{d}{ds} \Re f_s^{(-r')}(z) \bigg|_{s=t} \), then
\[
I_t^- = \left\{ w : (\Re w - U_t)^2 + \left( \Re w + \frac{1}{D_t} \right)^2 < \left( \frac{1}{D_t} \right)^2 \right\}, \quad (3.35)
\]
If \( r' = r + \tau \) necessarily, \( f_t^{(-r')}(z_1) \in I_t^\tau \). Therefore, for \( r' = r \) or \( r' = r + \tau \) if (i) \( f_t^{(-r')}(z_1) \in I_t^\tau \), for all \( t > 0 \) or \( f_t^{(-r')}(z_1) \in I_{t+t}^\tau \), for all \( t > 0 \), respectively, we are done. Otherwise, (ii) the stopping time
\[
\tau' := \begin{cases} 
\inf \{ t > 0 : f_t^{(-r')}(z_1) \notin I_t^\tau \} & , r' = r \\
\tau + \inf \{ t > 0 : f_{t+t}^{(-r')}(z_1) \notin I_{t+t}^\tau \} & , r' = r + \tau
\end{cases}
\]
is finite. Using \( f_t^{(-r')}(z_1) < y \) for all \( t > 0 \) and the fact that
\[
\frac{\partial}{\partial t} \left( \Re f_t^{(-r')}(z_1) - \Re f_t^{(-r')}(z_2) \right) > 0 \text{ for } t \in \begin{cases} [0, \tau') & , r' = r \\
[\tau, \tau') & , r' = r + \tau
\end{cases},
\]
geometric considerations reveal that
\[
\Re f_{\tau'}^{(-r')}(z_1) - \Re f_{\tau'}^{(-r')}(z_2) > \begin{cases} 
\Re f_0^{(-r)}(z_1) - \Re f_0^{(-r)}(z_2) & , r' = r \\
\Re f_{\tau+t}^{(-r)}(z_1) - \Re f_{\tau+t}^{(-r)}(z_2) & , r' = r + \tau
\end{cases}.
\]
In either case, the claim (3.34) is fulfilled for \( t \leq \tau' \). For \( t \geq \tau' \), we can reuse our initial argument that follows after (3.31) with \( z_1 \) and \( z_2 \) interchanged; regardless of how many times we repeat the argument, (3.34) will hold for all \( t > 0 \), as claimed. The progression of the remaining two cases, \( z_1 \in R_0^\tau \) and \( z_1 \in I_0^\tau \), where
\[
R_t^- := \left\{ z \in \mathbb{H} : \frac{d}{ds} \Re f_s^{(-r-t)}(z) \bigg|_{s=t} > \frac{d}{dt} \Re f_s^{(-r)}(z_2) \bigg|_{s=t} > -\infty \right\},
\]
\[
I_t^- := \left\{ z \in \mathbb{H} : \frac{d}{ds} \Re f_s^{(-r-t)}(z) \bigg|_{s=t} < \frac{d}{dt} \Re f_s^{(-r)}(z_2) \bigg|_{s=t} < \infty \right\},
\]
proceed in the same manner, that is, by exchanging the roles of \( z_1 \) and \( z_2 \) and repeating the arguments above. Therefore, in all possible cases, (3.34) holds, and we may
conclude $F_\infty(w_1) \neq F_\infty(w_2)$. Since $w_1$ and $w_2$ were taken arbitrarily, this completes the proof of injectivity.

Our remaining task is surjectivity of $F_\infty^{(s)}$ onto $\mathbb{H} \setminus \hat{\gamma}(-\infty, s]$. It is enough to show that $F_\infty$ is a surjection onto $\mathbb{H} \setminus \hat{\gamma}(-\infty, 0]$.

Let $z \in \mathbb{H} \setminus \hat{\gamma}(-\infty, 0]$, and define

$$N(z) := \overline{N_{3z/2}(z)} = \left\{ w \in \mathbb{H} : |z - w| \leq \frac{3z}{2} \right\}.$$  

Proceeding as in the proof of Theorem 3.1.2, it can be shown that w.p.1 there exists $s > 0$ such that

$$\sup_{w \in N(z)} \left| F_\infty^{(-s)}(w) - w \right| = \sup_{w \in N(z)} \left| \lim_{t \to \infty} \left( f_t^{(-s)}(w - \frac{2}{\mu} \log s) + \frac{2}{\mu} \log(s + t) \right) - w \right| < \frac{3z}{4}. \quad (3.36)$$

Further, Theorem 3.1.2 implies that $\left\{ f_t^{(-s)}(w - \frac{2}{\mu} \log s) + \frac{2}{\mu} \log(s + t) \right\}_{t > 0}$ converges uniformly over $w \in N(z)$ as $t \to \infty$. This implies that the function $F_\infty^{(-s)} : N(z) \to \mathbb{H}$ is continuous, so $F_\infty^{(-s)}(N(z))$ is connected. Combining this fact with (3.36) yields $z \in F_\infty^{(-s)}(N(z))$; therefore, there exists $w \in N(z)$ such that $z = F_\infty^{(-s)}(w)$.

Note that, by construction, we have that

$$F_\infty^{(-s)} \left( \gamma^{(-s)}(0, s] + \frac{2}{\mu} \log^+(s) \right) = \hat{\gamma}(-s, 0].$$

Since we also have that $F_\infty^{(-s)}$ is injective, it follows that $z = F_\infty^{(-s)}(w) \in \hat{\gamma}(-s, 0]$ if and only if $w - \frac{2}{\mu} \log^+(s) \in \gamma^{(-s)}(0, s]$. Since we have assumed the negation of the former condition in the previous ‘if and only if’ statement, it follows $w - \frac{2}{\mu} \log^+(s) \notin \gamma^{(-s)}(0, s]$, so $w - \frac{2}{\mu} \log^+(s)$ is in the domain of $g^s_\infty$ and $w' := g^s_\infty \left( w - \frac{2}{\mu} \log^+(s) \right) \in \mathbb{H}$. Thus,

$$z = F_\infty^{(-s)}(w) = F_\infty^{(-s)}(f_s(w') - \frac{2}{\mu} \log^+(s)) = F_\infty(w')$$

for $w' \in \mathbb{H}$, and we have shown that every $z \in \mathbb{H} \setminus \hat{\gamma}(-\infty, 0]$ has a pre-image in $\mathbb{H}$ under $F_\infty$, as needed to complete the proof. \qed
We also need that $z \notin \hat{\gamma}(\mathbb{R})$ w.p.1, which follows easily from analogous properties of SLE$^\mu_\kappa$ but requires proof.

**Lemma 3.2.2.** Let $\kappa \in (0,4]$ and take $\hat{\gamma}$ as above. For all $z \in \mathbb{H}$,
\[ P[z \in \hat{\gamma}(\mathbb{R})] = 0. \]

**Proof.** Let $z \in \mathbb{H}$ and define
\[ N(z) := N_{3z/2}(z) = \left\{ w \in \mathbb{H} : |z - w| \leq \frac{3z}{2} \right\}. \]
As in the surjectivity part of the previous proof, w.p.1 we can choose $s > 0$ large enough that $z \in F^{(-s)}_{\infty}(N(z))$, and injectivity of $F^{(-s)}_{\infty}$ implies that any $w \in N(z)$ such that $F^{(-s)}_{\infty}(w) = z$ is unique. Further, the random pre-image $w$ depends only on \( \{ U_{t-s} - U_{-s} \}_{t \leq 0} \) and is independent of \( \{ U_{t-s} - U_{-s} \}_{t \geq 0} \). On the other hand, the event
\[ E := \left\{ w - \frac{2}{\mu} \log^+(s) \in \gamma^{(-s)}(0, \infty) \right\} \]
depends only on \( \{ U_{t-s} - U_{-s} \}_{t \geq 0} \), hence $E$ is independent of the value of $w$. We can then use the distributional equivalence $\gamma^{(-s)} - U_{-s} \overset{D}{=} \gamma$ and Theorem 2.2.1 to get
\[ P \left[ w - \frac{2}{\mu} \log^+(s) \in \gamma^{(-s)}(0, \infty) \right] = P \left[ w - \frac{2}{\mu} \log^+(s) - U_{-s} \in \gamma(0, \infty) \right] = 0. \]
Also, $F^{(-s)}_{\infty}$ is a bijection onto $\mathbb{H} \setminus \hat{\gamma}(\infty, -s]$, $z \in \hat{\gamma}(\infty, s]$ if and only if $w - \log^+(s) \in \gamma^{(-s)}(0, \infty)$, so $\hat{\gamma}(\infty, -s] \cap F^{(-s)}_{\infty}(N(z)) = \emptyset$ follows automatically from our assumption on $s$. Therefore,
\[ P[z \in \hat{\gamma}(\mathbb{R})] = P[z \in \hat{\gamma}(\infty, -s)] + P[z \in \hat{\gamma}(-s, \infty)] = 0 + P \left[ w - \frac{2}{\mu} \log^+(s) \in \gamma^{(-s)}(0, \infty) \right] = 0. \]
This completes the proof. \( \square \)

For $\kappa \in (0,4]$ and $\mu > 0$, we can now use the SLE$^\mu_\kappa$ crossing probability defined in Subsection 2.2.3 and the map $F_{\infty}$ to define crossing probabilities for $\hat{\gamma}$. We define
Let \( \hat{W}(z) = \begin{cases} 1 & \text{if } z \text{ is passed over the top by } \hat{\gamma} \\ 0 & \text{if } z \text{ is passed underneath by } \hat{\gamma} \end{cases} \).

We define the top crossing probability, \( \hat{p} \), of a point \( z \in \mathbb{H} \) as the function \( \hat{p} : \mathbb{H} \to [0, 1] \) that sends \( z \) to \( \hat{W}(z) = 1 \).

Our main proposal is that \( \hat{p} \) depends only on \( \Im(z) \). This statement may appear to be a direct consequence of our construction of the curve and the temporal stationarity properties stated at the end of the previous section, but some work is required to attain a rigorous proof. One should note that the convergence of \( \Im(\gamma_n(t)) \) and \( \gamma_n(t+\Delta) - \gamma_n(t) \) as \( n \to \infty \) can be accomplished by similar means to above even if the logarithmic correction term is not included in the definition of \( \gamma_n(t) \). However, the proof of spatial invariance, below, will depend on the convergence of \( \Re(\gamma_n(t)) \), a result that depends crucially on the inclusion of the logarithmic correction term in the definition of \( \gamma_n(t) \).

We first prove a technical lemma that gives uniform continuity of \( p(z) \) when the imaginary part of \( z \) is fixed.

**Lemma 3.2.3.** Let \( \kappa \in (0, 4] \) and \( \mu > 0 \). Define \( p : \mathbb{H} \to [0, 1] \) to be the left crossing probability for \( \text{SLE}_\kappa^\mu \). Let \( y > 0 \) and \( A_y := \{ z \in \mathbb{H} : \Im(z) = y, \Re(z) \geq 2y \} \). Then \( p \) is uniformly continuous when restricted to \( A_y \).

**Proof.** Let the antecedent hold. We need to show that for all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for all \( z \in A_y \) and all \( z_0 \in N_\delta(z) \), \( |p(z) - p(z_0)| < \epsilon \).

Let \( \epsilon > 0 \). First note that \( p \) is continuous on \( \mathbb{H} \) (Theorem 2.2.2), so \( p \) is uniformly continuous when restricted to the compact set \( V_y := \{ w : \Re(w) = 2y, 0 \leq \Im(w) \leq y \} \). Therefore, there exists \( \delta' > 0 \) such that for all \( z \in V_y \) and all \( z_0 \in N_{\delta'}(z) \), \( |p(z) - p(z_0)| < \epsilon \).
Take \( z \in A_y \). Let \( \tau := \inf \{ t > 0 : g_t(z) - U_t = 2y \} \). It follows that \( \tau \) is a stopping time, and \( P[\tau < \infty] = 1 \) is a direct consequence of the second item of Theorem 2.1.1. Our strategy requires detailed geometric considerations that will be used to characterize the directions of the vectors \( \frac{d}{dt} (g_t(z) - g_t(z_0)) \), so we proceed by first identifying the geometry of certain sets related to signs of the real and imaginary parts of these vectors.

We will use some of the same symbols used in the notation in the proof of Lemma 3.2.1, but our definitions of the notation here will not be the same. For each \( t > 0 \) the set

\[
R_t^\geq := \left\{ w \in \mathbb{H} : \left. \frac{d}{ds} \mathcal{R} g_s(z) \right|_{s=t} < \left. \frac{d}{dt} \mathcal{R} g_t(s)(w) \right|_{t=0} < \infty \right\}
\]

is given by the inside of the semicircle,

\[ \left\{ w \in \mathbb{H} : \frac{2}{w - U_t} = \left. \frac{d}{ds} \mathcal{R} g_s(z) \right|_{s=t} \right\}. \]

Explicitly, if \( C_t := \left. \frac{d}{ds} \mathcal{R} g_s(z) \right|_{s=t} \), then

\[
R_t^\geq = \left\{ w \in \mathbb{H} : \left( (\Re w - U_t) - \frac{1}{C_t} \right)^2 + (\Im w)^2 < \left( \frac{1}{C_t} \right)^2 \right\}. \tag{3.37}
\]

Similarly, the set

\[
R_t^\leq := \left\{ w : -\infty < \left. \frac{d}{dt} \mathcal{R} g_t(s)(w) \right|_{t=0} < \left. \frac{d}{ds} \mathcal{R} g_s(z) \right|_{s=t} \right\}
\]

is given by the outside of the aforementioned semicircle; that is,

\[
R_t^\leq = \left\{ w : \left( (\Re w - U_t) - \frac{1}{C_t} \right)^2 + (\Im w)^2 > \left( \frac{1}{C_t} \right)^2 \right\}. \tag{3.38}
\]

We pursue a similar geometric decomposition of the imaginary part of the vector field for \( g_t \). For each \( t > 0 \) the set

\[
I_t^\geq := \left\{ w \in \mathbb{H} : \left. \frac{d}{ds} \mathcal{I} g_s(z) \right|_{s=t} < \left. \frac{d}{dt} \mathcal{I} g_t(s)(w) \right|_{t=0} < 0 \right\}
\]
is given by the outside of the circle,

\[ \left\{ w : \Re \left( \frac{2}{w - U_t} \right) = \frac{d}{ds} \Im g_s(z) \right\} \big|_{s=t} \]

Explicitly, if \( D_t := \frac{d}{ds} \Im g_s(z) \big|_{s=t} \), then

\[ I_t^\geq = \left\{ w : (\Re w - U_t)^2 + (\Im w + \frac{1}{D_t})^2 > \left( \frac{1}{D_t} \right)^2 \right\} \]

Similarly, the set

\[ I_t^\leq := \left\{ w \in \mathbb{H} : -\infty < \frac{d}{dt} \Im g_t^{(s)}(w_0) \big|_{t=0} < \frac{d}{ds} \Im g_s(z) \big|_{s=t} \right\} \]

is given by the inside of the aforementioned circle; that is,

\[ I_t^\leq = \left\{ w : (\Re w - U_t)^2 + (\Im w + \frac{1}{D_t})^2 < \left( \frac{1}{D_t} \right)^2 \right\} \]

We will use these geometric characterizations of the sets \( R_t^\geq, R_t^\leq, I_t^\geq \) and \( I_t^\leq \) soon.

First note that, for any \( \delta > 0 \) and \( z_0 \in N_\delta(z) \), we have

\[ z_0 \in S_\delta(z) := \left\{ w : \Re w \in [\Re z - \delta, \Re z + \delta], \Im w \in [\Im z - \delta, \Im z + \delta] \right\} \]

We claim that, for \( \delta \) sufficiently small, \( z \in A_y \) and \( z_0 \in S_\delta(z) \) imply

\[ g_t(z_0) \in S_\delta(g_t(z)) \]

Clearly, for \( t < \tau, \Re g_t(z) - U_t > 2y \) and \( \Im g_t(z) < y \). Using the Loewner equation, we also find that,

\[ C_t := \left. \frac{d}{ds} \Re g_s(z) \right|_{s=t} = \Re \left( \frac{2}{g(t) - U_t} \right) < \frac{2}{\Re (g(t) - U_t)} < \frac{1}{2y} \]

\[ D_t := \left. \frac{d}{ds} \Im g_s(z) \right|_{s=t} = \Im \left( \frac{2}{g(t) - U_t} \right) > -\frac{2\Im g_t(z)}{(\Re (g(t) - U_t))^2} > -\frac{1}{4y} \]

Therefore,

\[ \left( \frac{1}{C_t} \right)^2 > (2y)^2 \text{ and } \left( \frac{1}{D_t} \right)^2 > (4y)^2 \]
for all $t < \tau$.

Take $\delta < \min \left\{ \frac{\eta}{4}, \frac{\delta'}{2} \right\}$; the first constraint in the minima and geometric considerations based on equations (3.37) through (3.40) reveal the following set containments:

\[
\{ w : \Re w = \Re g_t(z) - \delta, \Im g_t(z) \in [\Im g_t(z) - \delta, \Im g_t(z) + \delta] \} \subset R^\gamma_t \quad (3.42)
\]

\[
\{ w : \Re w = \Re g_t(z) + \delta, \Im g_t(z) \in [\Im g_t(z) - \delta, \Im g_t(z) + \delta] \} \subset R^\prec_t \quad (3.43)
\]

\[
\{ w : \Im w = \Im g_t(z) - \delta, \Re g_t(z) \in [\Re g_t(z) - \delta, \Re g_t(z) + \delta] \} \subset I^\gamma_t \quad (3.44)
\]

\[
\{ w : \Im w = \Im g_t(z) + \delta, \Re g_t(z) \in [\Re g_t(z) - \delta, \Re g_t(z) + \delta] \} \subset I^\prec_t. \quad (3.45)
\]

Let $z \in A_y$ and suppose $|z - z_0| < \delta$. Then, by definition of the Loewner flow, $|g_t(z) - g_0(z)| < \delta$ so $g_0(z_0) \in S_\delta(g_0(z))$. Using the continuity in time and space of the vector field over $(t, z) \in [0, \tau] \times S_\delta(g_t(z))$ given by $\frac{d}{dt} g_t$, we can than view $\Delta z_t := g_t(z_0) - g_t(z)$ as a dynamical system defined on $S_\delta(0)$ for which equations (3.42) through (3.45) represent restrictions on the behavior of the solution $\Delta z_t$ if $\Delta z_t$ nears the boundary $\partial S_\delta(0)$. In particular, these conditions imply that $\Delta z_t \in S_\delta(0)$ for all $t \in [0, \tau]$. Consequently, (3.41) holds, and, by definition of $\tau$, $g_\tau(z) - U_\tau \in V_y$. Since $S_\delta(g_\tau(z)) = S_{\delta'/2}(g_\tau(z)) \subset N_\delta(g_\tau(z))$, we also have that $g_\tau(z_0) - U_\tau \in N_\delta(g_\tau(z) - U_\tau)$.

Since $\tau$ is a stopping time, it follows from the strong Markov property that $U_{\tau+t} - U_\tau \overset{D}{=} U_t$. Then, since $U_{\tau+t} - U_\tau$ drives the processes $g_{\tau+t}(z)$ and $g_{\tau+t}(z_0)$, we have

\[
|p(z) - p(z_0)| = |P[\gamma \text{ passes left of } z] - P[\gamma \text{ passes left of } z_0]| = |P[\gamma^{(\tau)} \text{ passes left of } g_\tau(z)] - P[\gamma^{(\tau)} \text{ passes left of } g_\tau(z_0)]| = |p(\gamma \text{ passes left of } g_\tau(z) - U_\tau) - p(\gamma \text{ passes left of } g_\tau(z_0) - U_\tau)| < \epsilon,
\]

where the last inequality follows from our original assumption on $\delta'$ and the fact $g_\tau(z) - U_\tau \in V_y$. This completes the proof.

\[\square\]

**Theorem 3.2.1.** Let $\kappa \in (0, 4]$ and take $\hat{\gamma} : \mathbb{R} \to \mathbb{H}$ as in (3.23). Then the top-crossing probability of a point $z \in \mathbb{H}$ with respect to $\hat{\gamma}$ is invariant under horizontal spatial shifts; that is, if $z \in \mathbb{H}$ and $x \in \mathbb{R}$ then $\hat{p}(z) = \hat{p}(z + x)$. 

\textbf{Proof.} Let \( z \in \mathbb{H} \) and \( x > 0 \). Set \( z' = z + x \) and \( y = \Im z \). No generality is lost in requiring \( x \) to be positive. We will get the result by proving that for all \( \varepsilon > 0 \),
\[
|\hat{p}(z') - \hat{p}(z)| < \varepsilon.
\]

Let \( \varepsilon > 0 \). First note that Lemma 3.2.3 guarantees the existence of \( \delta_1 > 0 \) such that \( w \in A_y \) and \( |w - w_0| < \delta_1 \) imply that \( |p(w) - p(w_0)| < \frac{\varepsilon}{6} \).

For \( s < 0 \), and take \( F_s^{(s)} \) as defined in (3.30). We have that \( F_{\infty}^{(s)} : \mathbb{H} \to \mathbb{H} \) \( \gamma(-\infty, s) \) is a continuous bijection, so the function \( G_{\infty}^{(s)} \) defined by \( G_{\infty}^{(s)}(z) := [F_{\infty}^{(s)}]^{-1}(z) \) is a continuous bijection from \( \mathbb{H} \setminus \gamma(-\infty, s) \) to \( \mathbb{H} \). By definition, we have that \( \gamma \) passes over the top of \( z \) if and only if \( \gamma(s) \) passes left of \( G_{\infty}^{(s)}(z) - \frac{2}{\mu} \log^+(-s) \).

For each \( s < 0 \) and \( l > 0 \), define \( \tau_l(s) := \inf\{t > s : U_t + \log^+(-t) \geq \Re z - l\} \). Then \( \tau_l(s) \) is a stopping time of the process \( \{U_{t+s} - U_s\}_{t \geq 0} \). Since \( U_t + \frac{2}{\mu} \log^+(-t) \to -\infty \) as \( t \to -\infty \) w.p.1, it is also follows that for each \( l > 0 \), there exists \( S = S(l) < 0 \) such that
\[
P[\exists s < S \text{ s.t. } \tau_l(s) = s] = 1 - P[\forall s < S, U_s + \frac{2}{\mu} \log^+(-s) < l] < \frac{\varepsilon}{6} \quad \text{(3.46)}
\]

For the sake of notational brevity, set \( Z = Z(l, s; z) := G_{\infty}^{(\tau_l(s))}(z) \), and let \( P_Z \) denote the probability measure on \((\mathbb{H}, \mathcal{B}(\mathbb{H}))\) given by,
\[
P_Z(B) = P[Z \in B], \quad B \in \mathcal{B}(\mathbb{H}).
\]

Now, since \( \tau_l(s) \) is a stopping time, the strong Markov property allows us to write
\[
\hat{p}(z) = P[\gamma \text{ passes over the top of } z]
\]
\[
= P\left[\gamma^{(\tau_l(s))} \text{ passes left of } G_{\infty}^{(\tau_l(s))}(z) - \frac{2}{\mu} \log^+(-\tau_l(s))\right]
\]
\[
= \int_{\mathbb{H}} p \left((\xi - U_{\tau_l(s)}) - \frac{2}{\mu} \log^+(-\tau_l(s))\right) P_Z(d\xi)
\]
\[
= \int_{\mathbb{H}} p \left(\xi - \frac{2}{\mu} \log^+(-\tau_l(s)) - U_{\tau_l(s)}\right) P_Z(d\xi)
\]
\[
= \int_{\mathbb{H}} p \left(\xi - \frac{2}{\mu} \log^+(-\tau_l(s)) - [\Re z - l - \frac{2}{\mu} \log(-\tau_l(s))]\right) P_Z(d\xi)
\]
\[
= \int_{\mathbb{H}} p \left(l + iy - (z - \xi)\right) P_Z(d\xi) \quad \text{(3.47)}
\]
If we define $\tau'_l(s) := \inf\{t > s : U_s + \log^+(-s) = \Re z' - l\}$ and proceed in exactly the same way, we also find that,

$$\hat{p}(z') = \int_{\mathbb{H}} p(l + iy - (z' - \xi)) \, P_{Z'}(d\xi), \quad (3.48)$$

where $Z' = Z'(l, s; z) := G_{\infty}^{(\tau'_l(s))}(z')$ and $P_{Z'}$ denotes the probability measure on $(\mathbb{H}, \mathcal{B}(\mathbb{H}))$ given by,

$$P_{Z'}(B) = P[Z' \in B], \quad B \in \mathcal{B}(\mathbb{H}).$$

Using (3.47) and (3.48), we can now write

$$|\hat{p}(z') - \hat{p}(z)| = \left| \int_{\mathbb{H}} p(l + iy - (z' - \xi)) \, P_{Z'}(d\xi) - \int_{\mathbb{H}} p(l + iy - (z - \xi)) \, P_{Z}(d\xi) \right|$$

$$= \left| \int_{\mathbb{H}} p(l + iy - (z' - \xi)) \, P_{Z'}(d\xi) + p(l + iy) \int_{\mathbb{H}} P_{Z'}(d\xi) ight.$$  

$$- \left( \int_{\mathbb{H}} p(l + iy - (z - \xi)) \, P_{Z}(d\xi) - p(l + iy) \int_{\mathbb{H}} P_{Z}(d\xi) \right) \bigg|$$

$$< \int_{\mathbb{H}} |p(l + iy - (z' - \xi)) - p(l + iy)| \, P_{Z'}(d\xi)$$

$$+ \int_{\mathbb{H}} |p(l + iy - (z - \xi)) - p(l + iy)| \, P_{Z}(d\xi).$$

Our strategy is now to control the size of each of the last two integrals in the previous equation. We first consider

$$\int_{\mathbb{H}} |p(l + iy - (z - \xi)) - p(l + iy)| \, P_{Z}(d\xi). \quad (3.49)$$

We have that $\tau_l(s)$ is a (forward flow) stopping time of the process $\{U_{s+t} - U_s\}_{t>0}$. To achieve a bound on (3.49), we also use the (reverse flow) stopping-times of the process $\{U_{-t}\}_{t>0}$ given by $\sigma_t := \inf\{t > 0 : U_{-t} + \log^+(-t) = \Re z - l\}$. These stopping times satisfy $\sigma_t < \infty$ w.p.1 and it is immediate that $\tau_l(s) > s$ implies $\tau_l(s) < \sigma_l$. Also note that for a fixed element of the probability space, (w.p.1) $\sigma_l$ increases as $l$ increases. Moreover, for any fixed time $S < 0$,

$$P[\sigma_l < S] \to 1 \text{ as } l \to \infty. \quad (3.50)$$
Therefore, Theorem 3.1.2 implies that \( w.p.1 \) there exists \( l = l(z) > 0 \) such that, for all \( w \in \mathbb{H} \) with \( \Re w \geq \Re z \),

\[
P \left[ F^{(-\sigma_l)}(w) - w \right] < \delta_1 \right) > 1 - \frac{\epsilon}{6}
\]

Now if \( \tau_l(s) > s \) then \( U_{\tau_l(s) - t} < \Re z \) for all \( t > 0 \) so \( \Re G^{(\tau_l(s))}(z) \geq \Re z \) must hold; further, as noted above, \( \tau_l(s) < \sigma_l \) follows immediately from \( \tau_l(s) > s \). The previous equation then gives us that

\[
P \left[ |z - Z(l, s; z)| < \delta_1 \mid \tau_l(s) > s \right] = P \left[ F^{(\tau_l(s))} \circ G^{(\tau_l(s))} - G^{(\tau_l(s))}(z) < \delta_1 \mid \tau_l(s) > s \right]
\]

or, equivalently,

\[
P \left[ Z(l, s; z) \notin N_{\delta_1}(z) \mid \tau_l(s) > s \right] < \frac{\epsilon}{6}.
\]

Fix \( l \) large enough that the previous inequality holds and \( l + iy \in A_y \). Letting \( P_{Z|\tau > s} \) denote the conditional probability measure on \( (\mathbb{H}, \mathcal{B}(\mathbb{H})) \) given by,

\[
P_{Z|\tau > s}(B) = P[Z \in B \mid \tau_l(s) > s], \quad B \in \mathcal{B}(\mathbb{H}),
\]

it follows that,

\[
\int_{C} \left| p(l + iy - (z - \xi)) - p(l + iy) \right| P_{Z|\tau > s}(d\xi) < \sup_{\eta \in N_{\delta_1}(0)} \left| p(l + iy - \eta) - p(l + iy) \right| P \left[ Z(l, s; z) \in N_{\delta_1}(z) \mid \tau_l(s) > s \right] \]

\[
+ \sup_{\eta \in \mathbb{H}} \left| p(\eta) - p(l + iy) \right| P \left[ Z(l, s; z) \notin N_{\delta_1}(z) \mid \tau_l(s) > s \right] < \frac{\epsilon}{6} \cdot 1 + 1 \cdot \frac{\epsilon}{6} = \frac{\epsilon}{3}.
\]

Let \( P_{Z|\tau = 0} \) denote the conditional probability measure on \( (\mathbb{H}, \mathcal{B}(\mathbb{H})) \) given by,

\[
P_{Z|\tau = 0}(B) = P[Z \in B \mid \tau_l(s) = s], \quad B \in \mathcal{B}(\mathbb{H}).
\]

As stated above, we can now choose \( S = S(l) \) sufficiently large that (3.46) holds and, for all \( s < S \),

\[
P \left[ \tau_l(s) = s \right] < P \left[ \exists s < S \text{ s.t. } \tau_l(s) = s \right] < \frac{\epsilon}{6}.
\]
Then, for all $s < S$, we have that,

$$
\int_{\mathbb{H}} |p(l + iy - (z - \xi)) - p(l + iy)| P_Z(d\xi) \\
= \int_{\mathbb{H}} |p(l + iy - (z - \xi)) - p(l + iy)| \mathbb{P}[\tau_l(s) > s] P_{Z|\tau > s} \\
+ \int_{\mathbb{H}} |p(l + iy - (z - \xi)) - p(l + iy)| \mathbb{P}[\tau_l(s) = s] P_{Z|\tau = s}(d\xi) \\
< \mathbb{P}[\tau_l(s) > s] \int_{\mathbb{H}} |p(l + iy - (z - \xi)) - p(l + iy)| P_{Z|\tau > s}(d\xi) \\
+ \mathbb{P}[\tau_l(s) = s] \sup_{\eta \in \mathbb{H}} |p(\eta) - p(l + iy)| \int_{\mathbb{H}} P_{Z|\tau = s}(d\xi) \\
\leq \int_{\mathbb{H}} |p(l + iy - (z - \xi)) - p(l + iy)| P_{Z|\tau > s}(d\xi) + \mathbb{P}[\tau_l(s) = s] \\
< \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{\epsilon}{2}
$$

Now that we have control of the quantity (3.49), we repeat the details of the proof that follow (3.49) but with $z$ replaced by $z'$ to also get

$$
\int_{\mathbb{H}} |p(l + iy - (z' - \xi)) - p(l + iy)| P_{Z'}(d\xi) < \frac{\epsilon}{2}. \quad (3.51)
$$

Finally, we have that

$$
|\hat{p}(z') - \hat{p}(z)| < \int_{\mathbb{H}} |p(l + iy - (z' - \xi)) - p(l + iy)| P_{Z'}(d\xi) \\
+ \int_{\mathbb{H}} |p(l + iy - (z - \xi)) - p(l + iy)| P_Z(d\xi) \\
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
$$

as needed. Since $\epsilon$ was taken arbitrarily, we can take $\epsilon \to 0$ to complete the proof. \qed

Until now, we have not emphasized the dependence of $\hat{\gamma}$ on the parameter $\kappa$ explicitly in the notation, but the geometry of $\hat{\gamma}$ obviously depends on $\kappa$ and $\mu$, so one should write $\hat{\gamma} = \hat{\gamma}(t; \kappa, \mu)$. Technical difficulties complicate the proof of the extension of the pivotal Theorem 3.1.1 to values $\kappa > 4$ corresponding to the case for which the SLE$_{\kappa}^\mu$ generating curve touches the real axis w.p.1. If the limiting curve
\hat{\gamma} is constructed for \kappa > 4. Complications also arise when defining the top crossing probability in this more general context. However, we expect that most of the results in this chapter will hold in this greater generality.
4. PRESENT WORK

The Schramm Loewner equation was originally derived with the intention of describing conformally invariant scaling limits of critical systems from mathematical physics with generating curves of Loewner chains. Self-avoiding paths or interfaces in many of these critical systems also share the domain Markov property. Two consequences of conformal invariance, namely scale invariance and invariance with respect to conformal maps that unzip a finite history of the continuum interface, are then combined with the domain Markov property to conclude that the only possible driving function for a Loewner equation capable of producing such an interface is the Brownian motion $\sqrt{\kappa}W_t$, $\kappa > 0$.

Remarkably, if one disregards physical motivations for a moment and disposes of the scale invariance requirement, thus assuming only invariance of the process with respect to the Loewner chain maps $g_t$ (but not scale invariance), then it follows that the Brownian motion with linear drift, $U_t := \sqrt{\kappa}W_t + \mu t$, constitutes a two parameter family of driving functions whose corresponding Loewner chain satisfies the remaining two requirements. We have denoted such Loewner chains by $\text{SLE}_\kappa^\mu$.

Motivated purely by mathematical interest, one can investigate the $\text{SLE}_\kappa^\mu$ Loewner chains and quickly determine the associated generating curve, $\gamma$, is skewed towards the side of the real axis that agrees with the sign of $\mu$. Taking $\mu > 0$ with no loss of generality and taking $\kappa \leq 4$ to avoid technical difficulties, it was shown that $\Re \gamma(t) \to \infty$ as $t \to \infty$. One is then naturally lead to conjecture stationarity properties of the distribution of both $\Im \gamma(t)$ and the time increments of $\gamma(t)$ as $t \to \infty$.

Initially, precise statements of this nature appear daunting because of the complicated relationship between the Loewner flow and the generating curve and the intuitively obvious fact that the asymptotic behavior of $\gamma(t)$ only exists in a a purely distributional sense. However, we were able to harness the long term behavior of $\gamma(t)$
by relating it to an infinite limit of the logarithmically-corrected reverse Loewner flow, which was shown to converge to a (random) deterministic path in $\mathbb{H}$ w.p.1, at least for the case $\kappa \leq 4$. The two limits were associated with each other by way of Theorem 3.1.5.

The reverse flow limit constructed, $\hat{\gamma} : \mathbb{R} \to \mathbb{H}$, is the primary object of this thesis. Having described the motivation for constructing this interesting process, we also derived temporal stationarity properties of the process and, especially, invariance of the top-crossing probability of a point $z$ by $\hat{\gamma}$ with respect to spatial horizontal translations of the $\hat{\gamma}$ distribution, established in the final section. It our hope that these properties of $\hat{\gamma}$, along with those inherited from SLE$_\kappa$ by way of Girsanov’s theorem, merit further investigation of the process.

One specific future research avenue is the extension of all of the results in Chapter 3, including a well-defined top-crossing probability, to the cases $\kappa \in (4, 8)$ and $\kappa \geq 8$, the additional two phases of the SLE$_\kappa$ generating curve. The central obstacle of this research avenue is in the proof that $\Re \gamma(t) \to \infty$ as $t \to \infty$ in this more general context; if that result is established, the remaining proofs and definitions can be generalized quite easily by building on the basic results from Chapter 2.

In the context of statistical physics, one would ultimately like to view the interface $\hat{\gamma}$ and the connected component of $\mathbb{H} \setminus \hat{\gamma}$ ‘above’ $\hat{\gamma}$ as scaling limits of respective objects related to some statistical system. In principle, one could use the construction of $\hat{\gamma}$ to estimate top-crossing probabilities and spatial covariance structures of the top crossing probabilities then compare these estimates to their discrete analogues associated with a large scale simulation of specific discrete statistical systems, for example, the long term behavior of an off-critical interface of the loop-erased random walk. Numerical results that suggested convergence of any statistical system to $\hat{\gamma}$ would be interesting and could merit further mathematical investigation of a correspondence.
5. APPENDICES

5.1. Complex Analysis Background

As usual, \( \mathbb{C} \) denotes the complex plane and \( \mathbb{H} \) denotes the upper half plane with a complex structure, say \( \mathbb{H} := \{ x + iy : y > 0 \} \). We presume the reader is familiar with basic complex analysis including the theory of holomorphic functions. A conformal map is a holomorphic bijection; such maps are known to be angle-preserving. The Riemann mapping theorem gives us the existence of many such maps between subsets of the plane. We first state only the existence part of the classical theorem.

**Theorem 5.1.1. Riemann Mapping Theorem** Every simply connected domain \( G \) in the extended complex plane whose boundary contains more than one point can be conformally mapped onto the unit disc, \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \).

More conditions are needed for a uniqueness statement. Published statements of the theorem will often note that the uniqueness follows from requiring that the map sends a specific point, \( z_0 \in G \), to zero and has a positive derivative at \( z_0 \). We will use different uniqueness conditions for our purposes, and we will talk about conformal maps from simply connected domains \( G \) to \( \mathbb{H} \). First we introduce a definition used throughout the paper.

**Definition** A bounded set \( A \subset \mathbb{H} \) is called a compact \( \mathbb{H} \)-hull if \( A = \mathbb{H} \cap \bar{A} \) and \( \mathbb{H} \setminus A \) is simply connected.

Since \( \mathbb{D} \) is conformally equivalent \( \mathbb{H} \), it follows from the Riemann mapping theorem that if \( A \subset \mathbb{H} \) is a compact \( \mathbb{H} \)-hull, then there are many conformal maps \( g : \mathbb{H} \setminus A \to \mathbb{H} \). The following theorem gives conditions for uniqueness of such a map [17].
Theorem 5.1.2. Suppose $A \subset \mathbb{H}$ is a compact $\mathbb{H}$-hull. There exists a unique conformal map $g_A : \mathbb{H} \setminus A \to \mathbb{H}$ with

$$\lim_{z \to \infty} [g_A(z) - z] = 0$$

Proof. Let the antecedent hold. Both $\mathbb{H} \setminus A$ and $\mathbb{H}$ can be conformally mapped to $\mathbb{D}$, say $h_1 : \mathbb{H} \setminus A \to \mathbb{D}$ and $h_2 : \mathbb{H} \to \mathbb{D}$, and we may require that the extensions of $h_1$ and $h_2$ to their domain boundaries satisfy $h_1(\infty) = h_2(\infty) = -1 \in \partial \mathbb{D}$. It then follows that $g := h_2^{-1} \circ h_1 : \mathbb{H} \setminus A \to \mathbb{H}$ has $|g(z)| \to \infty$ as $z \to \infty$.

Now choose $r$ large enough to ensure $A \subset \{ z \in \mathbb{C} : |z| < r \}$. The Schwarz reflection principle allows us to extend $g$ to a conformal transformation of $\{ z \in \mathbb{C} : |z| > r \}$ with $g(\bar{z}) = \bar{g}(z)$; therefore, $g(1/z)$ is analytic in a neighborhood of 0 and diverges to $\infty$ as $z \to 0$. Thus $f(z) := 1/g(1/z)$ can be expanded about the origin.

Then, by algebra, we can get an expansion of $g$ in a neighborhood of $\infty$ given by,

$$g(z) = b_{-1}z + b_0 + b_1z^{-1} + ...$$

The continuous extension of $g$ to $\partial(\mathbb{H} \setminus A)$ satisfies $g((r, \infty)) \subseteq \mathbb{R}$; hence, $b_j \in \mathbb{R}$ for $j \geq -1$, and $g(\mathbb{H} \setminus A) \subseteq \mathbb{H}$. Since $|g(z)| \to \infty$ as $z \to \infty$, it is unique up to composition with a transformation of the form $l(z) = az + b$ with $a > 0$ and $b \in \mathbb{R}$, since these are the only conformal maps mapping $\mathbb{H}$ to $\mathbb{H}$ that send $\infty$ to $\infty$. The coefficients in the map $l$ are then uniquely determined by $a := \frac{1}{b_{-1}}$ and $b := -b_0$ in order to achieve the map $g_A := l \circ g : \mathbb{H} \setminus A \to \mathbb{H}$ with $\lim_{z \to \infty}[g_A(z) - z] = 0$. □

In our study, we will discuss families of such maps parametrized by $t \in \mathbb{R}^+$, say $g_t(z) := g_{A_t}(z)$, where $A_t \subseteq \mathbb{H}$ are compact $\mathbb{H}$-hulls. In general coefficients of Laurent series expansions of the functions $g_t(z)$ about some fixed point $z_0$ are useful characteristics of such functions. In particular, families $g_t(z)$ corresponding to compact $\mathbb{H}$-hulls $A_t$ will always have $z = \infty$ in their domain, so that we can take a Laurent Series expansion of each $g_t(z)$ about $\infty$. That is, $g_t(z)$ can be written as
\[ g_t(z) = b_{-1}(t)z + b_0(t) + b_1(t)z^{-1} + \ldots \] for some sequence \( \{b_n\}_{n=-1}^{\infty} \subset \mathbb{C} \). Our assumptions about \( g_t(z) \) require \( b_{-1}(t) = 1 \) and \( b_0(t) = 0 \). Further, the coefficients \( b_{-1}(t) \) of the maps \( g_t(z) \) are uniquely determined since the maps \( g_t(z) \) are each uniquely determined. Properties of the family \( g_t : \mathbb{H} \setminus A_t \rightarrow \mathbb{H} \) are related to properties of the coefficients \( b_{-1}(t) \). We will call \( b_{-1}(t_0) \) the half plane capacity of \( A_{t_0} \). The precise definition follows.

**Definition** If \( A \) is a compact \( \mathbb{H} \)-hull, the *half plane capacity (from infinity)* of \( A \) is defined by,

\[
\text{hcap}(A) = \lim_{z \to \infty} z[g_A(z) - z].
\]

The chordal Loewner equation defined in Section 1.2.1 defines a family of compact \( \mathbb{H} \)-hull, \( K_t \), with \( \text{hcap}(K_t) = 2t \). A thorough exposition of set capacities and their relationship to chordal Loewner equation and other versions of the Loewner equation can be found in [17].

### 5.2. Solution of Loewner Equation with Linear Driving Function

Here we will consider solutions, \( g_{t,\text{lin},(s)}(z) \), of the deterministic, linearly-driven Loewner equation. These easily derived exact solutions yield generating curves, \( \gamma_{\text{lin}} : [0, \infty) \rightarrow \overline{\mathbb{H}} \), that satisfy

\[ \Im \gamma_{\text{lin}}(t) \to \text{constant}, \quad t \to \infty. \]

The equation,

\[
\frac{\partial}{\partial t} g_{t,\text{lin},(s)}(z) = \frac{2}{g_{t,\text{lin},(s)}(z) - \mu(s + t)}; \quad g_{t,\text{lin},(s)}(0) = z,
\]

(5.1)
can be solved exactly by separation of variables. Letting \( g_{t,\text{lin}}(z) := g_{t,\text{lin},(0)}(z) \), this method yields the following implicit form of the solution,

\[
2 \log(2 - \mu(g_{t,\text{lin}}(z) - \mu t)) - (g_{t,\text{lin}}(z) - \mu t) = t + c.
\]
Applying $g^0_{\text{lin}}(z) = z$ gives the constant, and the equation for $g^t_{\text{lin}}(z)$ reduces to

$$g^t_{\text{lin}}(z) + z - 2\mu t = \frac{2}{\mu} \log \frac{2 - \mu (g^t_{\text{lin}}(z) - \mu t)}{2 - \mu z} \quad (5.2)$$

If $\gamma_{\text{lin}}(t)$ denotes the generating curve of this Loewner flow, we can use the previous equation and the relation $g^t_{\text{lin}}(z)(\gamma_{\text{lin}}(t)) = \mu t$ to get,

$$\gamma_{\text{lin}}(t) - \mu t = -\frac{2}{\mu} \log(1 - \frac{\mu}{2}\gamma_{\text{lin}}(t)), \quad (5.3)$$

Decomposing the previous equation into real and imaginary parts gives the system,

$$\Re \gamma_{\text{lin}}(t) = \mu t - \frac{2}{\mu} \ln |1 - \frac{\mu}{2}\gamma_{\text{lin}}(t)| \quad (5.4)$$

$$\Im \gamma_{\text{lin}}(t) = -\frac{2}{\mu} \arg(1 - \frac{\mu}{2}\gamma_{\text{lin}}(t)). \quad (5.5)$$

Note that when the Loewner flow is initialized at points $g^0_{\text{lin}}(z) = z \in \{z \in \mathbb{H} : \Re z < 0\}$; it follows that for all $t > 0$,

$$\frac{\partial}{\partial t} \Re g^t_{\text{lin}}(z) < 0 \text{ and } \mu t > 0;$$

therefore, $\mu t - \Re g^t_{\text{lin}}(z) > 0$ and we may conclude that $\Re \gamma_{\text{lin}}(t) > 0$ whenever $t > 0$.

When we accompany this fact with (5.4), it immediately follows that $\Re \gamma_{\text{lin}}(t) \to \infty$ as $t \to \infty$. It easily follows from (5.5) that $\Im \gamma_{\text{lin}}(t)$ is bounded; whence,

$$\arg(1 - (2/\mu)\gamma_{\text{lin}}(t)) \to -\pi \text{ as } t \to \infty;$$

therefore,

$$\Im \gamma_{\text{lin}}(t) \uparrow 2\pi/\mu \text{ as } t \to \infty.$$

We are also interested in how $\gamma_{\text{lin}}(t)$ is parametrized as $t \to \infty$. Precisely, we can prove that the sequence of functions $\{\gamma^t_{r_{\text{lin}}} : [0, \infty) \to \mathbb{H}\}_{r>0}$ given by $\gamma^t_{r_{\text{lin}}}(t) = \gamma_{\text{lin}}(r+t) - \gamma_{\text{lin}}(r)$ converges uniformly to $\hat{\gamma}(t) := \mu t$. This is achieved by using (5.3) to write,

$$\gamma_{\text{lin}}(r+t) - \gamma_{\text{lin}}(r) = \mu t + \frac{2}{\mu} \log \left( \frac{1 - \frac{\mu}{2}\gamma_{\text{lin}}(r)}{1 - \frac{\mu}{2}\gamma_{\text{lin}}(r+t)} \right), \quad (5.6)$$
so as to conclude that

$$\gamma_{\text{lin}}(r + t) - \gamma_{\text{lin}}(r) \sim \mu t + \frac{2}{\mu} \log \left( \frac{\gamma_{\text{lin}}(r)}{\gamma_{\text{lin}}(r + t)} \right) \text{ as } t \to \infty. \quad (5.7)$$

It is easily shown $\gamma_{\text{lin}}(r)\gamma_{\text{lin}}(r + t)^{-1} \to 1$ as $t \to \infty$, so $\gamma_{\text{lin}}(t) \to \hat{\gamma}(t)$ uniformly in $t$, as claimed.

The same sequence of functions, $\{\gamma_{\text{lin}}^r\}_{r > 0}$, can be extracted from the reverse Loewner flow as follows. Let $f_{t}^{\text{lin}(s)}(z)$ denote the reverse-flow of the linearly-driven Loewner equation defined by

$$\frac{\partial}{\partial t} f_{t}^{\text{lin}(s)}(z) = \frac{-2}{f_{t}^{\text{lin}(s)}(z) + \mu(s - t)}; \quad f_{0}^{\text{lin}(s)}(z) = z. \quad (5.8)$$

This equation can be solved exactly, as was done above for $g_{t}^{\text{lin}(s)}(z)$; however, this will not be necessary for our purposes. Rather, letting $f_{t}^{\text{lin}}(z) := f_{t}^{\text{lin}(0)}(z)$, note that

$$f_{r}^{\text{lin}}(\gamma(t)) = \gamma_{\text{lin}}(r)(r + t);$$

consequently,

$$f_{r}^{\text{lin}}(0) = \gamma_{\text{lin}}(r)(r)$$

where $\gamma_{\text{lin}}(s)(t)$ denotes the generating curve associated with the solution to (5.1). Moreover, for any $s < 0$,

$$f_{-s+r}^{\text{lin}}(\gamma_{\text{lin}}(t)) = f_{r}^{\text{lin}(s)}(\gamma_{\text{lin}}(s + t)) = \gamma_{\text{lin}}(s-r)(-s + r + t)$$

Now the generating curve $\gamma_{\text{lin}}(s-r)(-s + r + t)$ is given by the equation

$$\frac{\partial}{\partial t} g_{t}^{\text{lin}(s-r)}(z) = \frac{-2}{g_{t}^{\text{lin}(s-r)}(z) - \mu(s - r + t)}; \quad g_{0}^{\text{lin}(s-r)}(z) = z.$$ 

Let $h_{t}^{\text{lin}(s-r)}(z) := g_{t}^{\text{lin}(s-r)}(z) - \mu(s - r + t)$ and note that,

$$\frac{\partial}{\partial t} h_{t}^{\text{lin}(s-r)}(z) = \frac{-2}{h_{t}^{\text{lin}(s-r)}(z)} - \mu; \quad h_{0}^{\text{lin}(s-r)}(z) = z + \mu(-s + r).$$
Similarly, we find that $h^\text{lin}_t(z) := g^\text{lin}_t(z) - \mu t$ satisfies the initial value problem,

$$\frac{\partial}{\partial t} h^\text{lin}_t(z) = \frac{2}{h^\text{lin}_t(z)} - \mu; \ h^\text{lin}_0(z) = z.$$ 

Therefore, since $h^\text{lin}_0(z - \mu(-s + r)) = z = h^\text{lin}_0(z)$, we have

$$h^\text{lin}_t(z - \mu(-s + r)) = h^\text{lin}_t(z),$$

thus, by definition,

$$g^\text{lin}(s,r) = \gamma^\text{lin}(s-r)(z) + \mu(-s + r) = h^\text{lin}_t(z),$$

In terms of the generating curves of $g^\text{lin}(s,r)(z)$ and $g^\text{lin}_t(z)$, this means

$$\gamma^\text{lin}(s-r)(t) + \mu(-s + r) = \gamma^\text{lin}(t); \quad (5.9)$$

therefore,

$$\gamma^\text{lin}_r(t) := \gamma^\text{lin}(r + t) - \gamma^\text{lin}(r)$$

$$= (\gamma^\text{lin}(s-r)(r + t) + \mu(-s + r)) - (\gamma^\text{lin}(s-r)(r) + \mu(-s + r))$$

$$= \gamma^\text{lin}(s-r)(r + t) - \gamma^\text{lin}(s-r)(r).$$

If we consider $s < 0$ as a fixed time and $r$ as the index of a sequence (indexed by positive real numbers) of functions of $t \in [0, \infty)$, we now have two geometrical representations of the same sequence of functions that are very different: Taking $t \in [0,T]$ with $T < \infty$, the segment $\gamma^\text{lin}(r + t) - \gamma^\text{lin}(r)$ is visually the tail end of a generating curve growing into the half plane as $r$ increases, but the segment $\gamma^\text{lin}(s-r)(r + t) - \gamma^\text{lin}(s-r)(r)$ is the tail end of a (shifted) generating curve evolving under the reverse Loewner flow as $r$ increases. In Chapter 3, we will find that the sequence of functions extracted from SLE$^\mu$ that are analogous to $\gamma^\text{lin}(r + t) - \gamma^\text{lin}(r)$, $\gamma^\text{lin}(s-r)(r + t) - \gamma^\text{lin}(s-r)(r)$ are different sequences w.p.1; however, we can show that they are distributionally equivalent. Notably, the former sequence does not converge w.p.1, but the latter sequence will be shown to converge w.p.1.
WORKS CITED


