Short-time Asymptotic Analysis of the Manakov System

by

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A Dissertation Submitted to the Faculty of the

Graduate Interdisciplinary Program
in Applied Mathematics

In Partial Fulfillment of the Requirements
For the Degree of

Doctor of Philosophy

In the Graduate College

The University of Arizona

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The University of Arizona
Graduate College

As members of the Dissertation Committee, we certify that we have read the dissertation prepared by Jesús Adrián Espínola Rocha
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and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copies of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

24 April 2006

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Acknowledgments

There are lots of people I am in doubt with. This list is not exhaustive, and I apologize in advance to the people I missed their names.

I would like to thank Dr. K. McLaughlin for great discussions and useful comments, specially by the end of the thesis work. He was an authentic co-advisor. I also thank to the *Working Group on the NLS Equation* members, for their time and the feedback received. Thanks to Dr. P. Miller, for his professional criticism and suggestions to improve the thesis. Thanks to Dr. M. Brio, for discussions and his suggestions on the numerical analysis part of the work. Thanks to Dr. J. Lega, for all her inverse-"scattering" advises.

I am also very thankful with my friends for helping me go through all this, over all these years, in preparing papers, talks, research and project reports. I also thank to them hours of discussion about mathematics, professional life, and ... life by its own.

I also thank Dr. L. Hermi, not only his friendship, but also for his research collaborations and (implicitly) accepting to be my research "padrino" (okay, my mentor).

Thanks to Dr. Zakharov, for his inspirating lectures on Soliton Theory and General Relativity.

Thanks to Bob Condon, for his support on my early computational experience.

Thanks to Drs. Ize and Minzoni, for showing me what is the wonderful world of mathematics and physics during my years as student in México.

Many thanks to Christa, for cheering me up every Monday/Friday at (circa) 6pm (the "usual" meeting date and time).

But most importantly, I am deeply thankful to my advisor, Dr. Nicholas Ercolani, for his not only mathematical, but his overall and general guidance and support along all these years.

I thank to DGAPA-UNAM for the awarded fellowship during my early graduate student days. I also thank Dr. Tabor and the Program in Applied Mathematics for receiving me as their student, and also for the financial support received.

Dedication

I would like to dedicate this work to my mother María de los Angeles.

The number of people I dedicate the thesis is large. Due to space limitations, I only state the names, but not the reasons why I decided to include them. These reasons are obvious in most of the cases.

It is also dedicated to: mi papá, Gladys, Rosy, Gil, David, Humberto, Marcela, Diana, Andrés, Jorge y Luz.

To my very best friends Euclides y Luis. To the León Tellez family. To el Profe and el equipo de atletismo. To Dr. Barajas. To my Seattle's family (this list is *huge*, but *all of you* are included), who always receive me with lots of love and family warm.

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Abstract

The Manakov system appears in the physics of optical fibers, as well as in quantum mechanics, as multi-component versions of the Nonlinear Schrödinger and the Gross-Pitaevskii equations.

Although the Manakov system is completely integrable its solutions are far from being explicit in most cases. However, the Inverse Scattering Transform (IST) can be exploited to obtain asymptotic information about solutions.

This thesis will describe the IST of the Manakov system, and its asymptotic behavior at short times. I will compare the focusing and defocusing behavior, numerically and analytically, for squared barrier initial potentials. Finally, I will show that the continuous spectrum gives the dominant contribution at short-times.

Introduction.

The basic propagation equation of optical pulses in single-mode fibers is described by the equation [Agr]

$$i\frac{\partial A}{\partial z} + i\beta_1 \frac{\partial A}{\partial t} - \frac{1}{2}\beta_2 \frac{\partial^2 A}{\partial t^2} + i\frac{\alpha}{2}A + \gamma |A|^2 A = 0.$$

Here, $A = A(z,t)$ is a complex function whose magnitude, $|A(z,t)|$, represents the intensity of the electromagnetic field inside the fiber. The fiber loss is included through the parameter α, and the chromatic dispersion through the parameters β_1 and β_2, which take account of the group velocity $v_g = \frac{1}{\beta_1}$, and the group velocity dispersion β_2. The equation also includes the effects of fiber nonlinearity through γ. This constant γ represents the intensity dependence of the refractive index of the fiber.

By a change of variables, we can eliminate the constant β_1. If we also assume no fiber losses, we should set $\alpha = 0$, to then obtain the *Nonlinear Schrödinger equation* (NLS equation), which in standard form is

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \sigma |u|^2 u = 0, \tag{1}$$

where $u(x,t)$ is a complex function now representing the adimensional intensity of the field inside the fiber. The constant $\sigma = \pm 1$ corresponds, in optical terms, to the anomalous/normal dispersion of the fiber (focusing/defocusing case), respectively. The anomalous dispersion regime is of particular interest because it is in this regime that optical fibers support *optical solitons*, through the balance between the dispersive and the nonlinear effects.

In 1972, Zakharov and Shabat [ZS] found that this equation is solvable by the Method of Inverse Scattering (ISM), or the Inverse Scattering Transform (IST), which is a nonlinear analogous to the Fourier transform for linear problems. We say then that the NLS equation is *a completly integrable equation.*

When two optical pulses (described by the NLS equation) propagate along the fiber, they should interact and get coupled together. In this instance, the system is not completly integrable in general. A birefringent optical fiber is a device on which this phenomenon could happen [Boa], [Kat]. In an axisymmetric optical fiber, the fundamental mode consists of two copropagating and orthogonally polarized linear electric fields. Any imperfection to the guiding core of the fiber, such as eccentricity, or bending, results in an asymmetry of the fiber, introducing *birefringence*, which is a difference in the dispersion relations between the polarized waves. Furthermore, for a nonlinear optical fiber, the amplitude coupling across polarizations can cause an additional self-induced birefringence. The interaction between these linear and nonlinear birefringences (or cross-phase modulation) can result in a rotation in the polarization as the pulse propagates along the fiber.

For the birefringent fiber, the evolution of the linearly polarized envelopes of a quasi-monochromatic pulse is governed by the coupled NLS (CNLS) system of equations

$$\begin{aligned} i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - Ku + (|u|^2 + A|v|^2)u + Bv^2u^* &= 0, \\ i\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + Kv + (A|u|^2 + |v|^2)v + Bu^2v^* &= 0. \end{aligned} \tag{2}$$

Here, $u(x,t)$ and $v(x,t)$ denote the complex-valued, orthogonal electromagnetic fields whose orientation coincides with the principal axes of the linear birefringence. The constant K is the linear birefringence coefficient, and is the half-difference of the phase velocities. The axisymmetry of the nonlinear coupling requires

$$A + B = 1.$$

Typical values for these parameters are $A = \frac{2}{3}$ and $B = \frac{1}{3}$, which are specific for the Kerr nonlinearity.

The system has different variants. One of them is the following [Men1]

$$\begin{aligned}
i\left(\frac{\partial u}{\partial t}+\delta\frac{\partial u}{\partial x}\right)+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+(|u|^2+A|v|^2)u+Bv^2u^*e^{-iR\delta t} &= 0,\\
i\left(\frac{\partial v}{\partial t}-\delta\frac{\partial v}{\partial x}\right)+\frac{1}{2}\frac{\partial^2 v}{\partial x^2}+(A|u|^2+|v|^2)v+Bu^2v^*e^{iR\delta t} &= 0.
\end{aligned}$$

In experimental cases, the birefringent beat length, R^{-1}, is much smaller compared with the dispersion scale length, δ. This causes rapid oscillations of the exponential factors, which are in general neglected. We then obtain a simplified form of the previous equation [Men2]

$$\begin{aligned}
i\left(\frac{\partial u}{\partial t}+\delta\frac{\partial u}{\partial x}\right)+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+(|u|^2+A|v|^2)u &= 0,\\
i\left(\frac{\partial v}{\partial t}-\delta\frac{\partial v}{\partial x}\right)+\frac{1}{2}\frac{\partial^2 v}{\partial x^2}+(A|u|^2+|v|^2)v &= 0.
\end{aligned}$$

As far as I know, the first work on the CNLS system of equations is the work of A.L. Berkhoer and V.E. Zakharov [BZ] in 1970 on self-interacting waves. In this reference, the authors derive the following system of equations for the envelopes of polarized electromagnetic fields u and v

$$\begin{aligned}
i\left(\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}\right)+\widehat{L}u &= (Q|u|^2+P|v|^2)u,\\
i\left(\frac{\partial v}{\partial t}+c\frac{\partial v}{\partial x}\right)+\widehat{L}v &= (P|u|^2+Q|v|^2)v,
\end{aligned}$$

where

$$\widehat{L}=\frac{\omega_k''}{2}\frac{\partial^2}{\partial x^2}+\frac{\partial\omega}{\partial k}\Delta_\perp,$$

and

$$c=\frac{\partial\omega}{\partial k}.$$

Here, ω_k is the dispersion relation; ω and k are frequencies and wave numbers; and c is the dispersion velocity. P and Q are constants given in terms of ω, k and the dilelectric properties of the propagation medium.

We then see that the NLS and the CNLS equations are good models for propagation in optical fibers. These equations actually arise in Quantum Mechanics when they describe the *Bose-Einstein Condensation* (BEC)[1], which we now briefly describe.

Particles with integral angular momentum are called *bosons*. At very low temperatures, near the absolute zero, there is a phase transition in an ideal gas of identical bosons. Bosons are stimulated by other bosons in the lowest energy state (ground state), resulting a macroscopic occupation of a single quantum state. This phenomenon was predicted by Bose (1924) and Einstein (1925), and therefore named *Bose-Einstein Condensation* (BEC). There are good reviews of the subject. Two of them are [Par] and [Dal]. Gases of sodium, rubidium and helium are typical for the presence of BEC, and it also appears in semiconductors and superfluidity.

The equation which describes the quantum state of BEC is the *Gross-Pitaevskii (GP) equation*

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\Delta\Psi + V(x)\Psi + g|\Psi|^2\Psi,$$

first derived by Gross [Gro1], [Gro2], and Pitaevskii [Pit] in 1961. A rigorous derivation of the GP energy functional was performed by Lieb, Seiringer and Yngvason in 2000 [Lieb]. The authors present there a rigorous description of the ground-state of a BEC.

The term $V(x)$ describes the potential due to the magnetic trap and the optical devices for confinement and cooling for the creation of the BEC. The nonlinear term represents the interaction between bosons, which is assumed to be of short range, i.e., the interaction distance is much less than the distance between particles, resulting in a two-body potential. The parameter g describes the scattering properties of the system, which could be attractive ($g < 0$), or repulsive ($g > 0$).

There could also exist coupling of two ground states of BEC [Pot], [ZPMW]. They

[1]From the similarity to the linear Schrödinger equation of Quantum Mechanics, to write the NLS and CNLS equations with this notation is a standard practice.

are modeled by the two-component Gross-Pitaevskii equation

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\Psi &= -\frac{\hbar^2}{2m}\Delta\Psi + V(x)\Psi + g(|\Psi|^2 + |\Phi|^2)\Psi, \\
i\hbar\frac{\partial}{\partial t}\Phi &= -\frac{\hbar^2}{2m}\Delta\Phi + V(x)\Phi + g(|\Psi|^2 + |\Phi|^2)\Psi.
\end{aligned} \tag{3}
$$

Under some conditions, all these systems can be reduced to *the Manakov System*

$$
\begin{aligned}
i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \sigma(|u|^2 + |v|^2)u &= 0, \\
i\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + \sigma(|u|^2 + |v|^2)v &= 0,
\end{aligned} \tag{4}
$$

which is the subject of study in this dissertation. The CNLS system of equations of nonlinear optics (equations (3)), and the two-component Gross-Pitaevskii equation (equations (3)) can be considered as generalizations of the Manakov system. In equations (3), set $K = 0$ and $A = 1$ (therefore, $B = 0$). Now, in equations (3), consider the one-dimensional problem, i.e., $\Delta = \frac{\partial^2}{\partial x^2}$, and set $V(x) = 0$ (this physically corresponds to turn off the device once the BEC has reached.) Both cases will yield to the Manakov system (4). The parameter $\sigma = \pm 1$, as we previously mentioned, corresponds to the focusing/defocusing case in optical fibers, and to the attractive/repulsive attraction effect in BEC.

To understand nature, more sophisticated models are needed. However, the Manakov system is a good minimal model to describe and understand more complicated phenomena in nature, such as those previously shown: the birefringent optical fiber and the BEC. This is then the reason we are interested in the Manakov system. In addition, the Manakov system is also a completly integrable equation, i.e., it is solved by the IST [Ma].

The IST can be performed in principle, but is not so easy to do it in practice. Usually, the scattering problem (also called eigenvalue, or spectral problem) is very hard (or impossible) to solve explicitly. This is because the spectral problem is a first order system of differential equations with *variable coefficients* to which, in general, it cannot be found an explicit solution. In order to facilitate this task, we used a barrier type of initial conditions (also termed *initial potentials*), which leads to a constant coefficient system of equations. Physically, these initial potentials describe the diffraction of light by a slit [Ma].

A second reason to choose this kind of potentials is that they exhibit a Gibb's type phenomenon at short-times. We then would like to describe the behaviour of these solutions.

Now, the inverse problem is solved by the Riemann-Hilbert Problem (RHP) approach. When solving the RHP, a system of integral equations arises. The scattering data obtained from the spectral analysis make so difficult to find a closed form of the solutions of the integral equations. Therefore, an approximation of the solution is considered. In this instance, a small-time asymptotic analysis, inspired by the analysis of J. DiFranco and K. McLaughlin to the NLS equation [DiF], [DiF-McL], is applied to the *focusing* Manakov system.

The thesis is then structured in the following way. The numerical analysis of the Manakov system is studied in Chapter 1. We will extend the *Linearized Crank-Nicolson (LCN) scheme* for the NLS equation, developed by Chang *et. al.* [Cha], to the Manakov system. We will show that LCN is consistent and linearly stable, together with several numerical simulations that show it is convergent. The LCN scheme also satisfies a discrete conservation law, which we will prove it is so. Some particular initial conditions are considered to observe its numerical evolution in time. Initial potentials such as sech-profiles with equal and different amplitudes (NLS reduction) are used, and also an external potential is added to the system, then simulating oscillating BECs.

Finally, the Gibb's type phenomenon mentioned earlier is also numerically observed in the Manakov system at short times. When squared barrier potentials are considered, fast oscillations on one side of the jump, and exponentially decay on the other, appear at early times in the evolution of the Manakov system (focusing and defocusing). The goal of this thesis is to study this phenomenon, as well as giving an explicit formula that describes this behavior.

Since we have control on the support of the barriers, three different cases will arise: disjoint, overlapping (but different), and same supports. For the case when the supports overlap (and are different), we observed the Gibbs phenomenon acting independently on each component, i.e., the interaction of the two components is neglegible at short times. This result will be analytically presented in Chapter 6.

We also compare the behavior of the focusing and the defocusing Manakov system with these three different types of initial conditions (squared barriers). We observe a similar (qualitative) behavior at short times for both, the focusing and defocusing cases. These observations lead us to guess that there is no significant contribution of the discrete spectrum to the solution of the Manakov system when the time is small. This fact will be also remarked in Chapter 6.

Chapter 2 will be dedicated to the study of the spectral (scattering) problem associated to the focusing Manakov system, which we will term the *Manakov-Zakharov-Shabat (MZS) system*. This chapter will be more a review of the scattering theory of the MZS system, while in chapter 3, we will explicitly compute the scattering data associated to the MZS system with squared barrier potentials.

In the cases of equal, and disjoint supports, the eigenvalues of the MZS system are shown to be purely imaginary. This result extends those found by Klaus and Shaw [KS1], [KS2] on the eigenvalues of the Zakharov-Shabat (ZS) system for the NLS equation. This is the subject of study of chapterr 4. We also include a calculation to compute the threshold condition for the existence of eigenvalues of the MZS system for these two types of squared potentials

The final chapter, chapter 6, will contain the description of the solutions of the focusing Manakov system at short times. We follow the approach of DiFranco [DiF], but applied to the focusing Manakov system, this is to say, the asymptotic analysis at short times of the RHP approach to the IST. We will also point out how our analysis works for either, the focusing and the defocusing case. The IST via RHPs will be introduced in chapter 5.

At the end of the thesis, we state our conclusions and set the trends for future research. The thesis also contains two appendices. The first one includes how to find solutions of the Manakov system via the Darboux Transforms. These solutions are considered in Chapter 1 to test our numerical code to solve the Manakov system. They can be also used to study the linear stability of the Manakov system, as well as for a study of the stability via the Evans functions technique, which we do not consider here, leaving this project for future research.

In the second appendix we study the analytic properties of the *Jost functions*. The Jost functions are solutions of the MZS system with exponential behavior at $+\infty$ and $-\infty$. In this appendix, we prove that some of the Jost solutions can be analytically extended into the upper-half complex plane (when the spectral parameter of the MZS system is viewed as a variable), while some others can be analytically extended into the lower-half complex plane.

Evenmore, we also show that for compact supported potentials (as those considered here) the Jost functions can be analytically extended into the whole complex plane, i.e., they are analytic functions of the spectral parameter. This will be of fundamental importance in chapter 6, where these properties are employed to modify the RHP and extract the required asymptotic behaviour of the Manakov system.

Chapter 1

Numerical Analysis of the Manakov System.

1.0 Introduction

In this chapter, we will consider a numerical scheme to study the numerical behavior of the Manakov system. This discretization is termed *the Linearized Crank-Nicolson Scheme.* This is a linear version of the standard Crank-Nicolson Scheme and, although it was developed to solve the Nonlinear Schrödinger (NLS) equation [Cha], we employ the same scheme to solve the Manakov system. We will prove this scheme is consistent, of order $O((\Delta x)^2 + (\Delta t)^2)$, with the Manakov system, and that it is linearly stable.

I will show some numerical simulations of how this scheme works in some particular and simple examples, such as the sech-initial profiles, sech-profiles with a potential, and the squared barrier potentials. For the latter, we will see the Gibbs phenomenon, we will observe how the interaction of two potentials is negligible at short times, and also how the behaviour of the focusing and defocussing Manakov system is qualitatively the same (also at short times), leading to the question of the contribution of the discrete spectrum at early stages of evolution of the system, question addressed in Chapter 3.

1.1 The Linearized Crank-Nicolson Scheme

In 1999, to solve the Nonlinear Schrödinger (NLS) equation with a potential

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \sigma|u|^2 u = V(x)u,$$

Chang *et. al.* [Cha] developed the following numerical scheme

$$\begin{aligned} i(u_m^{n+1}-u_m^n) &= -\frac{\mu}{2}\left[\frac{(u_{m+1}^{n+1}-2u_m^{n+1}+u_{m-1}^{n+1})+(u_{m+1}^n-2u_m^n+u_{m-1}^n)}{2}\right] \\ &\quad -\sigma\Delta t\left[\frac{3}{2}|u_m^n|^2-\frac{1}{2}|u_m^{n-1}|^2\right]\left[\frac{u_m^{n+1}+u_m^n}{2}\right]+V_m\left[\frac{u_m^{n+1}+u_m^n}{2}\right], \end{aligned}$$

where

$$\mu \equiv \frac{\Delta t}{(\Delta x)^2},$$

and $V_m = V(x_m)$ is the external potential evaluated at the m^{th} point of the lattice: $x = x_m$. Chang and colleagues termed the scheme "Linearized Crank-Nicolson II". We will simply call it the *Linearized Crank-Nicolson* (LCN) scheme.

Since the natural two-component generalization of the NLS equation is the **Manakov System**

$$\begin{aligned} i\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+\sigma(|u|^2+|v|^2)u &= 0, \\ i\frac{\partial v}{\partial t}+\frac{1}{2}\frac{\partial^2 v}{\partial x^2}+\sigma(|u|^2+|v|^2)v &= 0, \end{aligned} \tag{1.1}$$

the natural generalization of the LCN scheme for the Manakov system can be written as follows

$$\begin{aligned} i(u_m^{n+1}-u_m^n) &= -\frac{\mu}{2}\left[\frac{(u_{m+1}^{n+1}-2u_m^{n+1}+u_{m-1}^{n+1})+(u_{m+1}^n-2u_m^n+u_{m-1}^n)}{2}\right] \\ &\quad -\sigma\Delta t\left[\frac{3}{2}(|u_m^n|^2+|v_m^n|^2)-\frac{1}{2}(|u_m^{n-1}|^2+|v_m^{n-1}|^2)\right]\left[\frac{u_m^{n+1}+u_m^n}{2}\right], \\ i(v_m^{n+1}-v_m^n) &= -\frac{\mu}{2}\left[\frac{(v_{m+1}^{n+1}-2v_m^{n+1}+v_{m-1}^{n+1})+(v_{m+1}^n-2v_m^n+v_{m-1}^n)}{2}\right] \\ &\quad -\sigma\Delta t\left[\frac{3}{2}(|u_m^n|^2+|v_m^n|^2)-\frac{1}{2}(|u_m^{n-1}|^2+|v_m^{n-1}|^2)\right]\left[\frac{v_m^{n+1}+v_m^n}{2}\right]. \end{aligned} \tag{1.2}$$

It is important to remark some interesting facts about this scheme. The **first** and *most important* fact is that, as opposed to the standard Crank-Nicolson scheme, this scheme is linear. I.e., in order to find u_m^{n+1} and v_m^{n+1}, we only need to invert a matrix. The regular Crank-Nicolson scheme requires the resolution of a system of algebraic equations for u_m^{n+1} and v_m^{n+1}.

Second fact. The term $|u|^2+|v|^2$ is approximated by the extrapolation formula, as opposed to the standard Crank-Nicolson Scheme. In the latter, the approximation at $t=t^{n+\frac{1}{2}}$ is given by

$$|u_m^{n+\frac{1}{2}}|^2+|v_m^{n+\frac{1}{2}}|^2 \sim \frac{(|u_m^{n+1}|^2+|v_m^{n+1}|^2)+(|u_m^n|^2+|v_m^n|^2)}{2},$$

while in the Linearized Crank-Nicolson we have

$$|u_m^{n+\frac{1}{2}}|^2+|v_m^{n+\frac{1}{2}}|^2 \sim \left[\frac{3}{2}(|u_m^n|^2+|v_m^n|^2)-\frac{1}{2}(|u_m^{n-1}|^2+|v_m^{n-1}|^2)\right].$$

We will show how this approximation works in the next section.

Third fact. The LCN scheme is a three-level system, u_m^{n+1}, u_m^n, u_m^{n-1}, as opposed to the regular Crank-Nicolson scheme, which is a (nonlinear) two-level system. This is a direct consequence of the extrapolation formula.

1.2 Proof of Consistency

In this section, we will prove the consistency of the LCN scheme with the Manakov system.

Proposition 1.1. Consistency of the LCN Scheme. *The Linearized Crank-Nicolson (LCN) Scheme*

$$\begin{aligned}
i(u_m^{n+1}-u_m^n) &= -\frac{\mu}{2}\left[\frac{(u_{m+1}^{n+1}-2u_m^{n+1}+u_{m-1}^{n+1})+(u_{m+1}^n-2u_m^n+u_{m-1}^n)}{2}\right] \qquad (1.3)\\
&\quad -\sigma\Delta t\left[\frac{3}{2}(|u_m^n|^2+|v_m^n|^2)-\frac{1}{2}(|u_m^{n-1}|^2+|v_m^{n-1}|^2)\right]\left[\frac{u_m^{n+1}+u_m^n}{2}\right]\\
&\quad +V_m\left[\frac{u_m^{n+1}+u_m^n}{2}\right],\\
i(v_m^{n+1}-v_m^n) &= -\frac{\mu}{2}\left[\frac{(v_{m+1}^{n+1}-2v_m^{n+1}+v_{m-1}^{n+1})+(v_{m+1}^n-2v_m^n+v_{m-1}^n)}{2}\right]\\
&\quad -\sigma\Delta t\left[\frac{3}{2}(|u_m^n|^2+|v_m^n|^2)-\frac{1}{2}(|u_m^{n-1}|^2+|v_m^{n-1}|^2)\right]\left[\frac{v_m^{n+1}+v_m^n}{2}\right]\\
&\quad +V_m\left[\frac{v_m^{n+1}+v_m^n}{2}\right].
\end{aligned}$$

is consistent of order $O((\Delta x)^2 + (\Delta t)^2)$ with the Manakov system with an external potential

$$\begin{aligned} i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \sigma(|u|^2+|v|^2)u &= V(x)u, \\ i\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + \sigma(|u|^2+|v|^2)v &= V(x)v. \end{aligned} \tag{1.4}$$

Proof: The idea in the Crank-Nicolson scheme is to approximate each term at $t = t^{n+\frac{1}{2}}$, where

$$t^{n+\frac{1}{2}} = t^n + \frac{\Delta t}{2}.$$

Using the Taylor series (about t^{n+1} and t^n) of a continuosly differentiable function $f(t)$, we can check that

$$f(t^{n+\frac{1}{2}}) = \frac{f(t^{n+1}) + f(t^n)}{2} + O(\Delta t^2). \tag{1.5}$$

with $f(t^{n+\frac{1}{2}}) = f(t^n + \frac{\Delta t}{2})$. The expansion of the second spatial derivative of another function g (with continuous second derivatives) is the standard one

$$g_{xx}(x_m) = \frac{g(x_m+\Delta x) - 2g(x_m) + g(x_m - \Delta x)}{\Delta x^2} + O(\Delta x^2). \tag{1.6}$$

We start with an approximation centered at $\tilde{t}$ for the time derivative u_t, which is given by

$$u_t(\tilde{t}) = \frac{u(\tilde{t} + \frac{\Delta t}{2}) - u(\tilde{t} - \frac{\Delta t}{2})}{\Delta t} + O(\Delta t^2).$$

We can check that this last approximation is valid, by using the Taylor expansions of $u(\tilde{t} + \frac{\Delta t}{2})$ and $u(\tilde{t} - \frac{\Delta t}{2})$ about $\tilde{t}$. Put $\tilde{t} = t^n + \frac{\Delta t}{2}$, to obtain

$$u_t(t^n + \frac{\Delta t}{2}) = \frac{u(t^n + \Delta t) - u(t^n)}{\Delta t} + O(\Delta t^2). \tag{1.7}$$

To discretize the second spatial derivative u_{xx}, at $t^{n+\frac{1}{2}}$, use the time discretization, equation (1.5)

$$u_{xx}(x_m, t^{n+\frac{1}{2}}) = \frac{u_{xx}(x_m, t^{n+1}) + u_{xx}(x_m, t^n)}{2} + O(\Delta t^2),$$

and also the spatial discretization, equation (1.6), to obtain

$$\begin{aligned} u_{xx}(x_m, t^{n+\frac{1}{2}}) &= \frac{1}{\Delta x^2}\left[\frac{(u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}) + (u_{m+1}^{n} - 2u_m^{n} + u_{m-1}^{n})}{2}\right] + \\ &\quad + O(\Delta x^2) + O(\Delta t^2). \end{aligned} \tag{1.8}$$

The potential is approximated using equation (1.5)

$$V(x_m, t^{n+\frac{1}{2}}) = \frac{V(x_m, t^{n+1}) + V(x_m, t^n)}{2} + O((\Delta t)^2). \tag{1.9}$$

If V is time-independent

$$V(x_m, t^{n+\frac{1}{2}}) = V(x_m).$$

The term that requires good imagination to approximate is, obviously, the nonlinear term, and therefore there are many ways to do this. The *interpolation formula* is a way to do this, and it was developed by Chang *et.al.* [Cha]. We now describe this approximation. We denote the nonlinear term by

$$q(\xi(t))u,$$

where $\xi(t) = |u|^2 + |v|^2$. The function $q = q(\xi)$ is then a function of the norm of the field (u, v) only, and it implicitly depends on time since u and v depend on time.

The extrapolation formula approximates the value of a given function $q(\xi)$ at the point $\xi = \xi^{n+\frac{1}{2}}$, using the grid points ξ^{n-1} and ξ^n. Consider the line passing through the points $(\xi^{n-1}, q(\xi^{n-1}))$ and $(\xi^n, q(\xi^n))$. This line will pass through the point $(\xi^{n+\frac{1}{2}}, y^{n+\frac{1}{2}})$, with $y^{n+\frac{1}{2}} = \frac{3}{2}q(\xi^n) - \frac{1}{2}q(\xi^{n-1})$ is the approximation of $q(\xi^{n+\frac{1}{2}})$, with an error of the order

$$y^{n+\frac{1}{2}} - q(\xi^{n+\frac{1}{2}}) = O((\Delta\xi)^2).$$

Since $\xi = \xi(t)$ is a function of time, and we have a discretization for the time, we can verify that $\Delta\xi = \xi^n - \xi^{n-1} = O(\Delta t)$. Therefore

$$y^{n+\frac{1}{2}} - q(\xi^{n+\frac{1}{2}}) = O((\Delta t)^2),$$

i.e.

$$q(\xi^{n+\frac{1}{2}}) = \left[\frac{3}{2}q(\xi^n) - \frac{1}{2}q(\xi^{n-1})\right] + O((\Delta t)^2).$$

In our case $q(\xi) = \xi$. Because $\xi = |u|^2 + |v|^2$, and from the previous equation we get

$$|u^{n+\frac{1}{2}}|^2 + |v^{n+\frac{1}{2}}|^2 = \left[\frac{3}{2}(|u^n|^2 + |v^n|^2) - \frac{1}{2}(|u^{n-1}|^2 + |v^{n-1}|^2)\right] + O((\Delta t)^2).$$

Since $u_m^{n+\frac{1}{2}} = \frac{u_m^{n+1}+u_m^n}{2} + O(\Delta t^2)$ (by equation (1.5), we obtain the desired approximation of the nonlinear term

$$\begin{aligned}(|u_m^{n+\frac{1}{2}}|^2 + |v_m^{n+\frac{1}{2}}|^2)u_m^{n+\frac{1}{2}} &= \\ &= \frac{1}{2}\left[3(|u_m^n|^2 + |v_m^n|^2) - (|u_m^{n-1}|^2 + |v_m^{n-1}|^2)\right]\frac{u_m^{n+1} + u_m^n}{2} + O((\Delta t)^2) \quad (1.10)\end{aligned}$$

We repeat the same procedure with the second equation of the system (1.1). Putting together equations (1.7), (1.8), (1.9), and (1.10), and the respective equations for v_m^n, we can conclude that

$$(\text{Manakov system}) - (\text{LCN scheme}) = O((\Delta x)^2) + O((\Delta t)^2),$$

and this difference goes to zero, as $\Delta x \to 0$ and $\Delta t \to 0$, and this concludes the proof.

Q.E.D.

1.3 Proof of Linear Stability

Here, we prove the following

Proposition 1.2. Linear Stability of the LCN Scheme. *Assume that the potential V_m is constant: $V_m = V_0$. Then, the Linearized Crank-Nicolson scheme, equation (1.3), is linearly stable.*

Proof: On the scheme, we consider plane wave solutions, which have the form

$$\begin{aligned}u_m^n &= e^{im\theta}r^n \\ v_m^n &= e^{im\theta}\rho^n.\end{aligned}$$

Substitute into equation (1.3) to obtain

$$\begin{aligned} i u_m^n (r-1) &= -\frac{\mu}{2} u_m^n \left[r \frac{(e^{i\theta} - 2 + e^{-i\theta}) + (e^{i\theta} - 2 + e^{-i\theta})}{2} \right] \\ &\quad -\sigma \Delta t \left[\frac{3}{2}(r^{2n} + \rho^{2n}) - \frac{1}{2}(r^{2(n-1)} + \rho^{2(n-1)}) \right] u_m^n \left(\frac{r+1}{2} \right) + \\ &\quad + V_0\, u_m^n \left(\frac{r+1}{2} \right). \end{aligned}$$

Now, using the trigonometry identity, $e^{i\theta} - 2 + e^{-i\theta} = -4 \sin^2 \frac{\theta}{2}$, we obtain

$$i(r-1) = 2\mu(r+1) \sin^2 \left(\frac{\theta}{2} \right) - \sigma \Delta t \left(\frac{r+1}{2} \right) A^n + \left(\frac{r+1}{2} \right) V_0, \tag{1.11}$$

where

$$A^n = \frac{3}{2}(r^{2n} + \rho^{2n}) - \frac{1}{2}(r^{2(n-1)} + \rho^{2(n-1)}).$$

From equation (1.11) now follows that

$$r = \frac{i + B^n}{i - B^n},$$

with

$$B^n = \frac{1}{2} \left\{ 4\mu \sin^2 \frac{\theta}{2} - \sigma \Delta t\, A^n + V_0 \right\}.$$

Notice that A^n is a real number, and so B^n is also a real number. Therefore, it follows that

$$|r| = 1.$$

A similar argument works for the second equation in (1.3) of the LCN scheme, to obtain: $|\rho| = 1$. We say then that the scheme is *unconditionally* linearly stable. Explicitly, we have that $A^n = 2$ and $B^n = 2\mu \sin^2 \frac{\theta}{2} - \sigma \Delta t + \frac{1}{2} V_0$.

Q.E.D.

Remark 1: It is clear that linear stability is not enough to establish nonlinear stability of the scheme. To assess nonlinear stabilty, we probably need some energy methods, or to follow the proofs given in [Cha2] or [Zh] to construct a similar proof of nonlinear stability of the LCN scheme. This obviously should include the case when the potential is a function of $x = x_m$, i.e. $V_m = V(x_m)$. The numerical experiments in section 1.5.3 support the hope such stability will hold in the appropriate context. This is another project for future research.

Remark 2: For the *energy method*, we might need the following three pieces. 1) To construct a discrete energy funtional. 2) Find stationary points of this functional (solutions of the LCN scheme). And, 3) check if these solutions are minimizers of the energy functional.

Remark 3: The above proof of stability demonstrates that the LCN scheme supports exact plane wave solutions of arbitrary wavenumber that oscillate but do not grow in amplitude. However, in the nonlinear setting more should be checked. For instance, the focusing NLS equation is known to support plane wave soluitons which are modulationally unstable. Probably the same is true for the LCN scheme in the focusing case. If the perturbation of one of the exact plane-wave solutions (for $V_m = V_0$) one may expect that in the linearization of the scheme about the exact solution, perturbations of nearby wave numbers (nearby values of θ) will grow exponentially. This is an interesting question to be studied in future research. I thank Peter Miller for pointing out this issue to me [Mi2].

Remark 4: Notice that for constant potentials $V(x) = V_0$, the transformation $(u, v) \longrightarrow e^{V_0 t}(u, v)$ drops the potential term. Then, without loss of generality, we can put $V_0 = 0$ in the proof of the previous proposition.

1.4 A Discrete Conservation Law

It is a known fact that the Manakov system presents infinitely many conservation laws [Ma]. We might then expect that the LCN scheme also presents conservation laws.

Proposition 1.3. Conservation Laws of the LCN Scheme. *The Linearized Crank-Nicolson (LNC) Scheme shows the following discrete conservation laws*

$$\sum_{m=1}^{M} |u_m^n|^2 = const. \qquad\qquad \sum_{m=1}^{M} |v_m^n|^2 = const.$$

Proof: The proof is simple. Define the inner product

$$(f_m^n, g_m^n) = \sum_{m=1}^{M} f_m^n \, \overline{g_m^n},$$

where the bar denotes complex conjugation. Now, the discretization, in short form, is

$$i(u_t)_m^{n+\frac{1}{2}} = -\frac{1}{2}(u_{xx})_m^{n+\frac{1}{2}} - q\left(\xi_m^{n+\frac{1}{2}}\right) u_m^{n+\frac{1}{2}} + V(x) u_m^{n+\frac{1}{2}},$$

where q is a real function on the variable $\xi_m^{n+\frac{1}{2}} = \left|u_m^{n+\frac{1}{2}}\right|^2 + \left|v_m^{n+\frac{1}{2}}\right|^2$, same as in the previous section. Consider now the inner product

$$\left(i(u_t)_m^{n+\frac{1}{2}}, u_m^{n+\frac{1}{2}}\right) = -\frac{1}{2}\left((u_{xx})_m^{n+\frac{1}{2}}, u_m^{n+\frac{1}{2}}\right) - \left(q\left(\xi_m^{n+\frac{1}{2}}\right) u_m^{n+\frac{1}{2}}, u_m^{n+\frac{1}{2}}\right) + \left(V_m u_m^{n+\frac{1}{2}}, u_m^{n+\frac{1}{2}}\right).$$

Let us analize each of these terms. Notice that

$$\begin{aligned} \left(V_m u_m^{n+\frac{1}{2}}, u_m^{n+\frac{1}{2}}\right) &= V_m \left(u_m^{n+\frac{1}{2}}, u_m^{n+\frac{1}{2}}\right) \\ &= V_m \left|u_m^{n+\frac{1}{2}}\right|^2 \end{aligned}$$

and

$$\begin{aligned} \left(q\left(\xi_m^{n+\frac{1}{2}}\right) u_m^{n+\frac{1}{2}}, u_m^{n+\frac{1}{2}}\right) &= q\left(\xi_m^{n+\frac{1}{2}}\right) \left(u_m^{n+\frac{1}{2}}, u_m^{n+\frac{1}{2}}\right) \\ &= q\left(\xi_m^{n+\frac{1}{2}}\right) \left|u_m^{n+\frac{1}{2}}\right|^2 \end{aligned}$$

are real numbers, since $V(x)$ and $q(\xi)$ are real functions.

Now, using the "*Summation by parts formula*" [APT], we obtain

$$\begin{aligned} -\left((u_{xx})_m^{n+\frac{1}{2}}, u_m^{n+\frac{1}{2}}\right) &= \left((u_x)_m^{n+\frac{1}{2}}, (u_x)_m^{n+\frac{1}{2}}\right) \\ &= \left|(u_x)_m^{n+\frac{1}{2}}\right|^2, \end{aligned}$$

which also is real.

We finally consider the term

$$\begin{aligned}\left(i(u_t)_m^{n+\frac{1}{2}}, u_m^{n+\frac{1}{2}}\right) &= \frac{i}{2\Delta t}\sum_{m=1}^{M}\left(u_m^{n+1}-u_m^n\right)\overline{\left(u_m^{n+1}+u_m^n\right)} \\ &= \frac{i}{2\Delta t}\sum_{m=1}^{M}\left(\left|u_m^{n+1}\right|^2-|u_m^n|^2\right)-\frac{1}{\Delta t}\sum_{m=1}^{M}\mathrm{Im}\left(u_m^{n+1}\overline{(u_m^n)}\right).\end{aligned}$$

This is the only term which is not real. Then, its imaginary part should be identically zero

$$\sum_{m=1}^{M}(|u_m^{n+1}|^2-|u_m^n|^2)=0.$$

Therefore

$$\sum_{m=1}^{M}|u_m^{n+1}|^2=\sum_{m=1}^{M}|u_m^n|^2,$$

for all n. Then

$$\sum_{m=1}^{M}|u_m^{n+1}|^2=\text{const.} \tag{1.12}$$

and similarly for the equation of v_m^n

$$\sum_{m=1}^{M}|v_m^{n+1}|^2=\text{const.} \tag{1.13}$$

Q.E.D.

These are the discrete equivalents to the conservation laws of the Manakov system

$$\begin{aligned}\int_{-\infty}^{\infty}|u(x,t)|^2\,dx &= \text{const.}\\ \int_{-\infty}^{\infty}|v(x,t)|^2\,dx &= \text{const.}\end{aligned}$$

The Manakov system, as well as the NLS equation, have *infinitely many conserved quantities*, “enough” to “*completly integrate*” the equations. Here, we mean that the exact solutions of the Manakov system and the NLS equation can be solved via the Inverse Scattering Method [Dra], [Ma], [NMPZ], [New], [To]. The three principal conserved quantites are the mass (or intensity), momentum, and energy

$$\begin{aligned}
I[u,v] &= \frac{1}{2}\int_{-\infty}^{\infty} (|u(x,t)|^2 + |v(x,t)|^2)\,dx \\
P[u,v] &= \frac{1}{2}\int_{-\infty}^{\infty} (u^\star i\frac{\partial u}{\partial x} + v^\star i\frac{\partial v}{\partial x})\,dx \\
E[u,v] &= \frac{1}{2}\int_{-\infty}^{\infty} (|u_x|^2 + |v_x|^2) + \sigma(|u|^2 + |v|^2)^2\,dx.
\end{aligned}$$

The rest of the conservation laws can be found using asymptotic techniques [Ma], [NMPZ], [ZS]. However, it is not known that the discrete version of these integrals are conservation laws of the LCN scheme.

It is important to say here that, in the numerical examples shown in next section, the discrete conservation quantites were numerically computed. The precision in the computations is of the order 10^{-13}, i.e.

$$\left|\sum_{m=1}^{M}\left(|u_m^{n+1}|^2 + |v_m^{n+1}|^2\right) - \sum_{m=1}^{M}\left(|u_m^0|^2 + |v_m^0|^2\right)\right| \leq k \times 10^{-13},$$

where k is some constant.

Remark: The stability of a numerical scheme is just uniform boundness for any fixed time with respect to n. Define

$$U = (u_1^n, v_1^n, u_2^n, v_2^n, \ldots, u_M^n, v_M^n)^t.$$

Then, the scheme is stable if

$$||U^n|| \leq C(T)||U^0||,$$

for every n and t in $[0,T]$, and any fixed, but arbitrary, T. However, our discrete conservation laws (equations (1.12) and (1.13)) tell us that $C(T) = 1$, and that the inequality is actually an equality, namely

$$||U^n|| = ||U^0||.$$

This proves the nonlinear stability of the LCN scheme, including the case of nonconstant potentials. I thank Dr. M. Brio for pointing out this issue to me.

1.5 Numerical Examples

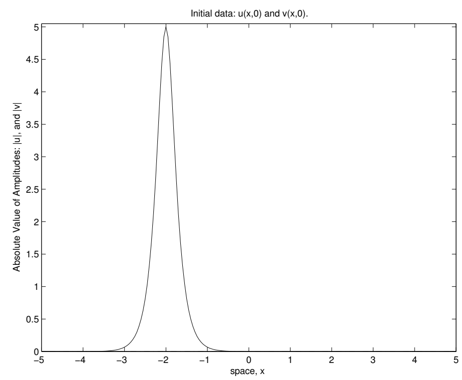

FIGURE 1.1. Initial data $u(x, 0)$ of a one-soliton solution.

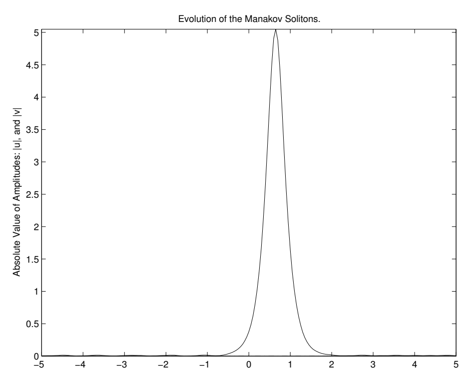

FIGURE 1.2. Evolution of a soliton after a reduction to the NLS equation of the Manakov system, at $t = 0.5$.

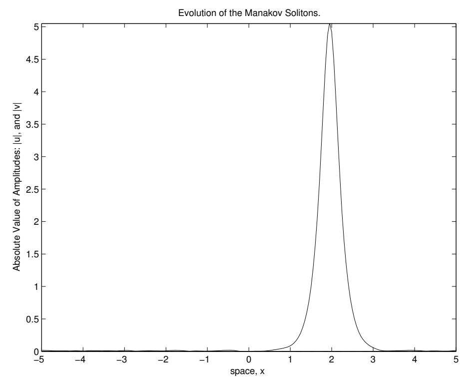

FIGURE 1.3. Evolution of a soliton after a reduction to the NLS equation of the Manakov system, at $t = 1.0$.

In this section, we test the LCN numerical scheme in different contexts. We will test it for the case $v(x,t) = 0$, as well when $u(x,t) = v(x,t) = k\ \mathrm{sech}(kx)e^{ik^2t}$; in both instances, the Manakov system reduces to the NLS equation.

We will also consider the stationary solutions of the Manakov system, found by the Darboux Transform (see appendix A)

$$\begin{aligned} u_0(x) &= 2\sqrt{|k_2^2 - k_1^2|}\frac{k_1 f_2}{[f_1, f_2]} \\ v_0(x) &= 2\sqrt{|k_2^2 - k_1^2|}\frac{k_2 f_1}{[f_1, f_2]}, \end{aligned} \tag{1.14}$$

where

$$\begin{aligned} f_1(x) &= 2\cosh k_1 x, \\ f_2(x) &= 2\sinh k_2(x-a); \end{aligned}$$

$$\begin{aligned} k_1^2 &\equiv \omega_1 \\ k_1^2 &\equiv \omega_2, \end{aligned}$$

with a an arbitrary constant, and $[f_1, f_2]$ represents the wronskian of f_1 and f_2. We will then add a parabolic external potential to Manakov, and we will observe how these solutions oscillate about the minimum of the potential.

We will also perform some numerical simulations for the case of squared barrier potentials. We define our basic barrier as follows

$$F_L(x) = H(x+L)H(x-L) = \begin{cases} 1, & |x| \leq L \\ 0, & |x| > L, \end{cases}$$

with $H(x)$ being the Heaviside function. We will then consider the evolution of the following initial conditions under the Manakov system. The constants p_0 and q_0 are taken to be real.

(a) Potentials with same support:

$u(x,0) = p_0 F_L(x)$,

$v(x,0) = q_0 F_L(x)$.

(b) Potentials with overlapping support:

$u(x,0) = p_0 F_{L_1}(x)$,

$v(x,0) = q_0 F_{L_2}(x)$, with $L_1 \neq L_2$.

(c) Potentials with disjoint support:

$u(x,0) = p_0 F_L(x - x_0)$,

$v(x,0) = q_0 F_L(x + x_0)$, where $x_0 > L$.

Notice that on (c), $v(x,0) = 0$ wherever $u(x,0) \neq 0$, and vice versa. I.e. $\mathrm{supp}(u(x,0)) \bigcap \mathrm{supp}(v(x,0)) = \emptyset$. In Chapter 2, we will prove that for cases (a) and (c), the Manakov-Zakharov-Shabat system has purely imaginary eigenvalues, while in Chapter 3, we will give the asymptotic solution (at small times) of the Manakov system for the same initial data.

1.5.1 Potentials with Same Amplitude: Reduction to the NLS Equation

If we set $v(x,t) = 0$ in the Manakov system, we obtain the NLS equation. We just need to put $v_m^0 = 0$, for all m, and we obtain the numerical solution of the NLS.

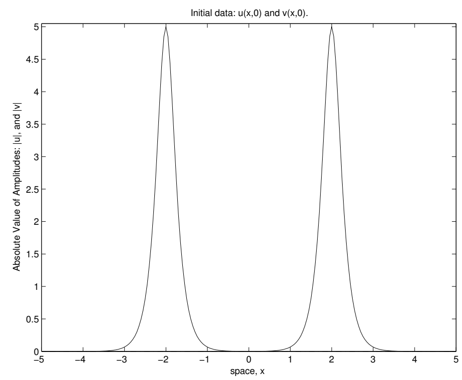

FIGURE 1.4. Initial data $u(x,0)$ and $v(x,0)$ after a NLS reduction of the Manakov system.

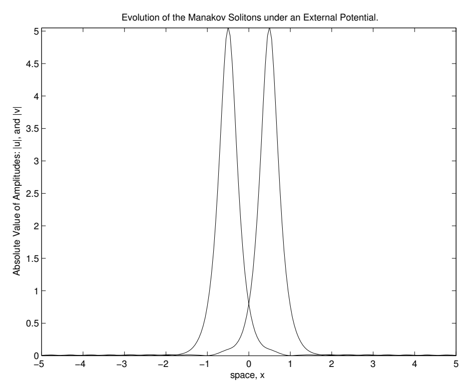

FIGURE 1.5. Evolution of two Manakov solitons after a NLS reduction of the Manakov system: $u(x,t) = v(x,t)$, at $t = 0.47$.

See sequence of figures from (1.1) through (1.3). They show the evolution of an NLS soliton

$$u(x,t) = k\ \mathrm{sech}(k(x - v_0 t))e^{-iv_0 kx/2}e^{ik^2 t},$$

where v_0 is the velocity of propagation, and k is the amplitude of the soliton.

Now set $u(x,t) = v(x,t)$, we again obtain the NLS equation. We need to consider the following discrete initial conditions $u_m^0 = v_m^0 = k\ \mathrm{sech}(kx_m)$. In the sequence of figures (1.4) to (1.8), we show the collision of two Manakov solitons with same

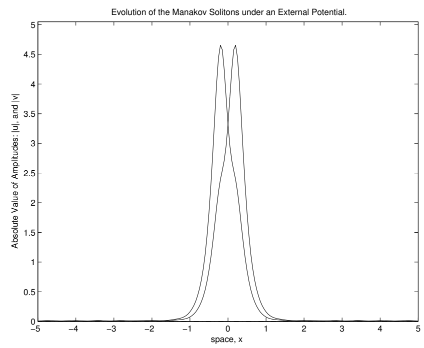

FIGURE 1.6. Collission between two Manakov solitons with same amplitudes, $t = 0.5$.

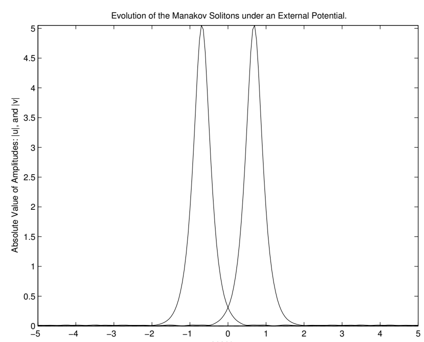

FIGURE 1.7. Collission between two Manakov solitons with same amplitudes, $t = 0.56$.

amplitudes. The initial data are

$$\begin{aligned} u(x,0) &= k \operatorname{sech}(k(x-x_0))e^{iv_0kx/2} \\ v(x,0) &= k \operatorname{sech}(k(x+x_0))e^{-iv_0kx/2} \end{aligned}$$

where $x_0 = \pm 2$ denotes the initial location of the solitons, with initial velocities $v_0 = \pm 5$. This waves will propagate, interact and then finally separate, as shown in the figures (1.4) through (1.8). The solutions behave like two independent solitons

$$\begin{aligned} u(x,t) &= k \operatorname{sech}(k(x-x_0-v_0t))e^{iv_0kx/2} \\ v(x,t) &= k \operatorname{sech}(k(x+x_0+v_0t))e^{-iv_0kx/2} \end{aligned}$$

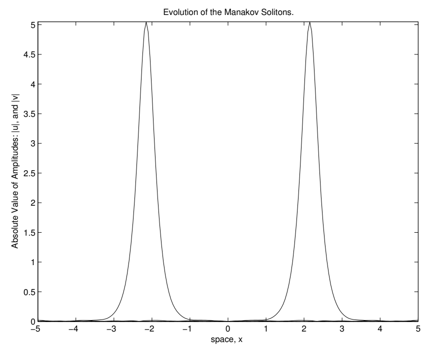

FIGURE 1.8. Collission between two Manakov solitons with same amplitudes, $t = 0.63$.

1.5.2 Potentials with Different Amplitudes

Now we consider the solutions given by the following initial data (see appendix A)

$$\begin{aligned} u_0(x) &= 2\sqrt{|k_2^2 - k_1^2|}\frac{k_1 f_2}{[f_1, f_2]} e^{iv_0 k_2 x/2} \\ v_0(x) &= 2\sqrt{|k_2^2 - k_1^2|}\frac{k_2 f_1}{[f_1, f_2]} e^{-iv_0 k_1 x/2}, \end{aligned} \tag{1.15}$$

where

$$\begin{aligned} f_1(x) &= 2\cosh k_1(x+a), \\ f_2(x) &= 2\sinh k_2(x-a); \end{aligned}$$

$$\begin{aligned} k_1^2 &\equiv \omega_1 \\ k_1^2 &\equiv \omega_2, \end{aligned}$$

with a, v_0 are arbitrary constants, and $[f_1, f_2]$ represents the wronskian of f_1 and f_2. The exact solutions are given by

$$\begin{aligned} u(x,t) &= 2\sqrt{|k_2^2 - k_1^2|}\frac{k_1 F_2}{[F_1, F_2]} e^{iv_0 k_2 x/2} e^{ik_2^2 t} \\ v(x,t) &= 2\sqrt{|k_2^2 - k_1^2|}\frac{k_2 F_1}{[F_1, F_2]} e^{-iv_0 k_1 x/2} e^{ik_1^2 t}, \end{aligned} \tag{1.16}$$

but here

$$
\begin{aligned}
F_1(x,t) &= 2\cosh k_1(x+a+v_0t), \\
F_2(x,t) &= 2\sinh k_2(x-a-v_0t).
\end{aligned}
$$

From figure (1.9) to (1.14), we observe the collision between these two solitons.

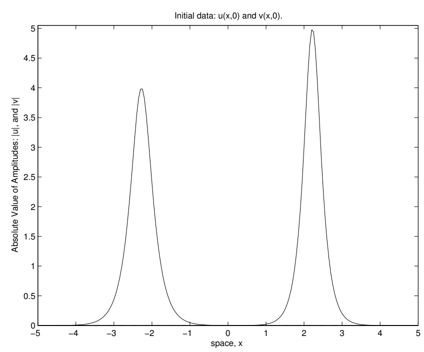

FIGURE 1.9. Initial data $u(x,0)$ and $v(x,0)$.

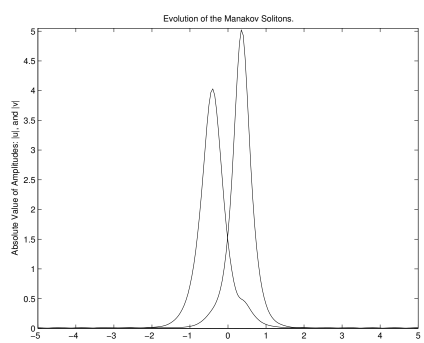

FIGURE 1.10. Collission between two Manakov solitons with different amplitudes, at $t = 0.44$.

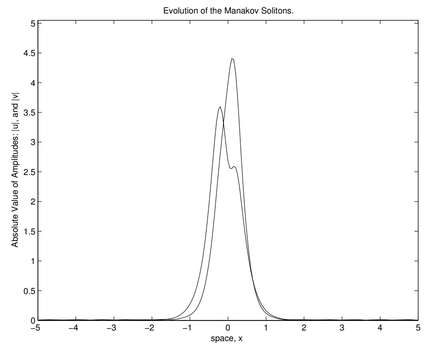

FIGURE 1.11. Collission between two Manakov solitons with different amplitudes, at $t = 0.5$.

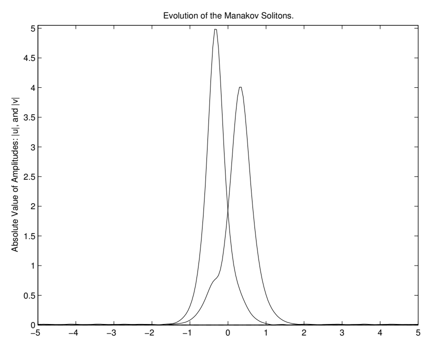

FIGURE 1.12. Collission between two Manakov solitons with different amplitudes, at $t = 0.56$.

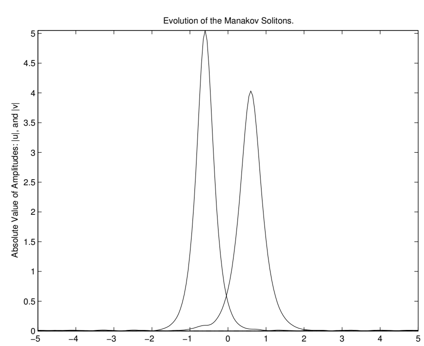

FIGURE 1.13. Collission between two Manakov solitons with different amplitudes, at $t = 0.63$.

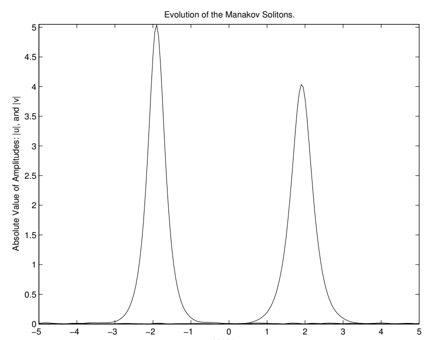

FIGURE 1.14. Collission between two Manakov solitons with different amplitudes, at $t = 1.0$.

1.5.3 Numerical Solutions with an External Potential

What it follows is a sequence of figures (from figure (1.15) through figure (1.23)) which shows the interaction between two Manakov solitons under the influence of an external potential. This dynamics for the NLS equation was studied by J. Bronski and R. Jerrard, [Bro], and L. Jonsson *et.al.* [Jon]. As far as one can tell, there is no analytical description of this phenomenon of interaction between two Manakov solitons, together with the corresponding oscillations in the potential.

To set ideas, we use a parabolic potential

$$V(x) = h^2 x^2,$$

with $h = \frac{1}{10}$. If we think of this potential as a classical harmonic oscillator, the frequency and the period are very easily computed

$$\text{frequency} = \sqrt{\frac{2}{\text{mass}}}\, h, \qquad\qquad \text{period} = \frac{2\pi}{\text{frequency}}.$$

Actually, and in analogy with the quantum model, the mass turn to be $\frac{1}{2}$, and the period is 31.4159. However, we observe in Figure (1.23), that the pulses return to their initial position after $t \approx 30$, not at $t = 31.4159$. This is an effect of the nonlinear interactions, causing a "speedup" of the solitons, and a shortening in the period[1]. For wide enough potentials (compared with the soliton width), the phase shift can be exactly (analytically) and numerically computed. Then compare this values and check if the numerical and the "nonlinear" periods coincide.

As it was already mentioned, this has not been investigated yet, and could be a project for further research. For the description of this model in the NLS case, look at [Bro] and [Jon], and references therein. It would be interesting to extend the analysis presented in these two references to the Manakov system.

[1] Thanks to P. Miller for this observation [Mi2].

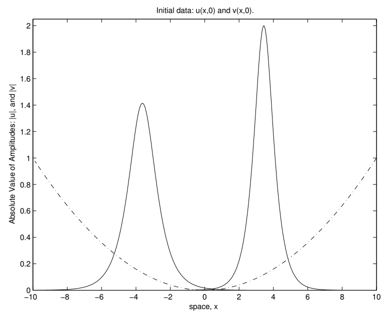

FIGURE 1.15. Initial data $u(x, 0)$ and $v(x, 0)$ under an external potential.

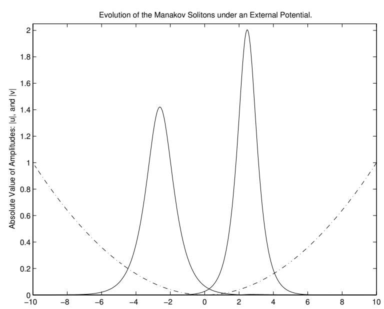

FIGURE 1.16. Evolution of the Manakov solitons under an External Potential, at $t = 3.75$.

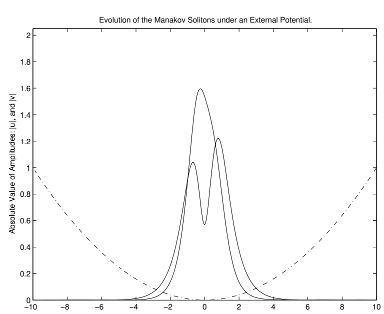

FIGURE 1.17. Evolution of the Manakov solitons under an External Potential, at $t = 7.50$.

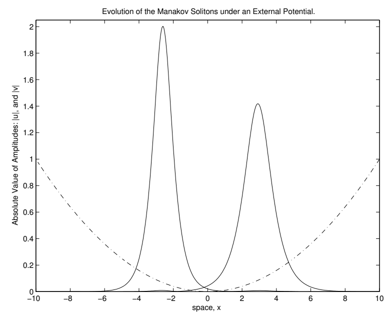

FIGURE 1.18. Evolution of the Manakov solitons under an External Potential, at $t = 11.25$.

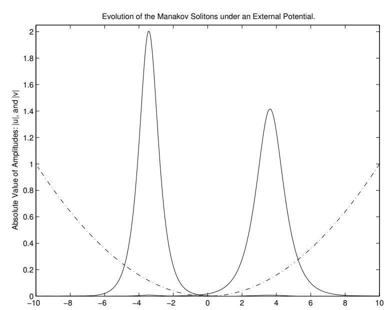

FIGURE 1.19. Evolution of the Manakov solitons under an External Potential, at $t = 15.00$.

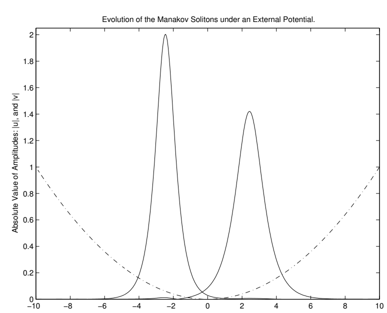

FIGURE 1.20. Evolution of the Manakov solitons under an External Potential, at $t = 18.75$.

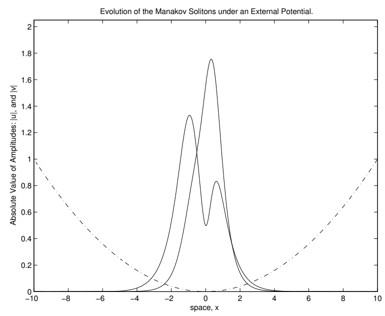

FIGURE 1.21. Evolution of the Manakov solitons under an External Potential, at $t = 22.50$.

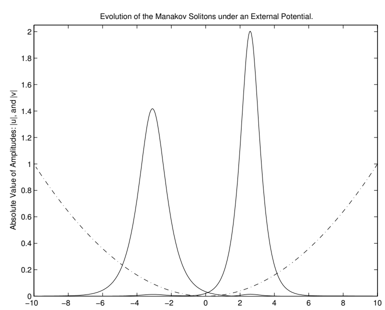

FIGURE 1.22. Evolution of the Manakov solitons under an External Potential, at $t = 26.25$.

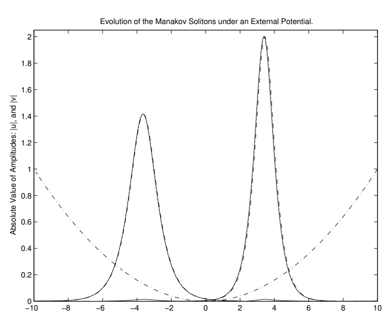

FIGURE 1.23. Evolution of the Manakov solitons under an External Potential, at $t = 30.00$.

1.6 The Gibbs Phenomenon in the Focusing Manakov System

The Gibbs phenomenon is an effect that appears in Fourier series expansions at simple discontinuities, like jumps. In the case of the defocusing Nonlinear Schrödinger equation, it appears in the case of squared barrier potentials [DiF]. Here, we show several numerical examples of the same phenomenon, but in the focusing Manakov system.

To finish our list of numerical solutions of the Manakov system, we include initial conditions of the squared barrier form. The basic form is

$$F_L(x) = H(x+L)H(x-L) = \begin{cases} 1, & |x| \leq L \\ 0, & |x| > L, \end{cases}$$

where $H(x)$ is the Heaviside function. We will then consider the evolution of the following initial conditions under the Manakov system. The constants p_0 and q_0 are real numbers.

For all the numerical experiments shown in the current and next sections (sections 1.6 and 1.7), the initial conditions are smoothed at neighborhoods of the jumps by means of the *Friedrichs mollifier*:

$$FM(x) = e^{1/b^2} e^{-1/(b^2-x^2)}.$$

It actually turns to happen that the Friedrichs mollifier also shows a Gibbs-type phenomenon behavior. See Figures 1.24 and 1.25.

1.6.1 Potentials with Same Support

In this subsection, we observe the evolution of the initial conditions

$$\begin{aligned} u(x,0) &= p_0 F_L(x), \\ v(x,0) &= q_0 F_L(x). \end{aligned}$$

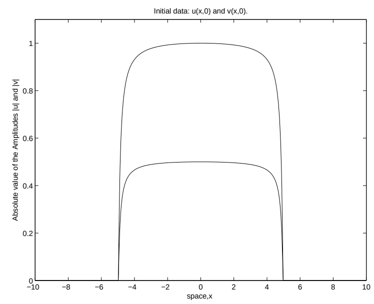

FIGURE 1.24. Friedrichs mollifier initial conditions: $FM(x) = e^{1/b^2}e^{-1/(b^2-x^2)}$. Here: $b = 5$

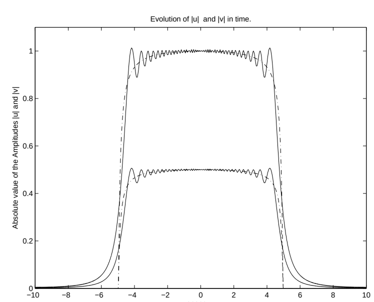

FIGURE 1.25. Evolution of the Friedrichs mollifier at $t = 0.1$

After $t = 10^{-4}$, the evolution of this potentials is shown in figure (1.27). The initial condition is plotted in figure (1.26). The asymptotic description of this solution will be studied in chapter 6.

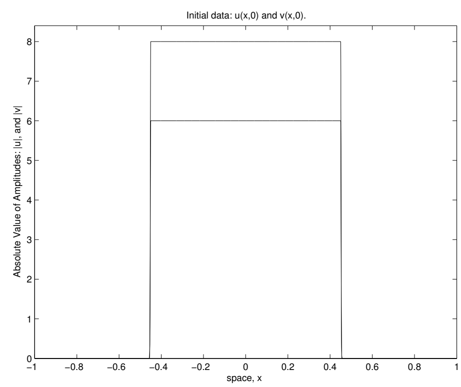

FIGURE 1.26. Initial data of the potentials $u(x, 0)$ and $v(x, 0)$ with same support. Here, $p_0 = 1.5$, $q_0 = 1$, $L = 0.45$

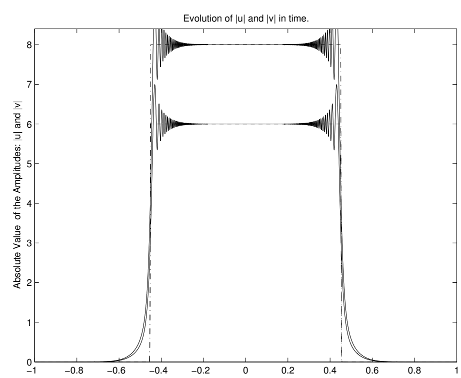

FIGURE 1.27. Evolution of the potentials $u(x, 0)$ and $v(x, 0)$ with same support, at $t = 10^{-4}$.

1.6.2 Potentials with Disjoint Supports

Consider know the case when the two initial potentials have disjoint supports

$$u(x,0) = p_0 F_L(x - x_0)$$
$$v(x,0) = q_0 F_L(x + x_0),$$

where $x_0 > L$. See figure (1.28). Once the pulses evolve in time, they also show a Gibbs phenomenon on each pulse, figure (1.29). Since the pulses a far away during the first moments of evolution, they can be approximated by the NLS approximation [DiF]. We briefly mention the NLS asymptotics of this phenomenon also in Chapter 3.

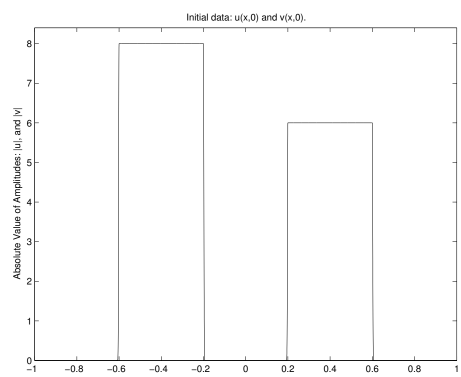

FIGURE 1.28. Initial data of the potentials $u(x,0)$ and $v(x,0)$ with disjoint support. Here, $p_0 = 1.5$, $q_0 = 1$, $L = 0.2$ and $x_0 = 0.4$

1.6.3 Potentials with Overlapping Supports

Take one of the barriers in the previous section, and change its support. We do this for the potential $v(x,0)$, to obtain

$$\begin{aligned} u(x,0) &= p_0 F_{L_1}(x) \\ v(x,0) &= q_0 F_{L_2}(x), \end{aligned}$$

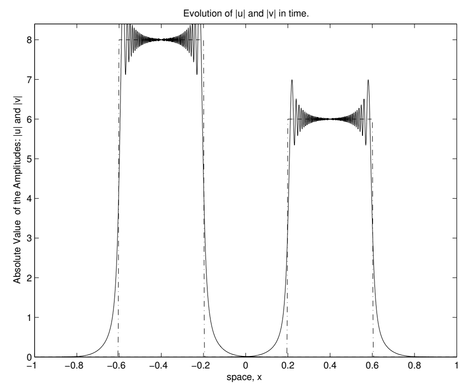

FIGURE 1.29. Evolution of the potentials $u(x, 0)$ and $v(x, 0)$ with disjoint support, at $t = 10^{-4}$.

with $L_1 \neq L_2$. It is a remarkable fact that, although it is a coupled system of equations, there is no interation between the pulses at short times. In figure (1.30), we observe the initial data, and figure (1.31) shows that there is no interaction between the pulses. We first observed this phenomenon through this numerical simulation, and then we performed the analytical study. We include this asymptotic analysis in Chapter 6.

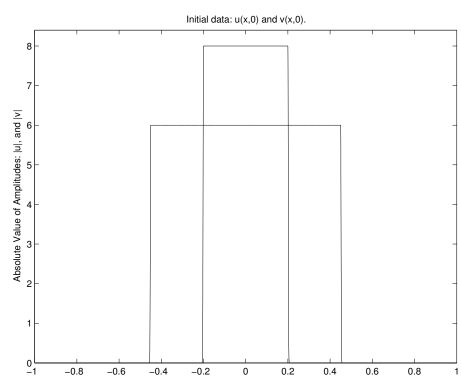

FIGURE 1.30. Initial data of the potentials $u(x, 0)$ and $v(x, 0)$ with overlapping support. Here, $p_0 = 1.5$, $q_0 = 1$, and $L_1 = 0.2$ and $L_2 = 0.45$

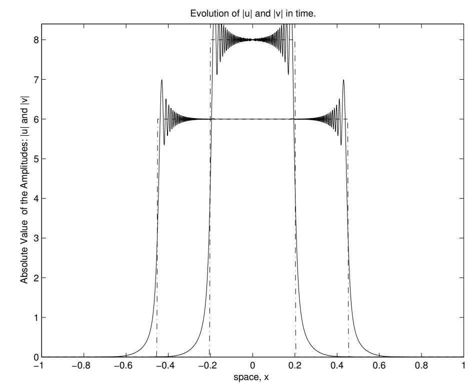

FIGURE 1.31. Evolution of the potentials $u(x, 0)$ and $v(x, 0)$ with overlapping support, at $t = 10^{-4}$.

1.7 Numerical Behaviour of the Defocusing Manakov System.

In this section, we consider the *defocusing Manakov system*, $\sigma = -1$, in equations (1.1). The initial conditions considered in this section are given in figures (1.26), (1.30), and (1.28), where the initial potentials have: same, overlapping and disjoint supports, respectively. In figures (1.32), (1.33) and (1.34), the evolution of these initial potentials is plotted for $t = 10^{-4}$.

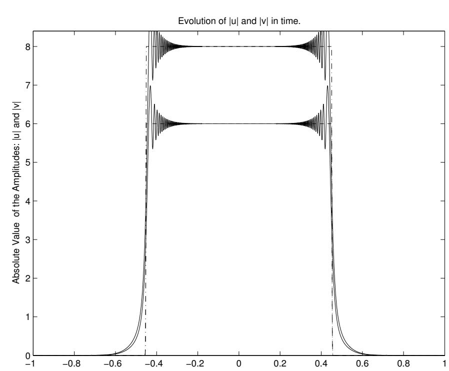

FIGURE 1.32. Evolution of the squared barrier initial potentials (with same supports) under the *defocusing* Manakov system, at $t = 10^{-4}$.

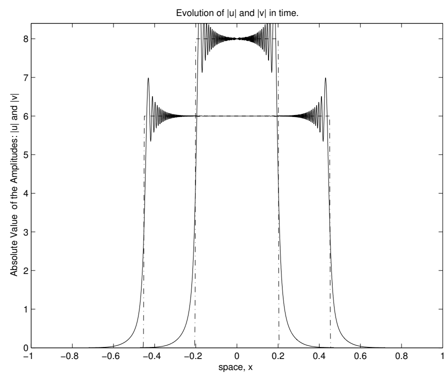

FIGURE 1.33. Evolution of the squared barrier initial potentials (with overlapping supports) under the *defocusing* Manakov system, at $t = 10^{-4}$.

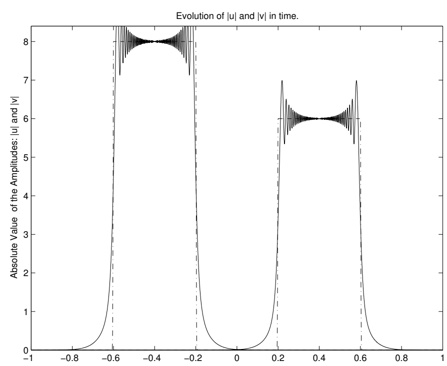

FIGURE 1.34. Evolution of the squared barrier initial potentials (with disjoint supports), under the *defocusing* Manakov system, at $t = 10^{-4}$.

We observe that the behaviour is similar as in the focusing Manakov system, shown in figures (1.27), (1.31), and (1.29). These numerical observations led us to conjecture that the discrete spectrum does not have a predominant role in the solutions of the focusing Manakov system. In chapter 6, we will analytically prove this statement.

1.8 Why do I believe in these Numerical Observations?

In this section, we have a sequence of *close-ups* to figure 1.27, which we reproduce here, in figure 1.35. As we previously said, in this numerical experiment the final time is 10^{-4}. the time and space step are $\Delta t = 10^{-7}$ and $\Delta x = 3.1623 \times 10^{-4}$ respectively.

In this sequence of pictures, we get closer and closer to the oscillations. Each figure has the appropriate scales. We observe that the oscillations evolves very smoothly, and the only figure that shows pieces of lines (due to the numerical approximation) is figure 1.41. We can observe that the range of the period in the oscillations varies from about 0.025 (see figures 1.38 and 1.39) to 0.005 (see figure 1.41).

Some other simulations were performed: at $t = 0.00005$ and at $t = 0.00001$. Both of them also showed the same type of behavior as the behaviour presented here. Although it is missing a comparison between the numerical solutions and the asymptotic *Erf*-type of solutions, it is a detail that should be reviewed and checked carefully. We expect that the "difference" between the numerical an dthe asymptotic solutions should be "small", of the order $\sqrt{t}$.

A natural question to be asked here is why one should choose the LCN scheme as the scheme to numerically study the Gibbs-type-phenomenon, as opposed to some other numerical schemes. My reason for choosing LCN was simply that it was a very easy scheme to implement. It gets realied as a linear system of equations, and just a matrix has to be inverted at each time-step. This is what is called a semi-implicit method. In the comparison paper by [Cha], we find that the LCN was as fast and precise (order (2,2)) as the regular Crank-Nicolson scheme. It turned to be faster and more precise than some other numerical schemes.

However, it would be interesting to use some other numerical schemes like regular Crank-Nicolson, pseudo-spectral Crank-Nicolson (Ceniceros-Tian [CT]) or the traditional split-step method, and see if one out of the several numerical schemes describe better the Gibbs-type phenomenon, or some other phenomena with discontinuities.

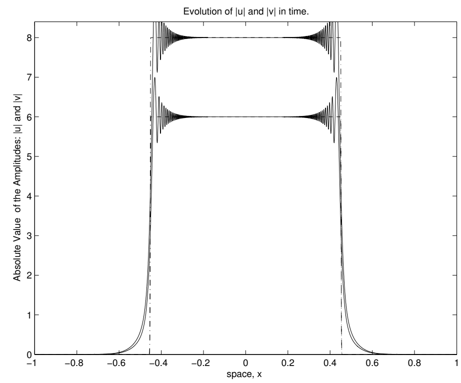

FIGURE 1.35. Evolution of the potentials $u(x, 0)$ and $v(x, 0)$ with same support, at $t = 10^{-4}$.

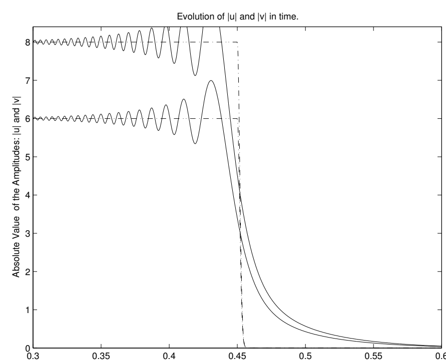

FIGURE 1.36. Close-up, at $t = 10^{-4}$.

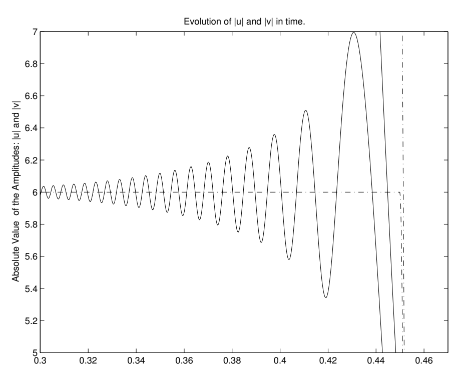

FIGURE 1.37. Close-up, at $t = 10^{-4}$.

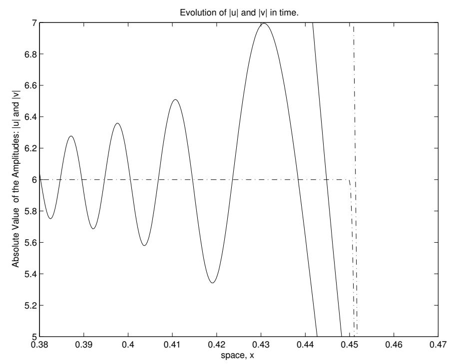

FIGURE 1.38. Close-up, at $t = 10^{-4}$.

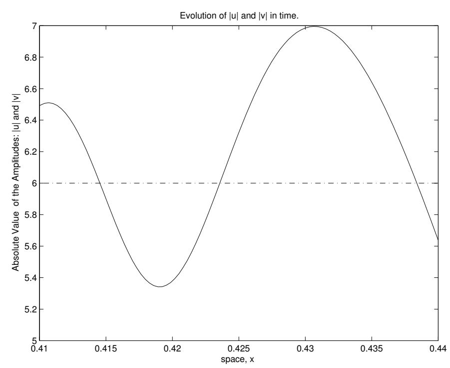

FIGURE 1.39. Close-up, at $t = 10^{-4}$.

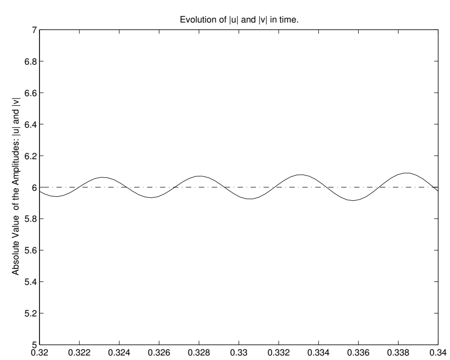

FIGURE 1.40. Close-up, at $t = 10^{-4}$.

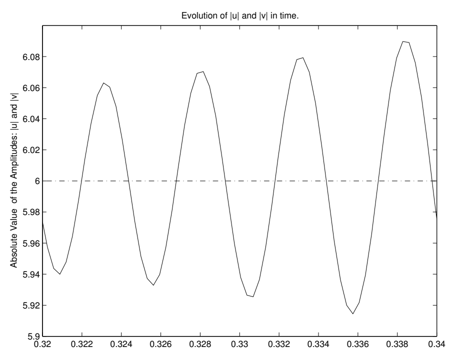

FIGURE 1.41. Close-up, at $t = 10^{-4}$.

Chapter 2

The Direct Scattering (Spectral) Problem

2.0 Introduction

For some classes of nonlinear Partial Differential Equations (PDEs), called *evolution equations*, a technique has been invented to exactly solve these classes of equations. In the literature, this technique is called the Inverse Scattering Method (ISM) [AKNS], [AS], [GGKM], [New], [NMPZ], [ZS]. This method involves a transformation between solutions of a PDE, and the spectral data of an associated linear eigenvalue problem. Nonlinear evolution equations can then be managed in a manner that parallels the treatment of linear PDEs by the Fourier Transform. Actually, the Fourier Transform is obtained from the Inverse Scattering correspondence in the limit of small amplitudes. For this reason, the ISM is viewed as an extension of the Fourier Transform to nonlinear problems, and the method is also called the Inverse Scattering Transform (IST). The evolution equations solved by the IST are named *integrable soliton equations.*

It is well known that the NLS equation and the Manakov system are equations of evolution, completly integrable by the IST [Ma], [ZS], among several other equations [AKNS], [AKNS2], [AS], [APT], [NMPZ]. In this Chapter, as well as in Chapter 5, the Scattering and Inverse Scattering associated to the Manakov system are considered. This will extend the method of solution of the NLS equation to the Manakov system..

For the Manakov system, and the NLS equation as well, the inverse problem is performed by the Riemann-Hilbert problem approach (chapter 5). Here, we will describe the general setting of the direct scattering (or spectral) problem. Next chapter will be dedicated to find the spectral information associated to squared barrier potentials with same, overlapping and disjoint supports. Chapter 6 will consider the short-time asymptotics of the Manakov system via the RHP approach.

2.1 The Manakov-Zakharov-Shabat System: The Spectral Problem

In 1974, S.V. Manakov showed [Ma] that the system

$$\begin{aligned} iu_t + \frac{1}{2}u_{xx} + \sigma(|u|^2 + |v|^2)u &= 0 \\ iv_t + \frac{1}{2}v_{xx} + \sigma(|u|^2 + |v|^2)v &= 0, \end{aligned} \tag{2.1}$$

is completly integrable. Manakov proved this assertion [Ma] by showing that the system (2.1) has the following *associated pair* of differential equations

$$\frac{\partial}{\partial x}\Psi = \mathcal{L}\Psi, \tag{2.2}$$

$$i\frac{\partial}{\partial t}\Psi = \mathcal{B}\Psi, \tag{2.3}$$

where Ψ is the 3×3 fundamental matrix solution of the system. The operators $\mathcal{L}$ and $\mathcal{B}$ are 3×3 linear operators given by

$$\mathcal{L} \equiv iED + N, \tag{2.4}$$

$$\mathcal{B} \equiv iE\mathcal{L} + \frac{1}{2}D(N_x - N^2), \tag{2.5}$$

where E is the spectral parameter, and D and N are 3×3 matrices given by

$$\begin{aligned} D &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ N = N(x,t) &= \begin{pmatrix} 0 & u(x,t) & v(x,t) \\ -\sigma u^\star(x,t) & 0 & 0 \\ -\sigma v^\star(x,t) & 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.6}$$

Here, $u^*(x,t)$, $v^*(x,t)$ are the complex conjugates of the solutions of the Manakov system. The operators $\mathcal{L}$ and $\mathcal{B}$ are known as **the Lax pair** of the Manakov system.

The system (2.2) and (2.3) is an overdetermined system for Ψ. We should then have an extra condition on them, a compatibility condition between the equations, namely

$$\frac{\partial}{\partial x}\frac{\partial}{\partial t}\Psi = \frac{\partial}{\partial t}\frac{\partial}{\partial x}\Psi.$$

Using (2.2), (2.3), and provided that the spectral parameter is time-independent, $\frac{d}{dt}E = 0$, this condition becomes

$$i\mathcal{L}_t - \mathcal{B}_x + [\mathcal{L}, \mathcal{B}] = 0, \tag{2.7}$$

where $[\mathcal{L}, \mathcal{B}] = \mathcal{L}\mathcal{B} - \mathcal{B}\mathcal{L}$ is the *commutator* of $\mathcal{L}$ and $\mathcal{B}$. This last equation is named *the Lax equation.* The Lax equation (2.7), in this instance, is equivalent to

$$iN_t - \frac{1}{2}DN_{xx} + DN^3 = 0,$$

which is the matrix form of the Manakov system. We then see that the Manakov system is the compatibility condition of the equations (2.2) and (2.3). On the other hand, the fact that the spectral parameter is time-independent is the reason this transformation is called an *isospectral transformation.*

Equation (2.2) is of fundamental importance, since it yields to the required information (namely, the Scattering Data) to solve the Manakov system. This equation can be put in the form

$$\frac{\partial}{\partial x}\Psi = iED\Psi + N\Psi, \tag{2.8}$$

which we will term **the Manakov-Zakharov-Shabat** (MZS) system, which is the 3×3 matrix genrealization of the Zakharov-Shabat (ZS) system associated to the NLS equation. Equation (2.3) is usually refered as the *t-flow equation* or the *time-dependence* equation.

Note on nomenclature. We alternatively use the term *potentials* of the MZS system, for the solutions of the Manakov system. This is due to the fact that $N(x,t)$ plays the same role of a potential in the MZS system, as the potential does in the linear Schrödinger equation of quantum mechanics.

In this dissertation, we concentrate on the focusing Manakov system, $\sigma = +1$. In this instance, the operator $\mathcal{L}$ is non self-adjoint, and its eigenvalues could be located anywhere in the E-complex plane (as oppossed to the defocusing case, on which the eigenvalues are located on the real axis, due to the fact that the operator $\mathcal{L}$ is self-adjoint.)

Let us assume $u(x,t)$ and $v(x,t)$ to be integrable functions that decay fast enough when $|x| \to \infty$, i.e, $N(x,t) \to 0$ as $|x| \to \infty$ rapidly. Then, for $E \in \mathbb{R}$, it can be shown that there are two different sets of solutions of the MZS system, $\Psi(x,E)$ and $\Phi(x,E)$, according to their asymptotic behavior

$$\Psi(x,E) \sim e^{iEDx}, \qquad \text{as} \qquad x \to +\infty \tag{2.9}$$

$$\Phi(x,E) \sim e^{iEDx}, \qquad \text{as} \qquad x \to -\infty. \tag{2.10}$$

The matrix functions $\Psi(x,E)$ and $\Phi(x,E)$ are called **the Jost solutions** of the MZS system. These fundamental solutions are actually interdependent, and they are related by the **Scattering Matrix**, $S(E)$

$$\Phi(x,E) = \Psi(x,E)S(E). \tag{2.11}$$

Now, an **eigenvalue**, E_k, is defined as a complex number such that there exists a *solution of the MZS system* $\phi(x,E_k)$ (corresponding to E_k), such that

$$\lim_{x\to\pm\infty} \phi(x,E_k) = 0, \tag{2.12}$$

which is called an **eigenfunction** of the MZS system. What this means is the following. We would like to find a value of E, namely, $E = E_k$, such that it matches some solution of the MZS that decays at $-\infty$, with some other solution that decays at $+\infty$. Shortly, we will see how this solution ϕ is given in terms of columns of Φ and Ψ.

The asymptotic behaviour (2.9) and (2.10), together with the definition of the scattering matrix, equation (2.11), and the definition of eigenvalue (2.12), give rise to the equation

$$S_{11}(E_k) = 0,$$

where $S_{11}(E)$ is the first entry of the scattering matrix. We can also check that the *eigenfunction* $\phi(x, E_j)$, corresponding to E_j, is actually the first column of the 3×3 fundamental matrix solution $\Phi(x, E_j)$. Let us do this in detail.

To this end, we now proceed to precisely define the concept of *eigenvalue* of the MZS system, i.e., we will give a precise form to compute the eigenvalues. Using the fact that the columns of $\Psi(x, E)$ are linearly independent, and using equation (2.11) we obtain that

$$\begin{aligned} \det(\Phi_1, \Psi_2, \Psi_3) &= \det(S_{11}(E)\Psi_1 + S_{21}(E)\Psi_2 + S_{31}(E)\Psi_3, \Psi_2, \Psi_3) \\ &= S_{11}(E) \det(\Psi), \end{aligned}$$

and from this equation, we find that the first entry of the Scattering matrix can be written as follows

$$S_{11}(E) = \frac{1}{\det(\Psi)} \det(\Phi_1, \Psi_2, \Psi_3). \tag{2.13}$$

In the appendix B, we will show that the columns Φ_1, Ψ_2, and Ψ_3 can be analytically continued into the upper-half complex plane, $\mathbb{C}^+$. (For this issue, [BC] or [DZ] can be also checked, and the references therein). Also, $\det(\Psi) = e^{iEx}$ is an analytic function of E. Hence, $S_{11}(E)$ is an analytic function of E in $\mathbb{C}^+$.

Now, if $E = E_k$ (with $\text{Im}(E_k) \geq 0$) is a solution of the equation

$$S_{11}(E_k) = 0,$$

then, the columns Φ_1, Ψ_2, and Ψ_3 are linearly dependent. This introduces the existence of two constants, C_k and D_k such that

$$\Phi_1(x; E_k) = C_k \Psi_2(x; E_k) + D_k \Psi_3(x; E_k). \tag{2.14}$$

These constants are known under the name of **proportionality** (or **normalization**) **constants**. Using this equation, and asumming that $\mathrm{Im}(E_k) > 0$, we verify that

$$\lim_{x \to \pm\infty} \Phi_1(x; E_k) = 0, \tag{2.15}$$

i.e., $\Phi_1(x; E_k)$ is eigenfunction with corresponding eigenvalue E_k. Observe that equation (2.14) is the matching we were refering above, after equation (2.12).

We then state the precise definition of eigenvalue

Definition 2.1. *We say that $E = E_k$ is an* **eigenvalue** *of the MZS system, if $E = E_k$ with $Im(E_k) > 0$ solves the equation*

$$S_{11}(E_k) = 0, \tag{2.16}$$

We can also have that $\mathrm{Im}(E_k) = 0$. However, the asymptotic behaviour in equation (2.15) fails to be true. We say we have a *limiting* or *emerging eigenvalue*, but we will not consider this type of eigenvalues in this thesis.

The number of zeros of the equation (2.16) depends on the type of potentials of the MZS system. Requiring that these potentials decay very fast (exponentially or faster), it can be proved that the number of eigenvalues is finite: $k = 1, 2, 3, \ldots, n$. In particular, this happens for squared barrier potentials (see chapter 3).

The quantity $1/S_{11}(E)$ is known as the *transmission coefficient*. For simplicity, however, we will use this name for the first entry of the scattering matrix, namely, $S_{11}(E)$.

The eigenvalues E_k, the constants of proportionality C_k and D_k (k =1, 2, ..., n), together with the reflection coefficients (defined for $E \in \mathbb{R}$)

$$R_1(E) \equiv \frac{S_{21}(E)}{S_{11}(E)} \qquad \text{and} \qquad R_2(E) \equiv \frac{S_{31}(E)}{S_{11}(E)}, \tag{2.17}$$

are called **the Scattering Data (SD)**. The Scattering Data evolve in time, and this evolution can be easily computed (see ahead in section 2.2). From this information, the *potentials* $u(x,t)$ and $v(x,t)$ of the MZS system can be recovered. This is to say, we have solved the Manakov system from the information given by the SD. This will be performed in Chapter 5, by solving a Riemann-Hilbert Problem.

As we previously mentioned, the eigenvalues are time-independent, i.e., they are constants of motion

$$\frac{d}{dt}E_j = 0.$$

However, the Scattering matrix is in fact time-*dependent*

$$\Phi(x,t;E) = \Psi(x,t;E)S(E;t), \tag{2.18}$$

whose temporal evolution can be determinded using the t-flow equation. This allows us to know how the reflection coefficients (2.17), as well as the proportionality constansts, change in time: $R_1 = R_1(E;t)$ and $R_2 = R_2(E;t)$, $C_k = C_k(t)$ $D_k = D_k(t)$ $(k = 1, 2, \ldots, n)$. The form they evolve in time is found in the following section.

Properties of the Scattering and the Fundamental Matrices: Focusing Case It is important to mention here that the Scattering matrix has the following two important properties

$$\begin{aligned} \det S(E;t) &= 1, \\ S(E;t)S^*(E;t) &= I. \end{aligned}$$

This is to say, the Scattering matrix is *unimodular* and *unitary.*

Similarly, the fundamental matrices Ψ and Φ are also unitary matrices, but not unimodular

$$\begin{aligned} \det \Psi(x,t;E) &= e^{iEx}, \\ \det \Phi(x,t;E) &= e^{iEx}, \\ \Psi(x,t;E)\Psi^*(x,t;E) &= I, \\ \Phi(x,t;E)\Phi^*(x,t;E) &= I. \end{aligned}$$

Properties of the Scattering and the Fundamental Matrices: Defocusing Case On the other hand, in the **defocusing** *case*, the Scattering matrix and the fundamental matrices have similar properties

$$\begin{aligned} \det S(E;t) &= 1, \\ S(E;t)\, D\, S^*(E;t) &= D, \\ \det \Psi(x,t;E) &= e^{iEx}, \\ \det \Phi(x,t;E) &= e^{iEx}, \\ \Psi(E;t)\, D\, \Psi^*(E;t) &= D, \\ \Phi(E;t)\, D\, \Phi^*(E;t) &= D. \end{aligned}$$

The matrix D is defined by the equation (2.6).

2.2 Evolution in Time of the Scattering Matrix

The goal of this section will be to compute the Scattering matrix at any time t. Once this is done, the evolution in time of the reflection coefficients immediatly follows. The computation of the evolution of hte proportionality constants is performed similarly.

2.2.1 Solving the t-Flow Equation

Let us assume that we have two solutions of the MZS system for $t > 0$, $\Psi(x,t;E)$ and $\Phi(x,t;E)$, with the asymptotic behavior

$$\Psi(x,t;E) \sim e^{iEDx}, \qquad \text{as} \qquad x \to +\infty \tag{2.19}$$

$$\Phi(x,t;E) \sim e^{iEDx}, \qquad \text{as} \qquad x \to -\infty, \tag{2.20}$$

with $\Psi(x,0;E) = \Psi(x)$, $\Phi(x,0;E) = \Phi(x)$ are the functions we found in the previous section.

We also know that $\Psi(x,t;E)$ and $\Phi(x,t;E)$ are related by the time dependent Scattering matrix $S(E;t)$

$$\Phi(x,t;E) = \Psi(x,t;E)S(E;t). \tag{2.21}$$

Our task is to determine the dependence of $S(E;t)$ with respect to time t, in terms of the Scattering matrix $S(E)$ found in the previous section.

We start with the observation that, since $\Psi(x,t;E)$ solves the MZS system then the following matrix

$$\widehat{\Psi}(x,t;E) = \Psi(x,t;E)\widehat{C}(t) \tag{2.22}$$

also solves the MZS system, where $\widehat{C}(t)$ is a nondegenerate matrix. We now postulate that this function $\widehat{\Psi}(x,t;E)$ solves the t-flow equation, and try to find the appropriate $\widehat{C}(t)$ that allows us to do this. Therefore, $\widehat{\Psi}(x,t;E)$ will satisfy the MZS system and the t-flow equation simultaneously (i.e., equations (2.2) and (2.3).)

Caution: Here, we should be aware that the function $\widehat{\Psi}(x,t;E)$ does not satisfies the asymptotics (2.19).

Now, using equation (2.22), substitute $\widehat{\Psi}(x,t;E)$ into equation (2.3)

$$i\frac{\partial}{\partial t}\widehat{\Psi} = \mathcal{B}\widehat{\Psi},$$

to obtain

$$i\frac{\partial}{\partial t}\left(\Psi(x,t;E)\widehat{C}(t)\right) = \mathcal{B}\left(\Psi(x,t;E)\widehat{C}(t)\right).$$

Using the asymptotic behaviour of $\Psi(x,t;E)$ (equation (2.19)), and letting $x \to \infty$ in the last equation, we obtain

$$i\frac{\partial}{\partial t}\left(e^{iEDx}\widetilde{C}(t)\right) = -E^2D\left(e^{iEDx}\widetilde{C}(t)\right),$$

where we used the fact that

$$\begin{aligned}
\mathcal{L} &= iED + N(x,t) \longrightarrow iED, \\
\mathcal{B} &= iE\mathcal{L} + \frac{1}{2}D(N_x - N^2) \longrightarrow -E^2D,
\end{aligned}$$

when $|x| \to \infty$, since $N(x,t) \to 0$ as $|x| \to 0$ (fast enough).

Now the equation for $\widehat{C}(t)$ is

$$\frac{d}{dt}\widehat{C}(t) = iE^2 D\widehat{C}(t),$$

i.e.

$$\widehat{C}(t) = e^{iE^2 Dt}.$$

With a similar argument, we find that

$$\widehat{\Phi}(x,t;E) = \Phi(x,t;E)\widehat{K}(t) \tag{2.23}$$

solves the system of equations (2.2) and (2.3), with

$$\widehat{K}(t) = e^{iE^2 Dt}.$$

Caution: Again, notice that the functions $\widehat{\Psi}(x,t;E)$ and $\widehat{\Phi}(x,t;E)$ do **not** satisfy the asymptotics (2.19) and (2.20).

On the other hand, since $\widehat{\Psi}(x,t;E)$ and $\widehat{\Phi}(x,t;E)$ solve the MZS system, they should also be related

$$\widehat{\Phi}(x,t;E) = \widehat{\Psi}(x,t;E)\widehat{S}(E;t), \tag{2.24}$$

where $\widehat{S}(E;t)$ is a scattering matrix. Compute the time derivative in each member of this equation, and multiply by i. We get

$$i\frac{\partial\widehat{\Phi}}{\partial t} = i\frac{\partial\widehat{\Psi}}{\partial t}\widehat{S} + \widehat{\Psi}i\frac{\partial\widehat{S}}{\partial t}.$$

We know that $\widehat{\Psi}(x,t;E)$ and $\widehat{\Phi}(x,t;E)$ solve the t-flow equation. Then

$$\mathcal{B}\widehat{\Phi} = \mathcal{B}\widehat{\Psi}\widehat{S} + \widehat{\Psi}i\frac{\partial}{\partial t}\widehat{S},$$

i.e.

$$\mathcal{B}(\widehat{\Phi} - \widehat{\Psi}\widehat{S}) = \widehat{\Psi}i\frac{\partial}{\partial t}\widehat{S}.$$

But the left-hand-side is zero by virtue of equation (2.24). Then

$$\frac{\partial}{\partial t}\widehat{S} = 0,$$

and

$$\widehat{S}(E;t) = \widehat{S}(E;0).$$

Finally, solve equations (2.22), and (2.23) for Ψ and Φ, and subtitute them into equation (2.21). It follows that

$$\widehat{\Phi}(x,t;E)\widehat{K}^{-1}(t) = \widehat{\Psi}(x,t;E)\widehat{C}^{-1}(t)S(E;t).$$

Use equation (2.24) in the previous equation, to obtain

$$\widehat{\Psi}(x,t;E)\widehat{S}(E;0)\widehat{K}^{-1}(t) = \widehat{\Psi}(x,t;E)\widehat{C}^{-1}(t)S(E;t),$$

i.e.

$$S(E;t) = \widehat{C}(t)\widehat{S}(E;0)\widehat{K}^{-1}(t).$$

then $\widehat{S}(E;0) = S(E;0) = S(E)$ is the scattering matrix at $t = 0$ (found in the previous section). So we obtain

$$\begin{aligned} S(E;t) &= C(t)S(E;0)K^{-1}(t), \\ &= e^{iE^2Dt}S(E;0)e^{-iE^2Dt}, \end{aligned} \tag{2.25}$$

i.e., the evolution in time of the Scattering matrix is just conjugation with respect to the complex exponential e^{iE^2t}.

It is important to say here that this argument is valid for any initial data, provided $N(x,t) \to 0$ fast enough as $|x| \to 0$.

From equation (2.25), we can verify that the first entry does not depend on time, and

$$\begin{aligned} S_{11}(E;t) &= S_{11}(E), \\ S_{21}(E;t) &= S_{21}(E)e^{2iE^2t}, \\ S_{31}(E;t) &= S_{31}(E)e^{2iE^2t}. \end{aligned}$$

The reflection coefficients are, therefore

$$\begin{aligned} R_1(E;t) &= \frac{S_{21}(E)}{S_{11}(E)} e^{2iE^2t}, \\ R_2(E;t) &= \frac{S_{31}(E)}{S_{11}(E)} e^{2iE^2t}. \end{aligned} \tag{2.26}$$

2.2.2 Evolution in Time of the Proportionality Constants

The calculation of the evolution in time of the proportionality constants parallels the computation of the evolution of the reflection coefficients, but it is slightly different in the sense that here we are evaluating the Jost solutions at the eigenvalues $E = E_k^*$.

Remember equation (2.14), but here we allow the time dependence in the Jost solutions and in the proportionality constants

$$\Phi_1(x,t;E_k) = C_k(t)\Psi_2(x,t;E_k) + D_k(t)\Psi_3(x,t;E_k). \tag{2.27}$$

Earlier in this section, we assumed that $\widehat{\Psi}(x,t;E) = \Psi(x,t;E)e^{iE^2Dt}$ (equation 2.22) and $\widehat{\Phi}(x,t;E) = \Phi(x,t;E)e^{iE^2Dt}$ (equation 2.23) solve the MZS sytem and the t-flow equation (equations (2.2) and (2.3)). In particular, $\widehat{\Phi}_1$, $\widehat{\Psi}_2$ and $\widehat{\Psi}_3$ solve the t-flow equation

$$\begin{aligned} i\frac{\partial}{\partial t}\big(\Phi_1 e^{iE^2t}\big) &= \mathcal{B}\big(\Phi_1 e^{iE^2t}\big), \\ i\frac{\partial}{\partial t}\big(\Psi_2 e^{iE^2t}\big) &= \mathcal{B}\big(\Psi_2 e^{iE^2t}\big), \\ i\frac{\partial}{\partial t}\big(\Psi_3 e^{iE^2t}\big) &= \mathcal{B}\big(\Psi_3 e^{iE^2t}\big), \end{aligned} \tag{2.28}$$

Multiply equation (2.27) by $e^{-iE_k^2t}$

$$\Phi_1(x,t;E_k)e^{-iE_k^2t} = C_k(t)e^{-2iE_k^2t}\big(\Psi_2(x,t;E_k)e^{iE_k^2t}\big) + D_k(t)e^{-2iE_k^2t}\big(\Psi_3(x,t;E_k)e^{iE_k^2t}\big).$$

Apply the operator $i\frac{\partial}{\partial t}$ to the last equation

$$\begin{aligned} i\frac{\partial}{\partial t}\big(\Phi_1 e^{-iE_k^2t}\big) &= i\frac{\partial}{\partial t}\big(C_k(t)e^{-2iE_k^2t}\big)\big(\Psi_2 e^{iE_k^2t}\big) + i\frac{\partial}{\partial t}\big(D_k(t)e^{-2iE_k^2t}\big)\big(\Psi_3 e^{iE_k^2t}\big) \\ &\quad + \big(C_k(t)e^{-2iE_k^2t}\big)i\frac{\partial}{\partial t}\big(\Psi_2 e^{iE_k^2t}\big) + \big(D_k(t)e^{-2iE_k^2t}\big)i\frac{\partial}{\partial t}\big(\Psi_3 e^{iE_k^2t}\big). \end{aligned}$$

Using equations (2.28)

$$\begin{aligned}\mathcal{B}\big(\Phi_1 e^{-iE_k^2 t}\big) &= i\frac{\partial}{\partial t}\big(C_k(t)e^{-2iE_k^2 t}\big)\big(\Psi_2 e^{iE_k^2 t}\big) + i\frac{\partial}{\partial t}\big(D_k(t)e^{-2iE_k^2 t}\big)\big(\Psi_3 e^{iE_k^2 t}\big) \\ &\quad + \big(C_k(t)e^{-2iE_k^2 t}\big)\mathcal{B}\big(\Psi_2 e^{iE_k^2 t}\big) + \big(D_k(t)e^{-2iE_k^2 t}\big)\mathcal{B}\big(\Psi_3 e^{iE_k^2 t}\big).\end{aligned}$$

Rearranging terms

$$\begin{aligned} i\frac{\partial}{\partial t}\big(C_k(t)e^{-2iE_k^2 t}\big)\big(\Psi_2 e^{iE_k^2 t}\big) + i\frac{\partial}{\partial t}\big(D_k(t)e^{-2iE_k^2 t}\big)\big(\Psi_3 e^{iE_k^2 t}\big) &= \\ = \mathcal{B}\left[\Phi_1 - C_k(t)\Psi_2 - D_k(t)\Psi_3\right] e^{-iE_k^2 t}. \end{aligned}$$

Now, by equation (2.27), the right-hand-side of the previous equation vanishes

$$i\frac{\partial}{\partial t}\big(C_k(t)e^{-2iE_k^2 t}\big)\Psi_2 + i\frac{\partial}{\partial t}\big(D_k(t)e^{-2iE_k^2 t}\big)\Psi_3 = 0.$$

Now, since the columns $\Psi_2(x,t;E)$ and $\Psi_3(x,t;E)$ are linearly independent, the corresponding coefficients should be zero

$$\begin{aligned} i\frac{\partial}{\partial t}\big(C_k(t)e^{-2iE_k^2 t}\big) &= 0, \\ i\frac{\partial}{\partial t}\big(D_k(t)e^{-2iE_k^2 t}\big) &= 0, \end{aligned}$$

i.e.

$$\begin{aligned} C_k(t)e^{-2iE_k^2 t} &= C_k(0), \\ D_k(t)e^{-2iE_k^2 t} &= D_k(0), \end{aligned}$$

to finally obtain

$$C_k(t) = C_k(0)e^{2iE_k^2 t}, \tag{2.29}$$

$$D_k(t) = D_k(0)e^{2iE_k^2 t}. \tag{2.30}$$

2.3 The Scattering Data depending on Time

We have now all the necessary information of perform the inverse scattering of the MZS system. Given some initial conditions, we compute the eigenvalues, the proportionality constants and the reflection coefficients at $t = 0$. Now, we know that the eigenvalues $E = E_k$ $(k = 1, 2, 3, \ldots, n)$ are time-independent

$$E_k(t) = E_k.$$

In general, the spectral parameter is time independent: $E(t) = E(0)$. The proportionality constants, as well as the reflection coefficients, evolve in time in a very easy way

$$\begin{aligned}
R_1(E;t) &= \frac{S_{21}(E)}{S_{11}(E)} e^{2iE^2 t}, \\
R_2(E;t) &= \frac{S_{31}(E)}{S_{11}(E)} e^{2iE^2 t}. \\
C_k(t) &= C_k(0) e^{2iE_k^2 t}, \\
D_k(t) &= D_k(0) e^{2iE_k^2 t}.
\end{aligned}$$

With this information, a Riemann-Hilbert Problem can be formulated. We will do this in chapter 5. Next chapter will be dedicated to compute the scattering data when the potentials are squared barriers.

2.4 Complex Conjugates of the Eigenvalues

As we mentioned in section 2.1, the columns $\Phi_1(x;E)$, $\Psi_2(x;E)$, and $\Psi_3(x;E)$ can be analytically continued into the E upper-half complex plane. Similarly, the columns $\Psi_1(x;E)$, $\Phi_2(x;E)$, and $\Phi_3(x;E)$ can be analytically continued into the E lower-half complex plane. Now, using the scattering relation equation (2.11), we compute the following determinant

$$\begin{aligned}
\det(\Psi_1, \Phi_2, \Phi_3) &= \det\left(\Psi_1, \Psi_1(S_{12}, S_{13}) + (\Psi_2, \Psi_3)\begin{pmatrix} S_{22} & S_{23} \\ S_{32} & S_{33} \end{pmatrix}\right) \\
&= \det\left(\Psi_1, (\Psi_2, \Psi_3)\begin{pmatrix} S_{22} & S_{23} \\ S_{32} & S_{33} \end{pmatrix}\right),
\end{aligned}$$

since there are multiples of the same column, Ψ_1. It follows

$$\begin{aligned}
\det(\Psi_1, \Phi_2, \Phi_3) &= \det\left((\Psi_1, \Psi_2, \Psi_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & S_{22} & S_{23} \\ 0 & S_{32} & S_{33} \end{pmatrix} \right), \\
&= \det(\Psi_1, \Psi_2, \Psi_3) \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & S_{22} & S_{23} \\ 0 & S_{32} & S_{33} \end{pmatrix}, \\
&= \det(\Psi_1, \Psi_2, \Psi_3) \det \begin{pmatrix} S_{22} & S_{23} \\ S_{32} & S_{33} \end{pmatrix}, \\
&= e^{iEDx} \det \begin{pmatrix} S_{22} & S_{23} \\ S_{32} & S_{33} \end{pmatrix},
\end{aligned}$$

From this, we can say that this determinant can be analytically continued into $\mathbb{C}^-$. Using now the fact that the scattering matrixis unitary, $S = (S^*)^{-1}$, the right-hand-side of the previous equation can be written as

$$\det(\Psi_1, \Phi_2, \Phi_3) = e^{iEDx} \overline{S_{11}(E^*)}.$$

Evaluating at $E = E_k^*$ (with $E = E_k$ eigenvalue of the MZS system), the right-hand-side is zero

$$\det(\Psi_1(x; E_k^*), \Phi_2(x; E_k^*), \Phi_3(x; E_k^*)) = 0,$$

which implies that $\Psi_1(x; E_k^*)$, $\Phi_2(x; E_k^*)$ and $\Phi_3(x; E_k^*)$ are linearly dependent, and also implies the existence of two constants $\widehat{C}_k$ and $\widehat{D}_k$ such that

$$\Psi_1(x; E_k^*) = \widehat{C}_k \Phi_2(x; E_k^*) + \widehat{D}_k \Phi_3(x; E_k^*). \tag{2.31}$$

It turns that: $\widehat{C}_k$ and $\widehat{D}_k$ are the complex conjugates of the proportionality constants

$$\widehat{C}_k = C_k^* \qquad \text{and} \qquad \widehat{D}_k = D_k^*.$$

Using equation (2.31), together with the asymptotic properties of the Jost functions, equations (2.9) and (2.9), and the fact that $\mathrm{Im}(E_k^*) < 0$, we can check that

$$\lim_{x \to \pm\infty} \Psi_1(x; E_k^*) = 0,$$

i.e., $\Psi_1(x; E_k^*)$ is also eigenfunction with eigenvalue E_k^*. Therefore, the solutions of the equation $S_{11}(E) = 0$, namely $E = E_k$, $k = 1, 2, 3, \ldots, n$, plus its complex conjugates $E = E_k^*$ conform the eigenvalues of the MZS system.

Chapter 3
Scattering of Squared Barrier Potentials

3.0 Introduction

In this chapter, we compute the Scattering matrix and the corresponding scattering data for the three different kinds of squared barrier potentials considered in this thesis. The basic squared barrier is given by

$$F_L(x) = \begin{cases} 1, & |x| \leq L \\ 0, & |x| > L, \end{cases}$$

with $L > 0$. We will compute the Scattering matrix for following three types of initial potentials: squared barriers with same, overlapping and disjoint supports. In all these cases, we consider p_0 and q_0 to be positive constants.

We also compute the conditions to have eigenvalues with for these initial data, and we realize that a threshold exists. Chapter 4 studies the location of the eigenvalues of the MZS system (equation (2.8)) with. squared barrier potentials.

Once this information is computed, we employ the theory to be developed in chapter 5 to describe the short-time asymptotics of the Manakov system with these initial conditions in chapter 6.

3.1 Potentials with Same Support

3.1.1 The Scattering Matrix

Here, we consider the initial data given by

$$\begin{aligned} u(x,0) &= p_0 F_L(x), \\ v(x,0) &= q_0 F_L(x), \end{aligned} \tag{3.1}$$

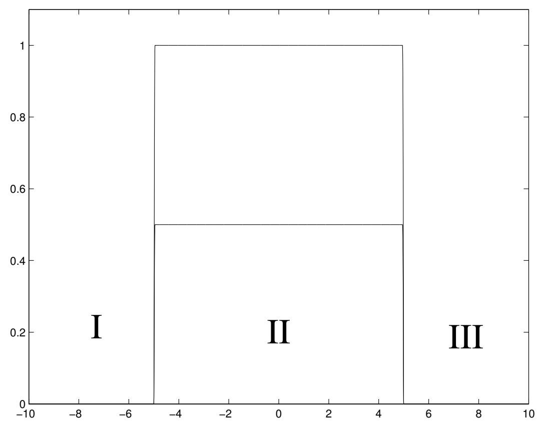

FIGURE 3.1. Typical initial data for $u(x,0)$ and $v(x,0)$ with same support. Here, $p_0 = 1$, $q_0 = \frac{1}{2}$, $L = 5$.

i.e., both potentials have the same support. See figure (3.1). The MZS system, equation (2.8), then becomes

$$\begin{aligned}
\frac{\partial}{\partial x}\Psi_I &= iED\Psi_I, && x < -L,\\
\frac{\partial}{\partial x}\Psi_{II} &= iED\Psi_{II} + N_0\Psi, && -L \le x \le L,\\
\frac{\partial}{\partial x}\Psi_{III} &= iED\Psi_{III}, && L < x,
\end{aligned}$$

with

$$N_0 = \begin{pmatrix} 0 & p_0 & q_0 \\ -p_0 & 0 & 0 \\ -q_0 & 0 & 0 \end{pmatrix}. \tag{3.2}$$

The convenience of using piecewise constant potentials is that the MZS system is very easy to solve

$$\begin{aligned}
\Psi_I(x) &= e^{iEDx}\,C_I, && x < -L,\\
\Psi_{II}(x) &= e^{(iED+N_0)x}\,C_{II}, && -L \le x \le L,\\
\Psi_{III}(x) &= e^{iEDx}\,C_{III}, && L < x,
\end{aligned}$$

Here, C_I, C_{II} and C_{III} are constant matrices.

We also assume continuity of Ψ. Then

$$\begin{aligned}\Psi_I(-L) &= \Psi_{II}(-L),\\ \Psi_I(L) &= \Psi_{III}(L).\end{aligned}$$

From this, we conclude that two of the constant matrices can be put in terms of C_{III}

$$\begin{aligned}C_I &= e^{iEDL}e^{-2(iED+N_0)L}e^{iEDL}\,C_{III},\\ C_{II} &= e^{-(iED+N_0)L}e^{iEDL}\,C_{III}.\end{aligned}$$

Setting $C_{III} = I$, where I is the identity matrix, $\Psi(x)$ is completly defined, including the condition $\Psi(x) \sim e^{iEDx}$, when $x \to \infty$.

Similarly, we find the solution $\Phi(x)$

$$\begin{aligned}\Phi_I(x) &= e^{iEDx}\,K_I, && x < -L,\\ \Phi_{II}(x) &= e^{(iED+N_0)x}\,K_{II}, && -L \leq x \leq L,\\ \Phi_{III}(x) &= e^{iEDx}\,K_{III}, && L < x,\end{aligned}$$

with the condition $\Phi(x) \sim e^{iEDx}$, when $x \to -\infty$. To this end, we put $K_I = I$. Now, from continuity assumptions we get

$$\begin{aligned}K_{II} &= e^{(iED+N_0)L}e^{-iEDL},\\ K_{III} &= e^{-iEDL}e^{2(iED+N_0)L}e^{-iEDL}.\end{aligned}$$

If we compare $\Phi(x)$ with $\Psi(x)$, we obtain the scattering matrix

$$S(E) = e^{-iEDL}e^{2(iED+N_0)L}e^{-iEDL}. \tag{3.3}$$

Now, as we studied in section 2.2, the scattering matrix depending on time is just a conjugation by the exponential matrix e^{iE^2Dt}

$$S(E;t) = e^{i(E^2t-EL)D}e^{2(iED+N_0)L}e^{-i(E^2t+EL)D}. \tag{3.4}$$

This is the *time-dependent Scattering matrix.* From the Scattering matrix, we can now compute the reflexion coefficient, and the eigenvalues of the MZS system.

3.1.2 The Transmission and Reflection Coefficients

In this section, we explicitly write down the expressions of the reflection and transmission coefficients for the barrier considered in the previous section. We only need to compute the entries $S_{11}(E;t)$, $S_{21}(E;t)$, and $S_{31}(E;t)$, of the scattering matrix $S(E;t)$.

A diagonalization process will allow us to compute

$$e^{(iED+N_0)x} = \frac{1}{\nu(E)} \begin{pmatrix} c_{11}(x) & c_{12}(x) & c_{13}(x) \\ c_{21}(x) & c_{22}(x) & c_{23}(x) \\ c_{31}(x) & c_{32}(x) & c_{33}(x) \end{pmatrix} \tag{3.5}$$

with

$$\nu(E) = \sqrt{E^2 + \sigma(|p_0|^2 + |q_0|^2)},$$

and

$$\begin{aligned}
c_{11}(x) &= \nu(E)\cos(\nu(E)x) - iE\sin(\nu(E)x) \\
c_{21}(x) &= -\sigma p_0^\star \sin(\nu(E)x) \\
c_{31}(x) &= -\sigma q_0^\star \sin(\nu(E)x) \\
c_{12}(x) &= p_0 \sin(\nu(E)x) \\
c_{22}(x) &= \left(\nu(E)\cos(\nu(E)x) + iE\sin(\nu(E)x) + \nu(E)e^{iEx}\right)\left(\frac{|q_0|^2}{|p_0|^2+|q_0|^2}\right) \\
c_{32}(x) &= \left(\nu(E)\cos(\nu(E)x) + iE\sin(\nu(E)x) - \nu(E)e^{iEx}\right)\left(\frac{p_0 q_0^\star}{|p_0|^2+|q_0|^2}\right) \\
c_{13}(x) &= q_0 \sin(\nu(E)x) \\
c_{23}(x) &= \left(\nu(E)\cos(\nu(E)x) + iE\sin(\nu(E)x) - \nu(E)e^{iEx}\right)\left(\frac{p_0^\star q_0}{|p_0|^2+|q_0|^2}\right) \\
c_{33}(x) &= \left(\nu(E)\cos(\nu(E)x) + iE\sin(\nu(E)x) + \nu(E)e^{iEx}\right)\left(\frac{|p_0|^2}{|p_0|^2+|p_0|^2}\right)
\end{aligned}$$

After matrix multiplication of the exponentials in the Scattering matrix

$$S(E;t) = e^{iE^2Dt}e^{-iEDL}e^{2(iED+N_0)L}e^{-iEDL}e^{-iE^2Dt},$$

we obtain the entries $S_{11}(E)$, $S_{21}(E;t)$, and $S_{31}(E;t)$

$$\begin{aligned} S_{11}(E) &= \frac{e^{2iEL}}{\nu(E)}\Big(\nu(E)\cos(2\nu(E)L) - iE\sin(2\nu(E)L)\Big), \\ S_{21}(E;t) &= -\sigma\frac{p_0^\star}{\nu(E)}\sin(2\nu(E)L)e^{2iE^2t}, \\ S_{31}(E;t) &= -\sigma\frac{q_0^\star}{\nu(E)}\sin(2\nu(E)L)e^{2iE^2t}. \end{aligned}$$

Therefore, the Reflection Coefficients, equation (2.17) are given by

$$R_1(E;t) = p_0^\star R(E;t), \tag{3.6}$$

$$R_2(E;t) = q_0^\star R(E;t), \tag{3.7}$$

where

$$R(E;t) = -\sigma\frac{\sin(2\nu(E)L)}{\nu(E)\cos(2\nu(E)L) - iE\sin(2\nu(E)L)}e^{-2iEL}e^{2iE^2t}. \tag{3.8}$$

Now, the eigenvalues are the solutions of the following (trascendental) equation

$$S_{11}(E_j) = 0, \tag{3.9}$$

i.e.

$$\frac{1}{\nu(E_j)}\Big(\nu(E_j)\cos(2\nu(E_j)L) - iE_j\sin(2\nu(E_j)L)\Big) = 0, \tag{3.10}$$

(where we have divided both sides by e^{2iE_jL}.)

The defocusing case ($\sigma = -1$) was studied by J. DiFranco [DiF]. For that case, there are no eigenvalues, since the operator $\mathcal{L}$ is self-adjoint. Here, we consider the focusing case: $\sigma = +1$.

Remember that p_0, q_0 are positive constants and $\sigma = 1$ for the focusing case, which is the case considered here. Then

$$\nu(E) = \sqrt{E^2 + (p_0^2 + q_0^2)}. \tag{3.11}$$

Similarly for all the coefficients $c_{ij}(x)$, and in all the previous formulas.

In his original paper [Ma], Manakov assumed, *without proof*, that the eigenvalues of the MZS system are *purely imaginary*. We will prove his assertion (for squared barrier potentials with same and disjoint support) in chapter 4. Starting here, and in the forthcoming sections, we will assume this is true.

3.1.3 Threshold for the Creation of Eigenvalues

It is known that not all the trascendental equations have a solution. It turns out that equation (3.10) does not always have a solution for any value of the constants p_0, q_0 and L. In this section, we will study the necessary condition to have solutions of this equation, and we will see there is a threshold for that.

We therefore have to prove the following proposition:

Proposition 3.1. THERESHOLD FOR THE CREATION OF THE EIGENVALUES. *For initial conditions given as in (3.1), the eigenvalues of the MZS system (associated to the focusing Manakov system) are located at the segment*

$$\{E \in \mathbb{C}/\ E = i\,\eta\,P_0, 0 < \eta < 1\},$$

and we have a finite number of them. The condition to have n eigenvalues is given in equation (3.18).

Consider the case when $\text{Im}(E) > 0$. This is because $S_{11}(E)$ is valid in $\mathbb{C}^+$, i.e., $\text{Im}(E) > 0$. A similar argument works when $\text{Im}(E) < 0$. If $\nu(E) \longrightarrow 0$, then

$$\frac{1}{\nu(E)}\Big(\nu(E)\cos(2\nu(E)L) - iE\sin(2\nu(E)L)\Big) \longrightarrow 1 - 2iEL.$$

Hence, if $E \longrightarrow i\sqrt{p_0^2 + q_0^2}$, then $\nu(E) \longrightarrow 0$, and so

$$\frac{1}{\nu(E)}\Big(\nu(E)\cos(2\nu(E)L) - iE\sin(2\nu(E)L)\Big) \longrightarrow 1 + 2L\sqrt{p_0^2 + q_0^2} \neq 0.$$

This means that $E = i\sqrt{p_0^2 + q_0^2}$ is not an eigenvalue.

Consider here the assumption made by Manakov, that **all** the eigenvalues are purely imaginary. This is to say, all the eigenvalues have the form

$$E = i\eta P_0,$$

where η is a *nonnegative* parameter, $\eta \geq 0$, and

$$P_0 \equiv \sqrt{p_0^2 + q_0^2},$$

Therefore, by the previous argument, we can verify that $\nu(E) \neq 0$ whenever $\eta \neq 1$. Hence, we can multiply equation (3.10) by $\nu(E)$ without loss of information

$$\nu(E)\cos(2\nu(E)L) - iE\sin(2\nu(E)L) = 0. \tag{3.12}$$

This equation defines the eigenvalues of the MZS system with squared barrier potentials. In terms of $E = i\eta P_0$, this equation could be written in the form

$$\sqrt{1-\eta^2}\,\cos\left(2P_0L\sqrt{1-\eta^2}\right) + \eta\,\sin\left(2P_0L\sqrt{1-\eta^2}\right) = 0.$$

This equation can be qualitatively studied.

For $\eta > 1$, we have

$$\nu(E) = i\xi P_0, \tag{3.13}$$

with

$$\xi \equiv \sqrt{\eta^2 - 1}. \tag{3.14}$$

Notice that $\xi > 0$, since $\eta > 1$. Then

$$\begin{aligned}\nu(E)\cos(2\nu(E)L) - iE\sin(2\nu(E)L) &= iP_0\left(\frac{\xi}{2L}\cosh\xi + \sqrt{\xi^2+1}\sinh\xi\right)\\ &\neq 0.\end{aligned}$$

Therefore, for $\eta > 1$, we do not have eigenvalues.

Noticing that $\eta = 1$ is a branch point of $\sqrt{\eta^2 - 1}$, one may wonder if the square root ($\xi = \sqrt{\eta^2 - 1}$) could just as well be defined to be negative upon increasing η through $\eta = 1$. Then, there might be roots corresponding to additional eigenvalues. This potential difficulty is resolved just by looking at equation (3.10), which is the important one. If we switch ξ by $-\xi$, it follows from equation (3.13) that $\nu(E)$ should be replaced by $-\nu(E)$ in (3.10). However, this transformation keeps invariant the left-hand-side of equation (3.10).

It remains to study the case when

$$0 \leq \eta < 1.$$

In this instance

$$\nu(E) = \sqrt{1-\eta^2}\ \ P_0,$$

and also

$$\sqrt{1-\eta^2}\ \cos\left(2P_0L\sqrt{1-\eta^2}\right) + \eta\ \sin\left(2P_0L\sqrt{1-\eta^2}\right) = 0. \tag{3.15}$$

If $\eta \neq 0$, i.e., $0 < \eta < 1$, consider equation (3.15). This equation can be re-written in the following form

$$\text{Arccos}(-\eta) + n\pi = 2P_0L\sqrt{1-\eta^2}, \tag{3.16}$$

for $n = 0, 1, 2, \ldots$, and Arccos denotes the principal branch of the Arc-cosine function. We can check that the condition to have n zeros is

$$(2n-1)\frac{\pi}{2} < 2\sqrt{p_0^2+q_0^2}L < (2n+1)\frac{\pi}{2}. \tag{3.17}$$

Then, n is the *number of eigenvalues.*

All of this (including equation (3.17)) can be clearly seen by graphing the functions appearing in equation (3.16). The left-hand-side is a trigonometric function, shifted by $n\pi$. The right-hand-side is an ellipsoid. In Figure 3.2, we observe three intersections between these two curves, corresponding to the case when $n = 3$. In this instance, $L = 1$, and $P_0 = \sqrt{p_0^2+q_0^2} = 5$. Hence, equation (3.17) becomes

$$(2n-1)\frac{\pi}{2} = 7.8539 < 2\sqrt{p_0^2+q_0^2}L = 10 < (2n+1)\frac{\pi}{2} = 10.9955.$$

This exactly corresponds to the case $n = 3$.

Equation (3.17) can be equivalently written as

$$\frac{2}{\pi}\sqrt{p_0^2+q_0^2}\ \ L - \frac{1}{2} < n < \frac{2}{\pi}\sqrt{p_0^2+q_0^2}\ \ L + \frac{1}{2}. \tag{3.18}$$

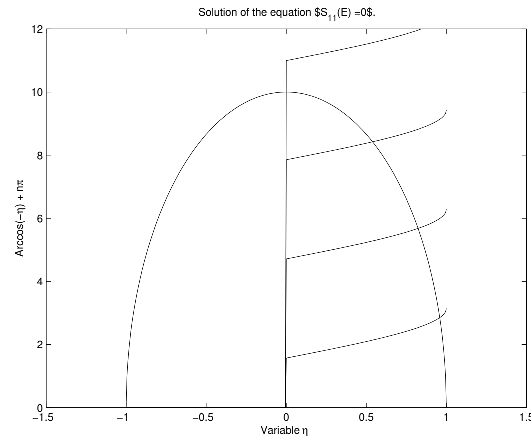

FIGURE 3.2. Graphs of the functions $y = \text{Arccos}(-\eta) + n\pi$ and $2P_0L\sqrt{1-\eta^2}$,

Then, $j = 1, 2, \ldots, n$ in equation (3.9). In all these cases $0 < \eta < 1$. this equation (3.18) is very helpful if we wish to count the number of eigenvalues.

Now, if instead of inequalities, an equality holds in (3.17), we can check that $E = 0$ (i.e., $\eta = 0$), which is a *marginal* (or emerging) *eigenvalue.* Say that the inequality on the right in equation (3.17) becomes an equality, this is to say,

$$2\sqrt{p_0^2 + q_0^2}L = (2n+1)\frac{\pi}{2}. \tag{3.19}$$

Graphically, this corresponds (see Figure 3.2) to the fact that the intersection of the arccos with the ellipsoid is exactly at the vertical axis. We graphically see that those are the points at which new eigenvalues emerge.

The condition to have *just one* intersection at the vertical axis, between the arccos and the ellipsoid, is setting $n = 0$ in (3.19):

$$2\sqrt{p_0^2 + q_0^2}\ \ L = \frac{\pi}{2}, \tag{3.20}$$

and this is the *threshold condition for the realization of eigenvalues.*

Finally, let us see how to match equation (3.15) with the results we have just stated. If $\eta = 0$ in equation (3.15), it follows that $\cos(2P_0L) = 0$, which implies

$$2P_0L = (2n+1)\frac{\pi}{2}, \tag{3.21}$$

which is the result enunciated above, equation (3.19).

This completes the proof of the proposition.

3.2 Potentials with Disjoint Support

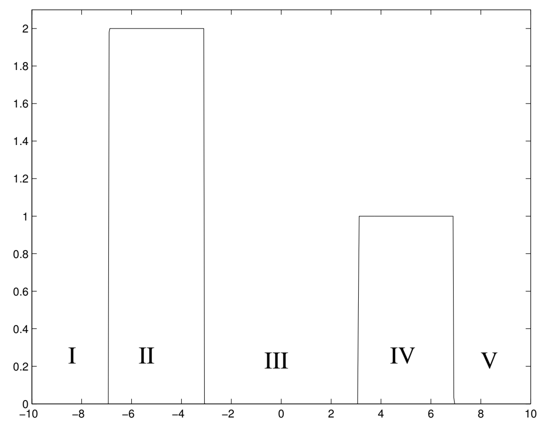

FIGURE 3.3. Typical initial data for $u(x, 0)$ and $v(x, 0)$ with disjoint support. Here, $p_0 = 2$, $q_0 = 1$, $L = 1$ and $\alpha = 5$

3.2.1 The Scattering Matrix

Here, the initial potentials are given by

$$\begin{aligned} u(x, 0) &= p_0 F_L(x - \widehat{\alpha}), \\ v(x, 0) &= q_0 F_L(x + \widehat{\alpha}), \end{aligned} \tag{3.22}$$

with $L < \widehat{\alpha}$.

The procedure is the same as in the previous section, and it won't be repeated again. We only state the final answer. The Scattering matrix is, in this instance

$$S(E; t) = e^{iE^2Dt} e^{-iED(\widehat{\alpha}+L)} e^{2(iED+N_2)L} e^{2iED(\widehat{\alpha}-L)} e^{2(iED+N_1)L} e^{-iED(\widehat{\alpha}+L)} e^{-iE^2Dt}, \tag{3.23}$$

with

$$N_1 = \begin{pmatrix} 0 & p_0 & 0 \\ -p_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad N_2 = \begin{pmatrix} 0 & 0 & q_0 \\ 0 & 0 & 0 \\ -q_0 & 0 & 0 \end{pmatrix}. \tag{3.24}$$

3.2.2 The Transmission and Reflection Coefficients

If $q_0 = 0$ in equation (3.2), then N_0 reduces to N_1, and also the exponential matrix in (3.5) reduces to

$$e^{(iED+N_1)x} \equiv \frac{1}{\nu_1(E)} \begin{pmatrix} a_1(x) & a_2(x) & 0 \\ a_3(x) & a_4(x) & 0 \\ 0 & 0 & a_5(x) \end{pmatrix}, \tag{3.25}$$

$$\equiv \frac{1}{\nu_1(E)} \begin{pmatrix} \nu_1(E)\cos(\nu_1(E)x) - iE\sin(\nu_1(E)x) & p_0\sin(\nu_1(E)x) & 0 \\ -\sigma p_0^\star \sin(\nu_1(E)x) & \nu_1(E)\cos(\nu_1(E)x) + iE\sin(\nu_1(E)x) & 0 \\ 0 & 0 & \nu_1(E)e^{iEx} \end{pmatrix}$$

where $\nu_1(E) = \sqrt{E^2 + p_0^2}$.

Now, set $p_0 = 0$ in (3.2), to obtain

$$e^{(iED+N_2)x} \equiv \frac{1}{\nu_1(E)} \begin{pmatrix} b_1(x) & 0 & b_2(x) \\ 0 & b_3(x) & 0 \\ b_4(x) & 0 & b_5(x) \end{pmatrix}, \tag{3.26}$$

$$= \frac{1}{\nu_2(E)} \begin{pmatrix} \nu_2(E)\cos(\nu_2(E)x) - iE\sin(\nu_2(E)x) & 0 & q_0\sin(\nu_2(E)x) \\ 0 & \nu_2(E)e^{iEx} & 0 \\ -\sigma q_0^\star \sin(\nu_2(E)x) & 0 & \nu_2(E)\cos(\nu_2(E)x) + iE\sin(\nu_2(E)x) \end{pmatrix},$$

with $\nu_2(E) = \sqrt{E^2 + q_0^2}$.

We now use these matrices to compute explicitly the Scattering matrix, and consequently, the Transmission and Reflection coefficients.

The initial conditions considered here are

$$
\begin{aligned}
u(x,0) &= \begin{cases} p_0, & -L_2 \le x \le -L_1 \\ 0, & \text{otherwise} \end{cases}, \\
v(x,0) &= \begin{cases} q_0, & L_1 \le x \le L_2 \\ 0, & \text{otherwise} \end{cases}.
\end{aligned}
$$

We found the scattering matrix in section 3.2.1, equation (3.23)

$$S(E;t) = e^{iE^2Dt}e^{-iEDL_2}e^{(iED+N_2)(L_2-L_1)}e^{2iEDL_1}e^{2(iED+N_1)(L_2-L_1)}e^{-iEDL_2}e^{-iE^2Dt}.$$

Please note the change in notation. Here: $L_1 = \widehat{\alpha}-L$ and $L_2 = \widehat{\alpha}+L$. The coefficients of the Scattering matrix that we need are

$$
\begin{aligned}
S_{11}(E) &= \frac{1}{\nu_1(E)\nu_2(E)} a_1(\widehat{L})b_1(\widehat{L})e^{2iE\widehat{L}}, && (3.27) \\
S_{21}(E;t) &= \frac{1}{\nu_1(E)\nu_2(E)} a_2(\widehat{L})b_2(\widehat{L})e^{2iEL_1}e^{2iE^2t}, \\
S_{31}(E;t) &= \frac{1}{\nu_1(E)\nu_2(E)} a_1(\widehat{L})b_3(\widehat{L})e^{-2iEL_1}e^{2iE^2t}, && (3.28)
\end{aligned}
$$

where

$$
\begin{aligned}
\widehat{L} &= L_2 - L_1, \\
\nu_1(E) &= \sqrt{E^2 + p_0^2}, \\
\nu_2(E) &= \sqrt{E^2 + q_0^2},
\end{aligned}
$$

and

$$
\begin{aligned}
a_1(\widehat{L}) &= \nu_1(E)\cos(\nu_1(E)\widehat{L}) - iE\sin(\nu_1(E)\widehat{L}) \\
a_2(\widehat{L}) &= -p_0\sin(\nu_1(E)\widehat{L}) \\
b_1(\widehat{L}) &= \nu_2(E)\cos(\nu_2(E)\widehat{L}) - iE\sin(\nu_2(E)\widehat{L}) \\
b_2(\widehat{L}) &= \nu_2(E)e^{iE\widehat{L}} \\
b_3(\widehat{L}) &= -q_0\sin(\nu_2(E)\widehat{L}).
\end{aligned}
$$

Therefore, the reflection coefficients are

$$
\begin{aligned}
R_1(E;t) &= \frac{S_{21}(E;t)}{S_{11}(E;t)} = \frac{a_2(\widehat{L})b_2(\widehat{L})}{a_1(\widehat{L})b_1(\widehat{L})} e^{2iE(L_1-\widehat{L})} e^{2iE^2 t}, \\
R_2(E;t) &= \frac{S_{31}(E;t)}{S_{11}(E;t)} = \frac{b_3(\widehat{L})}{b_1(\widehat{L})} e^{-2iEL_2} e^{2iE^2 t}.
\end{aligned}
\tag{3.29}
$$

3.2.3 Threshold for the Creation of Eigenvalues

Using similar arguments as those we employed in subsection 3.1.3, the eigenvalues are then defined by the solutions of the equation, $S_{11}(E) = 0$, i.e.

$$\left\{\nu_1 \cos(\nu_1 \widehat{L}) - iE \sin(\nu_1 \widehat{L})\right\} \cdot \left\{\nu_2 \cos(\nu_2 \widehat{L}) - iE \sin(\nu_2 \widehat{L})\right\} = 0. \tag{3.30}$$

In chapter 4, we will prove that the solutions to this equation are purely imaginary. Under this assumption, we can say the following. Suppose $q_0 < p_0$. Then, the threshold for the creation of eigenvalues is

$$p_0 \widehat{L} = \frac{\pi}{2}, \tag{3.31}$$

since this implies

$$\nu_1 \cos(\nu_1 \widehat{L}) - iE \sin(\nu_1 \widehat{L}) = 0,$$

i.e., $S_{11}(E) = 0$[1]. If $q_0 > p_0$, then just substitute q_0 in equation (3.31).

On the other hand, if n_1 and n_2 are integers such that

$$
\begin{aligned}
(2n_1 - 1)\frac{\pi}{2} < p_0 \widehat{L} < (2n_1 + 1)\frac{\pi}{2}, \\
(2n_2 - 1)\frac{\pi}{2} < q_0 \widehat{L} < (2n_2 + 1)\frac{\pi}{2},
\end{aligned}
$$

then, the total number of eigenvalues is $n = n_1 + n_2$, corresponding to the number of zeros of each factor in equation (3.30).

[1]Notice that $\nu_2 \cos(\nu_2(E)\widehat{L}) - iE \sin(\nu_2 \widehat{L}) \neq 0$ (since $q_0 < p_0$ implies $q_0 \widehat{L} < \frac{\pi}{2}$), but this does not really matter, since $S_{11}(E) = 0$ anyway, if $p_0 \widehat{L} = \frac{\pi}{2}$.

3.3 Potentials with overlapping Supports

3.3.1 The Scattering Matrix

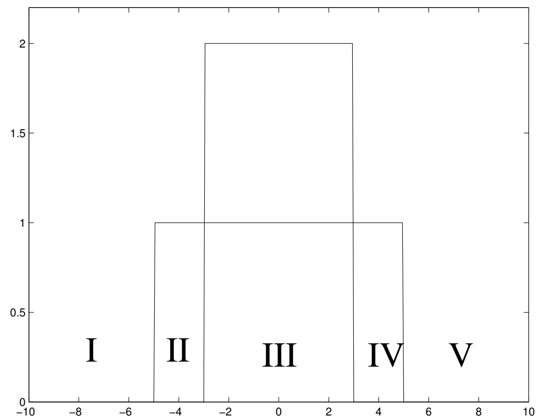

FIGURE 3.4. Typical initial data for $u(x,0)$ and $v(x,0)$ with overlapping support. Here, $p_0 = 2$, $q_0 = 1$, $L_1 = 3$ and $L_2 = 5$

Here, we compute the Scattering matrix for squared barrier potentials with overlapping supports. Namely, the initial conditions are

$$\begin{aligned} u(x,0) &= p_0 F_{L_1}(x), \\ v(x,0) &= q_0 F_{L_2}(x). \end{aligned} \tag{3.32}$$

Here, we continue assuming p_0 and q_0 to be positive constants. We consider the case $L_1 < L_2$, just to set ideas.

The MZS system is still very easily solved, since the potentials $u(x,0)$, $v(x,0)$ are again piecewise constants. The solutions are given by

$$\begin{aligned} \Psi_I(x) &= e^{iEDx}\, C_I, && x < -L_2, \\ \Psi_{II}(x) &= e^{(iED+N_2)x}\, C_{II}, && -L_2 \le x < -L_1, \\ \Psi_{III}(x) &= e^{(iED+N_0)x}\, C_{III}, && -L_1 \le x \le L_1, \\ \Psi_{IV}(x) &= e^{(iED+N_2)x}\, C_{IV}, && L_1 < x \le L_2, \\ \Psi_V(x) &= e^{iEDx}\, C_V, && L_2 < x, \end{aligned}$$

with
$$N_0 = \begin{pmatrix} 0 & p_0 & q_0 \\ -p_0 & 0 & 0 \\ -q_0 & 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad N_2 = \begin{pmatrix} 0 & 0 & q_0 \\ 0 & 0 & 0 \\ -q_0 & 0 & 0 \end{pmatrix},$$
as already defined in equations 3.2) and (3.24). Again, C_I, C_{II}, C_{III}, C_{IV}, and C_V are some constant matrices.

Similarly

$$\begin{aligned}
\Phi_I(x) &= e^{iEDx}\, K_I, && x < -L_2, \\
\Phi_{II}(x) &= e^{(iED+N_2)x}\, K_{II}, && -L_2 \le x < -L_1, \\
\Phi_{III}(x) &= e^{(iED+N_0)x}\, K_{III}, && -L_1 \le x \le L_1, \\
\Phi_{IV}(x) &= e^{(iED+N_2)x}\, K_{IV}, && L_1 < x \le L_2, \\
\Phi_V(x) &= e^{iEDx}\, K_V, && L_2 < x,
\end{aligned}$$

and K_I, K_{II}, K_{III}, K_{IV}, K_V are again constant matrices. Also, the asymptotics

$$\begin{aligned}
\lim_{x\to+\infty} \Psi(x;E)e^{-iEDx} &= I, \\
\lim_{x\to-\infty} \Phi(x;E)e^{-iEDx} &= I,
\end{aligned}$$

imply

$$C_V = I \qquad \text{and} \qquad K_I = I.$$

The continuity conditions at $x = -L_2, -L_1, L_1, L_2$, allows us to express C_I in terms of C_V, and K_V in terms of K_I. Now, we compute the scatterring matrix at $t = 0$, $S(E;0)$, from the relation $\Phi(x,0;E) = \Psi(x,0;E)S(E;0)$. Then, the Scattering matrix, $S(E;t)$, for these potentials is

$$S(E;t) = e^{iE^2Dt}e^{-iEDL_2}e^{(iED+N_2)(L_2-L_1)}e^{2(iED+N_0)L_1}e^{(iED+N_2)(L_2-L_1)}e^{-iEDL_2}e^{-iE^2Dt}. \tag{3.33}$$

3.3.2 The Transmission and Reflection Coefficients

The scattering matrix corresponding to initial data

$$\begin{aligned} u(x,0) &= p_0 F_{L_1}(x), \\ v(x,0) &= q_0 F_{L_2}(x), \end{aligned}$$

with $L_1 < L_2$ was given in equation (3.33)

$$S(E;t) = e^{iE^2Dt} e^{-iEDL_2} e^{(iED+N_2)(L_2-L_1)} e^{2(iED+N_0)L_1} e^{(iED+N_2)(L_2-L_1)} e^{-iEDL_2} e^{-iE^2Dt}.$$

The computation of the reflection coefficients is similar to the computation we did in the previous section. We obtain

$$\begin{aligned} S_{11}(E) &= \frac{1}{\nu(E)\nu_2^2(E)} A(E) e^{2iEL_2}, \\ S_{21}(E;t) &= \frac{1}{\nu(E)\nu_2^2(E)} B(E) e^{2iE^2t}, \\ S_{31}(E;t) &= \frac{1}{\nu(E)\nu_2^2(E)} C(E) e^{2iE^2t}, \end{aligned}$$

where

$$\begin{aligned} \nu(E) &= \sqrt{E^2 + p_0^2 + q_0^2}, \\ \nu_2(E) &= \sqrt{E^2 + q_0^2}, \end{aligned}$$

and

$$\begin{aligned}
A(E) &= [\nu_2 \cos(\nu_2 \widehat{L}) - iE \sin(\nu_2 \widehat{L})] \cdot [\nu \cos(2\nu L_1) - iE \sin(2\nu L_1)] \cdot \\
&\quad \cdot [\nu_2 \cos(\nu_2 \widehat{L}) - iE \sin(\nu_2 \widehat{L})] + \\
&\quad + [\nu_2 \cos(\nu_2 \widehat{L}) - iE \sin(\nu_2 \widehat{L})] \cdot (-q_0^2) \sin(2\nu L_1) \sin(\nu_2 \widehat{L}) + \\
&\quad + (-q_0^2) \sin(\nu_2 \widehat{L}) \sin(2\nu L_1) \cdot [\nu_2 \cos(\nu_2 \widehat{L}) - iE \sin(\nu_2 \widehat{L})] + \\
&\quad + (-q_0^2) \sin(\nu_2 \widehat{L}) \frac{q_0^2 \sin(\nu_2 \widehat{L})}{p_0^2 + q_0^2} \cdot [\nu \cos(2\nu L_1) + iE \sin(2\nu L_1) + \nu \frac{p_0^2}{q_0^2} e^{2iEL_1}], \qquad (3.34) \\
B(E) &= -p_0 \sin(2\nu L_1) \nu_2 e^{iE\widehat{L}} \cdot [\nu_2 \cos(\nu_2 \widehat{L}) - iE \sin(\nu_2 \widehat{L})] \\
&\quad + \frac{-p_0\, q_0^2}{p_0^2 + q_0^2} \sin(\nu_2 \widehat{L}) \nu_2 e^{iE\widehat{L}} \cdot [\nu \cos(2\nu L_1) + iE \sin(2\nu L_1) - \nu e^{2iEL_1}], \\
C(E) &= -q_0 \sin(\nu_2 \widehat{L}) \cdot [\nu \cos(2\nu L_1) - iE \sin(2\nu L_1)] \cdot \\
&\quad \cdot [\nu_2 \cos(\nu_2 \widehat{L}) - iE \sin(\nu_2 \widehat{L})] \\
&\quad + q_0\, q_0^2 \sin(2\nu L_1) \sin^2(\nu_2 \widehat{L}) \\
&\quad + (-q_0)[\nu_2 \cos(\nu_2 \widehat{L}) + iE \sin(\nu_2 \widehat{L})] \sin(2\nu L_1) [\nu_2 \cos(\nu_2 \widehat{L}) - iE \sin(\nu_2 \widehat{L})] \\
&\quad + (-q_0) \cdot [\nu_2 \cos(\nu_2 \widehat{L}) + iE \sin(\nu_2 \widehat{L})] \cdot \frac{q_0^2 \sin(\nu_2 \widehat{L})}{p_0^2 + q_0^2} \cdot \\
&\quad \cdot [\nu \cos(2\nu L_1) + iE \sin(2\nu L_1) + \nu \frac{p_0^2}{q_0^2} e^{2iEL_1}],
\end{aligned}$$

where $\widehat{L} = L_2 - L_1$.

The reflection coefficients are the given by

$$R_1(E;t) = \frac{B(E)}{A(E)} e^{-2iEL_2} e^{2iE^2 t}, \qquad (3.35)$$

$$R_2(E;t) = \frac{C(E)}{A(E)} e^{-2iEL_2} e^{2iE^2 t}. \qquad (3.36)$$

The expressions of $A(E)$, $B(E)$ and $C(E)$ can be simplified under some asymptotic approximations, but we will do this in Chapter 3.

3.3.3 Threshold for the creation of Eigenvalues

The eigenvalues are defined by the equation

$$A(E_j) = 0. \tag{3.37}$$

However, this equation is too complicated (see equation (3.34)) to perform a clear analysis as the analysis performed in sections 3.1.3 and 3.2.3. Then, we will not study the roots of this equation (3.37) here. However, in chapter 6, we will assume that the number of eigenvalues of this equation is finite.

Chapter 4

On the Location of the Eigenvalues of the Manakov-Zakharov-Shabat System

4.0 Introduction

For the focusing NLS equation, Zakharov and Shabat [ZS] suggested that if the initial data are real, then all the eigenvalues of the Zakharov-Shabat system lie on the imaginary axis. Later, Satsuma and Yajima [SY] postulated that the condition of symmetric initial data should be added. Both statements are, actually, false. In a recent paper [KS1], Klaus and Shaw showed examples of real, nonnegative, symmetric potentials with non-purely imaginary eigenvalues. In a more recent paper [KS2], the same autors proved that real, nonegative, integrable, single-lobe potentials have *purely imaginary* eigenvalues.

Here, I will prove that the eigenvalues of the MZS system are purely imaginary for the cases of squared barrier potentials with same support (section 4.3), and with disjoint support (section 4.4). We will follow the same approach as Shaw and Klaus [KS2], and the notation of Satsuma and Yajima [SY], but applied to the MZS system.

4.1 Preliminary Remarks

We know that the Manakov-Zakharov-Shabat (MZS) system is given by

$$\frac{\partial}{\partial x}\phi = iED\phi + N\phi,$$

Here, $\phi = \phi_k$ is a vector-eigenfunction, and $E = E_k$ is the corresponding eigenvalue. Through this chapter, we will drop the subscript k to simplify notation. As mentioned in chapter 2, the funcition $\phi = \phi_k$ correspond to the first column of the Jost solution $\Phi(x; E_k)$, i.e.: $\phi(x) = \phi_k(x) = \Phi_1(x; E_k)$

The matrices D and N are defined in the usual way as in chapter 2

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad N = N(x,t) = \begin{pmatrix} 0 & u(x,t) & v(x,t) \\ -\sigma u^\star(x,t) & 0 & 0 \\ -\sigma v^\star(x,t) & 0 & 0 \end{pmatrix}.$$

Now, define the matrix

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Multiply the MZS system from the left by $-i\sigma_1 D$

$$-i\sigma_1 D\frac{\partial}{\partial x}\phi + i\sigma_1 DN\phi = E\sigma_1\phi,$$

and then multiply by ϕ from the left, using the inner product

$$-(\phi, i\sigma_1 D\frac{\partial}{\partial x}\phi) + (\phi, i\sigma_1 DN\phi) = E(\phi, \sigma_1\phi). \tag{4.1}$$

The inner product is defined in the usual way

$$(f, g) = \int_{-\infty}^{\infty} f^\star(x)g(x)\, dx,$$

and $f^\star$ denotes the complex conjugate of f.

Since ϕ is eigenfunction (i.e. $|\phi| \to 0$ as $x \to \infty$), once we integrate by parts the first term in equation (4.1), we can check that $(\phi, i\sigma_1 D\frac{\partial}{\partial x}\phi)$ is purely imaginary, since $i\sigma_1 D\frac{\partial}{\partial x}$ is a skew-adjoint operator. It is also easy to check that $(\phi, \sigma_1\phi)$ is real, because σ_1 is self-adjoint. If $(\phi, i\sigma_1 DN\phi)$ is purely imaginary, and $(\phi, \sigma_1\phi) \neq 0$, we can then conclude that E is purely imaginary. For the focusing NLS equation, we immediately have that $(\phi, i\sigma_1 DN\phi)$ is purely imaginary for real initial data, but in our Manakov system, this statement is not straightforward.

Actually, if we define

$$A_r = \frac{1}{2}\begin{pmatrix} -2\mathrm{Im}(u+v) & 0 & 0 \\ 0 & 2\mathrm{Im}(u) & i(u^* - v) \\ 0 & -i(u - v^*) & 2\mathrm{Im}(u) \end{pmatrix},$$

and

$$A_i = \frac{1}{2}\begin{pmatrix} -2\mathrm{Re}(u+v) & 0 & 0 \\ 0 & -2\mathrm{Re}(u) & -(u^*+v) \\ 0 & -(u+v^*) & -2\mathrm{Re}(u) \end{pmatrix},$$

we can write

$$(\phi, i\sigma_1 DN\phi) = (\phi, A_r\phi) + i(\phi, A_i\phi),$$

with $(\phi, A_r\phi) = \mathrm{Re}(\phi, i\sigma_1 DN\phi)$ and $(\phi, A_i\phi) = \mathrm{Im}(\phi, i\sigma_1 DN\phi)$. Notice that the operators A_r and A_i are self-adjoint. The equation (4.1) can be written in the form

$$-(\phi, i\sigma_1 D\frac{\partial}{\partial x}\phi) + i(\phi, A_i\phi) = E(\phi, \sigma_1\phi) - (\phi, A_r\phi).$$

If we prove that $(\phi, A_r\phi) = 0$ and $(\phi, \sigma_1\phi) \neq 0$, we can conclude that E is purely imaginary.

Remember that the eigenvalues E are constant of motion. Then, all this calculations can be performed at $t = 0$. Let us consider the case when the initial data are real functions, i.e., $\mathrm{Im}(u(x,0)) = \mathrm{Im}(v(x,0)) = 0$. Then, the inner product $(\phi, A_r\phi)$ is reduced to

$$(\phi, A_r\phi) = \frac{i}{2}\int_{-\infty}^{\infty} (u(x,0) - v(x,0))(\phi_2^*\phi_3 - \phi_3^*\phi_2)\, dx. \tag{4.2}$$

Here, ϕ_2 and ϕ_3 denote the second and third components of the vector-eigenfunction ϕ. In the following sections, we are going to prove that this integral is zero for squared barrier potentials with the same support, and with disjoint supports, but before we will see how this integral is zero when the Manakov system is reduced to the NLS equation.

4.2 Reduction to the NLS Case

Before we do the analysis for the Manakov system, in this section we will see how the integral in equation (4.2) is automatically zero under the reduction to the NLS equation. Therefore, E would be purely imaginary, according to the results of Klaus and Shaw [KS2]. We then see why $(\phi, A_r\phi)$ is not analyzed by Klaus and Shaw. This is an extra term that appears in the Manakov system, that the authors did not have to concern about.

We know that if $u(x,0) \equiv v(x,0)$, the Manakov system reduces to the NLS equation. We also automatically have that

$$\begin{aligned}(\phi, A_r\phi) &= \frac{i}{2}\int_{-\infty}^{\infty}(u(x,0)-v(x,0))(\phi_2^*\phi_3 - \phi_3^*\phi_2)\,dx \\ &= 0.\end{aligned}$$

Then, this is Klaus and Shaw result.

If $v(x,0) \equiv 0$, the Manakov system reduces again to the NLS equation. Here, the integral in equation (4.2) becomes

$$(\phi, A_r\phi) = \frac{i}{2}\int_{-\infty}^{\infty} u(x,0)(\phi_2^*\phi_3 - \phi_3^*\phi_2)\,dx.$$

On the other hand, the third component of the MZS system is

$$\frac{\partial}{\partial x}\phi_3 = iE\phi_3,$$

whose solution is $\phi_3(x) = \phi_3(0)e^{iEx}$. Assume that Im$(E)>0$. Since ϕ is eigenfunction, we should have $|\phi_3| \to 0$ as $x \to \infty$. Now

$$\begin{aligned}|\phi_3(x)| = e^{-\operatorname{Im}(E)x}|\phi_3(0)| &\longrightarrow 0 \qquad \text{as} \qquad x \to +\infty \\ &\longrightarrow \infty \qquad \text{as} \qquad x \to -\infty,\end{aligned}$$

unless $\phi_3(0) = 0$. Then $\phi_3(x) \equiv 0$, and so

$$\begin{aligned}(\phi, A_r\phi) &= \frac{i}{2}\int_{-\infty}^{\infty} u(x,0)(\phi_2^*\phi_3 - \phi_3^*\phi_2)\,dx \\ &= 0.\end{aligned}$$

Then, E should be purely imaginary, again in agreement with the result of Klaus and Shaw. The case $u(x, 0) \equiv 0$ is also a reduction of Manakov to NLS, and the same argument works, but in that instance $\phi_2(x) \equiv 0$.

One more (quite general) reduction to the NLS equation is considered next. For $\alpha, \beta \in \mathbb{C}$, such that

$$|\alpha|^2 + |\beta|^2 = 1,$$

the transformation

$$\begin{aligned} u(x,t) &= \alpha U(x,t), \\ v(x,t) &= \beta U(x,t), \end{aligned}$$

reduces the Manakov system into the NLS equation

$$i\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial x^2} + \sigma |U|^2 U = 0.$$

Under this transformation, the MZS system becomes

$$\begin{aligned} \frac{\partial \phi_1}{\partial x} &= -iE\phi_1 + \alpha U\phi_2 + \beta U\phi_3 \\ \frac{\partial \phi_2}{\partial x} &= iE\phi_2 - \sigma\alpha^* U^*\phi_1 \\ \frac{\partial \phi_3}{\partial x} &= iE\phi_3 - \sigma\beta^* U^*\phi_1. \end{aligned}$$

Multiply second and third equations by α and β respectively, and add them:

$$\frac{\partial}{\partial x}(\alpha\phi_2 + \beta\phi_3) = iE(\alpha\phi_2 + \beta\phi_3) - \sigma U^*\phi_1,$$

where we used the fact that $|\alpha|^2 + |\beta|^2 = 1$. Define

$$\widetilde{\phi}_2 \equiv \alpha\phi_2 + \beta\phi_3.$$

We then obtain

$$\begin{aligned} \frac{\partial \phi_1}{\partial x} &= -iE\phi_1 + U\widetilde{\phi}_2 \\ \frac{\partial \widetilde{\phi}_2}{\partial x} &= iE\widetilde{\phi}_2 - \sigma U^*\phi_1, \end{aligned}$$

which is the Zakharov-Shabat system of the NLS equation, and then the Klaus-Shaw result applies.

4.3 Initial Data with Same Support

The following two subsections are dedicated to the proof of the following:

Proposition 4.1. *(*PURELY IMAGINARY EIGENVALUES OF THE MANAKOV-ZAKHAROV-SHABAT (MZS) SYSTEM*) Consider the initial potentials with same support*

$$\begin{aligned} u(x,0) &= p_0 F_L(x), \\ v(x,0) &= q_0 F_L(x), \end{aligned}$$

or with disjoint support

$$\begin{aligned} u(x,0) &= \begin{cases} p_0, & -L_2 \le x \le -L_1 \\ 0, & otherwise \end{cases}, \\ v(x,0) &= \begin{cases} q_0, & L_1 \le x \le L_2 \\ 0, & otherwise \end{cases}, \end{aligned}$$

with p_0 and q_0 positive numbers. Then, the eigenvalues of the MZS system (equation (2.8))

$$\frac{\partial}{\partial x}\phi = iED\phi + N\phi,$$

are **purely imaginary.**

Proof: In this section we prove the proposition for the case of same support. The case of disjoint support is considered in the next section.

To prove that the eigenvalues, E, of the MZS are purely imaginary, it suffices to prove that $(\phi, A_r\phi) = 0$ and $(\phi, \sigma_1\phi) \neq 0$.

The Inner Product $(\phi, A_r\phi)$ Vanishes. For the initial data with same support, the MZS system has the following solution

$$\begin{aligned}
\phi_I(x) &= e^{iEDx}\,\alpha, && x<-L,\\
\phi_{II}(x) &= e^{(iED+N_0)x}\,\beta, && |x|\le L,\\
\phi_{III}(x) &= e^{iEDx}\,\gamma, && L<x,
\end{aligned}$$

where α, β and γ are constant vectors in $\mathbb{C}^3$. The continuity conditions $\phi_I(-L) = \phi_{II}(-L)$ and $\phi_{II}(L) = \phi_{III}(L)$ imply

$$\begin{aligned}
\beta &= e^{(iED+N_0)L}e^{-iEDL}\alpha\\
\gamma &= e^{-iEDL}e^{2(iED+N_0)L}e^{-iEDL}\alpha.
\end{aligned}$$

Since ϕ is eigenfunction, $|\phi(x)| \to 0$ as $|x| \to \infty$. Now, this asymptotic behavior implies (for $\mathrm{Im}(E) > 0$)

$$\alpha = \begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad \text{and} \qquad \gamma = \begin{pmatrix}0\\\gamma_2\\\gamma_3\end{pmatrix}.$$

Now, since $u(x,0) = v(x,0) = 0$ for $L < |x|$, the inner product in equation (4.2) simplifies to

$$(\phi, A_r\phi) = \frac{i}{2}(p_0 - q_0)\int_{-L}^{L}\Big((\phi_{II})_2^*(\phi_{II})_3 - (\phi_{II})_3^*(\phi_{II})_2\Big)\,dx.$$

Now

$$\begin{aligned}
\phi_{II}(x) = e^{(iED+N_0)x}\,\beta &= e^{(iED+N_0)x}e^{(iED+N_0)L}e^{-iEDL}\alpha,\\
&= e^{(iED+N_0)(x+L)}e^{iEL}\begin{pmatrix}1\\0\\0\end{pmatrix},
\end{aligned}$$

i.e.

$$\phi_{II}(x) = \frac{e^{iEL}}{\nu(E)}\begin{pmatrix}\nu(E)\cos\Big(\nu(E)(x+L)\Big) - iE\sin\Big(\nu(E)(x+L)\Big)\\ -p_0\sin\Big(\nu(E)(x+L)\Big)\\ -q_0\sin\Big(\nu(E)(x+L)\Big)\end{pmatrix}. \tag{4.3}$$

Then, for $|x| \leq L$

$$
\begin{aligned}
(\phi_{II})_2^*(\phi_{II})_3 - (\phi_{II})_3^*(\phi_{II})_2 &= \left|\frac{e^{iEL}}{\nu(E)}\right|^2 p_0 q_0 \left(|\sin(\nu(x+L))|^2 - |\sin(\nu(x+L))|^2\right) \\
&= 0.
\end{aligned}
$$

Therefore

$$(\phi, A_r\phi) = 0.$$

The Inner Product $(\phi, \sigma_1\phi) \neq 0$ *is not Zero.* Now, we should check that $(\phi, \sigma_1\phi) \neq 0$. We have that

$$(\phi, \sigma_1\phi) = \int_{-\infty}^{\infty} \phi_1^*(\phi_2 + \phi_3) + \phi_1(\phi_2^* + \phi_3^*)\, dx,$$

In the interval $x < -L$, $\phi_I = e^{iEx}(1,0,0)^t$. Then: $(\phi_I)_2 = (\phi_I)_3 = 0$. Similarly, in the interval $L < x$, $\phi_{III} = e^{-iEx}(0,\gamma_2,\gamma_3)^t$. Then: $(\phi_{III})_1 = 0$. Hence

$$(\phi, \sigma_1\phi) = \int_{-L}^{L} (\phi_{II})_1^*\Big((\phi_{II})_2 + (\phi_{II})_3\Big) + (\phi_{II})_1\Big((\phi_{II})_2^* + (\phi_{II})_3^*\Big)\, dx.$$

Using the explicit form of ϕ_{II} given in equation (4.3), we observe that

$$(\phi_{II})_1^*\Big((\phi_{II})_2 + (\phi_{II})_3\Big) + (\phi_{II})_1\Big((\phi_{II})_2^* + (\phi_{II})_3^*\Big) =$$

$$= -(p_0+q_0)\left|\frac{e^{iEL}}{\nu(E)}\right|^2 \left\{ \begin{array}{c} \overline{\nu(E)\cos\Big(\nu(E)(x+L)\Big)}\sin\Big(\nu(E)(x+L)\Big)+ \\ +\nu(E)\cos\Big(\nu(E)(x+L)\Big)\overline{\sin\Big(\nu(E)(x+L)\Big)}+ \\ +2\,\mathrm{Im}(E)|\sin\Big(\nu(E)(x+L)\Big)|^2 \end{array} \right\}.$$

We can identify an exact derivative

$$(\phi_{II})_1^*\Big((\phi_{II})_2 + (\phi_{II})_3\Big) + (\phi_{II})_1\Big((\phi_{II})_2^* + (\phi_{II})_3^*\Big) =$$

$$= -(p_0+q_0)\left|\frac{e^{iEL}}{\nu(E)}\right|^2 \left\{ \begin{array}{c} \frac{d}{dx}\left(\overline{\sin\Big(\nu(E)(x+L)\Big)}\sin\Big(\nu(E)(x+L)\Big)\right)+ \\ +2\,\mathrm{Im}(E)\Big|\sin\Big(\nu(E)(x+L)\Big)\Big|^2 \end{array} \right\}.$$

Once we integrate from $-L$ to L

$$(\phi,\sigma_1\phi) = -(p_0+q_0)\left|\frac{e^{iEL}}{\nu(E)}\right|^2 \left\{ \begin{array}{c} \left|\sin\Big(2\nu(E)L\Big)\right|^2 + \\ +2\,\mathrm{Im}(E)\int\limits_{-L}^{L} |\sin\Big(\nu(E)(x+L)\Big)|^2\,dx \end{array} \right\}.$$

Since $\mathrm{Im}(E) > 0$, then

$$(\phi,\sigma_1\phi) \neq 0.$$

This implies that E **is purely imaginary** when the initial potentials have the same support.

4.4 Initial Data with Disjoint Supports

In this section, we continue the proof of Proposition 4.1, for the case of initial potentials with disjoint supports.

The Inner Product $(\phi, A_r\phi)$ *Vanishes.* As in the preceeding section, we have to prove that $(\phi, A_r\phi) = 0$ and $(\phi,\sigma_1\phi)\neq 0$ for potentials with same support. In this instance, the inner product $(\phi,\sigma_1\phi)$ becomes

$$\begin{aligned}(\phi, A_r\phi) \quad = \quad & \frac{i}{2}p_0\int\limits_{-L_2}^{-L_1}\Big((\phi_{II})_2^*(\phi_{II})_3-(\phi_{II})_3^*(\phi_{II})_2\Big)\,dx \\ & -\frac{i}{2}q_0\int\limits_{L_1}^{L_2}\Big((\phi_{IV})_2^*(\phi_{IV})_3-(\phi_{IV})_3^*(\phi_{IV})_2\Big)\,dx.\end{aligned} \tag{4.4}$$

where the solution $\phi(x)$ to the MZS system is given by

$$\begin{aligned}\phi_I(x) &= e^{iEDx}\,\alpha, && x<-L_2,\\ \phi_{II}(x) &= e^{(iED+N_1)x}\,\beta, && -L_2\le x<-L_1,\\ \phi_{III}(x) &= e^{iEDx}\,\gamma, && -L_1\le x\le L_1,\\ \phi_{IV}(x) &= e^{(iED+N_2)x}\,\delta, && L_1<x\le L_2,\\ \phi_V(x) &= e^{iEDx}\,\epsilon, && L_2<x,\end{aligned} \tag{4.5}$$

with α, β, γ, δ, and ϵ are constant vectors in $\mathbb{C}^3$. We proceed as in the previous section. We should determine the vectors β, and δ, since they determine $\phi_{II}(x)$ and $\phi_{IV}(x)$.

By the continuity conditions, we find

$$\begin{aligned} \beta &= e^{(iED+N_1)L_2} e^{-iEDL_2} \alpha, \\ \gamma &= e^{iEDL_1} e^{-(iED+N_1)L_1} e^{(iED+N_1)L_2} e^{-iEDL_2} \alpha \\ \delta &= e^{-(iED+N_2)L_2} e^{iEDL_2} S(E) \alpha, \\ \epsilon &= S(E)\alpha, \end{aligned} \tag{4.6}$$

with $S(E)$ the Scattering matrix found in section 3.2. Note again, that here there is a change in notation: $L_1 = L - \widehat{\alpha}$ and $L_2 = L + \widehat{\alpha}$.

The asymptotic behavior $|\phi| \to 0$ as $|x| \to \infty$, implies

$$\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \epsilon = S(E)\alpha = \begin{pmatrix} 0 \\ S_{21}(E) \\ S_{31}(E) \end{pmatrix}.$$

(Remeber that $S_{11}(E) = 0$ at the eigenvalues $E = E_k$.)

We can now compute $\phi_{II}(x)$

$$\begin{aligned} \phi_{II}(x) &= e^{(iED+N_1)x}\,\beta \\ &= \frac{e^{iEL_2}}{\nu_1(E)} e^{(iED+N_1)(x+L_2)} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \\ &= \frac{e^{iEL_2}}{\nu_1(E)} \begin{pmatrix} b_{11}(x+L_2) \\ b_{21}(x+L_2) \\ b_{31}(x+L_2) \end{pmatrix}, \end{aligned}$$

where $b_{11}(x+L_2)$, $b_{21}(x+L_2)$ and $b_{31}(x+L_2)$ are the corresponding entries of $e^{(iED+N_1)(x+L_2)}$. We have that $b_{31} = 0$ (see section 2.3, equation (3.25)), i.e., $(\phi_{II})_3(x) = 0$. Then

$$\int_{-L_2}^{-L_1} \Big((\phi_{II})_2^* (\phi_{II})_3 - (\phi_{II})_3^* (\phi_{II})_2 \Big)\, dx = 0. \tag{4.7}$$

Similarly

$$\begin{aligned}
\phi_{IV}(x) &= e^{(iED+N_2)x}\,\delta \\
&= e^{(iED+N_2)(x-L_2)}e^{iEDL_2}S(E)\alpha, \\
&= e^{iEL_2}\begin{pmatrix} c_{13}(L_2-x)S_{31}(E) \\ c_{22}(L_2-x)S_{21}(E) \\ c_{33}(L_2-x)S_{31}(E) \end{pmatrix},
\end{aligned}$$

where $c_{13}(L_2-x)$, $c_{22}(L_2-x)$ and $c_{33}(L_2-x)$ are the corresponding entries of $e^{(iED+N_2)(x-L_2)}$ (see section 2.3, equation (3.26)).

It is not important to know what $S_{21}(E)$ exactly is (*cf.* equation (3.28)), since all the information we need is contained in $S_{31}(E)$

$$S_{31}(E) = b_{11}(E)c_{31}(E).$$

Now, remember that (*cf.* equation (3.27))

$$S_{11}(E) = \frac{b_{11}(E)c_{11}(E)}{\nu_1(E)\nu_2(E)}e^{2iE(L_2-L_1)},$$

with

$$\begin{aligned}
b_{11}(E) &= \nu_1(E)\cos\Big(\nu_1(E)(L_2-L_1)\Big) - iE\sin\Big(\nu_1(E)(L_2-L_1)\Big), \\
c_{11}(E) &= \nu_2(E)\cos\Big(\nu_2(E)(L_2-L_1)\Big) - iE\sin\Big(\nu_2(E)(L_2-L_1)\Big).
\end{aligned}$$

Notice that $b_{11}(E)$ appears in both, $S_{11}(E)$ and $S_{31}(E)$.

Since E is an eigenvalue, then $S_{11}(E) = 0$. This implies either $b_{11}(E) = 0$ or $c_{11}(E) = 0$. If $b_{11}(E) = 0$, then $S_{31}(E) = 0$ and so $(\phi_{IV})_3 = 0$.

If $c_{11}(E) = 0$, we have

$$\nu_2(E)\cos\Big(\nu_2(E)(L_2-L_1)\Big) - iE\sin\Big(\nu_2(E)(L_2-L_1)\Big) = 0.$$

The threshold for this to be true is

$$\frac{\pi}{2} = q_0(L_2-L_1).$$

Assume for a moment that $p_0 > q_0$. Then

$$\frac{\pi}{2} = q_0(L_2 - L_1) < p_0(L_2 - L_1).$$

which implies $b_{11}(E) = 0$, since the threshold was also reached. Hence $(\phi_{IV})_3 = 0$.

In either case, $b_{11}(E) = 0$ or $c_{11}(E) = 0$, so we obtain

$$(\phi_{IV})_3 = 0,$$

whenever $p_0 > q_0$. Hence

$$\int_{L_1}^{L_2} \Big((\phi_{IV})_2^* (\phi_{IV})_3 - (\phi_{IV})_3^* (\phi_{IV})_2 \Big) \, dx = 0. \tag{4.8}$$

Substitute the values of the integrals in equations (4.8) and (4.7), into the integral in equation (4.4). We obtain that the inner product vanishes, $(\phi, A_r\phi) = 0$, for p_0, q_0 real and positive numbers, and $p_0 > q_0$.

For the case $p_0 < q_0$, we use the symmetries of the Manakov system. If $p_0 < q_0$, then use the fact that the transformations $x \longleftrightarrow -x$, and $u(x,t) \longleftrightarrow v(x,t)$ leave the Manakov system unchanged. Then define $\widetilde{q}_0 \equiv p_0$ and $\widetilde{p}_0 \equiv q_0$. The initial conditions then become

$$\begin{aligned} u(x,0) &= \begin{cases} \widetilde{p}_0, & -L_2 \le x \le -L_1 \\ 0, & \text{otherwise} \end{cases}, \\ v(x,0) &= \begin{cases} \widetilde{q}_0, & L_1 \le x \le L_2 \\ 0, & \text{otherwise} \end{cases}, \end{aligned}$$

with $\widetilde{p}_0 > \widetilde{q}_0$. Repeat the previous argument, and find that the inner product $(\phi, A_r\phi) = 0$ vanishes for $\widetilde{p}_0 > \widetilde{q}_0$. We can conclude that the inner product $(\phi, A_r\phi)$ vanishes

$$(\phi, A_r\phi) = 0 \tag{4.9}$$

for **any** p_0, q_0 positive constants.

The Inner Product $(\phi, \sigma_1\phi) \neq 0$ *is not Zero.* It remains to check that $(\phi, \sigma_1\phi) \neq 0$. This is similar to what we did in section 4.3. We have

$$(\phi, \sigma_1\phi) = \int_{-\infty}^{\infty} \phi_1^*(\phi_2 + \phi_3) + \phi_1(\phi_2^* + \phi_3^*)\, dx,$$

Since ϕ is eigenfunction

$$\begin{aligned} \phi_I(x) &= e^{-iEx}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad x < -L_2 \\ \phi_V(x) &= e^{iEx}\begin{pmatrix} 0 \\ S_{21}(E) \\ S_{31}(E) \end{pmatrix}, \qquad x > L_2 \end{aligned}$$

Hence

$$\begin{aligned} (\phi, \sigma_1\phi) &= \int_{-L_2}^{L_2} \phi_1^*(\phi_2 + \phi_3) + \phi_1(\phi_2^* + \phi_3^*)\, dx, \\ &= \int_{-L_2}^{-L_1} + \int_{-L_1}^{L_1} + \int_{L_1}^{L_2} \phi_1^*(\phi_2 + \phi_3) + \phi_1(\phi_2^* + \phi_3^*)\, dx, \end{aligned}$$

For $x \in [-L_2, -L_1]$, we use equations (4.5) and (4.6), to get

$$\begin{aligned} \phi_{II}(x) &= e^{iEL_2} e^{(iED+N_1)(x+L_2)} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \frac{e^{iEL_2}}{\nu_1(E)} \begin{pmatrix} a_1(x+L_2) \\ a_3(x+L_2) \\ 0 \end{pmatrix}, \end{aligned}$$

where using equations (3.25), we have

$$\begin{aligned} a_1 = a_1(x+L_2) &= \nu_1(E)\cos(\nu_1(E)x) - iE\sin(\nu_1(E)x) \\ a_3 = a_3(x+L_2) &= -p_0 \sin(\nu_1(E)x) \end{aligned}$$

Hence

$$\begin{aligned}
\phi_1^*(\phi_2+\phi_3)+\phi_1(\phi_2^*+\phi_3^*) &= (\phi_{II})_1^*\Big((\phi_{II})_2+(\phi_{II})_3\Big)+(\phi_{II})_1\Big((\phi_{II})_2^*+(\phi_{II})_3^*\Big)\\
&= (\phi_{II})_1^*(\phi_{II})_2+(\phi_{II})_1(\phi_{II})_2^*\\
&= \left|\frac{e^{iEL_2}}{\nu_1(E)}\right|^2\left(\overline{a_1}a_3+a_1\overline{a_3}\right)\\
&= -p_0\left|\frac{e^{iEL_2}}{\nu_1(E)}\right|^2\left[\frac{d}{dx}\left(\Big|\sin(\nu_1(E)(x+L_2))\Big|^2\right)+\right.\\
&\qquad\left.+\,2\,\mathrm{Im}(E)\Big|\sin(\nu_1(E)(x+L_2))\Big|^2\right].
\end{aligned}$$

Integrating over $[-L_2,-L_1]$

$$\begin{aligned}
\int_{-L_2}^{-L_1}\phi_1^*\phi_2+\phi_1\phi_2^*\,dx &= -p_0\left|\frac{e^{iEL_2}}{\nu_1(E)}\right|^2\left[\Big|\sin(\nu_1(E)(L_2-L_1))\Big|^2\right.\\
&\qquad\left.+\,2\,\mathrm{Im}(E)\int_{-L_2}^{-L_1}\Big|\sin(\nu_1(E)(x+L_2))\Big|^2\,dx\right].
\end{aligned}$$

Since $\mathrm{Im}(E)>0$, it follows that

$$\int_{-L_2}^{-L_1}\phi_1^*\phi_2+\phi_1\phi_2^*\,dx<0.$$

For $x\in[L_1,L_2]$, we use equations (4.5) and (4.6), and also equation (3.26) to get

$$\begin{aligned}
\phi_{IV}(x) &= e^{iEL_2}\begin{pmatrix} b_2(x-L_2)S_{31}(E)\\ b_3(x-L_2)S_{21}(E)\\ b_5(x-L_2)S_{31}(E)\end{pmatrix}\\
&= e^{iEL_2}\begin{pmatrix} 0\\ b_3(x-L_2)S_{21}(E)\\ 0\end{pmatrix},
\end{aligned}$$

since $S_{31}(E)=$ if $p_0>q_0$, as we mentioned earlier in this section. Then

$$\int_{L_1}^{L_2}\phi_1^*(\phi_2+\phi_3)+\phi_1(\phi_2^*+\phi_3^*)\,dx=0.$$

Using the simmetries $x \longleftrightarrow -x$, and $u(x,t) \longleftrightarrow v(x,t)$, of the Manakov system, and arguing the same way we did before equation 4.9, we obtain the same result for $p_0 < q_0$l.

For $x \in [-L_1, L_1]$, by equations (4.5) and (4.6), and also (3.26)

$$\begin{aligned} \phi_{III}(x) &= e^{iEDx}\gamma \\ &= -p_0 e^{iE(x+L_1+L_2)} \begin{pmatrix} 0 \\ \frac{\nu_2(E)\sin\big((L_2-L_1)\nu_1(E)\big)}{\nu_1(E)} \\ 0 \end{pmatrix} \end{aligned}$$

Then $(\phi_{III})_1(x) = 0$. Therefore

$$\int_{-L_1}^{L_1} \phi_1^*(\phi_2+\phi_3) + \phi_1(\phi_2^*+\phi_3^*)\,dx = 0.$$

We then have

$$\int_{-L_2}^{-L_1} + \int_{-L_1}^{L_1} + \int_{L_1}^{L_2} \phi_1^*(\phi_2+\phi_3) + \phi_1(\phi_2^*+\phi_3^*)\,dx < 0,$$

i.e.

$$(\phi, \sigma_1\phi) < 0.$$

We now can conclude that the eigenvalues of the Manakov-Zakharov-Shabat system are *purely imaginary,* under the hypothesis mentioned in proposition 4.1.

Q.E.D.

As a final remark, just let me say that, in a recent paper [Kla], M. Klaus proved that the condition for the *absence* of eigenvalues is

$$\int_{-\infty}^{\infty} \sqrt{u^2(x,0)+v^2(x,0)}\,dx < \frac{\pi}{2}.$$

This result obviously coincides with the threshold conditions found in this thesis.

CHAPTER 5

THE INVERSE SCATTERING PROBLEM: THE RIEMANN-HILBERT PROBLEM APPROACH

5.0 Introduction

This chapter is dedicated to the theory of the Inverse Scattering of the *focusing* Manakov system. In different words, we will describe how to recover the solutions $u(x,t)$ and $v(x,t)$ of the Manakov system, potentials of the Manakov-Zakharov-Shabat (MZS) system, equation (2.8)

$$\frac{\partial}{\partial x}\Psi = iED\Psi + N\Psi.$$

The technique to do this is the Riemann-Hilbert Problem (RHP). We formulate a *meromorphic Riemann-Hilbert Problem* (mRHP) in terms of the Scattering Data (SD) of the MZS system at $t = 0$. Since the SD evolution in time is simple, then a mRHP at any time t is automatically established.

Now, we do know how to solve RHPs. For this reason, we transform the mRHP into a regular *holomorphic Riemann-Hilbert Problem* (hRHP), with additional jump matrices along some specified curves in the upper and lower half complex planes. The interior of these curves actually contain the eigenvalues of the MZS system. We will be following the ideas given by Miller [Mi], and Kamvisis, McLaughlin and Miller [KMM], but applied to the MZS system.

We will also briefly describe how to obtain the solution of the hRHP (by the use of integral equations), and from this how to find the solutions of the focusing Manakov system. For details, we will refer the reader to [DZ] and [DiF] for the defocusing NLS case. For the Manakov system, see [Shc-Dok].

This last step, the inverse scattering step is, in general, not amenable of a closed and nice form. This is the reason we employ approximation theory given by asymptotic analysis. The required information and the problem will be left ready, at the end of this chapter, to perform the short-time asymptotics in the next one.

I also want to remark here that the transformation of the mRHP into a hRHP can be performed more generally (see [KMM] and [Mi] for the focusing NLS equation). However, the transformation given in section 5.8 (equations (5.38), (5.39) and (5.40)) is enough for our purposes. In addition, this transformation is given in terms of a *contour deformation*, (which can be thought as a nonlinear analogous of the steepest descent method for linear problems; see for example [DZ] and the references therein). This contour deformation has the particularity that not only removes the poles, but also removes a segment in the real line containg the origin. This will be particularly useful in chapter 6, since the reflection coefficients will be small for $t \sim 0$ along the new contours (since we removed a singularity at $E = 0$), so that we can approximate the solution of the Riemann-Hilbert Problem by Neuman series.

In conclusion of these introductory paragraphs, it should be remarked that the concept of a Riemann-Hilbert Problem formulated in application to integrable systems was first proposed by V.E. Zakharov and A.B Shabat in 1979 [ZS2].

5.1 The Meromorphic Riemann-Hilbert Problem.

As it is well known [Ma], the focusing Manakov system can support solitons, which are defined by the poles of the transmision coefficients, $\frac{1}{S_{11}(E)}$ (equation 2.16), associated to the scattering problem of the MZS system. The reflection coefficients, the proportionality constants, and the residues at the poles, all together will define the **meromorphic Riemann-Hilbert Problem** (mRHP).

In this section, we state the meromorphic Riemann-Hilbert Problem (mRHP) associated to the MZS system. The mRHP, as defined here, is an extension to the Manakov system of the work done by Kamvissis, McLaughlin and Miller [KMM] on the focusing NLS equation. The forthcoming sections will describe in detail how to contruct this mRHP out of the Scattering Data (SD) of the MZS system.

Problem 1. Meromorphic Riemann-Hilbert Problem *(mRHP). Fix x and t. Given the discrete data $\{E_k\}_{k=1}^N$, $\{C_k(t), D_k(t)\}_{k=1}^N$, together with their complex conjugates, and the jump matrix $V(E) = V(E; x, t)$ along $E \in \mathbb{R}$, find a matrix $m(E) = m(x, t; E)$ such that:*

1. **Rationality:** *The matrix $m(E)$ is a meromorphic function of E, with simple poles located at the eigenvalues of the MZS system, $\{E_k\}_{k=1}^N$, and at its complex conjugates. The residues at the poles are given by*

$$\operatorname*{Res}_{E=E_k} m(E) = \lim_{E\to E_k} m(E) \begin{pmatrix} 0 & 0 & 0 \\ c_k(x,t) & 0 & 0 \\ d_k(x,t) & 0 & 0 \end{pmatrix}, \tag{5.1}$$

$$\operatorname*{Res}_{E=E_k^*} m(E) = \lim_{E\to E_k^*} m(E) \begin{pmatrix} 0 & -c_k^*(x,t) & -d_k^*(x,t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.2}$$

for $k = 1, 2, \ldots, n$, with

$$c_k(x,t) \equiv \frac{C_k(t)}{S_{11}'(E_k)} e^{2iE_k x}, \tag{5.3}$$

$$d_k(x,t) \equiv \frac{D_k(t)}{S_{11}'(E_k)} e^{2iE_k x}, \tag{5.4}$$

and where $C_k(t)$ and $D_k(t)$ are the proportionality constants.

2. **Jump matrix:** *On the real axis, there is a jump relation*

$$m_+(E) = m_-(E)V(E), \quad E \in \mathbb{R}, \tag{5.5}$$

with

$$m_\pm(E) = \lim_{\epsilon \searrow 0} m(E \pm i\epsilon),$$

where $V_r(E) = V_r(E;x,t)$ is the **jump matrix** *(constructed from the scattering matrix), and explicitly depends on space, x, and time, t*

$$V_r(E) = \begin{pmatrix} 1 + \left|\frac{S_{21}(E;t)}{S_{11}(E)}\right|^2 + \left|\frac{S_{31}(E;t)}{S_{11}(E)}\right|^2 & \frac{\overline{S_{21}(E;t)}}{\overline{S_{11}(E)}} e^{-2iEx} & \frac{\overline{S_{31}(E;t)}}{\overline{S_{11}(E)}} e^{-2iEx} \\ \frac{S_{21}(E;t)}{S_{11}(E)} e^{2iEx} & 1 & 0 \\ \frac{S_{31}(E;t)}{S_{11}(E)} e^{2iEx} & 0 & 1 \end{pmatrix} \tag{5.6}$$

3. **Normalization:** *The matrix $m(E)$ satisfies the following limit on each plane*

$$m(E) \longrightarrow I, \quad \text{as} \quad E \longrightarrow \infty.$$

We proceed now to describe how construct this mRHP out of the Scattering Data.

5.2 The Jump Matrix.

The ideas presented in this section are those presented by Shchesnovich and Doktorov in [Shc-Dok], and Novikov, *et.al* [NMPZ].

Remember that the MZS system, equation (2.8), has two sets of linearly independent solutions, $\Psi(x;E)$ and $\Phi(x;E)$, which satisfy the following limits

$$\begin{aligned} \lim_{x\to+\infty} \Psi(x;E)e^{-iEDx} &= I, \\ \lim_{x\to-\infty} \Phi(x;E)e^{-iEDx} &= I, \end{aligned}$$

where I is the 3×3 identity matrix. The matrices Ψ and Φ are related by the scattering matrix, $S(E)$, equation (2.18)

$$\Phi(x;E) = \Psi(x;E)S(E). \tag{5.7}$$

Let us write the matrices Ψ and Φ in columns

$$\begin{aligned} \Psi &= (\Psi_1, \Psi_2, \Psi_3), \\ \Phi &= (\Phi_1, \Phi_2, \Phi_3). \end{aligned}$$

In the appendix B, we will show that the columns Φ_1, Ψ_2, Ψ_3 are analytic in the upper-half complex plane, when considered as functions of the complex variable E. On the other hand, Ψ_1, Φ_2, Φ_3 are analytic in the lower-half plane of E. Now, the goal is to construct a matrix $M(E)$, meromorphic in the complex plane, satisfying a jump relation on the real axis. To do this, we define the following auxiliary, lower triangular, matrix

$$S_L(E) \equiv \begin{pmatrix} 1 & 0 & 0 \\ \frac{S_{21}(E)}{S_{11}(E)} & 1 & 0 \\ \frac{S_{31}(E)}{S_{11}(E)} & 0 & 1 \end{pmatrix}. \tag{5.8}$$

Using the scattering relation, equation (5.7), we find

$$\Psi(x;E)S_L(E) = \left(\frac{\Phi_1(x;E)}{S_{11}(E)}, \Psi_2(x;E), \Psi_3(x;E) \right), \tag{5.9}$$

which is a complex matrix function, meromorphic in the upper semi-plane, with poles at $\{E_k\}_{k=1}^n$. We then define the following matrix for $\mathrm{Im}(E) > 0$

$$M_{\mathrm{up}}(E;x) \equiv \Psi(x;E)S_L(E) = \left(\frac{\Phi_1(x;E)}{S_{11}(E)}, \Psi_2(x;E), \Psi_3(x;E) \right), \tag{5.10}$$

We also define, for $\mathrm{Im}(E) < 0$

$$M^{-1}_{\mathrm{down}}(E;x) \equiv S^*_L(E)\Psi^{-1}(x;E) = S^*_L(E)\Psi^*(x;E) = \begin{pmatrix} \frac{1}{\overline{S_{11}(E^*)}}\Phi^*_1 \\ \Psi^*_2 \\ \Psi^*_3 \end{pmatrix}, \tag{5.11}$$

where we used the fact that Ψ is a unitary matrix, $\Psi\Psi^* = I$, and we also used the scattering relation equation (5.7). This matrix $M_-^{-1}(E;x)$ is also meromorphic, but now in the lower-half complex plane, with poles at the points $\{E^*_k\}_{k=1}^n$, zeroes of the equation $\overline{S_{11}(E^*)} = 0$.

We then define the matrix $M(E) = M(E;x)$

$$M(E;x,t) \equiv \begin{cases} M_{\mathrm{up}}(x;E), & \text{if} \quad \mathrm{Im}(E) > 0, \ \text{and} \quad E \neq E_k, \quad k = 1,2,\dots,n \\ M_{\mathrm{down}}(x;E), & \text{if} \quad \mathrm{Im}(E) < 0, \ \text{and} \quad E \neq E^*_k, \quad k = 1,2,\dots,n, \end{cases} \tag{5.12}$$

Because $M(E)$ is a linear combination of columns of Ψ and Φ, $M(E)$ also satisfies the MZS system

$$\frac{\partial}{\partial x} M = \mathcal{L} M.$$

We now proceed to define a candidate matrix, $m(x;E)$, solution of the mRHP, problem 1. This new matrix is defined as

$$m(E;x) \equiv M(E;x)e^{-iEDx}. \tag{5.13}$$

We can check that, for $E \in \mathbb{C}^+$, $m(E;x)$ has the following properties:

(1) $m(E;x) \to I$ as $x \to +\infty$,

(2) $m(E;x)$ is bounded as $x \to -\infty$, and

(3) it solves the differential equation

$$\frac{\partial}{\partial x}m = \mathcal{L}m - iEmD,$$

i.e.

$$\frac{\partial}{\partial x}m = iE[D,m] + Nm. \tag{5.14}$$

Finally, we define the boundary values at the real axis, $E \in \mathbb{R}$

$$m_\pm(E;x) \equiv \lim_{\epsilon \searrow 0} m_\pm(E \pm i\epsilon; x).$$

We are now in good shape to define the **jump matrix**, $V(E) = V(E;x)$. Along the real axis, we have:

$$\begin{aligned} m_-^{-1}(E;x)m_+(E;x) &= e^{iEDx}M_-^{-1}M_+e^{-iEDx} \\ &= e^{iEDx}(S_L^*\Psi^{-1})(\Psi S_L)e^{-iEDx} \\ &= e^{iEDx}(S_L^*S_L)e^{-iEDx} \\ &= V_r(E), \end{aligned}$$

where we defined

$$V_r(E) \equiv e^{iEDx}(S_L^*S_L)e^{-iEDx} \tag{5.15}$$

Explicitly

$$V_r(E) = \begin{pmatrix} 1 + \left|\frac{S_{21}}{S_{11}}\right|^2 + \left|\frac{S_{31}}{S_{11}}\right|^2 & \frac{\overline{S_{21}}}{\overline{S_{11}}}e^{-2iEx} & \frac{\overline{S_{31}}}{\overline{S_{11}}}e^{-2iEx} \\ \frac{S_{21}}{S_{11}}e^{2iEx} & 1 & 0 \\ \frac{S_{31}}{S_{11}}e^{2iEx} & 0 & 1 \end{pmatrix} \tag{5.16}$$

There are two facts we should point out. First, notice that

$$\det V_r(E) = 1.$$

This follows from the factorization of $V_r(E)$, equation (5.15), and the fact that S_L is diagonal with determinant equal to 1: $\det S_L = 1$.

Second, if we define $v_r(E) = S_L^* S_L$, we observe that this matrix is x-independent, as opposed to $V_r(E;x)$, which depends explicitly on x. The complex exponential terms do not appear in $v_r(E)$. This $v_r(E)$ is the jump relation of $M(E)$

$$\begin{aligned} M_-^{-1}(E;x)M_+(E;x) &= (S_L^*\Psi^{-1})(\Psi S_L) \\ &= S_L^* S_L \\ &\equiv v_r(E). \end{aligned}$$

Since $M_{\text{up}}(E)$ and $M_{\text{down}}(E)$ solve the MZS system, then $M_+(E)$ and $M_-(E)$ also do along $\mathbb{R}$. We have two sets of solutions of the MZS system along the real axis. Therefore, they should be interdependent. Their dependence is given by the jump $v_r(E)$.

Now, the definition (5.15) already provides a factorization of the jump matrix. We will be using the following factorization

$$\begin{aligned} V_r(E) &= e^{iEDx}(S_L^* S_L)e^{-iEDx} \\ &= (e^{iEDx}S_L^* e^{-iEDx})(e^{iEDx}S_L e^{-iEDx}). \end{aligned} \tag{5.17}$$

Explicitly we have

$$V_r(E) = \begin{pmatrix} 1 & \frac{\overline{S_{21}}}{\overline{S_{11}}}e^{-2iEx} & \frac{\overline{S_{31}}}{\overline{S_{11}}}e^{-2iEx} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{S_{21}}{S_{11}}e^{2iEx} & 1 & 0 \\ \frac{S_{31}}{S_{11}}e^{2iEx} & 0 & 1 \end{pmatrix} \tag{5.18}$$

Let us define, for $E \in \mathbb{R}$

$$b_-^{-1}(E) \equiv e^{iEDx}S_L^* e^{-iEDx}, \tag{5.19}$$

$$b_+(E) \equiv e^{iEDx}S_L e^{-iEDx}. \tag{5.20}$$

Then, from equation (5.17), we have

$$V_r(E) = b_-^{-1}(E)b_+(E). \tag{5.21}$$

This is the required factorization of the jump matrix $V_r(E)$. Here, we are using the same notation for the factorization as Deift and Zhou used in his paper [DZ]

$$b_+(E) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{S_{21}}{S_{11}}e^{2iEx} & 1 & 0 \\ \frac{S_{31}}{S_{11}}e^{2iEx} & 0 & 1 \end{pmatrix}, \tag{5.22}$$

$$b_-(E) = \begin{pmatrix} 1 & -\overline{\frac{S_{21}}{S_{11}}}e^{-2iEx} & -\overline{\frac{S_{31}}{S_{11}}}e^{-2iEx} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.23}$$

We also define the following matrices. For $\mathrm{Im}(E) > 0$, with $E \neq E_k$

$$m_{\mathrm{up}}(E;x) \equiv M_{\mathrm{up}}(E;x)e^{-iEDx} = \left(\frac{\Phi_1}{S_{11}}, \Psi_2, \Psi_3\right) e^{-iEDx}, \tag{5.24}$$

and for $\mathrm{Im}(E) < 0$ with $E \neq E_k^*$

$$m_{\mathrm{down}}^{-1}(E;x) \equiv e^{iEDx}M_{\mathrm{down}}^{-1}(E;x) = e^{iEDx}\begin{pmatrix} \frac{1}{S_{11}^*}\Phi_1^* \\ \Psi_2^* \\ \Psi_3^* \end{pmatrix}. \tag{5.25}$$

These matrices will be of particular importance in section 5.8, where we will remove the poles of the matrix $m(E)$. With this, we have completed the description of the jump condition of the mRHP, Problem 1, at $t = 0$.

5.3 The Residues

The residues at the poles E_k, E_k^* of the matrix $m(E)$ are given by equations (5.1) and (5.2), and we write them here again

$$\operatorname*{Res}_{E=E_k} m(E) = \lim_{E\to E_k} m(E) \begin{pmatrix} 0 & 0 & 0 \\ c_k(x,t) & 0 & 0 \\ d_k(x,t) & 0 & 0 \end{pmatrix},$$

$$\operatorname*{Res}_{E=E_k^*} m(E) = \lim_{E\to E_k^*} m(E) \begin{pmatrix} 0 & -c_k^*(x,t) & -d_k^*(x,t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for $k = 1, 2, \ldots, n$, with

$$\begin{aligned} c_k(x,t) &\equiv \frac{C_k(t)}{S'_{11}(E_k)} e^{2iE_k x}, \\ d_k(x,t) &\equiv \frac{D_k(t)}{S'_{11}(E_k)} e^{2iE_k x}. \end{aligned}$$

In this section, I would like to describe how this definition is related with the usual definition of residues, together with the result found by Manakov in his paper [Ma].

Since the columns $m_2(E;x,t)$ and $m_3(E;x,t)$ of $m(E;x,t)$ are analytic in the upper-half complex plane, and $m_1(E;x,t)$ is analytic in the lower-half complex plane [Ma] (see section 5.2 for explicit expressions of $m_1(E;x,t)$, $m_2(E;x,t)$ and $m_3(E;x,t)$), we have

$$\operatorname*{Res}_{E=E_k} m(E) = \lim_{E \to E_k} (E - E_k) \left(m_1(E;x,t), \;\; 0, \;\; 0\right)$$

Now, by equations (5.10), (5.12) and (5.13)

$$\begin{aligned} &= \lim_{E \to E_k} (E - E_k) \left(\frac{\Phi_1(x;E)}{S_{11}(E)} e^{iEx}, \;\; 0, \;\; 0 \right) \\ &= \lim_{E \to E_k} \left(\frac{\Phi_1(x;E)}{S_{11}(E)/(E - E_k)} e^{iEx}, \;\; 0, \;\; 0 \right) \\ &= \left(\frac{\Phi_1(x;E_k)}{S'_{11}(E_k)} e^{iE_k x}, \;\; 0, \;\; 0 \right) \end{aligned}$$

and by the definition of the proportionality constants, equation (2.14), it follows

$$= \left(C_k \frac{\Psi_2(x;E_k)}{S'_{11}(E_k)} e^{iE_k x} + D_k \frac{\Psi_3(x;E_k)}{S'_{11}(E_k)} e^{iE_k x}, \;\; 0, \;\; 0 \right),$$

which coincides with Manakov result (*cf.* [Ma]). Let us continue with the computation. Using equations (5.3) and (5.4)

$$
\begin{aligned}
&= \left(c_k(x)\Psi_2(x;E_k)e^{-iE_kx} + d_k(x)\Psi_3(x;E_k)e^{-iE_kx},\ \ 0,\ \ 0\right), \\
&= \left(0,\ \ \Psi_2(x;E_k)e^{-iE_kx},\ \ \Psi_3(x;E_k)e^{-iE_kx}\right)\begin{pmatrix} 0 & 0 & 0 \\ c_k(x) & 0 & 0 \\ d_k(x) & 0 & 0 \end{pmatrix}, \\
&= \lim_{E\to E_k}\left(0,\ \ \Psi_2(x;E)e^{-iEx},\ \ \Psi_3(x;E)e^{-iEx}\right)\begin{pmatrix} 0 & 0 & 0 \\ c_k(x) & 0 & 0 \\ d_k(x) & 0 & 0 \end{pmatrix}, \\
&= \lim_{E\to E_k}\left(\frac{\Phi_1(x;E)}{S_{11}}e^{iEx},\ \ \Psi_2(x;E)e^{-iEx},\ \ \Psi_3(x;E)e^{-iEx}\right)\begin{pmatrix} 0 & 0 & 0 \\ c_k(x) & 0 & 0 \\ d_k(x) & 0 & 0 \end{pmatrix}, \\
&= \lim_{E\to E_k} m(E)\begin{pmatrix} 0 & 0 & 0 \\ c_k(x) & 0 & 0 \\ d_k(x) & 0 & 0 \end{pmatrix},
\end{aligned}
$$

which is equation (5.1).

Also

$$\operatorname*{Res}_{E=E_k^*} m(E;x) = \lim_{E\to E_k^*}(E-E_k^*)\left(0,\ \ m_2(E;x),\ \ m_3(E;x)\right)$$

since $m_1(E;x)$ is analytic in the lower half-plane. Now, for $\mathrm{Im}(E) < 0$, $m(E;x) = M_{\mathrm{down}}(E;x)e^{-iEDx} = \Psi(x;E)(S_L^*(E))^{-1}e^{-iEDx}$, then by equation (5.8)

$$
\begin{aligned}
&= \operatorname*{Res}_{E=E_k^*} \Psi(x;E)(S_L^*(E))^{-1}e^{-iEDx} \\
&= \lim_{E\to E_k^*}(E-E_k^*)\left(0,\ \ \frac{-\overline{S_{21}}\Psi_1}{\overline{S_{11}}}e^{-iEx} + \Psi_2 e^{-iEx},\ \ \frac{-\overline{S_{31}}\Psi_1}{\overline{S_{11}}}e^{-iEx} + \Psi_3 e^{-iEx}\right)
\end{aligned}
$$

Now consider a cut-off of the support of a general potential with no compact support. Let us assume that the cut-off is $[-L, L]$, so that we have now compact supported potentials. Then, the scattering matrix, as well as the Jost solutions Ψ_2 and Ψ_3, are analytic (see Appendix 2). Then,

$$
\begin{aligned}
&= \lim_{E\to E_k^*}(E-E_k^*)\left(0,\ \ \frac{-\overline{S_{21}}\Psi_1}{\overline{S_{11}}}e^{-iEx} + \Psi_2 e^{-iEx},\ \ \frac{-\overline{S_{31}}\Psi_1}{\overline{S_{11}}}e^{-iEx} + \Psi_3 e^{-iEx}\right) \\
&= \lim_{E\to E_k^*}(E-E_k^*)\left(0,\ \ \frac{-\overline{S_{21}(E;L)}\Psi_1(x;L)}{\overline{S_{11}(E^*;L)}}e^{-iEx},\ \ \frac{-\overline{S_{31}(E;L)}\Psi_1(x;L)}{\overline{S_{11}(E^*;L)}}e^{-iEx}\right).
\end{aligned}
$$

In particular, $S_{21}(E)$ and $S_{31}(E)$ are analytic. We can then evaluate these entries of $S(E)$ in the lower half-plane, and also at $E = E_k^*$, and take the limit

$$\begin{aligned} &= \lim_{E\to E_k^*}\left(0,\ \frac{-\overline{S_{21}(E;L)}\Psi_1}{\overline{S_{11}(E^*;L)}/(E-E_k^*)}e^{-iEx},\ \frac{-\overline{S_{31}(E;L)}\Psi_1}{\overline{S_{11}(E^*;L)}/(E-E_k^*)}e^{-iEx}\right)\\ &= \left(0,\ \frac{-C_k^*(L)\Psi_1}{\overline{S'_{11}(E_k^*;L)}}e^{-iE_k^*x},\ \frac{-D_k^*(L)\Psi_1}{\overline{S'_{11}(E_k^*;L)}}e^{-iE_k^*x}\right),\end{aligned}$$

where $C_k(L) = S_{21}(E_k^*;L)$ and $D_k(L) = S_{31}(E_k^*;L)$ are the proportionality constants corresponding to this cut-off of the potential. Take now the limit $L \to \infty$ to obtain

$$= \left(0,\ \frac{-C_k^*\Psi_1(x;E_k^*)}{\overline{S'_{11}(E_k^*)}}e^{-iE_k^*x},\ \frac{-D_k^*\Psi_1(x;E_k^*)}{\overline{S'_{11}(E_k^*)}}e^{-iE_k^*x}\right). \tag{5.26}$$

This result coincides with the result obtained by Manakov (*cf.* [Ma].) Thie previous limit is not so obvious. The existence of these limits $S_{21}(E_k^*;L) \to C_k^*$ and $S_{31}(E_k^*;L) \to D_k^*$ requires a detailed analysis that I will not address here. However, for the compact supported potentials that I consider in this thesis, we actually have $S_{21}(E_k^*) = C_k^*$ and $S_{31}(E_k^*) = D_k^*$, and these limits are not necessary at all.

We can write equation (5.26) as

$$\begin{aligned} &= \left(0,\ -c_k^*(x)\Psi_1(x;E_k^*)e^{iE_k^*x},\ -d_k^*(x)\Psi_1(x;E_k^*)e^{iE_k^*x}\right)\\ &= e^{iE_k^*x}\left(\Psi_1(x;E_k^*),\ 0,\ 0\right)\begin{pmatrix}0 & -c_k^*(x) & -d_k^*(x)\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\\ &= \lim_{E\to E_k^*} e^{iEx}\left(\Psi_1(x;E),\ 0,\ 0\right)\begin{pmatrix}0 & -c_k^*(x) & -d_k^*(x)\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\\ &= \lim_{E\to E_k^*} e^{iEx}\left(\Psi_1,\quad \Psi_2 - \frac{\overline{S_{21}(E)}\Psi_1}{\overline{S_{11}(E^*)}},\quad \Psi_3 - \frac{\overline{S_{31}(E)}\Psi_1}{\overline{S_{11}(E^*)}}\right)\begin{pmatrix}0 & -c_k^* & -d_k^*\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\end{aligned}$$

i.e.,

$$
\begin{aligned}
&= \lim_{E \to E_k^*} (\Psi_1,\ \Psi_2,\ \Psi_3)\,(S_L^*)^{-1} e^{-iEDx} \begin{pmatrix} 0 & -c_k^*(x) & -d_k^*(x) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \lim_{E \to E_k^*} \left(e^{iEDx} S_L^* \Psi^{-1}\right)^{-1} \begin{pmatrix} 0 & -c_k^*(x) & -d_k^*(x) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \lim_{E \to E_k^*} m(E) \begin{pmatrix} 0 & -c_k^*(x) & -d_k^*(x) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned}
$$

which is equation (5.2). Then, all these definitions coincide.

For the pure soliton case, this definition should be slightly modified[1].

[1]**Remark.** For the pure soliton case, the reflection coefficients are identically zero. Here, we need to define

$$\Omega^{-1}(E) \equiv \overline{S_{11}(E^*)}$$

Notice that the zeroes of $\Omega^{-1}(E)$ are $E = E_k^*$

$$\Omega^{-1}(E_k^*) = \overline{S_{11}(E_k)} = 0.$$

This notation ($\Omega^{-1}(E)$) coincides with Manakov's notation. In his paper [Ma], Manakov denotes by $\Omega_{\alpha\beta}$ the following matrix

$$\Omega_{\alpha\beta} = \begin{pmatrix} S_{22} & S_{23} \\ S_{32} & S_{33} \end{pmatrix}^{-1}.$$

He shows that

$$\det \begin{pmatrix} S_{22} & S_{23} \\ S_{32} & S_{33} \end{pmatrix} = \overline{S_{11}(E^*)},$$

which coincides with our definition of Ω^{-1}

$$\Omega^{-1} \equiv \det \Omega_{\alpha\beta}^{-1} = \overline{S_{11}(E^*)},$$

For the case of pure solitons, the residues are differently defined, so that

$$
\begin{aligned}
c_k(x,t) &\equiv C_k(t) \frac{\Omega^{-1}(E_k)}{S'_{11}(E_k)} e^{2iE_k x}, \\
d_k(x,t) &\equiv D_k(t) \frac{\Omega^{-1}(E_k)}{S'_{11}(E_k)} e^{2iE_k x}.
\end{aligned}
$$

Compare with [KMM].

5.4 The Normalization

We can also check that the matrix $m(E)$, as defined in equation (5.13), is normalized to the identity matrix, I, at ∞

$$m(E) \longrightarrow I \qquad \text{as} \qquad E \longrightarrow \infty. \tag{5.27}$$

See, for example, Deift and Zhou [DZ], or Novikov *et. al* [NMPZ].

5.5 The mRHP at t=0

So far, we have checked that, given the Scattering Data: the eigenvalues E_k, the proportionality constants C_k and D_k, and their complex conjugates, together with the reflection coefficients, $R_1(E) = \frac{S_{21}(E)}{S_{11}(E)}$ and $R_2(E) = \frac{S_{31}(E)}{S_{11}(E)}$, we cah establish the following mRHP (which is Problem 1, at $t = 0$):

Problem 2. MEROMORPHIC RIEMANN-HILBERT PROBLEM AT $t = 0$. *(mRHP-$t = 0$). Fix x. Given the discrete data $\{E_k\}_{k=1}^N$, $\{C_k, D_k\}_{k=1}^N$, together with their complex conjugates, and the jump matrix $V_r(E) = V_r(E; x)$ along $E \in \mathbb{R}$, find a matrix $m(E) = m(x; E)$ such that:*

1. **Rationality:** *The matrix $m(E)$ is a meromorphic function of E, with simple poles located at the eigenvalues of the MZS system, $\{E_k\}_{k=1}^N$, and at its complex conjugates. The residues at the poles are given by*

$$\operatorname*{Res}_{E=E_k} m(E) = \lim_{E\to E_k} m(E) \begin{pmatrix} 0 & 0 & 0 \\ c_k(x) & 0 & 0 \\ d_k(x) & 0 & 0 \end{pmatrix}, \tag{5.28}$$

$$\operatorname*{Res}_{E=E_k^*} m(E) = \lim_{E\to E_k^*} m(E) \begin{pmatrix} 0 & -c_k^*(x) & -d_k^*(x) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.29}$$

for $k = 1, 2, \ldots, n$, with

$$c_k(x) \equiv \frac{C_k}{S_{11}'(E_k)} e^{2iE_kx}, \tag{5.30}$$

$$d_k(x) \equiv \frac{D_k}{S_{11}'(E_k)} e^{2iE_kx}, \tag{5.31}$$

and where C_k and D_k are the proportionality constants.

2. **Jump matrix:** *On the real axis, there is a jump relation*

$$m_+(E) = m_-(E)V_r(E), \quad E \in \mathbb{R}, \tag{5.32}$$

with

$$m_\pm(E) = \lim_{\epsilon \searrow 0} m(E \pm i\epsilon),$$

where $V_r(E) = V_r(E;x)$ is the **jump matrix** *(constructed from the scattering matrix), and explicitly depends on space x,*

$$V_r(E) = \begin{pmatrix} 1 + \left|\frac{S_{21}(E)}{S_{11}(E)}\right|^2 + \left|\frac{S_{31}(E)}{S_{11}(E)}\right|^2 & \overline{\frac{S_{21}(E)}{S_{11}(E)}} e^{-2iEx} & \overline{\frac{S_{31}(E)}{S_{11}(E)}} e^{-2iEx} \\ \frac{S_{21}(E)}{S_{11}(E)} e^{2iEx} & 1 & 0 \\ \frac{S_{31}(E)}{S_{11}(E)} e^{2iEx} & 0 & 1 \end{pmatrix} \tag{5.33}$$

3. **Normalization:** *The matrix $m(E)$ satisfies the following limit on each plane*

$$m(E) \longrightarrow I, \quad \text{as} \quad E \longrightarrow \infty.$$

So, let the time now run: $t > 0$. We obtain the the mRHP, Problem 1, since we know the evolution in time of the Scattering data (see section 2.3). However, we do not know how to solve a mRHP. The goal in section 5.8 will be tranform the mRHP into a holomorphic Riemann-Hilbert Problem (hRHP). Then, in section 5.11, we will sketch the procedure of resolution of a hRHP. But before, we will establish some analytic properties of the Scattering matrix in the following two sections.

5.6 On the Analyticity of the Scattering Matrix: Integrable Potentials

It can be proved that for potentials $u(x,t)$, $v(x,t)$ in $L^1(\mathbb{R})$, which decay fast enough at infinity ($u(x,t)$, $v(x,t) \to 0$ as $|x| \to \infty$), some columns of the Jost functions allow an analytic extension in E, to the upper-half complex plane, $\mathbb{C}^+$. Namely, Φ_1, Ψ_2, Ψ_3

can be analytically extended in E to $\mathbb{C}^+$, and Ψ_1, Φ_2, Φ_3 can be analytically extended in E to $\mathbb{C}^-$. See, for example, [BC], or [DZ]. Similarly, $\Phi_1^*, \Psi_2^*, \Psi_3^*$ can be analytically extended to $\mathbb{C}^-$, and $\Psi_1^*, \Phi_2^*, \Phi_3^*$ to $\mathbb{C}^+$.

The scattering relation $\Phi(x;E) = \Psi(x;E)S(E)$ allows us to write

$$S_{ij}(E) = \Psi_i^* \Phi_j, \qquad i, j = 1, 2, 3.$$

From this it follows that $S_{11}(E)$ can be analytically extended into $\mathbb{C}^+$, and $S_{22}(E)$, $S_{23}(E)$, $S_{32}(E)$, $S_{33}(E)$ into $\mathbb{C}^-$. However, $S_{12}(E), S_{13}(E), S_{21}(E), S_{31}(E)$ are, in general, only defined in $\mathbb{R}$. In the next section, we will consider the case of potentials with compact support, which induces a different behaviour.

5.7 On the Analyticity of the Scattering Matrix: Potentials with Compact Support

When the potentials $u(x,0)$ and $v(x,0)$ have compact support, it can be shown that the columns Φ_1, Ψ_2, Ψ_3 not only can be analytically extended in the upper-half plane, but in the whole complex plane! In general, the Jost solutions $\Psi(E;x)$ and $\Phi(E;x)$ are analytic functions of the spectral parameter E. This implies that the scattering matrix is also holomorphic, by virtue of the relation

$$S_{ij}(E) = \Psi_i^*(E;x)\Phi_j(E;x).$$

In particular, $S_{21}(E)$ and $S_{31}(E)$ are analytic functions of E. (As opposed to the case of $L^1(\mathbb{R})$ potentials, where they were only defined in $\mathbb{R}$).

Now, the first column in the equation $\Phi(x;E) = \Psi(x;E)S(E)$ is

$$\Phi_1(x;E) = S_{11}(E)\Psi_1(x;E) + S_{21}(E)\Psi_2(x;E) + S_{31}(E)\Psi_3(x;E).$$

Therefore, every element in this equation can be extended into $\mathbb{C}^+$. Evaluate then at any eigenvalue of the MZS system $E = E_k$

$$\Phi_1(x;E_k) = S_{11}(E_k)\Psi_1(x;E_k) + S_{21}(E_k)\Psi_2(x;E_k) + S_{31}(E_k)\Psi_3(x;E).$$

The first term is zero, since the eigenvalues are defined such that $S_{11}(E_k) = 0$. Hence

$$\Phi_1(x; E_k) = S_{21}(E_k)\Psi_2(x; E_k) + S_{31}(E_k)\Psi_3(x; E).$$

From this follows that the normalization constants are given by

$$C_k = S_{21}(E_k), \tag{5.34}$$

$$D_k = S_{31}(E_k). \tag{5.35}$$

This fact will be particular useful when removing the poles of the matrix $m(E)$, as we will see in the next section. The interpolation of the proportionality constants by $S_{21}(E)$ and $S_{31}(E)$ was also independently proposed by A Tovbis, S. Venakides, and X. Zhou, for the focusing NLS equation in [TVZ].

5.8 Conversion of the mRHP into a Holomorphic RHP: Removal of the Poles

We now have a mRHP associated with the focusing Manakov system. We will like to solve this mRHP in order to find the solutions of the Manakov system. There are two possible ways to proceed. One was developed by Manakov [Ma], and Novikov *et.al.* [NMPZ]. However, we will follow the ideas of Miller [Mi] and Kamvissis, McLaughlin and Miller [KMM]. They define a new matrix, $m_{\text{new}}(E)$, on a finite region, G (which contains the poles) and, in addition to this, a new jump matrix defined on the boundary of G, to match the new and the original matrices. We will describe the method in this section.

We know we have a finite number of eigenvalues. See Figure 3.1. We can choose the initial conditions such that all the eigenvalues do not lie on the real axis. Then, we define a curve γ with positive orientation such that it is the boundary of a simple connected domain G, which encloses all the eigenvalues. Similarly, we define G^* as a domain in the lower complex plane, which is the reflection of G along the real axis. The boundary of G^* is denoted by γ^*, also with positive orientation.

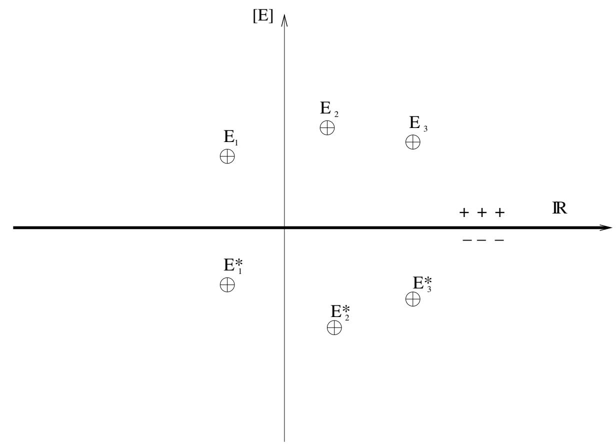

FIGURE 5.1. Poles of $m(E)$.

In [KMM] and [Mi], the authors require some polynomials of order n (the number of eigenvalues), $X(E)$ and $Y(E)$, such that they interpolate the proportionality contants at the eigenvalues

$$C_k = Pe^{X(E_k)},$$
$$D_k = Qe^{Y(E_k)},$$

where P, and Q are some constants. This can be done for $L^1(\mathbb{R})$ potentials, since polynomials are analytic functions and can be defined in $\mathbb{C}^+$. However, for compact supported potentials, we do not need any interpolant polynomials, since we have at our disposal the entries $S_{21}(E)$ and $S_{31}(E)$ of the scattering matrix. They are analytic in $\mathbb{C}$, and actually interpolate the proportionality constants at the eigenvalues, equations (5.34) and (5.35).

We now proceed to remove the poles of $m(E)$. First, we need to define two new regions in the complex plane

$$\begin{aligned} G &= \{E \in \mathbb{C} \mid \mathrm{Im}(E) > 0, \quad \text{and} \quad |E| < E_0\}, \\ G^* &= \{E \in \mathbb{C} \mid \mathrm{Im}(E) < 0, \quad \text{and} \quad |E| < E_0\}, \end{aligned}$$

where E_0 is a positive constant chosen such that all the eigenvalues are inside these regions

$$E_k \in G, \qquad \forall\, k = 1, 2, \ldots, n.$$
$$E_k^* \in G^*, \qquad \forall\, k = 1, 2, \ldots, n.$$

Now, the curve Σ is defined by the curves γ, $\widehat{\gamma}$, γ^*

$$\Sigma \equiv \gamma \cup \gamma^* \cup (\mathbb{R} \backslash \widehat{\gamma}), \tag{5.36}$$

with

$$\begin{aligned} \gamma &= \{E \in \mathbb{C} \mid \mathrm{Im}(E) > 0, \quad \text{and} \quad |E| = E_0\}, \\ \widehat{\gamma} &= \{E \in \mathbb{R} \mid |E| \leq E_0\}, \\ \gamma^* &= \{E \in \mathbb{C} \mid \mathrm{Im}(E) < 0, \quad \text{and} \quad |E| = E_0\}. \end{aligned} \tag{5.37}$$

They also define the boundaries of G and G^*

$$\begin{aligned} \partial G &= \gamma \cup \widehat{\gamma}; \\ \partial G^* &= \gamma^* \cup \widehat{\gamma}. \end{aligned}$$

These curves are oriented as follows

$$\begin{array}{ll} \gamma & \text{is clockwise oriented,} \\ \gamma^* & \text{is counter-clockwise oriented,} \\ \widehat{\gamma} & \text{is oriented as } \mathbb{R}. \end{array}$$

The positive side of these contours is on the left of them, as we move along with the given orientation. See Figure (5.2).

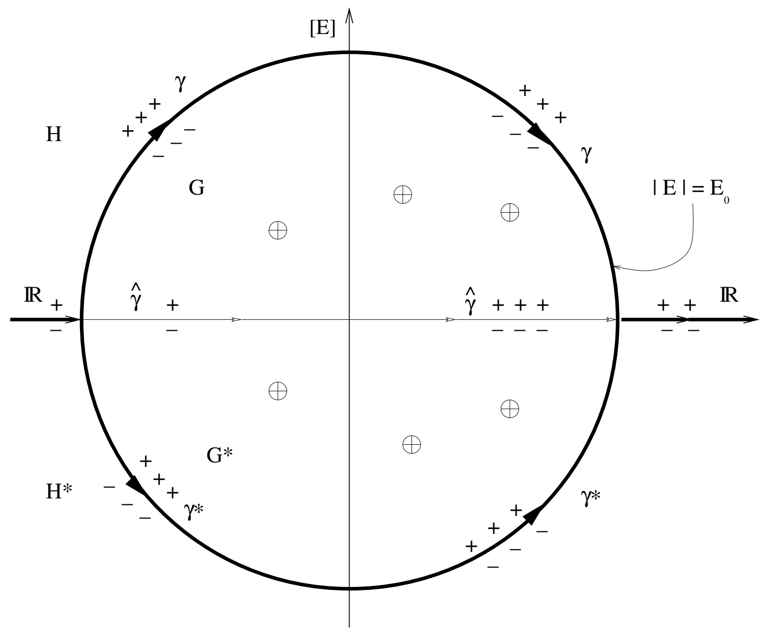

Figure 5.2. Opening the lenses.

We can now define the following **new** matrix $m_{\mathrm{new}}(E)$

$$m_{\mathrm{new}}(E) \equiv m(E)\begin{pmatrix} 1 & 0 & 0 \\ -\frac{S_{21}}{S_{11}}e^{2iEx} & 1 & 0 \\ -\frac{S_{31}}{S_{11}}e^{2iEx} & 0 & 1 \end{pmatrix}, \quad \text{for} \quad E \in G, \tag{5.38}$$

$$m_{\mathrm{new}}(E) \equiv m(E)\begin{pmatrix} 1 & \frac{S_{21}^*}{S_{11}^*}e^{-2iEx} & \frac{S_{31}^*}{S_{11}^*}e^{-2iEx} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{for} \quad E \in G^*. \tag{5.39}$$

$$m_{\mathrm{new}}(E) \equiv m(E), \quad \text{for} \quad E \in \mathbb{C}\backslash\overline{(G \cup G^*)}. \tag{5.40}$$

Compare with equation (2.17) in [KMM].

We also notice that

$$m_{\mathrm{new}}(E) \equiv m(E)b_+^{-1}(E), \quad \text{for} \quad E \in G, \tag{5.41}$$

$$m_{\mathrm{new}}(E) \equiv m(E)b_-^{-1}(E), \quad \text{for} \quad E \in G^*, \tag{5.42}$$

$$m_{\mathrm{new}}(E) \equiv m(E) \quad \text{for} \quad E \in \mathbb{C}\backslash\overline{(G \cup G^*)}, \tag{5.43}$$

where $b_+(E)$ and $b_-(E)$ define the factorization for $V_r(E)$, given in equations (5.20) and (5.19). (Or, alternatively, in equations (5.20) and (5.19)). We now proceed to check that $m_{new}(E)$ has no poles in the complex plane.

For $E \in \mathbb{C} \backslash \overline{(G \cup G^*)}$, $m_{\text{new}}(E)$ and $m(E)$ coincide. In these regions $m(E)$ has no poles, so $m_{\text{new}}(E)$ does not have any poles neither.

For $E \in G$, we have

$$\begin{aligned}
m_{\text{new}}(E) &= m(E) b_+^{-1}(E) \\
&= m_{\text{up}}(E) b_+^{-1}(E) \\
&= M_{\text{up}}(E) e^{-iEDx} b_+^{-1}(E), \qquad \text{by equation (5.24)} \\
&= \Psi(x;E) S_L(E) e^{-iEDx} b_+^{-1}(E), \qquad \text{by equation (5.10)} \\
&= \Psi(x;E) S_L(E) e^{-iEDx} \cdot (e^{iEDx} S_L^{-1}(E) e^{-iEDx}), \quad \text{by equation (5.20)} \\
&= \Psi(x;E) e^{-iEDx},
\end{aligned}$$

which is analytic function in G, since $\Psi(x;E)$ is analytic.

For $E \in G^*$, the calculation is similar

$$\begin{aligned}
m_{\text{new}}^{-1}(E) &= b_-(E) m^{-1}(E) \\
&= b_-(E) m_{\text{down}}^{-1}(E) \\
&= b_-(E) e^{iEDx} M_{\text{down}}^{-1}(E), \qquad \text{by equation (5.25)} \\
&= b_-(E) e^{iEDx} S_L^*(E) \Psi^*(x;E), \qquad \text{by equation (5.11)} \\
&= (e^{iEDx} (S_L^*)^{-1}(E) e^{-iEDx}) \cdot e^{iEDx} S_L^*(E) \Psi^*(x;E), \quad \text{by equation (5.19)} \\
&= e^{iEDx} \Psi^*(x;E),
\end{aligned}$$

which is analytic in G^*.

5.9 New Jumps

Up to this point, we have a sectionally analytic matrix $m_{\text{new}}(E)$, with jumps at the contours γ, γ^*, $\widehat{\gamma}$, and $\mathbb{R} \backslash \widehat{\gamma}$. In this section, we compute the jumps at each of these curves. Through those computations, we will make use of equations (5.41), (5.42) and (5.43).

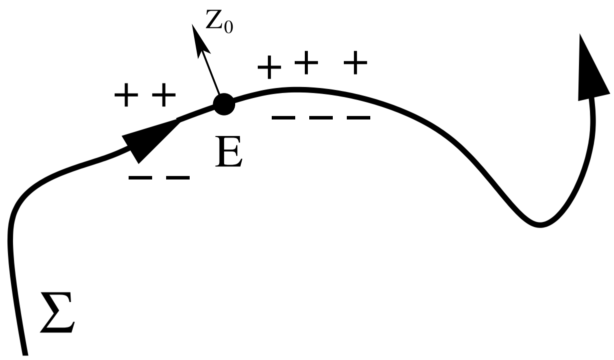

FIGURE 5.3. Boundary values of $m(E)$ at Σ.

Now, to compute the jumps, we need to define the boundary values of a matrix $m(E)$ along a curve. Given a curve, Σ, with some orientation (see Figure 5.3), consider a point $E \in \Sigma$. The $+$-side of Σ at E is on the left of Σ, as we move along the curve with the given orientation. The $-$-side in on the right. Consider now a "vector" z_0, orthogonal to this curve at the point E. The "vector" z_0 in on the $+$-side of Σ. See Figure 5.3 again. Now, the boundary values of $m(E)$ at Σ are denoted and defined as

$$m_{\pm}(E) = \lim_{\epsilon \downarrow 0} m(E \pm \epsilon z_0), \qquad E \in \Sigma.$$

This is an extension to any regular curve of the boundary values at the real axis. In that instance, $z_0 = i$. We now proceed to compute the jumps at the new contours.

For $E \in \mathbb{R} \backslash \widehat{\gamma}$, the jumps are the same as in $\mathbb{R}$, as we have already seen.

For $E \in \gamma$, we perform the following easy computation

$$\begin{aligned} (m_{\text{new}})_-^{-1}(E)(m_{\text{new}})_+(E) &= \left(m_{\text{up}}(E) b_+^{-1}(E) \right)^{-1} m_{\text{up}}(E) \\ &= b_+(E) m_{\text{up}}^{-1}(E) m_{\text{up}}(E) \\ &= b_+(E) \\ &\equiv V_\gamma(E). \end{aligned}$$

For $E \in \gamma^*$

$$\begin{aligned}
(m_{\text{new}})_-^{-1}(E)(m_{\text{new}})_+(E) &= m_{\text{down}}^{-1}(E)\Big(m_{\text{down}}(E)b_-^{-1}(E)\Big) \\
&= b_-^{-1}(E) \\
&\equiv V_{\gamma^*}(E).
\end{aligned}$$

Now, along $\widehat{\gamma}$

$$\begin{aligned}
(m_{\text{new}})_-^{-1}(E)(m_{\text{new}})_+(E) &= \Big(m_{\text{down}}(E)b_-^{-1}(E)\Big)^{-1}\Big(m_{\text{up}}(E)b_+^{-1}(E)\Big) \\
&= b_-(E)m_{\text{down}}^{-1}(E)m_{\text{up}}(E)b_+^{-1}(E) \\
&= b_-(E)V_r(E)b_+^{-1}(E), \qquad \text{by equation (5.5)} \\
&= b_-(E)\Big(b_-^{-1}(E)b_+(E)\Big)b_+^{-1}(E), \quad \text{by equation (5.21)} \\
&= I.
\end{aligned}$$

It is the identity matrix!!! So, there is a ficticious jump along the segment $(-E_0, E_0)$ in the real axis.

Therefore, the transformation $m(E) \longrightarrow m_{\text{new}}(E)$ (equations (5.41), (5.42), and (5.43)), allows:

1. to remove the poles of the matrix m(E), and

2. to remove the jump along $\widehat{\gamma}$.

Property 1 tells us that we can formulate a (holomorphic) RHP, and then use the formula (5.57) to recover m_{new}, solving the RHP. How to solve a RHP will be the topic of the following sections.

Property 2 will of particular importance in chapter 6, when the short-time asymptotic analysis is performed. We postpone the reason why this property is very important until section 6.3.

5.10 The Holomorphic Riemann-Hilbert Problem.

We are now in a good position to state the following:

Problem 3. THE HOLOMORPHIC RIEMANN-HILBERT PROBLEM *(hRHP). Fix x and t. Given the oriented countour $\Sigma = \gamma \cup \gamma^* \cup (\mathbb{R}\backslash\widehat{\gamma})$, find a matrix $m_{new}(E) = m_{new}(x,t;E)$ such that:*

1. **Analyticity:** *The matrix $m_{new}(E)$ is an analytic function of E in each component of $\mathbb{C}\backslash\Sigma$.*

2. **Boundary behavior:** *$m_{new}(E)$ assumes continuous values on Σ.*

3. **Jump conditions:** *The boundary values taken on Σ satisfy the jump relation*

$$(m_{new})_+(E) = (m_{new})_-(E)V(E), \tag{5.44}$$

   where V(E) is the jump matrix defined in equation (5.45) below.

4. **Normalization:** *The matrix $m_{new}(E)$ satisfies the following limit on each sector*

$$m_{new}(E) \longrightarrow I, \quad \textit{as} \quad E \longrightarrow \infty.$$

We define the jump along the curve Σ in the following way

$$V(E) \equiv \begin{cases} V_\gamma(E), & \text{for} \quad E \in \gamma \\ V_r(E), & \text{for} \quad E \in \mathbb{R}\backslash\widehat{\gamma} \\ V_{\gamma^*}(E), & \text{for} \quad E \in \gamma^* \end{cases}, \tag{5.45}$$

Each of these matrices are given by

$$V_\gamma(E) = b_+(E), \tag{5.46}$$

$$V_r(E) = b_-^{-1}(E)b_+(E), \tag{5.47}$$

$$V_{\gamma^*}(E) = b_-^{-1}(E), \tag{5.48}$$

where $b_+(E)$ and $b_-(E)$ are given in equations (5.20) and (5.19).

Notice that $V_r(E)$ is left untouched in $\mathbb{R}\backslash\widehat{\gamma}$, equation (5.47), so it is the same jump as given in equations (5.16) and (5.21).

Conclusions. If compare equation (5.45) with equation (174) in DiFranco's thesis [DiF], we see that we obtain the same form of the jump matrix for the focusing case.

We can then conclude the following:

1. The transformations (5.38), (5.39) and (5.40) suggested by DiFranco [DiF] remove the poles in the original mRHP, **Problem 1**.

2. The transformations, (5.38), (5.39) and (5.40), not only work for the defocusing NLS equation, but **also** for the focusing Manakov system. This is obviously true once the appropiate modifications due to the focusing character and the coupling in Manakov system are considered.

3. Through these transformations, we can go back and forth from the meromorphic and holomorphic RHPs, since $b_+(E)$ and $b_-(E)$ are invertible matrices.

4. Since the initial potentials $u(x,0)$, $v(x,0)$ have compact supports, we can apply these results when $t = 0$. For $t > 0$, the evolution of the scattering data is easy in time. Therefore, in the hRHP, **Problem 3**, we just have to substitute

$$\begin{aligned} S_{21}(E;t) &= S_{21}(E;0)e^{-2iE^2t}, \\ S_{31}(E;t) &= S_{31}(E;0)e^{-2iE^2t}, \end{aligned}$$

   to solve the Inverse Scattering Problem of the MZS system.

5. Given the success of this argument, we should be able to extend to general potentials with compact support and then, by limits, to the general case.

5.11 The Riemann-Hilbert Problem Approach to the Inverse Scattering.

In this section, we will follow the ideas by [DZ] and [DiF] applied to the case of 3×3 matrix systems, i.e., to the MZS system, equation (2.8). We will describe how to obtain the solution of the Manakov system, given a solution of the hRHP. We also will also briefly sketch how to solve the hRHP, Problem 3.

In chapter 2, we showed that the Manakov system, equation (2.1)

$$\begin{aligned} iu_t + \frac{1}{2}u_{xx} + \sigma(|u|^2 + |v|^2)u &= 0 \\ iv_t + \frac{1}{2}v_{xx} + \sigma(|u|^2 + |v|^2)v &= 0, \end{aligned}$$

has an associated Lax pair of linear operators, equations 2.4 and 2.5

$$\begin{aligned} \mathcal{L} &\equiv iED + N, \\ \mathcal{B} &\equiv iE\mathcal{L} + \frac{1}{2}D(N_x - N^2), \end{aligned}$$

such that the system (equations 2.2 and 2.3)

$$\begin{aligned} \frac{\partial}{\partial x}\Psi &= \mathcal{L}\Psi, \\ i\frac{\partial}{\partial t}\Psi &= \mathcal{B}\Psi, \end{aligned}$$

has the following compatibility condition

$$iN_t - \frac{1}{2}DN_{xx} + DN^3 = 0,$$

which is the matrix form of the Manakov system. This is true, whenever the spectral parameter is time independent

$$\frac{d}{dt}E = 0.$$

Following the Inverse Scatering Transform (IST), we consider eigenfunctions Ψ of equation (2.2), $\partial_x\Psi = \mathcal{L}\Psi$. In section 5.2, we studied how to construct a meromorphic matrix $m(E;x)$ out of these eigenfunctions Ψ. Also, in section 5.10, we contructed a new matrix $m_{\text{new}}(E;x)$, holomorphic in $\mathbb{C}$, with jump discontinuities in some given curve Σ. See Problem 3.

For fixed x, t, we considered the matrix

$$m_{\text{new}}(E) = m_{\text{new}}(x,t;E) \longrightarrow I, \tag{5.49}$$

as $E \to +\infty$. This matrix $m_{\text{new}}(E)$ has a power expansion in E^{-1}, which will play an important role in solving the Manakov system. The expansion will be

$$m_{\text{new}}(x,t;E) = I + \frac{(m_{\text{new}})_1(x,t)}{E} + \frac{(m_{\text{new}})_2(x,t)}{E^2} + O\left(\frac{1}{E^3}\right). \tag{5.50}$$

We know that $m_{\text{new}}(x,t;E)$ is analytic in $\mathbb{C}\backslash\Sigma$. Here, we will show how the IST works for the regular RHP.

Assume that we have solved the hRHP, Problem 3. Then, observe that

$$\frac{\partial}{\partial x} m_{\text{new}} = iE[D, m_{\text{new}}] + N m_{\text{new}}.$$

Substitute the expansion of $m_{\text{new}}(E)$ in powers of E^{-1} into the last equation. We obtain

$$N(x,t) = i[(m_{\text{new}})_1(x,t), D],$$

where $(m_{\text{new}})_1(x,t)$ is the first coefficient in the expansion of $m_{\text{new}}(E)$, equation (5.50). Now, this equation is also

$$\begin{pmatrix} 0 & u(x,t) & v(x,t) \\ -u^*(x,t) & 0 & 0 \\ -v^*(x,t) & 0 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & (m_1(x,t))_{12} & (m_2(x,t))_{12} \\ (m_1(x,t))^*_{12} & 0 & 0 \\ (m_1(x,t))^*_{13} & 0 & 0 \end{pmatrix}.$$

This is, $(m_{\text{new}})_1(x,t)$ provides the solution of the Manakov system. We actually would have to compute the limit

$$N(x,t) = \lim_{E\to\infty} i\left[m_{\text{new}}(x,t;E) - I, D\right] E.$$

In the rest of the section, we sketch how to solve the hRHP. First, we need a factorization of the jump matrix

$$V(E) = b^{-1}_{\text{minus}}(E) b_{\text{plus}}(E), \tag{5.51}$$

such that $b^{-1}_{\text{minus}}(E)$ and $b_{\text{plus}}(E)$ are two matrices, upper and lower triangular respectively, with $\det b^{-1}_{\text{minus}}(E) = \det b_{\text{plus}}(E) = 1$. Then, a new matrix w is defined

$$w(E) = b_{\text{plus}}(E) - b_{\text{minus}}(E), \tag{5.52}$$

together with the Cauchy-type operators

$$C_w[f](E) = C_+[fw_-](E) + C_-[fw_+](E),$$

and

$$C_\pm[g](E) = \frac{1}{2\pi i} \lim_{\substack{z' \to E \\ z' \in \{\pm \text{ side of } \Sigma\}}} \int_\Sigma \frac{g(\zeta)}{\zeta - z'}\, d\zeta, \quad \text{for} \quad E \in \Sigma.$$

Here, $w_+(E)$, $w_-(E)$ are two auxiliary matrices

$$w_+(E) = b_{\text{plus}}(E) - I, \tag{5.53}$$

$$w_-(E) = I - b_{\text{minus}}(E). \tag{5.54}$$

Note that, from equation (5.52)

$$w(E) = w_+(E) + w_-(E). \tag{5.55}$$

If $w_\pm \in L^\infty \cap L^2$, we now let $\mu(E)$ be a matrix, $\mu(E) \in C_w I + L^2$, such that it solves the integral equation

$$(I - C_w)[\mu](E) = I, \quad \text{for} \quad E \in \Sigma. \tag{5.56}$$

Then, the solution $m(x,t;E)$ to the RHP, Problem 3, is given by (see details in [DZ] and [DiF])

$$m_{\text{new}}(x,t;E) \equiv I + \frac{1}{2\pi i} \int_\Sigma \frac{\mu(\zeta) w(\zeta)}{\zeta - E}\, d\zeta, \quad E \in \mathbb{C} \backslash \Sigma. \tag{5.57}$$

This last equation is exact. If we let $E \to \infty$, and we find the coefficient of the $\frac{1}{E}$, we have solved the Manakov system. However, the integral equation (5.56) is in general hard to solve. Here it is where we need an approximation theory to learn something about the behaviour of the solutions.

Assume then that we can approximate the solutions of the equation (5.56) by Neuman series. Observe that, at leading order

$$\mu(E) \sim I, \tag{5.58}$$

so

$$m_{\text{new}}(x,t;E) \sim I + \frac{1}{2\pi i}\int\limits_{\Sigma} \frac{w(\zeta)}{\zeta - E}\, d\zeta.$$

Now, for large values of E follows that

$$m_{\text{new}}(x,t;E) \sim I + \left\{ \frac{1}{2\pi i}\int\limits_{\Sigma} w(\zeta)\, d\zeta \right\} \frac{1}{E} + \cdots ,$$

i.e.

$$(m_{\text{new}})_1(x,t) \sim \frac{1}{2\pi i}\int\limits_{\Sigma} w(\zeta)\, d\zeta, \tag{5.59}$$

at leading order. This integral is the right expression to use when we approximate the solutions of the Manakov system.

5.12 Approximation by Neumann Series

In this section, I will describe why approximation in equation (5.58) might be true. I will not estimate the errors here. I will give a heuristic description of the leading order terms, and obtain bound for the error in chapter 6. In particular, look at section 6.2.

There are two ways to approach this approximation.

First approach. Notice that the integral equation in equation (5.56) is linear in w. Then, *if the linear operator $C_w[f]$ is small in some norm*, then we can approximate the solution $\mu(E)$ by

$$\mu(E) = I + C_w[I](E) + C_w^2[I](E) + \ldots$$

By the linearity of $C_w[f]$ in w, if $w(E)$ is small along the curve Σ, we can the perform this approximation. Observe that at leading order, we get equation (5.58).

Second approach. Consider equation (5.44)

$$(m_{\text{new}})_+(E) = (m_{\text{new}})_-(E)V(E).$$

This can be written in the following way

$$(m_{\text{new}})_+(E) - (m_{\text{new}})_-(E) = (m_{\text{new}})_-(E)\Big(V(E) - I\Big). \tag{5.60}$$

Now, from complex analysis, we know that a function $F(E)$

$$F(E) = \frac{1}{2\pi i}\int_{\Sigma} \frac{f(\zeta)}{\zeta - E}\, d\zeta,$$

is an analytic function of E, for $E \notin \Sigma$, and $f(\zeta)$ is analytic along Σ. The boundary values of $F(E)$ at Σ exist, but they are different. The Plemelj-Sokhotsky formula tell us

$$F_+(E) - F_-(E) = f(E), \quad \text{for} \quad E \in \Sigma.$$

Then, $f(E)$ is the jump of $F(E)$ across the curve Σ. We can now define $m_{\text{new}}(E)$ solution of the hRHP

$$m_{\text{new}}(E) = I + \frac{1}{2\pi i}\int_{\Sigma} \frac{(m_{\text{new}})_-(\zeta)\Big(V(\zeta) - I\Big)}{\zeta - E}\, d\zeta, \tag{5.61}$$

which is an analytic function of E, with nice boundary values at Σ, and satisfies the jump condition (5.60). Also $m_{\text{new}}(E) \to I$ as $E \to \infty$.

The only defect ot this equation is that it is implicit in $m_{\text{new}}(E)$. However, we can solve this issue. Take a limit in (5.61) to get the boundary value at the $-$side of Σ

$$(m_{\text{new}})_-(E) = I + \frac{1}{2\pi i}\int_{\Sigma} \frac{(m_{\text{new}})_-(\zeta)\Big(V(\zeta) - I\Big)}{\zeta - E}\, d\zeta. \tag{5.62}$$

This is an integral equation for $(m_{\text{new}})_-$. We solve it, substitute in (5.61), and we obtain a representation for $m_{\text{new}}(E)$.

However, in general, the integral equation (5.62) is hard to solve. Again, it can be approximated by Neumann series. Notice that if the jump matrix is close to the identity (in some norm)

$$V(E) - I \sim 0,$$

We can approximate the integral equation by Neumann series.

Matching the approaches. Notice that in the first approach, if we say that w is small along Σ, it is equivalent to say that $V(E) - I \sim 0$ along Σ: if we choose $b_{\text{minus}}^{-1}(E) = I$, then equation (5.56) clearly becomes equation (5.62). Evenmore, it turns out that:

$$w(E) = V(E) - I,$$

and

$$\mu(E) = (m_{\text{new}})_{-}(E).$$

The short-time asymptotic analysis is supported on this argument. We will consider the appropiate change of variables, such that the jump matrix is close to the identity along the contour Σ. We will study the right change of variables, and show that the jump matrix is close to the identity along the curve of jump. We are going to do this in next chapter, chapter 6.

Chapter 6
Short-Time Asymptotics

6.0 Introduction

This chapter is dedicated to the short-time asymptotics analysis of the Manakov system. The approximation is performed in the Inverse Scattering step, i.e., when the RHP approach is employed. With a scaling suggested by J. DiFranco and McLaughlin [DiF], [DiF-McL]. We prove that, at leading order, the main contribution comes from the continuous spectrum. Actually, this contribution comes from the solution of the linear part of the Manakov system. From this, we conclude that the behavior at short times of both, the focusing and the defocusing Manakov system, is basically the same. Therefore, same conclusion is also valid for the NLS, by the reduction $u(x,t) = v(x,t)$ or $v(x,t) = 0$. We show that the asymptotics of the Manakov system is, to leading order, given by the error function.

6.1 Preliminary Calculations

We already found that, whenever $V(E) \sim I$ (or alternatively $w(E) \sim 0$), at leading order we have

$$(m_{\text{new}})_1(x,t) \sim \frac{1}{2\pi i} \int_{\Sigma} w(\zeta)\, d\zeta, \tag{6.1}$$

(*cf.* equation (5.59)), and so

$$u(x,t) = 2i(m_{\text{new}1}(x,t))_{12}, \tag{6.2}$$

$$v(x,t) = 2i(m_{\text{new}1}(x,t))_{13}, \tag{6.3}$$

are approximations of the solutions to the Manakov system. In this section, we explicitly compute the matrix $w(E)$ along the curve Σ. Section 6.3 will be devoted to checking that, under the short-time approximation, the elements of $w(E)$, namely, the reflection coefficients are small of order $O(\sqrt{t})$.

We have three different pieces of $w(E)$, since $\Sigma = \gamma \cup \gamma^* \cup (\mathbb{R}\backslash\widehat{\gamma})$

$$w(E) \equiv \begin{cases} w_\gamma(E), & \text{for} \quad E \in \gamma \\ w_{\gamma^*}(E), & \text{for} \quad E \in \gamma^* \\ w_r(E), & \text{for} \quad E \in \mathbb{R}\backslash\widehat{\gamma} \end{cases}, \tag{6.4}$$

On $\mathbb{R}\backslash\widehat{\gamma}$, the factorization of the jump matrix, equation (5.21) is $V(E) = b_-^{-1}(E)b_+(E)$. Comparing with equation (5.51), we have that $b_{\text{plus}}(E) = b_+(E)$ and $b_{\text{minus}}(E) = b_-(E)$. Therefore, using equation (5.52)

$$w_r(E) = \begin{pmatrix} 0 & \frac{\overline{S_{21}}}{\overline{S_{11}}}e^{-2iEx} & \frac{\overline{S_{31}}}{\overline{S_{11}}}e^{-2iEx} \\ \frac{S_{21}}{S_{11}}e^{2iEx} & 0 & 0 \\ \frac{S_{31}}{S_{11}}e^{2iEx} & 0 & 0 \end{pmatrix}, \quad \text{for } E \in \mathbb{R}\backslash\widehat{\gamma}. \tag{6.5}$$

On γ, $V = V_\gamma(E)$ is given by equation (5.46), and it is already factored: $V_\gamma(E) = b_{\text{minus}}^{-1}(E)b_{\text{plus}}(E)$. In this instance

$$\begin{aligned} b_{\text{minus}}^{-1}(E) &= I, \\ b_{\text{plus}}(E) &= b_+(E). \end{aligned}$$

Hence

$$\begin{aligned} w_+(E) &= b_{\text{plus}}(E) - I = b_+(E) - I, \\ w_-(E) &= I - b_{\text{minus}}(E) = I - I = 0. \end{aligned}$$

Then, for $E \in \gamma$, we define: $w_\gamma(E) = w_+(E) + w_-(E) = b_+(E) - I$, i.e. by equation (5.22)

$$w_\gamma(E) \equiv \begin{pmatrix} 0 & 0 & 0 \\ \frac{S_{21}}{S_{11}}e^{2iEx} & 0 & 0 \\ \frac{S_{31}}{S_{11}}e^{2iEx} & 0 & 0 \end{pmatrix}, \quad \text{for } E \in \gamma. \tag{6.6}$$

Similarly, along γ^* the jump matrix is factored $V_{\gamma^*}(E) = b_{\text{minus}}^{-1}(E)b_{\text{plus}}(E)$, but now (see equation (5.48))

$$\begin{aligned} b_{\text{minus}}^{-1}(E) &= b_-^{-1}(E), \\ b_{\text{plus}}(E) &= I. \end{aligned}$$

So

$$
\begin{aligned}
w_+(E) &= b_{\text{plus}}(E) - I = I - I = 0, \\
w_-(E) &= I - b_{\text{minus}}(E) = I - b_-(E).
\end{aligned}
$$

Therefore, for $E \in \gamma^*$, and using equation (5.23), the matrix $w_{\gamma^*}(E) = w_+(E) + w_-(E) = I - b_-(E)$ is given by

$$
w_{\gamma^*}(E) \equiv \begin{pmatrix} 0 & \frac{\overline{S_{21}}}{\overline{S_{11}}} e^{-2iEx} & \frac{\overline{S_{31}}}{\overline{S_{11}}} e^{-2iEx} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{for } E \in \gamma^*. \tag{6.7}
$$

Equations (6.5), (6.6) and (6.7), define the matrix $w(E)$ (equation (6.4)), that appears in equation (5.59).

We want to know the behavior of $u(x,t)$ and $v(x,t)$. Using equations (6.1), (6.2), (6.3), and (6.4), and considering that $\Sigma = \gamma \cup \gamma^* \cup (\mathbb{R} \backslash \widehat{\gamma})$, we have at leading order

$$
\begin{aligned}
u(x,t) &\sim \frac{1}{\pi} \int\limits_{\Sigma} w_{12}(\zeta)\, d\zeta = \\
&= \frac{1}{\pi} \int\limits_{\gamma} w_{12}(\zeta)\, d\zeta + \frac{1}{\pi} \int\limits_{\gamma^*} w_{12}(\zeta)\, d\zeta + \frac{1}{\pi} \int\limits_{\mathbb{R}} w_{12}(\zeta)\, d\zeta, \\
v(x,t) &\sim \frac{1}{\pi} \int\limits_{\Sigma} w_{13}(\zeta)\, d\zeta = \\
&= \frac{1}{\pi} \int\limits_{\gamma} w_{12}(\zeta)\, d\zeta + \frac{1}{\pi} \int\limits_{\gamma^*} w_{12}(\zeta)\, d\zeta + \frac{1}{\pi} \int\limits_{\mathbb{R}} w_{12}(\zeta)\, d\zeta;
\end{aligned}
$$

i.e.

$$
\begin{aligned}
u(x,t) &\sim \frac{1}{\pi} \int\limits_{\gamma} (w_\gamma)_{12}(\zeta)\, d\zeta + \frac{1}{\pi} \int\limits_{\gamma^*} (w_{\gamma^*})_{12}(\zeta)\, d\zeta + \frac{1}{\pi} \int\limits_{(\mathbb{R} \backslash \widehat{\gamma})} (w_r)_{12}(\zeta)\, d\zeta, \\
v(x,t) &\sim \frac{1}{\pi} \int\limits_{\gamma} (w_\gamma)_{13}(\zeta)\, d\zeta + \frac{1}{\pi} \int\limits_{\gamma^*} (w_{\gamma^*})_{13}(\zeta)\, d\zeta + \frac{1}{\pi} \int\limits_{(\mathbb{R} \backslash \widehat{\gamma})} (w_r)_{13}(\zeta)\, d\zeta.
\end{aligned}
$$

From equation (6.6), which is the definition of w_γ, we see that $(w_\gamma)_{12}(\zeta) = 0$ and

$(w_\gamma)_{13}(\zeta) = 0$. Then, the last two equation are

$$u(x,t) \sim \frac{1}{\pi}\int_{\gamma^*} (w_{\gamma^*})_{12}(\zeta)\, d\zeta + \frac{1}{\pi}\int_{(\mathbb{R}\backslash\widehat{\gamma})} (w_r)_{12}(\zeta)\, d\zeta, \tag{6.8}$$

$$v(x,t) \sim \frac{1}{\pi}\int_{\gamma^*} (w_{\gamma^*})_{13}(\zeta)\, d\zeta + \frac{1}{\pi}\int_{(\mathbb{R}\backslash\widehat{\gamma})} (w_r)_{13}(\zeta)\, d\zeta. \tag{6.9}$$

Similarly

$$-u^*(x,t) \sim \frac{1}{\pi}\int_{\gamma} (w_{\gamma})_{21}(\zeta)\, d\zeta + \frac{1}{\pi}\int_{(\mathbb{R}\backslash\widehat{\gamma})} (w_r)_{21}(\zeta)\, d\zeta, \tag{6.10}$$

$$-v^*(x,t) \sim \frac{1}{\pi}\int_{\gamma} (w_{\gamma})_{31}(\zeta)\, d\zeta + \frac{1}{\pi}\int_{(\mathbb{R}\backslash\widehat{\gamma})} (w_r)_{31}(\zeta)\, d\zeta. \tag{6.11}$$

Finally, notice that

$$\begin{aligned}(w_\gamma)_{21}(\zeta) &= (w_r)_{21}(\zeta) = R_1(\zeta,t)e^{2i\zeta x},\\ (w_\gamma)_{31}(\zeta) &= (w_r)_{31}(\zeta) = R_2(\zeta,t)e^{2i\zeta x},\end{aligned}$$

where

$$\begin{aligned}R_1(\zeta,t) &= \frac{S_{21}(\zeta,t)}{S_{11}(\zeta)},\\ R_2(\zeta,t) &= \frac{S_{31}(\zeta,t)}{S_{11}(\zeta)}.\end{aligned}$$

Therefore, equation (6.10) and (6.11) become

$$-u^*(x,t) \sim \frac{1}{\pi}\int_{\gamma\cup(\mathbb{R}\backslash\widehat{\gamma})} R_1(\zeta,t)e^{2i\zeta x}\, d\zeta, \tag{6.12}$$

$$-v^*(x,t) \sim \frac{1}{\pi}\int_{\gamma\cup(\mathbb{R}\backslash\widehat{\gamma})} R_2(\zeta,t)e^{2i\zeta x}\, d\zeta. \tag{6.13}$$

Equations (6.12) and (6.13) will lead to the description of the short-time asymptotics of the solutions of the Manakov system. However, we still need to show that $w(E)$ is small at the short time asymptotics. In next section, we will define the *DiFranco's Transformation*. We will see how this scaling applies to the reflection coefficients and check that they are small of order $O(\sqrt{t})$.

6.2 DiFranco's Transformation.

In his thesis [DiF], J. DiFranco showed how to approximate the solution of the defocusing NLS equation, when the initial data is a squared barrier potential.

The first step is to figure out the transformation needed to reach the appropriate scaling. DiFranco starts by solving the linear Schrödinger equation of quantum mechanics with no potential

$$iu_t = -\frac{1}{2}u_{xx},$$

with same squared barrier initial data. Using steepest descent arguments, he found the solution to be

$$u_{\text{linear}}(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(2EL)e^{2i(E^2t+Ex)}}{E}\,dE. \tag{6.14}$$

He then proposes the following change of variables

$$E = \frac{\lambda}{\sqrt{t}}. \tag{6.15}$$

We will refer to it as *DiFranco's Transformation.* This transformation is constructed to achieve the following goal: to avoid exponential growth of the exponential factor e^{2iE^2t}. This is not selfevident in the context of the linear Schrödinger equation, However, we can give an intuitive argument why this should be true. The details will be given in next section 6.3.

To approximate the solutions of the integral equation (5.56) by Neumann series, we require $w(E) \sim 0$ for $E \in \Sigma$, whose form is given by the x-reflection coefficients $R_1(E;t)e^{2iEx} = R_1(E;0)e^{2iE^2t}e^{2iEx}$ and $R_2(E;t)e^{2iEx} = R_2(E;0)e^{2iE^2t}e^{2iEx}$, and they should be small. If we choose a very large E_0 (the radius of $\gamma \cup \gamma^*$) $E_0 \gg 1$, then $|E| \geq E_0 \gg 1$. In next section, we will observe that, along such contour, we have

$$R_1(E;0)e^{2iE^2t} = -\frac{1}{E}\sin(2EL)e^{2iE^2t} + O\left(\frac{1}{E^2}\right)$$

Then,

$$|R_1(E;0)e^{2iE^2t}| \leq \frac{K}{|E|}e^{-4\,\mathrm{Re}(E)\,\mathrm{Im}(E)t}, \tag{6.16}$$

for some constant K. Take $t \sim 0$, but fixed. Also, take E such that $\mathrm{Re}(E)\,\mathrm{Im}(E) < 0$.

Now, it does not matter how small t is. In the limit $E \to \infty$, the right-hand-side of euqation (6.16) will blow-up, loosing the desired property that the x-rcs should be small. This difficulty can be addressed if we lock E and t together:

$$E^2t = \lambda,$$

for λ fix, giving rise to the DiFranco's transformation. Actually, if E_0 is the radius of $\gamma \cup \gamma^*$, then we define λ_0 (fixed) the radius of the circle $\Gamma \cup \gamma^* = \lambda(\gamma) \cup \lambda(\gamma^*)$ in the *DiFranco's plane*. In this conection, the radius $E_0 = E_0(t)$ is time-dependent, and

$$E_0(t) = \frac{\lambda_0}{\sqrt{t}}. \tag{6.17}$$

We the obtain that the exponential

$$e^{2iE^2t} = e^{2iE^2t}$$

is bounded in the DiFranco's λ-plane. More precisely, it is bounded along the curve $\lambda(\Sigma) = \Gamma \cup \Gamma^* \cup (\mathbb{R}\backslash\widehat{\Gamma})$.

We also observe that the curve $\Sigma = \Sigma(t)$ is time dependent in the E-plane. However, $\lambda(\Sigma) = \Gamma \cup \Gamma^* \cup (\mathbb{R}\backslash\widehat{\Gamma})$ is *time-independent* in the λ-plane. Now, the Neumann series are better approximated along fixed contours rather than in curves depending on the asymptotic parameter. So, the Nuemann series is calculated along $\lambda(\Sigma)$ in the λ-plane.

Now, equation (6.17) is telling that the circle $\gamma \cup \gamma^*$, and any othe region in E-plane, is shrunk into the λ-plane. See Figure 6.1.

Now, using DiFranco's transformation, it can be proved that

$$||\ C_w[f]\ ||_{L^2} \leq\ \text{const.}\ \sqrt{t}\ \ ||f||_{L^2} \tag{6.18}$$

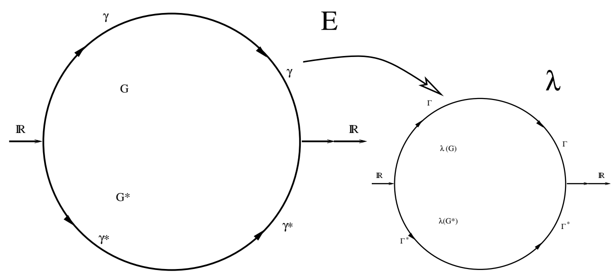

FIGURE 6.1. DiFranco's Transformation.

We learn

$$\mu(E) = I + C_w[I](E) + C_w^2[I](E) + \dots$$

is a convergent Neumann series in L^2. Then, the leading approximation along Σ is valid, and hence

$$u(x,t) \sim \frac{1}{\pi} \int\limits_{\gamma^* \cup (\mathbb{R} \setminus \widehat{\gamma})} R_1^*(E,t) e^{-2iEx} \, dE$$

Same idea will apply for the expression of $v(x,t)$.:

$$v(x,t) \sim \frac{1}{\pi} \int\limits_{\gamma^* \cup (\mathbb{R} \setminus \widehat{\gamma})} R_2^*(E,t) e^{-2iEx} \, dE.$$

This is to say, the approximation given in the previous chapter for $u(x,t)$ and $v(x,t)$ should be valid, provided that (6.18) is true.

A detailed proof of (6.18) is very complicated to follow. However, a quick argument can be given, whenever we avoid the difficult/technical step.

6.2.1 On the "smallness" of the norm of C_w

In section 6.3, we are going to prove the following:

Fact: For all $\lambda \in \Sigma$, and for all $t \sim 0$, there exists a constant K such that

$$||w_{\pm}(\lambda)|| \le K \frac{\sqrt{t}}{\lambda}. \tag{6.19}$$

This is not true for all x (the matrix $w_{\pm}$ depends on x), but we are not establishing any x interval in here. This will be worked out in the next section.

Now, the technical difficulty I mentioned earlier it is a non-trivial fact, that we will just announce here. For some contants $D_{\pm}$, we have

$$|||C_{\pm}|||_{L^2(\Sigma)\to L^2(\Sigma)} \leq D_{\pm}.$$

Therefore, from this inequality, we can deduce that

$$\begin{aligned}
||C_w[f]||_{L^2} &\leq ||C_+[fw_-]||_{L^2} + ||C_-[fw_+]||_{L^2} \\
&\leq D_+||fw_-||_{L^2} + D_-||fw_+||_{L^2} \\
&\leq D_+||f||_{L^2}||w_-||_{\infty} + D_-||f||_{L^2}||w_+||_{\infty} \\
&\leq \text{const. } \sqrt{t}\ \ ||f||_{L^2}
\end{aligned}$$

This proves (6.18), and with this the approximations (6.12) and (6.13) of $u(x,t)$ and $v(x,t)$ are justified. It remains now to check that (6.19) is true. Now, remember that $w(E)$ depends explicitly on the x-reflection coefficients $R_1(E;t)e^{2iEx}$ and $R_2(E;t)e^{2iEx}$. If we prove they satisfy the same norm, then equation (6.19) follows. So, we will be proving that the x-reflection coefficients are bounded by $\sqrt{t}$.

6.3 The Jump Matrix is Close to the Identity Matrix.

In this section, we will establish the conditions for the x-reflection coefficients (times an exponential), namely $\omega_{12}(\zeta)$ and $\omega_{13}(\zeta)$), to remain small along the countour Σ, as defined in 5.36. We will see why DiFranco's Transformation (6.15) is necessary to keep this coefficients bounded, so that the solutions to the integral equations (5.56) or (5.62) can be approximated by Neumann series. Under these conditions, $V(E) \sim I$ or $w(E) \sim 0$. In few words, we will be proving the following

Proposition 6.1. Small bounds for the reflection coefficients . *The reflection coefficient can be bounded along* Γ *by*

$$|\widetilde{R}(E;t)e^{2iEx}| \leq K\frac{\sqrt{t}}{|\lambda|}$$

for some constant K, and x in the interval

$$x_0 \leq x, \tag{6.20}$$

with x_0 given in equation (6.35). The interval (6.20) could be equivalently given by

$$-\frac{1}{2}\frac{\sqrt{t}}{\lambda_0} \leq x - L.$$

Consider the curve Σ as given by (5.36). Assume that the radius of $\gamma \cup \gamma^*$ is very large, i.e., assume

$$E_0 \gg 1.$$

Since $|E| \geq E_0$ along Σ, then

$$|E| \gg 1, \qquad \text{for} \qquad E \in \Sigma.$$

The reflection coefficients for potentials with the same support are given by equations (3.6), (3.7), and (3.8)

$$\begin{aligned} R_1(E;t) &= p_0^\star R(E;t), \\ R_2(E;t) &= q_0^\star R(E;t), \end{aligned} \tag{6.21}$$

where

$$R(E;t) = -\frac{\sin(2\nu(E)L)}{\nu(E)\cos(2\nu(E)L) - iE\sin(2\nu(E)L)} e^{-2iEL} e^{2iE^2 t}, \tag{6.22}$$

with $\nu(E)$ defined by

$$\nu(E) = \sqrt{E^2 + p_0^2 + q_0^2}. \tag{6.23}$$

Now, we should verify that the jump matrix is small along $\gamma \cup \mathbb{R} \backslash \widehat{\gamma}$. I.e., we should verify that

$$\frac{S_{21}(\zeta,t)}{S_{11}(\zeta)} e^{2iEx} = R_1(\zeta,t)e^{2iEx} = p_0 R(\zeta,t)e^{2iEx},$$

is small along Σ with $E_0 \gg 1$. If this is true, then the approximations (6.12) and (6.13) hold.

Let $E \to \infty$ in equation (6.23). We find that

$$\nu(E) \sim E,$$

and equation (6.22) is approximated in the following way

$$\begin{aligned} R(E;t) &\sim -\frac{\sin(2EL)}{E\cos(2\nu(E)L) - iE\sin(2\nu(E)L)} e^{-2iEL} e^{2iE^2 t} = \\ &= -\frac{\sin(2EL)}{Ee^{-2iEL}} e^{-2iEL} e^{2iE^2 t} = \\ &= -\frac{1}{E}\sin(2EL) e^{2iE^2 t}. \end{aligned}$$

Define

$$\widetilde{R}(E;t) \equiv -\frac{1}{E}\sin(2EL)e^{2iE^2 t} \tag{6.24}$$

It is clear that, for $E \in \mathbb{R}\backslash\widehat{\gamma}$

$$|\widetilde{R}(E;t)e^{2iEx}| \leq \frac{1}{|E|} \ll 1. \tag{6.25}$$

Now, for $E \in \gamma$, an equality like the previous one is true, but it is not so easy to see. Along γ, $\mathrm{Im}(E) > 0$. Hence,

$$\sin(2EL) = \frac{e^{2iEL} - e^{-2iEL}}{2} \sim -\frac{e^{-2iEL}}{2},$$

where e^{-2iEL} is the leading order term for for $E \in \gamma$ with $|E| \geq E_0 \gg 1$. Therefore,

$$\sin(2EL) = e^{-2iEL}\frac{e^{4iEL} - 1}{2},$$

and so

$$\begin{aligned} |\sin(2EL)| &\leq |e^{-2iEL}|\frac{|e^{4iEL}| + 1}{2}, \\ &\leq |e^{-2iEL}|\frac{1+1}{2} = \\ &= |e^{-2iEL}| \\ &= e^{2\mathrm{Im}(E)L}. \end{aligned}$$

Then, from (6.24)

$$|\widetilde{R}(E;t)e^{2iEx}| \leq \frac{1}{|E|}e^{-2\mathrm{Im}(E)(x-L)}|e^{2iE^2t}|. \tag{6.26}$$

Notice that, even for $t \sim 0$ but fixed, and $|E| = E_0$ large, the exponential $|e^{iE^2t}|$ is not necessarily small. For instance, in polar coordinates, for $E \in \gamma$, we have

$$\begin{aligned} E &= E_0 e^{i\theta} \\ E^2 &= E_0^2 e^{2i\theta} = E_0(\cos(2i\theta)) + i\sin(2i\theta)). \end{aligned}$$

Then

$$|e^{E^2t}| = e^{-E_0^2 t\sin(2\theta)} \longrightarrow \infty,$$

as $E_0 \to \infty$, when $\frac{\pi}{2} < \theta < \pi$.

Here is where DiFranco's transformation plays an important role. Keep λ fixed in equation $E = \lambda/\sqrt{t}$, i.e., in equation (6.15). Then, the limit $E \to \infty$ is equivalent to the short-time asymptotic limit $t \longrightarrow 0^+$. Hence,

$$|e^{iE^2t}| = e^{-\lambda_0\sin(2\theta)} = |e^{i\lambda^2}|,$$

is bounded for $|\lambda| = \lambda_0$ fixed. Here

$$\lambda_0 \equiv \sqrt{t}E_0 \tag{6.27}$$

is also a positive number.

From (6.26)

$$|\widetilde{R}(E;t)e^{2iEx}| \leq \frac{\sqrt{t}}{|\lambda|}e^{-2\mathrm{Im}(\lambda)(x-L)/\sqrt{t}}\,|e^{2i\lambda^2}| \tag{6.28}$$

If $x - L > 0$, then

$$|e^{-2\mathrm{Im}(\lambda)(x-L)/\sqrt{t}}| \leq 1.$$

Hence, for $x - L > 0$

$$|\widetilde{R}(E;t)e^{2iEx}| \leq \frac{\sqrt{t}}{|\lambda|}|e^{2i\lambda^2}|, \tag{6.29}$$

i.e., this inequality is valid at the right side of the jump. We would like to approximate at the left of the jump as well.

Let us asumme that we would like to approximate in the interval

$$x_0 \leq x \leq L,$$

for some x_0 to be determined. Here $x_0 - L \leq x - L \leq 0$. Then

$$|e^{-2\mathrm{Im}(\lambda)(x-L)/\sqrt{t}}| \leq |e^{-2\mathrm{Im}(\lambda)(x_0-L)/\sqrt{t}}|.$$

Using this inequality, equation (6.28) becomes

$$|\widetilde{R}(E;t)e^{2iEx}| \leq \frac{\sqrt{t}}{|\lambda|}e^{-2\,\mathrm{Im}(\lambda)(x_0-L)/\sqrt{t}}|e^{2i\lambda^2}|. \tag{6.30}$$

Define

$$f(t) \equiv \frac{\sqrt{t}}{|\lambda|}e^{-2\,\mathrm{Im}(\lambda)(x_0-L)/\sqrt{t}}, \tag{6.31}$$

for $t > 0$. Equation (6.30) now reads

$$|\widetilde{R}(E;t)e^{2iEx}| \leq f(t)\,|e^{2i\lambda^2}|. \tag{6.32}$$

Now, assuming that we can vary t, there should be an optimal $t_{\mathrm{optimal}} = t_{\mathrm{op}}$, such that $f(t)$ is minimized at $t = t_{\mathrm{op}}$. This is because $f(t) \to \infty$ as $t \to 0$ or $t \to \infty$, and $t \geq 0$. Then, it should have a minimum

$$f'(t) = \frac{t^{-1/2}}{|\lambda|}e^{-2\,\mathrm{Im}(\lambda)(x_0-L)/\sqrt{t}}\left(\frac{1}{2} + \mathrm{Im}(\lambda)\frac{(x_0-L)}{\sqrt{t}}\right).$$

The optimal $t = t_{\mathrm{op}}$ is such that $f'(t_{\mathrm{op}}) = 0$, which results

$$\mathrm{Im}(\lambda)\frac{(x_0-L)}{\sqrt{t_{\mathrm{op}}}} = -\,\frac{1}{2}. \tag{6.33}$$

Substitute t_{op} into the definition of $f(t)$, equation (6.31) to obtain

$$f(t_{\mathrm{op}}) = e\,\frac{\sqrt{t_{\mathrm{op}}}}{|\lambda|}.$$

At the same time, equation (6.32) is

$$|\widetilde{R}(E;t_{\text{op}})e^{2iEx}| \leq \ \ e\ \frac{\sqrt{t_{\text{op}}}}{|\lambda|}\ |e^{2i\lambda^2}|. \tag{6.34}$$

From equation (6.33), and dropping the subscript of t_{op}, we obtain the value of x_0

$$x_0 = -\ \frac{1}{2}\frac{\sqrt{t}}{\text{Im}(\lambda)} + L$$

Now, along γ, λ varies with $\text{Im}(\lambda) \leq \lambda_0$, where λ_0 is given by equation (6.27). Hence, the optimal x_0 is given by

$$x_0 = -\ \frac{1}{2}\frac{\sqrt{t}}{\lambda_0} + L. \tag{6.35}$$

Define

$$\Gamma \cup (\mathbb{R}\backslash\widehat{\Gamma}) \equiv \lambda(\gamma \cup (\mathbb{R}\backslash\widehat{\gamma})), \tag{6.36}$$

i.e., $\Gamma = \lambda(\gamma)$ and $\widehat{\Gamma} = \lambda(\widehat{\gamma})$ are the images of γ and $\widehat{\gamma}$ under DiFranco's transformation (6.15).

Combining equations (6.15) and (6.25), together with equations (6.29) and (6.34), we can obtain the statements of the proposition 6.1, concluding the proof.

Therefore, along $E \in \gamma \cup (\mathbb{R}\backslash\widehat{\gamma})$ with $E = \lambda/\sqrt{t}$, and for $x \geq L$ and a little bit to the left of the jump ($x_0 \leq x$), the "*x-reflection coefficients*"

$$R_x(E;t,x) = R(E;t)e^{2iEx},$$

can be approximated by

$$\widetilde{R}_x(E;t,x) = \widetilde{R}(E;t)e^{2iEx},$$

where $\widetilde{R}(E;t)$ is given by equation (6.24)

$$\widetilde{R}(E;t) = -\ \frac{\sqrt{t}}{\lambda}\sin(2\lambda L/\sqrt{t})e^{2i\lambda^2},$$

and $\widetilde{R}_x(E;t,x)$ is of the order $\sqrt{t}$

$$|\widetilde{R}(E;t)e^{2iEx}| \leq K\frac{\sqrt{t}}{|\lambda|}.$$

From this it follows that

$$||V(E) - I|| = O(\sqrt{t})$$

or

$$||w(E) - I|| = O(\sqrt{t})$$

and the approximations given in equations (6.12) and (6.13) are valid.

Remark. Remember that at the end of section 5.9, we stated a couple of properties that the transformation $m(E) \longrightarrow m_{\text{new}}(E)$ posseses (see page 126). Now, observe the following fact. For $E \in \widehat{\gamma}$, then:

$$|\widetilde{R}(E;t)e^{2iEx}| = \frac{1}{|E|}|\sin(2EL)| \leq 2L.$$

although this expression for $\widetilde{R}$ is nice and bounded, along $\widehat{\gamma}$, it is **not** necessarily small, since at $E = 0$

$$|\widetilde{R}(0;t)e^{2iEx}| = 2L.$$

Therefore, $w(E)$ cannot be small along $\widehat{\gamma}$ and the Nuemann series cannot be used to approximate the solution of the integral equation (5.56) along this contour. This is the reason why the removal of the segment $\widehat{\gamma}$ (Property 2, page 126) is fundamental in the contour deformation to obtain later a good asymptotic description.

6.4 Asymptotics of the Transmission Coefficient

In this section, we prove the following

Proposition 6.2. *The short-time asymptotics of the transmission coefficient is given, at leading order, by the following limit:*

$$\lim_{t \to 0^+} S_{11}(\lambda/\sqrt{t}) = 1. \tag{6.37}$$

Proof: Given

$$\nu(\zeta) = \sqrt{\zeta^2 + p_0^2 + q_0^2},$$

and

$$S_{11}(\zeta) = \frac{1}{\nu(\zeta)}\Big(\ \nu(\zeta)\cos(2\nu(\zeta)L) - i\zeta\sin(2\nu(\zeta)L)\ \Big)e^{2i\zeta L},$$

we use the transformation

$$\zeta = \frac{\lambda}{\sqrt{t}},$$

to obtain

$$\begin{aligned}
\nu\left(\frac{\lambda}{\sqrt{t}}\right) &= \sqrt{\frac{\lambda^2}{t} + p_0^2 + q_0^2}, \\
&= \frac{\lambda}{\sqrt{t}}\sqrt{1 + (p_0^2 + q_0^2)\frac{t}{\lambda^2}}.
\end{aligned}$$

When $t \to 0^+$

$$\nu\left(\frac{\lambda}{\sqrt{t}}\right) \sim \frac{\lambda}{\sqrt{t}},$$

i.e.

$$\lim_{t\to 0+} \frac{\sqrt{t}}{\lambda}\ \nu\left(\frac{\lambda}{\sqrt{t}}\right) = 1.$$

Therefore

$$\begin{aligned}
S_{11}\left(\frac{\lambda}{\sqrt{t}}\right) &\sim \frac{e^{2i\lambda L/\sqrt{t}}}{\frac{\lambda}{\sqrt{t}}}\left(\frac{\lambda}{\sqrt{t}}\cos\left(2\frac{\lambda}{\sqrt{t}}L\right) - i\frac{\lambda}{\sqrt{t}}\sin\left(2\frac{\lambda}{\sqrt{t}}L\right)\right), \\
&= \frac{e^{2i\lambda L/\sqrt{t}}}{\frac{\lambda}{\sqrt{t}}}\frac{\lambda}{\sqrt{t}}\left(\cos\left(2\frac{\lambda}{\sqrt{t}}L\right) - i\sin\left(2\frac{\lambda}{\sqrt{t}}L\right)\right), \\
&= \frac{e^{2i\lambda L/\sqrt{t}}}{\frac{\lambda}{\sqrt{t}}}\frac{\lambda}{\sqrt{t}}e^{-2i\lambda L/\sqrt{t}} \\
&= 1;
\end{aligned}$$

i.e.,

$$S_{11}\left(\frac{\lambda}{\sqrt{t}}\right) \longrightarrow 1,$$

when $t \to 0^+$.

Q.E.D.

This is actually a more general result. From the Manakov-Zakharov-Shabat system, and using a singular asymptotic analysis, we can check that the scattering matrix, $S(E;t)$, goes to the identity matrix, $S(E;t) \to I$, as $E \to \infty$. This implies $S_{11}(E,t) \to 1$ as $E \to \infty$.

From this argument, we can also say that the reflection coefficients behave like the entries $S_{21}(E)$ and $S_{31}(E)$ of the scattering matrix. Therefore, any contribution coming from the discrete spectrum is not important under the short-time approximation.

6.5 The Short-Time Approximation of the Reflection Coefficients: Squared Barrier Potentials with Same Support

Now that we know that the approximations in equations (6.12) and (6.12) are valid, we can evaluate these integrals to determine and describe the leading order behaviour of the solutions of the focusing Manakov system. This section will contain a description of the short-time asymptotics for potentials with same support.

In the previous section, once we used the DiFranco's transformation (6.15), and let $t \to 0^+$ we found

$$\begin{aligned} -u^*(x,t) &\sim \frac{1}{\pi} \int\limits_{\gamma\cup(\mathbb{R}\setminus\widehat{\gamma})} R_1(\zeta,t) e^{2i\zeta x}\, d\zeta, \\ &\sim \frac{p_0}{\pi} \int\limits_{\Gamma\cup(\mathbb{R}\setminus\widehat{\Gamma})} \widetilde{R}(\frac{\lambda}{\sqrt{t}}, t) e^{2i\lambda x/\sqrt{t}}\, \frac{d\lambda}{\sqrt{t}}, \end{aligned}$$

where $\Gamma \equiv \lambda(\gamma)$ and $\widehat{\Gamma} \equiv \lambda(\widehat{\gamma})$ are the images of γ and $\widehat{\gamma}$ under DiFranco's transformation. We then have

$$u^*(x,t) \sim \frac{p_0}{\pi} \int\limits_{\Gamma\cup(\mathbb{R}\setminus\widehat{\Gamma})} \frac{\sin(2\lambda L/\sqrt{t})}{\lambda} e^{2i\lambda^2} e^{2i\lambda x/\sqrt{t}}\, d\lambda. \tag{6.38}$$

Compare with (6.14). We observe this is the linear problem. We conclude that for the coupled system with coincident support, the leading order behavior of the solution is also the solution to the linear problem.

Following DiFranco [DiF], the integral (6.38) is given in terms of the *"Error function"*. The rest of the section will be stablishing to show this statement.

We can verify that the leading order term of $\sin(2\lambda L/\sqrt{t})$ is $-\frac{1}{2i}e^{-2i\lambda L/\sqrt{t}}$, for $|E| = |E_0| \gg 1$, along γ in the limit when t is small. We then repeat here the calculation in [DiF] and [DiF-McL]

$$\begin{aligned}
u^*(x,t) &\sim \frac{p_0}{\pi}\int\limits_{\Gamma\cup(\mathbb{R}\backslash\widehat{\Gamma})} \frac{\sin(2\lambda L/\sqrt{t})}{\lambda}e^{2i\lambda^2}e^{2i\frac{\lambda x}{\sqrt{t}}}\,d\lambda \\
&\sim \frac{p_0}{\pi}\int\limits_{\Gamma} \frac{\sin(2\lambda L/\sqrt{t})}{\lambda}e^{2i\lambda^2}e^{2i\frac{\lambda x}{\sqrt{t}}}\,d\lambda + \frac{p_0}{\pi}\int\limits_{\mathbb{R}\backslash\widehat{\Gamma}} \frac{\sin(2\lambda L/\sqrt{t})}{\lambda}e^{2i\lambda^2}e^{2i\frac{\lambda x}{\sqrt{t}}}\,d\lambda \\
&\sim \frac{p_0}{2\pi i}\int\limits_{\Gamma} \frac{-e^{-2i\frac{\lambda L}{\sqrt{t}}}}{\lambda}e^{2i\lambda^2}e^{2i\frac{\lambda x}{\sqrt{t}}}\,d\lambda + \frac{p_0}{2\pi i}\int\limits_{\mathbb{R}\backslash\widehat{\Gamma}} \frac{e^{2i\frac{\lambda L}{\sqrt{t}}}-e^{-2i\frac{\lambda L}{\sqrt{t}}}}{\lambda}e^{2i\lambda^2}e^{2i\frac{\lambda x}{\sqrt{t}}}\,d\lambda \\
&\sim -\frac{p_0}{2\pi i}\int\limits_{\Gamma\cup(\mathbb{R}\backslash\widehat{\Gamma})} \frac{1}{\lambda}e^{2i\lambda^2}e^{2i\frac{\lambda(x-L)}{\sqrt{t}}}\,d\lambda + \frac{p_0}{2\pi i}\int\limits_{\mathbb{R}\backslash\widehat{\Gamma}} \frac{e^{2i\frac{\lambda L}{\sqrt{t}}}}{\lambda}e^{2i\lambda^2}e^{2i\frac{\lambda x}{\sqrt{t}}}\,d\lambda \\
&\sim \frac{p_0}{2\pi i}\int\limits_{\Gamma\cup(\mathbb{R}\backslash\widehat{\Gamma})} \frac{1}{\lambda}e^{2i\lambda^2}e^{-2i\frac{\lambda(x-L)}{\sqrt{t}}}\,d\lambda - \frac{p_0}{2\pi i}\int\limits_{\mathbb{R}\backslash\widehat{\Gamma}} \frac{1}{\lambda}e^{2i\lambda^2}e^{-2i\frac{\lambda(x+L)}{\sqrt{t}}}\,d\lambda, \qquad (6.39)
\end{aligned}$$

where we changed $\lambda \to -\lambda$. Under the asymptotics $t \to 0^+$ the second integral in the last equation is of order $O(\sqrt{t})$. We can see this by using a stationary phase argument, to obtain

$$\frac{p_0}{2\pi i}\int\limits_{\mathbb{R}\backslash\widehat{\Gamma}} \frac{1}{\lambda}e^{2i\lambda^2}e^{-2i\frac{\lambda(x+L)}{\sqrt{t}}}\,d\lambda = -p_0\frac{\sqrt{t}\,e^{\frac{-i\pi}{4}}e^{-i\frac{(x+L)^2}{2t}}}{\sqrt{2\pi}(x+L)}\left(1+\frac{e^{i\frac{\pi}{2}}\,t}{(x+L)^2}+O(t^2)\right). \qquad (6.40)$$

Now, define

$$s \equiv \frac{x-L}{\sqrt{t}}, \qquad (6.41)$$

$$I(s) \equiv -\frac{1}{2\pi i}\int\limits_{\Gamma\cup(\mathbb{R}\backslash\widehat{\Gamma})} \frac{1}{\lambda}e^{2i\lambda^2}e^{-2is\lambda}\,d\lambda. \qquad (6.42)$$

Hence, using equations (6.39), (6.40) and (6.42), we obtain

$$u^*(x,t) \sim p_0 I(s) + O(\sqrt{t}). \tag{6.43}$$

Computing the derivative of $I(s)$, and by a contour deformation, we obtain

$$\begin{aligned}
\frac{dI}{ds} &= -\frac{1}{2\pi i} \int\limits_{\Gamma\cup(\mathbb{R}\setminus\widehat{\Gamma})} \frac{1}{\lambda} e^{2i\lambda^2}(-2i\lambda)e^{-2is\lambda}\, d\lambda, \\
&= \frac{1}{\pi} \int\limits_{\Gamma\cup(\mathbb{R}\setminus\widehat{\Gamma})} e^{2i\lambda^2} e^{-2is\lambda}\, d\lambda, \\
&= \frac{1}{\pi} \int\limits_{\mathbb{R}} e^{2i(\lambda-\frac{s}{2})^2}\, d\lambda\; e^{-2i\frac{s^2}{4}}.
\end{aligned}$$

Put $\xi = \lambda - \frac{s}{2}$

$$\frac{dI}{ds} = \frac{1}{\pi} \int\limits_{\mathbb{R}} e^{2i\xi^2}\, d\xi\; e^{-i\frac{s^2}{2}},$$

and now $\xi = i^{1/2}\eta/\sqrt{2}$

$$\begin{aligned}
\frac{dI}{ds} &= \frac{1}{\pi} \int\limits_{\mathbb{R}} e^{-\eta^2}\, d\eta\; \frac{i^{1/2}}{\sqrt{2}} e^{-i\frac{s^2}{2}}, \\
&= \frac{1}{\pi}\frac{\sqrt{\pi}}{\sqrt{2}} i^{1/2} e^{-i\frac{s^2}{2}}, \\
&= \frac{1}{\sqrt{2\pi}} e^{i\pi/4} e^{-i\frac{s^2}{2}}.
\end{aligned}$$

After integrating so that $I(s) \to 0$ as $s \to \infty$ and $I(s) \to 1$ as $s \to -\infty$, we find that

$$\begin{aligned}
I(s) &= \frac{e^{i\pi/4}}{\sqrt{2\pi}} \int\limits_{\infty}^{s} e^{-i\frac{\xi^2}{2}}\, d\xi \\
&= -\frac{e^{i\pi/4}}{\sqrt{2\pi}} \int\limits_{0}^{\infty} e^{-i\frac{\xi^2}{2}}\, d\xi + \frac{e^{i\pi/4}}{\sqrt{2\pi}} \int\limits_{0}^{s} e^{-i\frac{\xi^2}{2}}\, d\xi
\end{aligned}$$

After the substitution,

$$\xi = \sqrt{2}e^{-i\pi/4}\lambda,$$

we obtain

$$
\begin{aligned}
I(s) &= -\frac{1}{\sqrt{\pi}} \int_0^{+\infty} e^{-\lambda^2}\, d\lambda + \frac{1}{\sqrt{\pi}} \int_0^{\frac{e^{i\pi/4}}{\sqrt{2}}s} e^{-\lambda^2}\, d\lambda \\
&= -\frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{e^{i\pi/4}}{\sqrt{2}}s} e^{-\lambda^2}\, d\lambda.
\end{aligned}
$$

Now, the error function Erf is defined by

$$
\mathrm{Erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z e^{-\lambda^2}\, dz, \tag{6.44}
$$

we have

$$
I(s) = -\frac{1}{2} + \frac{1}{2}\,\mathrm{Erf}\left(\frac{se^{i\pi/4}}{\sqrt{2}}\right). \tag{6.45}
$$

From this, and from equation (6.43), we have

$$
u^*(x,t) \sim -\frac{p_0}{2}\left[1 - \mathrm{Erf}\left(\frac{se^{i\pi/4}}{\sqrt{2}}\right)\right] + O(\sqrt{t}), \tag{6.46}
$$

Therefore

$$
\begin{aligned}
u(x,t) &\sim -\frac{p_0}{2}\left[1 - \mathrm{Erf}\left(\frac{se^{-i\pi/4}}{\sqrt{2}}\right)\right] + O(\sqrt{t}), && (6.47)\\
v(x,t) &\sim -\frac{q_0}{2}\left[1 - \mathrm{Erf}\left(\frac{se^{-i\pi/4}}{\sqrt{2}}\right)\right] + O(\sqrt{t}). && (6.48)
\end{aligned}
$$

We have shown that the asymptotics to the solution of the focusing Manakov system is also described by the Error function, as is was in the defocusing Nonlinear Schrödinger equation.

We might notice a change of sign in the leading term, if we compare this equation with equation (22) in [DiF]. This just comes from the fact that, the reflection coefficient has a $\sigma = \pm 1$ coefficient. In his case, DiFranco considers the defocusing case $\sigma = -1$, as opposed to the focusing case $\sigma = 1$, which bring us a negative sign in the leading order term. This negative sign does not show up in our numerical experiments, since we are plotting the absolute value of the solutions $u(x,t)$ and $u(x,t)$.

Therefore, the leading behaviour is given by the Error function, no matter if we are considering the focusing or thee defocusing case. This is our analytical interpretation of the numerical observations made in Chapter 1.

So far, this is true for the case of coincident supports. We will establish the same result in the next section for potentials with disjoint and overlapping support.

6.6 The Short-Time Approximation of the Reflection Coefficients: Squared Barrier Potentials with Disjoint and Overlapping Supports

This section will consider a couple of surprising results. Here, we compute the asymptotics of the reflection coefficients at short-times, by the use of DiFranco's transformation. Although both reflection coefficients look very complicated in their explicit form as computed in chapter 3, when we study the short-time asymptotics, they turn out: 1) to be uncoupled, and 2) to have exactly the form as in the same support case.

6.6.1 Disjoint Support Potentials

In section 3.2, we found the reflection coefficients corresponding to barriers with disjoint support (equations (3.29))

$$\begin{aligned} R_1(E;t) &= \frac{S_{21}(E;t)}{S_{11}(E;t)} = \frac{a_2(E)b_2(E)}{a_1(E)b_1(E)} e^{2iE(L_1-\widehat{L})} e^{-2iE^2 t}, \\ R_2(E;t) &= \frac{S_{31}(E;t)}{S_{11}(E;t)} = \frac{b_3(E)}{b_1(E)} e^{-2iEL_2} e^{-2iE^2 t}, \end{aligned}$$

with

$$\begin{aligned} \widehat{L} &= L_2 - L_1, \\ \nu_1(E) &= \sqrt{E^2 + p_0^2}, \\ \nu_2(E) &= \sqrt{E^2 + q_0^2}, \end{aligned}$$

and

$$\begin{aligned}
a_1(E) &= \nu_1(E)\cos(\nu_1(E)\widehat{L}) - iE\sin(\nu_1(E)\widehat{L}) \\
a_2(E) &= -p_0\sin(\nu_1(E)\widehat{L}) \\
b_1(E) &= \nu_2(E)\cos(\nu_2(E)\widehat{L}) - iE\sin(\nu_2(E)\widehat{L}) \\
b_2(E) &= \nu_2(E)e^{iE\widehat{L}} \\
b_3(E) &= -q_0\sin(\nu_2(E)\widehat{L}).
\end{aligned}$$

Performing once again DiFranco's scaling, $E = \lambda/\sqrt{t}$, it is easy to check that

$$\begin{aligned}
a_1(E) &= \widetilde{a}_1\left(\frac{\lambda}{\sqrt{t}}\right) + O(1), \\
a_2(E) &= \widetilde{a}_2\left(\frac{\lambda}{\sqrt{t}}\right) + O(\sqrt{t}), \\
b_1(E) &= \widetilde{b}_1\left(\frac{\lambda}{\sqrt{t}}\right) + O(1), \\
b_2(E) &= \widetilde{b}_2\left(\frac{\lambda}{\sqrt{t}}\right) + O(1), \\
b_3(E) &= \widetilde{b}_3\left(\frac{\lambda}{\sqrt{t}}\right) + O(\sqrt{t}).
\end{aligned}$$

Here

$$\begin{aligned}
\widetilde{a}_1\left(\frac{\lambda}{\sqrt{t}}\right) &= \frac{\lambda}{\sqrt{t}}e^{-i\lambda\widehat{L}/\sqrt{t}}, \\
\widetilde{a}_2\left(\frac{\lambda}{\sqrt{t}}\right) &= -\sigma p_0^*\sin\left(\frac{\lambda}{\sqrt{t}}\widehat{L}\right), \\
\widetilde{b}_1\left(\frac{\lambda}{\sqrt{t}}\right) &= \frac{\lambda}{\sqrt{t}}e^{-i\lambda\widehat{L}/\sqrt{t}}, \\
\widetilde{b}_2\left(\frac{\lambda}{\sqrt{t}}\right) &= \frac{\lambda}{\sqrt{t}}e^{i\lambda\widehat{L}/\sqrt{t}}, \\
\widetilde{b}_3\left(\frac{\lambda}{\sqrt{t}}\right) &= -\sigma q_0^*\sin\left(\frac{\lambda}{\sqrt{t}}\widehat{L}\right).
\end{aligned}$$

From this, it follows that the reflection coefficients have the following approximations

$$\begin{aligned} R_1(E;t) &= \widetilde{R}_1(\lambda,t)+O(t), \\ R_2(E;t) &= \widetilde{R}_2(\lambda,t)+O(t), \end{aligned}$$

with

$$\begin{aligned} \widetilde{R}_1(\lambda,t) &= -\sigma p_0^* \frac{\sqrt{t}}{\lambda}\sin\left(\frac{\lambda}{\sqrt{t}}\widehat{L}\right) e^{i\lambda(L_1+L_2)/\sqrt{t}} e^{-2i\lambda^2} \\ \widetilde{R}_1(\lambda,t) &= -\sigma q_0^* \frac{\sqrt{t}}{\lambda}\sin\left(\frac{\lambda}{\sqrt{t}}\widehat{L}\right) e^{i\lambda(L_1+L_2)/\sqrt{t}} e^{-2i\lambda^2}. \end{aligned}$$

Although they look the same, this is due to the symmetry we chose for the initial potentials. If we choose $[-L_4,-L_3]$ and $[L_1,L_2]$ to be the supports of $u(x,0)$ and $v(x,0)$, respectively, we actually find that

$$\widetilde{R}_1(\lambda,t) = -\sigma p_0^* \frac{\sqrt{t}}{\lambda}\sin\left(\frac{\lambda}{\sqrt{t}}(L_4-L_3)\right) e^{i\lambda(L_3+L_4)/\sqrt{t}} e^{-2i\lambda^2}, \tag{6.49}$$

$$\widetilde{R}_1(\lambda,t) = -\sigma q_0^* \frac{\sqrt{t}}{\lambda}\sin\left(\frac{\lambda}{\sqrt{t}}(L_2-L_1)\right) e^{i\lambda(L_1+L_2)/\sqrt{t}} e^{-2i\lambda^2}. \tag{6.50}$$

Again, we observe that the reflection coefficients are independent at leading order in the short-time asymptotics. This is not a surprising fact. Since the initial potentials have disjoint supports, the effect of one potential on the other is exponentially small, because the exponential decay outside the support. So, at leading order *there is no interaction between the pulses at early evolution of the solutions of the Manakov system.* We can therefore apply again the approximations found in section 6.1, equations (6.12) and (6.13). This explains our numerical observations from chapter 1, Figures 1.28 and 1.29.

6.6.2 Overlapping Support Potentials

In section 3.3, we found the expressions of the reflection coefficients for squared barrier potentials with different, but overlapping supports. We found equations (3.35), and

(3.36)

$$
\begin{aligned}
R_1(E;t) &= \frac{B(E)}{A(E)} e^{-2iEL_2} e^{-2iE^2 t}, \\
R_2(E;t) &= \frac{C(E)}{A(E)} e^{-2iEL_2} e^{-2iE^2 t}.
\end{aligned}
$$

The expressions for $A(E)$, $B(E)$, and $C(E)$ are given by

$$
\begin{aligned}
A(E) &= [\nu_2 \cos(\nu_2 \widehat{L}) - iE \sin(\nu_2 \widehat{L})] \cdot [\nu \cos(2\nu L_1) - iE \sin(2\nu L_1)] \cdot \\
&\quad \cdot [\nu_2 \cos(\nu_2 \widehat{L}) - iE \sin(\nu_2 \widehat{L})] + \\
&\quad + [\nu_2 \cos(\nu_2 \widehat{L}) - iE \sin(\nu_2 \widehat{L})] \cdot (-\sigma |q_0|^2) \sin(2\nu L_1) \sin(\nu_2 \widehat{L}) + \\
&\quad + (-\sigma |q_0|^2) \sin(\nu_2 \widehat{L}) \sin(2\nu L_1) \cdot [\nu_2 \cos(\nu_2 \widehat{L}) - iE \sin(\nu_2 \widehat{L})] + \\
&\quad + (-\sigma |q_0|^2) \sin(\nu_2 \widehat{L}) \frac{|q_0|^2 \sin(\nu_2 \widehat{L})}{|p_0|^2 + |q_0|^2} \cdot \\
&\qquad \cdot [\nu \cos(2\nu L_1) + iE \sin(2\nu L_1) + \nu \frac{|p_0|^2}{|q_0|^2} e^{2iEL_1}], \\
B(E) &= -\sigma p_0^* \sin(2\nu L_1) \nu_2 e^{iE\widehat{L}} \cdot [\nu_2 \cos(\nu_2 \widehat{L}) - iE \sin(\nu_2 \widehat{L})] \\
&\quad + \frac{-\sigma p_0^* |q_0|^2}{|p_0|^2 + |q_0|^2} \sin(\nu_2 \widehat{L}) \nu_2 e^{iE\widehat{L}} \cdot [\nu \cos(2\nu L_1) + iE \sin(2\nu L_1) - \nu e^{2iEL_1}], \\
C(E) &= -\sigma q_0^* \sin(\nu_2 \widehat{L}) [\nu \cos(2\nu L_1) - iE \sin(2\nu L_1)] [\nu_2 \cos(\nu_2 \widehat{L}) - iE \sin(\nu_2 \widehat{L})] \\
&\quad + q_0^* |q_0|^2 \sin(2\nu L_1) \sin^2(\nu_2 \widehat{L}) \\
&\quad - \sigma q_0^* [\nu_2 \cos(\nu_2 \widehat{L}) + iE \sin(\nu_2 \widehat{L})] \sin(2\nu L_1) [\nu_2 \cos(\nu_2 \widehat{L}) - iE \sin(\nu_2 \widehat{L})] \\
&\quad - \sigma q_0^* [\nu_2 \cos(\nu_2 \widehat{L}) + iE \sin(\nu_2 \widehat{L})] \cdot \frac{|q_0|^2 \sin(\nu_2 \widehat{L})}{|p_0|^2 + |q_0|^2} \cdot \\
&\quad \cdot [\nu \cos(2\nu L_1) + iE \sin(2\nu L_1) + \nu \frac{|p_0|^2}{|q_0|^2} e^{2iEL_1}],
\end{aligned}
$$

where

$$
\begin{aligned}
\nu(E) &= \sqrt{E^2 + p_0^2 + q_0^2}, \\
\nu_2(E) &= \sqrt{E^2 + q_0^2}, \\
\widehat{L} &= L_2 - L_1.
\end{aligned}
$$

Performing the change of varibles $E = \lambda/\sqrt{t}$, and letting $t \to 0^+$, we find that the following proposition holds

Proposition 6.3. *When $t \to 0^+$, the factors $A(E)$, $B(E)$, and $C(E)$ of the reflections coefficients are approximated by*

$$\begin{aligned} B(E) &= \widetilde{B}\left(\frac{\lambda}{\sqrt{t}}\right) + O\left(\frac{1}{\sqrt{t}}\right), \\ C(E) &= \widetilde{C}\left(\frac{\lambda}{\sqrt{t}}\right) + O\left(\frac{1}{\sqrt{t}}\right), \\ A(E) &= \widetilde{A}\left(\frac{\lambda}{\sqrt{t}}\right) + O\left(\frac{1}{t}\right), \end{aligned}$$

for $E \in \gamma \cup (\mathbb{R}\backslash\widehat{\gamma})$, with γ and $\widehat{\gamma}$ defined in section 5.8, equation (5.37). Here

$$\begin{aligned} \widetilde{B}\left(\frac{\lambda}{\sqrt{t}}\right) &\equiv -\sigma p_0^* \frac{\lambda^2}{t} \sin\left(\frac{2\lambda L_1}{\sqrt{t}}\right), \\ \widetilde{C}\left(\frac{\lambda}{\sqrt{t}}\right) &\equiv -\sigma q_0^* \frac{\lambda^2}{t} \sin\left(\frac{2\lambda L_1}{\sqrt{t}}\right), \\ \widetilde{A}\left(\frac{\lambda}{\sqrt{t}}\right) &\equiv \frac{\lambda^3}{t^{\frac{3}{2}}} e^{-2i\lambda L_2/\sqrt{t}}. \end{aligned}$$

We now state a second proposition, which is an immediate consequence of proposition 6.3.

Proposition 6.4. *Under the same assumptions as in proposition 6.3, the* reflection coefficients *are approximated by*

$$\begin{aligned} R_1(E) &= \widetilde{R}_1(\lambda, t) + O(t) \\ R_2(E) &= \widetilde{R}_2(\lambda, t) + O(t), \end{aligned}$$

where

$$\widetilde{R}_1(\lambda, t) = \frac{\widetilde{B}\left(\frac{\lambda}{\sqrt{t}}\right)}{\widetilde{A}\left(\frac{\lambda}{\sqrt{t}}\right)} = -\sigma p_0^* \frac{\sqrt{t}}{\lambda} \sin\left(\frac{2\lambda L_1}{\sqrt{t}}\right) e^{-2i\lambda^2}, \tag{6.51}$$

$$\widetilde{R}_2(\lambda, t) = \frac{\widetilde{C}\left(\frac{\lambda}{\sqrt{t}}\right)}{\widetilde{A}\left(\frac{\lambda}{\sqrt{t}}\right)} = -\sigma q_0^* \frac{\sqrt{t}}{\lambda} \sin\left(\frac{2\lambda L_2}{\sqrt{t}}\right) e^{-2i\lambda^2}. \tag{6.52}$$

This is equivalent to Proposition 6.3.1 in [DiF]. Also notice that $\widetilde{R}_1$ and $\widetilde{R}_2$ are independent of each other. This coincides with the numerical observations shown in Figures 1.30 and 1.31.

Proposition 6.3 is a consequence of the facts that $e^{i\alpha\lambda/\sqrt{t}}$ is a bounded function along $\gamma \cup (\mathbb{R}\backslash\widehat{\gamma})$, and from the Taylor expansions (also along $\gamma \cup (\mathbb{R}\backslash\widehat{\gamma})$) of the sine and cosine trigonometric functions.

We notice that the approximations of the reflection coefficients of supports with disjoint and overlapping support have exactly the same form. The analysis performed in section 6.5 applies also here; i.e., the Error function is the correct special function to describe the short-time asymptotics of the solutions of the focusing/defocusing Manakov system, with squared barrier potentials.

We observe that these approximations of the reflection coefficients are independent of each other. This analytical result coincides with the numerical observations of chapter 1, namely, Figures 1.30, 1.31, 1.28 and 1.29. We observed that the potentials evolve in time independent of each other.

Conclusions

Out of the several kinds of coupled systems of Nonlinear Schrodinger (NLS) equations, the Manakov system turns to be a good model to study because its completely integrability allows one to study, in greater analytical depth, qualitative features that are common to more general models describing related physical phenomena.

In this thesis, we studied the evolution in time of some particular initial conditions under the *focusing* Manakov system. These initial conditions were the squared barrier potentials. There were two main reasons to choose this type of potentials. The first one was for convenience, and the second one was because of the interesting behavior of the solutions at early times.

The MZS system, for squared barrier potentials, turned to be a constant coefficient system of ordinary differential equations to which is easy to find a solution, as opposed to a variable coefficient system of equations, in general more complicated to solve.

Concerning the second reason, the evolution of these kinds of potentials at early times, i.e, when $t \sim 0$, is interesting by itself. A Gibbs-type phenomenon is presented at the jumps of the potentials: fast oscillations on one side of the jump, and exponential decay to the other side. We observed this phenomenon for different type of squared boxes in chapter 1, Figures 1.26 through 1.29.

The analysis presented in this thesis generalizes the work of J. DiFranco and K. McLaughlin [DiF], [DiF-McL], performed for the defocusing NLS equation. Two main differences exist between their work and my work. The Zakharov-Shabat (ZS) system associated to the NLS equation is a 2×2 system, as opposed to the MZS system which is a 3×3 system of differential equations.

Also, since we are dealing with the focusing Manakov system, solitons may arise. They come from the eigenvalues of the MZS system, as opposed to defocusing NLS equation which does not support solitons.

Considering this issue of solitons, we should know which conditions are required to generate solitons. At the end of chapter 3, we found the condition for the generation of solitons when the potentials have the same, and disjoint supports. The threshold for same support was

$$2\sqrt{p_0^2+q_0^2}\,L=\frac{\pi}{2},$$

and for disjoint support we have either

$$p_0\widehat{L}=\frac{\pi}{2} \qquad \text{or} \qquad q_0\widehat{L}=\frac{\pi}{2}.$$

In a recent paper [Kla], M. Klaus proved that the condition for the *absence* of eigenvalues is

$$\int_{-\infty}^{\infty}\sqrt{u^2(x,0)+v^2(x,0)}\;dx<\frac{\pi}{2}.$$

This result obviously coincides with the threshold conditions found in this thesis.

Continuing with the study of the eigenvalues of the MZS system, we could ask ourselves the question of the location of the eigenvalues in the complex plane. Chapter 4 aimed to be an extension of the results of Klaus and Shaw [KS1], [KS2] on the location of the eigenvalues of the ZS system associated to the focusing NLS equation. Their result was basically the following: for real, nonnegative, integrable potentials with a single maximum, the eigenvalues of the ZS system are located on the imaginary axis.

We proved a similar statement on the location of the eigenvalues of the MZS system. We found that, for squared barrier potentials with same and disjoint supports, the eigenvalues are located at the imaginary axis.

The previous result is just a step towards a generalization of the Klaus-Shaw result applied to the MZS system. Our current research focuses on finding the location of eigenvalues when the initial potentials have disjoint support, but they are not necessarily piecewise constant. This might give us an insight into the case of more general potentials.

A concluding chapter should say something about expectations that will guide future research. Since each of the squared potentials with disjoint support are single-lobe, I dare to state the following conjecture: If the initial potentials of the MZS system separately satisfy the conditions of the Klaus-Shaw theorem, then the eigenvalues of the MZS system are purely imaginary.

Once we knew something about the location of the eigenvalues, we proceeded to study the behaviour of the solutions of the Manakov system. We performed this study by a contour deformation of the original mRHP, which allows us: 1) to remove the eigenvalues of the mRHP, and 2) to keep small the reflection coefficients along the new contour. The latter also allowed us to approximate the solutions by Neuman series of an integral equation (see equation (5.56)).

We found that, under the short-time asymptotics, the solutions can be approximated by the special function $\mathrm{Erf}(z)$, the error function. It is important to remark here that the approximation shown in this thesis is not valid in the real line. According to the results found in section 6.3, this approximation is only valid for x in the intervals

$$\left(-\frac{1}{2}\frac{\sqrt{t}}{\lambda_0}+L, \quad \infty\right) \qquad \text{and} \qquad \left(-\infty, \quad \frac{1}{2}\frac{\sqrt{t}}{\lambda_0}-L\right).$$

This is to say, the approximation is valid a little bit left of the jump at $x = L$, and a little bit right of the jump at $x = -L$.

An approximation for $x \in (-L, L)$ for the defocusing NLS equation was performed by DiFranco and McLaughlin [DiF-McL]. In their paper, they use a different norm, namely, the Hölder continuity norm. For the case studied in this thesis, i.e., the Manakov system, a similar computation can be carried out. However, I have left this project for future work.

Another very remarkable result was the following. In all of the three cases (same, disjoint and overlapping supports), when $t \sim 0$, the reflection coefficients

$$R_1(E;t) = \frac{S_{21}(E;t)}{S_{11}(E)}, \qquad R_2(E;t) = \frac{S_{31}(E;t)}{S_{11}(E)}$$

turned out to be uncoupled, this is to say, they are independent in the short-time asymptotic approximation. See equations (6.21), (6.22), (6.49), (6.50), (6.51), and (6.52).

From this fact, and equations (6.12), (6.13), we can conclude that the behavior of the solutions of the Manakov system are independent of each other, i.e., there is no interaction at early stages of the evolution in time. This behavior was observed in chapter 1 through figures 1.29 to 1.30.

Comparing the results for the focusing case found in this thesis, with the results given in [DiF] and [DiF-McL], we observe that in both cases (focusing and defocusing) the behavior is the same, and it is described by the error function $\mathrm{Erf}(z)$. This fact is a consequence of the fact that the transmission coefficient goes to 1 in the short-time limit

$$\lim_{t\to 0} S_{11}(\lambda/\sqrt{t}) = 1, \tag{6.53}$$

while the eigenvalues (which appear only in the focusing case) solve the equation

$$S_{11}(E) = 0,$$

which is incompatible with (6.53).

To have a different way of observing this fact of similar behavior in the focusing and defocusing cases, we need to take a look at the transformation used to remove the poles, equations (5.38), (5.39) and (5.40). Considering that here we have 3×3 matrices, and up to a minus sign (because of the focusing caracter of the equation), we obtained the "same" transformation as in [DiF-McL]. See equation (3.14) in [DiF-McL]. Once this transformation is carried out, we only dealt with the reflection coefficients (which contain the information of the continuous spectrum), leaving out the discrete spectrum (the eigenvalues) because of the limit (6.53). Therefore, the continuous spectrum plays a predominant role over the discrete spectrum under the regime of small time.

In this work, I did not explicitly compute the terms of order $O(t)$ and higher. It would be also interesting to computing these higher order terms, and verify if some sort of interaction between the two channels appear at this higher order approximations. My guess is that they do appear.

Returning to the numerical analysis of the Manakov system, the Linearized Crank-Nicolson (LCN) scheme shown in chapter 1 happens to be time and computer-memory consuming. This is due to the fact that at the jumps the numerical scheme should use very small space-steps. I would also like to implement a numerical code that captures the behavior at the jumps at short-range, as well as the behavior at medium- and long-time limits. We might then observe the formation of solitons. The scheme to be implemented would be the proposed by Chang and Xu [Cha2]. The scheme shown in this paper obeys two discrete conservation laws, turns to be unconditionally stable, and convergent. It also takes care of the magnitude of the derivatives: the bigger derivatives will require small space-steps, while away from the jumps only require not so small space-steps.

For the long-time behaviour, we might need to implement "transparent" boundary conditions in the numerical code. Such conditions have been employed in diffusion type equations.

Having a code of this kind, will allow us to observe the evolution of the squared potentials at medium- and long-time, leading to a possible analytic/asymptotic description of the evolution.

Suggested by P. Miller, another problem to consider is variable amplitude potentials. The amplitude should be a function of the final small time t_f: the smaller the time t_f the higher the amplitude of the potentials, say $p_0 = 2/\sqrt{t_f}$ and $q_0 = 1/\sqrt{t_f}$. This might create a balance in the short-time asymptotics, and maybe a new special function would arise.

The last problem would be of particular interest for the focusing Manakov system since, as we could remember from our analysis done in chapter 3, the higher the potentials, bigger is the number of solitons appearing in the solutions. Here, it would be interesting to observe how the amplitude of the potentials, the short-time, and the number of solitons balance and interact with each other to obtain a (possibly) new interesting behavior.

Aside from the main ideas presented in this thesis, I would also be interested in studying the stability of solutions of the Manakov system via the Evans function. In appendix A, we computed exact solutions of the Manakov system which allows us to compute solutions of the Linearized Manakov system and therefore to compute the Evans function. This would be a first step to study more complicated models such as the coupled NLS system of equations of nonlinear optics, or coupled Gross-Pitaevskii equations in Bose-Einstein condensation.

Appendix A
Solutions of the Manakov System: The Bäcklund Transform

A.0 Introduction

In this chapter, we will find some particular solutions of the Manakov System in the real line by the method of the *Bäcklund tranforms*, also called *Darboux transforms*, *Additions*, or *The Commutation Method*. The contents of this chapter follows the ideas in [Erc1], [Erc2], and [Cr]. Two kind of solutions are found, corresponding to a one-soliton solution and a two-soliton solution.

We will check that these solutions decay exponentially to zero at $\pm\infty$. We will also check that the two-soliton solution (under a large initial dephase) behaves like two separated "*sech*" profiles, i.e., two separated solitons. We also study the case when the two-soliton solution approaches to a one-soliton solution under a special limiting case

A.1 Exact Solutions of the Manakov System

It is known that the Manakov system

$$\begin{aligned} i\frac{\partial u}{\partial t} &= -\frac{\partial^2 u}{\partial y^2} - 2(|u|^2 + |v|^2)u, \\ i\frac{\partial v}{\partial t} &= -\frac{\partial^2 v}{\partial y^2} - 2(|u|^2 + |v|^2)v \end{aligned} \tag{A.1}$$

has exact solutions [Ma]. We will look for solutions of the form

$$\begin{aligned} \widetilde{u_0}(y,t) &= u_0(y)e^{i\omega_1 t} \\ \widetilde{v_0}(y,t) &= v_0(y)e^{i\omega_2 t}. \end{aligned} \tag{A.2}$$

The functions $u_0(y)$ and $v_0(y)$ satisfy the stationary system

$$\begin{aligned} -\frac{d^2u_0}{dy^2} - 2(|u_0|^2 + |v_0|^2)u_0 + \omega_1 u_0 &= 0 \\ -\frac{d^2v_0}{dy^2} - 2(|u_0|^2 + |v_0|^2)v_0 + \omega_2 v_0 &= 0. \end{aligned} \tag{A.3}$$

Using the technique of the Darboux transform, we can contruct solutions of this system. In this chapter, we will show that, when $\omega_1 = \omega_2$, $u_0(y)$ and $v_0(y)$ are given by the standrad one-soliton solution

$$\begin{aligned} u_0(y) &= \cos\theta\; k\,\mathrm{sech}(ky) \\ v_0(y) &= \sin\theta\; k\,\mathrm{sech}(ky), \end{aligned} \tag{A.4}$$

with

$$k^2 = \omega \equiv \omega_1 = \omega_2,$$

and θ is a constant parameter. Notice that for $\theta \neq \frac{\pi}{2} + k\pi$

$$v_0(y) = \tan\theta\, u_0(y). \tag{A.5}$$

This is the reduction mentioned in seccion 4.2.

We will also show that, when $\omega_1 \neq \omega_2$, the solutions $u_0(y)$ and $v_0(y)$ are given by

$$\begin{aligned} u_0(y) &= 2\sqrt{|k_2^2 - k_1^2|}\frac{k_1 f_2}{[f_1, f_2]} \\ v_0(y) &= 2\sqrt{|k_2^2 - k_1^2|}\frac{k_2 f_1}{[f_1, f_2]}, \end{aligned} \tag{A.6}$$

where

$$\begin{aligned} f_1(y) &= 2\cosh k_1 y, \\ f_2(y) &= 2\sinh k_2(y-a); \end{aligned}$$

$$\begin{aligned} k_1^2 &\equiv \omega_1 \\ k_1^2 &\equiv \omega_2, \end{aligned}$$

and a is an arbitrary constant. The solutions presented in equation (A.6) are actually the solutions given by Ankiewicz *et.al* [AKA], and Akhmediev *et.al* [SA].

The expression $[f_1, f_2]$ represents the wronskian of f_1 and f_2

$$\begin{aligned}
[f_1, f_2] &= f_1 f_2' - f_1' f_2 \\
&= 4\{k_2 \cosh k_1 y \cosh k_2(y-a) - k_1 \sinh k_1 y \sinh k_2(y-a)\}.
\end{aligned}$$

A.2 Application of the Bäcklund Transform to the Manakov System

In this section we follow the ideas in [Cr], [Erc1], [Erc2]. The general details of these computations can be found in section A.4, at the end of this appendix.

Consider the following eigenvalue problem. The functions f_1 and f_2 denote eigenfunction with eigenvalues k_1^2 and k_2^2. Then

$$\begin{aligned}
(D^2 + V_0) f_1 &= k_1^2 f_1, \\
(D^2 + V_0) f_2 &= k_2^2 f_2.
\end{aligned} \tag{A.7}$$

Here $D \equiv \frac{d}{dx}$, and $V_0 = V_0(x)$ denotes a potential function. From these, a new eigenvalue problem can be formulated and solved

$$(D^2 + V_1) f_2^{(1)} = k_2^2 f_2^{(1)}, \tag{A.8}$$

(**Remark:** The superscripts do **not** denote derivatives.) The new potential $V_1 = V_1(x)$ is given by

$$V_1(x) = V_0(x) + 2D^2 \log f_1. \tag{A.9}$$

The eigenfunction $f_2^{(1)}$ to the new eigenvalue problem (A.8) is

$$f_2^{(1)} = \frac{[f_1, f_2]}{f_1}, \tag{A.10}$$

with eigenvalue k_2^2. We can repeat the argument one more time to generate a second potential $V_2 = V_2(x)$, but we will need to more eigenfunctions of equation (A.7), g_1

and g_2

$$\begin{aligned}(D^2+V_0)f_1 &= k_1^2 f_1,\\ (D^2+V_0)f_2 &= k_2^2 f_2.\\ (D^2+V_0)g_1 &= k_1^2 g_1,\\ (D^2+V_0)g_2 &= k_2^2 g_2.\end{aligned} \tag{A.11}$$

The new eigenfunctions are given by

$$\begin{aligned}f_1^{(1)} &= \frac{[f_1,g_1]}{f_1},\\ f_2^{(1)} &= \frac{[f_1,f_2]}{f_1},\\ f_2^{(1)} &= \frac{[f_1,g_2]}{f_1},\end{aligned}$$

with eigenvalues k_1^2, k_2^2 and k_2^2 respectively, corresponding to the potential $V_1(x) = V_0(x) + 2D^2 \log f_1$.

The new second potential, $V_2 = V_2(x)$, can be then generated in the following way

$$V_2(x) = V_1(x) + 2D^2 \log f_2^{(1)},$$

which can be simplified using the expression of V_1 and $f_2^{(1)}$, to obtain

$$V_2(x) = V_0(x) + 2D^2 \log[f_1, f_2].$$

The eigenfunctions for this potential are

$$\begin{aligned}f_1^{(2)} &= \frac{[f_2^{(1)}, f_1^{(1)}]}{f_2^{(1)}},\\ f_2^{(2)} &= \frac{[f_2^{(1)}, g_2^{(1)}]}{f_2^{(1)}},\end{aligned}$$

with eigenvalues k_1^2 and k_2^2. We can simplify these expresions to get

$$\begin{aligned}f_1^{(2)} &= \frac{[g_1,f_1,f_2]}{[f_1,f_2]},\\ f_2^{(2)} &= \frac{[g_2,f_1,f_2]}{[f_1,f_2]},\end{aligned}$$

These functions solve the new system

$$\begin{aligned}(D^2 + V_2)f_1^{(2)} &= k_1^2 f_1^{(2)},\\ (D^2 + V_2)f_2^{(2)} &= k_2^2 f_2^{(2)},\end{aligned}$$

Any multiple of them

$$\begin{aligned}F_1 = C_1 f_1^{(2)} &= C_1 \frac{[g_1, f_1, f_2]}{[f_1, f_2]},\\ F_2 = C_2 f_2^{(2)} &= C_2 \frac{[g_2, f_1, f_2]}{[f_1, f_2]},\end{aligned}$$

also solves the same system

$$\begin{aligned}(D^2 + V_2)F_1 &= k_1^2 F_1,\\ (D^2 + V_2)F_2 &= k_2^2 F_2,\end{aligned} \tag{A.12}$$

Now the compatibility condition between equations (A.3) and (A.12) is

$$V_2(x) = 2(F_1^2 + F_2^2),$$

with

$$\begin{aligned}u_0 &= F_1,\\ v_0 &= F_2.\end{aligned}$$

The factors C_1 and C_2 are constants to be chosen.

A.2.1 The One-Soliton Solution

To start with, let $V_0(x) \equiv 0$, equation (A.7) becomes

$$\begin{aligned}D^2 f_1 &= k_1^2 f_1,\\ D^2 f_2 &= k_2^2 f_2,\end{aligned}$$

with solutions

$$\begin{aligned} f_1(x) &= 2\cosh k_1(x-a_1), \\ f_2(x) &= 2\sinh k_2(x-a_2). \end{aligned}$$

Here, a_1 and a_2 are constant phase shifts. The new potential, according to equation (A.9), is

$$\begin{aligned} V_1(x) &= 2D^2 \log(f_1(x)) \\ &= 2k_1^2 \mathrm{sech}^2 k_1(x-a_1). \end{aligned}$$

To construct an eigenfunction (equation (A.10)) to this potential, we needed a second eigenfunction of the operator $D^2 + V_0(x)$.

Therefore, the solution to the eigenvalue problem (A.8), with potential $V_1(x)$, is given by (A.10), which in this case is

$$\begin{aligned} u(x) &= f_2^{(1)} = \frac{[f_1, f_2]}{f_1} = \frac{f_1 f_2' - f_1 f_2}{f_1} = \\ &= 4\frac{k_1 \cosh k_1(x-a_1)\cosh k_2(x-a_2) - k_2 \sinh k_1(x-a_1)\sinh k_2(x-a_2)}{2\cosh k_1(x-a_1)}. \end{aligned}$$

If $k \equiv k_1 = k_2$ (one-soliton solution) and $a \equiv a_1 = a_2$, we obtain

$$\begin{aligned} u_0^{(1)}(x) &= 4k\frac{(\cosh^2 k(x-a) - \sinh^2 k(x-a))}{2k\cosh k(x-a)} \\ &= 2k\mathrm{sech} k(x-a). \end{aligned}$$

This is the one-soliton solution (A.4), to system (A.3), in the case $\omega_1 = \omega_2 \equiv k^2$. In [Ya], the autor explicityly gives the solution we showed in equation (A.4). We can check that this is a solution to the Manakov system for any value of θ.

A.2.2 The Two-Soliton Solution

Consider now the case when $\omega_1 \neq \omega_2$. Define $k_1^2 \equiv \omega_1$ and $k_2^2 \equiv \omega_2$. We start with $V_0 \equiv 0$. Then, system (A.11) becomes

$$\begin{aligned} D^2 f_1 &= k_1^2 f_1, \\ D^2 f_2 &= k_2^2 f_2. \\ D^2 g_1 &= k_1^2 g_1, \\ D^2 g_2 &= k_2^2 g_2, \end{aligned}$$

with solutions

$$\begin{aligned} f_1(x) &= 2\cosh k_1(x-a_1), \\ f_2(x) &= 2\sinh k_2(x-a_2). \\ g_1(x) &= e^{k_1 x}, \\ g_2(x) &= e^{k_2 x}, \end{aligned}$$

with a_1 and a_2 constant phase shifts. The new potential is

$$V_2(x) = 2D^2 \log[f_1, f_2].$$

The solutions to the system (A.12) are (see section A.4 at end of appendix)

$$\begin{aligned} F_1 &= C_1 \frac{[e^{k_1 x}, f_1, f_2]}{[f_1, f_2]} \\ F_2 &= C_2 \frac{[e^{k_2 x}, f_1, f_2]}{[f_1, f_2]}. \end{aligned} \tag{A.13}$$

The functions F_1 and F_2 are the sought solutions to problem (A.12), provided

$$V_2(x) = 2(F_1^2 + F_2^2).$$

This is to say, we have to prove that

$$V_0(x) + 2D^2 \log([f_1, f_2]) = 2(F_1^2 + F_2^2),$$

i.e.:

$$D^2 \log([f_1, f_2]) = (F_1^2 + F_2^2), \tag{A.14}$$

since we chose $V_0(x) = 0$.

Let us start with left-hand-side of (A.14). We have to check that it can be written as the sum of two squares. We obtain[1]

$$D^2 \log([f_1, f_2]) = (k_2^2 - k_1^2) \left(\frac{f_1^2(f_2'^2 - k_2^2 f_2^2) + f_2^2(-f_1'^2 + k_1^2 f_1^2)}{[f_1, f_2]^2} \right).$$

Now,

$$\begin{aligned} f_2'^2 - k_2^2 f_2^2 &= 4k_2^2, \\ -f_1'^2 + k_1^2 f_1^2 &= 4k_1^2. \end{aligned}$$

Therefore,

$$\begin{aligned} D^2 \log([f_1, f_2]) &= (k_2^2 - k_1^2) \left(\frac{f_1^2(4k_2^2) + f_2^2(4k_1^2)}{[f_1, f_2]^2} \right) \\ &= 4(k_2^2 - k_1^2) \left\{ \left(\frac{k_2 f_1}{[f_1, f_2]} \right)^2 + \left(\frac{k_1 f_2}{[f_1, f_2]} \right)^2 \right\}. \end{aligned}$$

Comparing with equation (A.14), we should have

$$\tilde{F}_1 = 2\sqrt{|k_2^2 - k_1^2|} \frac{k_2 f_1}{[f_1, f_2]} \qquad \text{and} \qquad \tilde{F}_2 = 2\sqrt{|k_2^2 - k_1^2|} \frac{k_1 f_2}{[f_1, f_2]}.$$

We still have to check that $\tilde{F}_1$ and $\tilde{F}_2$ are eigenfunctions of the operator $D^2 + V_2(x)$. To do this, we will check that $\tilde{F}_1$ and $\tilde{F}_2$ are, in fact, F_1 and F_2 given in equation (A.13), for a suitable choice of C_1 and C_2.

---

[1] It is important to notice here the importance of the fact that

$$k_2^2 \neq k_1^2,$$

since this generates the new potential $V_2(x)$. If $k_2^2 = k_1^2$, automatically $V_2(x) = 0$, and we stay in the case of the one-soliton solution. However, under the limit $\frac{k_1}{k_2} \to 1$, with $k_1 \to \infty$ and $k_2 \to \infty$, this potential will generate two separated one-soliton profiles, which is the expected behaviour. We will show this later, in next section.

We know that

$$\begin{aligned} F_1 &= C_1 \frac{[e^{k_1 x}, f_1, f_2]}{[f_1, f_2]} \\ F_2 &= C_2 \frac{[e^{k_2 x}, f_1, f_2]}{[f_1, f_2]} \end{aligned}$$

are eigenfunctions of $D^2 + V_2(x)$ (see section A.4). Let us compute the first wronskian

$$\begin{aligned} [e^{k_1 x}, f_1, f_2] &= [e^{k_1 x}, e^{k_1 x} + e^{-k_1 x}, 2\sinh k_2(x - a_2)] \\ &= 2[e^{k_1 x}, e^{-k_1 x}, \sinh k_2(x - a_2)] \\ &= 2 \begin{vmatrix} e^{k_1 x} & e^{-k_1 x} & \sinh k_2(x - a_2) \\ k_1 e^{k_1 x} & -k_1 e^{-k_1 x} & k_2 \cosh k_2(x - a_2) \\ k_1^2 e^{k_1 x} & k_1^2 e^{-k_1 x} & k_2^2 \sinh k_2(x - a_2) \end{vmatrix} \\ &= 2(\sinh k_2(x - a_2) 2k_1^3 + 0 + k_2^2 \sinh k_2(x - a_2)(-2k_1)) \\ &= 4k_1 \sinh k_2(x - a_2)(k_2^2 - k_1^2) \\ &= -2(k_2^2 - k_1^2) k_1 f_2(x). \end{aligned}$$

Similarly,

$$[e^{k_2 x}, f_1, f_2] = 2(k_2^2 - k_1^2) k_2 f_1(x).$$

Hence,

$$\begin{aligned} F_1 &= -2C_1(k_2^2 - k_1^2) \frac{k_1 f_2}{[f_1, f_2]} \\ F_2 &= 2C_2(k_2^2 - k_1^2) \frac{k_2 f_1}{[f_1, f_2]}. \end{aligned}$$

are the eigenfunctions to the Schrödinger operator $D^2 + V_2(x)$. We can choose $C_1 = -1/\sqrt{|k_2^2 - k_1^2|}$ and $C_2 = 1/\sqrt{|k_2^2 - k_1^2|}$, so we finally get

$$F_1 = 2\sqrt{|k_2^2 - k_1^2|} \frac{k_1 f_2}{[f_1, f_2]} = u_0(x) \tag{A.15}$$

$$F_2 = 2\sqrt{|k_2^2 - k_1^2|} \frac{k_2 f_1}{[f_1, f_2]} = v_0(x), \tag{A.16}$$

which are the stationary solutions to the stationary Manakov system, equation (A.3), with $\omega_1 \equiv k_1^2$ and $\omega_2 \equiv k_2^2$.

Consider now the following change of variables

$$y = x - a_1,$$

and define

$$a \equiv a_1 - a_2.$$

We then obtain the solution as in equation (A.6)

$$\begin{aligned} u_0(y) &= 2\sqrt{|k_2^2 - k_1^2|}\frac{k_1 f_2}{[f_1, f_2]} \\ v_0(y) &= 2\sqrt{|k_2^2 - k_1^2|}\frac{k_2 f_1}{[f_1, f_2]}, \end{aligned}$$

where

$$\begin{aligned} f_1(y) &= 2\cosh k_1 y, \\ f_2(y) &= 2\sinh k_2 (y - a); \end{aligned}$$

It is important to note here, that these solutions were also found in [Ya] and [AKA].

A.3 Asymptotic Behaviour

In the present section, we will show the asymptotic behaviour of the solutions when independent variable goes to infinity. We also check that the two-soliton solution behaves like a one-soliton pulse when the initial phase shift is large. Finally, we also study the case of the limit when the two eigenvalues, k_1 and k_2 go to each other: $k_1/k_2 \to 1$. In this case, both solutions go to zero, but under the limit $k_1 \to \infty$ and $k_2 \to \infty$, the solution have a sech-profile.

A.3.1 Asymptotic Behaviour at Infinity

It is clear that the solutions $u_0(y) = \cos\theta\,\mathrm{sech}(ky)$ and $v_0(y) = \tan\theta\, u_0(y)$ go to zero as $|y| \to \infty$. This decay is exponential. Now, when $\omega_1 \neq \omega_2$, the solutions (A.6) also decay exponentially to zero. We will check this assertion.

To fix ideas, assume $k_2 > k_1 > 0$. Since $f_1(y)$ and $f_2(y)$ are the cosh and sinh functions, then

$$f_1(y) \sim e^{k_1 y}, \quad f_2(y) \sim e^{k_2 y}, \quad f_1'(y) \sim k_1 e^{k_1 y}, \quad f_2'(y) \sim k_2 e^{k_2 y},$$

and

$$[f_1, f_2] = f_1 f_2' - f_1' f_2 \sim (k_2 - k_1) e^{(k_1+k_2)y}.$$

Note that the factors k_1 and k_2 in the asymptotic behaviour f_1' and f_2' play an important role, since they make the wronskian $[f_1, f_2]$ not to vanish.

Hence

$$u_0(y) = 2\sqrt{|k_2^2 - k_1^2|}\frac{k_1 f_2}{[f_1, f_2]} \sim \frac{e^{k_2 y}}{(k_2 - k_1)e^{(k_1+k_2)y}},$$

i.e.

$$u_0(y) \sim \frac{e^{-k_1 y}}{(k_2 - k_1)} \to 0 \quad \text{as} \quad y \to \infty.$$

A similar computation shows that, as $y \to \infty$, $v_0(y) \to 0$ as well. Therefore, the solutions $u_0(y)$, $v_0(y)$ of the system (A.3) decay exponentially to zero, in the two-soliton solution case, $\omega_1 \neq \omega_2$.

A.3.2 Two Soliton Interaction

Choose the initial phase shift, "a", to be positive or negative, and with large values in absolute value, i.e., $|a|$ to be large. This will reflect the asymptotic behaviour of the solution before and after an interaction of the two solitons. Again, to fix ideas, we consider the case $k_2 > k_1 > 0$.

"Riding" on the First Soliton. Consider the first case when we are "*riding*" on the soliton $u_0(y)$. Here, we mean that $y = y_0$ is constant.

The limit $a \sim \infty$. Now, the limit $a \sim \infty$ means that the two pulses $u_0(y)$ and $v_0(y)$ are initially far away from each other.

We can easily verify that

$$\lim_{a\to\infty} \frac{f_2(y_0)}{2\cosh k_2(y_0 - a)} = -1. \tag{A.17}$$

This means

$$f_2(y_0) \sim -2\cosh k_2(y_0 - a), \qquad \text{as} \qquad a \sim \infty. \tag{A.18}$$

Using this asymptotic behaviour of f_2, the behaviour of the wronskian is

$$[f_1, f_2] \sim 4\cosh k_2(y_0 - a)(k_2 \cosh k_1 y_0 + k_1 \sinh k_1 y_0)$$

when $a \sim \infty$. Hence, the asymptotic behaviour of $u_0(y_0)$ is

$$u_0(y_0) \sim \frac{k_1\sqrt{k_2^2 - k_1^2}}{k_2 \cosh k_1 y_0 + k_1 \sinh k_1 y_0}$$

Express the cosh and sinh in terms of exponentials to obtain

$$u_0(y_0) \quad \sim \quad \frac{k_1}{\frac{1}{2}(e^{k_1 y_0 + b} + e^{-k_1 y_0 - b})},$$

where

$$e^b \equiv \frac{\sqrt{k_2 + k_1}}{\sqrt{k_2 - k_1}}.$$

Therefore, the $u_0(y_0)$ profile is

$$u_0(y_0) \sim k_1 \mathrm{sech}(k_1 + b), \qquad \text{as} \qquad a \sim \infty. \tag{A.19}$$

Now, the second component $v_0(y_0)$ has the following behavior

$$v_0(y_0) \sim 2\sqrt{k_2^2 - k_1^2}\frac{k_2 2\cosh k_1 y_0}{4\cosh k_2(y_0 - a)(k_2 \cosh k_1 y_0 + k_1 \sinh k_1 y_0)}, \tag{A.20}$$

and so

$$v_0(y_0) \to 0, \qquad \text{as} \qquad a \to \infty. \tag{A.21}$$

This means that the other soliton is negligible, or it disappears from the point of view of the first soliton.

The limit $a \sim -\infty$. This is what happens after an interaction between the two solitons. We know that

$$\lim_{a \to -\infty} \frac{f_2(y_0)}{2\cosh k_2(y_0 - a)} = 1, \tag{A.22}$$

i.e.,

$$f_2(y_0) \sim 2\cosh k_2(y_0 - a), \qquad \text{as} \qquad a \sim -\infty. \tag{A.23}$$

Therefore, the behaviour of the wronskian is now

$$[f_1, f_2] \sim 4\cosh k_2(y_0 - a)(k_2 \cosh k_1 y_0 - k_1 \sinh k_1 y_0)$$

when $a \sim -\infty$. Hence,

$$\begin{aligned} u_0(y_0) &\sim 2\sqrt{k_2^2 - k_1^2} \frac{k_1 2\cosh k_2(y_0 - a)}{4\cosh k_2(y_0 - a)(k_2 \cosh k_1 y_0 - k_1 \sinh k_1 y_0)} \\ &\sim \frac{k_1}{\frac{1}{2}(e^{k_1 y_0 - b} + e^{-k_1 y_0 + b})}, \end{aligned}$$

to finally get

$$u_0(y_0) \sim k_1 \operatorname{sech}(k_1 y_0 - b), \qquad \text{as} \qquad a \sim -\infty. \tag{A.24}$$

In equation (A.20), we observe that it does not matter if $a \sim \infty$ or $a \sim -\infty$, we obtain, in either case

$$v_0(y_0) \to 0, \qquad \text{as} \qquad a \to \pm\infty.$$

Comparing equations (A.19) and (A.24), we observe that there is a phase shift of Δy is

$$\Delta y = -b - b = -2b = -\log \frac{k_2 + k_1}{k_2 - k_1}. \tag{A.25}$$

This is a consecuence of the nonlinear interaction.

"Riding" on the Second Soliton. In this case, we are "*riding*" on the soliton $v_0(y)$. We will be considering that $Y_0 = y - a$ is constant, and let $y \to \pm\infty$. Therefore, $a \to \pm\infty$, respectively.

The limit $y \sim \infty$. This is equivalent to the limit $a \to \infty$, meaning that the two pulses $u_0(y)$ and $v_0(y)$ are initially far away from each other. We start with the eigenfunctions

$$\begin{aligned} f_1(y) &= 2\cosh k_1 y, \\ f_2(y) &= 2\sinh k_2 Y_0. \end{aligned}$$

Notice that

$$\lim_{y\to\infty} \frac{\cosh k_1 y}{\sinh k_1 y} = 1 \tag{A.26}$$

Hence,

$$\cosh k_1 y \sim \sinh k_1 y, \qquad \text{as} \qquad y \sim \infty,$$

and so

$$[f_1, f_2] \sim 4\cosh k_1 y \sqrt{k_2^2 - k_1^2}\cosh(k_2 Y_0 - b).$$

Therefore, when $y \sim \infty$

$$\begin{aligned} u_0(y) &= 2\sqrt{k_2^2 - k_1^2}\frac{k_1 f_2}{[f_1, f_2]} \\ &\sim \frac{k_1 \sinh(k_2 Y_0)}{\cosh k_1 y}\operatorname{sech}(k_2 Y_0 - b), \end{aligned}$$

and

$$\begin{aligned} v_0(y) &= 2\sqrt{k_2^2 - k_1^2}\frac{k_2 f_1}{[f_1, f_2]} \\ &\sim 2\sqrt{k_2^2 - k_1^2}\frac{k_2 2\cosh k_1 y}{4\cosh k_1 y\sqrt{k_2^2 - k_1^2}\cosh(k_2 Y_0 - b)}. \end{aligned}$$

Therefore, as $y \ \to \infty$

$$\begin{aligned} u_0(y) &\to 0, \\ v_0(y) &\sim k_2 \operatorname{sech}(k_2 Y_0 - b). \end{aligned} \tag{A.27}$$

Therefore, before the interaction, and "riding" on the v_0 soliton, we observe that this soliton initially has the sech profile, meanwhile the other soliton vanishes or it is negligible faraway from this one.

The limit $y \sim -\infty$. This corresponds to the limit: $a \sim -\infty$, describing the behaviour after a collision. Here, we have the limit

$$\lim_{y\to-\infty} \frac{\cosh k_1 y}{\sinh k_1 y} = -1 \tag{A.28}$$

Hence,

$$\cosh k_1 y \sim -\sinh k_1 y, \qquad \text{as} \qquad y \sim -\infty,$$

and the wronskian will also have a different behaviour

$$[f_1, f_2] \sim 4\cosh k_1 y\sqrt{k_2^2 - k_1^2}\cosh(k_2 Y_0 + b).$$

The behaviour for u_0 will be basically the same

$$\begin{aligned} u_0(y) &= 2\sqrt{k_2^2 - k_1^2}\frac{k_1 f_2}{[f_1, f_2]} \\ &\sim \frac{k_1 \sinh(k_2 Y_0)}{\cosh k_1 y}\operatorname{sech}(k_2 Y_0 + b), \end{aligned}$$

but not for v_0

$$\begin{aligned} v_0(y) &= 2\sqrt{k_2^2 - k_1^2}\frac{k_2 f_1}{[f_1, f_2]} \\ &\sim 2\sqrt{k_2^2 - k_1^2}\frac{k_2 2\cosh k_1 y}{4\cosh k_1 y\sqrt{k_2^2 - k_1^2}\cosh(k_2 Y_0 + b)} \end{aligned}$$

Hence, after the collision

$$\begin{aligned} u_0(y) &\to 0, \\ v_0(y) &\sim k_2 \operatorname{sech}(k_2 Y_0 + b), \qquad \text{as} \qquad y \sim -\infty, \end{aligned} \tag{A.29}$$

and the second soliton has a phase shift

$$\Delta y = b - (-b) = 2b = \log\frac{k_2 + k_1}{k_2 - k_1}, \tag{A.30}$$

by comparing equations (A.27) and (A.29).

A.3.3 Limiting Case: Two-Soliton to One-Soliton Solution

In this section, we expect to recover the solution (A.4)

$$\begin{aligned} u_0(y) &= \cos\theta\, k\,\mathrm{sech}(ky) \\ v_0(y) &= \sin\theta\, k\,\mathrm{sech}(ky), \end{aligned}$$

from solution (A.6)

$$\begin{aligned} u_0(y) &= 2\sqrt{|k_2^2 - k_1^2|}\frac{k_1 f_2}{[f_1, f_2]} \\ v_0(y) &= 2\sqrt{|k_2^2 - k_1^2|}\frac{k_2 f_1}{[f_1, f_2]}, \end{aligned}$$

with

$$\begin{aligned} f_1(y) &= 2\cosh k_1 y, \\ f_2(y) &= 2\sinh k_2(y-a), \end{aligned}$$

under the limit $k_2 \to k_1$. We will see that this limit is not enough, and one more condition is needed. We will show that it is also required that $k_1 \to \infty$ and $k_2 \to \infty$, meanwhile $k_2/k_1 \to 1$. This could be a good guess, noticing that

$$\begin{aligned} \frac{u_0(y)}{v_0(y)} &= \frac{k_1 f_2(y)}{k_2 f_1(y)} \\ &= \frac{k_1 \sinh k_2(y-a)}{k_2 \cosh k_1 y}. \end{aligned}$$

Now let $k_2 = k_1 \equiv k$

$$\frac{u_0(y)}{v_0(y)} = \frac{\sinh k(y-a)}{\cosh ky}.$$

Now, if $k \to \infty$

$$\frac{u_0(y)}{v_0(y)} \to\ 1,$$

which is the expected behaviour.

Before doing any computation, let me explain what the plan is. First, (and this will be the longest computation), we will construct an alternative expression for $u_0(y)$

and $v_0(y)$, *before* any limit to be taken. Second, and after these analytic expressions are found, we will consider the appropriate limits.

If we define

$$\phi_2(y) \equiv \text{cotanh} k_2(y-a),$$

the wronskian can be written as

$$[f_1, f_2] = 4\sinh k_2(y-a)\{k_2 \cosh k_1 y\, \phi_2(y) - k_1 \sinh k_1 y\}.$$

Define now $b_2(y)$ by

$$e^{-b_2(y)} \equiv \sqrt{\frac{k_2\phi_2 - k_1}{k_2\phi_2 + k_1}}.$$

Writing the hyperbolic functions in terms of exponential functions and using this expression of $b_2(y)$, we get

$$[f_1, f_2] = 4\sinh k_2(y-a)\sqrt{k_2^2\phi_2^2 - k_1^2}\cosh(k_1 y - b_2(y)),$$

Therefore

$$\begin{aligned} u_0(y) &= 2\sqrt{k_2^2 - k_1^2}\frac{k_1 f_2}{[f_1, f_2]} \\ &= k_1\sqrt{\frac{1-(\frac{k_1}{k_2})^2}{\phi_2^2 - (\frac{k_1}{k_2})^2}}\text{sech}(k_1 y - b_2(y)). \end{aligned} \tag{A.31}$$

This is exact. No limit has been taken yet. We proceed now to find a similar expression for $v_0(y)$. The process is similar but it is not the same, since we have to define a different function ϕ_1.

We start with a different factorization of the wronskian. Define

$$\phi_1(y) \equiv \tanh k_1 y. \tag{A.32}$$

Hence

$$[f_1, f_2] = 4\cosh k_1 y\,(k_2 \cosh k_2(y-a) - k_1\phi_1(y)\sinh k_2(y-a)).$$

Using again the definition of *cosh* and *sinh* in terms of exponentials and defining

$$e^{-b_1(y)} \equiv \sqrt{\frac{k_2 - k_1\phi_1(y)}{k_2 + k_1\phi_1(y)}},$$

We obtain

$$[f_1, f_2] = 4\cosh k_1 y \sqrt{k_2^2 - k_1^2\phi_1^2}\cosh(k_2(y-a) - b_1(y)).$$

Therefore,

$$\begin{aligned} v_0(y) &= 2\sqrt{k_2^2 - k_1^2}\frac{k_2 f_1}{[f_1, f_2]} \\ &= k_2\sqrt{\frac{(\frac{k_2}{k_1})^2 - 1}{(\frac{k_2}{k_1})^2 - \phi_1^2}}\operatorname{sech}(k_2(y-a) - b_1(y)). \end{aligned} \tag{A.33}$$

Hence, the solutions (equations (A.31) and (A.33) are

$$u_0(y) = k_1\sqrt{\frac{1 - (\frac{k_1}{k_2})^2}{\phi_2^2 - (\frac{k_1}{k_2})^2}}\operatorname{sech}(k_1 y - b_2(y)) \tag{A.34}$$

$$v_0(y) = k_2\sqrt{\frac{(\frac{k_2}{k_1})^2 - 1}{(\frac{k_2}{k_1})^2 - \phi_1^2}}\operatorname{sech}(k_2(y-a) - b_1(y)), \tag{A.35}$$

where ϕ_1, ϕ_2, b_1 and b_2 are defined by

$$\phi_1(y) \equiv \tanh k_1 y \tag{A.36}$$

$$\phi_2(y) \equiv \operatorname{cotanh} k_2(y-a) \tag{A.37}$$

$$e^{-b_1(y)} \equiv \sqrt{\frac{k_2 - k_1\phi_1}{k_2 + k_1\phi_1}} \tag{A.38}$$

$$e^{-b_2(y)} \equiv \sqrt{\frac{k_2\phi_2 - k_1}{k_2\phi_2 + k_1}}. \tag{A.39}$$

This expressions are exact. **No limit** has been considered yet.

Now consider the following limits. We will show that

$$\begin{aligned}\sqrt{\frac{1-(\frac{k_1}{k_2})^2}{\phi_2^2-(\frac{k_1}{k_2})^2}} &\longrightarrow 1\\ \sqrt{\frac{(\frac{k_2}{k_1})^2-1}{(\frac{k_2}{k_1})^2-\phi_1^2}} &\longrightarrow 1,\end{aligned}$$

when $k_1 \to \infty$ and $k_2 \to \infty$, with the condition $\frac{k_1}{k_2} \to 1$. In such case

$$u_0(y) \sim k_1 \operatorname{sech}(k_1 y - b_2(y)) \tag{A.40}$$

$$v_0(y) \sim k_2 \operatorname{sech}(k_2(y-a) - b_1(y)). \tag{A.41}$$

Since

$$\begin{aligned}\phi_1(y) &= \tanh k_1 y \sim 1, \qquad \text{as} \qquad k_1 \sim \infty,\\ \phi_2(y) &= \operatorname{cotanh} k_2(y-a) \sim 1, \qquad \text{as} \qquad k_2 \sim \infty,\end{aligned}$$

we get

$$\begin{aligned}\lim_{k_2\to\infty}\sqrt{\frac{1-(\frac{k_1}{k_2})^2}{\phi_2^2-(\frac{k_1}{k_2})^2}} &\sim \sqrt{\frac{1-(\frac{k_1}{k_2})^2}{1-(\frac{k_1}{k_2})^2}} = 1,\\ \lim_{k_1\to\infty}\sqrt{\frac{(\frac{k_2}{k_1})^2-1}{(\frac{k_2}{k_1})^2-\phi_1^2}} &\sim \sqrt{\frac{(\frac{k_2}{k_1})^2-1}{(\frac{k_2}{k_1})^2-1}} = 1.\end{aligned}$$

This is to say, the approximated solutions are given by equations (A.40) and (A.41)[2].

[2]Now, a problem arises for the limits of $b_1(y)$ and $b_2(y)$. Again,

$$\frac{k_1}{k_2}\phi_1(y) = \frac{k_1}{k_2}\tanh k_1 y \to 1,$$

when $k_1 \to \infty$, $k_2 \to \infty$ and $\frac{k_1}{k_2} \to 1$. Hence,

$$e^{-b_1(y)} \equiv \sqrt{\frac{1-\frac{k_1}{k_2}\phi_1(y)}{1+\frac{k_1}{k_2}\phi_1(y)}} \longrightarrow \frac{0}{\sqrt{2}} = 0.$$

A.4 Bäcklund Transforms

In this section, we introduce the general setting for the **Bäcklund transforms,** which are also called **Additions**. The contents are essentially the notes [Erc1] (we can compare with [Cr].)

Additions. Consider the Schrödinger operator

$$Q = D^2 + q(x), \tag{A.44}$$

And the linear operator

$$T_1 = D + u_1(x), \tag{A.45}$$

where $D = d/dx$ denotes the derivative with respect to x.

We would like to find another linear second order differential operator $\widetilde{Q}$, with the following property

$$T_1 Q = \widetilde{Q} T_1. \tag{A.46}$$

To find such operator, let us consider $f_1 \in \ker T_1$

$$T_1 f_1 = 0. \tag{A.47}$$

Therefore,

$$0 = \widetilde{Q} T_1 f_1 = T_1(Q f_1).$$

Hence $Q f_1 \in \ker T_1$ and so,

$$Q f_1 = \lambda_1 f_1, \tag{A.48}$$

But this implies that

$$b_1(y) \to \infty \quad !$$

Similarly,

$$b_2(y) \to \infty \quad !$$

Since $a + b_1(y) \sim b_1(y)$, $b_2(y) \sim b_1(y)$, and $k_2 \sim k_1$, equations (A.40) and (A.41) become

$$u_0(y) \sim k_1 \operatorname{sech}(k_1 y - b_2(y)) \quad \sim \quad k_1 \operatorname{sech}(k_1 y - b_1(y)) \tag{A.42}$$

$$v_0(y) \sim k_2 \operatorname{sech}(k_2(y - a) - b_1(y)) \quad \sim \quad k_1 \operatorname{sech}(k_1 y - b_1(y)), \tag{A.43}$$

which are the one-soliton solutions.

for some λ_1. This is true, since T_1 is first order, and the kernel should be one-dimensional.

Now, from (A.45) and (A.47)

$$(D + u_1(x))f_1 = 0,$$

and it follows

$$u_1(x) = \frac{-f_1'}{f_1} = -D \log f_1. \tag{A.49}$$

Now, we can factorize Q in the following way

$$Q - \lambda_1 = (D - u_1)(D + u_1)$$

Conjugating the previous equation by T_1, and using (A.45) and (A.46), we obtain

$$\begin{aligned} T_1(Q - \lambda_1)T_1^{-1} &= T_1(D - u_1)(D + u_1)T_1^{-1} \\ \widetilde{Q} - \lambda_1 &= T_1(D - u_1) \\ &= (D + u_1)(D - u_1). \end{aligned}$$

Therefore,

$$\begin{aligned} Q - \lambda_1 &= D^2 + u_1' - u_1^2 \\ \widetilde{Q} - \lambda_1 &= D^2 - u_1' - u_1^2. \end{aligned}$$

The new operator $\widetilde{Q}$ is then related with Q in the following way

$$\widetilde{Q} - Q = -2u_1'. \tag{A.50}$$

From this, follows that the new potential $\widetilde{q}(x)$ of $\widetilde{Q}$ is

$$\widetilde{q}(x) - q(x) = -2u_1'. \tag{A.51}$$

Using (A.49)

$$\widetilde{q}(x) = q(x) + 2D^2 \log f_1. \tag{A.52}$$

Question: *Which are the eigenfunctions of $\widetilde{Q}$?*

Let f_2 be an eigenfunction of Q

$$Qf_2 = \lambda_2 f_2.$$

Therefore,

$$\widetilde{Q}T_1 f_2 = T_1(Qf_2) = T_1(\lambda_2 f_2) = \lambda_2(T_1 f_2),$$

i.e., $T_1 f_2$ is eigenfunction of $\widetilde{Q}$, with eigenvalue λ_2, where

$$T_1 f_2 = (D + u_1(x))f_2 = \left(D - \frac{f_1'}{f_1}\right) f_2 = \frac{f_1 f_2' - f_1' f_2}{f_1},$$

i.e.,

$$T_1 f_2 = \frac{[f_1, f_2]}{f_1}, \tag{A.53}$$

and $[f_1, f_2] \equiv f_1 f_2' - f_1' f_2$ is the wronskian between the functions f_1 and f_2.

Consider a second eigenfunction, f_3, of Q, with eigenvalue λ_3[3]. Repeating same reasoning, $Qf_3 = \lambda_3 f_3$, implies

$$\widetilde{Q}(T_1 f_3) = \lambda_3(T_1 f_3),$$

meaning that the following function

$$T_1 f_3 = \frac{[f_1, f_3]}{f_1}$$

is an eigenfunction of $\widetilde{Q}$ with eigenvalue λ_3.

Now, we are going to repeat the process, to generate a second potential $\widetilde{\widetilde{Q}}$, out of $\widetilde{Q} = D^2 + \widetilde{q}(x)$.

Let

$$\widetilde{T}_1 = D + \widetilde{u}_1.$$

Problem: Find $\widetilde{\widetilde{Q}} = D^2 + \widetilde{\widetilde{q}}(x)$, such that

$$\widetilde{T}_1\widetilde{Q} = \widetilde{\widetilde{Q}}\widetilde{T}_1.$$

[3]At this moment, we do not need this eigenfunction, but we will be using it when we construct the eigenfunctions of a newer potential $\widetilde{\widetilde{Q}}$.

If $\widetilde{f}_1 \in \ker \widetilde{T}_1$, i.e., $\widetilde{T}_1 \widetilde{f}_1 = 0$, then

$$\widetilde{u}_1 = \frac{-\widetilde{f}_1'}{\widetilde{f}_1} = -D \log \widetilde{f}_1.$$

Also, $\widetilde{Q}\widetilde{f}_1 \in \ker \widetilde{T}_1$. Hence,

$$\widetilde{Q}\widetilde{f}_1 = \widetilde{\lambda}_1 \widetilde{f}_1,$$

for some $\widetilde{\lambda}_1$. But we know that $T_1 f_2$ is eigenfunction of $\widetilde{Q}$

$$\widetilde{Q}(T_1 f_2) = \lambda_2 (T_1 f_2).$$

Hence, we could consider $\widetilde{f}_1$ and $\widetilde{\lambda}_1$

$$\widetilde{f}_1 = T_1 f_2 \qquad \text{and} \qquad \widetilde{\lambda}_1 = \lambda_2. \tag{A.54}$$

Similarly,

$$\widetilde{\widetilde{q}} = \widetilde{q} - 2\widetilde{u}',$$

i.e.,

$$\widetilde{\widetilde{q}} = \widetilde{q} + 2D^2 \log \widetilde{f}_1$$

i.e.,

$$\widetilde{\widetilde{q}} = \widetilde{q} + 2D^2 \log(\text{Eigenfunction of } \widetilde{Q}).$$

Using the expression of $\widetilde{q}$ given in (A.52), and the fact that $\widetilde{f}_1 = T_1 f_2 = \frac{[f_1, f_2]}{f_1}$

$$\begin{aligned} \widetilde{\widetilde{q}} &= q(x) + 2D^2 \log f_1 + 2D^2 \log \frac{[f_1, f_2]}{f_1} \\ &= q(x) + 2D^2 \log f_1 + 2D^2 \log [f_1, f_2] - 2D^2 \log f_1. \end{aligned}$$

This is to say,

$$\widetilde{\widetilde{q}} = q(x) + 2D^2 \log [f_1, f_2].$$

The second potential $\widetilde{\widetilde{q}}$ is given in terms of eigenfunctions of our initial operator Q.

How to find eigenfunctions to $\widetilde{\widetilde{Q}} = D^2 + \widetilde{\widetilde{q}}$? Same way as before.

Consider the following eigenvalues and eigenfunctions of the operator Q[4]

$$Qf_1 = \lambda_1 f_1; \qquad Qf_2 = \lambda_1 f_2; \tag{A.55}$$

$$Qf_3 = \lambda_1 f_3; \qquad Qf_4 = \lambda_1 f_4; \tag{A.56}$$

Let $\widetilde{f}_2$ be eigenfunction of $\widetilde{Q}$

$$\widetilde{Q}\widetilde{f}_2 = \widetilde{\lambda}_2 \widetilde{f}_2.$$

Then, $\widetilde{T}_1\widetilde{f}_2$ is eigenfunction of $\widetilde{\widetilde{Q}}$ with eigenvalue $\widetilde{\lambda}_2$. This eigenfunction $\widetilde{T}_1\widetilde{f}_2$ can be computed explicitly

$$\widetilde{T}_1\widetilde{f}_2 = \frac{[\widetilde{f}_1, \widetilde{f}_2]}{\widetilde{f}_1},$$

with $\widetilde{f}_1$ and $\widetilde{f}_2$ eigenfunctions of $\widetilde{Q}$. We know that

$$\widetilde{f}_1 = T_1 f_2 = \frac{f_1, f_2]}{f_1},$$

and also construct $\widetilde{f}_2$ from f_3

$$\widetilde{f}_2 = T_1 f_3 = \frac{f_1, f_3]}{f_1}.$$

Hence[5]

$$\begin{aligned} \widetilde{T}_1\widetilde{f}_2 &= \frac{\left[\frac{[f_1,f_2]}{f_1}, \frac{[f_1,f_3]}{f_1}\right]}{\frac{[f_1,f_2]}{f_1}} = \frac{\left[\frac{[f_2,f_1]}{f_1}, \frac{[f_3,f_1]}{f_1}\right]}{\frac{[f_1,f_2]}{f_1}} + \frac{\frac{[f_2,f_3,f_1]}{f_1}}{\frac{[f_1,f_2]}{f_1}} \\ &= \frac{[f_2, f_3, f_1]}{[f_1, f_2]}. \end{aligned}$$

[4]We can consider $\lambda_3 = \lambda_1$ and $\lambda_4 = \lambda_2$ with corresponding eigenfunctions g_1 and g_2, but we are leaving this particular case until the end of the appendix.

[5]By induction, the wronskian identity

$$\frac{[f_1, f_2, \ldots, f_{n-1}, f_n]}{f_n} = \left[\frac{[f_1, f_n]}{f_n}, \frac{[f_2, f_n]}{f_n}, \ldots, \frac{[f_{n-1}, f_n]}{f_n}\right]$$

can be proved. The case $n = 3$

$$\left[\frac{[f_1, f_3]}{f_3}, \frac{[f_2, f_3]}{f_3}\right] = \frac{[f_1, f_2, f_3]}{f_3}$$

which is the one used here, can be easily checked.

Now, $\widetilde{f}_2 = T_1 f_3$ is eigenfunction of $\widetilde{Q}$ with eigenvalue $\widetilde{\lambda}_2 = \lambda_3$, since

$$\widetilde{Q}(T_1 f_3) = T_1(Q f_3) = T_1(\lambda_3 f_3) = \lambda_3(T_1 f_3).$$

Therefore, $\widetilde{T}_1 \widetilde{f}_2$ is an eigenfunction of $\widetilde{\widetilde{Q}}$, with eigenvalue λ_3

$$\widetilde{\widetilde{Q}}(\widetilde{T}_1 \widetilde{f}_2) = \lambda_3(\widetilde{T}_1 \widetilde{f}_2).$$

Similarly, if we define, $\widetilde{f}_3 = T_1 f_4$, $\widetilde{f}_3$ is an eigenfunction of $\widetilde{Q}$ with eigenvalue λ_4. Then $\widetilde{T}_1 \widetilde{f}_3$ is eigenfunction of $\widetilde{\widetilde{Q}}$ with eigenvalue λ_4. Here,

$$\widetilde{T}_1 \widetilde{f}_3 = \frac{[f_4, f_1, f_2]}{[f_1, f_2]}.$$

Summarizing: The eigenfunctions of $\widetilde{\widetilde{Q}} = D^2 + q(x) + 2D^2 \log[f_1, f_2]$ are

$$\widetilde{T}_1 \widetilde{f}_2 = \frac{[f_3, f_1, f_2]}{[f_1, f_2]} \qquad \text{and} \qquad \widetilde{T}_1 \widetilde{f}_3 = \frac{[f_4, f_1, f_2]}{[f_1, f_2]},$$

with eigenvalues λ_1 and λ_2, respectively.

Since Q has two linearly independent eigenfunctions, with λ_1, and two linearly independent eigenfunctions, with λ_2 (i.e., set $\lambda_3 = \lambda_1$, $\lambda_4 = \lambda_2$ and $g_1 = f_3$, $g_2 = f_4$ in equations (A.55) and (A.56)), we obtain

$$\begin{aligned} Q f_1 &= \lambda_1 f_1 \qquad \text{and} \qquad Q \widehat{f}_1 = \lambda_1 \widehat{f}_1, \\ Q f_2 &= \lambda_2 f_2 \qquad \text{and} \qquad Q \widehat{f}_2 = \lambda_2 \widehat{f}_2. \end{aligned}$$

The eigenfunctions for $\widetilde{\widetilde{Q}}$ are

$$F_1 = \frac{[g_1, f_1, f_2]}{[f_1, f_2]} \qquad \text{and} \qquad F_2 = \frac{[g_2, f_1, f_2]}{[f_1, f_2]}, \tag{A.57}$$

with eigenvalues λ_1 and λ_2, respectively. These are the two basic formulas used in the present appendix.

Appendix B

Analytic Extension of the Jost Solutions.

B.0 Introduction

The material contained in this appendix is based on a series of lectures given by Jason Newport to the Working Group on the Nonlinear Schrödinger equation, during the Fall Semester of 2005, in the Deparment of Mathematics at the University of Arizona.

In chapter 2, we mentioned that the Manakov system (2.1)

$$\begin{aligned} iu_t + \frac{1}{2}u_{xx} + \sigma(|u|^2 + |v|^2)u &= 0 \\ iv_t + \frac{1}{2}v_{xx} + \sigma(|u|^2 + |v|^2)v &= 0, \end{aligned}$$

has an associatted *linear system* of ordinary differential equations, namely, the Manakov-Zakharov-Shabat (2.8) system

$$\frac{\partial}{\partial x}\Psi = iED\Psi + N\Psi. \tag{B.1}$$

In this system, E is the spectral parameter and

$$\begin{aligned} D &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ N = N(x,t) &= \begin{pmatrix} 0 & u(x,t) & v(x,t) \\ -\sigma u^\star(x,t) & 0 & 0 \\ -\sigma v^\star(x,t) & 0 & 0 \end{pmatrix}, \end{aligned}$$

where $\sigma = \pm 1$, corresponding to the focusing/defocusing case, respectively. Here, we are studying the focusing case: $\sigma = 1$. Also, in all the computations in this appendix, we set $t = 0$.

The MZS system is a first order system of differential equations. Therefore, we need to set some "*initial conditions*" to uniquely determine the solutions. There are two different types of "*initial conditions*" that we are going to consider here.

First type. The "*initial conditions*" at $x = -\infty$ will be satisfied by a 3×3 matrix, denoted by $\Phi(x; E)$, such that

$$\Phi(x; E) \sim e^{iEDx}, \tag{B.2}$$

as $x \to -\infty$. The matrix $\Phi(x; E)$ is a complete system of solutions of the MZS system.

Second type. The other type of "*initial conditions*" are considered at $x = \infty$. They are

$$\Psi(x; E) \sim e^{iEDx}, \tag{B.3}$$

as $x \to \infty$. Similarly, $\Psi(x; E)$ is a matrix whose columns form a complete system of solutions of the MZS system.

Notice that, we require

$$N(x) \longrightarrow 0 \qquad \text{as} \qquad x \longrightarrow \infty,$$

so that these conditions $B.2$ and $B.3$ make sense.

We then have two complete systems of solutions to the MZS system, $\Phi(x; E)$ and $\Psi(x; E)$. These particular solutions are called the **Jost solutions** of the MZS, or the **Jost functions**.

Since the each of the Jost functions form a complete set of solutions, they should be dependent of each other. They are related by the **Scattering matrix**, $S(E)$,

$$\Phi(x; E) = \Psi(x; E)S(E).$$

In chapter 2, we mentioned that $\Phi(x; E)$ and $\Psi(x; E)$ satisfy some unitary properties. In particular, for the focusing case, we found that $S(E)$, $\Phi(x; E)$ and $\Psi(x; E)$ are unitary matrices. This allows to compute the entries of the scattering matrix in terms of $\Phi(x; E)$ and $\Psi(x; E)$

$$S_{ij}(E) = \Psi_i^*(x; E)\Phi_j(x; E),$$

where Ψ_i denotes the i^{th} column of Ψ and the star * denotes the transpose conjugate. From this we can determine the analytic properties of the scattering matrix in terms of the Jost functions.

In this appendix we will basically show two facts. First, for integrable potentials, the column $\Phi_1(x;E)$, which is defined for $E \in \mathbb{R}$, can be analytically extended into the upper-half complex plane: $E \in \mathbb{C}^+$. Same argument will work for $\Psi_2(x;E)$ and $\Psi_3(x;E)$. On the other hand, the columns $\Psi_1(x;E)$, $\Phi_2(x;E)$ and $\Phi_3(x;E)$ can be analytically extended into the lower-half complex plane: $E \in \mathbb{C}^-$. The proof of this fact is a slight variation (just by a negative sign) of the argument given here.

The second fact concerns the analyticity properties of the Jost solutions of the MZS system with potentials of compact support, like the potentials worked out in this thesis. We will show that, if $N(x)$ has compact support, then the Jost solutions $\Phi(x;E)$ and $\Psi(x;E)$ can be analytically extended in the whole complex plane $E \in \mathbb{C}$.

B.1 On the Analyticity of the Jost solutions: Integrable Potentials

We know that the Jost function $\Phi(x;E)$ satisfies the MZS system, equation (B.1),together with the condition (B.2). The MZS system can be then put in form of an integral equation

$$\Phi(x) = e^{iEDx} + \int_{-\infty}^{x} e^{iED(x-y)} N(y)\Phi(y)\, dy. \tag{B.4}$$

Define

$$\begin{aligned} \widetilde{\Phi}(x;E) &\equiv \Phi(x)e^{-iEDx} \\ R(x,y;E) &\equiv e^{iED(x-y)}N(y). \end{aligned}$$

From these definitions, the corresponding integral equation of $\widetilde{\Phi}(x)$ is

$$\widetilde{\Phi}(x) = I + \int_{-\infty}^{x} R(y)\widetilde{\Phi}(y)e^{-iED(x-y)}\, dy. \tag{B.5}$$

The first column of this equation is

$$\widetilde{\Phi}_1(x) = I + \int\limits_{-\infty}^{x} e^{iE(x-y)} R(y) \widetilde{\Phi}_1(y)\, dy. \tag{B.6}$$

Notices that the exponential is a scalar, not a matrix. The Jost solutions exists of $E \in \mathbb{R}$. We willprove that for potentials such that

$$\int\limits_{-\infty}^{\infty} \sqrt{|u(x)|^2 + |v(x)|^2}\, dy,$$

that decay at infinity: $|u(x)|, |v(x)| \to 0$ as $|x| \to$, the first column $\widetilde{\Phi}_1$ of $\widetilde{\Phi}$ can be analytically extended in the upper-half plane: $\Im(E) > 00$.

Equation (B.6) suggest the following iteration scheme (*Picard's iterations*)

$$w_{j+1} = I + \int\limits_{-\infty}^{x} e^{iE(x-y)} R(y) w_j\, dy, \tag{B.7}$$

with $w_0 = 0$, which implies $w_1 = I$. Notice that $w_{j+1} = w_{j+1}(x)$ are functions of x.

Now, the eigenvalues of R^*R are

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= e^{-2\,\mathrm{Im}(E)(x-y)}(|u(x)|^2 + |v(x)|^2) \\ \lambda_3 &= e^{2\,\mathrm{Im}(E)(x-y)}(|u(x)|^2 + |v(x)|^2), \end{aligned} \tag{B.8}$$

and

$$|e^{iE(x-y)}| = e^{-\,\mathrm{Im}(E)(x-y)}. \tag{B.9}$$

We will use these facts promtly.

Notice that, from (B.7) follows that

$$w_{j+1} - w_j = \int\limits_{-\infty}^{x} e^{iE(x-y)} R(y)(w_j - w_{j-1})\, dy.$$

Assume that

$$\mathrm{Im}(E) > 0.$$

Now, the Euclidean norm of $w_{j+1} - w_j$ is bounded by

$$\|w_{j+1} - w_j\| \leq \int_{-\infty}^{x} |e^{iE(x-y)}| \|R(y)\| \, \|w_j - w_{j-1}\| \, dy \tag{B.10}$$

Now, the matrix norm of R, induced by the Euclidean norm is

$$\|R\| = \sqrt{\max \sigma(R^*R)} = \sqrt{\max\{\lambda_1, \lambda_2, \lambda_3\}},$$

i.e.

$$\|R\| = \lambda_3 = e^{\mathrm{Im}(E)(x-y)} \sqrt{|u(y)|^2 + |v(y)|^2},$$

by equation (B.8). Then, from equation (B.10) follows

$$\begin{aligned} \|w_{j+1} - w_j\| &\leq \int_{-\infty}^{x} e^{-\mathrm{Im}(E)(x-y)} e^{\mathrm{Im}(E)(x-y)} \sqrt{|u(y)|^2 + |v(y)|^2} \, \|w_j - w_{j-1}\| \, dy \\ &= \int_{-\infty}^{x} \sqrt{|u(y)|^2 + |v(y)|^2} \, \|w_j - w_{j-1}\| \, dy. \end{aligned} \tag{B.11}$$

For $j = 1$

$$\begin{aligned} \|w_2 - w_1\| &\leq \int_{-\infty}^{x} \sqrt{|u(y)|^2 + |v(y)|^2} \, \|w_1 - w_0\| \, dy \\ &= C \int_{-\infty}^{x} \sqrt{|u(y)|^2 + |v(y)|^2} \, dy, \end{aligned}$$

where $C = \max_{y \in (-\infty, x)} \|w_1 - w_0\|(y)$.

For $j = 2$

$$\begin{aligned}
\|w_3 - w_2\| &\leq \int_{-\infty}^{x} \sqrt{|u(y)|^2 + |v(y)|^2}\, \|w_2 - w_1\|\, dy \\
&\leq C \int_{-\infty}^{x} \sqrt{|u(y)|^2 + |v(y)|^2} \int_{-\infty}^{y} \sqrt{|u(z)|^2 + |v(z)|^2}\, dz\, dy = \\
&= C \int_{-\infty}^{x} \frac{1}{2} \frac{d}{dy} \left(\int_{-\infty}^{y} \sqrt{|u(z)|^2 + |v(z)|^2}\, dz \right)^2 dy = \\
&= \frac{1}{2} C \left(\int_{-\infty}^{x} \sqrt{|u(y)|^2 + |v(y)|^2}\, dy \right)^2
\end{aligned}$$

In general, by an induction argument, we can prove that

$$\|w_{j+1} - w_j\| \leq \frac{1}{j!} C \left(\int_{-\infty}^{x} \sqrt{|u(y)|^2 + |v(y)|^2}\, dy \right)^j .$$

On the other hand

$$w_m = \sum_{j=0}^{m} w_{j+1} - w_j. \tag{B.12}$$

This implies

$$\begin{aligned}
\|w_m\| &\leq \sum_{j=0}^{m} \|w_{j+1} - w_j\| \\
&\leq \sum_{j=0}^{m} \frac{1}{j!} C \left(\int_{-\infty}^{x} \sqrt{|u(y)|^2 + |v(y)|^2}\, dy \right)^j \\
&\leq C \sum_{j=0}^{\infty} \frac{1}{j!} \left(\int_{-\infty}^{x} \sqrt{|u(y)|^2 + |v(y)|^2}\, dy \right)^j \\
&= C e^{\int_{-\infty}^{x} \sqrt{|u(y)|^2 + |v(y)|^2}\, dy} .
\end{aligned}$$

Hence, the following limit exists

$$\lim_{m \to \infty} \|w_m\| \leq C e^{\int_{-\infty}^{x} \sqrt{|u(y)|^2 + |v(y)|^2}\, dy} ,$$

and so the sequence $\{w_m\}$ converges to a finite limit. This proves that equation (B.5)has a unique solution for $\mathrm{Im}(E) \geq 0$. This implies that $\widetilde{\Phi}_1(x)$, and therefore $\Phi_1(x)$, can be analytically extended into the upper-half complex plane. The same argument works for $\Psi_2(x)$ and $\Psi_3(x)$.

If $\mathrm{Im}(E) \leq 0$, a similar argument works to prove that $\Psi_1(x)$, $\Phi_2(x)$, and $\Phi_3(x)$, can be analytically extended in the lower-half plane.

B.2 On the Analyticity of the Jost solutions: Potential with Compact Support

In the previous section, we showed that $\Phi_1(x)$ can be analytically continued for $\mathrm{Im}(E) \geq 0$, when the potentials are integrable. In this section, we will assume that the potentials have compact support

$$\begin{aligned} \mathrm{supp}(u(x)) &\subset (-L, L) \\ \mathrm{supp}(v(x)) &\subset (-L, L), \end{aligned}$$

for some positive constant L. Under this hypothesis, the potentials are integrable, then we can extend the column $\Phi_1(x)$ in the upper-half plane. Evenmore, $\Phi_1(x)$ can be analytically extended into the lower-half complex plane $\mathrm{Im}(E) \leq 0$. We will prove this statement in the following paragraphs.

Then, the main assumption in this section is

$$\mathrm{Im}(E) \leq 0.$$

We also have that equation (B.5) holds for $\widetilde{\Phi}(x)$, but here for compact support

$$\widetilde{\Phi}(x) = I + \int_{-L}^{x} R(y)\widetilde{\Phi}(y)e^{-iED(x-y)}\, dy. \tag{B.13}$$

In this instance, we need to consider three different cases

(a) $I_a = (-\infty, -L)$

(b) $I_b = (-L, L)$

(c) $I_c = (L, \infty)$.

If $x \in I_a$, denote by $\widetilde{\Phi}_a(x)$ the solution of (B.13). Similarly if $x \in I_b$ or $x \in I_c$.

Case (a). If $x \in I_a$, $N(x) = 0$, since the support of the potentials is compact. Then $R(y, x; E) = 0$. Therefore

$$\widetilde{\Phi}(x) = \widetilde{\Phi}_a(x) = I,$$

and this *matrix* can be analytically continued into the lower-half complex plane, $\mathrm{Im}(E) \leq 0$.

Case (b). This case requires a little bit of of argumente, but not too much, and it is similar to the argument given in section B.1.

The integral equation to be solved is (B.13),and the iteration scheme is basically the same as that shown in (B.7)

$$w_{j+1} = I + \int_{-L}^{x} R(y) w_j e^{-iED(x-y)}\, dy. \tag{B.14}$$

Also,

$$\|w_{j+1} - w_j\| \leq \int_{-L}^{x} \|R(y)\| \, \|w_j - w_{j-1}\| |e^{-iED(x-y)}| \, dy \tag{B.15}$$

Here is where we require $\mathrm{Im}(E) \leq 0$, to compute the norm of R

$$\|R\| = \lambda_2 = e^{-\mathrm{Im}(E)(x-y)} \sqrt{|u|^2 + |v|^2},$$

where we used equation (B.8). Then,

$$\begin{aligned} \|w_{j+1} - w_j\| &\leq \int_{-L}^{x} e^{-\mathrm{Im}(E)(x-y)} \sqrt{|u|^2 + |v|^2} \, \|w_j - w_{j-1}\| e^{-\mathrm{Im}(E)(x-y)} \, dy \\ &= \int_{L}^{x} e^{-2\,\mathrm{Im}(E)(x-y)} \sqrt{|u|^2 + |v|^2} \, \|w_j - w_{j-1}\| \, dy. \end{aligned}$$

Since $-L \leq y$, then $e^{-2\,\mathrm{Im}(E)(-y)} \leq e^{-2\,\mathrm{Im}(E)L}$. Therefore,

$$\|w_{j+1} - w_j\| \leq e^{-2\,\mathrm{Im}(E)(x+L)} \int_{-L}^{x} \sqrt{|u|^2 + |v|^2}\, \|w_j - w_{j-1}\|\, dy. \tag{B.16}$$

As we did in the previous section, iterating equation (B.16), we can check that

$$\|w_{j+1} - w_j\| \leq Ce^{-2\,\mathrm{Im}(E)(x+L)} \frac{e^{-4j\,\mathrm{Im}(E)L}}{j!} \left(\int_{-L}^{x} \sqrt{|u|^2 + |v|^2}\, dy \right)^j.$$

Using equation (B.12), and following the same argument as in the previous section, we find that

$$\begin{aligned} \|w_m\| &\leq Ce^{-2\,\mathrm{Im}(E)(x+L)} \sum_{j=0}^{\infty} \frac{1}{j!} \left(e^{-4\,\mathrm{Im}(E)L} \int_{-L}^{x} \sqrt{|u|^2 + |v|^2}\, dy \right)^j \\ &= Ce^{-2\,\mathrm{Im}(E)(x+L)} e^{e^{-4\,\mathrm{Im}(E)L} \int_{-L}^{x} \sqrt{|u|^2+|v|^2}\, dy}. \end{aligned} \tag{B.17}$$

If we let $m \to \infty$, the iteration (B.14) has a unique solution denoted by

$$\widetilde{\Phi}(x) = \widetilde{\Phi}_b(x),$$

for $\mathrm{Im}(E) \leq 0$. Notice that if we let $L \to \infty$ in equation (B.17) (i.e., in the limit the support is not compact), then the argument fails since the exponential blows up.

Case (c). We are almost done. In this case, $x \in I_c$. Then, the integral (B.13) becomes

$$\widetilde{\Phi}(x) = I + \int_{-L}^{L} R(y, x; E)\widetilde{\Phi}(y)e^{-iED(x-y)}\, dy,$$

for $\mathrm{Im}(E) \leq 0$. However, for $y \in I_b$, $\widetilde{\Phi}(y) = \widetilde{\Phi}_b(y)$. Therefore, the previous equation becomes

$$\widetilde{\Phi}_c(x) = I + \int_{-L}^{L} R(y, x; E)\widetilde{\Phi}_b(y)e^{-iED(x-y)}\, dy.$$

This is not an integral equation for $\widetilde{\Phi}_c(x)$ anymore. The matrix function $\widetilde{\Phi}_c(x)$ is exactly given by this equation. The right-hand-side of the equation can be analytically extended into $\mathrm{Im}(E) \leq 0$, since we can extend $\widetilde{\Phi}_b(y)$, as well as $R(y,x;E)$ and the exponential. Therefore $wt\Phi(x)$ and also $\Phi(x)$ can be analytically extended to $E \in \mathbb{C}^-$ as well. Therefore, $\Phi(x;E)$ is an analytic function of the spectral paramater. We can apply the same argument to $\Psi(x;E)$

We can conclude that the Jost functions $\Phi(x;E)$ and $\Psi(x;E)$ can be analytically extended into the whole complex plane, i.e., they are analytic functions of the variable E, when $N(x)$ has compact support.

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