COMPACT SYMMETRIC SPACES, TRIANGULAR FACTORIZATION, AND CAYLEY COORDINATES

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DEDICATION

This is for Oma.
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Let $X$ be a simply connected, compact Riemannian symmetric space. We can represent $X$ as the homogeneous space $U/K$, where $U$ is a simply connected compact Lie group, and $K$ is the fixed point set of an involution $\Theta$ of $U$. Let $G$ be the complexification of $U$. We consider the intersections of the image of the Cartan embedding

$$\phi : U/K \rightarrow U \subset G : uK \mapsto uu^{-\Theta} \quad (0.0.1)$$

with the strata of the Birkhoff (or triangular, or LDU) decomposition

$$G = \prod_{w \in W} \Sigma_w, \quad \Sigma_w^G = N^- w H N^+ \quad (0.0.2)$$

relative to a $\Theta$-stable decomposition of the Lie algebra, $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$.

For a generic element $g$ in this intersection, $g \in \phi(U/K) \cap \Sigma_1^G$, this yields a unique triangular factorization $g = ldu$. Our main contribution is to produce explicit formulas for the diagonal term $d$ in classical cases, using Cayley coordinates (this choice of coordinate is motivated by considerations beyond sheer convenience).

These formulas have several applications:
1) we can compute $\pi_0(\phi(U/K) \cap \Sigma_1^G)$ explicitly;
2) we can compute $\int_{\phi(U/K)} a_{\phi}^{-i\lambda}$ (where $a_{\phi}$ is the positive part of $d$) using elementary techniques in rank 1 cases;
3) they are useful in explicitly calculating Evens-Lu Poisson structures on $U/K$ (see [Caine(2006)]).

Our set-up involves choosing specific representations of the various $\mathfrak{u}$ in $\mathfrak{su}(n, \mathbb{C})$ that are compatible with $\Theta$; that is, $\Theta$ fixes each of the subspaces $\mathfrak{n}^-$, $\mathfrak{h}$, and $\mathfrak{n}^+$ which, in our setup, always consist of strictly lower triangular, diagonal, and strictly upper triangular matrices, respectively.

The formulas contain determinants such as $\det(1 + X)$, where $X$ is in $i\mathfrak{p}$, the $-1$-eigenspace of $\Theta$ acting on the Lie algebra $\mathfrak{u}$. Due to the relatively sparse nature of these matrices, these determinants are often easily calculable, and we illustrate this with many examples.
CHAPTER 1

INTRODUCTION

Let $K$ be a simply connected compact Lie group with complexification $G$. Given a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ of the Lie algebra of $G$, we have the triangular (or Birkhoff or LDU) decomposition of the group

$$G = \prod_{w \in W} \Sigma^G_w, \quad \Sigma^G_w = N^- w H N^+$$

where the components are indexed by elements of the Weyl group $W = N_G(H)/Z_G(H)$. The set of generic elements $\Sigma^G_1$ compose all but a set of measure zero, and for these elements the decomposition $g = ldu$ is unique, where $l \in N^-$ (lower triangular unipotent), $u \in N^+$ (upper triangular unipotent), and $d = ma \in H$ where $m \in H \cap K$ (unitary) and $a \in \exp(\mathfrak{h} \cap i\mathfrak{k})$ (positive). A formula of Harish-Chandra asserts that for $\lambda \in \mathfrak{h}_\mathbb{R}^*$

$$\int_K a^{-i\lambda} = \int_K e^{-i \lambda(\log a)} = c(2\delta - i\lambda) = \prod_{\alpha > 0} \frac{\langle 2\delta, \alpha \rangle}{\langle 2\delta - i\lambda, \alpha \rangle},$$

where the integral is with respect to normalized Haar measure, the product is over positive complex roots, and $2\delta$ is the sum of the positive complex roots. The so-called “$c$-function” of Harish-Chandra is of central importance in harmonic analysis on the noncompact symmetric space $G/K$ (See [Helgason(2000)] §5-6 of chapter IV for extensions and interpretations of the $c$-function).

Let $X$ be a compact symmetric space. Such a symmetric space can be realized as $U/K$ where $U$ is the universal covering space of $\text{Isom}(X)$ and $K$ is the stability subgroup of a chosen base-point $x_0 \in X$. Symmetric space theory also tells us that $K$ is the fixed point set of an involution $\Theta$ of $U$.

We consider an integral similar to (1.0.2) over a compact symmetric space. This generalization is achieved by first embedding $U/K$ into $G$, the complexification of $U$, as follows (this totally geodesic embedding is due to Cartan [Cartan(1983)])

$$\phi : U/K \hookrightarrow U \subset G : uK \mapsto uu^{-\Theta}$$

and then considering $\phi(U/K) \cap \Sigma^G_1$, the intersection of this image with the generic elements of $G$, relative to a $\Theta$-stable triangular decomposition of the Lie algebra of
When $\Theta$ is an inner automorphism, the formula has the form
\[
\int_{\phi(U/K)} a_{\phi}^{-i\lambda} = \frac{|W(K)|}{|W(U)|} \sum_{w} \prod_{\delta} \frac{\langle \delta, \alpha \rangle}{i \delta - i\lambda, \alpha}, \tag{1.0.4}
\]
where $a_{\phi}$ is the positive part of $d$, the integral is with respect to the unique normalized invariant measure, and $w$ are elements of order two in $\Gamma_{0} = T_{0} \cap K$ which index the connected components of $\phi(U/K) \cap \Sigma_{1}^{G}$ as shown in [Pickrell(2006)]. This is proven using the Duistermaat-Heckman exact stationary phase method of integration [Duistermaat and Heckman(1982)], applied to a symplectic structure studied by Evens and Lu [Evens and Lu(2001)], and Foth and Otto [Foth and Otto(2005)].

The main purpose of this paper is to develop elegant, explicit formulas for the diagonal term $d$ of a generic $g \in \phi(U/K)$ for classical compact symmetric spaces. This enables efficient calculation of the above integrals, and lets us explicitly compute $\pi_{0}(\phi(U/K) \cap \Sigma_{1}^{G})$, the connected components of the intersection. Determining $\pi_{0}(\phi(U/K) \cap \Sigma_{1}^{G})$ comes down to finding the possible $w$ (from (1.0.4)) that occur in $\phi(U/K)$, which we do explicitly.

Our set-up involves choosing a representation of $U$ inside $SU(n, \mathbb{C})$ such that the usual triangular decomposition is $\Theta$-stable; that is, $\Theta$ preserves diagonal, upper triangular, and lower triangular matrices. When $\Theta$ is an inner automorphism, we can parameterize $\phi(U/K)$ using the Cayley map
\[
\Phi : u \rightarrow \{ g \in U | -1 \notin \text{spec}(g) \} : X \mapsto g = \frac{1 - X}{1 + X}. \tag{1.0.5}
\]
If we restrict $\Phi$ to $i\Pi$, the $-1$-eigenspace of $\Theta$, then $\Phi_{|i\Pi}$ is a coordinate for almost all of $\phi(U/K)$. When $U$ is simply connected, the diagonal term of a generic $\Phi_{|i\Pi}(X) = g = tdu$ is
\[
d = \prod_{k=1}^{\text{rank}(U)} \left( \frac{\det(1 + I_{k}X)}{\det(1 + X)} \right)^{h_{k}} \in H \subset G \tag{1.0.6}
\]
where the $h_{k}$ are coroots for $U$, and $I_{k} = \begin{bmatrix} -1_{k \times k} & 0 \\ 0 & 1_{(n-k) \times (n-k)} \end{bmatrix}$. When $U$ is an orthogonal group, the formula appears slightly different to accommodate the root structure of $SO(n)$.

Using this formula, we can see that the condition for $g$ in (1.0.5) to belong to $\phi(U/K) \cap \Sigma_{1}^{G}$ (the top stratum) is that $\det(1 + I_{k}X)$ is not zero, for each $k$. This is what we use to determine the connected components of this top stratum (these factors also occur in the work [Caine(2006)], because these components are symplectic leaves for a natural Poisson structure on $U/K$).
It is a bit mysterious why Cayley coordinates work so well in this situation. Not only does it map \( n^- \rightarrow N^- \), \( n^+ \rightarrow N^+ \), and \( \mathfrak{h} \rightarrow H \), but it also maps \( \mathfrak{k} \rightarrow K \) and \( i\mathfrak{p} \) (the \(-1\) eigenspace of \( \Theta \)) into \( \phi(U/K) \), making it ideal for our purposes.

Although it might seem that the choice of Cayley coordinates is an artificial device, this is not the case. To develop a theory of harmonic analysis on infinite dimensional classical symmetric spaces, it turns out that the use of Cayley coordinates is essential (see [Pickrell(1989)]). We hope our formulas will be useful in understanding various issues which remain open for infinite dimensional symmetric spaces.

The computational advantages of these formulas should be mentioned. Using Cayley coordinates, one could use a computer algebra system (such as Maple) to take an arbitrary matrix \( X \in i\mathfrak{p} \subset \mathfrak{su}(n, \mathbb{C}) \) and factor \( (1 - X)(1 + X)^{-1} \) into \( ldu \), but inversion of matrices and LU factorization are expensive calculations involving Gaussian elimination and/or computation of many determinants. Furthermore, the resulting expressions are often intractibly complicated. By contrast, the formulas presented in this paper find \( d \) by computing at most \( n \) determinants of the form given in (1.0.6) above. Though determinants are not easily computable in general, the calculations for these symmetric spaces are quite speedy compared to the above brute force method, and the resulting formulas are much simpler. Perhaps this is due to the relatively sparse nature of the matrices involved. Regardless of the reason, computations of many specific examples show that the formulas presented here tend to make formerly expensive and messy calculations much quicker and cleaner.

The plan of this paper is as follows. In chapters 2 through 5 we review background material.

In chapter 2 we review the Birkhoff decomposition of a semisimple Lie group \( U \) and its relation to that of its complexification \( G \). Specifically, we show that \( \Sigma_w^G \) and \( \Sigma_w^G = \Sigma_w^G \cap U \) are homotopy equivalent.

In chapter 3 we consider \( \Theta \in Aut(U) \) of finite order and let \( K = U^\Theta \). Then we have a diagram

\[
\begin{array}{ccc}
G & \nearrow & U \\
\searrow & & \searrow \\
G_0 & \nearrow & U \\
& & \searrow \\
& & K
\end{array}
\]

(1.0.7)

where the arrows are inclusions. We also introduce the Cartan embedding, \( \phi : U/K \rightarrow U \subset G \).

In chapters 4 and 5 we review the background of symmetric spaces (\( \Theta \) is an involution). In this case, we can say much more about the Cartan embedding \( \phi \), and of its intersection with \( \Sigma_w^G \).
In chapter 6 we perform explicit calculations of the diagonal distribution of elements in $\phi(U/K) \cap \Sigma^G_1$ for many classical symmetric spaces (all such that $\Theta$ is an inner automorphism) of compact type using Cayley coordinates.

In chapter 7 we use the formulas derived in chapter 6 to compute some examples of the integral (1.0.4).
CHAPTER 2

THE BIRKHOFF DECOMPOSITION OF $G$

2.1 Factorization of Matrices

Every $n \times n$ matrix $X = (X_{ij})$ can be written uniquely as a sum

$$X = L + D + U$$ (2.1.1)

where $L$ is strictly lower triangular, $D$ is diagonal, and $U$ is strictly upper triangular. In terms of Lie algebra theory, this corresponds to the statement that there is a triangular decomposition

$$\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$ (2.1.2)

where $\mathfrak{n}^+$ (respectively $\mathfrak{n}^-$) is the nilpotent subalgebra consisting of strictly upper (resp. lower) triangular matrices, and $\mathfrak{h}$ is the Cartan subalgebra of diagonal matrices.

The group-theoretic analogue of this triangular decomposition is equivalent to the algebra underlying Gaussian elimination. Suppose that $g = (g_{ij})$ is an invertible $n \times n$ matrix. For a generic $g$ there is a unique LDU, or lower-diagonal-upper triangular, factorization

$$g = ldu,$$ (2.1.3)

where $l$ is lower triangular and unipotent, $d$ is diagonal, and $u$ is upper triangular and unipotent. The entries of these factors can be written explicitly as ratios of determinants:

$$d = \text{diag}(\det g_{(1)}, \det g_{(2)}/\det g_{(1)}, \det g_{(3)}/\det g_{(2)}, \ldots, \det g_{(n)}/\det g_{(n-1)})$$ (2.1.4)

where $g_{(k)}$ is the $k^{\text{th}}$ principal submatrix, $g_{(k)} = (g_{ij})_{1 \leq i, j \leq k}$. For $i > j$, we have

$$l_{ij} = \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} \\ a_{21} \\ \vdots \\ a_{j-1,1} & a_{j-1,j} \\ a_{i,1} & \cdots & a_{ij} \end{bmatrix} / \det g_{(j)}$$ (2.1.5)
and for $i < j$,

$$
\begin{align*}
  u_{ij} &= \det\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1,i-1} & a_{1,j} \\
  & \ddots & & \vdots & \vdots \\
  & & a_{i,1} & \ldots & a_{i,j}
\end{bmatrix} / \det g(i). \quad (2.1.6)
\end{align*}
$$

The meaning of generic is precisely that $\det g(k) \neq 0$ for $k = 1, \ldots, n - 1$.

2.2 Generalization to Lie Groups: The Birkhoff Decomposition

The following theorem shows when group elements can be analogously factored. Let $G$ be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$. We choose a Cartan subalgebra $\mathfrak{h}$ and a set of positive roots $\Delta^+ \subset \Delta \subset \mathfrak{h}^*_\mathbb{R}$. Let $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) denote the direct sum of the positive (resp. negative) root spaces $\mathfrak{g}_\alpha$ (resp. $\mathfrak{g}_{-\alpha}$) for $\alpha \in \Delta^+$. Then we have the triangular decomposition

$$
\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+. \quad (2.2.1)
$$

Let $H = \exp \mathfrak{h}$, $N^- = \exp \mathfrak{n}^-$, and $N^+ = \exp \mathfrak{n}^+$. Let $W(G, H) = N_G(H)/Z_G(H)$ be the Weyl group of $(G, H)$.

**Example 2.2.2.** Let $G = SL(n, \mathbb{C})$, $n \times n$ matrices with determinant 1, and $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, $n \times n$ matrices of trace 0. Choose $\mathfrak{h}$ to be diagonal matrices in $\mathfrak{g}$ and choose positive roots so that $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) is the subalgebra of strictly upper (resp. lower) triangular matrices. Then $\mathfrak{h}$ is spanned by $\{h_i = e_{ii} - e_{i+1,i+1} | 1 \leq i \leq n - 1\}$, and $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) is spanned by $\{e_{ij} | i < j$ (resp. $i > j$)$\}$ where $e_{ij}$ denotes the $n \times n$ matrix with $(i, j)^{th}$ entry 1 and the rest 0.

Many examples will be presented in this paper. When dealing with specific examples of classical Lie groups, we will always arrange for $\mathfrak{h}$ to consist of diagonal matrices, $\mathfrak{n}^+$ to be upper triangular, and $\mathfrak{n}^-$ to be lower triangular.

The following theorem is well known.

**Theorem 2.2.3.** The double cosets of $N^-\backslash G/HN^+$ are indexed by $W(G, H)$. We denote the double coset corresponding to $w \in W(G, H)$ by $\Sigma^G_w = N^-wHN^+$ where $w$ is any representative of $w$ in $N_G(H)$. Thus we obtain the Birkhoff (or triangular or LDU) decomposition

$$
G = \prod_{w \in W} \Sigma^G_w, \quad \Sigma^G_w = N^-wHN^+. \quad (2.2.4)
$$

Furthermore, each $\Sigma^G_w$ is diffeomorphic to $(N^- \cap wN^-w^{-1}) \times H \times N^+$. 
Remark 2.2.5. (a) Recall that a double coset of $H_1 \backslash G / H_2$ consists of all elements $h_1 gh_2$ for $h_1 \in H_1, h_2 \in H_2$, and $g \in G$.

(b) This decomposition is equivalent to the Bruhat decomposition. See chapter IX, section 2 of [Helgason(1978)].

(c) In the case when $w = 1$, $g \in \Sigma^G_1$ is called a generic element. The factorization of a generic element is unique, but for other $w \in W(G,H)$ there are different factorizations for each $g \in \Sigma^G_w$. The following examples illustrate this.

Example 2.2.6. Let $G = SL(2, C)$, with triangular decomposition of $G$ as above. Then $H$ is the subgroup of diagonal matrices of determinant 1, and $N^+$ (resp. $N^-$) the subgroup of upper (resp. lower) triangular unipotent matrices. The Weyl group is $W \cong \mathbb{Z}_2 = \{1, -1\}$, so the Birkhoff decomposition gives us two cosets, or “strata.” We choose representatives

$$w_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in w_1, \text{ and } w_{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in w_{-1}$$

in $N_G(H)$. Setting

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y & 0 \\ 0 & 1/y \end{bmatrix} \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} y & yz \\ xy & xyz + 1/y \end{bmatrix},$$

we see that only when $a \neq 0$ is such a matrix in $\Sigma^G_1$. Solving, we obtain

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/a \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}.$$ (2.2.7)

Doing the same for $w_{-1}$ we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y & 0 \\ 0 & 1/y \end{bmatrix} \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1/y \\ y & yz - x/y \end{bmatrix}.$$ (2.2.8)

We see that only when $a = 0$ (forcing $c \neq 0$ and $b = -1/c$) is such a matrix in $\Sigma^G_{-1}$. Solving, we obtain

$$\begin{bmatrix} 0 & -1/c \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & 1/c \end{bmatrix} \begin{bmatrix} 1 & d/c + t/c^2 \\ 0 & 1 \end{bmatrix},$$ (2.2.9)

where $t \in C$ is arbitrary. If we require that $\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \in (N^- \cap wN^-w^{-1})$ as in the last line of the theorem, we see that we have

$$\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \in wN^-w^{-1} = \{ \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} | \alpha \in C \}$$ (2.2.10)
if and only if \( t = 0 \), eliminating the ambiguity in the decomposition of a non-generic element.

**Example 2.2.13.** Let \( G = SL(3, \mathbb{C}) \) with the triangular decomposition of \( \mathfrak{sl}(n, \mathbb{C}) \) as above. We perform the same computation as above for just one double coset \( \Sigma^G_w \), the one corresponding to the \( w \) with representative

\[
\mathbf{w} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.
\]

We see that for a matrix \( g \) to be in \( \Sigma^G_w \), it will have a factorization of the form

\[
g = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ v & w & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} j & 0 & 0 \\ 0 & k/j & 0 \\ 0 & 0 & 1/k \end{bmatrix} \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} 0 & -k/j \\ -j & -jx - uk/j \\ -jw & -wzj - zk/j \end{bmatrix}.
\]

This will happen if and only if \( a = 0 \), \( b \neq 0 \), and \( d \neq 0 \). (Recall that \( g_{(2)} \) is the \( 2 \times 2 \) principal minor of \( g \)). The factorization of \( g \) is then

\[
\begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ h - c a / d^2 + t g / d & g / d & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -d & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 1/bd \end{bmatrix} \begin{bmatrix} 1 & \frac{e - t b}{d} & \frac{f - t c}{d} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

where \( t \in \mathbb{C} \) is arbitrary. If we require that the leftmost matrix be in \( (N^- \cap wN^-w^{-1}) \), then

\[
\begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ h - c a / d^2 + t g / d & g / d & 1 \end{bmatrix} \in wN^-w^{-1} = \left\{ \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ \beta & \gamma & 1 \end{bmatrix} \mid \alpha, \beta, \gamma \in \mathbb{C} \right\}.
\]

This happens if and only if \( t = 0 \), eliminating the ambiguity.

**Remark 2.2.20.** Though the factorization of each element \( g = twhu \) will depend on the choice of representative \( w \in w \), the decomposition of \( G \) into \( \coprod_{w \in W} \Sigma^G_w \) is independent of such a choice, since \( w \) is unique up to multiplication by \( h \in H \).
2.3 A Cellular Decomposition

Let $U$ be a compact, semisimple Lie group with Lie algebra $\mathfrak{u}$, and let $G$ and $\mathfrak{g}$ be their complexifications, respectively. Fix a maximal abelian subalgebra $\mathfrak{t} \subset \mathfrak{u}$ with corresponding maximal torus $T \subset U$. Then $\mathfrak{h} = \mathfrak{t}^C$ is a Cartan subalgebra of $\mathfrak{g}$. As in the last section, with a choice of positive roots in $\Delta \subset \mathfrak{h}_R^*$ (recall that $\mathfrak{h}_R = i\mathfrak{t}$), we obtain a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Note that the Weyl groups $W(G,H) \cong N_G(H)/H$ and $W(U,T) \cong N_U(T)/T$ are isomorphic. In this section we describe the relationship between $\Sigma^U_w := U \cap \Sigma^G_w$ and $\Sigma^G_w$ with respect to the inclusion $U \hookrightarrow G$.

Let $B^+$ (resp. $B^-$) = $HN^+$ (resp. $N^-H$), and let $q : G \to G/B^+$ denote the quotient map. The map $q$ applied to the Birkhoff decomposition induces the (more conventional) triangular stratification for the flag space,

$$ U/T \simeq G/B^+ = \coprod_w \Sigma^U_w, \quad \Sigma^U_w := N^- \cdot wB^+, $$

where each $\Sigma^U_w$ is a cell ($\simeq N^- \cap wN^-w^{-1}$). As a consequence, for the pieces of the induced decomposition for $U$, there are isomorphisms

$$ \Sigma^U_w = U \cap \Sigma^G_w \simeq \Sigma_w \times T. $$

The inclusions $\Sigma^U_w \to \Sigma^G_w$ are homotopy equivalences, because $T$ is homotopy equivalent to $B^+$:

$$ \begin{array}{ccc}
T & \to & \Sigma^U_w \\
\downarrow & & \downarrow q \\
B^+ & \to & \Sigma^G_w \\
\end{array} $$

(*2.3.3*)

**Example 2.3.4.** Let $U = SU(2,\mathbb{C})$ with complexification $G = SL(2,\mathbb{C})$, and maximal torus $T \cong U(1,\mathbb{C})$ realized as

$$ \{ \begin{bmatrix} e^{is} & 0 \\ 0 & e^{-is} \end{bmatrix} | s \in \mathbb{R} \}. $$

(*2.3.5*)

We keep the same triangular decomposition of $G$ as in the previous section, which gives us the following cellular decomposition of $U/T$:

$$ \Sigma_1 \cong N^- = \{ \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} | t \in \mathbb{C} \} \cong \mathbb{C}, $$

(*2.3.6*)

$$ \Sigma_{-1} \cong (N^- \cap \text{conj}([0 & -1 \\ 1 & 0])N^-) \cong \{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \}. $$

(*2.3.7*)

So the two cells are a plane and a point; therefore, $SU(2)/U(1) \cong S^2$. 


CHAPTER 3

AUTOMORPHISMS AND FIXED-POINT SETS

3.1 Homogeneous Spaces of the type $G/G^\sigma$

Let $G$ be a semisimple Lie group with Lie algebra $\mathfrak{g}$, and let $\sigma \in Aut(G)$ be an automorphism of finite order. In this section we consider the fixed point set $H = G^\sigma = \{ g \in G | \sigma(g) = g \}$, and the quotient space $G/H$.

Example 3.1.1. Fix some $g_0 \in U$, and define $\sigma(u) = g_0 u g_0^{-1}$. Then $K = Z_U(g_0)$. If $g_0 \in Z(U)$, then $K = U$ and $U/K$ is a point.

Example 3.1.2. Let $U = SU(2, \mathbb{C}) = \{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} | \|a\|^2 + \|b\|^2 = 1 \}$ and define $\sigma$ as conjugation by $I_1 := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Then we have

$$\sigma(u) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then requiring that $u = \sigma(u)$ gives us $K = \{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} | \|a\|^2 = 1 \} \cong U(1)$, which is homeomorphic to a circle. The quotient is $U/K \cong S^2$, as we showed in example 2.3.4.

It is also true that $SO(3)/SO(2) \cong S^2$, a fact we will justify in the next chapter. Observing that $SU(2)$ is the double cover of $SO(3)$ and that $U(1) \cong SO(2)$ should persuade the reader that this is at least plausible. We use this in the next example.

Example 3.1.5. Let $U = SO(3, \mathbb{R}) = \{ g \in SL(3, \mathbb{R}) | gg^t = 1 \}$, and define $\sigma$ as conjugation by $I_2 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, an inner automorphism. Looking at the action of $\sigma$ on the rotation matrices that generate $SO(3)$, we see it fixes rotations of the form

$$\begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and sends rotations

$$\begin{pmatrix} \cos \phi & 0 & -\sin \phi \\ 0 & 1 & 0 \\ \sin \phi & 0 & \cos \phi \end{pmatrix},$$

and
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \psi & -\sin \psi \\
0 & \sin \psi & \cos \psi
\end{bmatrix}
\]
to their respective inverses, thus fixing elements of order two
\[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]. So \( K \) is the subgroup of \( SO(3) \)

\[
K = \langle \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}, \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix}, \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix} \rangle
\]

\[\cong SO(2) \times Z(SO(2)) Z(SO(3)).\] (3.1.6)

Therefore, the quotient space \( U/K \) is

\[
U/K \cong SO(3)/SO(2) \times Z(SO(2)) Z(SO(3))
\]

\[\cong S^2/\mathbb{Z}_2\] (3.1.7)

\[\cong \mathbb{R}P^2.\] (3.1.8)

**Example 3.1.11.** Let \( U = SU(3, \mathbb{C}) \), and define \( \sigma \) as conjugation by the matrix

\[
P = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}.
\] (3.1.12)

To see what the space \( U/U^\sigma \) is for this automorphism of order 3, we look at \( d\sigma \) acting on the Lie algebra \( \mathfrak{u} \). The action of \( d\sigma \) is also conjugation by \( P \). Consider,

\[
X \in \mathfrak{su}(3, \mathbb{C}) = \{ \begin{bmatrix}
is & i(s + t) & x_{12} & x_{13} \\
-i(s + t) & 1 & x_{23} \\
0 & 0 & 1 & it
\end{bmatrix} | x_{ij} \in \mathbb{C}, s, t \in \mathbb{R} \}
\] (3.1.13)

Then requiring that \( X = X^\sigma \) gives us

\[
\begin{bmatrix}
is & x_{12} & x_{13} \\
-i(s + t) & x_{23} & x_{12} \\
-x_{13} & -x_{23} & it
\end{bmatrix} =
\begin{bmatrix}
-i(s + t) & x_{23} & -x_{12} \\
-x_{23} & it & -x_{13} \\
x_{12} & x_{13} & is
\end{bmatrix}
\] (3.1.14)

\[
\Rightarrow X = \begin{bmatrix}
0 & x & -\bar{x} \\
-\bar{x} & 0 & x \\
x & -\bar{x} & 0
\end{bmatrix}
\] (3.1.15)

The subalgebra consisting of all such elements is abelian, so \( U^\sigma \) is a 2 (real-) dimensional torus. This is an example of a flag manifold as described in section 2.3.
3.2 The Cartan Embedding

**Theorem 3.2.1.** The map

$$\phi : G/H \to G : gH \mapsto g\sigma(g^{-1})$$

(3.2.2)

is an embedding.

*Proof.* It suffices to prove that $\phi$ is well-defined and injective, since the map

$$G \to G : g \mapsto g\sigma(g^{-1})$$

(3.2.3)

is obviously smooth, and $G/H$ has the quotient topology. Suppose $\phi(g) = \phi(gg_0)$. Then

$$(gg_0)\sigma((gg_0)^{-1}) = gg_0\sigma(g_0^{-1})\sigma(g^{-1})$$

(3.2.4)

is equal to $g\sigma(g^{-1})$ exactly when $\sigma(g_0) = g_0$, and so $g_0 \in H$. Therefore, $\phi$ is well-defined and injective, and, hence, an embedding.

The geometry of these structures have been studied extensively in [Wolf and Gray(1968)]; however, it seems the relationship of the Cartan embedding with these geometries has not been studied beyond automorphisms of order 2 (symmetric spaces). It would be interesting to discover what geometric aspects are preserved under this embedding for automorphisms of other finite order.
CHAPTER 4

SYMMETRIC SPACES

Though we will be working with symmetric spaces as a homogeneous spaces, focusing on their algebraic characterization, in the literature they are generally defined by their geometric properties. For the sake of completeness, we provide some of that background here. The canonical source for Riemannian symmetric spaces is [Helgason(1978)], and information on non-Riemannian symmetric spaces can be found in [Flensted-Jensen(1986)]. The development of this subject is mainly due to the work of E. Cartan, Weyl, Harish-Chandra, and Helgason.

4.1 Symmetric Spaces as Homogeneous Spaces

Let $X$ be a $C^\infty$ manifold with an affine connection $\nabla$, and for any point $x \in X$, let $S_x$ denote the map that sends a point $y$ to its reflection across a geodesic through $y$ and $x$. That is, if $\gamma$ is a geodesic with $\gamma(0) = x$, then $S_x(\gamma(t)) = \gamma(-t)$. If $N$ is a normal neighborhood of $x$, then $S_x$ is a diffeomorphism of $N \setminus S_x(N)$.

**Definition 4.1.1.** (a) An **affine locally symmetric space** is such a manifold that for every $x \in X$, the local diffeomorphism $S_x$ is an affine map. If $S_x$ can be extended to a diffeomorphism of all of $X$, for every $x$, then $X$ is an **affine globally symmetric space**, or just a **symmetric space**.

(b) If $g$ is a (pseudo-) Riemannian metric on a symmetric space $X$ for which $\nabla$ is the Levi-Civita connection, that is

$$Y \langle Z_1, Z_2 \rangle = \langle \nabla_Y Z_1, Z_2 \rangle + \langle Z_1, \nabla_Y Z_2 \rangle$$

for vector fields $Y, Z_1, Z_2$, then $X$ is a **(pseudo-) Riemannian symmetric space**.

**Example 4.1.3.** Consider $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ with the standard Riemannian metric induced by the flat metric on $\mathbb{R}^3$, and choose a basepoint $x_0 = (0, 0, 1)$, the north pole. Then $S_{x_0}$ is reflection across great circles through the north pole. The north and south poles are fixed under this automorphism of $S^2$, and note that this is equivalent to the rigid rotation of $\pi$ radians about the $z$ axis. If $x$ is any other point on the sphere, and $\psi$ is a rigid rotation ($\psi \in SO(3)$) which sends $x_0$ to $x$, then $S_x = \psi S_{x_0} \psi^{-1}$. We will be returning often to this example of a compact Riemannian symmetric space.
Example 4.1.4. Consider the unit disk in the complex plane with the hyperbolic metric, otherwise known as the “Poincare disk,” denoted by $\Delta$. In this metric, a geodesic passing through two points is the arc of a circle passing through those points and intersecting the boundary of the disk at right angles. If our basepoint is $x_0 = 0$, then geodesics lie in diameters of the disk, and $S_{x_0}$ is usual Euclidean reflection through zero. Again, this is equivalent to rotation by $\frac{\pi}{2}$ about zero, and if $\psi$ is an isometry of the disk ($\psi \in SU(1, 1)$) such that $\psi(x_0) = x$, then $S_x = \psi S_{x_0} \psi^{-1}$.

Definition 4.1.5. Let $G$ be a connected Lie group, let $\Theta : G \to G$ be an involution, an automorphism of order two, and let $G^\Theta$ denote the subgroup that is the fixed point set of $\Theta$. Then if $H$ is a closed subgroup of $G$ with the property that $(G^\Theta)_0 \subseteq H \subseteq G^\Theta$, we call $(G, H)$ a symmetric pair.

Example 4.1.6. For any symmetric pair $(G, H)$ the homogeneous space $G/H$, with the bi-invariant metric (and its associated connection) induced by the quotient map, is a symmetric space. With respect to this metric, the transitive action induced by left translation

$$G \times G/H \to G/H : (g, g_0H) \mapsto gg_0H \quad (4.1.7)$$

is an isometry. Our chosen basepoint will be $eH$; note that the isotropy subgroup of this basepoint is $H$ itself. The reflection around the basepoint $eH$ is given by

$$S_{eH}(gH) = \Theta(g)H, \quad (4.1.8)$$

and for any $g' \in G$,

$$S_{g'H}(gH) = (L_{g'} \circ \Theta \circ L_{g'}^{-1}g)H \quad (4.1.9)$$

where $L_{g'}$ is left translation by $g'$.

Note that the first example in this section is a special case of this (see examples 2.3.4 and 3.1.2). In fact, as the next theorem states, every symmetric space can be realized in this way.

Theorem 4.1.10. Let $G$ be a group that acts transitively and isometrically on the (pseudo-) Riemannian symmetric space $X$. If $H$ is the isotropy subgroup of a basepoint $x_0 \in X$, then $(G, H)$ is a symmetric pair relative to the involution $\Theta(g) = S_{x_0} g S_{x_0}$, and

$$G/H \to X : gH \mapsto g(x_0) \quad (4.1.11)$$

is an isomorphism of symmetric spaces.

Proof. Since the action of $G$ on $X$ is transitive, the above map is surjective. If $g_1(x_0) = g_2(x_0)$, then $g_2^{-1}g_1$ fixes $x_0$, and so is in $H$. Therefore $g_2H = g_2g_2^{-1}g_1H = g_1H$, proving injectivity.
We must show that \((G, H)\) is a symmetric pair. Let \(g \in (G^\Theta)_0\), and consider the isometry \(S_{x_0}gS_{x_0}\) of \(X\). Then \(S_{x_0}g = gS_{x_0}\), and, in particular, \(S_{x_0}g(x_0) = gS_{x_0}(x_0) = g(x_0)\). Therefore, \(S_{x_0}\) fixes \(g(x_0)\). We would like to conclude that \(g(x_0) = x_0\), and we do so using connectedness. Since \(g \in (G^\Theta)_0\), we can connect \(g\) with the identity \(e\) by a path in \(G^\Theta\). We can use this to connect \(g(x_0)\) with \(e(x_0) = x_0\), and every point on the path is fixed by \(S_{x_0}\). But in a neighborhood of \(x_0\), the only point that \(S_{x_0}\) fixes is \(x_0\) itself. Therefore, \(g(x_0)\) must be \(x_0\), which implies that \(g \in H\); hence, \((G^\Theta)_0 \subseteq H\).

Now let \(g \in H\). The reflection \(S_{x_0}\) is defined by what it does on geodesics, so suppose we can connect \(x \in X\) with our chosen basepoint \(x_0\). Then since \(g\) moves geodesics to geodesics, it must move the reflection of \(x\) to the reflection of \(g(x)\) (see Figure 4.1); that is, \(gS_{x_0}(x) = S_{x_0}g(x)\), which implies that \(S_{x_0}gS_{x_0} = g\). Hence, \(g \in G^\Theta\), and so \(H \subseteq G^\Theta\). Therefore, \((G, H)\) is a symmetric pair.

To show that this map is an isometry, it suffices to show that it maps geodesics to geodesics. Since \(G\) acts transitively, we can write any point in \(X\) as \(x = g(x_0)\), then we have \(\Theta(g)(x_0) = S_{x_0}gS_{x_0}(x_0) = S_{x_0}g(x_0) = S_{x_0}(x)\). This shows that the reflection of \(gH\) in \(G/H\) gets sent to the reflection of \(g(x_0)\). Therefore, this is an isometry.

For the remainder of this paper, we shall identify symmetric spaces with their realizations as homogeneous spaces \(G/H\), where \(H\) is the fixed point set of an involution of \(G\). It is important to remember that these are concrete and often familiar spaces, such as spheres, hyperbolic space, projective space, and complex and real Grassmanians.

**Notation:** We will write \(g^\Theta\) for \(\Theta(g)\); this is common notation and, since \(\Theta\) commutes
with inverse map \( g \mapsto g^{-1} \) (\( \Theta \) is a group automorphism), there is no confusion when writing \( g^{-\Theta} = (g^{-1})^{\Theta} = (g^{\Theta})^{-1} \). When the group of isometries is compact, we shall generally denote it by \( U \), with subgroup \( K = U^{\Theta} \). Also, we will often be using matrices of the form

\[
I_k = \begin{pmatrix}
-1_{k\times k} & \frac{1}{1(n-k)\times(n-k)}
\end{pmatrix}
\]

(4.1.12)

where \( 0 \leq k \leq n \). The extreme cases are \( I_0 = 1 \) and \( I_n = -1 \). The size of the matrix should be understood from context.

*Example* 4.1.13. Let \( U = SU(2) \). This acts on \( \mathbb{C}^2 \) in the usual way (through its defining representation) and, since \( U \) is unitary, a transitive and isometric action on \( \mathbb{CP}^1 \) (diffeomorphic to \( S^2 \)) is induced via homogeneous coordinates. Consider the involution \( \text{conj}(I_1) \) of \( U \). The fixed point set \( K = U^\Theta \) consists of matrices in \( SU(2) \) of the form

\[
\left\{ \begin{pmatrix} a & b \\ -b & \bar{a} \end{pmatrix} \right\}^{\Theta} = \left\{ \begin{pmatrix} a & -b \\ b & \bar{a} \end{pmatrix}, \text{det} = 1 \right\} = \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \right\} | \text{det} = 1 \}
\]

(4.1.14)

which is isomorphic to \( U(1) \). This subgroup fixes the point \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), or the north pole upon identification with \( S^2 \). Therefore, \( U/K \) is isomorphic to \( \mathbb{CP}^1 \cong S^2 \) as a symmetric space.

*Example* 4.1.15. Let \( U = SU(4) \), and choose \( \Theta = \text{conj}(I_2) \). The our fixed point set is

\[
K = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\} \in SU(4)
\]

(4.1.16)

which means that \( A, B \in U(2) \). To see what this symmetric space is, note that \( g \in SU(4) \) acts isometrically on \( \mathbb{C}^4 \), and if \( g \in K \cong S(U(2) \times U(2)) \) then the action of \( g \) preserves the subspace given by the first two basis vectors. If we consider this subspace our fixed “point,” then our symmetric space is the so called Grassmanian manifold (or, simply, the Grassmanian) which consists of all \( 2 \) (complex) dimensional subspaces of \( \mathbb{C}^4 \), denoted \( Gr(2, \mathbb{C}^4) \). The previous example, \( \mathbb{CP}^4 \), is also a Grassmanian, namely \( Gr(1, \mathbb{C}^2) \); in general, we have \( \mathbb{CP}^n = Gr(1, \mathbb{C}^{n+1}) \).

### 4.2 A Rough Classification

The involution \( \Theta \) also acts on the Lie algebra through its differential \( d\Theta \), though we will also use \( \Theta \) to refer to this involution. This induces a decomposition of the Lie algebra into \( +1 \) and \( -1 \) eigenspaces, and the \( +1 \) eigenspace is the Lie algebra of the isotropy subgroup.
Let us consider the case where our group is compact. Then \( u \), the Lie algebra of \( U \), is split into a direct sum \( u = \mathfrak{k} \oplus \mathfrak{i} \mathfrak{p} \), where the +1 eigenspace, \( \mathfrak{k} \), is the Lie algebra of the isotropy group \( K \), and \( i \mathfrak{p} \) is the −1 eigenspace.

Let \( g_0 = \mathfrak{k} \oplus \mathfrak{p} \), and let \( g \) be the complexification of \( u \). Note that \( g = \mathfrak{k} \oplus \mathfrak{p} \oplus i \mathfrak{k} \oplus i \mathfrak{p} \) is also the complexification of \( g_0 \). Our setup is illustrated by the following diagram:

\[
\begin{array}{c}
\mathfrak{g} \\
\mathfrak{g}_0 \\
\mathfrak{u} \\
\mathfrak{k} \\
\end{array}
\]

where the arrows denote inclusion maps. Taking connected analytic groups corresponding to these Lie algebras, we get a corresponding diagram of groups:

\[
\begin{array}{c}
G \\
G_0 \\
U \\
G_0/K \\
K \\
U/K \\
\end{array}
\]

The two symmetric spaces \( U/K \) and \( G_0/K \) are considered dual to each other, one compact, and the other noncompact since \( K \) is a maximally compact subgroup in \( G_0 \). (Since \( i \mathfrak{p} \subset \mathfrak{u} \), which is compact, we have that \( \mathfrak{p} \) is noncompact.)

**Example 4.2.3.** Consider our previous example \( U = SU(2), K = U(1) \). The complexification of \( SU(2) \) is \( SL(2, \mathbb{C}) \), and \( G_0 = SU(1, 1) \).

\[
\begin{array}{c}
SL(2, \mathbb{C}) \\
SU(1, 1) \\
\Delta \\
U(1) \\
\mathbb{C}P^1 \\
\end{array}
\]

So the Poincare disk is the noncompact dual symmetric space to \( \mathbb{C}P^1 \). Notice that the upper pairs in the diagram are symmetric spaces, too. For \( g \in SL(2, \mathbb{C}) \), we have

\[
g^{-*} = g \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-*} = \begin{bmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

\[
\Rightarrow g = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in SU(2),
\]
and
\[ g^{-*\Theta} = g \Rightarrow \text{conj}(I_2) \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-*} = \begin{bmatrix} d & c \\ b & a \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]  
(4.2.7)
\[ \Rightarrow g = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \in SU(1,1). \]  
(4.2.8)

This generalizes as the following diagram illustrates:

\[ \begin{array}{ccc}
\begin{array}{c}
g^{-*\Theta} \\
G
\end{array} & \xrightarrow{\sim} & \begin{array}{c}
g^{-*} \\
G_0
\end{array} & \xleftarrow{\sim} & \begin{array}{c}
-X^{*\Theta} \\
-g_0
\end{array} & \xrightarrow{\sim} & \begin{array}{c}
-X^* \\
u
\end{array}
\end{array} \]

(4.2.9)

where \((\cdot)^{-*}\) denotes the Cartan involution of \(G\). Details about this special involution can be found in [Knapp(2002)], chapter VI. Here we present a brief description.

**Definition 4.2.10.** Let \(g_0\) be a real semisimple Lie algebra with Killing form \(B\). An involution \(\Theta\) is called a **Cartan involution** if the form \(B_{\Theta}\), defined by
\[ B_{\Theta}(X,Y) := -B(X,Y^\Theta), \]  
(4.2.11)
is positive definite.

The facts that we need to know about Cartan involutions are summarized in the following

**Theorem 4.2.12.** Let \(G_0\) be a semisimple Lie group, let \(g_0\) be its Lie Algebra viewed as a real Lie algebra, let \(B\) be the Killing form, and let \(k\) be the compact real form of \(g_0\) with corresponding (compact) analytic subgroup \(K\). Then

- \(g_0\) has a Cartan involution unique up to inner automorphism.
- \(k\) is the +1 eigenspace of any Cartan involution, and \(g_0 = k \oplus p\) is the unique Cartan decomposition of \(g_0\) into +1 and -1 eigenspaces.
- \([k,k] \subseteq k, \quad [k,p] \subseteq p, \quad [p,p] \subseteq k\).
- There is an involution of \(G_0\) whose differential is the Cartan involution, and its fixed point set is \(K\).
- The map \(K \times p \to G_0 : (g,X) \mapsto g \exp X\) is a diffeomorphism onto \(G_0\).
- \(G_0\) has finite center if and only if \(K\) is compact; if this is the case, then \(K\) is a maximal compact subgroup in \(G_0\).
We will continue to use the notation \((\cdot)^{-*}\) because for matrix groups, inverse conjugate transpose is a Cartan involution.

The examples above verify this theorem; \(SU(2)\) is maximally compact in \(SL(2, \mathbb{C})\). Note also that \(U(1)\) is maximally compact in \(SU(1, 1)\), so \(\text{conj}(I_1)\) is a Cartan involution for \(SU(1, 1)\). By the construction of our diagram (4.2.9), note that the upper right and lower left involutions will always be Cartan involutions. This leads to a rough classification of symmetric spaces. If \(\Theta\) is an involution of \(G\) with fixed subgroup \(H\), then

<table>
<thead>
<tr>
<th>Cartan inv. is trivial</th>
<th>(H, G) compact</th>
<th>(G/H) Riemannian and compact</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cartan inv. is (\Theta) (not triv.)</td>
<td>(H) compact, (G) not</td>
<td>(G/H) Riemannian and noncompact</td>
</tr>
<tr>
<td>Cartan inv. not trivial, not (\Theta)</td>
<td>(H, G) not compact</td>
<td>(G/H) non-Riemannian</td>
</tr>
</tbody>
</table>

Our examples illustrate this division; \(SU(2)/U(1) \cong \mathbb{C}\mathbb{P}^1\) is Riemannian and compact, and \(SU(1, 1)/U(1) \cong \Delta\) is Riemannian and noncompact.

4.3 Lie Groups are Symmetric Spaces

Let \(G\) be a semisimple Lie group, and consider the symmetric space \((G \times G)/G_d\) where \(G_d\) is the fixed point set of the involution defined by \(\Theta(g_1, g_2) = (g_2, g_1)\). Then we have that \(G_d\) is the subgroup consisting of the diagonal elements of \(G \times G\). Note that \((G \times G)/G_d\) is isomorphic to \(G\) as a group by the isomorphism

\[
\psi : (G \times G)/G_d \to G : (g_1, g_2) \mapsto g_1 g_2^{-1}
\]

This puts a symmetric space structure on \(G\). Reflection of \(g\) across the basepoint \(e\) is given by applying \(\Theta\) to a representative of \(\psi^{-1}g\) (see equation (4.1.8)). Noting that \(\psi[(g, e)] = g\), we have

\[
S_e(g) = \psi[\Theta(g, e)] = \psi[(e, g)] = g^{-1}.
\]

By applying (4.1.9), we have \(S_{g'}(g) = g'g^{-1}g'\).

To see what the geodesics are with respect to this structure, let \(\mathfrak{g}_d\) be the Lie algebra of \(G_d\). First we note that

\[
(X, Y) = \left(\frac{1}{2}(X + Y), \frac{1}{2}(X + Y)\right) + \left(\frac{1}{2}(X - Y), -\frac{1}{2}(X - Y)\right)
\]
is a decomposition of $\mathfrak{g} \oplus \mathfrak{g}$ into $+1$ and $-1$ eigenspaces of $\Theta$. Then we identify the tangent space at the identity of $(G \times G)/G_d$ with the $-1$ eigenspace of $\Theta$, which consists of elements of the form $(X, -X), X \in \mathfrak{g}$. Consider the following diagram.

\[
\begin{array}{c}
(X, -X) \xrightarrow{q} [(X, -X)] \xrightarrow{2X} G \times G \\
\mathfrak{g} \oplus \mathfrak{g} \xrightarrow{q} (\mathfrak{g} \oplus \mathfrak{g})/G_d \xrightarrow{d\psi} \mathfrak{g} \\
G \times G \xrightarrow{q} (G \times G)/G_d \xrightarrow{\psi} G
\end{array}
\]

\[
(\exp X, \exp -X) \xrightarrow{[\exp X, \exp -X]} (\exp X)(\exp -X)^{-1} = \exp(2X)
\]

The vertical maps are $(\exp, \exp)$, $[[\exp, \exp]]$, and $\exp$, respectively. The geodesics in $(G \times G)/G_d$ are induced by the exponential map from $\mathfrak{g} \oplus \mathfrak{g}$ to $G \times G$, so the left square of the diagram commutes. Since $X$ commutes with $-X$, we have $(\exp X)(\exp -X)^{-1} = \exp(2X)$, and so the right side commutes.

Therefore, the geodesics of $G$ under this symmetric space structure are translates of the one parameter subgroups.

4.4 The Cartan Embedding is Totally Geodesic

Consider the symmetric space $G/H$ where $G$ is semisimple, and $\Theta$ is the defining involution.

**Proposition 4.4.1.** (a) The Cartan embedding

\[\phi : G/H \to G : gH \mapsto gg^{-\Theta}\]  

is totally geodesic.

(b) $\text{Im}(\phi) \subset \{g \in G | g^{-1} = g^\circ\}$

**Proof.** For part (a), it suffices to show that the image of a geodesic in $G/H$ under $\phi$ is a geodesic in $G$. Identifying the tangent space of $G/H$ with $i\mathfrak{p}$, we can write a geodesic through $eH$ in $G/H$ as $g(t) = \exp(tX)H$, where $X \in i\mathfrak{p}$. Since $X^\Theta = -X,$
we have

\[ g(t)g(t)^{-\Theta} = \exp(tX) \exp(tX)^{-\Theta} \]  
\[ = \exp(tX) \exp(-tX)^{\Theta} \]  
\[ = \exp(tX) \exp(-tX^{\Theta}) \]  
\[ = \exp(tX) \exp(tX) = \exp(2tX). \]  

Part (b) is a straightforward computation:

\[ (gg^{-\Theta})(gg^{-\Theta})^{\Theta} = gg^{-\Theta}g^{\Theta}g^{-1} = gg^{-1} = e. \]
CHAPTER 5

THE INTERSECTION OF $\phi(U/K)$ WITH A TRIANGULAR DECOMPOSITION OF $G$

Most of this chapter is taken verbatim from section 1 of [Pickrell(2006)]. Throughout this chapter, $U$ will denote a simply connected compact Lie group with complexification $G$, $\Theta$ will denote an involution of $U$ with fixed point set $K$, and $X$ will denote the quotient, $U/K$. This implies that $K$ is connected and $X$ is simply connected (Theorem 8.2 in Chapter VII of [Helgason(1978)]).

5.1 Facts about the Cartan embedding

As mentioned in the previous chapter, we have a Lie algebra diagram

$$
\begin{array}{ccc}
g & = & u \oplus iu \\
g_0 & = & k \oplus p \\
u & = & k \oplus ip \\
k & \nearrow & \searrow \\
\end{array}
$$

where $\Theta$, acting on the Lie algebra level and extended complex linearly to $g$, is $+1$ on $k$ and $-1$ on $p$. We let $(\cdot)^{-\Theta}$ denote the (Cartan) involution for the pair $(G, U)$. The involution for the pair $(G, G_0)$ is given by $\sigma(g) = g^{-\Theta}$.

We have natural maps

$$
\begin{array}{ccc}
K & \rightarrow & U \\
\downarrow & & \downarrow \\
G_0 & \rightarrow & G
\end{array} \rightarrow \begin{array}{ccc}
U & \rightarrow & U/K \\
\downarrow & & \downarrow \\
G & \rightarrow & G/G_0
\end{array}
$$

(5.1.2)

The vertical arrows (given by inclusion) are homotopy equivalences; more precisely, there are diffeomorphisms (polar or Cartan decompositions)

$$
K \times p \rightarrow G_0, \quad U \times iu \rightarrow G, \quad U \times_K i\mathfrak{k} \rightarrow G/G_0.
$$

(5.1.3)

in each case given by the formula $(g, X) \rightarrow g \exp(X)$ (mod $G_0$ in the last case).
The Cartan embedding gives us totally geodesic embeddings of symmetric spaces

\[
\begin{align*}
U/K \xrightarrow{\phi} U &: gK \rightarrow gg^{-\Theta} \\
G/G_0 \xrightarrow{\phi} G &: gG_0 \rightarrow gg^{*\Theta} = gg^{-\sigma}
\end{align*}
\] (5.1.4)

Recall that the symmetric space structures on \(U\) and \(G\) are derived from the Killing form.

A group element \(g \in \phi(G/G_0)\) is of the form \(g = g_1g_1^{-\sigma}\), and satisfies the equation \(g^* = g^{\Theta}\) (i.e. \(gg^{\sigma} = 1\)). Also, \(g^* = g^{\Theta}\) implies that \(Ad(g) \circ \sigma\) is an antilinear involution, and if \(g = g_1g_1^{-\sigma}\), then \(Ad(g) \circ \sigma = Ad(g_1) \circ \sigma \circ Ad(g_1^{-1})\). Hence \(\sigma\) and \(Ad(g) \circ \sigma\) are inner conjugate. These considerations lead to the following well-known

**Proposition 5.1.5.** (a) In terms of \(g \in G\), the images of the symmetric spaces \(U/K\), \(G/G_0\), and \(U\) have the following forms when embedded in \(G\):

\[
\begin{align*}
\phi(U/K) &= \{g^{-1} = g^* = g^{\Theta}\}_0 \rightarrow U = \{g^{-1} = g^*\} \\
\phi(G/G_0) &= \{g^* = g^{\Theta}\}_0 \rightarrow G
\end{align*}
\] (5.1.6)

where \(\cdot\)_0 denotes the connected component containing the identity.

(b) The connected components of \(\{g^{-1} = g^* = g^{\Theta}\}\) are separated by the map which sends \(g\) to the inner conjugacy class of the involution \(\eta = Ad(g) \circ \Theta\), subject to the constraint that \(\eta\) equals \(\Theta\) in \(\text{Out}(U) = Ad(U) / \text{Aut}(U)\). A similar statement applies to \(\{g^* = g^{\Theta}\}\), with \(\sigma\) and antilinear automorphisms of \(G\) in place of \(\Theta\) and involutions of \(K\).

**Remark:** We do not claim that all such \(\eta\) are in the range of the map in part (b). In some cases \((Gr(n, \mathbb{C}^{n+m})\), for instance\) the connected components of \(\{g^{-1} = g^* = g^{\Theta}\}\) are in bijection with classes of involutions of \(U\) \((SU(n + m))\). But for other symmetric spaces \((Gr(p, \mathbb{H}^{p+q})\), for instance\) this is not the case. It is an interesting problem to more completely understand this relationship.

**Proof of 5.1.5.** We first recall why \(\{gg^{\sigma} = 1\}\) is smooth.

Consider the map \(\psi : G \rightarrow G : g \rightarrow gg^{\sigma}\). If we use right translation to identify the tangent space at any point of \(G\) with \(g\), the derivative at \(g\) is given by \(x \rightarrow x + Ad(g)[\sigma(x)]\). Thus \(ker(d\psi|_g)\) is identified with the \(-1\) eigenspace of \(Ad(g) \circ \sigma\) acting on \(g\).

Now suppose \(gg^{\sigma} = 1\). Since \(Ad(g) \circ \sigma\) is an involution, the spectrum of \(Ad(g) \circ \sigma\) is fixed. Thus the dimension of the \(-1\) eigenspace of \(Ad(g) \circ \sigma\) is constant on \(\{gg^{\sigma} = 1\}\). It follows that \(\psi\) has constant rank on the connected components of \(\psi^{-1}(1)\). Since
\( \psi \) is an algebraic map, this implies that \( \{ g^* = g^\Theta \} \) is an embedded submanifold. A similar argument applies to the intersection with \( U \).

The action

\[
G \times \{ gg^\sigma = 1 \} \to \{ gg^\sigma = 1 \} : (g, g_1) \to gg_1g^{*\Theta} \tag{5.1.7}
\]
is isometric (for the symmetric space structure). The constancy of the rank of \( \psi \) on connected components is equivalent to the statement that the dimension of the isotropy subgroup for the action of \( G \) is constant on connected components of \( \{ gg^\sigma = 1 \} \) (in fact this dimension is the same on all components). Then all (open) orbits in each connected component must have the same dimension, so there can be only one orbit per connected component. (If there were more than one orbit in a connected component, at least one of the orbits would be of a lower dimension (a boundary), contradicting the constancy of the dimension of the isotropy subgroup on connected components.) Hence the action of \( G \) must be transitive on connected components. The same applies to the same action of \( U \) on \( \{ g \in U : gg^\Theta = 1 \} \). This implies (a).

For the first part of (b), note that in fact the map

\[
\{ g \in U : g^{-1} = g^{\Theta} \} \to \{ \eta \in Aut(U)^{(2)} : Out(\eta) = Out(\Theta) \} : g \to Ad(g) \circ \Theta \tag{5.1.8}
\]
is a universal covering for each connected component in the image. (For the identity component this covering is understood more intelligibly by identifying the total space with \( U/K \):

\[
Z(U)/Z_K(U) \to U/K \xrightarrow{q} Ad(U) \cdot \Theta \tag{5.1.9}
\]
where \( q(g_1K) = Ad(g_1) \circ \Theta \circ Ad(g_1)^{-1} \); to obtain a similar picture for another component, we replace \( \Theta \) by \( Ad(g) \circ \Theta \), for some \( g \) in the component).

The second part of (b) is similar (We could also note that the inclusion \( \{ g^{-1} = g^* = g^{\Theta} \} \to \{ g^* = g^{\Theta} \} \) is a homotopy equivalence, since we know this is true for the identity component, and we are free to change \( \Theta \) to \( Ad(g) \circ \Theta \); the fact that the \( \pi_0 \)'s are the same is a reflection of the fact that classifying \( \Theta \)'s and classifying \( \sigma \)'s are canonically isomorphic problems (see e.g. 2. of \( \S 6 \), chapter 10 of [Helgason(1978)]).

Example 5.1.10. If \( X = S^n \), then we have

\[
\begin{align*}
G &= Spin(n + 1, \mathbb{C}) \\
G_0 &= Spin(1, n, \mathbb{R}) \\
U &= Spin(n + 1, \mathbb{R}) \\
K &= Spin(n, \mathbb{R})
\end{align*}
\tag{5.1.11}
\]

\( \square \)
where at the level of matrices, $\Theta$ is conjugation by $I_1$, and $g^*$ is the Hermitian conjugate of $g$.

In the special cases $n = 2, 3, 4$ this is equivalent to

$$G = SL(2, \mathbb{C})$$

$$G_0 = SU(1, 1) \quad \quad U = SU(2, \mathbb{C})$$

$$K = U(1)$$

(when we identify $S^2$ with $\mathbb{C}P^1$),

$$G = SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$$

$$G_0 = SL(2, \mathbb{C}) \quad \quad U = SU(2) \times SU(2)$$

$$K = SU(2)$$

(when we identify $S^3$ with $SU(2)$, $K$ is embedded diagonally, and $SL(2, \mathbb{C})$ is embedded in $G$ as $g \to (g, g^{-*})$), and

$$G = Sp(2, \mathbb{C})$$

$$G_0 = Sp(1, 1) \quad \quad U = Sp(2)$$

$$K = Sp(1) \times Sp(1)$$

(when we are identifying $S^4$ with $\mathbb{H}P^1$), respectively. Note that in this last case the 4 dimensional space $S^4$ is being geodesically embedded into the 10 dimensional, rank 2 space $Sp(2, \mathbb{C})/Sp(1, 1)$.

**Example 5.1.15.** Now we focus on $S^2$. Let $U = SU(2, \mathbb{C})$, and define $\Theta$ by $g^\Theta = \text{conj}(I_1)g$. Then $K = U(1)$, and $U/K \cong S^2$. Here we explicitly compute the image of the Cartan embedding.

$$\phi(U/K) = \{g^{-1} = g^* = \text{conj}(I_1)g\}.$$  

(5.1.16)
So,
\[
\begin{bmatrix}
  a & b \\
  -\bar{b} & \bar{a}
\end{bmatrix} \in \phi(U/K) \Rightarrow \begin{bmatrix}
  a & b \\
  -\bar{b} & \bar{a}
\end{bmatrix}^* = \begin{bmatrix}
  -1 & 0 \\
  0 & 1
\end{bmatrix} \begin{bmatrix}
  a & b \\
  -\bar{b} & \bar{a}
\end{bmatrix} \begin{bmatrix}
  1 & 0 \\
  0 & 1
\end{bmatrix} \quad (5.1.17)
\]
\[
\Rightarrow \begin{bmatrix}
  \bar{a} & -b \\
  \bar{b} & a
\end{bmatrix} = \begin{bmatrix}
  a & -b \\
  \bar{b} & \bar{a}
\end{bmatrix} \quad (5.1.18)
\]
\[
\Rightarrow \begin{bmatrix}
  a & b \\
  -\bar{b} & \bar{a}
\end{bmatrix} = \begin{bmatrix}
  z & x + iy \\
  -x + iy & z
\end{bmatrix}, \quad (5.1.19)
\]

where \(x, y, z \in \mathbb{R}\), and the determinant one condition implies that \(x^2 + y^2 + z^2 = 1\). This is clearly a copy of \(S^2\) inside \(SU(2)\).

### 5.2 A compatible Birkhoff decomposition

Fix a maximal abelian subalgebra \(t_0 \subset \mathfrak{k}\). We then obtain \(\Theta\)-stable Cartan subalgebras
\[
\mathfrak{h}_0 = Z_{\mathfrak{g}_0}(t_0) = t_0 \oplus \mathfrak{a}_0, \quad \mathfrak{t} = t_0 \oplus i\mathfrak{a}_0, \quad \text{and} \quad \mathfrak{h} = \mathfrak{h}_0^C \quad (5.2.1)
\]
for \(\mathfrak{g}_0, \mathfrak{u},\) and \(\mathfrak{g}\), respectively, where \(\mathfrak{a}_0 \subset \mathfrak{p}\) (see (6.60) of [Knapp(2002)]). We let \(T_0\) and \(T\) denote the maximal tori in \(K\) and \(U\) corresponding to \(t_0\) and \(\mathfrak{t}\), respectively.

Let \(\Delta\) denote the roots for \(\mathfrak{h}\) acting on \(\mathfrak{g}\); \(\Delta \subset \mathfrak{h}_\mathbb{R}^*\), where \(\mathfrak{h}_\mathbb{R} = \mathfrak{a}_0 \oplus i\mathfrak{t}_0\). We choose a Weyl chamber \(C^+\) which is \(\Theta\)-stable (to prove that \(C^+\) exists, we must show that \(i\mathfrak{t}_0\), the \(+1\) eigenspace of \(\Theta\) acting on \(\mathfrak{h}_\mathbb{R}\), contains a regular element of \(\mathfrak{g}\); this is equivalent to the fact that \(\mathfrak{h}_0\) in (5.2.1) is a Cartan subalgebra). Since \(\sigma = -\cdot^\Theta\) and \((-\cdot)^*\) is the identity on \(\mathfrak{h}_\mathbb{R}\), \(\sigma(C^+) = -C^+\).

Given our choice of \(C^+\), we obtain a \(\Theta\)-stable triangular decomposition \(\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+\), so that \(\sigma(n^\pm) = n^\mp\). Let \(N^\pm = \exp(n^\pm), H = \exp(\mathfrak{h})\), and \(B^\pm = HN^\pm\). We also let \(W = W(G, T)\) denote the Weyl group, \(W = N_U(T)/T \simeq N_G(H)/H\).

At the group level we have the Birkhoff or triangular or LDU decomposition for \(G\),
\[
G = \Pi_{w \in W} \Sigma_w^G, \quad \Sigma_w^G = N^-wHN^+, \quad (5.2.2)
\]
where \(\Sigma_w^G\) is diffeomorphic to \((N^- \cap wN^-w^{-1}) \times H \times N^+\). When we intersect this decomposition with \(\phi(G/G_0)\), and the other spaces in Proposition 5.1.5, we obtain various decompositions. We will first determine the structure of the pieces in the \(\{g^* = g^\Theta\}\) case (thus initially we ignore connectedness issues).

**Proposition 5.2.3.** Fix \(w \in W\).
(a) The intersection \( \{g^* = g^\Theta\} \cap \Sigma_w^G \) is nonempty if and only if there exists \( w \in w \subset N_U(T) \), such that \( w*\Theta = w \); \( w \) is unique modulo the action
\[
T \times \{w \in N_U(T) : w*\Theta = w\} \rightarrow \{w*\Theta = w\} : \lambda, w \rightarrow \lambda w*\Theta.
\]
(5.2.4)

(b) For the action \( B^- \times \{g^* = g^\Theta\} \rightarrow \{g^* = g^\Theta\} : b, g \rightarrow bgb*\Theta \), the stability subgroup is given by
\[
B_w = \{b : w^{-1}bw = \sigma(b)\}
\]
(5.2.5)
\[
\simeq \{l \in N^- : w^{-1}lw = \sigma(l) \in N^+\} \times \{h \in H : h^{w^{-1}} = \sigma(h)\}.
\]
(5.2.6)

(c) The orbits of \( B^- \) in \( \{g^* = g^\Theta\} \cap \Sigma_w^G \) are open and indexed by
\[
\pi_0(\{w \in w : w*\Theta = w\}) \simeq \{w \in w : w*\Theta = w\}/T,
\]
(5.2.7)
where \( T \) acts as in part (a).

(d) The map
\[
N^- \cap N^{-w} \times \{h \in H, L \in N^- \cap N^{+w} : h^{w^{-1}} = h^*, \sigma(L)^{wh} = L^{-1}\} \rightarrow \{g^* = g^\Theta\} \cap \Sigma_w^G
\]
(5.2.8)
given by \( l, h, L \rightarrow ll^{-1}wh(ll^{-1})*\Theta \) is a diffeomorphism onto the connected component containing \( w \). This component is homotopic to the torus \( \exp(\{Ad(w^{-1})\Theta\}_{l = -1}) \).

(e) In particular for \( w = 1 \), the map
\[
N^- \times (T_0^{(2)} \times_{exp(ia_0^{(2)})} exp(ia_0)) \times exp(it_0) \rightarrow \{g^* = g^\Theta\} \cap \Sigma_1^G
\]
(5.2.9)
\[
l, [w, m], a_\phi \rightarrow g = lwm_{a_\phi}l*\Theta
\]
(5.2.10)
is a diffeomorphism, so that the connected components for \( \{g^* = g^\Theta\} \cap \Sigma_1^G \) are indexed by \( T_0^{(2)}/exp(ia_0^{(2)}) \).

Proof. Suppose that \( g \in \Sigma_w^G \). We write \( g = lwhu \), for some \( l \in N^- \), \( w \in w \subset N_U(T) \), \( h \in exp(t_{\mathbb{R}}) \), \( u \in N^+ \). If we additionally require that \( l \in N^- \cap (N^-)^w \), then this decomposition is unique, but we will not require this at the outset.

We have \( g = g*\Theta \) if and only if
\[
lwhu = u^*\Theta(wh)^*\Theta l^*\Theta
\]
(5.2.11)
if and only if
\[
(wh)^*\Theta = (u^\sigma l)(wh)(ul^\sigma)
\]
(5.2.12)
\[
= \{(u^\sigma l) - (u^\sigma l)_{+}\}(wh)(ul^\sigma)
\]
(5.2.13)
\(L = L_- L_+\) denotes the decomposition induced by the diffeomorphism

\[N^- \cap (N^-)^w \times N^- \cap (N^+)^w \to N^- : L_-, L_+ \to L = L_- L_+\]  

(5.2.15)

Thus (5.2.14) holds if and only if

\[(u^\sigma l)_- = 1 = (u^\sigma l)_+^{(wh)^{-1}} u^\sigma\]  

(5.2.16)

and \((wh)^{\ast \Theta} = wh\), or, using the fact that \(h\) is real,

\[h^{\Theta} = h^w \quad \text{and} \quad ww^{\Theta} = 1.\]  

(5.2.17)

Consider part (a). If \(g\) is in the intersection, then we have just seen that \(w\) must satisfy \(w^{\ast \Theta} = w\). Conversely, given a unitary representative \(w\) for \(w\) satisfying \(w^{\ast \Theta} = w\), the intersection contains \(w\) and hence is nonempty. This proves (a).

Part (b) is straightforward.

Now consider (c). We first write \(g \in \{g^\ast = g^\Theta\} \cap \Sigma_w^G\) uniquely as \(lwu\), where \(l \in N^- \cap w N^- w^{-1}\), \(\omega = wh\), and \(u \in N^+\). For the first part of (c) we must prove that we can relax the constraint on \(l\) to arrange for \(u = l^{\ast \Theta}\). We can write

\[g = ll^{-1} \omega^{-1} \{(\omega L^{-1})u\},\]  

(5.2.18)

where \(L \in N^- \cap w N^+ w^{-1}\) is arbitrary. We must prove the existence of \(L\) such that \(ll^{-1} = w^{\ast \Theta} \omega \sigma (L)^{-1} \omega^{-1}\), or

\[u^\sigma l = \omega L^{-\sigma} \omega^{-1} L.\]  

(5.2.19)

The basic fact is that this equation has a unique solution \(L \in N^- \cap N^+\) satisfying \(\omega L^\sigma \omega^{-1} = L^{-1}\), namely \(L = (u^\sigma l)^{1/2}\) (square root has an unambiguous meaning in a simply connected nilpotent Lie group). To see this simply plug such an \(L\) into (5.2.19). We obtain the equation \(L^2 = u^\sigma l\). The fact that \(u^\sigma l\), and its square root, satisfy \(\omega L^\sigma \omega^{-1} = L^{-1}\) follows from (5.2.16), and uniqueness of the square root.

As we remarked previously, the existence of a solution \(L\) proves that \(B^-\) has open orbits. The rest of part (c) is relatively straightforward, using (5.2.17).

The uniqueness of the solution \(L\), subject to the constraint we imposed, implies the first part of (d). The second statement in (d) follows routinely from the first part.

For part (e), to clarify the statement, observe that

\[exp(i a_0)^{(2)} = K \cap exp(i a_0) = T_0^{(2)} \cap exp(i a_0).\]  

(5.2.20)
Now suppose that $w = 1$. In this case $w$ is in the kernel of the homomorphism $T \rightarrow T : w \rightarrow ww^\Theta$, and this equals the subgroup generated by $T_0^{(2)}$ and $exp(ia_0)$. We can modify $w$ by multiplying by something in the image of the homomorphism $T \rightarrow T : \lambda \rightarrow \lambda \lambda^{-\Theta}$. This image is $exp(ia_0)$. Therefore we can choose $w \in T_0^{(2)}$, but this choice is unique only modulo the intersection of $T_0^{(2)}$ and $exp(ia_0)$. This proves (e).

Example 5.2.21. Consider $S^2 \cong SU(2)/U(1)$ as in the previous example. Here we have

$$T_0^{(2)} = \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

(5.2.22)

and $a_0 = \emptyset$. Therefore, there are two connected components of $\phi(U/K) \cap \Sigma_1^G$. We can see this explicitly by factoring

$$\begin{bmatrix} z & x + iy \\ -x + iy & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (-x + iy)/z & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1/z \end{bmatrix} \begin{bmatrix} 1 & (x - iy)/z \\ 0 & 1 \end{bmatrix}$$

(5.2.23)

$$= \begin{bmatrix} 1 & 0 \\ (-x + iy)/z & 1 \end{bmatrix} \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} \begin{bmatrix} |z| & 0 \\ 0 & 1/|z| \end{bmatrix} \begin{bmatrix} 1 & (x - iy)/z \\ 0 & 1 \end{bmatrix}$$

(5.2.24)

So we see that generic elements are those with $|z| \neq 0$. The two connected components of the generic part are the upper and lower hemispheres.

Example 5.2.25. In the group case, $U = K \times K$, where $K$ embeds diagonally (see section 4.3). The image of $t_0$ inside $u$ is $\{ (X, X) : X \in t_0 \}$, while $ia_0 = \{ (X, -X) : X \in t_0 \}$. This implies that the quotient $T_0^{(2)}/exp(ia_0)^{(2)}$ is trivial. Thus in this group case, the set of generic elements (considered in part (e)) is connected, as we already know.

5.3 Identifying open orbits of $B^-$

Notation Given $w$ as in (c) of Proposition 5.2.3, we let $\Sigma_w^{\{g^* = g^\Theta\}}$ denote the corresponding connected component of $\{g^* = g^\Theta\} \cap \Sigma_w^G$ (the $B^-$-orbit of $w$, in the sense of (c) of 5.2.3). If $w \in \phi(G/G_0)$, then we will write $\Sigma_w^{\phi(G/G_0)}$ for this component. We also set $\Sigma_w^{\phi(U/K)} = \phi(U/K) \cap \Sigma_w^{\phi(G/G_0)}$.

Having understood the intersection of $\{g^* = g^\Theta\}$ with the triangular decomposition, we now want to specialize this to the identity component. In an abstract way this is answered by Proposition 5.1.5(b). Concerning open orbits, we have the following
Proposition 5.3.1. Suppose that $w \in N_U(T)$ satisfies $w^{-\Theta} = w$. The following are equivalent:

(a) $\sum_{w}^{g^\Theta} = \phi(G/G_0)$ is an open $B^-$-orbit in the identity component, $\phi(G/G_0)$.

(b) There exists $w_1 \in N_U(T_0)$ such that $\phi(w_1K) = w$.

Hence the open orbits can be parameterized by either $N_U(T_0)/N_K(T_0)$ (the intrinsic point of view), or the set of $w \in T^{(2)}_0/\exp(i\alpha_0)^{(2)}$ such that $Ad(w) \circ \Theta$ is equivalent to $\Theta$ in the sense of Proposition 5.1.5 (the nonintrinsic point of view, as in Proposition 5.2.3(e)).

In addition, the $w_1K$ are exactly the $T_0$ fixed points in $U/K$.

Proof. Determining the possible (open) $B^-$ orbits in $G/G_0$ is equivalent to determining the possible (open) $G_0$ orbits in $B^- \backslash G$. Thus the equivalence of (a) and (b) follows from Theorem 4.6 and its Corollaries in [Wolf and Gray(1968)]. The other statements are obvious. 

In general it apparently remains an open question to systematically obtain representatives for all $B^-$ orbits in $G/G_0$, from the intrinsic point of view (see [Wolf and Zierau(1997)] for the Hermitian symmetric case). In this regard the nonintrinsic point of view of Proposition 5.2.3 seems to have some utility. We exploit this in the next chapter.
CHAPTER 6

CAYLEY COORDINATE CALCULATIONS

In this section we use Cayley coordinates to give specific formulas which describe the intersection of the image of the Cartan embedding \( \phi(U/K) \) with the Birkhoff decomposition (\( \bigoplus \Sigma^G_w \)) of \( G \), when \( U/K \) is a compact Type I symmetric space.

Recall that we have \( K = U^\Theta \), \( \mathfrak{k} = u^\Theta \), and we identify the tangent space at a point of \( U/K \) with \( i\mathfrak{p} \), where \( \mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p} \). For each symmetric space we consider, we will choose a representation of \( U \) in \( SU(n, \mathbb{C}) \) and a corresponding representation of \( \mathfrak{u} \) in \( \mathfrak{su}(n, \mathbb{C}) \). This places \( \phi(U/K) \) inside \( SU(n, \mathbb{C}) \), and lets us use the following convenient coordinate for the symmetric space.

**Definition 6.0.1.** Let \( \Phi \) denote the Cayley map,

\[
\Phi : \mathfrak{u}(n, \mathbb{C}) \to \{ g \in U(n, \mathbb{C}) | -1 \notin \text{spec}(g) \} : X \mapsto g = \frac{1 - X}{1 + X}. \quad (6.0.2)
\]

This map is invertible,

\[
\Phi^{-1}(g) = X = \frac{1 - g}{1 + g}, \quad (6.0.3)
\]

so the image of \( \Phi \) is open and dense in \( U(n, \mathbb{C}) \). When \( \Theta \) can be represented as conjugation by a matrix (such as when \( \Theta \) is inner), we show that the Cayley map, restricted to \( i\mathfrak{p} \), is a coordinate for almost all of \( \phi(U/K) \). Then we give conditions on \( X \) for \( g \) to be a generic element, and explicitly compute \( d \) (as in \( g = ldu \)) in that case, working out many examples in detail.

A complication arises in the fact that the Cayley map does not necessarily send trace zero matrices to determinant one matrices. We show that when \( \Theta \) can be represented as conjugation by a matrix (such as when \( \Theta \) is inner), we show that the Cayley map, restricted to \( i\mathfrak{p} \), is a coordinate for almost all of \( \phi(U/K) \). Then we give conditions on \( X \) for \( g \) to be a generic element, and explicitly compute \( d \) (as in \( g = ldu \)) in that case, working out many examples in detail.

For these classical cases, \( \mathfrak{u} \) is either \( \mathfrak{su}(n, \mathbb{C}) \), \( \mathfrak{so}(n, \mathbb{C}) \), or \( \mathfrak{sp}(n, \mathbb{C}) \). Our convention through these calculations is to choose our representation of \( \mathfrak{u} \) so that \( \mathfrak{h} \) consists of diagonal matrices in \( \mathfrak{su}(n, \mathbb{C}) \), \( \mathfrak{n}^+ \) and \( \mathfrak{n}^- \) are strictly upper and lower triangular matrices, respectively, and \( \Theta \) preserves all three subspaces. The representations that achieve this are the following:

\[
\mathfrak{so}(n) \cong \{ X \in \mathfrak{su}(n, \mathbb{C}) | -X^r = X \}, \quad (6.0.4)
\]
\[ sp(n) \cong \{ X \in \mathfrak{su}(2n, \mathbb{C}) \mid -X^\tau = \text{conj}(I_n)X \}, \tag{6.0.5} \]

where \( \tau \) denotes antitranspose (reflection across the antidiagonal). See the appendix for more information about these representations.

### 6.1 Main Result

We begin with a lemma.

**Lemma 6.1.1.** Let \( f : \text{Mat}(n \times n) \to \text{Mat}(n \times n) \) be a differentiable function that respects addition of matrices and multiplication of scalars (so \( f(cg_1 + g_2) = cf(g_1) + f(g_2) \)). Then \( df_g(\cdot) = f(\cdot) \) for all \( g \in \text{Mat}(n \times n) \).

**Proof.** Let \( g \in \text{Mat}(n \times n), X \in T_g(\text{Mat}(n \times n)) \cong \text{Mat}(n \times n) \). Then

\[
\begin{align*}
   df|_g(X) &= \frac{d}{dt} \bigg|_{t=0} f(g + tX) \\
   &= \lim_{t \to 0} \frac{f(g + tX) - f(g)}{t} \\
   &= f(X)
\end{align*}
\tag{6.1.2}
\tag{6.1.3}
\tag{6.1.4}
\]

**Proposition 6.1.5.** Suppose we can represent \( \Theta \in \text{Aut}^2(U) \) as conjugation by a matrix. Then \( \Phi(i\mathfrak{p}) \subset \phi(U/K) \).

**Proof.** We must show that

\[
\Phi(i\mathfrak{p}) \subset \{ g \in U \mid g^\Theta = g^{-1} \}_0 = \phi(U/K). \tag{6.1.6}
\]

Let \( X \in i\mathfrak{p} \), and \( g = \Phi(X) \). Conjugation respects addition and scalar multiplication of matrices, so by Lemma 6.1.1, we have

\[
\begin{align*}
   \left( \frac{1 - X}{1 + X} \right)^\Theta &= \frac{(1 - X)^\Theta}{(1 + X)^\Theta} \\
   &= \frac{1^\Theta - X^\Theta}{1^\Theta + X^\Theta} \\
   &= \frac{1 + X}{1 - X} \\
   &= \left( \frac{1 - X}{1 + X} \right)^{-1}
\end{align*}
\tag{6.1.7}
\tag{6.1.8}
\tag{6.1.9}
\tag{6.1.10}
\]
Therefore, since $\Phi(\mathfrak{p})$ is connected, by continuity of $\Phi$ we have

$$\Phi(\mathfrak{p}) \subset \{ g \in U(n, \mathbb{C}) | g^\Theta = g^{-1} \}. \quad (6.1.11)$$

Furthermore since the determinant is fixed under conjugation, we have

$$\det(g) = \det(g^\Theta) = \det(g^{-1}) = \frac{1}{\det(g)} \quad (6.1.12)$$

which implies that $\det(g) = \pm 1$. By continuity of $\Phi$, and since $0 \in \mathfrak{p}$, we have $\det(g) = 1$. So,

$$\Phi(\mathfrak{p}) \subset \{ g \in SU(n, \mathbb{C}) | g^\Theta = g^{-1} \}. \quad (6.1.13)$$

All that remains to be shown is that $\Phi(\mathfrak{p}) \subset U$. In the case where $U = SU(n)$, we are done. For $U = SO(n)$, note that $\tau \in Aut^{(2)}(SU(n, \mathbb{C}))$ respects addition of matrices and multiplication of scalars, and $\mathfrak{so}(n, \mathbb{C})$ is the $-1$ eigenspace of $\tau$. Then Lemma 6.1.1 applies, and by (6.1.10) we have

$$\Phi(\mathfrak{p}) \subset \{ g \in SU(n, \mathbb{C}) | g^\tau = g^{-1} \} = U. \quad (6.1.14)$$

The case where $U = Sp(n)$ follows similarly, since $\mathfrak{sp}(n, \mathbb{C})$ is the $-1$ eigenspace of $\text{conj}(I_n) \circ \tau$. This completes the proof. \qed

Now that our coordinate is in place, we use it to examine the intersection of $\phi(U/K)$ and the Birkhoff decomposition.

\textbf{Notation}: Let $A$ be an $m \times n$ matrix, and let $1 \leq k \leq m, 1 \leq l \leq n$. Recall that we use $A_{i_1, \ldots, i_k, j_1, \ldots, j_l}$ to denote the $k \times l$ submatrix consisting of the intersection of rows $i_1, \ldots, i_k$ and columns $j_1, \ldots, j_l$ of $A$. We use the shorthand $A_{(k)} = A_{1, \ldots, k, 1, \ldots, k}$ for the principal blocks of $A$. Also, recall that

$$I_k := \begin{bmatrix} -1_{k \times k} & 0 \\ 0 & 1_{(n-k) \times (n-k)} \end{bmatrix}. \quad (6.1.15)$$

The size of $I_k$ will be understood from the context. It is useful to think of multiplication on the left (resp. right) by $I_k$ as changing the sign of the first $k$ rows (resp. columns) of the matrix.

\textbf{Theorem 6.1.16}. For $X \in \mathfrak{p} \subset \mathfrak{su}(n, \mathbb{C})$, its image $g := \frac{1-X}{1+X}$ under the Cayley map is in $\phi(U/K) \cap \Sigma^\Theta_1$ if and only if $\det(1 + I_k X) \neq 0$ for $1 \leq k \leq n - 1$. If this
is the case, then factoring \( g = ldu \), we have

\[
d = \begin{bmatrix}
\frac{\det(1+I_1X)}{\det(1+X)} & \frac{\det(1+I_2X)}{\det(1+X)} & \cdots & \frac{\det(1+I_nX)}{\det(1+X)} \\
\frac{\det(1+I_2X)}{\det(1+X)} & \frac{\det(1+I_3X)}{\det(1+X)} & \cdots & \frac{\det(1+I_{n-1}X)}{\det(1+X)} \\
\frac{\det(1+I_3X)}{\det(1+X)} & \frac{\det(1+I_4X)}{\det(1+X)} & \cdots & \frac{\det(1+I_nX)}{\det(1+X)} \\
\frac{\det(1+I_{n-1}X)}{\det(1+X)} & \frac{\det(1+I_nX)}{\det(1+X)} & \cdots & \frac{\det(1+I_nX)}{\det(1+X)} \\
\end{bmatrix}
\]

(6.1.17)

**Proof.** Recall that \( g(k) \) denotes the \( k \times k \) principal block of \( g \), and that \( \det g = 1 \). By Gaussian elimination we have

\[
d = \begin{bmatrix}
\det(g(1)) & \frac{\det(g(2))}{\det(g(1))} & \cdots & \frac{\det(g(n))}{\det(g(n-1))} \\
\frac{\det(g(1))}{\det(g(2))} & \frac{\det(g(3))}{\det(g(2))} & \cdots & \frac{\det(g(n))}{\det(g(n-1))} \\
\frac{\det(g(1))}{\det(g(2))} & \frac{\det(g(3))}{\det(g(2))} & \cdots & \frac{\det(g(n))}{\det(g(n-1))} \\
\frac{\det(g(1))}{\det(g(2))} & \frac{\det(g(3))}{\det(g(2))} & \cdots & \frac{\det(g(n))}{\det(g(n-1))} \\
\end{bmatrix},
\]

(6.1.18)

and since the denominators that appear in the entries of \( l \) and \( u \) also consist of \( \det(g(k)) \) for various \( k \), we have that \( g \) is generic exactly when \( \det(g(k)) \) are all non-zero (see section 2.1). Therefore, it suffices to show that

\[
\det(g(k)) = \frac{\det(1+I_kX)}{\det(1+X)}
\]

(6.1.19)

for \( 1 \leq k \leq n \). For all such \( k \), we have

\[
\frac{\det(1+I_kX)}{\det(1+X)} = \det\left(\frac{1+I_kX}{1+X}\right)
\]

(6.1.20)

\[
= \det\left(\frac{1}{\det(1+X)}(1+I_kX)\ adj(1+X)\right)
\]

(6.1.21)

\[
= \frac{1}{\det(1+X)^{n-k}} \ det\ ((1+I_kX)\ adj(1+X)).
\]

(6.1.22)

The product \((1+I_kX)\ adj(1+X)\) is a matrix that has the first \( k \) rows equal to those of \((1-X)\ adj(1+X)\), and the last \( n-k \) rows equal to those of \((1+X)\ adj(1+X)\) =
\[
\det(1 + X) \cdot 1_{n \times n}. \text{ So, (6.1.22) equals}
\]
\[
\frac{1}{(\det(1 + X))^n} \det \left[ \begin{array}{c|c}
(1 - X) \adj(1 + X) & * \\
0 & \det(1 + X) \cdot 1_{n-k \times n-k}
\end{array} \right] \quad (6.1.23)
\]
\[
= \det \left[ \begin{array}{c|c}
\frac{1-X}{1+X} & * \\
0 & 1_{n-k \times n-k}
\end{array} \right] \quad (6.1.24)
\]
\[
= \det g(k) \quad (6.1.25)
\]

\[\square\]

It should be noted that we could have proven the theorem by examining \((\adj(1 + X)) (1 + XI_k)\). We would then have spoken of expanding determinants along the appropriate columns instead of rows.

This theorem says that the diagonal entries of \(d\) are ratios of determinants expressed in terms of \(X\). For some examples, we can write these determinants as a sum of smaller determinants. From Fredholm theory, we have the following equality (see page 33 of [Simon(2005)]).

\[
\det(1 + A) = \sum_{j=0}^{\infty} \Tr(\land^j(A)) \quad (6.1.26)
\]

This formula is for trace class operators on a separable Hilbert space, possibly infinite dimensional. For us, this sum terminates. When \(j = 0\), the summand is 1, and for \(1 \leq j \leq n\) we have

\[
\Tr(\land^j(A)) = \sum_{1 \leq i_1 < \cdots < i_j \leq n} \langle \land^j(A)e_{i_1} \wedge \cdots \wedge e_{i_j}, e_{i_1} \wedge \cdots \wedge e_{i_j} \rangle \quad (6.1.27)
\]
\[
= \sum_{1 \leq i_1 < \cdots < i_j \leq n} \det A_{i_1 \cdots i_j, i_1 \cdots i_j} \quad (6.1.28)
\]

This says the individual entries of \(d\) are ratios of sums of determinants of submatrices of \(X\). Specifically,

\[
d_{kk} = \frac{\det(1 + I_kX)}{\det(1 + I_{k-1}X)} \quad (6.1.29)
\]
\[
= \frac{1 + \sum_{j=1}^{n} \sum_{1 \leq i_1 < \cdots < i_j \leq n} \det(I_kX)_{i_1 \cdots i_j, i_1 \cdots i_j}}{1 + \sum_{j=1}^{n} \sum_{1 \leq i_1 < \cdots < i_j \leq n} \det(I_{k-1}X)_{i_1 \cdots i_j, i_1 \cdots i_j}} \quad (6.1.30)
\]
and, as a special case,\[\det(1 + X) = 1 + \sum_{j=1}^{n} \sum_{1 \leq i_1 < \cdots < i_j \leq n} \det X_{i_1 \ldots i_j, i_1 \ldots i_j}. \tag{6.1.31}\]

Note that each \(\det(1 + I_k X)\) has essentially the same summands as \(\det(1 + X)\). The presence of \(I_k\) merely changes the sign of some of the summands.

**Example 6.1.32.** Let \(U/K = SU(2)/U(1) \cong S^2\). Our coordinate is\[ip = \left\{ \begin{bmatrix} 0 & -z \\ z & 0 \end{bmatrix} \right\}, \tag{6.1.33}\]
and from (6.1.30) above, we have\[d = \begin{bmatrix} \frac{1-|z|^2}{1+|z|^2} & 0 \\ 0 & \frac{1+|z|^2}{1-|z|^2} \end{bmatrix}. \tag{6.1.34}\]

From example 5.2.21, we saw that there are two connected components of \(\phi(U/K) \cap \Sigma_1^G\), indexed by\[T_0^{(2)} = \{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \}. \tag{6.1.35}\]
We can use our coordinate to pick out these \(w\). Obviously, \(\Phi(0) = +1\), but one can see that \(-1\) is not in the image of \(\Phi\). However, letting \(|z|\) tend to infinity, we see that\[\lim_{|z| \to \infty} \Phi \left( \begin{bmatrix} 0 & -z \\ z & 0 \end{bmatrix} \right) = \lim_{|z| \to \infty} \begin{bmatrix} \frac{1-|z|^2}{1+|z|^2} & 0 \\ 0 & \frac{1+|z|^2}{1-|z|^2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{6.1.36}\]

We can make this calculation for two reasons. First, elements of \(T_0^{(2)}\) are themselves diagonal, so we can just consider the diagonal term in the limit. Second, \(\phi(U/K)\) is compact, so \(\lim_{|z| \to \infty} \Phi(X)\) exists, and is in \(\phi(U/K)\).

We use this idea to prove the following theorem, which is motivated by Proposition 5.2.3. This will allow us to identify the connected components of \(\phi(U/K) \cap \Sigma_1^G\).

**Theorem 6.1.37.** The elements of \(T_0^{(2)}\) present in \(\phi(U/K) \cap \Sigma_1^G\) are exactly those obtained in the asymptotics of the Cayley coordinate by letting each summand of \(\det(1 + X)\) dominate. That is,\[T_0^{(2)} = \{ \lim_{t \to \infty} \Phi(X_t) \} \tag{6.1.38}\]
provided the limit exists, where \(X_t \in ip\) has some subset of the entries equal to \(t\), and the rest zero.
Proof. First we will show which asymptotics yield which elements of $T^{(2)}_0$. Then we show that there are no others in $\phi(U/K) \cap \Sigma_1^G$.

Let $\Phi(X) = ldu$. Then $d_{kk}$ is given by equation (6.1.30). Suppose we are able to allow the summand $\det X_{i_1\ldots i_j, i_1\ldots i_j}$ dominate for a fixed choice of $\{i_1, \ldots, i_j\}$. Since $I_k X$ differs from $I_{k-1} X$ only in the sign of the entries involving the $k^{th}$ row, the sign of $\det(I_k X)_{i_1\ldots i_j, i_1\ldots i_j}$ will differ from that of $\det(I_{k-1} X)_{i_1\ldots i_j, i_1\ldots i_j}$ if and only if $k \in \{i_1, \ldots, i_j\}$. Therefore the element in

$$T^{(2)}_0 \subset \left\{ \begin{pmatrix} \pm 1 & \cdots & \pm 1 \\ \vdots & \ddots & \vdots \\ \pm 1 & \cdots & \pm 1 \end{pmatrix} \right\}$$

(6.1.39)

that arises from letting this summand dominate is the diagonal matrix $w$ with

$$w_{ll} = \begin{cases} -1 & \text{if } l \in \{i_1, \ldots, i_j\} \\ 1 & \text{otherwise.} \end{cases}$$

(6.1.40)

To see that other asymptotics do not yield elements of $T^{(2)}_0$, suppose two terms of the same total degree satisfy

$$c = \det X_{i_1\ldots i_j, i_1\ldots i_j} = \lambda \det X_{i_1\ldots i_j, l_1\ldots l_j}$$

(6.1.41)

where $\{i_1, \ldots, i_j\} \neq \{l_1, \ldots, l_j\}$, and $\lambda \neq 0$. Let $k \in \{i_1, \ldots, i_j\}\backslash\{l_1, \ldots, l_j\}$. Then, letting $t \to \infty$, we have

$$d_{kk} \approx \frac{c + \lambda c}{c + \lambda c} \to \frac{1 + \lambda}{1 + \lambda} \neq \pm 1.$$ 

(6.1.42)

Many examples will be presented in the following section.

6.2 Explicit Calculations of $d$ When $\Theta$ Is Inner

We will now compute $\det(1 + I_k X)$ for $X \in iP$, for each class of classical compact Type I symmetric spaces where $\Theta$ is inner.

Note: When we write these inner automorphisms as conjugation by a matrix in $U$, for ease of computation, we will not generally rescale the matrix, since

$$\text{conj}(\lambda A) X = \lambda A X \lambda^{-1} A^{-1} = AXA^{-1} = \text{conj}(A) X.$$ 

(6.2.1)
**Symmetric space.** \( SU(n + m)/S(U(n) \times U(m)) \cong Gr(n, \mathbb{C}^{n+m}) \)

Involution: \( \Theta : X \mapsto \text{conj}(I_n)(X) \) (Inner)

First noting that \( \det(1 + I_nX) = \det(1 - X) = \det(1 + X) \), we easily obtain an intrinsic formula for \( d \) in the coordinate \( X \).

\[
d = \prod_{k=1}^{n+m-1} \left( \frac{\det(1 + I_kX)}{\det(1 + X)} \right)^{h_k} 
\]

where \( h_k = e_{kk} - e_{k+1,k+1} \) are coroots in \( \mathfrak{h}_\mathbb{R} \). Note also that

\[
d = (\det(1 + X))^{-\sum h_k} \cdot \prod_{k=1}^{n+m-1} (\det(1 + I_kX))^{h_k},
\]

emphasizing the role of \( \det(1 + I_kX) \). It is the vanishing of these determinants that distinguish generic elements from non-generic, because \( \det(1 + X) \) is never zero (\( X \) does not have a \(-1\) eigenvalue).

Now we take a closer look at the matrices involved. In this case \( \Theta \) is conjugation by \( I_n = \begin{bmatrix} -1_{n \times n} & 0 \\ 0 & 1_{m \times m} \end{bmatrix} \). The fixed point set \( K \) consists of determinant 1 matrices of the form

\[
\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \quad \text{where } A \in U(n, \mathbb{C}), B \in U(m, \mathbb{C}).
\]  

(6.2.4)

Obviously, \( K \cong S(U(n) \times U(m)) \). Then we have \( \mathfrak{g} \) as trace=0 matrices of the form

\[
\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \quad \text{where } M_1 \in u(n, \mathbb{C}), M_2 \in u(m, \mathbb{C}),
\]

(6.2.5)

and \( \mathfrak{p} \) consists of

\[
X = \begin{bmatrix} 0 & -Z^* \\ Z & 0 \end{bmatrix} \quad \text{where } Z \in \text{Mat}_{m \times n}(\mathbb{C}) \cong \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m).
\]  

(6.2.6)

It should be noted that this \( Z \) is the graph coordinate for \( Gr(n, \mathbb{C}^{n+m}) \).

Since \( X \) has this block form with diagonal blocks equal to 0, we have \( \det(1 + X) = \det(1 + ZZ^*) \), and

\[
\det(1 + I_kX) = \begin{cases} 
\det(1 + ZI_kZ^*) & \text{if } 1 \leq k \leq n \\
\det(1 - I_{k-n}ZZ^*) & \text{if } n \leq k \leq n + m
\end{cases}
\]

(6.2.7)

The submatrices \( X_{i_1\ldots i_j, i_1\ldots i_j} \) also have diagonal block equal to 0, say

\[
X_{i_1\ldots i_j, i_1\ldots i_j} = \begin{bmatrix} 0 & -Y^* \\ Y & 0 \end{bmatrix},
\]

(6.2.8)
where $Y$ is a square submatrix of $Z$. This yields the following formula:

$$\det(1 + I_k X) = 1 + \sum_Y \pm \det YY^*,$$

(6.2.9)

where $Y$ ranges over all square submatrices of $Z$. The sign of the summand is negative if and only if an odd number of rows of the submatrix $X_{i_1\ldots i_j, i_1\ldots i_j}$ are among the first $k$ rows of $X$. In particular,

$$\det(1 + X) = 1 + \sum_Y \det YY^*.$$

(6.2.10)

To more clearly illustrate the pattern of the signs of these summands, we look at some specific examples.

**Example 6.2.11.** $Gr(1, \mathbb{C}^{n+1}) \cong \mathbb{CP}^n$

Here, $X$ has the form

$$X = \begin{bmatrix}
0 & -\bar{z}_{11} \cdots - \bar{z}_{n1} \\
\bar{z}_{11} & \ddots \\
\vdots & \ddots & 0 \\
\bar{z}_{n1} & & & \\
\end{bmatrix}$$

(6.2.12)

Applying formula (6.2.9) above, we see that

$$\det(1 + I_k X) = 1 + \sum_{i=1}^{k-1} z_{i1} \bar{z}_{i1} - \sum_{j=k}^{n} z_{j1} \bar{z}_{j1},$$

(6.2.13)

and so

$$d_{kk} = \begin{cases}
1 - \frac{\sum_{i=1}^{n} |z_{i1}|^2}{\sum_{j=1}^{n} |z_{j1}|^2}, & k = 1 \\
1 + \frac{\sum_{i=1}^{k-1} |z_{i1}|^2 - \sum_{j=k}^{n} |z_{j1}|^2}{\sum_{j=k-1}^{n} |z_{j1}|^2}, & 2 \leq k \leq n
\end{cases}$$

(6.2.14)

For $\mathbb{CP}^1 \cong S^2$, we have seen that $a = d_{11}$ is the absolute value of the height function of the unit sphere under the standard embedding into $\mathbb{R}^3$. In Cayley coordinates, $\mathbb{HP} \cong \mathbb{C}$, and the above formula yields $a = (1 - |z|^2)/(1 + |z|^2)$, which is the height function in stereographic coordinates (under projection from the south pole) or the $z$ coordinate on the Riemann sphere.
Example 6.2.15. $Gr(2, \mathbb{C}^5)$

Here, $X$ has the form

$$X = \begin{bmatrix}
0 & -\tilde{z}_{11} & -\tilde{z}_{21} & -\tilde{z}_{31} \\
-\tilde{z}_{12} & -\tilde{z}_{22} & -\tilde{z}_{32} \\
\tilde{z}_{11} & \tilde{z}_{12} & 0 \\
\tilde{z}_{21} & \tilde{z}_{22} & 0 \\
\tilde{z}_{31} & \tilde{z}_{32} & 0
\end{bmatrix} \quad (6.2.16)$$

Applying formula (6.2.9) we have

$$d_{11} = \frac{1 + \left( -|\tilde{z}_{11}|^2 + |\tilde{z}_{12}|^2 \right) - \det Z_{12,12} Z_{12,12}^*}{1 + \left( +|\tilde{z}_{11}|^2 + |\tilde{z}_{12}|^2 \right) + \det Z_{12,12} Z_{12,12}^*} \quad (6.2.17)$$

$$d_{22} = \frac{1 + \left( -|\tilde{z}_{21}|^2 - |\tilde{z}_{22}|^2 \right) - \det Z_{13,12} Z_{13,12}^*}{1 + \left( +|\tilde{z}_{21}|^2 + |\tilde{z}_{22}|^2 \right) + \det Z_{13,12} Z_{13,12}^*} \quad (6.2.18)$$

$$d_{33} = \frac{1 + \left( +|\tilde{z}_{11}|^2 + |\tilde{z}_{12}|^2 \right) - \det Z_{13,12} Z_{13,12}^*}{1 + \left( +|\tilde{z}_{21}|^2 - |\tilde{z}_{22}|^2 \right) + \det Z_{13,12} Z_{13,12}^*} \quad (6.2.19)$$

$$d_{44} = \frac{1 + \left( +|\tilde{z}_{11}|^2 + |\tilde{z}_{12}|^2 \right) - \det Z_{13,12} Z_{13,12}^*}{1 + \left( +|\tilde{z}_{21}|^2 - |\tilde{z}_{22}|^2 \right) + \det Z_{13,12} Z_{13,12}^*} \quad (6.2.20)$$
The summands $|z_{ij}|^2$ are arranged to illustrate the pattern that $\det(1 + I_k X)$ follows. When $k = 0$, we have $\det(1 + X)$; all signs are positive. As $k$ increases from 1 to $n$, all summands involving $k^{th}$ column of $Z$ (the $k^{th}$ row of $-\hat{Z}^*$) change sign. Then, as $k$ increases from $n + 1$ to $n + m$, all summands involving the $(n - k)^{th}$ row of $Z$ change sign. This is due to the alternating property of the determinant.

As we have seen, the elements of $T_0^{(2)}$ present in $\phi(U/K)$ (modulo $\exp(iA_0)^{(2)}$) index the connected components of $\phi(U/K) \cap \Sigma^c_1$. This notion of watching which summands change sign as we increment $k$ is helpful in finding these elements, as seen in Theorem 6.1.37. This lets us explicitly identify these $w \in T_0^{(2)}$.

It is obvious that $\Phi(0) = 1_{(n+m)\times(n+m)}$ is always present. The others can be found by letting each summand of $\det(1 + I_k X)$ in turn dominate the others. This can be done because $U/K$ is compact, and because $\Phi$ maps onto almost all of $\phi(U/K)$.

Applying Theorem 6.1.37 to $SU(n + m)/S(U(n) \times U(m)) \cong Gr(n, \mathbb{C}^{n+m})$, we see the elements of $T_0^{(2)}$ in $\phi(U/K)$ are those with the same number of $-1$’s in the upper right block as in the upper left block. This is a consequence of the block structure of $X$. The only summands that are non-zero are $\det(I_k X)_{i_1,\ldots,i_j}$ where an identical number of $\{i_1,\ldots,i_j\}$ are between 1 and $n$, as are between $n + 1$ and $n + m$.

This also explains why all summands are of even degree. For the above condition to hold, $j$ must be even. We can be very specific about this process. To obtain the element in $T_0^{(2)}$ with $-1$ in the $i_1,\ldots,i_j$ diagonal entries and $+1$ elsewhere, we just let the summand $\det(I_k X)_{i_1,\ldots,i_j}$ dominate. This can be done in a number of ways, for instance, by letting the entries

$$t = z_{i_1/2+1} = z_{i_1/2+1} = \cdots = z_{i_j/2}$$

(6.2.22)

tend to infinity equally. For the example $Gr(2, \mathbb{C}^5)$ above, by setting $z_{11} = z_{32}$ and letting them tend to infinity, $\det Z_{13,12} Z_{13,12}^*$ dominates, and we can see that

$$d = \begin{bmatrix} -1 & \cdot \cdot \cdot & -1 \\ \cdot\cdot\cdot & -1 \\ \cdot\cdot\cdot & 1 & -1 \end{bmatrix} \in T_0^{(2)}$$

(6.2.23)
is present in $\phi(U/K)$. So in this example, $\phi(Gr(2, \mathbb{C}^5) \cap \Sigma_I^G$ has 10 connected components.

The next two classes of symmetric spaces are subspaces of the complex Grassmanian. Therefore, the involution is the same (though $I_n$ may need to be scaled to be in the group), and the formulas are similar.

**Symmetric space.** $SO(2n)/U(n)$

Involution: $\Theta : X \mapsto \text{conj}(I_n)(X)$  (Inner)

To obtain an intrinsic formula for $d$, we need to account for the different pattern of the coroots in the orthogonal case. We have

$$d = \prod_{k=1}^{n-1} \left( \frac{\det(1 + I_k X)}{\det(1 + X)} \right)^{h_k} \left( \frac{\det(1 + I_n X)}{\det(1 + X)} \right)^{\frac{1}{2}(-h_{n-1} + h_n)}$$  (6.2.24)

where

$$h_k = \begin{cases} e_{k,k} - e_{k+1,k+1} + e_{2n-k,2n-k} - e_{2n-k+1,2n-k+1}, & 1 \leq k < n \\ e_{k-1,k-1} + e_{kk} - e_{k+1,k+1} - e_{k+2,k+2}, & k = n \end{cases}$$  (6.2.25)

are coroots in $\mathfrak{h}_R$. In matrix form,

$$h_k = \begin{bmatrix} \ddots & & & & 1 & -1 \\ & \ddots & & & -1 \\ & & \ddots & & 1 & -1 \\ & & & \ddots & 1 & -1 \\ & & & & \ddots \end{bmatrix}, \quad h_n = \begin{bmatrix} \ddots & & & & 1 & -1 \\ & \ddots & & & 1 & -1 \\ & & \ddots & & 1 & -1 \\ & & & \ddots & 1 & -1 \end{bmatrix}. \quad (6.2.26)$$

This is a subspace of $Gr(n, \mathbb{C}^{2n})$, so the involution (of $\mathfrak{so}(2n)$) is still $\Theta : X \mapsto \text{conj}(I_n)(X)$. Indeed, since $\mathfrak{so}(2n) \cong \{X \in \mathfrak{su}(n, \mathbb{C})| -X^\tau = X\}$ we see that

$$i \mathfrak{p} = \{X = \begin{bmatrix} 0 & -Z^\tau \\ Z & 0 \end{bmatrix} | Z = -Z^\tau \}. \quad (6.2.27)$$

So this symmetric space has dimension $n(n-1)/2$.

**Example 6.2.28.** Let $n = 3$, then $U/K \cong SO(6)/U(3)$. 
In this case, we have

\[
X = \begin{bmatrix}
0 & -z_{11} & -z_{21} & 0 \\
-z_{12} & 0 & z_{21} & 0 \\
z_{11} & z_{12} & 0 & 0 \\
z_{21} & 0 & -z_{12} & 0 \\
0 & -z_{21} & -z_{11} & 0 \\
\end{bmatrix},
\]  

(6.2.29)

and using formula (6.2.9), our result simplifies to

\[
d = \text{diag} \left( \frac{1 - |z_{11}|^2 + |z_{12}|^2 - |z_{21}|^2}{1 + |z_{11}|^2 + |z_{12}|^2 + |z_{21}|^2}, \right.
\]

(6.2.30)

\[
\left. (1 + |z_{11}|^2 + |z_{12}|^2 - |z_{21}|^2)(1 + |z_{11}|^2 - |z_{12}|^2 - |z_{21}|^2), \right.
\]

(6.2.31)

\[
\left. \frac{1 - |z_{11}|^2 - |z_{12}|^2 - |z_{21}|^2}{1 + |z_{11}|^2 + |z_{12}|^2 - |z_{21}|^2}, \right.
\]

(6.2.32)

\[
\left. 1 + |z_{11}|^2 + |z_{12}|^2 - |z_{21}|^2, \right.
\]

(6.2.33)

\[
\left. \frac{1 - |z_{11}|^2 + |z_{12}|^2 - |z_{21}|^2}{1 + |z_{11}|^2 + |z_{12}|^2 + |z_{21}|^2}, \right.
\]

(6.2.34)

\[
\left. \frac{1 + |z_{11}|^2 + |z_{12}|^2 + |z_{21}|^2}{1 - |z_{11}|^2 - |z_{12}|^2 - |z_{21}|^2}, \right.
\]

(6.2.35)

Applying Theorem 6.1.37 to \(SO(2n)/U(n)\), the elements of \(T_0^{(2)}\) in \(\phi(U/K)\) are those with the same number of \(-1\)'s in the upper right block as in the upper left block.

As with the complex Grassmanian, when \(j\) is odd, all summands arising from determinants of submatrices \(X_{i_1...i_j,j_1...j_k}\) are zero, due to the diagonal block structure. Furthermore, since the entries on the antidiagonal are zero (corresponding to non-compact roots), every parameter of \(X\) contributes to four entries of \(X\) due to the symmetry of \(Sp(n)\), and therefore yields a term of degree 4. Hence, the only non-zero determinants are of submatrices of size \(4m \times 4m\), giving us elements of \(T_0^{(2)}\) with \(4m\) entries of \(-1\), an equal number in each diagonal block.

**Symmetric space.** \(Sp(n)/U(n)\)

Involution: \(\Theta: X \mapsto \text{conj}(I_n)(X)\) \hspace{1cm} (Inner)

Here, elements of \(h\) satisfy \(-X^\tau = \text{conj}(I_n)X\), so

\[
d = \prod_{k=1}^n \left( \frac{\det(1 + I_k X)}{\det(1 + X)} \right)^{h_k} \]

(6.2.36)
where
\[ h_k = \begin{cases} 
  e_{kk} - e_{k+1,k+1} + e_{2n-k,2n-k} - e_{2n-k+1,2n-k+1}, & 1 \leq k < n \\
  e_{kk} - e_{k+1,k+1}, & k = n 
\end{cases} \quad (6.2.37) \]
are coroots in \( h_\mathbb{R} \).

This is another subspace of \( Gr(n, \mathbb{C}^{2n}) \). Since \( sp(n) \cong \{ X \in su(2n, \mathbb{C}) | -X^\tau = \text{conj}(I_n)X \} \), we have
\[
i p = \{ X = \begin{bmatrix} 0 & -Z^* \\ Z & 0 \end{bmatrix} | Z = Z^\tau \}. \quad (6.2.38)\]
So this symmetric space has dimension \( n(n+1)/2 \).

**Example 6.2.39.** Let \( n = 2 \), then \( U/K \cong Sp(2)/U(2) \), and
\[
X = \begin{bmatrix} 0 & -\bar{z}_{11} & -\bar{z}_{21} \\ \bar{z}_{11} & \bar{z}_{12} & -\bar{z}_{21} \\ \bar{z}_{21} & \bar{z}_{11} & 0 \end{bmatrix}, \quad (6.2.40)
\]
and using formula (6.2.9), we have
\[
d = \text{diag}
\begin{pmatrix}
1 + |z_{12}|^2 - |z_{21}|^2 - \det ZZ^*, \\
1 + 2|z_{11}|^2 + |z_{12}|^2 + |z_{21}|^2 + \det ZZ^*, \\
1 - 2|z_{11}|^2 - |z_{12}|^2 - |z_{21}|^2 + \det ZZ^*, \\
1 + |z_{12}|^2 - |z_{21}|^2 - \det ZZ^*, \\
1 - 2|z_{11}|^2 - |z_{12}|^2 - |z_{21}|^2 + \det ZZ^*, \\
1 + 2|z_{11}|^2 + |z_{12}|^2 + |z_{21}|^2 + \det ZZ^* \\
1 + |z_{12}|^2 - |z_{21}|^2 - \det ZZ^*
\end{pmatrix}, \quad (6.2.41) - (6.2.44)
\]

Applying Theorem 6.1.37, we see that all elements of \( T_0^{(2)} \) in \( \phi(Sp(n)/U(n)) \) are present. Unlike in \( SO(2n)/U(n) \), we have non-zero entries on the antidiagonal of \( X \) (corresponding to the presence of non-compact roots), so our summands contain determinants of submatrices of size \( j \times j \) for all even \( j \). Therefore, all \( w \) with an even number of \( -1 \) entries that are symmetric across the antidiagonal occur.

**Symmetric space.** \( Sp(p + q)/Sp(p) \times Sp(q) \cong Gr(p, \mathbb{H}^{p+q}) \)

**Involution:** \( \Theta : X \mapsto \text{conj}(I)X \quad (\text{Inner}) \)
where
\[ \hat{I} = \begin{bmatrix} 1_{p \times p} & 0 & 0 & 0 \\ 0 & -1_{q \times q} & 0 & 0 \\ 0 & 0 & -1_{q \times q} & 0 \\ 0 & 0 & 0 & 1_{p \times p} \end{bmatrix}. \] (6.2.45)

Again, elements in \( h_\mathbb{R} \) satisfy \(-X^\tau = \text{conj}(I_n)X\), so
\[ d = \prod_{k=1}^{p+q} \left( \frac{\det(1 + I_kX)}{\det(1 + X)} \right)^{h_k} \] (6.2.46)

where
\[ h_k = \begin{cases} e_{kk} - e_{k+1,k+1} + e_{2n-k,2n-k} - e_{2n-k+1,2n-k+1}, & 1 \leq k < p + q \\ e_{kk} - e_{k+1,k+1}, & k = p + q \end{cases} \] (6.2.47)

are coroots in \( h_\mathbb{R} \).

So, this makes
\[ \mathfrak{e} = \left\{ \begin{bmatrix} A & 0 & 0 & B \\ 0 & C & D & 0 \\ 0 & -D^* & -C^* & 0 \\ -B^* & 0 & 0 & -A^* \end{bmatrix} \mid A = -A^*, B = B^*, C = -C^*, D = D^* \right\}. \] (6.2.48)

That is, \([A \begin{bmatrix} B \\ -B^* \\ -A^* \end{bmatrix}] \in \mathfrak{sp}(p)\), and \([C \begin{bmatrix} D \\ -D^* \\ -C^* \end{bmatrix}] \in \mathfrak{sp}(q)\). So \( \mathfrak{e} \cong \mathfrak{sp}(p) \oplus \mathfrak{sp}(q) \).

Then we have
\[ \mathfrak{i}_p = \left\{ \begin{bmatrix} 0 & Z_1 & Z_2 & 0 \\ -Z_1^* & 0 & 0 & Z_2^* \\ -Z_2^* & 0 & 0 & -Z_1^* \\ 0 & -Z_2^{*\tau} & Z_1^{*\tau} & 0 \end{bmatrix} \mid Z_1, Z_2 \in \text{Mat}_{p \times q}(\mathbb{C}) \cong \mathcal{L}(\mathbb{C}^p, \mathbb{C}^p) \right\} \] (6.2.49)

Example 6.2.50. Let \( p = q = 1 \); then
\[ X = \begin{bmatrix} 0 & z_1 & z_2 & 0 \\ -z_1^* & 0 & 0 & -z_2^* \\ -z_2^* & 0 & 0 & -z_1^* \\ 0 & -z_2^{*\tau} & z_1^{*\tau} & 0 \end{bmatrix} \] (6.2.51)
and
\[
d = \text{diag} \left( \frac{1 - |z_1|^2 - |z_2|^2}{1 + |z_1|^2 + |z_2|^2}, \frac{1 + |z_1|^2 - |z_2|^2 + (|z_1|^2 + |z_2|^2)^2}{1 - (|z_1|^2 + |z_2|^2)^2}, \frac{1 + |z_1|^2 - |z_2|^2 + (|z_1|^2 + |z_2|^2)^2}{1 - (|z_1|^2 + |z_2|^2)^2}, \frac{1 + |z_1|^2 + |z_2|^2}{1 - |z_1|^2 - |z_2|^2} \right). \tag{6.2.52}
\]

To establish the reality of \(d\) for arbitrary \(p\) and \(q\), we will use the following lemma.

**Lemma 6.2.56.** If \(X \in \mathbb{R}^{ip}\) is of the form
\[
\begin{bmatrix}
0 & Z_1 & 0 \\
-Z_1^* & 0 & Z_2 \\
0 & -Z_2^* & 0
\end{bmatrix}, \tag{6.2.57}
\]
then \(d\) is real.

**Proof.** It suffices to prove that the determinant of every submatrix \(X_{i_1...i_j, i_1...i_j}\) is real. Such a submatrix is necessarily of the block form
\[
\begin{bmatrix}
0 & A_{a \times b} & 0 \\
-A_{b \times a}^* & 0 & B_{b \times c} \\
0 & -B_{c \times b}^* & 0
\end{bmatrix}, \quad a, b, c \leq 0, a + b + c = j. \tag{6.2.58}
\]

This comes from selecting the first \(a\) rows of \(\{i_1...i_j\}\) from the first block rows of \(X\), the next \(b\) rows from the middle block, and the last \(c\) rows from the last block. Then we have
\[
\det \begin{bmatrix}
0 & A_{a \times b} & 0 \\
-A_{b \times a}^* & 0 & B_{b \times c} \\
0 & -B_{c \times b}^* & 0
\end{bmatrix} = \pm \det \begin{bmatrix}
0 & 0 & A_{a \times b} \\
-A_{b \times a}^* & B_{b \times c} & 0 \\
0 & 0 & -B_{c \times b}^*
\end{bmatrix} = \pm \det \begin{bmatrix}
-A_{b \times a}^* & B_{b \times c} & 0 \\
0 & 0 & A_{a \times b} \\
0 & 0 & -B_{c \times b}^*
\end{bmatrix}. \tag{6.2.59}
\]
This determinant is non-zero if and only if \( a + c = b \). Furthermore, letting \( Y \) be the square matrix \([-A^*_{b \times a}|B_{b \times c}]\), this determinant is equal to
\[
\pm \begin{vmatrix} Y & 0 \\ 0 & -Y^* \end{vmatrix} = \pm |YY^*| \in \mathbb{R},
\]
(6.2.62)
since \( YY^* \) is positive definite.

It follows immediately that \( d \) is real in the case of \( Sp(p+q)/Sp(p) \times Sp(q) \).

Applying Theorem 6.1.37, we see that the elements of \( T^{(2)}_0 \) present are those with an equal even number of \(-1\) in the center \( Sp(q) \) part as in the outer \( Sp(p) \) part, and symmetric across the anti-diagonal. This follows from a similar line of reasoning as in the proof of Lemma 6.2.56 above.

**Symmetric space.** \( SO(p+q)/SO(p) \times SO(q) \cong Gr(p, \mathbb{R}^{p+q}) \)

**Involution:** \( \Theta : X \mapsto conj(\hat{I})X \) (Inner or Outer depending on the parity of \( p \) and \( q \)).

The form of \( \hat{I} \) also depends on the parity of \( p \) and \( q \). This will be explained in two cases.

**Case 1:** \( p \) and \( q \) are not both odd.

Without loss of generality, assume \( p \) is even. Then \( \hat{I} \) has the block form
\[
\hat{I} = \begin{bmatrix}
-1_{\frac{p}{2} \times \frac{p}{2}} & 0 & 0 \\
0 & 1_{q \times q} & 0 \\
0 & 0 & -1_{\frac{p}{2} \times \frac{p}{2}}
\end{bmatrix}.
\]
(6.2.63)
The involution is inner in this case, since \( \hat{I} \in SO(p+q) \). Now we compute our intrinsic formula. The formula depends on the parity of \( q \). If \( q \) is even (and so \( p+q \) is even) we have
\[
d = \prod_{k=1}^{\frac{p+q}{2}-1} \left( \frac{\det(1+I_kX)}{\det(1+X)} \right)^{h_k} \left( \frac{\det(1+I_{\frac{p+q}{2}}X)}{\det(1+X)} \right)^{\frac{1}{2}(-h(\frac{p+q}{2} - 1) + h \frac{p+q}{2})}
\]
(6.2.64)
where coroots in \( \mathfrak{h}_\mathbb{R} \) are equal to
\[
h_k = \begin{cases} 
  e_{kk} - e_{k+1,k+1} + e_{p+q-k,p+q-k} - e_{p+q-k+1,p+q-k+1}, & 1 \leq k < \frac{p+q}{2} \\
  e_{k-1,k-1} + e_{kk} - e_{k+1,k+1} - e_{k+2,k+2}, & k = \frac{p+q}{2}.
\end{cases}
\]
(6.2.65)
As matrices, the \( h_k \) look as in (6.2.26). If \( p+q \) is odd, the formula is
\[
d = \prod_{k=1}^{\frac{p+q}{2}-1} \left( \frac{\det(1+I_kX)}{\det(1+X)} \right)^{h_k} \left( \frac{\det(1+I_{\frac{p+q}{2}}X)}{\det(1+X)} \right)^{\frac{1}{2}h(\frac{p+q}{2})}
\]
(6.2.66)
and the coroots are
\[ h_k = \begin{cases} 
  e_{kk} - e_{k+1,k+1} + e_{p+q-k,p+q-k} - e_{p+q-k+1,p+q-k+1}, & 1 \leq k < \left\lfloor \frac{p+q}{2} \right\rfloor, \\
  2e_{kk} - 2e_{k+2,k+2}, & k = \left\lfloor \frac{p+q}{2} \right\rfloor. 
\end{cases} \]

Note that, when \( p + q \) is odd, the last root always has the form
\[ h_{\left\lfloor \frac{p+q}{2} \right\rfloor} = \begin{bmatrix} \ddots & 2 \\ & 0 \\ & & -2 \\ & & & \ddots \end{bmatrix}. \]

Conjugating by \( \hat{I} \) in \( \mathfrak{so}(p + q) \cong \{ X \in \mathfrak{su}(p + q, \mathbb{C}) | X = -X^\tau \} \) we get
\[ \mathfrak{k} = \left\{ \begin{bmatrix} A & 0 & B \\ 0 & C & 0 \\ -B^* & 0 & -A^* \end{bmatrix} | A = -A^*, B = -B^\tau, C = -C^* = -C^\tau \right\}. \]

That is, \( C \in \mathfrak{so}(q) \), and \( \begin{bmatrix} A & B \\ -B^* & -A^* \end{bmatrix} \in \mathfrak{so}(p) \). So \( \mathfrak{k} \cong \mathfrak{so}(p) \oplus \mathfrak{so}(q) \). Then we have
\[ i\mathfrak{p} = \left\{ \begin{bmatrix} 0 & Z & 0 \\ -Z^* & 0 & -Z^\tau \\ 0 & Z^{\tau*} & 0 \end{bmatrix} | Z \in \mathbb{Mat}_2^{\times q}(\mathbb{C}) \cong \mathcal{L}(\mathbb{C}^q, \mathbb{C}^q) \right\}. \]

**Example 6.2.71.** Let \( q = 1 \); then \( U/K \cong SO(2n + 1)/SO(1) \times O(2n) \cong \mathbb{RP}^{2n} \).

Here, \( X \) has the form
\[ X = \begin{bmatrix} \vdots & \vdots \\ z_1 & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ 0_{n \times n} & \vdots & 0_{n \times n} \\ -z_n & \ldots & -z_1 \\ \xi_1 & \vdots & \xi_1 \\ 0_{n \times n} & \vdots & 0_{n \times n} \\ \xi_n & \vdots & \xi_n \end{bmatrix}. \]

This is similar to \( \mathbb{CP}^n \), but each column and row is “doubled,” so each summand will appear twice, perhaps with some sign changes depending on \( k \). The result is
this:

\[
\det(1 + I_k X) = \begin{cases} 
1 + 2 \sum_{i=1}^{n-k} z_i \bar{z}_i, & 1 \leq k \leq n \\
1, & k = n + 1 \\
1 + 2 \sum_{i=1}^{k-(n+1)} z_i \bar{z}_i, & n + 1 \leq k \leq 2n + 1.
\end{cases}
\] (6.2.73)

For \( \mathbb{RP}^6 \), we have

\[
d = diag\left(\frac{1 + 2|z_1|^2 + 2|z_2|^2}{1 + 2|z_1|^2 + 2|z_2|^2}, \frac{1}{1 + 2|z_1|^2}, 1, \frac{1 + 2|z_1|^2}{1 + 2|z_1|^2}, \frac{1 + 2|z_1|^2 + 2|z_2|^2}{1 + 2|z_1|^2}, \frac{1 + 2|z_1|^2 + 2|z_2|^2 + 2|z_3|^2}{1 + 2|z_1|^2 + 2|z_2|^2}\right).
\] (6.2.74)

Example 6.2.81. Let \( p = 4 \) and \( q = 2 \); then \( U/K = SO(6)/SO(4) \times SO(2) \). Here, \( X \) has the form

\[
X = \begin{bmatrix}
0 & 0 & z_{11} & z_{12} & 0 & 0 \\
0 & 0 & z_{21} & z_{22} & 0 & 0 \\
-z_{11} & -z_{21} & 0 & 0 & -z_{22} & -z_{12} \\
-z_{12} & -z_{22} & 0 & 0 & -z_{21} & -z_{11} \\
0 & 0 & z_{12} & z_{11} & 0 & 0 \\
0 & 0 & z_{21} & z_{22} & 0 & 0
\end{bmatrix}.
\] (6.2.82)

It follows from Lemma 6.2.56 that \( d \) is real for \( SO(p+q)/SO(p) \times SO(q) \) in this inner case.

Applying Theorem 6.1.37, we see that the elements of \( T_0^{(2)} \) present are those with an equal even number of \(-1\) in the center \( SO(q) \) part as in the outer \( SO(p) \) part, and symmetric across the anti-diagonal. This follows from a similar line of reasoning as in the proof of Lemma 6.2.56.

We have seen that for \( SO(2n+1)/SO(1) \times O(2n) \cong \mathbb{RP}^{2n} \), the middle term of \( d \) is always 1. Since there must be an equal number of \(-1\) in the inner \( O(1) \) block
as in the outer $O(2n)$ block, there can be no $-1$ at all. Thus, the only $w$ present for $\mathbb{RP}^{2n}$ is the identity matrix, and so there is only one connected component for its generic part.

Case 2: $p$ and $q$ are both odd.

Now $\tilde{I}$ has the form

$$\tilde{I} = \begin{bmatrix}
\frac{1}{2} \times \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} \times \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2} \times \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} \times \frac{1}{2}
\end{bmatrix}.$$  \hspace{1cm} (6.2.83)

This is a special case. The automorphism $\Theta$ is an outer automorphism, since $\lambda \tilde{I} \not\in \text{SO}(p+q)$ for all $\lambda \in \mathbb{C}$. Nevertheless, since we can represent $\Theta$ as conjugation by a matrix, we still have $\Phi(ip) \subset \phi(U/K)$ by the same argument as in the proof of Proposition 6.1.5.

Here, $p+q$ is even, so for our intrinsic formula we have

$$d = \prod_{k=1}^{\frac{p+q}{2}-1} \left( \frac{\det(1 + I_k X)}{\det(1 + X)} \right)^{h_k} \left( \frac{\det(1 + I_{\frac{p+q}{2}} X)}{\det(1 + X)} \right)^{\frac{1}{2}(-h(p+q-1)+h_{\frac{p+q}{2}})} \hspace{1cm} (6.2.84)$$

where coroots in $\mathfrak{h}_\mathbb{R}$ are equal to

$$h_k = \begin{cases} 
\varepsilon_{kk} - \varepsilon_{k+1,k+1} + \varepsilon_{p+q-k,p+q-k} - \varepsilon_{p+q-k+1,p+q-k+1}, & 1 \leq k < \frac{p+q}{2} \\
\varepsilon_{k-1,k-1} + \varepsilon_{k,k} - \varepsilon_{k+1,k+1} - \varepsilon_{k+2,k+2}, & k = \frac{p+q}{2}.
\end{cases} \hspace{1cm} (6.2.85)$$

As matrices, the $h_k$ look as in (6.2.26). This intrinsic formula looks the same as when $p$ and $q$ are both even, but the coordinate $X$ will look much different since $\Theta$ is an outer automorphism, so $\det(1 + I_k X)$ takes on quite a different form. The
fixed point set is

\[
\mathfrak{t} = \left\{ \begin{array}{cccccc}
A & 0 & u & u & 0 & B \\
0 & C & -v & v & D & 0 \\
-u^* & v^* & 0 & 0 & -v^\tau & -u^\tau \\
-u^* & -v^* & 0 & 0 & v^\tau & -u^\tau \\
0 & -D^* & v^{*\tau} & -v^{*\tau} & -C^\tau & 0 \\
-B^* & 0 & u^{*\tau} & u^{*\tau} & 0 & -A^{*\tau}
\end{array} \right\} \tag{6.2.86}
\]

\[|A = -A^*, C = -C^*, B = -B^{\tau}, D = -D^{\tau}|. \tag{6.2.87}\]

That is,

\[
\begin{bmatrix}
A & u & B \\
-u^* & 0 & -u^{\tau} \\
-B^* & u^{*\tau} & -A^{*\tau}
\end{bmatrix} \in \mathfrak{so}(p), \quad \text{and} \quad
\begin{bmatrix}
C & v & D \\
-v^* & 0 & -v^{\tau} \\
-D^* & v^{*\tau} & -C^{\tau}
\end{bmatrix} \in \mathfrak{so}(q).
\]

Then we have

\[
i\mathfrak{p} = \left\{ \begin{array}{cccccc}
0 & Z_1 & w_1 & -w_1 & Z_2 & 0 \\
-Z_1^* & 0 & w_2 & w_2 & 0 & -Z_2^{*\tau} \\
-w_1^* & -w_2^* & i\sigma & 0 & -w_2^{*\tau} & w_1^{*\tau} \\
-w_1^* & -w_2^* & 0 & -i\sigma & -w_2^{*\tau} & -w_1^{*\tau} \\
-Z_2^* & 0 & w_2^{*\tau} & w_2^{*\tau} & 0 & -Z_1^{*\tau} \\
0 & Z_2^{*\tau} & -w_1^{*\tau} & w_1^{*\tau} & Z_1^{*\tau} & 0
\end{array} \right\} \tag{6.2.88}
\]

\[|Z_1, Z_2 \in \text{Mat}_{\frac{p-1}{2}\times\frac{q-1}{2}}(\mathbb{C}), \ w_1 \in \mathbb{C}_{-\frac{p-1}{2}}, \ w_2 \in \mathbb{C}_{-\frac{q-1}{2}}|. \tag{6.2.89}\]
Example 6.2.90. Let $q = 1$; then $U/K \cong SO(2n+2)/S(O(1) \times O(2n+1)) \cong \mathbb{R}P^{2n+1}$. Here, $X \in \mathfrak{sp}$ has the form

\[
X = \begin{bmatrix}
0_{n \times n} & z_n & -z_n \\
 0_{n \times n} & : & : \\
 -z_n & \cdots & -z_n \\
 z_n & \cdots & z_n \\
 -z_n & \cdots & -z_n \\
-\bar{z}_n & \cdots & -\bar{z}_n \\
\end{bmatrix}
\]  

(6.2.91)

This is similar to the even-dimensional projective plane $\mathbb{R}P^{2n}$, but because of the duplication of $z_1, \ldots, z_n$, each $|z_j|^2$ has a coefficient of 4 when it appears. The presence of the phase is due to the fact that $\Theta$ is an outer automorphism. The result is

\[
\det(1 + I_k X) = \begin{cases}
1 + s^2 + 4 \sum_{i=1}^{n-k} z_i \bar{z}_i, & 1 \leq k < n \\
(1 + is)(1 - is) = 1 + s^2, & k = n \\
(1 - is)(1 - is) = 1 - 2is - s^2, & k = n+1 \\
(1 - is)(1 + is) = 1 + s^2, & k = n+2 \\
1 + s^2 + 4 \sum_{i=1}^{k-(n+2)} z_i \bar{z}_i, & n + 2 < k \leq 2n + 2.
\end{cases}
\]  

(6.2.92)

For $\mathbb{R}P^{5}$, we have

\[
d = \text{diag}\left(\begin{array}{c}
\frac{1 + is^2 + 4|z_1|^2}{1 + s^2 + 4|z_1|^2 + 4|\bar{z}_2|^2}, \\
\frac{1 + s^2}{1 + s^2 + 4|z_1|^2}, \\
1 + is, \\
1 - is, \\
\frac{1 + is}{1 - is}, \\
\frac{1 + s^2 + 4|z_1|^2}{1 + s^2}, \\
\frac{1 + s^2 + 4|z_1|^2 + 4|z_2|^2}{1 + s^2 + 4|z_1|^2}
\end{array}\right)
\]  

(6.2.93) (6.2.94) (6.2.95) (6.2.96) (6.2.97) (6.2.98)

The lowest dimensional example that is not a real projective space is $SO(6)/SO(3) \times SO(3)$, and already $\det(1 + I_k X)$ turns out to be quite compli-
cated. It is easy enough to compute with a program such as Maple, but a concise presentation is elusive.

The elements of $T^{(2)}_0$ in $\phi(U/K)$ in this case are difficult to see, but, according to Proposition 5.2.3, the connected components of $\phi(U/K) \cap \Sigma^G_1$ are indexed by $T^{(2)}_0/\exp(i\mathbf{a}_0)^{(2)}$. This is the first symmetric space we have encountered so far where $i\mathbf{a}_0 \cap i\mathbf{p} \neq \emptyset$.

So, ignoring contributions from the middle two rows (and columns), we can assume that the middle two diagonal entries are $+1$ for all $w$ we consider. By Theorem 6.1.37, this means when we observe the behavior of $\det X_{i_1\ldots i_j, i_1\ldots i_j}$ we assume that the middle two rows (columns) are not among $\{i_1, \ldots, i_j\}$. This puts us in the same situation as in the previous case, and so the elements in $T^{(2)}_0/\exp(i\mathbf{a}_0)^{(2)}$ are those with the same number of $-1$ in the inner $O(q)$ part (excepting the middle two entries) as in the outer $O(p)$ part, and that are symmetric across the anti-diagonal.

For $SO(2n+2)/S(O(1) \times O(2n+1)) \cong \mathbb{R}P^{2n+1}$, this has a similar effect as with $\mathbb{R}P^{2n}$; there is one connected component of $\phi(\mathbb{R}P^{2n+1}) \cap \Sigma^G_1$.

6.3 The Case when $\Theta$ is an Outer Automorphism

These two symmetric spaces are $SU(n)/SO(n)$ and $SU(2n)/Sp(n)$. (We dealt with $SO(p+q)/SO(p) \times SO(q)$ when $p$ and $q$ are both odd in the previous section.) The issue here is that Proposition 6.1.5 does not apply, since for these spaces, we cannot say that $\det(g) = 1$ for $g \in \Phi(i\mathbf{p})$. So, in general, the Cayley map does send $i\mathbf{p}$ into $\phi(SU(n)/K)$.

Nevertheless, if we relax the determinant condition, the formula could be applied to the homogeneous spaces $U(n)/SO(n)$ and $U(2n)/Sp(n)$. Perhaps this could be of some use to studying the large rank limit of these spaces.
CHAPTER 7
APPLICATIONS TO INTEGRATION

Let $U/K$ be a compact Hermitian symmetric space, and consider the integral

$$\int_{\phi(U/K)} a_{\phi}^{-i\lambda} (7.0.1)$$

where $\lambda \in \mathfrak{h}_R^*$, relative to the decomposition $\mathfrak{u}^C = \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. It has been shown that this can be evaluated via the Duistermaat-Heckman method by taking advantage of the Poisson structure on the dual symmetric space $G_0/K$. However, our formulas for $a_{\phi}$ allow us to evaluate this integral using elementary techniques.

As an example, we do this for $\mathbb{CP}^n$. First consider $U/K = SU(2) \cong \mathbb{CP}^1$. When $\Phi(X) = g$, and $g \in \phi(U/K)$ is generic, we have

$$a_{\phi}(g) = \left( \frac{\det(1 + I_1X)}{\det(1 + X)} \right)^{h_1} = \left( \begin{array}{cc} 1 - |z|^2 & 0 \\ 0 & 1 + |z|^2 \end{array} \right)^{1/|z|^2} (7.0.2)$$

Since the rank of $SU(2)$ is 1, we have $\lambda = s\alpha$ where

$$\alpha(h) = 2, \quad h = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right). (7.0.4)$$

So this integral, where $dV$ is induced from normalized Haar measure, is

$$\int_{\phi(U/K)} a_{\phi}(g)^{-i\lambda} dV = \int_{\phi(U/K)} a_{\phi}(g)^{-i\alpha} dV (7.0.5)$$

Restricting ourselves to $\Sigma^1_{\phi(U/K)} = \Phi(\{|z|^2 < 1\})$, the connected component of $\phi(U/K) \cap \Sigma^C_1$ containing the identity, we have

$$\int_{\Sigma^1_{\phi(U/K)}} a_{\phi}(g)^{-i\alpha} dV = \int_{|z|^2 < 1} \left( \frac{1 - |z|^2}{1 + |z|^2} \right)^{h_1^{-i\alpha}} (1 + |z|^2)^c d\mu(z) (7.0.7)$$

$$= \int_{|z|^2 < 1} (1 + |z|^2)^{-2is} \left( \frac{c}{1 + |z|^2} \right)^2 d\mu(z). (7.0.8)$$
where $dm$ is Lebesgue measure on $i\mathbb{P} \cong \mathbb{C}$, and $c$ is the normalization constant $\frac{2}{\pi} = \frac{1}{2\pi}$. The factor $4/(1 + |z|^2)^2$ comes from the “round metric” on $\mathbb{CP}^1$, which one can compute by stereographic projection, and $2\pi$ is half of the volume (surface area) of the Riemann sphere realized as the unit sphere. (The upper hemisphere is $\Sigma^\phi(U/K)$.) After some substitutions, we have

$$c \int_{|z|^2 < 1} \left( \frac{1 - |z|^2}{1 + |z|^2} \right)^{-2is} \frac{1}{(1 + |z|^2)^2} dm(z)$$

(7.0.9)

$$= c \int_0^1 \int_0^{2\pi} \left( \frac{1 - r^2}{1 + r^2} \right)^{-2is} \frac{1}{(1 + r^2)^2} r d\theta dr$$

(7.0.10)

$$= 2\pi c \int_0^1 \left( \frac{1 - r^2}{1 + r^2} \right)^{-2is} \frac{1}{(1 + r^2)^2} r dr$$

(7.0.11)

$$= \pi c \int_0^1 \left( \frac{1 - u}{1 + u} \right)^{-2is} \frac{1}{(1 + u)^2} du$$

(7.0.12)

$$= \frac{\pi c}{2} \int_0^1 v^{-2is} dv$$

(7.0.13)

$$= \frac{1}{1 - 2is} v^{1 - 2is} |_0^1$$

(7.0.14)

$$= \frac{1}{1 - 2is} = \langle \delta, \alpha \rangle.$$  

(7.0.15)

This is the result we expect. The reason for the intermediate substitution in (7.0.12) will become clear in the induction step.

Now we proceed by induction on $n$. For $X \in i\mathbb{P}$, we have

$$X = \begin{bmatrix} 0 & -Z^* \\ Z & 0 \end{bmatrix} = \begin{bmatrix} 0 & -z_1 & \cdots & -z_n \\ z_1 & \vdots \\ \vdots & 0 \\ z_n \end{bmatrix},$$

(7.0.16)

and so, when $\Phi(X) = g$, and $g \in \phi(U/K)$ is generic, we have

$$a_\phi(g) = \prod_{k=1}^n \left( \frac{\det(1 + I_kX)}{\det(1 + X)} \right)^{h_k}$$

(7.0.17)

$$= \det(1 + I_1X)^{h_1} \cdots \det(1 + I_nX)^{h_n} \det(1 + X)^{-\Sigma h_i}$$

(7.0.18)
where

\[ h_1 = \begin{bmatrix} 1 & -1 \\ 0 & \ddots \\ \vdots & \ddots & 0 \end{bmatrix}, \ldots, h_n = \begin{bmatrix} 0 & \ddots \\ \vdots & \ddots & 0 \\ & & 1 & -1 \end{bmatrix}. \]  

Choosing positive roots \( \alpha_j \) such that \( \alpha_j(h_j) = 2 \), and taking \( \lambda = \sum s_j \alpha_j \), we have

\[
\int_{\phi(U/K)} a_\phi(g)^{-i\lambda} \, dV = \int_{\phi(U/K)} a_\phi(g)^{-i\sum s_j \alpha_j} \, dV 
\]  

(7.0.20)

In Cayley coordinates, the measure \( dV \) on \( \mathbb{C}P^n \) is

\[
dV = \frac{c_1}{\det(1 + X)^{1+n}} \, dm(Z) 
\]  

(7.0.21)

up to a normalization constant \( c_1 \). Restricting to \( \Sigma_{\phi(U/K)}^1 \), we have

\[
\int_{\Sigma_{\phi(U/K)}^1} a_\phi(g)^{-i\sum s_j \alpha_j} \, dV 
\]  

(7.0.22)

\[
= c_1 \int_{\phi^{-1}(\Sigma_{\phi(U/K)}^1)} \det(1 + I_1 X)^{-2i s_1} \ldots \det(1 + I_n X)^{-2i s_n} \det(1 + X)^{-1-n+2i \sum_{j=1}^n s_j} \, dm(Z) 
\]  

(7.0.23)

\[
= c_1 \int_{|Z|<1} \left( 1 - \sum_{j=1}^n |z_j|^2 \right)^{-2i s_1} \left( 1 + |z_1|^2 - \sum_{j=2}^n |z_j|^2 \right)^{-2i s_2} \ldots 
\]  

(7.0.24)

\[
\ldots \left( 1 + \sum_{j=1}^{n-1} |z_j|^2 - |z_n|^2 \right)^{-2i s_n} \left( 1 + |Z|^2 \right)^{-1-n+2i \sum_{j=1}^n s_j} \, dm(Z) 
\]  

(7.0.25)

Now substitute \( u_j = |z_j|^2 \), and integrate over the \( \theta_j \)'s.

\[
= c_2 \int_{u_j > 0, \sum u_j > 1} \left( 1 - \sum_{j=1}^n u_j \right)^{-2i s_1} \left( 1 + u_1 - \sum_{j=2}^n u_j \right)^{-2i s_2} \ldots 
\]  

(7.0.26)

\[
\ldots \left( 1 + \sum_{j=1}^{n-1} u_j - u_n \right)^{-2i s_n} \left( 1 + \sum_{j=1}^n u_j \right)^{-1-n+2i \sum_{j=1}^n s_j} \, du_n \ldots du_1 
\]  

(7.0.27)
We introduce two more substitutions: $s = u_{n-1} + u_n$, $t = u_{n-1} - u_n$. Notice $t$ only appears in the factor with exponent $-2is_n$. The region of integration is now described by $u_j > 0$, $\sum_{j=1}^{n-2} u_j + s < 1$, and $-s < t < s$. Integrating with respect to $t$, we have

\[
= c_3 \int \ldots \left(1 - \sum_{j=1}^{n-2} u_j - s\right)^{-2is_1} \ldots \left(1 + \sum_{j=1}^{n-2} u_j - s\right)^{-2is_{n-1}} \ldots \\
\ldots \left(1 + \sum_{j=1}^{n-2} u_j + t\right)^{-2is_n} \left(1 + \sum_{j=1}^{n-2} u_j + s\right)^{-1-n+2i \sum_{j=1}^{n} s_j} dt \, ds \, \prod_{j=1}^{n-2} du_j \, \ldots \, du_1
\]

(7.0.28)

\[
= c_3 \int_{u_j, s > 0, \sum u_j + s < 1} \left(1 - \sum_{j=1}^{n-2} u_j - s\right)^{-2is_1} \ldots \left(1 + \sum_{j=1}^{n-2} u_j - s\right)^{-2is_{n-1}} \ldots \\
\ldots \left(1 + \sum_{j=1}^{n-2} u_j + t\right)_{t=s}^{t=-s}^{-2is_n} \left(1 + \sum_{j=1}^{n-2} u_j + s\right)^{-1-n+2i \sum_{j=1}^{n} s_j} ds \, \prod_{j=1}^{n-2} du_j \, \ldots \, du_1
\]

(7.0.29)

\[
= \frac{c_3}{1 - 2is_n} \int_{u_j, s > 0, \sum u_j + s < 1} \left(1 - \sum_{j=1}^{n-2} u_j - s\right)^{-2is_1} \ldots \left(1 + \sum_{j=1}^{n-3} u_j - u_{n-2} - s\right)^{-2is_{n-2}} \ldots \\
\ldots \left(1 + \sum_{j=1}^{n-2} u_j + s\right)^{-1-n+2i \sum_{j=1}^{n-1} s_j} ds \, \prod_{j=1}^{n-2} du_j \, \ldots \, du_1
\]

(7.0.30)
\[- \frac{c_3}{1 - 2i s_n} \int_{u_j, s > 0, \sum u_j + s < 1} \left( 1 - \sum_{j=1}^{n-2} u_j - s \right)^{-2i s_1} \ldots \left( 1 + \sum_{j=1}^{n-3} u_j - u_{n-2} - s \right)^{-2i s_{n-2}} \]

\[
\ldots \left( 1 + \sum_{j=1}^{n-2} u_j - s \right)^{-2i (s_{n-1} + s_n)} \left( 1 + \sum_{j=1}^{n-2} u_j + s \right)^{-1 - n + 2i \sum_{j=1}^{n} s_j} ds \, du_{n-2} \ldots du_1
\]

Both of these integrals can be evaluated by the induction hypothesis. So, after obtaining a common denominator and taking account of the normalization, we have our result.

\[
\frac{c_3}{1 - 2i s_n} \left( \frac{1}{(1 - 2i s_1)(2 - i(s_1 + s_2)) \ldots (n - 1 - 2i(s_1 + \ldots s_{n-1}))} \right)
\]

\[
= \frac{1}{(1 - 2i s_1) \ldots (n - 2 - 2i(s_1 + \ldots + s_{n-2}))(n - 1 - (-1 + 2i(s_1 + \ldots + s_n)))}
\]

\[
= \frac{c_3}{(1 - 2i s_1)(2 - 2i(s_1 + s_2)) \ldots (n - 2i(s_1 + \ldots s_n))}
\]

\[= \prod_{\alpha_j > 0} \frac{\langle \delta, \alpha_j \rangle}{\langle \delta - i\lambda, \alpha_j \rangle}
\]
APPENDIX A

REPRESENTATIONS OF $\mathfrak{so}(n)$ AND $\mathfrak{sp}(n)$

We are motivated to use the following representations of $\mathfrak{so}(n)$ and $\mathfrak{sp}(n)$ in $\mathfrak{su}(n, \mathbb{C})$ because they are the fixed point sets of involutions that preserve the triangular decomposition of $\mathfrak{sl}(n, \mathbb{C}) = n^- \oplus \mathfrak{h} \oplus n^+$ where $\mathfrak{h}$ consists of diagonal matrices, and $n^+$ ($n^-$) consist of upper (lower) triangular matrices.

Lemma A.0.1. Let $\tau : \mathfrak{sl}(n, \mathbb{C}) \to \mathfrak{sl}(n, \mathbb{C})$ be given by reflection across the anti-diagonal. Then $X \mapsto -X^\tau$ is a Lie algebra involution that stabilizes the above triangular decomposition, and it is also an involution of $\mathfrak{su}(n, \mathbb{C}) \subset \mathfrak{sl}(n, \mathbb{C})$.

Proof. That $-(-)^\tau$ stabilizes the triangular decomposition is obvious, and that it preserves $\mathfrak{su}(n, \mathbb{C})$ follows from the fact that it commutes with $-(-)^*$. To show that it respects the bracket $[X, Y] = XY - YX$, we show that $X^\tau Y^\tau = (YX)^\tau$, from which the result follows. Since the $(i, j)^{th}$ entry of $X^\tau$ is given by

$$ (X^\tau)_{ij} = x_{n+1-j, n+1-i} \quad (A.0.2) $$

we have

$$ (X^\tau Y^\tau)_{ij} = \sum_{k=1}^{n} (X^\tau)_{ik} (Y^\tau)_{kj} \quad (A.0.3) $$

$$ = \sum_{k=1}^{n} x_{n+1-k, n+1-i} y_{n+1-j, n+1-k} \quad (A.0.4) $$

$$ = \sum_{k=1}^{n} y_{n+1-j, n+1-k} x_{n+1-k, n+1-i} \quad (A.0.5) $$

$$ = \sum_{l=n}^{n} y_{n+1-j, l} x_{l, n+1-i} \quad \text{(substituting } l = n+1-k) \quad (A.0.6) $$

$$ = (YX)_{n+1-j, n+1-i} \quad (A.0.7) $$

$$ = ((YX)^\tau)_{ij} \quad (A.0.8) $$

\[ \square \]

Proposition A.0.9. $\mathfrak{so}(n) \cong \{ X \in \mathfrak{su}(n) | -X^\tau = X \}$, and $\mathfrak{sp}(n) \cong \{ X \in \mathfrak{su}(2n) | -X^\tau = \text{conj}(I_n)X \}$. 
Proof. We present the proof for \( \mathfrak{so}(n) \); the proof for \( \mathfrak{sp}(n) \) proceeds similarly. Let \( X \in \mathfrak{su}(n, \mathbb{C}) \) such that \( -X^\tau = X \). It suffices to show that the set of such matrices is conjugate to the set of skew symmetric matrices in \( \mathfrak{su}(n, \mathbb{C}) \). Let \( J = \begin{bmatrix} 1 & & \cdots & 1 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \) be an \( n \times n \) matrix with ones on the antidiagonal and zeros elsewhere. The following equality, which can be shown by an easy calculation, will be of use to us.
\[
(A^\tau)^\tau = JAJ = (A^\tau)^t, \quad \text{for any } A \in \mathfrak{gl}(n, \mathbb{C}), \quad (A.0.10)
\]

Now consider
\[
\text{conj}(\frac{1}{\sqrt{2}}(J + i1)(X)) = \frac{1}{\sqrt{2}}(J - i1)X \frac{1}{\sqrt{2}}(J + i1) \quad (A.0.11)
\]
\[
= \frac{1}{2}(JX - iX)(J + i1) \quad (A.0.12)
\]
\[
= \frac{1}{2}(JXJ - iXJ + iJX + X). \quad (A.0.13)
\]
This matrix is skew symmetric:
\[
-\frac{1}{2}(JXJ - iXJ + iJX + X)^t = \frac{1}{2}(-JX^tJ + iJX^t - iX^tJ - X^t) \quad (A.0.14)
\]
\[
= \frac{1}{2}(-X^\tau + iX^\tau J - iJX^\tau - JX^\tau J) \quad (A.0.15)
\]
\[
= \frac{1}{2}(X - iXJ + iJX + JXJ) \quad (A.0.16)
\]
\[
= \frac{1}{2}(JXJ - iXJ + iJX + X). \quad (A.0.17)
\]
This shows that this conjugate of this representation (skew-symmetric across the anti-diagonal) is included in the standard representation (skew-symmetric across the diagonal). A nearly identical computation shows the reverse inclusion. \( \square \)

Note: This representation of \( \mathfrak{so}(n) \) is the space of infinitesimal isometries of the following \( n \) (real) dimensional subspace of \( \mathbb{C}^n \):
\[
\left\{ \begin{bmatrix} x_n - ix_1 \\ \vdots \\ x_1 - ix_n \end{bmatrix} \mid x_j \in \mathbb{R} \right\} \cong \mathbb{R}^n. \quad (A.0.18)
\]
REFERENCES


