Periodic Ising Correlations

by

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Dedication

to

* my parents Anne and Petter Hystad

* my sister Inger Lise Hystad

* my brothers Svein Martin Hystad and Alf Magnus Hystad

Table of Contents

Table of Contents—*Continued*

List of Figures

Abstract

We consider the finite two-dimensional Ising model on a lattice with periodic boundary conditions. Kaufman determined the spectrum of the transfer matrix on the finite, periodic lattice, and her derivation was a simplification of Onsager's famous result on solving the two-dimensional Ising model. We derive and rework Kaufman's results by applying representation theory, which give us a more direct approach to compute the spectrum of the transfer matrix. We determine formulas for the spin correlation function that depend on the matrix elements of the induced rotation associated with the spin operator. The representation of the spin matrix elements is obtained by considering the spin operator as an intertwining map. We wrap the lattice around the cylinder taking the semi-infinite volume limit. We control the scaling limit of the multi-spin Ising correlations on the cylinder as the temperature approaches the critical temperature from below in terms of a Bugrij-Lisovyy conjecture for the spin matrix elements on the finite, periodic lattice. Finally, we compute the matrix representation of the spin operator for temperatures below the critical temperature in the infinite-volume limit in the pure state defined by plus boundary conditions.

1. Introduction

1.1. Introduction to the Two-Dimensional Ising Model

The Ising model is one of the most studied models in modern physics. Since its introduction in 1925 by the Germans E. Ising and W. Lenz, more than a thousand research papers have been published on the subject. The model has had great success in shedding light on the existence of phase transitions at a finite temperature T_C (critical temperature). The simplicity of the model made it possible to obtain exact mathematical results in the thermodynamic limit of statistical mechanics. The Ising model was originally proposed as a model for ferromagnetism and experimentally we have the following situation (see [Th72]): A magnet is placed in a magnetic field H at sufficiently low temperatures T ($T < T_C$) such that the magnet develops a magnetization $M(H, T)$. If the magnetic field is turned off, a spontaneous magnetization still remains for $T < T_C$ and is defined as

$$M_0(T) := \lim_{H \to 0^+} M(H, T).$$

For $T \geq T_C$, the spontaneous magnetization is zero vanishing abruptly at the Curie point $T = T_C$. This situation is shown in the figure below.

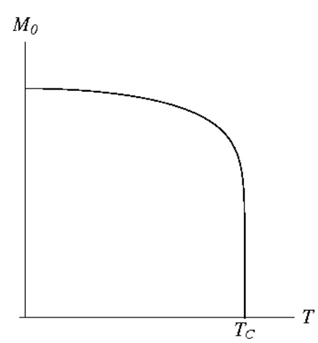

Figure 1.1. The figure shows the spontaneous magnetization M_0 as a function of temperature T.

Consider a finite subset Λ of the two-dimensional lattice

$$\mathbb{Z}^2 = \{(m,n) : m, n \in \mathbb{Z}\}$$

and a configuration of spins $\{\sigma\}$ located at the vertices of Λ. The spins take the value of $+1$ or -1 which corresponds to spin up or spin down. The spins interact with their nearest neighbors and the total interaction energy is given by

$$E_\Lambda(\sigma) = -\sum_{(i,j)\in\Lambda, |i-j|=1} J_{ij}\sigma_i\sigma_j - H\sum_{i\in\Lambda}\sigma_i.$$

In the first term, the sum is over all nearest-neighbors (i,j) in Λ and the second term corresponds to the interaction of the spins with the magnetic field $H \geq 0$. Here J_{ij} is a real number and is the interaction constant for a pair of sites (i,j). When $J_{ij} > 0$ we say that the interaction is ferromagnetic. In this thesis, we are interested in the case $H = 0$, and most of the work that is done on the Ising model is carried out in the absence of an external magnetic field. The partition function is defined as

$$Z_\Lambda = \sum_{\sigma\in\Omega_\Lambda} \exp\left(-\tfrac{E_\Lambda(\sigma)}{k_B T}\right),$$

where k_B is the Boltzmann constant and Ω_Λ is the set of all possible configurations on Λ. For sites $i_1, ..., i_n$ in Λ, the correlation function is given by

$$\langle\sigma_{i_1}\cdot\cdots\cdot\sigma_{i_n}\rangle_\Lambda = \frac{1}{Z_\Lambda}\sum_{\sigma\in\Omega_\Lambda}\sigma_{i_1}\cdot\cdots\cdot\sigma_{i_n}\exp\left(\tfrac{-E_\Lambda(\sigma)}{k_B T}\right).$$

We shall have more to say about this in Section 2.1.

1.2. Brief History and Background Information

There are several review articles and books covering the early history of the Ising model, including the well-known book *The Two-Dimensional Ising model [MW73] by McCoy and Wu* and the papers *History of the Lenz-Ising Model 1920-1950* [NI05] by M. Niss and *On the Theory of the Ising Model of Ferromagnetism* [NM53] by G. F. Newell and E. W. Montroll. Lenz who was Ising's thesis advisor proposed the model as a model for ferromagnetism. He assigned Ising the task performing the mathematical calculations of his model. The result was given in Ising's doctorate thesis of 1924 and published in 1925. Ising proved that the Ising model does not display ferromagnetism in one dimension. He incorrectly extended this result to also be true for two and three dimensions. Ising's contemporaries more or less neglected the model in the 30's and early 40's. There are a variety of suggested reasons for that (see [NI05]). One reason was Ising's negative conclusion about ferromagnetism in two and three dimensions. Some physicists also argued that the model was in conflict with the Heisenberg model, which was considered the leading model of ferromagnetism during that time. Others concluded that the Ising model was in conflict with the new area of quantum mechanics. In 1936 (see [NI05]) Peierls proved that the two and three- dimensional Ising models display spontaneous magnetization at sufficiently low temperatures. However, he dismissed the model as a realistic model of ferromagnetism. In the 30's and 40's, the development in the field of binary alloys and adsorption showed a mathematical analogy to the ferromagnetism in the Ising model. For example, Peirles proved that the mathematical equations involved in describing a transition point are equivalent in the Ising model and in the theory of adsorption. Even though, the Ising model did not get much attention as a model for ferromagnetism, its mathematical equivalents were frequently used in the theory of binary alloys in the 30's and 40's. G. F. Newell and E. W. Montroll [NM53] reviewed the equivalence of the Ising model of a ferromagnet to that of a binary alloy and to a simplified model of a gas and liquid. In the early 40's, the development of the Ising model was characterized by emphasis of the mathematics behind a transition point rather than the physical aspect of the model.

Montroll, Kramer, Wannier and Onsager made important contributions to the Ising model and laid the foundation for further study of the model. Peirles proof of the occurrence of spontaneous magnetization in the two-dimensional Ising model suggested the existence of a transition point in it. H. A. Kramer and G. H. Wannier [KW41-1] and [KW41-2] studied the Ising model in the absence of an external magnetic field, since the mathematics involved simplifies in this case. However, following this strategy made it impossible to directly compute the magnetization, so they focused instead of computing the energy and the specific heat. They were unable to prove the existence of a Curie point, but they showed that if the transition point T_C exists, it satisfies the relation $\sinh^2\left(\frac{2J}{k_B T_C}\right) = 1$, where J is the interaction strength between neighboring spins and k_B is the Boltzmann constant.

In 1942, the Norwegian-born L. Onsager determined the exact value of the specific heat as a function of temperature in the thermodynamic limit of the two-dimensional Ising model in the absence of an external magnetic field. He published his result in 1944 in the paper, *Crystal Statistics. I. A Two-Dimensional Model with an Order-Disorder Transition* [Ons44], which is one of the most important papers in modern physics. The transition point is a singularity either in the specific heat or in the first derivative of the specific heat. Onsager proved that the specific heat is logarithmic divergent as T approaches the critical temperature T_C. He used the transfer matrix approach introduced by Kramers, Wannier, and Montroll to determine an exact expression for the partition function. He showed that the partition function can be approximated by the largest eigenvalue of the transfer matrix on the lattice. The mathematics involved in Onsager's calculations is extremely complicated. In 1949, Onsager's student, B. Kaufman, simplified Onsager's calculations considerably by using Lie group theory and spinor analysis. She published her result in the paper, *Crystal Statistics. II. Partition Function Evaluated by Spinor Analysis* [Kau49]. Kaufman [Kau49] showed that by realizing the 2^M dimensional transfer matrix as a spin representation of a $2M$ dimensional rotation, the eigenvalues of the transfer matrix can be found from the angles of rotation of the rotation matrix. She found the spectrum of the transfer matrix on a lattice, with N rows and M sites per row, to be given by the two sets,

$$\exp[\tfrac{1}{2}(\pm\gamma_0 \pm \gamma_2 \pm ... \pm \gamma_{2M-2})], \tag{1.1}$$

$$\exp[\tfrac{1}{2}(\pm\gamma_1 \pm \gamma_3 \pm ... \pm \gamma_{2M-1})], \tag{1.2}$$

where $\gamma_k := \gamma(\frac{\pi k}{M})$ and where the function $\gamma(q)$ is the positive root of

$$\cosh\gamma(q) = \cosh(2K_2^*)\cosh(2K_1) - \sinh(2K_2^*)\sinh(2K_1)\cos(q). \tag{1.3}$$

Here $\mathcal{K}_1$ and $\mathcal{K}_2$ are the vertical and horizontal couplings constant respectively and the dual $\mathcal{K}_2^*$ of $\mathcal{K}_2$ is defined as,

$$\sinh(2\mathcal{K}_2)\sinh(2\mathcal{K}_2^*) = 1.$$

In (1.1), there is an even number of minus signs for $T < T_C$ and an odd number for $T > T_C$ while in (1.2) there is an even number of minus signs in both cases. The largest eigenvalue in each of the spectra is then

$$\exp(\tfrac{1}{2}[\gamma_1 + \gamma_3 + ... + \gamma_{2M-1}])$$

and

$$\exp(\tfrac{1}{2}[\gamma_0 + \gamma_2 + ... + \gamma_{2M-2}]).$$

For large M, the following approximations take place, $\gamma_{2r} \approx \gamma_{2r-1}$ for $1 \leq r \leq M-1$ while

$$\gamma_0 = \begin{cases} \gamma_1 & \text{for } T < T_C; \\ -\gamma_1 & \text{for } T > T_C. \end{cases}$$

Kaufman [Kau49] and Onsager [Ons44] then determined that for large N and M, the partition function Z on the rectangular lattice is given by

$$(2\sinh(2K_2))^{-\frac{NM}{2}} Z \approx 2\exp(\tfrac{N}{2}[\gamma_1 + \gamma_3 + ... + \gamma_{2M-1}]) \quad \text{for} \quad T < T_C,$$

$$(2\sinh(2K_2))^{-\frac{NM}{2}} Z \approx \exp(\tfrac{N}{2}[\gamma_1 + \gamma_3 + ... + \gamma_{2M-1}]) \quad \text{for} \quad T > T_C.$$

By using the full spectrum of eigenvalues, Kaufman [Kau49] found that the exact value of the partition function Z can be written as

$$Z = \frac{1}{2}(2\sinh(2\mathcal{K}_2))^{\frac{NM}{2}} \Bigg(\prod_{k=1}^{M} \big(2\cosh(\tfrac{N}{2}\gamma_{2k-1})\big) + \prod_{k=1}^{M} \big(2\sinh(\tfrac{N}{2}\gamma_{2k-1})\big) + \\ + \prod_{k=1}^{M} \big(2\cosh(\tfrac{N}{2}\gamma_{2k})\big) + \prod_{k=1}^{M} \big(2\sinh(\tfrac{N}{2}\gamma_{2k})\big) \Bigg).$$

In a third paper [KO49], Kaufman and Onsager determined the two-point correlation for the five shortest distances and correlations for sites lying within the same row for large N and M. Onsager never compared his results with experiments. His focus was on the mathematics behind transitions point and not on the physical interpretations of the Ising model. This was a development that became mainly the norm in the area after Onsager's solution of the two-dimensional Ising model. Onsager also determined an exact expression for the spontaneous magnetization. Since the definition of the spontaneous magnetization requires the magnetic field to be nonzero, Onsager calculated the value of it by using an alternative definition at zero external field. However, he never published his result. The result was first published in 1952 by C. N. Yang [Ya52]. Yang based his method on the matrix problem solved by Onsager [Ons44] and Kaufman [Kau49] and reduced the calculation to an eigenvalue problem. He used an elliptic transformation analogous to the one used by Onsager [Ons44] in order to determine these eigenvalues. The Onsager-Yang formula for the spontaneous magnetization for $T < T_C$ is

$$\langle \sigma \rangle = (1 - k^2)^{\frac{1}{8}},$$

where $k = \frac{1}{\sinh(2\mathcal{K}_1)\sinh(2\mathcal{K}_2)}$. Yang's method is very complicated despite the relative simple expression of the result. In [MPW65], Montroll, Potts and Ward derived the Onsager-Kaufman formulas for the correlations by writing it in terms of Pfaffians. They also derived the Onsager-Yang formula for the spontaneous magnetization by writing it in terms of the Potts-Ward formula for the two-point function, which subsequently can be written in terms of a Toeplitz determinant. The limit of this determinant was evaluated by using a version of Szegö's limit theorem.

Since we are interested in the correlation functions on the finite, periodic lattice, we focus next on some of the work that have been done in this field. Bugrij [Bug01] considered the two-point spin correlation function for the two dimensional Ising model on the finite, periodic lattice wrapped on a cylinder. He expressed the two-point function as a Toeplitz determinant and found a form factor representation of it for temperatures both below and above the critical temperature. The expression he obtained for the isotropic Ising model was the following [Bug01]:

$$\langle \sigma_{r_1} \sigma_{r_2} \rangle = \xi \xi_T e^{-\frac{r}{\Lambda}} \sum_n g_n(r),$$

$$g_n(r) = \frac{e^{\frac{-n}{\Lambda}}}{n!(M)^n} \sum_q \prod_{i=1}^{n} \left(\frac{e^{-r\gamma_i - \eta_i}}{\sinh \gamma_i} \right) F_n^2(q),$$

where

$$F_n(q) = \prod_{i<j} \frac{\sin(\frac{q_i - q_j}{2})}{\sinh(\frac{\gamma_i + \gamma_j}{2})},$$

ξ is a constant, and where the sum is over $q = \frac{\pi l}{M}$ for integer l. The spins in the formula are located within the same row with a distance r from each other. The functions ξ_T, Λ and η_i depend on the temperature and on M, where M is the number of spins on the cylinder circumference. For temperatures below T_C, Bugrij calculated the determinant of the Toeplitz operator by using Wiener-Hopf sum technique. (See [MW73] for a description of the Wiener-Hopf sum technique). He factored the kernel of the Toeplitz matrix into two functions; one analytic inside the unit circle, the other one analytic outside the unit circle. By expressing these functions in terms of projection operators, Bugrij determined the inverse of the Toeplitz matrix by reducing the problem to a system of Wiener-Hopf sum equations. The determinant calculation could then be reduced to a problem of calculating the trace, which involves the inverse of the Toeplitz matrix. However, the factorization he found for the inverse of the Toeplitz matrix is complicated and lacks proof for one of the steps involved (see A30, p29, [Bug01]). The pointwise scaling limit of the two-point correlation function on the cylinder is also given in [Bug01]. The multi-point correlation function on the cylinder and the torus can be evaluated in terms of spin matrix elements in the orthonormal basis of transfer matrix eigenstates. Bugrij and Lisovyy [BL03] proposed a formula for the spin matrix elements on the finite, periodic two-dimensional Ising model for the isotropic case and later [BL04] for the anisotropic case. We will take a closer look at this formula in Section 3.2. The proposed formula for the anisotropic case only involved redefining the function $\gamma(q)$ such that the function depends on the coupling constants in both the vertical and the horizontal direction as given in 1.3. The expression for the spin matrix elements is then analogous to the isotropic case and only involves an extra factor that depends on the two coupling constants. In the papers [GIPST07] and [GIPST07II], G. von Gehleny, N. Iorgovz, S. Pakuliak, V. Shaduraz and Yu Tykhyyz provide proof for the Bugrij-

Lisovyy proposed formula. The calculations involved are extremely complicated. J. Palmer in his book *Planar Ising Correlations* [Pal06] determined the thermodynamic limit of the multi-spin correlation functions for the two-dimensional Ising model in the pure state defined by plus boundary conditions. He also controlled the scaling limit of the multi-point Ising correlation functions in the infinite-volume limit. We will adapt several of his techniques in our calculation of the correlation functions.

1.3. Summary of Thesis

There is a 2×2 complex matrix $T_z(V)$ whose entries are rational functions of $z \in \mathbb{C}$ which completely determines the transfer matrix V on the finite periodic lattice. The matrix $T_z(V)$ is called the induced rotation associated with the transfer matrix. The set of pairs (λ, z) such that $\det(\lambda - T_z(V)) = 0$ is an elliptic curve $\mathcal{M}$ which is important for the spectral analysis of the transfer matrix. We will introduce this curve in Section $C.1$. In particular, the map $\mathcal{M} \ni (\lambda, z) \to z \in \mathbb{P}^1$ is a two fold covering, and there are two cycles $\mathcal{M}_{\pm}$ on $\mathcal{M}$ which cover the circle $\mathbb{S}^1 = \{z : |z| = 1\}$. On the cycle $\mathcal{M}_+$ we have $\lambda < 1$, and on the cycle $\mathcal{M}_-$, we have $\lambda > 1$. Just which points $z_j \in \mathbb{S}^1$ are relevant for the spectral analysis depend on the boundary conditions for the model. For spin periodic boundary conditions on the lattice, the $(2M+1)^{th}$ roots of unity, $z^{2M+1} = 1$, are relevant as are the $(2M+1)^{th}$ roots of -1, $z^{2M+1} = -1$. We will refer to these two finite sets as the periodic spectrum Σ_P and the anti-periodic spectrum Σ_A. In the infinite-volume limit all the points $z \in \mathbb{S}^1$ are relevant.

In Chapter 2, we derive Kaufman's [Kau49] results given in the paper, *Crystal Statistics. II. Partition Function Evaluated by Spinor Analysis*, which was a simplification of the derivation of Onsager's [Ons44] result as described in the previous chapter, using representation theory. This includes an alternative derivation of the induced rotation associated with the transfer matrix as well as a more direct approach of computing the spectrum of the transfer matrix.

As described in Section 1.1, Kaufman [Kau49] analyzed the transfer matrix as an element in a spin representation of the orthogonal group on the finite, periodic lattice. She showed that the space in which the transfer matrix acts, can be divided into two invariant subspaces, which we will denote by $(U = 1)$ and $(U = -1)$. Here U is the product of the basis elements of the finite sequence space $W := l^2(-M, ..., M, \mathbb{C}^2)$ we are working in, which are certain representations of the Clifford relations. The notation $(U = \pm 1)$ is the short-hand notation for the $\pm$ eigenspaces of U. We will use representation theory to show that the $+1$ eigenspace of U is isomorphic to the subspace of even elements of the alternating tensor algebra of a subspace of W in the anti-periodic Fourier representation, and to the subspace of odd elements in the periodic Fourier representation. We show that the -1 eigenspace of U is isomorphic to the subspace of odd elements of the alternating tensor algebra of a subspace of W in the anti-periodic representation, and to the subspace of even elements in the periodic representation. These results are summarized in Theorem 2.3.

In Section 2.6, we compute the matrix elements of the induced rotation associated with the spin operator by restricting the spin operator to be a map from the Fourier space with periodic boundary conditions to the Fourier space with anti-periodic boundary conditions. This result is given in Proposition 2.6. Let us denote the matrix of the induced rotation of the spin operator by

$$s := \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, D are matrix elements for s in a polarization of W. In order to understand the semi-infinite volume limit of the two point correlation for the spin operator, we want to know what happens to the eigenvector associated with the largest eigenvalue for the transfer matrix V under the action of the spin operator. To compute this we need to find a formula for D^{-1}.

The spin matrix elements on the finite, periodic lattice can be written in terms of the $D^{-\tau}$, BD^{-1}, $D^{-1}C$ matrix elements of the induced rotation associated with the spin operator. This result is given in Proposition 3.2 and Theorem 3.3. In Section 3.2, we introduce the proposed Bugrij-Lisovyy formula [BL03] for the spin matrix elements on the finite, periodic lattice in the orthonormal basis of transfer matrix eigenstates. The Bugrij-Lisovyy formula provides a conjecture for the inverse D^{-1}.

In Sections 3.5 and 3.6 we control the scaling limit of the multi-spin correlation functions on the cylinder as the temperature approaches the critical temperature from below and this leads to Lemmas 3.6 and 3.7. The scaling limit is calculated in terms of the Bugrij-Lisovyy conjecture for the spin matrix elements on the finite, periodic lattice. We provide Pfaffian formulas for the spin correlations and the result is given in Theorem 3.10. In Chapter 5, we provide formulas for the Green function for the Dirac operator on the cylinder with one branch point as given in [Lis05]. We show the connection between these formulas and the scaling limit calculations. In Section 4.7 we exhibit the 'new' elements V_+ and V_- in the Bugrij-Lisovyy formula as part of a holomorphic factorization of the periodic and anti-periodic summability kernels on the spectral curve. This result is given in Proposition 4.3.

In Appendix B, we represent the matrix elements for the Fock representation of an element g in the Clifford group as Pfaffians of a skew symmetric matrix whose entries are given in terms of the matrix elements of the induced rotation associated with g. This result is given in Lemma B.1 and Theorem B.2.

In Section $C.2$, we derive the spin matrix elements in the infinite-volume limit in the pure state defined by plus boundary conditions for temperatures below the critical temperature. The result is given in Theorem C.2. The spin matrix elements are well-known in the physics literature, but we are not aware of any mathematical proofs of those formulas. Some features of this calculation give insight into what happens on the finite periodic lattice.

2. Calculation of the two-point spin correlation function by representation theoretic methods

2.1. The Two-Dimensional Ising Model

Let $\mathbb{Z}^2$ denote the two-dimensional integer lattice. We are interested in a finite subset of $\mathbb{Z}^2$, so for positive integers M and N introduce

$$\Lambda = \{(j_1, j_2) \in \mathbb{Z}^2 : |j_1| \leq M, |j_2| \leq N\}.$$

Each vertex is assigned a spin value of $+1$ (spin up) or -1 (spin down). A configuration σ is a particular assignments of spin values to the vertices, i.e a spin configuration is a map

$$\sigma : \Lambda \to \{+1, -1\}.$$

Each spin interacts with its nearest neighbors; in a configuration σ the interaction energy in the absence of an external magnetic field is defined by

$$E_\Lambda(\sigma) = -\sum_{(i,j)\in\Lambda, |i-j|=1} J_{ij}\sigma_i\sigma_j.$$

The sum is over all nearest-neighbors (i, j) in Λ, i.e, the sites i and j are one lattice unit apart either in the horizontal or vertical direction. Here, J_{ij} is the interaction constant which is a real valued function of the pairs (i, j). In our calculations, we are interested in the particular choice $J_{ij} = J_1 > 0$ if the sites are horizontally separated, and $J_{ij} = J_2 > 0$ if the sites are vertically separated. The probability for a given configuration is proportional to the Boltzmann weight

$$w(\sigma) := \exp\Bigg(- \tfrac{E_\Lambda(\sigma)}{k_B T}\Bigg),$$

where T is the temperature and k_B is the Boltzmann constant. The total weight is given in terms of the partition function

$$Z_\Lambda = \sum_{\sigma\in\Omega_\Lambda} \exp\Bigg(- \tfrac{E_\Lambda(\sigma)}{k_B T}\Bigg),$$

where Ω_Λ is the set of all possible configurations on Λ. The correlation function is the expected value of a product of spin variables at sites $i_1, ..., i_n$ in Λ. It is defined by

$$\langle \sigma_{i_1} \cdot \cdots \cdot \sigma_{i_n} \rangle_\Lambda = \frac{1}{Z_\Lambda} \sum_{\sigma \in \Omega_\Lambda} \sigma_{i_1} \cdot \cdots \cdot \sigma_{i_n} \exp\left(\frac{-E_\Lambda(\sigma)}{k_B T} \right).$$

The mathematical formulation of a phase transition is a nonanalytic point of the canonical free energy or the grand-canonical potential as a function of, for example, temperature in the infinite-volume limit $\Lambda \to \mathbb{Z}^2$ [Th72]. The partition function Z_Λ and the spin correlation function $\langle \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_n} \rangle_\Lambda$ are both analytic functions of the temperature for $T \neq 0$, so in order to determine a phase transition point, one has to consider the thermodynamic limit $\Lambda \to \mathbb{Z}^2$. In 1944, Onsager showed that the infinite-volume limit of the free energy per site is a nonanalytic function of the temperature. The temperature at which a phase transition occurs is called the critical temperature T_C. The Ising model can be analyzed in more detail than any other local model in a neighborhood around the critical temperature and has therefore become one of the most studied models in modern physics.

We say that interactions with $J_{ij} > 0$ are ferromagnetic. In a ferromagnetic Ising model, each nonzero term $-J_{ij}\sigma_i\sigma_j$ is negative when the spins σ_i and σ_j are aligned and positive when σ_i and σ_j are unaligned. According to the Boltzmann distribution, there is a higher probability for configurations in which there are many nearest neighbors that are aligned (ordered configurations) compared to configurations in which there are many nearest neighbors that are unaligned (disordered configurations). The temperature in the Boltzmann distribution has the effect of giving more weights to ordered configurations compared to disordered ones at low temperature. As the temperature increases, the difference between ordered and disordered configurations gets smaller, and in the limit $T \to \infty$ the configurations are given equal weight. In the infinite-volume limit, ordered configurations are favored for temperatures below T_C and disordered configurations are favored for temperatures above T_C. Palmer [Pal06] considered the infinite-volume limit of correlation functions for which the sum over configurations only included spins that take values of $+1$ on the boundary of Λ. This type of boundary is referred to as + boundary conditions. He showed that the infinite-volume limit for the one-point function $\langle \sigma_i \rangle_\Lambda$ is strictly positive for $T < T_C$ and zero for $T > T_C$. For boundary values of -1, i.e. the spins take value of -1 on the boundary of Λ, the infinite-volume limit of $\langle \sigma_i \rangle_\Lambda$ is strictly negative at $T < T_C$ [Pal06]. In this work we are interested in the Ising model with periodic boundary conditions. In this case, the infinite-volume limit of the one-point function is zero for all temperatures [Pal06]. It is shown in [LML72] that for $T > T_C$ the infinite-volume limit of correlations does not depend on the boundary conditions.

2.2. The Transfer Matrix

In this section we follow the analysis in Kaufman [Kau49] to show that the transfer matrix for the periodic Ising model on a finite lattice can be expressed as an element in a spin representation of the orthogonal group.

Let $\Omega_\Lambda(\text{row})$ denote the space of configurations of a row. In other words, a i^{th} row configuration is a map

$$\sigma^i : \{-M, ..., M\} \to \{-1, 1\}.$$

Since each site takes the value of $+1$ or -1, there are 2^{2M+1} possible configurations for each row. For $i = -N, ..., N$, we denote a collection of configurations $\sigma^i \in \Omega_\Lambda(\text{row})$ as $\sigma^i_j := \sigma_{ij}$. Thus, the spin variable σ_{ij} is located at the site j in the i^{th} row. The configuration of the lattice is then given by the set $\{\sigma^{-N}, ..., \sigma^N\}$. We denote the interaction energy between spins within a row by $E_\Lambda(\sigma^i)$ and the interaction energy between two adjacent rows by $E_\Lambda(\sigma^i, \sigma^{i+1})$. Then the energy of a configuration σ is given by

$$E_\Lambda(\sigma) = \sum_{i=-N}^{N} E_\Lambda(\sigma^i) + \sum_{i=-N}^{N} E_\Lambda(\sigma^i, \sigma^{i+1}).$$

Since we assume periodic boundary conditions on the lattice, the $(N+1)^{th}$ row interacts with the $-N^{th}$ row, and the $(M+1)^{th}$ column interacts with the $-M^{th}$ column. For

$\sigma, \tau \in \Omega_\Lambda(\text{row})$, we define the 2^{2M+1} dimensional matrices,

$$V_1(\sigma) := \exp(\sum_{j=-M}^{M} \mathcal{K}_1 \sigma_j \sigma_{j+1}),$$
$$V_2(\sigma, \tau) := \exp(\sum_{j=-M}^{M} \mathcal{K}_2 \sigma_j \tau_j),$$

where $\mathcal{K}_l := \frac{J_l}{k_B T}$ for $l = 1, 2$. We assume $\sigma_{M+1} = \sigma_{-M}$. Here, $V_1(\sigma^i)$ represents the Boltzmann weight associated with the horizontal interaction between the spins in the i^{th} row for the configuration σ and $V_2(\sigma^i, \sigma^{i+1})$ represents the Boltzmann weight associated with the vertical interaction between the spins in the i^{th} and $(i+1)^{th}$ row. The partition function can be written as a sum of matrix products [Kau49]:

$$\begin{aligned} Z_\Lambda = & \sum_{\substack{\sigma^i \in \Omega_\Lambda(\text{row}) \\ i=-N,\dots,N}} V_2(\sigma^{-N}, \sigma^{-(N+1)}) V_1(\sigma^{-N}) \times \dots \times V_1(\sigma^N) V_2(\sigma^N, \sigma^{-N}) \\ = & \ \text{tr}(V_1 V_2)^{2N+1}, \end{aligned} \tag{2.1}$$

where we used the notation

$$\sum_{\substack{\sigma^i \in \Omega_\Lambda(\text{row}) \\ i=-N,\dots,N}} := \sum_{\sigma^{-N}} \sum_{\sigma^{-N+1}} \cdots \sum_{\sigma^N}$$

and where each sum on the right is over all possible configurations in $\Omega_\Lambda(\text{row})$. Introduce the tensor product

$$\bigotimes_{j=-M}^{M} \mathbb{C}_j^2 = \mathbb{C}_{-M}^2 \otimes \dots . \otimes \mathbb{C}_M^2,$$

where $\mathbb{C}_j^2 = \mathbb{C}^2$ for each j. This vector space has a basis that is indexed by $\Omega_\Lambda(\text{row})$, i.e. we have the map

$$\Omega_\Lambda(\text{row}) \ni \sigma \to e_\sigma := \bigotimes_{j=-M}^{M} \begin{pmatrix} \frac{1+\sigma_j}{2} \\ \frac{1-\sigma_j}{2} \end{pmatrix}$$

(see [Pal06]). The dimension of the space $\bigotimes_{j=-M}^{M} \mathbb{C}_j^2$ is 2^{2M+1} since there are 2^{2M+1} elements in $\Omega_\Lambda(\text{row})$. Introduce the matrices

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and define the $2^{2M+1} \times 2^{2M+1}$ matrices

$$\sigma_j := \underbrace{I \otimes ... \otimes I}_{M+j} \otimes \sigma \otimes I \otimes .... \otimes I \tag{2.2}$$

$$C_j := \underbrace{I \otimes ... \otimes I}_{M+j} \otimes C \otimes I \otimes .... \otimes I, \tag{2.3}$$

where σ and C are located in the j^{th} position. The matrices σ_j and C_j act on the tensor product space $\bigotimes_{j=-M}^{M} \mathbb{C}_j^2$, and for $\sigma \in \Omega_\Lambda(\text{row})$, the action of the spin operator σ_j is given by (2.2) (see [Pal06]). It is convenient to write the matrices given in (2.2) and (2.3) as

$$\sigma_j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_j, \quad C_j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_j \tag{2.4}$$

in the tensor product space. It is shown in [Ons44] and [Kau49] that the transfer matrices can be written as

$$\begin{array}{l} V_1 = \exp\left(\sum_{j=-M}^{M} \mathcal{K}_1 \sigma_j \sigma_{j+1}\right), \\ V_2 = (2\sinh(2\mathcal{K}_2))^{M+\frac{1}{2}} \exp\left(\sum_{j=-M}^{M} \mathcal{K}_2^* C_j\right), \end{array} \tag{2.5}$$

where σ_j and C_j are given in (2.4). The dual interaction constants $\mathcal{K}_j^*$ are defined by the relation

$$\sinh(2\mathcal{K}_j^*)\sinh(2\mathcal{K}_j) = 1 \quad \text{for} \quad j = 1, 2.$$

We now redefine V_2 as

$$V_2 = \exp\left(\sum_{j=-M}^{M} \mathcal{K}_2^* C_j\right) \tag{2.6}$$

such that the partition function in (2.1) can be written

$$Z_\Lambda = (2\sinh(2\mathcal{K}_1)^{M+\frac{1}{2}} \operatorname{tr}(V_1 V_2)^{2N+1}.$$

For $-M \leq k \leq M$, define

$$p_k := \underbrace{C \otimes ... \otimes C}_{M+k} \otimes \sigma \otimes I \otimes .... \otimes I,$$
$$q_k := \underbrace{C \otimes ... \otimes C}_{M+k} \otimes - i\sigma C \otimes I \otimes .... \otimes I.$$

The operators p_k and q_k satisfy the commutator relations,

$$p_kp_l + p_lp_k = 2\delta_{kl}, \quad q_kq_l + q_lq_k = 2\delta_{kl}, \quad p_kq_l + q_lp_k = 0, \tag{2.7}$$

so they are representations of the Clifford relations. Introduce the vector space W of complex linear combinations of q_k and p_k. An orthonormal basis of W is given by the set $\{\frac{q_k}{\sqrt{2}}, \frac{p_k}{\sqrt{2}}\}_{k=-M}^{M}$ such that for an element $w \in W$, we have

$$w = \sum_{k=-M}^{M} x_k(w)\tfrac{q_k}{\sqrt{2}} + y_k(w)\tfrac{p_k}{\sqrt{2}}. \tag{2.8}$$

We define the distinguished nondegenerate complex bilinear form $(\cdot, \cdot)$ on W by

$$(u, v) = \sum_{k=-M}^{M} x_k(u)x_k(v) + y_k(u)y_k(v).$$

As noted in Appendix A, we have the following definitions which can also be found in [Pal06].

Definition 2.1. *The Clifford algebra* $\text{Cliff}(W)$ *over the orthogonal space* W *is the associative algebra with multiplicative unit* e *that satisfies the relations*

$$uv + vu = (u, v)e \quad \textit{for all} \quad u, v \in W,$$

where $(\cdot, \cdot)$ *is a distinguished nondegenerate complex bilinear form.*

The matrices p_k and q_k generate an irreducible representation of the Clifford algebra $\text{Cliff}(W)$ on $\bigotimes_{j=-M}^{M} \mathbb{C}_j^2$ (see [BW35]).

Definition 2.2. *A linear transformation* V *on* $\otimes_{j=-M}^{M}\mathbb{C}_j^2$ *is an element of the Clifford group if there exists a linear transformation*

$$T(V) : W \to W,$$

such that for all $w \in W \subseteq \text{Cliff}(W)$*, we have*

$$VwV^{-1} = T(V)w.$$

The operator $T(V)$ *is called the induced rotation associated with* V*.*

The induced rotation $T(V)$ is complex orthogonal with respect to the bilinear form $(\cdot,\cdot)$ and it determines V up to a scalar multiple [Pal06]. The exponent in the transfer matrix is a quadratic element of the Clifford algebra. We write the maps C_j and σ_j in terms of the vectors p_k and q_k. Using the definitions of p_k and q_k we obtain

$$\begin{aligned}
&C_j = ip_jq_j,\\
&\sigma_j = C_{-M}C_{-M+1}\cdots C_{j-1}p_j,\\
&\sigma_j\sigma_{j+1} = iq_jp_{j+1} = -ip_{j+1}q_j \quad \text{for} \quad -M \le j \le M-1,\\
&\sigma_{-M}\sigma_M = ip_{-M}q_MC_{-M}C_{-M+1}\cdots C_M := ip_{-M}q_MU.
\end{aligned} \tag{2.9}$$

In the last line, we used

$$U := C_{-M}\cdots C_M = \prod_{k=-M}^{M} ip_kq_k.$$

We will see below that the 'volume element' U play a central role in the analysis of the transfer matrix. Using the relations in (2.9), the transfer matrices V_1 in (2.5) and V_2 in (2.6) can be written

$$\begin{aligned}
V_1 &= \left(\textstyle\prod_{j=-M}^{M-1}\exp\left(-i\mathcal{K}_1p_{j+1}q_j\right)\right)\exp\left(i\mathcal{K}_1p_{-M}q_MU\right),\\
V_2 &= \textstyle\prod_{j=-M}^{M}\exp\left(i\mathcal{K}_2^*p_jq_j\right),
\end{aligned} \tag{2.10}$$

where (see [Th72])

$$\exp\left(i\mathcal{K}_1p_{-M}q_MU\right) = \frac{1}{2}(I+U)\exp\left(i\mathcal{K}_1p_{-M}q_M\right) + \frac{1}{2}(I-U)\exp\left(-i\mathcal{K}_1p_{-M}q_M\right). \tag{2.11}$$

It is easy to check that $U^2 = 1$ and that U commutes with even elements in the Clifford algebra. From this it follows immediately that V_1 and V_2 leave invariant the ±1 eigenspaces for U. Notice that U satisfies the relations

$$\left(\tfrac{I+U}{2}\right)^2 = \left(\tfrac{I+U}{2}\right), \quad \left(\tfrac{I-U}{2}\right)^2 = \left(\tfrac{I-U}{2}\right), \quad \left(\tfrac{I+U}{2}\right)\left(\tfrac{I-U}{2}\right) = 0,$$

so $\left(\frac{I\pm U}{2}\right)$ are the orthogonal projections onto the ±1 eigenspaces of U. Let $(U=\pm1)$ denote the ±1 eigenspaces of U. Define

$$V := V_1V_2$$

and

$$V^A := \frac{1}{2}(I+U)\Big(\prod_{j=-M}^{M} \exp i\mathcal{K}_2^* p_j q_j\Big)\Big(\prod_{j=-M}^{M-1} \exp(-i\mathcal{K}_1 p_{j+1} q_j)\Big) \exp(i\mathcal{K}_1 p_{-M} q_M), \tag{2.12}$$

$$V^P := \frac{1}{2}(I-U)\Big(\prod_{j=-M}^{M} \exp(i\mathcal{K}_2^* p_j q_j)\Big)\Big(\prod_{j=-M}^{M} \exp(-i\mathcal{K}_1 p_{j+1} q_j)\Big). \tag{2.13}$$

Then we have (see [Kau49])

$$V = V^A \oplus V^P,$$

where

$$V^A = V|_{(U=1)} \quad \text{and} \quad V^P = V|_{(U=-1)}.$$

The exponential factors in V^A and V^P are elements in a spin representation of the orthogonal group [Kau49]. Notice that the exponent in the last factor in V^A differ from the other ones by a minus sign. In the calculation of the induced rotation associated with the transfer matrix, we align this factor with the other, by extending the sequences x and y in (2.8) to be $(2M+1)$ anti-periodic on the invariant subspace for V_1 where $(U = 1)$. On the invariant subspace for V_1 where $(U = -1)$, the sequences x and y are extended to be $(2M+1)$ periodic. The letters P and A refer to periodic and anti-periodic. The distinction between periodic and anti-periodic boundary conditions will play an important role in the calculation of the spin operator. We will see in Section 2.6 that it is natural to let the spin operator be a map from a space with periodic boundary conditions to a space with anti-periodic boundary conditions (or the other way around).

2.3. The Induced Rotation associated with the Transfer Matrix

In this section we adapt the technique introduced in [Pal06] to calculate the induced rotation associated with the transfer matrix. We compute the induced rotation for the symmetrical operator $V := V_2^{\frac{1}{2}} V_1 V_2^{\frac{1}{2}}$ in order to obtain a real, symmetric matrix.

According to Definition (2.2), we must find $T(V_1)$ and $T(V_2)$ such that

$$V_1 w V_1^{-1} = T(V_1)w \quad \text{and} \quad V_2 w V_2^{-1} = T(V_2)w \quad \text{for all} \quad w \in W \subseteq \text{Cliff}(W).$$

Using the Taylor series expansion (for $\lambda \in \mathbb{C}$) (see page 12 of [Pal06]),

$$\exp(\lambda X)v\exp(-\lambda X) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \operatorname{ad}^n(X)v, \tag{2.14}$$

where $\operatorname{ad}^0(X)$ is the identity, $\operatorname{ad}(X)$ is the commutator,

$$\operatorname{ad}(X)v = [X, v] = Xv - vX,$$

and

$$\operatorname{ad}^n(X)v = \underbrace{[X, ..., [X}_{\text{n}}, v], ...],$$

we can compute $T(V_1)$ and $T(V_2)$. We start by computing $T(V_2)$. Using the Clifford relations (2.7), we have for
$k = -M, ..., M$,

$$\left[\frac{i}{2}\mathcal{K}_2^* \sum_{j=-M}^{M} p_j q_j, q_k\right] = i\mathcal{K}_2^* p_k, \tag{2.15}$$

$$\left[\frac{i}{2}\mathcal{K}_2^* \sum_{j=-M}^{M} p_j q_j, p_k\right] = -i\mathcal{K}_2^* q_k. \tag{2.16}$$

From (2.14), (2.15), and (2.16) it follows that

$$\begin{aligned} V_2^{\frac{1}{2}} q_k V_2^{-\frac{1}{2}} &= \cosh(\mathcal{K}_2^*)q_k + i\sinh(\mathcal{K}_2^*)p_k \\ V_2^{\frac{1}{2}} p_k V_2^{-\frac{1}{2}} &= -i\sinh(\mathcal{K}_2^*)q_k + \cosh(\mathcal{K}_2^*)p_k \end{aligned} \tag{2.17}$$

and hence the rotation matrix $T(V_2^{\frac{1}{2}})$ with respect to the ordered orthogonal basis $\{q_k, p_k\}_{k=-M}^{M}$ is given by

$$T(V_2^{\frac{1}{2}}) = \begin{pmatrix} \cosh(\mathcal{K}_2^*) & -i\sinh(\mathcal{K}_2^*) & & & & \\ i\sinh(\mathcal{K}_2^*) & \cosh(\mathcal{K}_2^*) & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \cosh(\mathcal{K}_2^*) & -i\sinh(\mathcal{K}_2^*) \\ & & & & i\sinh(\mathcal{K}_2^*) & \cosh(\mathcal{K}_2^*) \end{pmatrix}. \tag{2.18}$$

(Here we have dropped the factor $\frac{1}{\sqrt{2}}$, which makes the basis an orthonormal basis, since the norm does not play a role in this section.) Define

$$V_1^P := \prod_{j=-M}^{M} \exp\left(-i\mathcal{K}_1 p_{j+1} q_j\right), \tag{2.19}$$

$$V_1^A := \prod_{j=-M}^{M-1} \exp\left(-i\mathcal{K}_1 p_{j+1} q_j\right) \exp\left(i\mathcal{K}_1 p_{-M} q_M\right). \tag{2.20}$$

We compute the rotation matrices $T(V_1^P)$ and $T(V_1^A)$. Again using the Clifford relations, we have for $k = -M, ..., M$,

$$\left[-i\mathcal{K}_1 \sum_{j=-M}^{M} p_{j+1} q_j, q_k \right] = -2i\mathcal{K}_1 p_{k+1}, \tag{2.21}$$

$$\left[-i\mathcal{K}_1 \sum_{j=-M}^{M} p_{j+1} q_j, p_{k+1} \right] = 2i\mathcal{K}_1 q_k. \tag{2.22}$$

Now consider the exponent of the last factor of V_1^A. We have the relations

$$[i\mathcal{K}_1(p_{-M} q_M), q_M] = 2i\mathcal{K}_1 p_{-M}, \tag{2.23}$$

$$[i\mathcal{K}_1(p_{-M} q_M), p_{-M}] = -2i\mathcal{K}_1 q_M. \tag{2.24}$$

It follows from (2.14), (2.21), (2.22), (2.23), and (2.24) that

$$\begin{array}{ll} V_1^P q_k (V_1^P)^{-1} & = \cosh(2\mathcal{K}_1) q_k - i \sinh(2\mathcal{K}_1) p_{k+1} \\ V_1^P p_{k+1} (V_1^P)^{-1} & = i \sinh(2\mathcal{K}_1) q_k + \cosh(2\mathcal{K}_1) p_{k+1}, \\ V_1^A q_M (V_1^A)^{-1} & = \cosh(2\mathcal{K}_1) q_M + i \sinh(2\mathcal{K}_1) p_{-M}, \\ V_1^A p_{-M} (V_1^A)^{-1} & = -i \sinh(2\mathcal{K}_1) q_M + \cosh(2\mathcal{K}_1) p_{-M}. \end{array} \tag{2.25}$$

For $j = 1, 2$, introduce the notation

$$c_j := \cosh(2\mathcal{K}_j) \quad \text{and} \quad s_j := \sinh(2\mathcal{K}_j),$$

for the hyperbolic parametrization of the Boltzmann weights. For $j = 1, 2$, we write

$$c_j^* := \cosh(2\mathcal{K}_j^*) = \frac{c_j}{s_j} \quad \text{and} \quad s_j^* := \sinh(2\mathcal{K}_j^*) = \frac{1}{s_j}$$

for the dual interactions. Then using (2.25) we have the following rotation matrices with respect to the ordered orthogonal basis $\{q_k, p_k\}_{k=-M}^{M}$:

(We show here the case $M = 2$)

$$T(V_1^A) = \begin{pmatrix} c_1 & & & is_1 & & & & & & \\ & c_1 & & & & & & & is_1 & \\ & & c_1 & & & is_1 & & & & \\ -is_1 & & & c_1 & & & & & & \\ & & & & c_1 & & & is_1 & & \\ & & -is_1 & & & c_1 & & & & \\ & & & & & & c_1 & & & is_1 \\ & & & & -is_1 & & & c_1 & & \\ & -is_1 & & & & & & & c_1 & \\ & & & & & & -is_1 & & & c_1 \end{pmatrix}$$

$$T(V_1^P) = \begin{pmatrix} c_1 & & & is_1 & & & & & & \\ & c_1 & & & & & & & -is_1 & \\ & & c_1 & & & is_1 & & & & \\ -is_1 & & & c_1 & & & & & & \\ & & & & c_1 & & & is_1 & & \\ & & -is_1 & & & c_1 & & & & \\ & & & & & & c_1 & & & is_1 \\ & & & & -is_1 & & & c_1 & & \\ & is_1 & & & & & & & c_1 & \\ & & & & & & -is_1 & & & c_1 \end{pmatrix}.$$

Now we compute the action of the induced rotation in the Fourier transform and find a common representation for $T(V_1^P)$ and $T(V_1^A)$ which we continue to denote by $T(V_1)$. Let $w = \sum_{j=-M}^{M} x_j q_j + y_j p_j$ be an element in W. Using (2.25), one finds

$$\begin{aligned} & V_1^P w (V_1^P)^{-1} \\ &= \textstyle\sum_{j=-M}^{M} (x_j, y_j) \begin{pmatrix} c_1 q_j - is_1 p_{j+1} \\ c_1 p_j + is_1 q_{j-1} \end{pmatrix} \\ &= \textstyle\sum_{j=-M}^{M} c_1 x_j q_j - i x_j s_1 p_{j+1} + y_j c_1 p_j + i y_j s_1 q_{j-1}. \end{aligned} \tag{2.26}$$

If we extend the series w to be $(2M+1)$ periodic, i.e. $x_{j+2M+1} = x_j$ and $y_{j+2M+1} = y_j$, the last equation in (2.26) can be written

$$\sum_{j=-M}^{M} (c_1 y_j - i x_{j-1} s_1) p_j + (x_j c_1 + i y_{j+1} s_1) q_j. \tag{2.27}$$

We compute $V_1^A w (V_1^A)^{-1}$ in a similar way and obtain that this expression is again given by (2.27) if we extend the series w to be (2M+1) anti-periodic, i.e. $x_{j+2M+1} = -x_j$ and $y_{j+2M+1} = -y_j$. For $k = -M, ..., M$, introduce the notation

$$\theta_k^P = \frac{2\pi i k}{2M+1}, \quad \theta_k^A = \frac{2\pi i(k+\frac{1}{2})}{2M+1}$$

and

$$z_P := z_P(k) = e^{i\theta_k^P}, \quad z_A := z_A(k) = e^{i\theta_k^A}.$$

We refer to the set of $(2M+1)^{th}$ roots of unity, $z_P^{2M+1} = 1$, as the periodic spectrum, Σ_P. The set of $(2M+1)^{th}$ roots of -1, $z_A^{2M+1} = -1$, we refer to as the anti-periodic spectrum, Σ_A. By specializing the finite Fourier series

$$x(z) = \frac{1}{\sqrt{2M+1}} \sum_{k=-M}^{M} x_j z^j$$

to $z \in \Sigma_P$ on the subspace $(U = -1)$ or to $z \in \Sigma_A$ on the subspace $(U = 1)$, we obtain a common representation for $T(V_1)$. The inverse transform is given by

$$x_j = \frac{1}{\sqrt{2M+1}} \sum_z x(z) z^{-k},$$

where $z \in \Sigma_A$ or $z \in \Sigma_P$ as appropriate. Define

$$x'_j := c_1 x_j + i s_1 y_{j+1},$$

$$y'_j := c_1 y_j - i s_1 x_{j-1}$$

in the expression for (2.27). Now using

$$\sum_{j=-M}^{M} x'_j z^j = \sum_{j=-M}^{M} x'_j z^{j+1} z^{-1}, \tag{2.28}$$

$$\sum_{j=-M}^{M} y'_j z^j = \sum_{j=-M}^{M} y'_j z^{j-1} z, \tag{2.29}$$

(2.26) and (2.27) we obtain

$$\begin{pmatrix} x'(z) \\ y'(z) \end{pmatrix} = \begin{pmatrix} c_1 & i s_1 z^{-1} \\ -i s_1 z & c_1 \end{pmatrix} \begin{pmatrix} x(z) \\ y(z) \end{pmatrix}, \tag{2.30}$$

where $z \in \Sigma_A$ or $z \in \Sigma_P$. From (2.18) and (2.30) it follows that the action on Fourier transform coordinates is given by

$$T(V_2^{\frac{1}{2}}) \begin{pmatrix} x(z) \\ y(z) \end{pmatrix} = \begin{pmatrix} \cosh(\mathcal{K}_2^*) & -i\sinh(\mathcal{K}_2^*) \\ i\sinh(\mathcal{K}_2^*) & \cosh(\mathcal{K}_2^*) \end{pmatrix} \begin{pmatrix} x(z) \\ y(z) \end{pmatrix},$$

$$T(V_1) \begin{pmatrix} x(z) \\ y(z) \end{pmatrix} = \begin{pmatrix} c_1 & is_1 z^{-1} \\ -is_1 z & c_1 \end{pmatrix} \begin{pmatrix} x(z) \\ y(z) \end{pmatrix}$$

with $z \in \Sigma_A$ or $z \in \Sigma_P$ as appropriate. We define

$$T(V) := T(V_2^{\frac{1}{2}})T(V_1)T(V_2^{\frac{1}{2}})$$

which is real and symmetric, and hence, also diagonalizable as we will see below. One finds

$$T(V) = \begin{pmatrix} c(z) & b(z) \\ \bar{b}(z) & c(z) \end{pmatrix}, \tag{2.31}$$

where

$$\begin{aligned} c(z) &:= c_2^* c_1 - s_2^* s_1 \left(\tfrac{z+z^{-1}}{2}\right), \\ b(z) &:= -ic_1 s_2^* + ic_2^* s_1 \left(\tfrac{z+z^{-1}}{2}\right) + s_1 \left(\tfrac{z-z^{-1}}{2i}\right). \end{aligned} \tag{2.32}$$

A simple calculation gives that

$$c^2 - |b|^2 = 1,$$

and it follows that there exists a real-valued function $\gamma(z) > 0$ and a $\mathbb{S}^1$-valued function $w(z)$ such that

$$\begin{aligned} c(z) &= \cosh\gamma(z) = c_2^* c_1 - s_2^* s_1 \left(\tfrac{z+z^{-1}}{2}\right), \\ b(z) &= w(z)\sinh\gamma(z) = -ic_1 s_2^* + ic_2^* s_1 \left(\tfrac{z+z^{-1}}{2}\right) + s_1 \left(\tfrac{z-z^{-1}}{2i}\right). \end{aligned}$$

Thus, for $T < T_C$ the induced rotation $T(V)$ associated with V is given by multiplication with

$$\begin{pmatrix} \cosh\gamma(z) & w(z)\sinh\gamma(z) \\ \overline{w}(z)\sinh\gamma(z) & \cosh\gamma(z) \end{pmatrix} = \exp\left[\gamma \begin{pmatrix} 0 & w \\ \overline{w} & 0 \end{pmatrix}\right], \tag{2.33}$$

where $z \in \Sigma_A$ or $z \in \Sigma_P$ as appropriate. Define as in [Pal06]

$$\alpha_1 := (c_1^* - s_1^*)(c_2 + s_2) = e^{2(K_2 - K_1^*)}$$

$$\alpha_2 := (c_1^* + s_1^*)(c_2 + s_2) = e^{2(K_2 + K_1^*)}.$$

When $T < T_C$, we have that $1 < \alpha_1 < \alpha_2$ (see [Pal06]). It follows from (2.32) that

$$w^2(z) = \frac{b(z)}{\bar{b}(z)} = -\frac{b(z)}{b(z^{-1})}.$$

We can factor $b(z)$ in the following way:

$$b(z) = is_1 \frac{(c_2^* - 1)}{2} z^{-1}(z - \alpha_1)(z - \alpha_2).$$

This implies that

$$w^2(z) = -z^{-2} \frac{(\alpha_1 - z)(\alpha_2 - z)}{(\alpha_1 - z^{-1})(\alpha_2 - z^{-1})}. \tag{2.34}$$

Define

$$\mathcal{A}_j(z) := \sqrt{\alpha_j - z} \quad \text{for} \quad j = 1, 2,$$

where the square root is chosen to have positive real part, i.e. $\mathcal{A}_j(1) > 0$.

Since $1 < \alpha_1 < \alpha_2$ for $T < T_C$, we observe that $z \to (\mathcal{A}_j(z))^{\pm}$ is analytic for z in a neighborhood of the unit disc ($|z| < \alpha_j$) while $z \to (\mathcal{A}_j(z^{-1}))^{\pm}$ is analytic in a neighborhood of the exterior of the unit disc ($|z| > \alpha_j^{-1}$). Since $b(z)$ is a positive multiple of i for $z = 1$ and $T < T_C$, it follows that $w(1) = i$. Thus, the appropriate square root in (2.34) is

$$w(e^{i\theta}) = ie^{-i\theta} e^{\beta(\theta)},$$

where

$$2\beta(\theta) = \log \frac{(\alpha_1 - e^{i\theta})(\alpha_2 - e^{i\theta})}{(\alpha_1 - e^{-i\theta})(\alpha_2 - e^{-i\theta})}. \tag{2.35}$$

Hence,

$$w(z) = iz^{-1} \frac{\mathcal{A}_1(z)\mathcal{A}_2(z)}{\mathcal{A}_1(z^{-1})\mathcal{A}_2(z^{-1})} \tag{2.36}$$

for $|z| = 1$. One can achieve a simple form of the factorization in (2.36) in the uniformization parameter of the spectral curve associated with the induced rotation $T(V)$ for the transfer matrix as we will see in Section 4.7.

2.4. Fock Representations

In this section we recall the definitions of the Fock representations of the Clifford algebra, which are explained in more details in Appendix A and in [Pal06].
Assume W is an even-dimensional complex vector space with a distinguished nondegenerate complex bilinear form $(\cdot,\cdot)$. We say that a subspace V of W is isotropic if the bilinear form $(\cdot,\cdot)$ vanishes identically for all elements in V. A direct sum decomposition

$$W = W_+ \oplus W_-$$

where both W_+ and W_- are isotropic subspaces of W is called an isotropic splitting of W. An isotropic splitting is also referred to as a polarization. Introduce the map

$$Qv = v_+ - v_-,$$

where the components of v are given by $v = v_+ + v_-$ relative to the isotropic splitting $W = W_+ \oplus W_-$. The operator Q is also referred to as a polarization and gives a parametrization of the splitting $W = W_+ \oplus W_-$. Introduce the Fermion Fock space,

$$\mathrm{Alt}(W_+) = \bigoplus_{k=0}^{n} \mathrm{Alt}^k(W_+),$$

where $\mathrm{Alt}(W_+)$ is the alternating tensor algebra, $\mathrm{Alt}^k(W_+)$ is the space of alternating k tensors over W_+, and $n = \dim(W_+)$. We define $\mathrm{Alt}^0(W_+) = \mathbb{C}$ and we have that $\mathrm{Alt}^1(W_+) = W_+$. Recall that $\dim \mathrm{Alt}^k(W_+) = \binom{n}{k}$. The Fock representation F_Q of the Clifford algebra associated with the polarization Q acting on $\mathrm{Alt}(W_+)$ is defined by

$$F_Q(w) = c(w_+) + a(w_-),$$

where $w = w_+ + w_- \in W_+ \oplus W_-$. Here W_- is identified with the dual W_+^* via the nondegenerate complex bilinear form $W_+ \ni w_+ \mapsto (w_+, w_-)$ for $w_- \in W_-$. The creation operator $c(w_+)$ associated with $w_+ \in W_+$ acts on $\mathrm{Alt}^k(W_+)$ in the following way,

$$\mathrm{Alt}^k(W_+) \ni v \mapsto c(w_+)v = w_+ \wedge v \in \mathrm{Alt}^{k+1}(W_+). \tag{2.37}$$

The annihilation operator $a(w_-)$ associated with $w_- \in W_-$ acts on a vector $v = v_1 \wedge v_2 \wedge .... \wedge v_k$ in $\text{Alt}^k(W_+)$ in the following way:

$$a(w_-)v = \sum_{j=1}^{n} (-1)^{j-1}(w_-, v_j) v_1 \wedge v_2 \wedge ... \wedge v_{j-1} \wedge v_{j+1} \wedge ..... \wedge v_k. \tag{2.38}$$

Here for $w \in W_-$, we have $a(w) = c^\tau(w)$, where τ denotes the transpose of $c(w)$ with respect to the complex bilinear form $(\cdot, \cdot)$. It can be checked that a and c satisfy the anticommutation relations,

$$\begin{array}{l} c(x)c(y) + c(y)c(x) = 0 \quad \text{for} \quad x, y \in W_+, \\ a(x)a(y) + a(y)a(x) = 0 \quad \text{for} \quad x, y \in W_-, \\ a(x)c(y) + c(y)a(x) = (x, y)e \quad \text{for} \quad x \in W_- \quad \text{and} \quad y \in W_+. \end{array} \tag{2.39}$$

Thus, it follows that F_Q satisfies the generator relations for the Clifford algebra of W,

$$F_Q(x)F_Q(y) + F_Q(y)F_Q(x) = (x, y)I$$

for all $x, y \in W$. If W_+ and W_- are orthogonal with respect to the Hermitian inner product given by

$$\langle x, y \rangle = (\bar{x}, y),$$

where $x \mapsto \bar{x}$ is a conjugation on W, we refer to the isotropic splitting $W_+ \oplus W_-$ of W as a Hermitian polarization. With this splitting, we define the Fock representation $F_Q(w)$ of $\text{Cliff}(W)$ associated with the polarization Q to be

$$F_Q(w) = a^*(w_+) + a(\bar{w}_-),$$

where $w_+ \in W_+$ and $w_- \in W_-$. Here we have $\bar{w}_- \in W_+$ and $a(w) = (a^*(w))^*$ for $w \in W_+$ such that $(a^*(w))^*$ is the adjoint of $a^*(w)$ with respect to the Hermitian inner product on $\text{Alt}(W_+)$. It can be checked that by using the Clifford relations and the isotropic splitting $W = W_+ \oplus W_-$, we have

$$F_Q(x)F_Q(y) + F_Q(y)F_Q(x) = (x, y)I \quad \text{for all} \quad x, y \in W.$$

The last Fock representation will play a role in the calculation of the spin matrix elements in the infinite-volume limit under the pure state defined by + boundary conditions for temperatures below T_C. The vacuum vector

$$0 := 1 \oplus 0 \oplus .... \oplus 0$$

is the unique vector in $\mathrm{Alt}(W_+)$ that satisfies

$$a(w)0 \equiv 0$$

for all $w \in W_-$ (see [Pal06] and [GJ81]).

2.5. The Spectrum of the Induced Rotation and the Transfer Matrix

In this section we determine the eigenvalues and eigenvectors of $T(V)$. We also determine the spectra of the transfer matrices V^A and V^P. Kaufman [Kau49] showed that the eigenvalues of the transfer matrix can be found from the angles of rotation of the rotation matrix by realizing the 2^{2M+1} dimensional transfer matrix as a spin representation of the $(2M+1)$ dimensional rotation matrix. In this section we will rework some of Kaufman's [Kau49] results in terms of representation theory. Using representation theory, we will also have a simpler method of determining the eigenvalues of the transfer matrix.

Recall that W is the complex vector space $l^2[-M, ..., M, \mathbb{C}^2]$ with the distinguished nondegenerate complex bilinear form $(\cdot, \cdot)$ given by

$$(u, v) = \sum_{k=-M}^{M} x_k(u)x_k(v) + y_k(u)y_k(v)$$

for

$$W \ni u = \sum_{k=-M}^{M} x_k(u)\frac{q_k}{\sqrt{2}} + y_k(u)\frac{p_k}{\sqrt{2}},$$

and with the Hermitian inner product given by $\langle u, v\rangle = (\bar{u}, v)$. Here $u \mapsto \bar{u}$ denotes the conjugation of u. Recall that the Fourier series of $\begin{bmatrix} x(z) \\ y(z) \end{bmatrix}$ is given by

$$\begin{bmatrix} x(z) \\ y(z) \end{bmatrix} = \frac{1}{\sqrt{2M+1}} \sum_{k=-M}^{M} \begin{bmatrix} x_k \\ y_k \end{bmatrix} z^k, \tag{2.40}$$

where z is in the periodic spectrum Σ_P or in the anti-periodic spectrum Σ_A. After Fourier transform, the Hermitian innerproduct becomes

$$\sum_{k=-M}^{M} \bar{x}(z_k)x'(z_k) + \bar{y}(z_k)y'(z_k)$$

and the complex bilinear form becomes

$$\sum_{k=-M}^{M} x(z_k)x'(z_k^{-1}) + y(z_k)y'(z_k^{-1}), \tag{2.41}$$

where the conjugation is given by $x(z) \mapsto \bar{x}(z^{-1})$. Let T^A denote the induced rotation associated with V^A, and T^P the induced rotation associated with V^P, where V^A and V^P are given in (2.12) and (2.13). We are interested in the isotropic splittings

$$W = W_+^A \oplus W_-^A \quad \text{and} \quad W = W_+^P \oplus W_-^P,$$

where W_+^A and W_+^P are the span of all eigenvectors of T^A and T^P respectively associated with the eigenvalues less than one, and W_-^A and W_-^P are the span of all eigenvectors of T^A and T^P respectively associated with the eigenvalues greater than one. From (2.33) we see that $W_\pm^A$ are the ± 1 eigenspaces of the polarization given by the multiplication operators in the anti-periodic Fourier representation,

$$Q(z_A) = -\begin{pmatrix} 0 & w(z_A) \\ \bar{w}(z_A) & 0 \end{pmatrix} \tag{2.42}$$

and $W_\pm^P$ are the ± 1 eigenspaces of the polarization given by the multiplication operators in the periodic Fourier representation,

$$Q(z_P) = -\begin{pmatrix} 0 & w(z_P) \\ \bar{w}(z_P) & 0 \end{pmatrix}. \tag{2.43}$$

Let Q^A denote multiplication by $Q(z_A)$ and let Q^P denote multiplication by $Q(z_P)$. Define

$$Q_\pm^A := \tfrac{1}{2}(I \pm Q^A) \quad \text{and} \quad Q_\pm^P := \tfrac{1}{2}(I \pm Q^P).$$

Then

$$W_\pm^A = Q_\pm^A W \quad \text{and} \quad W_\pm^P = Q_\pm^P W$$

so $Q_\pm^A$ and $Q_\pm^P$ are orthogonal projections on $W_\pm^A$ and $W_\pm^P$ respectively. Define

$$a(z) := \sqrt{\frac{\mathcal{A}_1(z)\mathcal{A}_2(z)}{\mathcal{A}_1(z^{-1})\mathcal{A}_2(z^{-1})}}, \tag{2.44}$$

such that we have

$$w(z) = ia^2(z)z^{-1},$$

and where we recall that $\mathcal{A}_j = \sqrt{\alpha_j - z}$ for $z = \Sigma_A$ or $z \in \Sigma_P$ as appropriate. The appropriate square root in (2.44) is

$$a(e^{i\theta}) = e^{\frac{1}{2}\beta(\theta)},$$

where $\beta(\theta)$ is defined in (2.35). It can be checked that the eigenvalues of T^A are given by $e^{-\gamma(z_A(k))}$ and $e^{\gamma(z_A(k))}$ with corresponding eigenvectors

$$e^A_{+,k}(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} a(z) \\ iza(z)^{-1} \end{pmatrix} \delta_{z_A(k)}(z) \tag{2.45}$$

and

$$e^A_{-,k}(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} a(z) \\ -iza(z)^{-1} \end{pmatrix} \delta_{z_A(-k)}(z) \tag{2.46}$$

respectively for $z \in \Sigma_A$ and $k = -M, ..., M$. The eigenvalues of T^P are given by $e^{-\gamma(z_P(k))}$ and $e^{\gamma(z_P(k))}$ with corresponding eigenvectors

$$e^P_{+,k}(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} a(z) \\ iza(z)^{-1} \end{pmatrix} \delta_{z_P(k)}(z) \tag{2.47}$$

and

$$e^P_{-,k}(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} a(z) \\ -iza(z)^{-1} \end{pmatrix} \delta_{z_P(-k)}(z) \tag{2.48}$$

respectively for $z \in \Sigma_P$ and $k = -M, ..., M$. The sets of eigenvectors $\{e^A_{+,k}\}$ and $\{e^A_{-,k}\}$ are normalized with respect to the inner product $\langle \cdot, \cdot \rangle$, and dual with respect to the bilinear form $(\cdot, \cdot)$. The same applies for the sets $\{e^P_{\pm,k}\}$. Recall that $\gamma(e^{i\theta})$ is given by the positive root of

$$\cosh(\gamma(e^{i\theta})) = c_1 c_2^* - s_1 s_2^* \cos(\theta)$$

which implies that $\theta \mapsto \cosh(\gamma(e^{i\theta}))$ is a strictly increasing function for $\theta \in (0, \pi)$. Hence $\theta \mapsto \gamma(e^{i\theta})$ is a strictly increasing function for $\theta \in (0, \pi)$. We also observe that $\gamma(e^{i\theta})$ is an even function of θ. It follows that the eigenvalues $e^{\pm\gamma\left(e^{i\theta^P_k}\right)}$ with

$k = 1, ..., M$ for T^P, and the eigenvalues $e^{\pm\gamma\left(e^{i\theta_k^A}\right)}$ with $k = 0, .., M-1$ for T^A, occur with multiplicity two, while the eigenvalues $e^{\pm\gamma(1)}$ for T^P and $e^{\pm\gamma(-1)}$ for T^A have multiplicity one.

Recall that $V = V^A \oplus V^P$, where $V^A = V_{|(U=1)}$ and $V^P = V_{|(U=-1)}$, and where $U = \prod_{k=-M}^{M} ip_k q_k$. Define

$$\mathrm{Alt}_{\mathrm{even}}(W_+) := \bigoplus_{0\le 2k\le n} \mathrm{Alt}^{2k}(W_+) \quad \text{and} \quad \mathrm{Alt}_{\mathrm{odd}}(W_+) = \bigoplus_{0\le 2k+1\le n} \mathrm{Alt}^{2k+1}(W_+),$$

where $n = \dim(W_+)$.

In the next proposition we will use representation theory to show that the $+1$ eigenspace of U is isomorphic to the subspace of even elements of the alternating tensor algebra in the anti-periodic representation, and to the subspace of odd elements in the periodic representation. We show that -1 eigenspace of U is isomorphic to the subspace of odd elements of the alternating tensor algebra in the anti-periodic representation, and to the subspace of even elements in the periodic representation. We prove the following.

Theorem 2.3. *Consider the isotropic splittings $W = W_+^A \oplus W_-^A$ and $W = W_+^P \oplus W_-^P$ associated with the polarizations defined in (2.42) and (2.43). Let $(U = \pm 1)$ denote the ± 1 eigenspaces of U, where $U = \prod_{k=-M}^{M} ip_k q_k$. Then*

$$\mathrm{Alt}_{\mathrm{even}}(W_+^A) \simeq (U=1), \quad \mathrm{Alt}_{\mathrm{even}}(W_+^P) \simeq (U=-1),$$

$$\mathrm{Alt}_{\mathrm{odd}}(W_+^A) \simeq (U=-1), \quad \mathrm{Alt}_{\mathrm{odd}}(W_+^P) \simeq (U=1).$$

Proof. We start by making a change of basis $\{\frac{q_k}{\sqrt{2}}, \frac{p_k}{\sqrt{2}}\}_{k=-M}^{M}$ to a basis $\left\{\frac{1}{\sqrt{2}}q(z(k)), \frac{1}{\sqrt{2}}p(z(k))\right\}_{k=-M}^{M}$, where $q(z(k))$ and $p(z(k))$ are real with respect to the conjugation $v(z) \mapsto \bar{v}(z^{-1})$, and where z is an element in Σ_P or in Σ_A. The basis elements $q(z(k))$ and $p(z(k))$ will be defined below. We denote the linear transformation that sends $\frac{q_k}{\sqrt{2}}$ to $\frac{1}{\sqrt{2}}q(z_P(k))$ and $\frac{p_k}{\sqrt{2}}$ to $\frac{1}{\sqrt{2}}p(z_P(k))$ by R^P, and the linear transformation that sends $\frac{q_k}{\sqrt{2}}$ to $\frac{1}{\sqrt{2}}q(z_A(k))$ and $\frac{p_k}{\sqrt{2}}$ to $\frac{1}{\sqrt{2}}p(z_A(k))$ by R^A, where R^P and R^A are elements in the orthogonal group. Define the 'volume elements'

$$U^A := \prod_{k=-M}^{M} ip(z_A(k))q(z_A(k)) \quad \text{and} \quad U^P := \prod_{k=-M}^{M} ip(z_P(k))q(z_P(k))$$

in the Clifford algebra. The transformations R^A and R^P induce an automorphism of the Clifford algebra such that

$$U = \det(R^A)U^A \quad \text{and} \quad U = \det(R^P)U^P. \tag{2.49}$$

We show below that

$$\det R^P = -1 \quad \text{and} \quad \det R^A = 1. \tag{2.50}$$

Let $\mathcal{N}^P$ and $\mathcal{N}^A$ denote the number operators in $\mathrm{Alt}(W_+^P)$ and $\mathrm{Alt}(W_+^A)$ respectively, which are defined as

$$\mathcal{N}^P | \mathrm{Alt}^k(W_+^P) = k \quad \text{and} \quad \mathcal{N}^A | \mathrm{Alt}^k(W_+^A) = k.$$

Let F^P and F^A denote the Fock representations associated with the Clifford relations acting on $\mathrm{Alt}(W_+^P)$ and $\mathrm{Alt}(W_+^A)$. Then

$$(-1)^{\mathcal{N^P}} F^P(v)(-1)^{\mathcal{N^P}} = -F^P(v) \quad \text{for} \quad v \in \mathrm{Alt}(W_+^P).$$

Similarly, we have

$$(-1)^{\mathcal{N^A}} F^A(v)(-1)^{\mathcal{N^A}} = -F^A(v) \quad \text{for} \quad v \in \mathrm{Alt}(W_+^A).$$

It can later be checked that $(U^P)^2 = 1$ and

$$U^P p(z_P(k))(U^P)^{-1} = -p(z_P(k)) \quad \text{and} \quad U^P q(z_P(k))(U^P)^{-1} = -q(z_P(k)),$$

so U^P is an element of the Clifford group with induced rotation -1 on W^P. Similarly, U^A is an element of the Clifford group with induced rotation -1 on W^A. It follows that $F^P(U^P)$ and $(-1)^{\mathcal{N}^P}$, and $F^A(U^A)$ and $(-1)^{\mathcal{N}^A}$ have the same induced rotation in the Fock representation. Since we also have $(-1)^{2\mathcal{N}^P} = (-1)^{2\mathcal{N}^A} = 1$, it follows that $F^P(U^P) = l(-1)^{\mathcal{N}^P}$ and $F^A(U^A) = l(-1)^{\mathcal{N}^A}$ for $l = \pm 1$. We show that $l = 1$, i.e. that

$$U^P = (-1)^{\mathcal{N}^P}, \tag{2.51}$$

$$U^A = (-1)^{\mathcal{N}^A}. \tag{2.52}$$

Then it follows from (2.49), (2.50), (2.51) and (2.52) that

$$U = -(-1)^{\mathcal{N}^P} \quad \text{on} \quad \mathrm{Alt}(W_+^P)$$

and

$$U = (-1)^{\mathcal{N}^A} \quad \text{on} \quad \mathrm{Alt}(W_+^A).$$

Since $(-1)^{\mathcal{N}_P} = 1$ on $\mathrm{Alt}_{\mathrm{even}}(W_+^P)$, we have

$$\mathrm{Alt}_{\mathrm{even}}(W_+^P) \simeq (U = -1).$$

A similar argument shows that

$$\mathrm{Alt}_{\mathrm{even}}(W_+^A) \simeq (U = 1).$$

It follows that

$$\mathrm{Alt}_{\mathrm{odd}}(W_+^A) \simeq (U = -1) \quad \text{and} \quad \mathrm{Alt}_{\mathrm{odd}}(W_+^P) \simeq (U = 1).$$

Define

$$a_k^* := \tfrac{1}{2}(p_k + iq_k) \quad \text{and} \quad a_k := \tfrac{1}{2}(p_k - iq_k)$$

such that $W = W_+ \oplus W_-$, where $W_+ = \mathrm{span}\{a_k^*\}$ and $W_- = \mathrm{span}\{a_k\}$. It can be checked that a^* and a satisfy the anticommutative relations

$$\begin{aligned} &a_k^* a_l^* + a_l^* a_k^* = 0, \\ &a_k a_l + a_l a_k = 0, \\ &a_k a_l^* + a_l^* a_k = \delta_{kl}, \end{aligned} \tag{2.53}$$

where we used the Clifford relations (2.7). Let $\mathcal{N}$ denote the number operator in $\mathrm{Alt}(W_+)$. In the Fock representation associated with the polarization above, the number operator $\mathcal{N}$ is given by $\mathcal{N} = \sum_{k=-M}^{M} a_k^* a_k$. Using the fact that $(a_k^* a_k)^2 = a_k^* a_k$, we obtain

$$(-1)^{\mathcal{N}} = \prod_{k=-M}^{M} e^{i\pi a_k^* a_k} = \prod_{k=-M}^{M} (1 - 2a_k^* a_k) = \prod_{k=-M}^{M} ip_k q_k = U. \tag{2.54}$$

We now define the basis elements $q(z_P(k))$ and $p(z_P(k))$.

In the polarization $W = W_+^P \oplus W_-^P$, define the creation operators as

$$a_k^{P*} := c(e_{+,k}^P)$$

and the annihilation operators as

$$a_k^P := a(e_{-,k}^P),$$

where c and a are defined in (2.37) and (2.38), and $\{e_{+,k}^P\}$ and $\{e_{-,k}^P\}$ are the eigenvectors for the induced rotation $T(V)$ as defined in (2.47) and (2.48). Since $\bar{a}(z) = a(z^{-1})$ in (2.44), we find that $e_{+,k}^P(z)$ is the conjugate of $e_{-,k}^P(z)$, i.e. $e_{+,k}^P(z) \mapsto \bar{e}_{+,k}^P(z^{-1}) = e_{-,k}^P(z)$. It follows that $(c(e_{+,k}^P))^* = a(e_{-,k}^P)$. In the z_P coordinates we define,

$$p(z_P(k)) := (a_k^{P*} + a_k^P) \quad \text{and} \quad q(z_P(k)) := i(a_k^P - a_k^{P*}).$$

We have that $\bar{p}(z^{-1}) = p(z)$ and $\bar{q}(z^{-1}) = q(z)$ so $p(z(k))$ and $q(z(k))$ are real with respect to conjugation. By using the anticommutative relations given in (2.39), it can be checked that $p(z_P)$ and $q(z_P)$ satisfy the Clifford relations,

$$\begin{aligned} p(z_P(k))p(z_P(l)) + p(z_P(l))p(z_P(k)) &= 2\delta_{kl}, \\ q(z_P(k))q(z_P(l)) + q(z_P(l))q(z_P(k)) &= 2\delta_{kl}, \\ p(z_P(k))q(z_P(l)) + q(z_P(l))p(z_P(k)) &= 0. \end{aligned} \tag{2.55}$$

and that $(p, p) = (q, q) = 2$. Now, we can do exactly the same calculations as the one given in (2.54), and we obtain

$$(-1)^{\mathcal{N}^P} = \prod_{k=-M}^{M} ip(z_P(k))q(z_P(k)).$$

Next we compute the determinant of R^P. It's linear transformation consists of a compositions of transformations,

$$R^P : \mathbb{C}^{2(2M+1)} \xrightarrow{\mathcal{F}_P \oplus \mathcal{F}_P} \mathbb{C}^{2(2M+1)} \xrightarrow{R_1^P} \mathbb{C}^{2(2M+1)} \xrightarrow{R_2^P} \mathbb{C}^{2(2M+1)} \xrightarrow{x \mapsto \sqrt{2}x} \mathbb{C}^{2(2M+1)},$$

where $\mathcal{F}_P$ is the finite inverse Fourier transform in the periodic representation, R_1^P is the transformation from the Fourier series representation to the basis $\{e^P_{+,k}, e^P_{-,-k}\}_{k=-M}^M$, while R_2^P is the transformation from $\{a_k^{P*}, a_{-k}^P\}_{k=-M}^M$ to $\{q(z_P(k)), p(z_P(k))\}_{k=-M}^M$. Since $\{q(z_P(k)), p(z_P(k))\}$ is not the normalized basis for W, but $\left\{\frac{1}{\sqrt{2}}q(z_P(k)), \frac{1}{\sqrt{2}}p(z_P(k))\right\}$ is, the transformation $x \mapsto \sqrt{2}x$ takes the first basis to the second. Let us denote the matrix of this transformation by R_3. (Note that the factor of $\sqrt{2}$ is incorporated in our definitions of the Clifford relations: $xy + yx = (x, y)e$ for $x, y \in W$, where the more standard definition is $xy + yx = 2(x, y)e$.) The coordinate basis of $\{q(z_P(k))\}$ in $\mathbb{R}^{2(2M+1)}$ is given by $\{e_k\}$ and the coordinate basis of $\{p(z_P(k))\}$ in $\mathbb{R}^{2(2M+1)}$ is given by $\{e_{k+1}\}$, where $\{e_k\}$ is the standard basis. We have

$$R_3 = \begin{pmatrix} \sqrt{2} & 0 & & & & \\ 0 & \sqrt{2} & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \sqrt{2} & 0 \\ & & & & 0 & \sqrt{2} \end{pmatrix},$$

where

$$\det(R_3) = 2^{2M+1}. \tag{2.56}$$

The inverse Fourier transform

$$\mathcal{F}_P : \mathbb{C}^{2M+1} \to \mathbb{C}^{2M+1}$$

in the z_P representation has the following matrix,

$$\mathcal{F}_P := \frac{1}{\sqrt{2M+1}} \begin{pmatrix} z_{P,-M}^{-M} & \cdots & \cdots & z_{P,M}^{-M} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ z_{P,-M}^{M} & \cdots & \cdots & z_{P,M}^{M} \end{pmatrix}, \tag{2.57}$$

where $z_{P,k}^l := z_P^l(k) = e^{\frac{2\pi i k l}{2M+1}}$ for $k, l = -M, ..., M$. Including the normalization factor $\frac{1}{\sqrt{2M+1}}$ makes the linear transformation unitary. The determinant of $\mathcal{F}_P$ is the product of its eigenvalues, where the eigenvalues are given by ± 1 and $\pm i$. The multiplicities of the eigenvalues λ are given in [MP72]: For the discrete inverse Fourier transform with transform size N, where N is odd, the multiplicities of its eigenvalues λ are the following:

Size N	$\lambda = 1$	$\lambda = -1$	$\lambda = i$	$\lambda = -i$
$4m+1$	$m+1$	m	m	m
$4m+3$	$m+1$	$m+1$	$m+1$	m

, (2.58)

where m is an integer. Using (2.58) it follows that

$$\det(\mathcal{F}_P) = (-i)^M. \tag{2.59}$$

Let $\mathcal{F}_P \oplus \mathcal{F}_P$ act on the vector $\begin{pmatrix} x_{-M} \\ y_{-M} \\ .. \\ .. \\ x_M \\ y_M \end{pmatrix}$. Then

$$\mathcal{F}_P \oplus \mathcal{F}_P = \frac{1}{2M+1} \begin{pmatrix} z_{P,-M}^{-M} & 0 & z_{P,-M+1}^{-M} & 0 & \ldots\ldots & z_{P,M}^{-M} & 0 \\ 0 & z_{P,-M}^{-M} & 0 & z_{P,-M+1}^{-M} & \ldots. & 0 & z_{P,M}^{-M} \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \\ z_{P,-M}^{M} & 0 & z_{P,-M+1}^{M} & 0 & \ldots\ldots & z_{P,M}^{M} & 0 \\ 0 & z_{P,-M}^{M} & 0 & z_{P,-M+1}^{M} & \ldots. & 0 & z_{P,M}^{M} \end{pmatrix}.$$

It can be checked that by interchanging rows and columns an even number of times in the matrix above, we have

$$\det(\mathcal{F}_P \oplus \mathcal{F}_P) = \det \begin{pmatrix} \mathcal{F}_P & 0 \\ 0 & \mathcal{F}_P \end{pmatrix} = (-i)^{2M} = (-1)^M, \tag{2.60}$$

where $\mathcal{F}_P$ is given in (2.57).

Now let us calculate the matrix of the transformation R_1^P where $e_{+,k}^P(z)$ and $e_{-,-k}^P(z)$ are located in the k^{th} slot corresponding to the k^{th} Fourier coefficient. Notice here that we have ordered the eigenvector basis in the order $e_{+,k}^P e_{-,-k}^P$. The reason for this ordering is that the determinant becomes much easier to calculate compared for the one with the 'standard ordering' $e_{+,k}^P, e_{-,k}^P$. Define

$$R_{1,k}^P := \frac{1}{\sqrt{2}} \begin{pmatrix} a(z_P(k)) & a(z_P(k)) \\ iz_P(k)a(z_P(k))^{-1} & -iz_P(k)a(z_P(k))^{-1} \end{pmatrix}$$

The matrix of R_1^P is given by

$$R_1^P := \left(\bigoplus_{k=-M}^{M} R_{1,k}^P \right)^{-1},$$

where

$$\det R_1^P = \left((-i)^{2M+1} \prod_{k=-M}^{M} z_P(k) \right)^{-1} = i^{2M+1}. \tag{2.61}$$

The matrix of R_2^P that goes from $\{a_k^*, a_{-k}\}_{k=-M}^{M}$ to $\{q(z_P(k), p(z_P(k)\}_{k=-M}^{M}$ is given by

$$R_2^P = \begin{pmatrix} -i & 1 & 0 & 0 & .. & & & & .. & 0 & 0 \\ 0 & 0 & & & & & & & & i & 1 \\ .. & & -i & 1 & & & & & & & .. \\ .. & & & & & & i & 1 & & & .. \\ & & & & & -i & 1 & & & & \\ & & & & & i & 1 & & & & \\ & & & & & & -i & 1 & & & \\ .. & & i & 1 & & & & & & & .. \\ 0 & 0 & & & & & & & & -i & 1 \\ i & 1 & .. & & & & & & .. & 0 & 0 \end{pmatrix}^{-1}.$$

By interchanging the rows, we obtain,

$$\det R_2^P = (-1)^M \det \begin{pmatrix} -i & 1 & & & & \\ i & 1 & & & & \\ & & .. & & & \\ & & & .. & & \\ & & & & -i & 1 \\ & & & & i & 1 \end{pmatrix}^{-1} = (-1)^M \frac{1}{2^{2M+1}} i^{2M+1}. \tag{2.62}$$

It follows from (2.56), (2.60), (2.61) and (2.62) that

$$\det R^P = \det(R_3) \det(\mathcal{F}_P \oplus \mathcal{F}_P) \det R_1^P \det R_2^P = -1.$$

Now, in the polarization $W = W_+^A \oplus W_-^A$, we define

$$a_k^{A*} := c(e_{+,k}^A) \quad \text{and} \quad a_k^P := a(e_{-,k}^A),$$

where $e_{+,k}^A$ and $e_{-,k}^A$ are given in (2.45) and (2.46). In the z_A coordinates we define

$$p(z_A(k)) := (a_k^{A*} + a_k^A) \quad \text{and} \quad q(z_A(k)) := i(a_k^A - a_k^{A*}).$$

Again, a similar calculation to the one given in (2.54) gives

$$(-1)^{\mathcal{N}_A} = \prod_{k=-M}^{M} ip(z_A(k))q(z_A(k)).$$

The calculation of $\det R^A$ is similar. The linear transformation R^A is a composition of transformations

$$R^A : \mathbb{C}^{2(2M+1)} \xrightarrow{\mathcal{F}_A \oplus \mathcal{F}_A} \mathbb{C}^{2(2M+1)} \xrightarrow{R_1^A} \mathbb{C}^{2(2M+1)} \xrightarrow{R_2^A} \mathbb{C}^{2(2M+1)} \xrightarrow{x \mapsto \sqrt{2}x} \mathbb{C}^{2(2M+1)},$$

where $\mathcal{F}_A$ is the finite inverse Fourier transform in the anti-periodic representation, R_1^A is the transformation from the Fourier series representation to the basis $\{e^A_{+,k}, e^A_{-,-k}\}_{k=-M}^M$, while R_2^A is the transformation from $\{a_k^{A*}, a^A_{-k}\}_{k=-M}^M$ to $\{q(z_A(k)), p(z_A(k))\}_{k=-M}^M$. The matrix of $\mathcal{F}_A : \mathbb{C}^{2M+1} \to \mathbb{C}^{2M+1}$ is given by

$$\mathcal{F}_A := \frac{1}{\sqrt{2M+1}} \begin{pmatrix} z_{A,-M}^{-M} & .. & .. & z_{A,M}^{-M} \\ .. & & & .. \\ .. & & & .. \\ z_{A,-M}^{M} & .. & .. & z_{A,M}^{M} \end{pmatrix}, \tag{2.63}$$

where $z_{A,k}^l := z_A^l(k) = e^{\frac{2\pi i\left(k+\frac{1}{2}\right)l}{2M+1}}$ for $k, l = -M, ..., M$. We recognize this matrix as the Vandermonde matrix. The determinant of the Vandermonde matrix is well-known and can be found for example in [Si96] on page 238. Using this, we obtain

$$\begin{aligned} \det \mathcal{F}_A &= \tfrac{1}{\sqrt{2M+1}^{2M+1}} (z_{A,-M}^{-M} .... z_{A,M}^{-M})^{2M+1} \begin{vmatrix} 1 & z^1_{A,-M} & z^2_{A,-M} & .... & z^{2M}_{A,-M} \\ .. & & & & .. \\ .. & & & & .. \\ 1 & z^1_{A,M} & z^2_{A,M} & .... & z^{2M}_{A,M} \end{vmatrix} \\ &= (-1)^M \frac{1}{\sqrt{2M+1}^{2M+1}} \prod_{-M\le j<k\le M} (w^{k+\frac{1}{2}} - w^{j+\frac{1}{2}}), \end{aligned} \tag{2.64}$$

where $w^k = e^{\frac{2\pi i k}{2M+1}}$. The right hand side of the equation in (2.64) can be written

$$(-1)^M \tfrac{1}{\sqrt{2M+1}^{2M+1}} (-1)^M \prod_{-M\le j<k\le M} (w^k - w^j), \tag{2.65}$$

where we used that

$$\prod_{-M\le j<k\le M} w^{\frac{1}{2}} = (-1)^{\binom{2M+1}{2}} = (-1)^M.$$

Since $\mathcal{F}_P$ clearly also is a Vandermonde matrix, we have

$$\det \mathcal{F}_P = \frac{1}{\sqrt{2M+1}^{2M+1}} \prod_{-M \le j < k \le M} (w^k - w^j). \tag{2.66}$$

It follows from (2.64), (2.65) and (2.66) that

$$\det \mathcal{F}_A = \det \mathcal{F}_P = (-i)^M. \tag{2.67}$$

Let

$$\mathcal{F}_A \oplus \mathcal{F}_A : \mathbb{C}^{2(2M+1)} \longrightarrow \mathbb{C}^{2(2M+1)}$$

act on the vector $\begin{pmatrix} x_{-M} \\ y_{-M} \\ .. \\ .. \\ x_M \\ y_M \end{pmatrix}$. We obtain

$$\det(\mathcal{F}_A \oplus \mathcal{F}_A) = \det \begin{pmatrix} \mathcal{F}_A & 0 \\ 0 & \mathcal{F}_A \end{pmatrix} = (-1)^M. \tag{2.68}$$

Define

$$R^A_{1,k} := \frac{1}{\sqrt{2}} \begin{pmatrix} a(z_A(k)) & a(z_A(k)) \\ i z_A(k) a(z_A(k))^{-1} & -i z_A(k) a(z_A(k))^{-1} \end{pmatrix}.$$

Then the matrix of R^A_1 is given by

$$R^A_1 := \left(\bigoplus_{k=-M}^{M} R^A_{1,k} \right)^{-1},$$

where

$$\det R^A_1 = \left((-i)^{2M+1} \prod_{k=-M}^{M} z_A(k) \right)^{-1} = (-i)^{2M+1}. \tag{2.69}$$

Furthermore, we have

$$\det R^A_2 = \det R^P_2 = (-1)^M \frac{1}{2^{2M+1}} i^{2M+1}. \tag{2.70}$$

Combining (2.56), (2.68), (2.69) and (2.70) we obtain

$$\det R^A = \det(R_3) \det(\mathcal{F}_A \oplus \mathcal{F}_A) \det R^A_1 \det R^A_2 = 1$$

and the proposition is proved.

□

We want to choose a representation of the transfer matrix V^A in the Fock representation $\mathrm{Alt}(W_+^A)$ such that the vacuum vector 0_A in the isotropic splitting of $W = W_+^A \oplus W_-^A$ is an eigenvector for V^A associated with its largest eigenvalue. Similarly we want to choose a representation of the transfer matrix V^P in the Fock representation $\mathrm{Alt}(W_+^P)$ such that the vacuum vector 0_P in the isotropic splitting of $W = W_+^P \oplus W_-^P$ is an eigenvector for V^P corresponding to its largest eigenvalue. By Theorem (2.3), we can write V as the map,

$$V^A \oplus V^P : \mathrm{Alt}_{\mathrm{even}}(W_+^A) \oplus \mathrm{Alt}_{\mathrm{even}}(W_+^P) \to \mathrm{Alt}_{\mathrm{even}}(W_+^A) \oplus \mathrm{Alt}_{\mathrm{even}}(W_+^P),$$

where

$$V^A \simeq V|_{\mathrm{Alt}_{\mathrm{even}}(W_+^A)} \quad \text{and} \quad V^P \simeq V|_{\mathrm{Alt}_{\mathrm{even}}(W_+^P)}.$$

Let T_+^A denote the restriction of T^A to W_+^A, and let T_+^P denote the restriction of T^P to W_+^P. Define as in [Pal06] the linear transformations $\Gamma(T_+^A)$ and $\Gamma(T_+^P)$ acting on $\mathrm{Alt}(W_+^A)$ and $\mathrm{Alt}(W_+^P)$ respectively as

$$\Gamma(T_+^A) = 1 \oplus T_+^A \oplus (T_+^A \otimes T_+^A) \oplus \cdots \oplus (\underbrace{T_+^A \otimes \cdots \otimes T_+^A}_{\text{2M+1}}). \tag{2.71}$$

and

$$\Gamma(T_+^P) = 1 \oplus T_+^P \oplus (T_+^P \otimes T_+^P) \oplus \cdots \oplus (\underbrace{T_+^P \otimes \cdots \otimes T_+^P}_{\text{2M+1}}). \tag{2.72}$$

For example, we have

$$(T_+^A \otimes T_+^A)(e_{+,k}^A \wedge e_{+,l}^A) = e^{-\gamma(z_A(k))} e^{-\gamma(z_A(l))} (e_{+,k}^A \wedge e_{+,l}^A).$$

It can be checked that

$$T(\Gamma(T_+^A)) = T^A \quad \text{and} \quad T(\Gamma(T_+^P)) = T^P.$$

It follows that there exists two real numbers λ_0^A and λ_0^P such that the representations of V^A and V^P in the Fock representations $\mathrm{Alt}(W_+^A)$ and $\mathrm{Alt}(W_+^P)$ respectively are given by

$$V^A = \lambda_0^A \Gamma(T_+^A)|_{\mathrm{Alt}_{\mathrm{even}}(W_+^A)} \quad \text{and} \quad V^P = \lambda_0^P \Gamma(T_+^P)|_{\mathrm{Alt}_{\mathrm{even}}(W_+^P)}. \tag{2.73}$$

The spectra of the induced rotations T^A and T^P do not contain 1 so both T_+^A and T_+^P are strict contractions. Then it follows from (2.71) that the largest eigenvalue of $\Gamma(T_+^A)$ is 1 with the unique eigenvector given by the vacuum vector 0_A in $\text{Alt}(W_+^A)$. Hence, λ_0^A is the largest eigenvalue of V^A with corresponding eigenvector 0_A. A similar argument shows that λ_0^P is the largest eigenvalue of V^P with corresponding eigenvector 0_P. We now determine the eigenvalues for the transfer matrices V^A and V^P. We start by proving the following lemma.

Lemma 2.4. *We have*

$$\det(\Gamma(T_+^P)) = (\det T_+^P)^{2^{2M}} \quad \textit{and} \quad \det(\Gamma(T_+^A)) = (\det T_+^A)^{2^{2M}} \quad \textit{for} \quad M \geq 0, \quad (2.74)$$

where $\Gamma(T_+^A)$ *and* $\Gamma(T_+^P)$ *are defined in (2.71) and (2.72).*

Proof. We prove the first equation since the proof of the second one is exactly the same. Let us consider the tensor product $(T_+^P)^{\otimes k}$ which acts on $\text{Alt}^k(W_+^P)$. We have that $\dim(\text{Alt}^k(W_+^P)) = \binom{2M+1}{k}$ for $k = 1, ..., 2M+1$. The number of eigenvalues for T_+^P is $2M+1$. We want to determine the multiplicity of $e^{-\gamma(n)}$ in the product of eigenvalues for $(T_+^P)^{\otimes k}$ acting on $\text{Alt}^k(W_+^P)$. Thus, we want to figure out the number of ways we can combine $e^{-\gamma(n)}$ with the remaining factors $e^{-\gamma(l)}$ for $l = -M, ..., M$ and $n \neq l$ in the product $e^{-\gamma(l_1)-\gamma(l_2)...-\gamma(l_k)}$, where $-M \leq l_1 < l_2 < ... < l_k \leq M$. Since we already have 'used up' $e^{-\gamma(n)}$, this is equivalent to the number of ways we can pick out $(k-1)$ elements of $e^{-\gamma(l)}$ from $2M$ elements. This number is $\binom{2M}{k-1}$. We obtain,

$$\det(\underbrace{T_+^P \otimes \cdots \otimes T_+^P}_{\text{k}}) = \left(e^{-\gamma(\theta_{-M}^P)} e^{-\gamma(\theta_{-M+1}^P)} \cdots e^{-\gamma(\theta_M^P)}\right)^{\binom{2M}{k-1}} = (\det(T_+^P))^{\binom{2M}{k-1}}. \quad (2.75)$$

It follows from (2.75) that

$$\begin{aligned} \det(\Gamma(T_+^P)) &= (\det T_+^P)^{\binom{2M}{0}} (\det T_+^P)^{\binom{2M}{1}} \cdots (\det T_+^P)^{\binom{2M}{j}} \cdots (\det T_+^P)^{\binom{2M}{2M}} \\ &= [\det(T_+^P)]^{\sum_{j=0}^{2M} \binom{2M}{j}}. \end{aligned}$$

It is well-known that

$$\sum_{j=0}^{2M} \binom{2M}{j} = 2^{2M}$$

which can be proved by induction and using the fact that for $n, j \in \mathbb{N}$ and $n > j$,

$$\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}. \tag{2.76}$$

Hence, the lemma is proved. □

Recall that Σ_A and Σ_P denote the anti-periodic and periodic spectrum respectively.

Proposition 2.5. *The eigenvalues of the transfer matrices V^A and V^P are given by*

$$\exp\left[\tfrac{1}{2}(\pm\gamma(z_A(-M)) \pm .... \pm \gamma(z_A(M)))\right] \tag{2.77}$$

and

$$\exp\left[\tfrac{1}{2}(\pm\gamma(z_P(-M)) \pm .... \pm \gamma(z_P(M)))\right], \tag{2.78}$$

respectively, where γ is the positive root of

$$\cosh\gamma(z) = c_1 c_2^* - s_1 s_2^* \frac{z+z^{-1}}{2} \quad \textit{for} \quad z \in \Sigma_A \quad \textit{and} \quad z \in \Sigma_P.$$

There is an even number of minus signs in both spectra for $T < T_C$. The largest eigenvalues of the transfer matrices V^A and V^P are given by

$$\prod_{z_A \in \Sigma_A} e^{\frac{\gamma(z_A)}{2}} \quad \textit{and} \quad \prod_{z_P \in \Sigma_P} e^{\frac{\gamma(z_P)}{2}}$$

respectively.

Proof. Recall that the transfer matrices V^A and V^P given in (2.12) and (2.13) have factors of the form $\prod e^{pq}$. Recall the well-known formula,

$$\det(e^A) = e^{\mathrm{tr}(A)},$$

for a complex $n \times n$ matrix A, where $\mathrm{tr}(A)$ is the trace of A. It follows that

$$\det e^{\sum pq} = e^{\sum \mathrm{tr}(pq)}.$$

Using the Clifford relations and properties of trace, we have

$$\operatorname{tr}(pq) = \operatorname{tr}(qp) = -\operatorname{tr}(pq)$$

which implies that $\operatorname{tr}(pq) = 0$. Hence $\det e^{\sum pq} = 1$ which implies that $\det V^A = \det V^P = 1$. From (2.73) we have the identities

$$\det V^A = \det[\lambda_0^A \, \Gamma(T_+^A)] \quad \text{and} \quad \det V^P = \det[\lambda_0^P \, \Gamma(T_+^P)]. \tag{2.79}$$

Since $\operatorname{Alt}(W_+^A)$ and $\operatorname{Alt}(W_+^P)$ each has dimension 2^{2M+1}, and $\Gamma(T_+^A)$ and $\Gamma(T_+^P)$ act on $\operatorname{Alt}(W_+^A)$ and $\operatorname{Alt}(W_+^P)$ respectively, it follows from (2.79) that

$$1 = (\lambda_0^A)^{2^{2M+1}} \det(\Gamma(T_+^A)) \quad \text{and} \quad 1 = (\lambda_0^P)^{2^{2M+1}} \det(\Gamma(T_+^P)). \tag{2.80}$$

From Lemma 2.4, we have

$$\det \Gamma(T_+^A) = (\det T_+^A)^{2^{2M}} \quad \text{and} \quad \det \Gamma(T_+^P) = (\det T_+^P)^{2^{2M}},$$

and it follows from (2.80) that

$$\lambda_0^A = \frac{1}{\sqrt{\det(T_+^A)}} \quad \text{and} \quad \lambda_0^P = \frac{1}{\sqrt{\det(T_+^P)}}.$$

Since the eigenvalues of T_+^A and T_+^P are given by the sets $\{e^{-\gamma(z_A)}\}$ and $\{e^{-\gamma(z_P)}\}$ respectively, we obtain

$$\lambda_0^A = \prod_{z_A \in \Sigma_A} e^{\frac{\gamma(z_A)}{2}} \quad \text{and} \quad \lambda_0^P = \prod_{z_P \in \Sigma_P} e^{\frac{\gamma(z_P)}{2}}.$$

The eigenvectors of $\Gamma(T_+^A)$ restricted to $\operatorname{Alt}_{\text{even}}(W_+^A)$ are given by $\{0_A\}$, $\{0 \oplus 0 \oplus e_{+,k_1}^A \wedge e_{+,k_2}^A \oplus 0 \oplus \cdots \oplus 0\}_{-M \le k_1 < k_2 \le M}, \ldots,$ $\{0 \oplus \cdots \oplus 0 \oplus e_{+,k_1}^A \wedge \ldots\ldots \wedge e_{+,k_{2M-1}}^A \wedge e_{+,k_{2M}}^A\}_{-M \le k_1 < \ldots. < k_{2M-1} < k_{2M} \le M}$, and it follows from (2.73) that the eigenvalues of V^A are given by (2.77). A similar argument shows that the eigenvalues of V^P are given by (2.78).

□

2.6. Induced Rotation associated with the Spin Operator

In this section we compute the induced rotation associated with the spin operator. We determine the matrix corresponding to this rotation and the orthonormal basis of eigenvectors of $T(V)$. Recall that

$$U = \prod_{k=-M}^{M} i p_k q_k.$$

The spin operators $\sigma_j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_j$ anti-commute with U, so we have

$$\sigma_j \tfrac{1}{2}(I - U) = \tfrac{1}{2}(I + U)\sigma_j.$$

Thus, σ_j map the -1 eigenspace for U into the $+1$ eigenspace for U. We therefore restrict the spin operator as a map from the z_P representation to the z_A representation (or the other way around). Then it follows from Theorem 2.3, that the spin operators σ_j are given by the maps

$$\sigma_j : \mathrm{Alt}_{\mathrm{even}}(W_+^P) \to \mathrm{Alt}_{\mathrm{even}}(W_+^A).$$

We now compute the induced rotation associated with the spin operator. Using the Clifford relations, we have for $-M \leq k, j \leq M$,

$$\begin{aligned} \sigma_j p_k \sigma_j^{-1} &= -\operatorname{sgn}(k - j - 1) p_k, \\ \sigma_j q_k \sigma_j^{-1} &= -\operatorname{sgn}(k - j) q_k \end{aligned}$$

where

$$\operatorname{sgn}(x) = \begin{cases} +1 & \text{if } x \geq 0; \\ -1 & \text{if } x < 0. \end{cases}$$

We are interested in locating the spin operator at the center in a row. We therefore formulate the matrix representation of the induced rotation associated with σ_0. Let s_0 denote multiplication by $-\operatorname{sgn}(k)$ and s_1 denote multiplication by $-\operatorname{sgn}(k-1)$. Then the induced rotation s for the spin operator σ_0 is given by

$$s := T(\sigma_0) = \begin{pmatrix} s_0 & 0 \\ 0 & s_1 \end{pmatrix}.$$

In the Fourier representation we have the following matrix representation for s_0:

$$s_0 f(z_A) = \frac{1}{2M+1} \sum_{z_P \in \Sigma_P} \frac{2z_P}{z_A - z_P} f(z_P). \tag{2.81}$$

The matrix representation for s_1 is given by

$$s_1 f(z_A) = \frac{1}{2M+1} \sum_{z_P \in \Sigma_P} \frac{2z_A}{z_A - z_P} f(z_P). \tag{2.82}$$

We include the calculation for (2.81). We have

$$\begin{aligned}
s_0 f(z_A) &= \sum_{k=-M}^{M} \frac{1}{2M+1} \sum_{z_P \in \Sigma_P} f(z_P) z_P^{-k} (-\operatorname{sgn}(k)) z_A^k \\
&= \frac{1}{2M+1} \sum_{z_P \in \Sigma_P} f(z_P) \sum_{k=-M}^{-1} (\tfrac{z_A}{z_P})^k - \frac{1}{2M+1} \sum_{z_P \in \Sigma_P} f(z_P) \sum_{k=0}^{M} (\tfrac{z_A}{z_P})^k \\
&= \frac{1}{2M+1} \sum_{z_P \in \Sigma_P} f(z_P) \left(\frac{(\frac{z_A}{z_P})^0 - (\frac{z_A}{z_P})^{-M}}{(\frac{z_A}{z_P} - 1)} + \frac{-(\frac{z_A}{z_P})^{M+1} + (\frac{z_A}{z_P})^0}{(\frac{z_A}{z_P} - 1)} \right) \\
&= \frac{1}{2M+1} \sum_{z_P \in \Sigma_P} f(z_P) \frac{2z_P}{z_A - z_P},
\end{aligned}$$

where we used that $-z_A^{M+1} = z_A^{-M}$, $z_P^{M+1} = z_P^{-M}$ and the convention $\operatorname{sgn}(0) = 1$. We define the matrix of the operator s to be given by

$$s := \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C and D are given by the maps

$$\begin{array}{llll} A: & W_+^P \to W_+^A, & B: & W_-^P \to W_+^A, \\ C: & W_+^P \to W_-^A, & D: & W_-^P \to W_-^A. \end{array} \tag{2.83}$$

We compute the matrices corresponding to A, B, C and D and the orthonormal bases $\{e^A_{\pm,k}\}$ and $\{e^P_{\pm,k}\}$, where $e^A_{\pm,k}(z)$ and $e^P_{\pm,k}(z)$ are given in (2.45), (2.46), (2.47) and (2.48), and where $(e^A_{-,k}, e^A_{+,l}) = \delta_{kl}$ and $(e^P_{-,k}, e^P_{+,l}) = \delta_{kl}$. We show how to compute the D matrix elements. The computation of the other elements are similar. We have for $W \ni e_k(z) := e^P_{+,k}(z) + e^P_{-,k}(z)$,

$$Q_-^A s Q_-^P e_k = Q_-^A s e^P_{-,k} = \sum_{l=-M}^{M} D_{lk} e^A_{-,l}.$$

Then

$$
\begin{aligned}
D_{lk} &= (e^A_{+,l}, s\, e^P_{-,k}) \\
&= \frac{1}{2} a(z_A(l)) s_0(\overline{z_A}(l), \overline{z_P}(k)) a^{-1}(z_P(k)) + \\
&+ \frac{1}{2} a^{-1}(z_A(l)) z_A(l) s_1(\overline{z_A}(l), \overline{z_P}(k)) z_P^{-1}(k) a(z_P(k)) \\
&= \frac{1}{(2M+1)} \frac{z_A(l)}{z_P(k) - z_A(l)} [a(z_A(l)) a^{-1}(z_P(k)) + a^{-1}(z_A(l)) a(z_P(k))].
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
A_{lk} &= (e^A_{-,l}, s\, e^P_{+,k}) \\
&= \frac{1}{2} a^{-1}(z_A(l)) s_0(z_A(l), z_P(k)) a(z_P(k)) + \\
&+ \frac{1}{2} a(z_A(l)) z_A^{-1}(l) s_1(z_A(l), z_P(k)) z_P(k) a^{-1}(z_P(k)) \\
&= \frac{1}{(2M+1)} \frac{z_P(k)}{z_A(l) - z_P(k)} [a^{-1}(z_A(l)) a(z_P(k)) + a(z_A(l)) a^{-1}(z_P(k))],
\end{aligned}
$$

$$
\begin{aligned}
B_{lk} &= (e^A_{-,l}, s\, e^P_{-,k}) \\
&= \frac{1}{2} a^{-1}(z_A(l)) s_0(z_A(l), \overline{z_P}(k)) a^{-1}(z_P(k)) + \\
&- \frac{1}{2} a(z_A(l)) z_A^{-1}(l) s_1(z_A(l), \overline{z_P}(k)) z_P^{-1}(k) a(z_P(k)) \\
&= \frac{1}{(2M+1)} \frac{1}{z_A(k) z_P(k) - 1} [a^{-1}(z_A(l)) a^{-1}(z_P(k)) - a(z_A(l)) a(z_P(k))],
\end{aligned}
$$

$$
\begin{aligned}
C_{lk} &= (e^A_{+,l}, s\, e^P_{+,k}) \\
&= \frac{1}{2} a(z_A(l)) s_0(\overline{z_A}(l), z_P(k)) a(z_P(k)) + \\
&- \frac{1}{2} a^{-1}(z_A(l)) z_A(l) s_1(\overline{z_A}(l), z_P(k)) z_P(k) a^{-1}(z_P(k)) \\
&= \frac{1}{(2M+1)} \frac{z_A(l) z_P(k)}{1 - z_P(k) z_A(l)} [a(z_A(l)) a(z_P(k)) - a^{-1}(z_A(l)) a^{-1}(z_P(k))],
\end{aligned}
$$

where $a(z)$ is defined in (2.44). Thus, we have proved the following:

Proposition 2.6. *The matrix of the induced rotation s associated with the spin operator σ_0 computed in terms of the orthonormal bases $\{e^A_{\pm,k}\}$ and $\{e^P_{\pm,k}\}$ defined in (2.45), (2.46), (2.47) and (2.48) is given by*

$$
s = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
$$

where A, B, C and D are given by the maps in (2.83) and have the following matrix elements for $k, l = -M, ..., M$:

$$A_{lk} = \frac{1}{(2M+1)} \frac{z_P(k)}{z_A(l) - z_P(k)} [a^{-1}(z_A(l))a(z_P(k)) + a(z_A(l))a^{-1}(z_P(k))], \tag{2.84}$$

$$B_{lk} = \frac{1}{(2M+1)} \frac{1}{z_A(k)z_P(k) - 1} [a^{-1}(z_A(l))a^{-1}(z_P(k)) - a(z_A(l))a(z_P(k))], \tag{2.85}$$

$$C_{lk} = \frac{1}{(2M+1)} \frac{z_A(l)z_P(k)}{1 - z_P(k)z_A(l)} [a(z_A(l))a(z_P(k)) - a^{-1}(z_A(l))a^{-1}(z_P(k))], \tag{2.86}$$

$$D_{lk} = \frac{1}{(2M+1)} \frac{z_A(l)}{z_P(k) - z_A(l)} [a(z_A(l))a^{-1}(z_P(k)) + a^{-1}(z_A(l))a(z_P(k))], \tag{2.87}$$

where

$$a(z) = \sqrt{\frac{\mathcal{A}_1(z)\mathcal{A}_2(z)}{\mathcal{A}_1(z^{-1})\mathcal{A}_2(z^{-1})}} \quad \text{and} \quad \mathcal{A}_j = \sqrt{\alpha_j - z}, \qquad j = 1, 2$$

for $z \in \Sigma_A$ or $z \in \Sigma_P$.

In order to make the matrix elements of D and A real and the matrix elements of B and C pure imaginary, we multiply $e^A_{+,k}(z)$ and $e^P_{+,k}(z)$ by a factor of $\sqrt{\frac{i}{z}}$, and $e^A_{-,k}(z)$ and $e^P_{-,k}(z)$ by a factor of $\sqrt{zi}$ for $z \in \Sigma_A$ and $z \in \Sigma_P$ respectively. Here the square root is taken in the right half plane with positive real part. It follows then that the matrix elements of s are given by:

$$A_{lk} = \frac{1}{(2M+1)} \frac{i\sqrt{z_P(k)}\sqrt{z_A(l)}}{z_A(l) - z_P(k)} [a^{-1}(z_A(l))a(z_P(k)) + a(z_A(l))a^{-1}(z_P(k))], \tag{2.88}$$

$$B_{lk} = \frac{1}{(2M+1)} \frac{i\sqrt{z_P(k)}\sqrt{z_A(l)}}{z_A(k)z_P(k) - 1} [a^{-1}(z_A(l))a^{-1}(z_P(k)) - a(z_A(l))a(z_P(k))], \tag{2.89}$$

$$C_{lk} = \frac{1}{(2M+1)} \frac{i\sqrt{z_P(k)}\sqrt{z_A(l)}}{1 - z_P(k)z_A(l)} [a(z_A(l))a(z_P(k)) - a^{-1}(z_A(l))a^{-1}(z_P(k))], \tag{2.90}$$

$$D_{lk} = \frac{1}{(2M+1)} \frac{i\sqrt{z_P(k)}\sqrt{z_A(l)}}{z_P(k) - z_A(l)} [a(z_A(l))a^{-1}(z_P(k)) + a^{-1}(z_A(l))a(z_P(k))]. \tag{2.91}$$

Thus, we notice that $A_{lk} = -D_{lk}$ and $B_{lk} = C_{lk}$.

Let F^P and F^A denote the Fock representations associated with the Clifford relations acting on $\mathrm{Alt}(W_+^P)$ and $\mathrm{Alt}(W_+^A)$ respectively. We have

$$F^P(x) = c(x_+) + a(x_-), \tag{2.92}$$

where $x_+ \in W_+^P$ and $x_- \in W_-^P$. Here W_-^P is identified with the dual W_+^{P*} through the bilinear form $W_+^P \ni x_+ \to (x_+, x_-)$ for $x_- \in W_-^P$. The representation p_k, q_k acting on $\otimes_{j=-M}^M \mathbb{C}_j^2$ is an irreducible $*$ representation of the Clifford algebra $\mathrm{Cliff}(W)$ on $\otimes_{j=-M}^M \mathbb{C}_j^2$ [Pal06]. Consider the isotropic splitting

$$W = \mathrm{span}\{p_k + iq_k\} \oplus \mathrm{span}\{p_k - iq_k\}$$

with vacuum vector

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes ... \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Let F denote the Fock representation associated with the Clifford relations acting on $\mathbb{C}^{2(2M+1)}$. Since $\sigma_0 x \sigma_0^{-1} = sx$ for $x \in W$, where $\sigma_0 : \mathbb{C}^{2(2M+1)} \to \mathbb{C}^{2(2M+1)}$, it follows that

$$\sigma_0 F(x) = F(s(x))\sigma_0. \tag{2.93}$$

The Fock representation defined in (2.92) is an irreducible $*$ representation of the Clifford algebra. Any irreducible $*$ representations of the Clifford algebra over W are unitarily equivalent [BW35]. Thus, it follows that there exists a unitary map

$$U_P : \mathbb{C}^{2(2M+1)} \to \mathrm{Alt}(W_+^P)$$

such that

$$F^P(x) = U_P F(x) U_P^{-1} \tag{2.94}$$

and a unitary map

$$U_A : \mathbb{C}^{2(2M+1)} \to \mathrm{Alt}(W_+^A)$$

such that

$$F^A(s(x)) = U_A F(s(x)) U_A^{-1}. \tag{2.95}$$

We represent these maps in the following diagram:

$$\begin{array}{ccc}
\mathrm{Alt}(W_+^A) & \xrightarrow{F^A \circ s} & \mathrm{Alt}(W_+^A) \\
\uparrow{\scriptstyle U_A} & & \uparrow{\scriptstyle U_A} \\
\mathbb{C}^{2(2M+1)} & \xrightarrow{F \circ s} & \mathbb{C}^{2(2M+1)} \\
\uparrow{\scriptstyle \sigma_0} & & \uparrow{\scriptstyle \sigma_0} \\
\mathbb{C}^{2(2M+1)} & \xrightarrow{F} & \mathbb{C}^{2(2M+1)} \\
\uparrow{\scriptstyle U_P^{-1}} & & \uparrow{\scriptstyle U_P^{-1}} \\
\mathrm{Alt}(W_+^P) & \xrightarrow{F^P} & \mathrm{Alt}(W_+^P).
\end{array}$$

It follows from (2.93), (2.94) and (2.95) that

$$U_A \sigma_0 U_P^{-1} F^P(x) = F^A(s(x)) U_A \sigma_0 U_P^{-1}.$$

The map

$$\sigma := U_A \sigma_0 U_P^{-1} : \mathrm{Alt}(W_+^P) \to \mathrm{Alt}(W_+^A)$$

is an intertwining map, and is unique up to a constant by Schur's lemma. In the Fourier series representation, s is given by $s = s_0 \oplus s_1$, where s_0 and s_1 are given in (2.81) and (2.82).

Thus, we have the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Alt}(W_+^P) & \xrightarrow{\sigma} & \mathrm{Alt}(W_+^A) \\
\uparrow{\scriptstyle F^P} & & \uparrow{\scriptstyle F^A \circ s} \\
\mathrm{Alt}(W_+^P) & \xrightarrow{\sigma} & \mathrm{Alt}(W_+^A),
\end{array}$$

where σ satisfies the intertwining relation

$$\sigma F^P(x) = F^A(s(x)) \sigma. \tag{2.96}$$

We will use this relation in Section 3.1 when we find an expression for the spin matrix elements. We are interested in understanding the $N \to \infty$ limit of the two point correlation for the spin operator. As we will see in the next section, the eigenvector associated with the largest eigenvalue for the transfer matrix will dominate in this limit under the action of the spin operator σ. In order to compute this limit, we will need to know the spin matrix elements for σ.

2.7. Two-Point Correlation Function

Recall from (2.1) that $Z_\Lambda = \mathrm{tr}(V^{2N+1})$. It can be checked (see [Kau49] and [Th72]) that for $n > m$,

$$\langle \sigma_{mi}\sigma_{nj}\rangle_\Lambda = \frac{\mathrm{tr}(\sigma_i V^{n-m}\sigma_j V^{(2N+1)-n+m})}{\mathrm{tr}(V^{2N+1})}. \tag{2.97}$$

The largest eigenvalue of the transfer matrix is the one in the anti-periodic spectrum. This is found to be $\lambda_0^A := e^{\frac{1}{2}\sum_{j=-M}^{M}\gamma(e^{i\theta_j^A})}$ (see 2.77).

Recall that in the last section we found a Fock representation of the transfer matrix in which the vacuum vector 0_A in $\mathrm{Alt}(W_+^A)$ is the eigenvector corresponding to the largest eigenvalue λ_0^A of the transfer matrix. Now we divide by λ_0^A in the expression for the two-point correlation function in (2.97) and obtain

$$\langle \sigma_{mi}\sigma_{nj}\rangle_\Lambda = \frac{\mathrm{tr}\left(\sigma_i\big(\frac{V}{\lambda_0^A}\big)^{n-m}\sigma_j\big(\frac{V}{\lambda_0^A}\big)^{(2N+1)-n+m}\right)}{1+\big(\frac{\lambda_0^P}{\lambda_0^A}\big)^{2N+1}+\big(\frac{\lambda_1}{\lambda_0^A}\big)^{2N+1}+\ldots+\big(\frac{\lambda_{4M}}{\lambda_0^A}\big)^{2N+1}},$$

where we denoted the eigenvalues of $T(V)$ by λ_k for $k = 1, \ldots, 4M$. Since λ_0^A is the largest eigenvalue of the transfer matrix, we have

$$\lim_{N\to\infty}\big(\frac{\lambda_j}{\lambda_0^A}\big)^{(2N+1)-n+m} = 0$$

and

$$\lim_{N\to\infty}\big(\frac{\lambda_0^P}{\lambda_0^A}\big)^{(2N+1)-n+m} = 0.$$

Thus, in the semi-infinite volume limit, $N \to \infty$, only the terms which involve the vacuum vector 0_A survives. Recall that

$$V = V^A \oplus V^P,$$

where

$$V^A = \lambda_0^A \Gamma(T_+^A)|_{\mathrm{Alt_{even}}(W_+^A)}, \quad V^P = \lambda_0^P \Gamma(T_+^P)|_{\mathrm{Alt_{even}}(W_+^P)},$$

and where $\Gamma(T_+^A)$ and $\Gamma(T_+^P)$ are defined in (2.71) and (2.72). Using this and the fact that

$$\Gamma(T_+^A)0_A = 0_A,$$

we obtain

$$\begin{aligned}
\lim_{N\to\infty} \langle \sigma_{mi}\sigma_{nj}\rangle_\Lambda &= \langle 0_A, \sigma_i \big(\tfrac{V}{\lambda_0^A}\big)^{n-m} \sigma_j 0_A\rangle \\
&= \sum_{K=1}^{M} \sum_{\substack{-M\le k_1<\dots\\ \dots<k_{2K}\le M}} \langle 0_A, \sigma_i \big(\tfrac{\lambda_0^P}{\lambda_0^A}\big)^{n-m} \Gamma^{n-m}(T_+^P) e_{+,k_1}^P \wedge \cdots \wedge e_{+,k_{2K}}^P\rangle \times \\
&\times \langle e_{+,k_1}^P \wedge \cdots \wedge e_{+,k_{2K}}^P \sigma_j, 0_A\rangle \\
&= e^{\frac{1}{2}(n-m)\sum_{j=-M}^{M}\left[\gamma\left(e^{i\theta_j^P}\right)-\gamma\left(e^{i\theta_j^A}\right)\right]} \times \\
&\times \sum_{K=1}^{M} \sum_{\substack{-M\le k_1<\dots\\ \dots<k_{2K}\le M}} \langle 0_A, \sigma_i e_{+,k_1}^P \wedge \cdots \wedge e_{+,k_{2K}}^P\rangle \times \\
&\times e^{-(n-m)\sum_{j=1}^{2K}\gamma\left(e^{i\theta_{k_j}^P}\right)} \langle e_{+,k_1}^P \wedge \cdots \wedge e_{+,k_{2K}}^P \sigma_j, 0_A\rangle.
\end{aligned} \tag{2.98}$$

On the torus, where both M and N are fixed, we have the following expression for the two-point correlation function,

$$\begin{aligned}
\langle \sigma_{mi}\sigma_{nj}\rangle_\Lambda = &\sum_{L=1}^{M}\sum_{K=1}^{M} \sum_{\substack{-M\le l_1<\dots\\ \dots<l_{2L}\le M}} \sum_{\substack{-M\le k_1<\dots\\ \dots<k_{2K}\le M}} \langle e_{+,l_1}^A \wedge \cdots \wedge e_{+,l_{2L}}^A, \sigma_i e_{+,k_1}^P \wedge \cdots \wedge e_{+,k_{2K}}^P\rangle \times \\
&\times \langle e_{+,k_1}^P \wedge \cdots \wedge e_{+,k_{2K}}^P \sigma_j, e_{+,l_1}^A \wedge \cdots \wedge e_{+,l_{2L}}^A\rangle \times \\
&\times \Bigg(\Bigg[e^{(n-m)\left[\frac{1}{2}\sum_{j=-M}^{M}\left[\gamma\left(e^{i\theta_j^P}\right)-\gamma\left(e^{i\theta_j^A}\right)\right]\right]} e^{-(n-m)\sum_{j=1}^{2K}\gamma\left(e^{i\theta_{k_j}^P}\right)} \times \\
&\times e^{-(2N+1-n+m)\sum_{j=1}^{2L}\gamma\left(e^{i\theta_{l_j}^A}\right)} \Bigg] + \\
&+ \Bigg[e^{(2N+1-n+m)\left[\frac{1}{2}\sum_{j=-M}^{M}\left[\gamma\left(e^{i\theta_j^P}\right)-\gamma\left(e^{i\theta_j^A}\right)\right]\right]} e^{-(2N+1-n+m)\sum_{j=1}^{2K}\gamma\left(e^{i\theta_{k_j}^P}\right)} \times \\
&\times e^{-(n-m)\sum_{j=1}^{2L}\gamma\left(e^{i\theta_{l_j}^A}\right)} \Bigg]\Bigg) [\mathrm{tr}(\tfrac{V}{\lambda_0^A})^{2N+1}]^{-1}.
\end{aligned}$$

Because of the translation invariance, it is enough to calculate

$$\langle e^{A}_{+,l_1} \wedge \cdots \wedge e^{A}_{+,l_L}, \sigma_{00} e^{P}_{+,k_1} \wedge \cdots \wedge e^{P}_{+,k_K} \rangle \tag{2.99}$$

for $L+K =$ even. In other words, by knowing the formula for the spin matrix elements we will have an expression for the two-point correlation function for the finite periodic Ising model and for the Ising model where the periodic lattice is wrapped on a cylinder. We will prove in Section 3.1 that the spin matrix elements can be expressed in terms of D^{-1}, $D^{-\tau}$, BD^{-1} and $D^{-1}C$. In Section 3.4, we show that the scaling limit of the two-point correlation function can be written as the Pfaffian of a matrix involving D^{-1}. We are therefore particular interested in knowing the inverse of D in (2.91). Bugrij-Lisovyy [BL03] proposed a formula for the spin matrix elements in (2.99) on the finite periodic lattice in the orthonormal basis of transfer matrix eigenstates. In Section 3.3 we show the connection between the Bugrij-Lisovyy formula and the inverse of D which will lead us to a conjecture for D^{-1}.

3. Spin matrix elements on the finite, periodic lattice

3.1. Spin Matrix Elements on the Finite, Periodic Lattice

Recall that the matrix of the induced rotation s associated with the spin operator σ relative to the isotropic splitting $s : W_+^P \oplus W_-^P \to W_+^A \oplus W_-^A$ is denoted by

$$s := \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, D are complex functions. In this section we show that the spin matrix elements computed in the orthonormal basis of transfer matrix eigenstates can be written in terms of the matrix elements of $D^{-\tau}$, BD^{-1} and $D^{-1}C$. Here BD^{-1} and $D^{-1}C$ are skew symmetric matrices provided D is invertible. Since the matrix of the Fock representation of the spin operator σ is an element in the Clifford group $\mathcal{G}$, we can use Theorem (B.2) to express the kernel of the spin operator as the exponential of a skew symmetric matrix whose entries are the $D^{-\tau}$, BD^{-1}, D^{-1}, and $D^{-1}C$ matrix elements.

Recall that the sets $\{e_{+,j}^A\}$ and $\{e_{+,j}^P\}$ given in (2.45) and (2.47) are orthonormal bases for W_+^A and W_+^P respectively with respect to the Hermitian inner product. Their corresponding dual bases $\{e_{-,j}^A\}$ and $\{e_{-,j}^P\}$ for W_-^A and W_-^P are given in (2.46) and (2.48). The sets $\{e_{+,j}^A\}$ and $\{e_{-,j}^A\}$ are dual with respect to the complex bilinear form $(\cdot,\cdot)$; that is $(e_{-,j}^A, e_{+,k}^A) = \delta_{jk}$. We define

$$e_{\pm,J}^A := e_{\pm,J_1}^A \wedge e_{\pm,J_2}^A \wedge ... \wedge e_{\pm,J_k}^A$$

and

$$e_{\pm,J}^P := e_{\pm,J_1}^P \wedge e_{\pm,J_2}^P \wedge ... \wedge e_{\pm,J_k}^P$$

for $1 \leq J_1 < J_2 < ... < J_k \leq M$, where the subsets of integers are indexed in natural order. The sets $\{e_{\pm,J}^A\}$ and $\{e_{\pm,J}^P\}$ are then orthonormal bases for $\mathrm{Alt}(W_\pm^A)$ and $\mathrm{Alt}(W_\pm^P)$ respectively. Let $\mathcal{P}$ denote the collection of subsets of $\{1, ..., M\}$. For an element J in $\mathcal{P}$, we write

$$J = \{J_1, J_2, ..., J_k\} \quad \text{with} \quad J_1 < J_2 < .... < J_k.$$

We write $\#J = k$ for the number of elements in J. If R is a $2M \times 2M$ matrix, we let $R_{I,J}$ denote the $(\#I + \#J) \times (\#I + \#J)$ sub matrix of R made from the rows and columns of R indexed by I and J respectively. We write the elements in $\mathrm{Alt}(W_{\pm}^A)$ as

$$G(e_{\pm}^A) = \sum_{I \in \mathcal{P}} G_I e_{\pm,I}^A,$$

where

$$\mathcal{P} \ni I \to G_I \in \mathbb{C}.$$

Here $G(e_{\pm}^A)$ can be thought of as a 'polynomial' in the elements $e_{\pm,j}^A$ in the exterior algebra. Recall from Section 2.6, that the operator σ is a map

$$\sigma : \mathrm{Alt}(W_+^P) \to \mathrm{Alt}(W_+^A),$$

where

$$\sigma e_{+,J}^P = \sum_{I \in \mathcal{P}} \sigma_{I,J} e_{+,I}^A$$

and the matrix representation in the bases $\{e_{+,I}^A\}$ and $\{e_{+,J}^P\}$ is given by

$$\sigma_{I,J} = \langle e_{+,I}^A, \sigma e_{+,J}^P \rangle.$$

The sum of the number of elements in I and the number of elements in J is even for the spin matrix elements $\sigma_{I,J}$. (If the number of elements is odd, $\sigma_{I,J} = 0$). Define the skew symmetric matrix

$$R = \begin{pmatrix} BD^{-1} & D^{-\tau} \\ -D^{-1} & D^{-1}C \end{pmatrix}.$$

We will show that

$$\sigma_{I,J} = \langle 0_A, \sigma 0_P \rangle \, \mathrm{Pf}(R_{I,J}),$$

where $R_{I,J}$ is the $(\#I + \#J) \times (\#I + \#J)$ matrix

$$R_{I,J} = \begin{pmatrix} BD^{-1}_{I \times I} & D^{-\tau}_{I \times J} \\ -D^{-1}_{J \times I} & D^{-1}C_{J \times J} \end{pmatrix}$$

with matrix elements

$$(R_{I,J})_{\alpha,\beta} = \begin{cases} BD^{-1}_{I_\alpha,I_\beta} & \text{for } 1 \le \alpha < \beta \le \#I; \\ D^{-\tau}_{I_\alpha,J_\beta} & \text{for } 1 \le \alpha \le \#I \quad \text{and} \quad 1 \le \beta \le \#J; \\ -D^{-1}_{J_\beta,I_\alpha} & \text{for } 1 \le \alpha \le \#I \quad \text{and} \quad 1 \le \beta \le \#J; \\ D^{-1}C_{J_\alpha,J_\beta} & \text{for } 1 \le \alpha < \beta \le \#J. \end{cases}$$

Recall from Section 2.6 that the spin operator σ satisfies the intertwining relation

$$\sigma F^P(x) = F^A(sx)\sigma, \tag{3.1}$$

for $x \in W$, where F^P and F^A are the Fock representations associated with the Clifford relations acting on $\mathrm{Alt}(W_+^P)$ and $\mathrm{Alt}(W_+^A)$ respectively. Recall that the Fock representation F^A is given by

$$F^A(x) = c(x_+) + a(x_-),$$

where $x_+ \in W_+^A$ and $x_- \in W_-^A$. Here W_-^A is identified with the dual $(W_+^A)^*$ via the nondegenerate complex bilinear form $W_+^A \ni x_+ \mapsto (x_+, x_-)$ for $x_- \in W_-^A$. Now define the annihilation operator a_H as the hermitian adjoint

$$a_H(x) := c^*(x)$$

for $x \in W_+^A$, where

$$a_H(x) : \mathrm{Alt}(W_+^A) \to \mathrm{Alt}(W_+^A).$$

Then the relation between this annihilation operator and the one defined as the transpose $a(x) = c^\tau(x)$ for $x \in W_-^A$ is

$$a_H(x) = a(\bar{x})$$

for $x \in W_+^A$. The following lemma characterize the action of the spin operator σ on the vacuum state 0_P in $\mathrm{Alt}(W_+^P)$.

Lemma 3.1. *Let $\{e_{+,i}^A\}$ denote an orthonormal basis for W_+^A. The spin operator σ acts on the vacuum state 0_P in $\mathrm{Alt}(W_+^P)$ as follows*

$$\sigma 0_P = \langle 0_A, \sigma\, 0_P \rangle e^{\frac{1}{2}\sum_{j,k}(BD^{-1})_{jk} c(e_{+,j}^A) c(e_{+,k}^A)} 0_A, \tag{3.2}$$

where 0_A is the vacuum state in $\mathrm{Alt}(W_+^A)$, $\langle 0_A, \sigma 0_P \rangle$ is the one-point function, and

$$s := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is the matrix of the induced rotation $s : W_+^P \oplus W_-^P \to W_+^A \oplus W_-^A$ associated with σ.

Proof. Let

$$X := \frac{1}{2} \sum_{j,k=-M}^{M} (BD^{-1})_{jk} c(e_{+,j}^A) c(e_{+,k}^A).$$

We will show that for a nonzero constant m we have

$$\sigma 0_P = m \exp(X) 0_A.$$

For $x \in W_-^P$, we have

$$\sigma F^P \begin{pmatrix} 0 \\ x \end{pmatrix} 0_P = 0$$

which by the intertwining relation in (3.1) can be written

$$0 = F^A s \begin{pmatrix} 0 \\ x \end{pmatrix} \sigma 0_P = (c(Bx) + a(Dx)) \sigma 0_P. \tag{3.3}$$

Suppose $Dx = y$, where D is a map from W_-^P to W_-^A. Assuming D is invertible, this implies $x = D^{-1}y$. Equation (3.3) can then be written

$$(c(BD^{-1}y) + a(y)) \sigma 0_P = 0 \quad \text{for all} \quad y \in W_-^A.$$

In the basis $\{e_{-,l}^A\}$ for W_-^A, we have

$$(c(BD^{-1}e_{-,l}^A) + a(e_{-,l}^A)) \sigma 0_P = 0. \tag{3.4}$$

Up to a constant multiplier these relations determine $\sigma 0_P$. We show that $\sigma 0_P$ is given by (3.2) by demonstrating that this expression satisfies Equation (3.4). Using (2.14) with $\lambda = 1$ we start by showing that

$$e^X a(e_{-,l}^A) e^{-X} = c(BD^{-1}e_{-,l}^A) + a(e_{-,l}^A) : \tag{3.5}$$

We calculate the commutator $[X, a(e^A_{-,l})]$. We have

$$\begin{aligned}
&\left[c(e^A_{+,j})c(e^A_{+,k}), a(e^A_{-,l})\right] \\
&= c(e^A_{+,j})c(e^A_{+,k})a(e^A_{-,l}) - a(e^A_{-,l})c(e^A_{+,j})c(e^A_{+,k}) \\
&= c(e^A_{+,j})(e^A_{-,l}, e^A_{+,k}) - c(e^A_{+,j})a(e^A_{-,l})c(e^A_{+,k}) - a(e^A_{-,l})c(e^A_{+,j})c(e^A_{+,k}) \\
&= c(e^A_{+,j})(e^A_{-,l}, e^A_{+,k}) - (e^A_{-,l}, e^A_{+,j})c(e^A_{+,k}).
\end{aligned}$$

It follows that

$$\begin{aligned}
[X, a(e^A_{-,l})] &= \frac{1}{2}\sum_{j=-M}^{M}(BD^{-1})_{jl}c(e^A_{+,j}) - \frac{1}{2}\sum_{k=-M}^{M}(BD^{-1})_{lk}c(e^A_{+,k}) \\
&= \sum_{k=-M}^{M}(BD^{-1})_{kl}c(e^A_{+,k}) \\
&= c(BD^{-1}e^A_{-,l}),
\end{aligned}$$

where we in the second equation used the fact that BD^{-1} is skew symmetric. It can be checked that the higher order terms $\mathrm{ad}^n(X)a(e^A_{-,l})$ is zero for $n > 1$, and (3.5) follows. From (3.4) and (3.5) we then obtain

$$(c(BD^{-1}e^A_{-,l}) + a(e^A_{-,l}))\sigma 0_P = e^X a(e^A_{-,l})e^{-X} m e^X 0_A = m e^X a(e^A_{-,l})0_A = 0. \qquad (3.6)$$

Since X^* is the sum of products of annihilation operators, only the first term in the Taylor series expansion of e^{X^*} survives under the action on 0_A. It follows that

$$\langle 0_A, \sigma 0_P\rangle = \langle 0_A, m\exp(X)0_A\rangle = m\langle (e^X)^* 0_A, 0_A\rangle = m\langle 0_A, 0_A\rangle = m,$$

since $\langle 0_A, 0_A\rangle = 1$, and the lemma is proved. We notice that $\sigma 0_P$ also could have been calculated directly by using the holomorphic representation of the creation and annihilation operators as given in Appendix B (see B.19). □

In our calculations of the spin matrix elements, we substitute for $\sigma 0_P$ the expression

$$\langle 0_A, \sigma\, 0_P\rangle e^{\frac{1}{2}\sum_{j,k}(BD^{-1})_{jk}c(e^A_{+,j})c(e^A_{+,k})}0_A.$$

We use the Clifford relations to move the annihilation operators to the right until they hit the vacuum state 0_A and give zero. When the annihilation operator anticommutes with the creation operator, there is a possible pairing. The matrix elements of the spin operator can be written as a sum of all possible pairings. We prove the following.

Proposition 3.2. *Let* $\{e^P_{+,k}\}$ *and* $\{e^A_{+,j}\}$ *denote orthonormal bases for* W^P_+ *and* W^A_+ *respectively. Let*

$$s := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

denote the matrix of the induced rotation $s : W^P_+ \oplus W^P_- \to W^A_+ \oplus W^A_-$ *associated with* σ*. For* $-M \leq k < j \leq M$*, we have*

$$\frac{\langle e^A_{+,k}, \sigma\, e^P_{+,j}\rangle}{\langle 0_A, \sigma\, 0_P\rangle} = D^{-\tau}_{kj}, \tag{3.7}$$

$$\frac{\langle e^A_{+,k} e^A_{+,j}, \sigma\, 0_P\rangle}{\langle 0_A, \sigma\, 0_P\rangle} = (BD^{-1})_{kj} \tag{3.8}$$

$$\frac{\langle 0_A, \sigma\, e^P_{+,k} e^P_{+,j}\rangle}{\langle 0_A, \sigma\, 0_P\rangle} = (D^{-1}C)_{kj}, \tag{3.9}$$

where 0_P *and* 0_A *are the vacuum states in* $\mathrm{Alt}(W^P_+)$ *and* $\mathrm{Alt}(W^A_+)$ *respectively.*

Proof. We first prove (3.7). We have

$$\begin{aligned} \langle e^A_{+,k}, \sigma\, e^P_{+,j}\rangle &= \langle c(e^A_{+,k})0_A, \sigma e^P_{+,j}\rangle \\ &= \langle 0_A, a_H(e^A_{+,k})\sigma e^P_{+,j}\rangle \\ &= \langle 0_A, a(\overline{e^A_{+,k}})\sigma c(e^P_{+,j})\rangle \\ &= \langle 0_A, a(e^A_{-,k})\sigma F^P(e^P_{+,j})0_P\rangle. \end{aligned} \tag{3.10}$$

We rewrite this expression using the intertwining relation (3.1),

$$\sigma F^P(x) = F^A(s(x))\sigma = (c(Ax) + a(Cx))\sigma. \tag{3.11}$$

Inserting (3.11) with $x = e^P_{+,j}$ into (3.10) we obtain

$$\langle e^A_{+,k}, \sigma\, e^P_{+,j}\rangle = \langle 0_A, a(e^A_{-,k})[c(Ae^P_{+,j}) + a(Ce^P_{+,j})]\sigma 0_P\rangle. \tag{3.12}$$

Using the Clifford relations we obtain for the first part of the right hand side of (3.12)

$$\langle 0_A, a(e^A_{-,k})c(Ae^P_{+,j})\sigma 0_P\rangle = \langle 0_A, (e^A_{-,k}, Ae^P_{+,j})\sigma 0_P\rangle = A_{kj}\langle 0_A, \sigma 0_P\rangle, \tag{3.13}$$

where

$$\langle 0_A, (c(Ae^P_{+,j})| = \langle 0_A a(\overline{Ae^P_{+,j}})| = 0.$$

Now we consider the second part of the right hand side of (3.12) and the action of σ on 0_P in (3.2). Using the Clifford relations one finds

$$\begin{aligned}
&a(e^A_{-,k})a(Ce^P_{+,j})c(e^A_{+,j'})c(e^A_{+,k'})0_A \\
&= -(e^A_{-,k}, e^A_{+,j'})(Ce^P_{+,j}, e^A_{+,k'})0_A \\
&+ (e^A_{-,k}, e^A_{+,k'})(Ce^P_{+,j}, e^A_{+,j'})0_A \\
&= (-\delta_{kj'}C_{k'j} + \delta_{kk'}C_{j'j})0_A.
\end{aligned}$$

Combining the calculation above with (3.2), we obtain for the second part of the right hand side of (3.12)

$$\begin{aligned}
&a(e^A_{-,k})a(Ce^P_{+,j})\langle 0_A, \sigma\, 0_P\rangle \tfrac{1}{2} \sum_{j',k'=-M}^{M} (BD^{-1})_{j'k'} c(e^A_{+,j'})c(e^A_{+,k'})0_A \\
&= \langle 0_A, \sigma\, 0_P\rangle \tfrac{1}{2}\Big(- \sum_{k'=-M}^{M} (BD^{-1})_{kk'}C_{k'j} + \sum_{j'=-M}^{M} (BD^{-1})_{j'k}C_{j'j}\Big)0_A.
\end{aligned}$$

Since BD^{-1} is skew symmetric, the expression above can be written

$$\langle 0_A, \sigma\, 0_P\rangle (C^\tau BD^{-1})_{jk} 0_A. \tag{3.14}$$

Combining (3.13) and (3.14), we obtain

$$\begin{aligned}
\langle e^A_{+,k}, \sigma\, e^P_{+,j}\rangle &= \langle 0_A, \sigma 0_P\rangle (A^\tau + C^\tau BD^{-1})_{jk} \\
&= \langle 0_A, \sigma 0_P\rangle D^{-\tau}_{kj},
\end{aligned}$$

where we in the last line used (B.5). To prove (3.8), a similar calculation gives

$$
\begin{aligned}
&\langle 0_A, a(e^A_{-,j})a(e^A_{-,k})\sigma 0_P\rangle \\
&= -\tfrac{1}{2}\langle 0_A, \sigma\, 0_A\rangle\Big\langle 0_A, \sum_{j',k'=-M}^{M}(BD^{-1})_{j'k'}(e^A_{-,j}, e^A_{+,j'})(e^A_{-,k}, e^A_{+,k'})0_A\Big\rangle \\
&+ \tfrac{1}{2}\langle 0_A, \sigma\, 0_P\rangle\Big\langle 0_A, \sum_{j',k'=-M}^{M}(BD^{-1})_{j'k'}(e^A_{-,j}, e^A_{+,k'})(e^A_{-,k}, e^A_{+,j'})0_A\Big\rangle \\
&= \langle 0_A, \sigma\, 0_P\rangle(BD^{-1})_{kj}.
\end{aligned}
$$

For the Equation (3.9), we have

$$
\begin{aligned}
&\langle 0_A, \sigma\, e^P_{+,k}e^P_{+,j}\rangle \\
&= \langle 0_A, \sigma F^P(e^P_{+,k})F^P(e^P_{+,j})0_P\rangle.
\end{aligned}
$$

By the identity (3.11), the right hand side of the equation above can be written

$$
\langle 0_A, [c(Ae^P_{+,k}) + a(Ce^P_{+,k}))][c(Ae^P_{+,j}) + a(Ce^P_{+,j}))]\sigma 0_P\rangle. \tag{3.15}
$$

One finds

$$
\begin{aligned}
&a(Ce^P_{+,k})a(Ce^P_{+,j})c(e^A_{+,j'})c(e^A_{+,k'})0_A \\
&= [-(Ce^P_{+,k}, e^A_{+,j'})(Ce^P_{+,j}, e^A_{+,k'})) + (Ce^P_{+,j}, e^A_{+,j'})(Ce^P_{+,k}, e^A_{+,k'}))]0_A \\
&= [-C_{j'k}C_{k'j} + C_{j'j}C_{k'k}]0_A.
\end{aligned} \tag{3.16}
$$

Combining (3.15) with (3.2) and using the calculation in (3.16), we obtain

$$
\begin{aligned}
&a(Ce^P_{+,k})a(Ce^P_{+,j})\frac{1}{2}\sum_{j',k'=-M}^{M}(BD^{-1})_{j'k'}c(e^A_{+,j'})c(e^A_{+,k'})0_A \\
&= \frac{1}{2}\langle 0_A, \sigma 0_P\rangle\Big(-\sum_{j',k'=-M}^{M}(BD^{-1})_{j'k'}C_{j'k}C_{k'j} + \sum_{j',k'=-M}^{M}(BD^{-1})_{j'k'}C_{j'j}C_{k'k}\Big)0_A \\
&= \frac{1}{2}\langle 0_A, \sigma 0_P\rangle\Big(\sum_{k'=-M}^{M}(BD^{-1}C)^{\tau}_{kk'}C_{k'j} + \sum_{k'=-M}^{M}(C^{\tau}BD^{-1})_{jk'}C_{k'k}\Big)0_A \\
&= \langle 0_A, \sigma 0_P\rangle(C^{\tau}BD^{-1}C)_{jk}.
\end{aligned} \tag{3.17}
$$

Using the Clifford relations, we obtain

$$\begin{aligned}\langle 0_A, a(Ce^P_{+,k})c(Ae^P_{+,j})\sigma 0_P\rangle &= \langle 0_A, (Ce^P_{+,k}, Ae^P_{+,j})\sigma 0_P\rangle \\ &= (A^\tau C)_{jk}\langle 0_A, \sigma 0_P\rangle. \end{aligned} \tag{3.18}$$

Combining (3.17) and (3.18), we obtain

$$\langle 0_A, \sigma\, e^P_{+,k} e^P_{+,j}\rangle = \langle 0_A, \sigma 0_P\rangle (C^\tau B D^{-1} C + A^\tau C)_{jk} = \langle 0_A, \sigma 0_P\rangle (D^{-1}C)_{jk},$$

where we in the last equation used (B.5). □

We prove the following theorem.

Theorem 3.3. *Let $\{e^A_{+,i}\}$ and $\{e^P_{-,j}\}$ denote orthonormal bases for W^A_+ and W^P_- respectively and define*

$$e^A_{+,I} := e^A_{+,I_1} \wedge ... \wedge e^A_{+,I_k} \quad \textit{for} \quad 1 \le I_1 < ... < I_k \le M$$

and

$$e^P_{-,J} := e^P_{-,J_1} \wedge ... \wedge e^P_{-,J_k} \quad \textit{for} \quad 1 \le J_1 < ... < J_k \le M.$$

Let

$$s := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

denote the matrix of the induced rotation $s : W^P_+ \oplus W^P_- \to W^A_+ \oplus W^A_-$ associated with σ. Suppose $\langle 0_A, \sigma 0_P\rangle \neq 0$. Then the kernel $\sigma(e^A_+, e^P_-)$ of σ can be written

$$\sigma(e^A_+, e^P_-) = \langle 0_A, \sigma 0_P\rangle \sum_{I,J\in\mathcal{P}} \mathrm{Pf}(R_{I,J}) e^A_{+,I} \wedge e^P_{-,J},$$

where

$$R_{I,J} = \begin{pmatrix} BD^{-1}_{I\times I} & D^{-\tau}_{I\times J} \\ -D^{-1}_{J\times I} & D^{-1}C_{J\times J} \end{pmatrix}.$$

The spin matrix elements $\sigma_{I,J}$ are given by

$$\sigma_{I,J} = \langle 0_A, \sigma 0_P\rangle\, \mathrm{Pf}(R_{I,J}). \tag{3.19}$$

The sum is over all such I and J with $\#I + \#J$ even.

Proof. Since σ satisfies the intertwining relation (3.1), the proof follows from Theorem B.2 and Proposition 3.2. □

3.2. Bugrij-Lisovyy Formula for the Spin Matrix Elements

Recall that the sets $\{e^A_{\pm,k}\}_{k=-M}^M$, where

$$e^A_{+,k}(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} a(z) \\ iza(z)^{-1} \end{pmatrix} \delta_{z_A(k)}(z)$$

and

$$e^A_{-,k}(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} a(z) \\ -iza(z)^{-1} \end{pmatrix} \delta_{z_A(-k)}(z)$$

for $z \in \Sigma_A$ are orthonormal bases of $W^A_\pm$. Here

$$z_A(k) = e^{i\theta^A_k}, \quad \theta^A_k = \frac{2\pi\left(k+\frac{1}{2}\right)}{2M+1},$$

$$a(z) = \sqrt{\frac{\mathcal{A}_1(z)\mathcal{A}_2(z)}{\mathcal{A}_1(z^{-1})\mathcal{A}_2(z^{-1})}} \quad \text{and} \quad \mathcal{A}_j(z) = \sqrt{\alpha_j - z}$$

with

$$\alpha_1 := (c_1^* - s_1^*)(c_2 + s_2) = e^{2(\mathcal{K}_2 - \mathcal{K}_1^*)},$$

$$\alpha_2 := (c_1^* + s_1^*)(c_2 + s_2) = e^{2(\mathcal{K}_2 + \mathcal{K}_1^*)}.$$

The orthonormal bases $\{e^P_{\pm,k}\}_{k=-M}^M$ of $W^P_\pm$ are defined similarly with $z_P(k) = e^{i\theta^P_k}$ and $\theta^P_k = \frac{2\pi k}{2M+1}$. We use the short-hand notation $\theta \in \Sigma_P$ for $z = e^{i\theta} \in \Sigma_P$, where $z^{2M+1} = 1$. Similarly, we use the notation $\theta \in \Sigma_A$ for $z = e^{i\theta} \in \Sigma_A$, where $z^{2M+1} = -1$. We now consider the isotropic case, and define the interaction constant to be $\mathcal{K} := \mathcal{K}_1 = \mathcal{K}_2$. The function $\gamma(\theta)$ is defined as the positive root of the equation

$$\cosh(\gamma(\theta)) = \sinh(2\mathcal{K}) + \sinh(2\mathcal{K})^{-1} - \cos(\theta).$$

A. I. Bugrij and O. Lisovyy [BL03] proposed the following formula for spin matrix elements on the finite, periodic lattice for the isotropic case in the orthonormal basis of transfer matrix eigenstates,

$$\begin{aligned}
&\langle e^A_{+,l_1} \wedge e^A_{+,l_2} \wedge \dots \wedge e^A_{+,l_m}, \sigma e^P_{+,k_1} \wedge e^P_{+,k_2} \wedge \dots \wedge e^P_{+,k_{m'}} \rangle \\
&= \sqrt{\xi \xi_T} \prod_{i=1}^{m} \frac{e^{\frac{1}{2} v\left(e^{i\theta^A_{l_i}}\right)}}{\sqrt{(2M+1)\sinh \gamma(\theta^A_{l_i})}} \prod_{j=1}^{m'} \frac{e^{-\frac{1}{2} v\left(e^{i\theta^P_{k_j}}\right)}}{\sqrt{(2M+1)\sinh \gamma(\theta^P_{k_j})}} \times \\
&\times \prod_{1 \le i < i' \le m} \frac{\sin \frac{\theta^A_{l_i} - \theta^A_{l_{i'}}}{2}}{\sinh \frac{\gamma(\theta^A_{l_i}) + \gamma(\theta^A_{l_{i'}})}{2}} \prod_{1 \le j < j' \le m'} \frac{\sin \frac{\theta^P_{k_j} - \theta^P_{k_{j'}}}{2}}{\sinh \frac{\gamma(\theta^P_{k_j}) + \gamma(\theta^P_{k_{j'}})}{2}} \times \\
&\times \prod_{1 \le i \le m, 1 \le j \le m'} \frac{\sinh \frac{\gamma(\theta^A_{l_i}) + \gamma(\theta^P_{k_j})}{2}}{\sin \frac{\theta^A_{l_i} - \theta^P_{k_j}}{2}},
\end{aligned} \tag{3.20}$$

where

$$\xi = |1 - (\sinh(2\mathcal{K}))^{-4}|^{\frac{1}{4}}.$$

The cylindrical parameters ξ_T and $v(e^{i\theta})$ are defined as

$$\xi_T^4 = \frac{\prod_{\theta' \in \Sigma_P} \prod_{\theta \in \Sigma_A} \sinh^2\left(\frac{\gamma(\theta') + \gamma(\theta)}{2}\right)}{\prod_{\theta' \in \Sigma_P} \prod_{\theta \in \Sigma_P} \sinh\left(\frac{\gamma(\theta') + \gamma(\theta)}{2}\right) \prod_{\theta \in \Sigma_A} \prod_{\theta' \in \Sigma_A} \sinh\left(\frac{\gamma(\theta) + \gamma(\theta')}{2}\right)} \tag{3.21}$$

and

$$v(e^{i\theta}) = \log \frac{\prod_{\theta' \in \Sigma_A} \sinh\left(\frac{\gamma(\theta) + \gamma(\theta')}{2}\right)}{\prod_{\theta' \in \Sigma_P} \sinh\left(\frac{\gamma(\theta) + \gamma(\theta')}{2}\right)}, \tag{3.22}$$

where $\theta = \theta^A_{l_i}$ or $\theta = \theta^P_{k_j}$. Now we rewrite $v(e^{i\theta})$. We have

$$\begin{aligned}
\log \prod_{\theta' \in \Sigma_A} \sinh\left(\frac{\gamma(\theta') + \gamma(\theta)}{2}\right) e^{-\gamma(\theta)} &= \sum_{\theta' \in \Sigma_A} \log \left[(e^{\frac{\gamma(\theta)}{2}} e^{-\frac{\gamma(\theta')}{2}} (e^{\gamma(\theta')} - e^{-\gamma(\theta)})) e^{-\gamma(\theta)} \frac{1}{2} \right] \\
&= \frac{1}{2} \sum_{\theta' \in \Sigma_A} \log \left[e^{-\frac{\gamma(\theta)}{2}} e^{-\frac{\gamma(\theta')}{2}} (e^{\gamma(\theta')} - e^{-\gamma(\theta)}) \frac{1}{2} \right] + \\
&+ \frac{1}{2} \sum_{\theta' \in \Sigma_A} \log \left[e^{-\frac{\gamma(\theta)}{2}} e^{\frac{\gamma(\theta')}{2}} (1 - e^{-\gamma(\theta')} e^{-\gamma(\theta)}) \frac{1}{2} \right]
\end{aligned}$$

which implies that

$$v(e^{i\theta}) = \log \frac{\prod_{\theta' \in \Sigma_A} e^{\frac{\gamma(\theta')}{2}} (1 - e^{-\gamma(\theta')} e^{-\gamma(\theta)})}{\prod_{\theta' \in \Sigma_P} e^{\frac{\gamma(\theta')}{2}} (1 - e^{-\gamma(\theta')} e^{-\gamma(\theta)})} \tag{3.23}$$

or alternatively

$$v(e^{i\theta}) = \log \frac{\prod_{\theta' \in \Sigma_A} e^{-\frac{\gamma(\theta')}{2}} (e^{\gamma(\theta')} - e^{-\gamma(\theta)})}{\prod_{\theta' \in \Sigma_P} e^{-\frac{\gamma(\theta')}{2}} (e^{\gamma(\theta')} - e^{-\gamma(\theta)})}. \tag{3.24}$$

Define

$$V_+(e^{i\theta}) := e^{\frac{v(e^{i\theta})}{2}} = \sqrt{\frac{e^{-\frac{\gamma(\pi)}{2}}(\lambda^{-1}(e^{i\theta}) - \lambda(-1))}{e^{-\frac{\gamma(0)}{2}}(\lambda^{-1}(e^{i\theta}) - \lambda(1))} \frac{\prod_{\theta' > 0 \in \Sigma_A} e^{-\frac{\gamma(\theta')}{2}} (\lambda^{-1}(e^{i\theta}) - \lambda(e^{i\theta'}))}{\prod_{\theta' > 0 \in \Sigma_P} e^{-\frac{\gamma(\theta')}{2}} (\lambda^{-1}(e^{i\theta}) - \lambda(e^{i\theta'}))}} \tag{3.25}$$

and

$$V_-(e^{i\theta}) := e^{-\frac{v(e^{i\theta})}{2}} = \sqrt{\frac{e^{\frac{\gamma(0)}{2}}(\lambda(e^{i\theta}) - \lambda^{-1}(1))}{e^{\frac{\gamma(\pi)}{2}}(\lambda(e^{i\theta}) - \lambda^{-1}(-1))} \frac{\prod_{\theta' > 0 \in \Sigma_P} e^{\frac{\gamma(\theta')}{2}} (\lambda(e^{i\theta}) - \lambda^{-1}(e^{i\theta'}))}{\prod_{\theta' > 0 \in \Sigma_A} e^{\frac{\gamma(\theta')}{2}} (\lambda(e^{i\theta}) - \lambda^{-1}(e^{i\theta'}))}}, \tag{3.26}$$

where $\lambda(e^{i\theta}) = e^{\gamma(\theta)}$. Here the square roots are taken with positive real parts. The function $V_+(e^{i\theta})$ is analytic in a neighborhood of the part of the spectral curve where $|\lambda| \leq 1$ while $V_-(e^{i\theta})$ is analytic in a neighborhood of the part of the spectral curve where $|\lambda| \geq 1$. Alternatively, the formulas in (3.21) and (3.22) can be written (see [BL03])

$$\ln \xi_T = \frac{(2M+1)^2}{2\pi^2} \int_0^\pi \int_0^\pi \frac{d\theta \, d\theta' \, \gamma'(\theta)\gamma'(\theta')}{\sinh[(2M+1)\gamma(\theta)] \sinh[(2M+1)\gamma(\theta')]} \ln \left| \frac{\sin((\theta+\theta')/2)}{\sin((\theta-\theta')/2)} \right|, \tag{3.27}$$

where $\gamma'(\theta)$ denotes the derivative of $\gamma(\theta)$ with respect to θ, and

$$v(\theta) := v(e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta' \sinh \gamma(\theta)}{\cosh \gamma(\theta) - \cos(\theta')} \ln \coth((2M+1)\gamma(\theta')/2). \tag{3.28}$$

We show that the formulas in (3.23), (3.24) and (3.28) are identical. We introduce the short-hand notation $\gamma_p := \gamma(p)$. We have

$$2(\cosh \gamma(p) - \cos(\theta')) = (e^{\gamma_p} - z)(1 - z^{-1} e^{-\gamma_p}), \tag{3.29}$$

where $z = e^{i\theta'}$. Using this identity and the identities

$$\prod_{\theta' \in \Sigma_P} (x - e^{i\theta'}) = (x^{2M+1} - 1)$$

and

$$\prod_{\theta' \in \Sigma_A} (x - e^{i\theta'}) = (x^{2M+1} + 1),$$

we have

$$2^{2M+1} \prod_{\theta' \in \Sigma_P} (\cosh \gamma_p - \cos \theta') = e^{(2M+1)\gamma_p} (1 - e^{-(2M+1)\gamma_p})^2 \quad \text{and} \tag{3.30}$$

$$2^{2M+1} \prod_{\theta' \in \Sigma_A} (\cosh \gamma_p - \cos \theta') = e^{(2M+1)\gamma_p} (1 + e^{-(2M+1)\gamma_p})^2. \tag{3.31}$$

Hence

$$\coth^2 \left(\frac{(2M+1)\gamma_p}{2} \right) = \frac{\prod_{\theta' \in \Sigma_A} (\cosh \gamma_p - \cos \theta')}{\prod_{\theta' \in \Sigma_P} (\cosh \gamma_p - \cos \theta')}. \tag{3.32}$$

Using the identity

$$\cosh \gamma_p - \cos \theta' = \cosh \gamma_{\theta'} - \cos p,$$

(3.30) and (3.31), the equation in (3.32) can be written

$$\coth^2 \left(\frac{(2M+1)\gamma_p}{2} \right) = \frac{\prod_{\theta' \in \Sigma_A} (\cosh \gamma_{\theta'} - \cos p)}{\prod_{\theta' \in \Sigma_P} (\cosh \gamma_{\theta'} - \cos p)} = \frac{\prod_{\theta' \in \Sigma_A} (e^{\gamma_{\theta'}} - e^{ip})(1 - e^{-\gamma_{\theta'}} e^{-ip})}{\prod_{\theta' \in \Sigma_P} (e^{\gamma_{\theta'}} - e^{ip})(1 - e^{-\gamma_{\theta'}} e^{-ip})}.$$

From the equation above we have

$$\begin{aligned} \log \coth^2 \left(\frac{(2M+1)\gamma_p}{2} \right) &= \sum_{\theta' \in \Sigma_A} [\log(e^{\gamma_{\theta'}} - e^{ip}) + \log(1 - e^{-\gamma_{\theta'}} e^{-ip})] \\ &\quad - \sum_{\theta' \in \Sigma_P} [(\log(e^{\gamma_{\theta'}} - e^{ip}) + \log(1 - e^{-\gamma_{\theta'}} e^{-ip}))] \end{aligned}$$

Here $\log(e^{\gamma_{\theta'}} - z)$ is analytic for $|z| < 1$ while $\log(1 - e^{-\gamma_{\theta'}} z^{-1})$ is analytic for $|z| > 1$. Thus, by a residue calculation we obtain

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sinh \gamma_\theta \, dp}{\cosh \gamma_\theta - \cos p} \log(e^{\gamma_{\theta'}} - e^{ip}) = \log(e^{\gamma_{\theta'}} - e^{-\gamma_\theta}).$$

and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sinh \gamma_\theta \, dp}{\cosh \gamma_\theta - \cos p} \log(1 - e^{-\gamma_{\theta'}} e^{-ip}) = \log(1 - e^{-\gamma_{\theta'}} e^{-\gamma_\theta}),$$

and the formula is proved. It is convenient to write the formulas ξ_T and $v(e^{i\theta})$ in terms of the integral formulas (3.27) and (3.28) when we calculate the scaling limit. For the anisotropic Ising model ($\mathcal{K}_1 \neq \mathcal{K}_2$), Bugrij and Lisovyy [BL04] proposed the following formula for the spin matrix elements,

$$|\langle e^A_{+,l_1} \wedge e^A_{+,l_2} \wedge ... \wedge e^A_{+,l_m}, \sigma e^P_{+,k_1} \wedge e^P_{+,k_2} \wedge ... \wedge e^P_{+,k_{m'}} \rangle|^2$$
$$= \xi\xi_T \prod_{i=1}^{m} \frac{e^{v\left(e^{i\theta^A_{l_i}}\right)+Q}}{(2M+1)\sinh\gamma(\theta^A_{l_i})} \prod_{j=1}^{m'} \frac{e^{-v\left(e^{i\theta^P_{k_j}}\right)-Q}}{(2M+1)\sinh\gamma(\theta^P_{k_j})} \times$$
$$\times \prod_{1\leq i<i'\leq m} \frac{\sin^2 \frac{\theta^A_{l_i}-\theta^A_{l_{i'}}}{2}}{\sinh^2 \frac{\gamma(\theta^A_{l_i})+\gamma(\theta^A_{l_{i'}})}{2}} \prod_{1\leq j<j'\leq m'} \frac{\sin^2 \frac{\theta^P_{k_j}-\theta^P_{k_{j'}}}{2}}{\sinh^2 \frac{\gamma(\theta^P_{k_j})+\gamma(\theta^P_{k_{j'}})}{2}} \times$$
$$\times \left[\frac{\tanh(\mathcal{K}_2)(1-\tanh^2(\mathcal{K}_1))}{\tanh(\mathcal{K}_1)(1-\tanh^2(\mathcal{K}_2))}\right]^{\frac{(m-m')^2}{2}} \prod_{1\leq i\leq m, 1\leq j\leq m'} \frac{\sinh^2 \frac{\gamma(\theta^A_{l_i})+\gamma(\theta^P_{k_j})}{2}}{\sin^2 \frac{\theta^A_{l_i}-\theta^P_{k_j}}{2}}, \tag{3.33}$$

where

$$Q = \tfrac{1}{2} \sum_{k=-M}^{M} [\gamma(e^{i\theta^A_k}) - \gamma(e^{i\theta^P_k})]$$

and where $\gamma(\theta)$ here is the positive root of the equation,

$$\cosh\gamma(\theta) = c_2^* c_1 - s_2^* s_1 \cos(\theta).$$

3.3. Pfaffian Formalism and the Bugrij-Lisovyy Formula

In the previous chapter we gave the Bugrij-Lisovyy formula proposed for the spin matrix elements on the finite periodic lattice between the transfer matrix eigenstates. From Proposition 3.2, we have that for $-M \leq k < j \leq M$,

$$\frac{\langle e^A_{+,k}, \sigma\, e^P_{+,j} \rangle}{\langle 0_A, \sigma\, 0_P \rangle} = D^{-\tau}_{kj}, \quad \frac{\langle e^A_{+,k} e^A_{+,j}, \sigma\, 0_P \rangle}{\langle 0_A, \sigma\, 0_P \rangle} = (BD^{-1})_{kj} \quad \frac{\langle 0_A, \sigma\, e^P_{+,k} e^P_{+,j} \rangle}{\langle 0_A, \sigma\, 0_P \rangle} = (D^{-1}C)_{kj}.$$

Using (3.20), these calculations lead us to the following conjecture for the isotropic Ising model:

Conjecture. *The matrix elements of BD^{-1} is given by,*

$$BD^{-1}_{i,i'}$$
$$\propto \frac{V_+(z_A(i))}{\sqrt{(2M+1)\sinh\gamma(\theta^A_i)}} \frac{V_+(z_A(i'))}{\sqrt{(2M+1)\sinh\gamma(\theta^A_{i'})}} \frac{\sin \frac{\theta^A_i - \theta^A_{i'}}{2}}{\sinh \frac{\gamma(\theta^A_i)+\gamma(\theta^A_{i'})}{2}}. \tag{3.34}$$

The matrix elements of $D^{-\tau}$ is given by,

$$D^{-\tau}_{i,j} \propto \frac{V_+(z_A(i))}{\sqrt{(2M+1)\sinh\gamma(\theta_i^A)}} \frac{V_-(z_P(j))}{\sqrt{(2M+1)\sinh\gamma(\theta_j^P)}} \frac{\sinh\frac{\gamma(\theta_i^A)+\gamma(\theta_j^P)}{2}}{\sin\frac{\theta_i^A-\theta_j^P}{2}}. \tag{3.35}$$

The matrix elements of $D^{-1}C$ is given by,

$$\begin{aligned} & D^{-1}C_{j,j'} \\ & \propto \frac{V_-(z_P(j))}{\sqrt{(2M+1)\sinh\gamma(\theta_j^P)}} \frac{V_-(z_P(j'))}{\sqrt{(2M+1)\sinh\gamma(\theta_{j'}^P)}} \frac{\sin\frac{\theta_j^P-\theta_{j'}^P}{2}}{\sinh\frac{\gamma(\theta_j^P)+\gamma(\theta_{j'}^P)}{2}}, \end{aligned} \tag{3.36}$$

where $\propto$ denotes proportional to.

Here we have extended the Bugrij-Lisovyy conjecture to include the elements $\langle e^A_{+,k}, \sigma\, e^P_{+,j}\rangle$ which are not 'physical elements'. By applying Wick's Theorem or Theorem B.2, we can express the spin matrix elements $\langle e^A_{+,l_1}\wedge\ldots\wedge e^A_{+,l_m}, \sigma e^P_{+,k_1}\wedge\ldots\wedge e^P_{+,k_{m'}}\rangle$ in terms of Pfaffians of a skew symmetric matrix, whose elements are $\langle e^A_{+,l_i}\wedge e^A_{+,l_{i'}}\sigma 0_P\rangle$, $\langle 0_A, \sigma e^P_{+,k_j}\wedge e^P_{+,k_{j'}}\rangle$ and $\langle e^A_{+,l_i}, \sigma e^P_{+,k_j}\rangle$. By using the proposed Bugrij-Lisovyy formula for these elements, we show that the Pfaffian of the matrix can be written as a product of the Jacobian elliptic functions in the uniformization parameter u. We find that up to a constant, this product is the same as the one given in the Bugrij-Lisovyy formula. This supports the Bugrij-Lisovyy conjecture. We start by writing the factor

$$\frac{\sinh(\frac{\gamma(\theta)+\gamma(\theta')}{2})}{\sin(\frac{\theta-\theta'}{2})}$$

in (3.35) in terms of the uniformization parameters by performing the elliptic substitutions

$$e^{\theta i} = z(u,a), \quad e^{\theta' i} = z(u',a),$$

where

$$\Im u = \Im u' = \frac{K'}{2} \quad \text{and} \quad \Re u \in (0, 2K).$$

Then the Uniformization Theorem (C.1) implies

$$e^{\frac{\gamma+\theta i}{2}} = \sqrt{k}\,\mathrm{sn}(u-ia), \tag{3.37}$$

$$e^{\frac{-\gamma+\theta i}{2}} = \sqrt{k}\,\mathrm{sn}(u+ia). \tag{3.38}$$

We have

$$\frac{\sinh(\frac{\gamma(\theta)+\gamma(\theta')}{2})}{\sin(\frac{\theta-\theta'}{2})} = i\frac{e^{\frac{\gamma+\gamma'}{2}} - e^{-\frac{\gamma+\gamma'}{2}}}{e^{\frac{\theta i-\theta' i}{2}} - e^{-\frac{\theta i-\theta' i}{2}}} = i\frac{e^{\frac{\gamma+\theta i}{2}} e^{\frac{\gamma'+\theta' i}{2}} - e^{-\frac{\gamma+\theta i}{2}} e^{-\frac{\gamma'+\theta' i}{2}}}{e^{\theta i} - e^{\theta' i}}. \tag{3.39}$$

Thus the numerator of (3.39) becomes

$$ik[\operatorname{sn}(u-ia)\operatorname{sn}(u'-ia) - \operatorname{sn}(u'+ia)\operatorname{sn}(u+ia)].$$

Using the identity

$$\operatorname{sn}(u-ia)\operatorname{sn}(u+ia) = \frac{\operatorname{sn}^2(u)-\operatorname{sn}^2(ia)}{1-k^2\operatorname{sn}^2(u)\operatorname{sn}^2(ia)}$$

which follows from the addition formulas (C.4) and (C.5), the denominator becomes

$$k\left(\frac{\operatorname{sn}^2(u)-\operatorname{sn}^2(ia)}{1-k^2\operatorname{sn}^2(u)\operatorname{sn}^2(ia)}\right) - k\left(\frac{\operatorname{sn}^2(u')-\operatorname{sn}^2(ia)}{1-k^2\operatorname{sn}^2(u')\operatorname{sn}^2(ia)}\right).$$

After some simplifications, the expression in (3.39) becomes

$$\begin{aligned} &\frac{-2i[\operatorname{cn}(ia)\operatorname{dn}(ia)\operatorname{sn}(ia)][\operatorname{sn}(u)\operatorname{cn}(u')\operatorname{dn}(u') + \operatorname{sn}(u')\operatorname{cn}(u)\operatorname{dn}(u)]}{(\operatorname{sn}^2(u)-\operatorname{sn}^2(u'))(1-k^2\operatorname{sn}^4(ia))} \\ &= s_1\frac{1}{\operatorname{sn}(u-u')}. \end{aligned} \tag{3.40}$$

In the isotropic case we have $s_1^{-1} = \sqrt{k}$. If we substitute $v := -u + \frac{iK'}{2}$ and $v' := -u' - \frac{iK'}{2}$ in (3.40), and apply the identity (C.8), we obtain

$$s_1\frac{1}{\operatorname{sn}(u-u')} = -\sqrt{k}\operatorname{sn}(v-v').$$

Now we will make the following elliptic substitutions

$$\begin{aligned} e^{i\theta^A_{l_j}} &= k\operatorname{sn}(u_{l_j}-ia)\operatorname{sn}(u_{l_j}+ia) \quad \text{for} \quad 1 \le j \le m, \\ e^{i\theta^P_{k_j}} &= k\operatorname{sn}(u_{k_j}-ia)\operatorname{sn}(u_{k_j}+ia) \quad \text{for} \quad 1 \le j \le m', \end{aligned}$$

followed by the translations

$$\begin{aligned} v_{l_j} &:= -u_{l_j} + \frac{iK'}{2} \quad \text{for} \quad 1 \le j \le m, \\ v_{k_j} &:= -u_{k_j} - \frac{iK'}{2} \quad \text{for} \quad 1 \le j \le m'. \end{aligned}$$

Then using Theorem B.2, we obtain for $m + m' = \text{even}$,

$$\begin{aligned}
&\langle e^A_{+,l_1} \wedge e^A_{+,l_2} \wedge \dots \wedge e^A_{+,l_m}, \sigma e^P_{+,k_1} \wedge e^P_{+,k_2} \wedge \dots \wedge e^P_{+,k_{m'}} \rangle \\
&= \langle 0_A, \sigma 0_P \rangle \, \mathrm{Pf}(r),
\end{aligned}$$

where r is the skew symmetric matrix with matrix elements above the diagonal given by: For $1 \le i < i' \le m$,

$$\begin{aligned}
r_{l_i, l_{i'}} &= \frac{\langle e^A_{+,l_i} e^A_{+,l_{i'}}, \sigma 0_P \rangle}{\langle 0_A, \sigma 0_P \rangle} \\
&\propto \frac{V_+(z_A(l_i))}{\sqrt{(2M+1)\sinh\gamma(\theta^A_{l_i})}} \frac{V_+(z_A(l_{i'}))}{\sqrt{(2M+1)\sinh\gamma(\theta^A_{l_{i'}})}} (-\sqrt{k}\,\mathrm{sn}(v_{l_i} - v_{l_{i'}})),
\end{aligned}$$

for $1 \le i \le m \quad \text{and} \quad 1 \le j \le m'$,

$$\begin{aligned}
r_{l_i, m+k_j} &= \frac{\langle e^A_{+,l_i}, \sigma e^P_{+,k_j} \rangle}{\langle 0_A, \sigma 0_P \rangle} \\
&\propto \frac{V_+(z_A(l_i))}{\sqrt{(2M+1)\sinh\gamma(\theta^A_{l_i})}} \frac{V_-(z_P(k_j))}{\sqrt{(2M+1)\sinh\gamma(\theta^P_{k_j})}} (-\sqrt{k}\,\mathrm{sn}(v_{l_i} - v_{k_j}))
\end{aligned}$$

and for $1 \le j < j' \le m'$,

$$\begin{aligned}
r_{m+k_j, m+k_{j'}} &= \frac{\langle 0_A, \sigma e^P_{+,k_j} e^P_{+,k_{j'}} \rangle}{\langle 0_A, \sigma 0_P \rangle} \\
&\propto \frac{V_-(z_P(k_j))}{\sqrt{(2M+1)\sinh\gamma(\theta^P_{k_j})}} \frac{V_-(z_P(k_{j'}))}{\sqrt{(2M+1)\sinh\gamma(\theta^P_{k_{j'}})}} (-\sqrt{k}\,\mathrm{sn}(v_{k_j} - v_{k_{j'}})).
\end{aligned}$$

The proportionality constants in each of the terms above are the same and therefore omitted from the calculation. Define E to be the $(m+m') \times (m+m')$ diagonal matrix

$$E = \begin{pmatrix} e^A & 0 \\ 0 & e^P \end{pmatrix}$$

with matrix elements

$$e^A_{l_i, l_i} = \frac{V_+(z_A(l_i))}{\sqrt{(2M+1)\sinh\gamma(\theta^A_{l_i})}} \quad \text{for} \quad 1 \le i \le m,$$

$$e^P_{m+k_j, m+k_j} = \frac{V_-(z_P(k_j))}{\sqrt{(2M+1)\sinh\gamma(\theta^P_{k_j})}} \quad \text{for} \quad 1 \le j \le m'.$$

Define s to be the $(m+m') \times (m+m')$ skew symmetric matrix with matrix elements above the diagonal given by

$$\begin{aligned} s_{l_i,l_{i'}} &= -\sqrt{k}\,\mathrm{sn}(v_{l_i} - v_{l_{i'}}) \quad \text{for} \quad 1 \le i < i' \le m, \\ s_{l_i,m+k_j} &= -\sqrt{k}\,\mathrm{sn}(v_{l_i} - v_{k_j}) \quad \text{for} \quad 1 \le i \le m, \quad 1 \le j \le m', \\ s_{m+k_j,m+k_{j'}} &= -\sqrt{k}\,\mathrm{sn}(v_{k_j} - v_{k_{j'}}) \quad \text{for} \quad 1 \le j < j' \le m'. \end{aligned}$$

Then from Equation (C.21) and Lemma (C.3), we have

$$\begin{aligned} \mathrm{Pf}(r) &= \mathrm{Pf}(EsE^\tau) \\ &= (\det E)\,\mathrm{Pf}(s) \\ &= \prod_{i=1}^{m} \frac{V_+(z_A(l_i))}{\sqrt{(2M+1)\sinh\gamma(\theta^A_{l_i})}} \prod_{j=1}^{m'} \frac{V_-(z_P(k_j))}{\sqrt{(2M+1)\sinh\gamma(\theta^P_{k_j})}} \times \\ &\times \prod_{1\le i<i'\le m} -\sqrt{k}\,\mathrm{sn}(v_{l_i} - v_{l_{i'}}) \prod_{1\le j<j'\le m'} -\sqrt{k}\,\mathrm{sn}(v_{k_j} - v_{k_{j'}}) \times \\ &\times \prod_{1\le i\le m, 1\le j\le m'} -\sqrt{k}\,\mathrm{sn}(v_{l_i} - v_{k_j}), \end{aligned}$$

which up to a constant is the same product formula as given in [BL03].

3.4. Numerical Calculations

We have numerically compared the calculation of our BD^{-1}, $D^{-\tau}$, and $D^{-1}C$ matrix elements with the corresponding terms in the Bugrij and Lisovyy formula (3.20). What we find is, that as we let the temperature approach T_C from below, our results are close to theirs, but there is a discrepancy. We are uncertain of the reason for this. We also find that in the scaling regime, i.e., as we let M get bigger while we keep the temperature close to T_C, the precision improves. We compared the formulas in MATLAB by creating our $(2M+1) \times (2M+1)$ matrix D as given in (2.91). Then we multiplied the transpose of this matrix with the matrix with elements $\langle e^A_{+,l}, \sigma e^P_{+,j} \rangle$ for $-M \le l, j \le M$ as given in the Bugrij-Lisovyy formula (3.35). If this had returned the identity matrix, it would have shown numerically that the Bugrij-Lisovyy formula provides an inverse for our matrix D^τ.

3.5. Scaling Limits

In this section we calculate the scaling limit of the correlation functions as the temperature approaches the critical temperature T_C from below. We look at the correlation functions at the scale of the correlation length in a neighborhood of the critical point. Recall that $\mathcal{K}_j$ is defined by $\mathcal{K}_j = \frac{J_j}{k_B T}$, where k_B is the Boltzmann constant, T is the temperature, and $J_i > 0$ is the interaction constant. We consider the isotropic case and define $\mathcal{K} := \mathcal{K}_1 = \mathcal{K}_2$ and $J := J_1 = J_2$. Let the temperature T vary such that $0 < T < T_C$, where the critical temperature T_C is defined through the relation

$$\sinh(\tfrac{2J}{k_B T_C})^2 = 1.$$

Recall that the function $\gamma(\theta)$ is defined as the positive root of

$$\cosh(\gamma(\theta)) = \sinh(2\mathcal{K}) + \sinh(2\mathcal{K})^{-1} - \cos(\theta).$$

The correlation length μ_T^{-1} is defined as the reciprocal of

$$\mu_T := \gamma(0) = 2(\mathcal{K} - \mathcal{K}^*).$$

For convenience we write $\mu := \mu_T$. Notice that μ is positive when the temperature T is below the critical temperature. Since $\mathcal{K}^* = \mathcal{K}$ at $T = T_C$, we have that $\mu \to 0$ as $T \uparrow T_C$. Thus, the correlation length μ^{-1} tends to infinity as T approaches the critical point T_C from below. Fix the horizontal side length of the lattice to be $2L$ for $L > 0$ and define $M_T \in \mathbb{Z}$ so that

$$|2L\mu^{-1} - (2M_T + 1)| < 1. \tag{3.41}$$

Notice that when $2L\mu^{-1}$ is an even integer, the inequality cannot be strict so in this case we choose one of the two possible values of M_T. Define

$$L_T := \frac{\mu(2M_T + 1)}{2}. \tag{3.42}$$

Then

$$|L - L_T| < \frac{\mu}{2}$$

so $L_T \to L$ as $T \uparrow T_C$. Thus, we have that $2L_T > L$ which is an estimate that we will use in our calculations. We typically use subscript T to index sets that depends on the temperature. For convenience we summarize the sets we will introduce in this chapter in the following table:

T-dependent sets	T-independent sets
$\Gamma_T^x = \{x \in \mu\{-M_T, ..., M_T\}\}$	$\Gamma^x = \{x \in \mathbb{R} : -L \leq x \leq L\}$
$\Gamma_T^y = \{y \in \mu\mathbb{Z}\}$	$\Gamma^y = \{y \in \mathbb{R}\}$
$\Gamma^*_{P,T,L_T} = \{p \in \frac{\pi}{L_T}\{-M_T, ..., M_T\}\}$	$\Gamma^*_P = \{p \in \frac{\pi}{L}\mathbb{Z}\}$
$\Gamma^*_{A,T,L_T} = \{p \in \frac{\pi}{L_T}\{-M_T + \frac{1}{2}, ..., M_T + \frac{1}{2}\}\}$	$\Gamma^*_A = \{p \in \frac{\pi}{L}(\mathbb{Z} + \frac{1}{2})\}$
$\Gamma^*_{P,T} = \{p \in \frac{\pi}{L}\{-M_T, ..., M_T\}\}$	
$\Gamma^*_{A,T} = \{p \in \frac{\pi}{L}\{-M_T + \frac{1}{2}, ..., M_T + \frac{1}{2}\}\}$	
$\hat{\Gamma}^*_{A,T} = \{p \in \frac{\pi}{L}\{-M_T + \frac{1}{2}, ..., M_T - \frac{1}{2}\}\}$	
$\Gamma^*_{P,T,L'} = \{p \in \frac{\pi}{L}\mu L'\{-M_T, ..., M_T\}\}$	
$\Sigma_{P,T} = \{\theta \in \frac{2\pi}{2M_T+1}\{-M_T, ..., M_T\}\}$	
$\Sigma_{A,T} = \{\theta \in \frac{2\pi}{2M_T+1}\{-M_T + \frac{1}{2}, ..., M_T + \frac{1}{2}\}\}$	
$\Sigma_{P,\infty} = \{\theta \in \frac{2\pi}{2M_T+1}\mathbb{Z}\}$	
$\Sigma_{P,\mu,T} = \{\theta \in \frac{2\pi}{2M_T+1}\frac{L_T}{L}\mu L'\{-M_T, ..., M_T\}\}$	
$\Sigma_{A,T,L_T} = \{\theta \in \frac{2\pi}{2M_T+1}\frac{L_T}{L}\{-M_T + \frac{1}{2}, ..., M_T - \frac{1}{2}\}\}$	

We give here a short description of the sets in the table. The elements of the sets Γ_T^x and Γ_T^y are the horizontal and vertical coordinates of the lattice points respectively, which converge to Γ^x and Γ^y in the continuum limit $\mu \to 0$. We use the hyperbolic representation for the kernels of D^{-1}, $D^{-\tau}$, BD^{-1} and $D^{-1}C$ in order to control the scaling limit of these operators. The periodic and anti-periodic spectra are here denoted by $\Sigma_{P,T}$ and $\Sigma_{A,T}$, where the integer M_T depends on the temperature. To calculate the scaling limit, we introduce the scaling variables $p = \mu^{-1}\theta$ which are elements of the sets Γ^*_{P,T,L_T} and Γ^*_{A,T,L_T} for θ in the periodic or anti-periodic spectrum. Using the hyperbolic representations, we introduce a scaled version of the function γ which allow us to focus on the analysis of this function in the scaling limit. We define a bijection $p \to [p]$ of the set Γ^*_{P,T,L_T} onto $\Gamma^*_{P,T}$, where $\Gamma^*_{P,T}$ consists of the points $[p]$ closest to p in Γ^*_{P,T,L_T}. A similar bijection can be found between Γ^*_{A,T,L_T} and $\Gamma^*_{A,T}$. We introduce isometric embeddings of the temperature dependent spaces $l^2(\Gamma^*_{P,T})$ and $l^2(\Gamma^*_{A,T})$ into the temperature independent spaces $l^2(\Gamma^*_P)$ and $l^2(\Gamma^*_A)$ respectively in order to analyze the scaling behavior. We will later suppose that a function $f \in l^2(\Gamma_P^*)$ is in the dense set of compactly supported functions. Then there exists an $L' > 0$ such that f has finite support contained in the set $\Gamma^*_{P,T,L'}$. This set along with the sets $\Sigma_{P,\mu,T}$ and Σ_{A,T,L_T} will be involved in the calculation of the strong convergence of $D^{-\tau}$ and D^{-1} in the continuum limit.

The horizontal coordinates of the lattice points are elements of the set

$$\Gamma_T^x := \{x \in \mu\{-M_T, -M_T+1, ..., M_T\}\} \tag{3.43}$$

which converges to the set

$$\Gamma_T^x \to \Gamma^x := \{x \in \mathbb{R} : -L \leq x \leq L\} \tag{3.44}$$

in the continuum limit $\mu \to 0$. In the vertical direction we take the thermodynamic limit before the scaling limit, so we define the vertical coordinates of the lattice points to be elements of the set

$$\Gamma_T^y := \{y \in \mu\mathbb{Z}\}. \tag{3.45}$$

In the continuum limit $\mu \to 0$ we have

$$\Gamma_T^y \to \Gamma^y := \{y \in \mathbb{R}\}. \tag{3.46}$$

For $k = 1, ..., n$ and $n \leq 2M + 1$, let

$$a_k := (x_k, y_k) \in \Gamma^x \times \Gamma^y,$$

with $y_k < y_{k+1}$ for all $k \in \{1, ..., n\}$. Let $[a_k] \in \Gamma_T^x \times \Gamma_T^y$ denote the point which is closest to a_k, i.e $[a_k] \to a_k$ as $T \uparrow T_C$. For $T < T_C$, define

$$\tau([a];T) := \frac{\langle \sigma(\mu^{-1}[a_n]) \cdots \sigma(\mu^{-1}[a_1]) \rangle_{M_T}}{\langle 0_A, \sigma 0_P \rangle_{M_T}^n},$$

where $[a] = ([a_1], [a_2], ..., [a_n])$, $\mu^{-1}[a_k] = (\mu^{-1}[x_k], \mu^{-1}[y_k])$ and the spin correlations are evaluated at temperature T in $\mathcal{K} = \frac{J}{k_B T}$ with J held fixed. The scaling limit of the n-point correlation function from below T_C at $a = (a_1, a_2, ..., a_n)$ is defined by

$$\tau^{[-]}(a) := \lim_{T \uparrow T_C} \tau([a];T).$$

We invoke the notation $[-]$ for the limit taken from below the critical temperature. The points in the periodic spectrum are elements of the set

$$\Sigma_{P,T} := \left\{ \theta \in \frac{2\pi}{2M_T+1}\{-M_T, ..., M_T\} \right\}$$

and the points in the anti-periodic spectrum are elements of the set

$$\Sigma_{A,T} := \left\{ \theta \in \frac{2\pi}{2M_T+1}\{ - M_T + \tfrac{1}{2}, ..., M_T + \tfrac{1}{2}\} \right\}.$$

Introduce the scaling variables,

$$\begin{cases} p_k^P = \frac{\theta_k^P}{\mu} = \frac{\pi k}{L_T} & \text{for } -M_T \leq k \leq M_T; \\ p_k^A = \frac{\theta_k^A}{\mu} = \frac{\pi\left(k+\frac{1}{2}\right)}{L_T} & \text{for } -M_T \leq k \leq M_T. \end{cases}$$

The dual lattice to Γ_T^x is given by

$$\Gamma^*_{P,T,L_T} := \left\{ p \in \frac{\pi}{L_T}\{-M_T, ..., M_T\} \right\}$$

which converges to

$$\Gamma^*_P := \left\{ p \in \frac{\pi}{L}\mathbb{Z} \right\} \tag{3.47}$$

as $T \uparrow T_C$. Define

$$\Gamma^*_{A,T,L_T} := \left\{ p \in \frac{\pi}{L_T}\{-M_T + \tfrac{1}{2}, ..., M_T + \tfrac{1}{2}\} \right\}$$

which converges to

$$\Gamma^*_A := \left\{ p \in \frac{\pi}{L}\left(\mathbb{Z} + \tfrac{1}{2}\right) \right\} \tag{3.48}$$

as $T \uparrow T_C$. For $T < T_C$, introduce the scaled function

$$\gamma_T(p) := \gamma(p\mu) \quad \text{for} \quad |\mu p| < \pi.$$

The scaled function γ_T plays a central role in the control of the scaling limits for D^{-1}, $D^{-\tau}$, BD^{-1} and $D^{-1}C$. Define

$$\omega(p) := \sqrt{1+p^2}.$$

From [Pal06] we have the following lemma which is the key ingredient in controlling the scaling limit:

Lemma 3.4. *[Pal06] For each $\epsilon > 0$, there exists an interval $[T_0, T_C]$ with $0 < T_0 < T_C$ and constants $c > 0$ and $C > 0$ independent of p and T such that*

$$c\omega(p) \leq \mu^{-1}\gamma_T(p) \leq C\omega(p) \quad \textit{for} \quad T \in [T_0, T_C] \quad \textit{and} \quad |\mu p| \leq \pi$$

and

$$(\tfrac{2}{\pi} - \epsilon)\omega(p) \leq \mu^{-1}\sinh(\gamma_T(p)) \leq C\omega(p) \quad \textit{for} \quad T \in [T_0, T_C] \quad \textit{and} \quad |\mu p| \leq \pi.$$

Furthermore,

$$\lim_{T \uparrow T_C} \mu^{-1}\gamma_T(p) = \omega(p), \tag{3.49}$$

$$\lim_{T \uparrow T_C} \mu^{-1}\sinh\gamma_T(p) = \omega(p), \tag{3.50}$$

where the limits are uniform for p in any compact subset of $\mathbb{R}$.

In [BL03], the scaling limit of the matrix elements of the spin operator is given. Since the proof is not given in [BL03] we provide it here. Introduce the notation,

$$p = \frac{\theta}{\mu} \quad \text{and} \quad p' = \frac{\theta'}{\mu}$$

for θ and θ' in $\Sigma_{A,T}$ or $\Sigma_{P,T}$. Recall that the proposed Bugrij-Lisovyy formula [BL03] was given by

$$\begin{aligned}
&\langle e^A_{+,i_1} \wedge e^A_{+,i_2} \wedge \ldots \wedge e^A_{+,i_m}, \sigma e^P_{+,j_1} \wedge e^P_{+,j_2} \wedge \ldots \wedge e^P_{+,j_{m'}} \rangle \\
&= \sqrt{\xi\xi_T} \prod_{i=1}^{m} \frac{e^{\frac{1}{2}v(z_A(i))}}{\sqrt{(2M_T+1)\sinh\gamma(\theta^A_i)}} \prod_{j=1}^{m'} \frac{e^{-\frac{1}{2}v(z_P(j))}}{\sqrt{(2M_T+1)\sinh\gamma(\theta^P_j)}} \times \\
&\times \prod_{1 \leq i < i' \leq m} \frac{\sin\frac{\theta^A_i - \theta^A_{i'}}{2}}{\sinh\frac{\gamma(\theta^A_i)+\gamma(\theta^A_{i'})}{2}} \prod_{1 \leq j < j' \leq m'} \frac{\sin\frac{\theta^P_j - \theta^P_{j'}}{2}}{\sinh\frac{\gamma(\theta^P_j)+\gamma(\theta^P_{j'})}{2}} \times \\
&\times \prod_{1 \leq i \leq m, 1 \leq j \leq m'} \frac{\sinh\frac{\gamma(\theta^A_i)+\gamma(\theta^P_j)}{2}}{\sin\frac{\theta^A_i - \theta^P_j}{2}},
\end{aligned} \tag{3.51}$$

where

$$\xi = |1 - (\sinh(2\mathcal{K}))^{-4}|^{\frac{1}{4}}.$$

Let $\gamma'(\theta)$ denote the derivative of γ with respect to θ. The cylindrical parameters ξ_T and $v(z)$ are given by

$$\begin{aligned} &\ln \xi_T \\ &= \frac{(2M_T+1)^2}{2\pi^2} \int_0^\pi \int_0^\pi \frac{d\theta d\theta' \gamma'(\theta)\gamma'(\theta')}{\sinh[(2M_T+1)\gamma(\theta)]\sinh[(2M_T+1)\gamma(\theta')]} \ln \left| \frac{\sin((\theta+\theta')/2)}{\sin((\theta-\theta')/2)} \right|, \end{aligned} \tag{3.52}$$

$$v(\theta) := v(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta' \sinh\gamma(\theta)}{\cosh\gamma(\theta) - \cos(\theta')} \ln \coth((2M_T+1)\gamma(\theta')/2). \tag{3.53}$$

Define

$$\tilde{v}(p) := \lim_{T\uparrow T_C} v(p\mu) \quad \text{and} \quad \ln \tilde{\xi}_T := \lim_{T\uparrow T_C} \ln \xi_T.$$

Lemma 3.5. *For $T < T_C$, the scaling limit $T \uparrow T_C$ of (3.51) becomes*

$$\begin{aligned} &\sqrt{\xi\tilde{\xi}_T} \prod_{i=1}^{m} \frac{e^{\frac{1}{2}\tilde{v}(p_i^A)}}{\sqrt{2L\omega(p_i^A)}} \prod_{j=1}^{m'} \frac{e^{-\frac{1}{2}\tilde{v}(p_j^P)}}{\sqrt{2L\omega(p_j^P)}} \prod_{1\le i\le i'\le m} \frac{p_i^A - p_{i'}^A}{\omega(p_i^A)+\omega(p_{i'}^A)} \times \\ &\times \prod_{1\le j\le j'\le m'} \frac{p_j^P - p_{j'}^P}{\omega(p_j^P)+\omega(p_{j'}^P)} \prod_{1\le i\le m, 1\le j\le m'} \frac{\omega(p_i^A)+\omega(p_j^P)}{p_i^A - p_j^P}, \end{aligned}$$

where

$$\tilde{v}(p) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dp'\,\omega(p)}{p^2 + p'^2 + 1} \ln(\coth(L\omega(p'))) \tag{3.54}$$

for $p \in \Gamma_P^*$ or $p \in \Gamma_A^*$ and

$$\ln \tilde{\xi}_T = \frac{2L^2}{\pi^2} \int_0^\infty \int_0^\infty \frac{dp\,dp'\,\omega'(p)\omega'(p')}{\sinh(2L\omega(p))\sinh(2L\omega(p'))} \ln \left| \frac{p+p'}{p-p'} \right|, \tag{3.55}$$

where ω' denotes the derivative of ω.

Proof. For $\theta = p\mu$, we have

$$\begin{aligned} \lim_{T\uparrow T_C} \frac{1}{(2M_T+1)\sinh[\gamma(\theta)]} &= \lim_{T\uparrow T_C} \frac{1}{2L_T\mu^{-1}\sinh\left[\gamma_T(p)\right]} \\ &= \frac{1}{2L\omega(p)}, \end{aligned} \tag{3.56}$$

where we in the second equation used (3.50). We use the dominated convergence theorem to prove (3.54). After substituting $\theta = p\mu$ and $\theta' = p'\mu$ into (3.53), we obtain

$$v(p\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dp'\mu \sinh \gamma(p\mu) \chi_{\left[-\frac{\pi}{\mu}, \frac{\pi}{\mu}\right]}(p')}{\cosh \gamma(p\mu) - \cos(p'\mu)} \ln \coth(L_T \mu^{-1} \gamma(p'\mu)), \tag{3.57}$$

where $\chi_{\left[-\frac{\pi}{\mu}, \frac{\pi}{\mu}\right]}(p')$ denotes the characteristic function of the interval, $[-\frac{\pi}{\mu}, \frac{\pi}{\mu}]$. We show that the integrand in (3.57) is bounded uniformly in μ by an integrable function. From [Bug01] we have the relation

$$4 \sinh^2 \frac{\gamma(p)}{2} = \mu^2 + 4 \sin^2 \frac{p}{2}. \tag{3.58}$$

This implies

$$\cosh \gamma(p) - \cos p' = (1 - \cos p) + (1 - \cos p') + \frac{\mu^2}{2}. \tag{3.59}$$

Observe that

$$\cos \theta \leq 1 - \frac{2}{\pi^2} \theta^2 \quad \text{for} \quad |\theta| \leq \pi.$$

Using this identity and equation (3.59), we have

$$\frac{1}{\mu^{-2}(\cosh \gamma(p\mu) - \cos(p'\mu))} \leq \frac{1}{\frac{2}{\pi^2}\left(p^2 + p'^2 + (\frac{\pi}{2})^2\right)} \quad \text{for} \quad |p| \leq \tfrac{\pi}{\mu} \quad \text{and} \quad |p'| \leq \tfrac{\pi}{\mu}. \tag{3.60}$$

Choose T_0' such that $0 < T_0' < T_C$. Then by Lemma (3.4), there exists for all $T \in [T_0', T_C]$, constants, C_1, $C_2 > 0$ independent of p and T such that

$$\mu^{-1} \sinh \gamma_T(p) \leq C_1 \omega(p) \tag{3.61}$$

and

$$C_2 \omega(p) \leq \mu^{-1} \gamma_T(p) \tag{3.62}$$

for $|p\mu| \leq \pi$. Now using (3.62), the facts that $2L_T > L$ and $\coth(\mu^{-1}\gamma_T(p'))$ is a decreasing function of p' bounded away from zero for $T < T_C$, we have

$$\coth[L_T \mu^{-1} \gamma_T(p')] \leq \coth\left[\tfrac{1}{2} L C_2 \omega(p')\right]. \tag{3.63}$$

Using (3.60), (3.61) and (3.63) we obtain the following bound for the integrand in the formula for $v(p\mu)$ in (3.57):

$$\begin{aligned}&\mu\frac{\sinh\gamma_T(p)}{[\cosh\gamma_T(p)-\cos(p'\mu)]}\ln\coth\left[L_T\mu^{-1}\gamma_T(p')\right]\\ &\leq\frac{\frac{\pi^2}{2}C_1\omega(p)}{p'^2+p^2+(\frac{\pi}{2})^2}\ln\coth\left[\tfrac{1}{2}LC_2\omega(p')\right] \qquad (3.64)\end{aligned}$$

for p' restricted to the interval $[-\pi\mu^{-1},\pi\mu^{-1}]$. Again using the fact that $2L_T>L$, the last expression is bounded by an integrable function, independent of the temperature. We have

$$\begin{aligned}&\lim_{\mu\to 0}\frac{1}{\mu^2}\left[(1-\cos(p\mu))+(1-\cos(p'\mu))+\frac{\mu^2}{2}\right]\\ &=\tfrac{1}{2}(p^2+p'^2+1). \qquad (3.65)\end{aligned}$$

By the dominated convergence theorem, (3.49), (3.50) and (3.65) we obtain

$$\begin{aligned}\tilde{v}(p)&=\lim_{T\uparrow T_C}v(p\mu)\\ &=\lim_{T\uparrow T_C}\frac{1}{\pi}\int_0^{\frac{\pi}{\mu}}\mu\frac{dp'\,\sinh\gamma_T(p)}{[\cosh\gamma_T(p)-\cos(p'\mu)]}\ln\coth\left[L_T\mu^{-1}\gamma_T(p')\right]\\ &=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dp'\,\omega(p)}{p'^2+p^2+1}\ln(\coth(L\omega(p'))). \qquad (3.66)\end{aligned}$$

To prove (3.55) we again use the dominated convergence theorem. After substituting $\theta=p\mu$ and $\theta'=p'\mu$ into (3.52), we obtain

$$\begin{aligned}&\ln\xi_T\\ &=\frac{(2\mu^{-1}L_T)^2}{2\pi^2}\int_0^\infty\int_0^\infty\frac{dp\,dp'\,\mu^2\gamma_T'(p)\gamma_T'(p')\chi_{\left[-\frac{\pi}{\mu},\frac{\pi}{\mu}\right]}(p)\chi_{\left[-\frac{\pi}{\mu},\frac{\pi}{\mu}\right]}(p')}{\sinh(\mu^{-1}2L_T\gamma_T(p))\sinh(\mu^{-1}2L_T\gamma_T(p'))}\ln\left|\frac{\sin(\frac{p\mu+p'\mu}{2})}{\sin(\frac{p\mu-p'\mu}{2})}\right|.\end{aligned}$$

We have

$$\frac{d}{dp}\gamma(p)=\frac{\sin(\frac{p}{2})\cos(\frac{p}{2})}{\sqrt{1+\sinh^2\gamma(\frac{p}{2})}\sqrt{\frac{\mu^2}{4}+\sin^2(\frac{p}{2})}}.$$

Then

$$\begin{aligned}\lim_{u\to 0}\gamma_T'(p) &= \lim_{u\to 0}\frac{\sin(\frac{pu}{2})\cos(\frac{pu}{2})}{\cosh\gamma(\frac{pu}{2})\sqrt{\frac{\mu^2}{4}+\sin^2(\frac{pu}{2})}}\\ &= \frac{p}{\sqrt{1+p^2}}\\ &= \omega'(p). \end{aligned} \tag{3.67}$$

We have the following upper bound for $\gamma_T'(p)$:

$$|\gamma_T'(p)| \le C_1\left|\frac{\sin(pu)}{\sinh(\gamma_T(p))}\right| \le C_2\omega'(p)$$

for some constants C_1 and C_2. We obtain,

$$\begin{aligned}\ln\tilde{\xi}_T &= \lim_{T\uparrow T_C}\frac{(2L_T)^2}{2\pi^2}\int_0^{\frac{\pi}{\mu}}\int_0^{\frac{\pi}{\mu}} dp'\,dp\,\frac{\gamma_T'(p')\gamma_T'(p)}{\sinh(2L_T\mu^{-1}\gamma_T(p')\sinh(2L_T\mu^{-1}\gamma_T(p))}\\ &\quad\times\ln\left|\frac{\sin(\frac{p\mu}{2}+\frac{p'\mu}{2})}{\sin(\frac{p\mu}{2}-\frac{p'\mu}{2})}\right|\\ &= \frac{2L^2}{\pi^2}\int_0^\infty\int_0^\infty\frac{dp'\,dp\,\omega'(p')\omega'(p)}{\sinh(2L\omega(p'))\sinh(2L\omega(p))}\ln\left|\frac{p+p'}{p-p'}\right|.\end{aligned}$$

Using (3.50) and (3.56) the lemma is proved. □

Define

$$X_{T,1} := D^{-\tau}$$

which is the map

$$X_{T,1} : l^2(\Sigma_{P,T}) \to l^2(\Sigma_{A,T}),$$

and where $D^{-\tau}$ is conjectured to be given by (3.35). We conjugate $X_{T,1}$ by functions O_T^A and O_T^P, i.e.

$$X_{T,1} \mapsto \hat{X}_{T,1} := O_T^A X_{T,1} O_T^{*P},$$

such that we have the following commutative diagram:

$$\begin{array}{ccc} l^2(\Sigma_{A,T}) & \xrightarrow{O_T^A} & l^2(\Gamma_A^*) \\ {\scriptstyle X_{T,1}}\uparrow & & \uparrow{\scriptstyle \hat{X}_{T,1}} \\ l^2(\Sigma_{P,T}) & \xrightarrow{O_T^P} & l^2(\Gamma_P^*) \end{array}\quad .$$

The functions O_T^A and O_T^P will be defined below. We start by embedding the temperature dependent space $l^2(\Sigma_{P,T})$ into the temperature independent space $l^2(\Gamma_P^*)$. We do this in the following way: Introduce the temperature dependent set

$$\Gamma^*_{P,T} := \left\{p \in \frac{\pi}{L}\{-M_T, ..., M_T\}\right\}.$$

We notice that $|\mu p| < \pi$ for $p \in \Gamma^*_{P,T}$. We rewrite a function in $l^2(\Sigma_{P,T})$ in terms of the T dependent scaling variables and define a bijection from the set Γ^*_{P,T,L_T} onto $\Gamma^*_{P,T}$. Then we introduce an embedding O_T^P of $l^2(\Gamma^*_{P,T})$ into the temperature independent space $l^2(\Gamma^*_P)$. Introduce the scaled transformation

$$f(\theta) \to f(\mu p)$$

as a unitary map

$$l^2(\Sigma_{P,T}) \to l^2(\Gamma^*_{P,T,L_T}). \tag{3.68}$$

Define a bijection

$$\Gamma^*_{P,T,L_T} \ni p \xrightarrow{i_T^P} [p] \in \Gamma^*_{P,T}\ ,$$

where $[p]$ is the point in $\Gamma^*_{P,T}$ closest to p in Γ^*_{P,T,L_T}. Now embedd (3.68) into $l^2(\Gamma^*_P)$ by the map O_T^P defined by

$$l^2(\Sigma_{P,T}) \ni f(\theta) \mapsto O_T^P f(p) := \chi_T^P(p) f(\mu p) \in l^2(\Gamma^*_P), \tag{3.69}$$

where χ_T^P is given by the map,

$$\chi_T^P : \Gamma^*_P \to \mathbb{C},$$

and where

$$\chi_T^P(p) = \begin{cases} 1 & \text{if } p \in i_T^P(\Gamma^*_{P,T,L_T}); \\ 0 & \text{if } p \notin i_T^P(\Gamma^*_{P,T,L_T}). \end{cases}$$

The embedding of the temperature dependent space $l^2(\Sigma_{A,T})$ into the temperature independent space $l^2(\Gamma^*_A)$ goes in a similar fashion. Introduce the transformation

$$f(\theta) \to f(\mu p)$$

as a unitary map

$$l^2(\Sigma_{A,T}) \to l^2(\Gamma^*_{A,T,L_T}). \tag{3.70}$$

Define

$$\Gamma^*_{A,T} := \Big\{ p \in \frac{\pi}{L}\{-M_T + \tfrac{1}{2}, ..., M_T + \tfrac{1}{2}\} \Big\}.$$

We notice here that $|\mu p| < \pi$ for $\Gamma^*_{A,T} \ni p = \frac{\pi k}{L}$ and $k = -M_T + \frac{1}{2}, ..., M_T - \frac{1}{2}$. When $k = M_T + \frac{1}{2}$, we do not necessarily have $|\mu p| < \pi$, so Lemma 3.4 can no longer be applied in this case for proving uniform bound. This is an inconvenience that we will take into account in some of our calculations. Define the bijection

$$\Gamma^*_{A,T,L_T} \ni p \xrightarrow{i^A_T} [p] \in \Gamma^*_{A,T} \ ,$$

where $[p]$ is the point in $\Gamma^*_{A,T}$ closest to p in Γ^*_{A,T,L_T}. We embedd (3.70) into $l^2(\Gamma^*_A)$ by the map O^A_T defined by

$$l^2(\Sigma_{A,T}) \ni f(\theta) \mapsto O^A_T f(p) := \chi^A_T(p) f(\mu p) \in l^2(\Gamma^*_A), \tag{3.71}$$

where χ^A_T is given by the map,

$$\chi^A_T : \Gamma^*_A \to \mathbb{C},$$

where

$$\chi^A_T(p) = \begin{cases} 1 & \text{if } p \in i^A_T(\Gamma^*_{A,T,L_T}); \\ 0 & \text{if } p \notin i^A_T(\Gamma^*_{A,T,L_T}). \end{cases}$$

Introduce the set

$$\Sigma_{P,\infty} := \Big\{ \theta \in \frac{2\pi}{2M_T + 1}\mathbb{Z} \Big\}.$$

We have that $O^{*P}_T(f(\theta)) = O^{*P}_T(\theta) f(\mu^{-1}\theta \frac{L_T}{L})$, where $O^{*P}_T(\theta) := \chi^P_T(\theta)$. Here χ^P_T is given by the map $\chi^P_T : \Sigma_{P,\infty} \to \mathbb{C}$, where

$$\chi^P_T(\theta) = \begin{cases} 1 & \text{if } \theta \in \Sigma_{P,T}; \\ 0 & \text{if } \theta \in \Sigma_{P,\infty} \backslash \Sigma_{P,T}. \end{cases}$$

We have

$$O^{P*}_T(\mu(i^P_T(p))^{-1}) = O^P_T(p)^{-1}$$

for $p \in i_T^P(\Gamma^*_{P,T,L_T})$. We can now substitute for $X_{T,1}$ its isometric image

$$\hat{X}_{T,1} := O_T^A X_{T,1} O_T^{*P} : l^2(\Gamma^*_P) \to l^2(\Gamma^*_A),$$

where

$$\hat{X}_{T,1} f(p) = \sum_{p' \in \Gamma^*_P} \chi_T^A(p) X_{T,1}(\mu p, \mu p') \chi_T^P(p') f(p'),$$

and where

$$\begin{aligned} X_{T,1}(\mu p, \mu p') = & \frac{e^{\frac{1}{2}v(p\mu)}}{\sqrt{2L_T\mu^{-1}\sinh\gamma_T(p)}} \frac{e^{-\frac{1}{2}v(p'\mu)}}{\sqrt{2L_T\mu^{-1}\sinh\gamma_T(p')}} \\ & \times \frac{\sinh\frac{1}{2}(\gamma_T(p)+\gamma_T(p'))}{\sin(\frac{1}{2}(p\mu - p'\mu)} \end{aligned} \tag{3.72}$$

restricted to $p' \in \Gamma^*_{P,T}$ and $p \in \Gamma^*_{A,T}$. For $\alpha > 0$ and $\beta > 0$ define the operator

$$X_{T,1}^{\alpha,\beta} := e^{-\alpha\mu^{-1}\gamma_T} \hat{X}_{T,1} e^{-\beta\mu^{-1}\gamma_T}$$

with kernel

$$X_{T,1}^{\alpha,\beta}(p,p') = e^{-\alpha\mu^{-1}\gamma_T(p)} \chi_T^A(p) X_{T,1}(\mu p, \mu p') \chi_T^P(p') e^{-\beta\mu^{-1}\gamma_T(p')}.$$

The smoothing properties of the exponential factors are important for the control of the convergence of the scaling limit. Introduce the operator,

$$X_1^{\alpha,\beta} : l^2(\Gamma^*_P) \to l^2(\Gamma^*_A),$$

where

$$X_1^{\alpha,\beta} f(p) = \sum_{p' \in \Gamma^*_P} X_1^{\alpha,\beta}(p,p') f(p'), \tag{3.73}$$

and where the kernel $X_1^{\alpha,\beta}(p,p')$ is given by

$$X_1^{\alpha,\beta}(p,p') = e^{-\alpha\omega(p)} e^{\frac{1}{2}\tilde{v}(p)} \frac{1}{\sqrt{2L\omega(p)}} \frac{1}{\sqrt{2L\omega(p')}} \frac{\omega(p)+\omega(p')}{p-p'} e^{-\frac{1}{2}\tilde{v}(p')} e^{-\beta\omega(p')} \tag{3.74}$$

for $p \in \Gamma^*_A$. The pointwise limit,

$$\lim_{T \uparrow T_C} X_{T,1}^{\alpha,\beta}(p,p') = X_1^{\alpha,\beta}(p,p')$$

is proved in Lemma 3.5. We prove the following.

Lemma 3.6. *For $\alpha,\beta>0$, the operator $X^{\alpha,\beta}_{T,1}$ converges strongly to the operator $X^{\alpha,\beta}_{1}$: $l^2(\Gamma^*_P)\to l^2(\Gamma^*_A)$ as $T\uparrow T_C$.*

Proof. Since the integrand in $v(p'\mu)$ is greater or equal to zero for all p', we have $e^{-\frac{v(p'\mu)}{2}}\le 1$. We will first show that $e^{\frac{v(p\mu)}{2}}$ is bounded by a constant uniformly in μ. Choose T_0' such that $0<T_0'<T_C$. We have from the proof of Lemma 3.5 that there exists for all $T\in[T_0',T_C]$, constants, C_1, $C_2>0$ independent of p and T such that

$$\begin{aligned} v(p\mu) &\le \frac{1}{\pi}\int_0^{\frac{\pi}{\mu}} \frac{dp'\frac{\pi^2}{2}C_1\omega(p)}{p'^2+p^2+(\frac{\pi}{2})^2}\ln\coth\left[\tfrac{1}{2}LC_2\omega(p')\right] \\ &\le \frac{\pi}{2}KC_1\omega(p)\int_0^{\infty}\frac{dp'}{(p'^2+p^2+(\frac{\pi}{2})^2)} \\ &= \frac{\pi}{2}K\,C_1\frac{\omega(p)}{\sqrt{p^2+(\frac{\pi}{2})^2)}}\frac{\pi}{2} \\ &\le \frac{\pi^2}{4}KC_1 \end{aligned} \tag{3.75}$$

where we in the evaluation of the integral used the substitution $p'=\sqrt{p^2+(\frac{\pi}{2})^2}\tan\theta$ for $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, and where K is a constant greater than zero. We notice that for $p=\frac{\pi}{L}(M_T+\frac{1}{2})$, we can find a bound for $v(p\mu)$ by using the fact that $\gamma_T(p)$ is periodic. The value of $\gamma_T(p)$ at that point is the same as the one for some p with $|p\mu|\le\pi$, and we can then apply Lemma 3.4.

The rest of the proof is inspired by the analysis as given in [Pal06], where the infinite-volume case was considered. Using $\sinh\gamma=2\sinh\frac{\gamma}{2}\cosh\frac{\gamma}{2}$, the action of $X^{\alpha,\beta}_{T,1}$ on $f\in l^2(\Gamma^*_P)$ is given by

$$\begin{aligned} &X^{\alpha,\beta}_{T,1}f(p) \\ &=\frac{1}{2}e^{-\alpha\mu^{-1}\gamma_T(p)}e^{\frac{1}{2}v(p\mu)}\frac{1}{\sqrt{2L_T\mu^{-1}\sinh(\gamma_T(p))}}\frac{1}{\sqrt{2L_T\mu^{-1}\sinh(\gamma_T(p'))}} \\ &\times[\sinh(\tfrac{1}{2}\gamma_T(p))\,\widehat{s}_T\,(\sinh(\tfrac{1}{2}\gamma_T(p')))^{-1}+\cosh(\tfrac{1}{2}\gamma_T(p))\,\widehat{s}_T\,(\cosh(\tfrac{1}{2}\gamma_T(p')))^{-1}] \\ &\times\sinh\gamma_T(p')e^{-\frac{1}{2}v(p'\mu)}e^{-\beta\mu^{-1}\gamma_T(p')}f(p') \\ &:=e^{\frac{1}{2}v(p\mu)}(a_{1,T}\,\widehat{s}_T\,a_{2,T}+a_{3,T}\,\widehat{s}_T\,a_{4,T})e^{-\frac{1}{2}v(p'\mu)}f(p'), \end{aligned}$$

where

$$\left.\begin{array}{c} a_{1,T}(p) \\ a_{3,T}(p) \end{array}\right\} = e^{-\alpha\frac{\gamma_T(p)}{\mu}}\frac{1}{\sqrt{\mu^{-1}\sinh(\gamma_T(p))}}\left\{\begin{array}{l} 2\mu^{-1}\sinh\frac{1}{2}\gamma_T(p) \\ \cosh\frac{1}{2}\gamma_T(p) \end{array}\right. ,$$

$$\left.\begin{array}{l} a_{2,T}(p) \\ a_{4,T}(p) \end{array}\right\} = e^{-\beta\frac{\gamma_T(p)}{\mu}}\sqrt{\mu^{-1}\sinh(\gamma_T(p))}\left\{\begin{array}{l} (2\mu^{-1}\sinh\frac{1}{2}\gamma_T(p))^{-1} \\ (\cosh\frac{1}{2}\gamma_T(p))^{-1} \end{array}\right.$$

restricted to the domain $\Gamma^*_{P,T}$ or $\Gamma^*_{A,T}$ as appropriate, and where

$$\hat{s}_T f(p) = \frac{1}{2L_T}\frac{1}{2}\sum_{p'\in\Gamma^*_P}\frac{\mu\,\chi^A_T(p)\chi^P_T(p')}{\sin\frac{1}{2}(p\mu - p'\mu)}f(p'). \tag{3.76}$$

Let a_j denote multiplication by $a_j(p)$ on $l^2(\Gamma^*_P)$ or $l^2(\Gamma^*_A)$ as appropriate. Using Lemma 3.4 and the fact that $\lim_{T\uparrow T_C}\cosh(\frac{\gamma_T(p)}{2}) = 1$, the pointwise limit of the multiplication operators $a_{j,T}$ for $j = 1, 2, 3, 4$ is given by

$$\lim_{T\uparrow T_C} a_{j,T}(p) = a_j(p),$$

where

$$\begin{aligned} a_1(p) &= e^{-\alpha\omega(p)}\sqrt{\omega(p)} \\ a_2(p) &= e^{-\beta\omega(p)}(\sqrt{\omega(p)})^{-1} \\ a_3(p) &= e^{-\alpha\omega(p)}(\sqrt{\omega(p)})^{-1} \\ a_4(p) &= e^{-\beta\omega(p)}\sqrt{\omega(p)}. \end{aligned}$$

Let $\tilde{T}_0$ be such that $0 < \tilde{T}_0 < T_C$. From Lemma 3.4, we have that for all T in the interval $[\tilde{T}_0, T_C]$ there exists constants $c > 0$ and $C > 0$, such that

$$|a_{j,T}(p)| \leq C\sqrt{\omega(p)}^{\pm 1}e^{-cr\omega(p)},$$

where $r = \min\{\alpha, \beta\}$ and for the appropriate choice of $\pm$. (We notice that for $p = \frac{\pi}{L}(M_T + \frac{1}{2})$, we can find a bound for $a_{j,T}(p)$ by using the fact that $\gamma_T(p)$ is periodic. The value of $\gamma_T(p)$ at that point is the same as the one for some p with $|p\mu| \leq \pi$, and we can then apply lemma 3.4).

The functions $C\sqrt{\omega(p)}^{\pm 1}e^{-cr\omega(p)}$ are even, positive and decreasing for large enough p. This implies that the series,

$$\sum_{p\in\Gamma^*_P} C\sqrt{\omega(p)}^{\pm 1}e^{-cr\omega(p)} \quad \text{and} \quad \sum_{p\in\Gamma^*_A} C\sqrt{\omega(p)}^{\pm 1}e^{-cr\omega(p)}$$

converge by comparing the series with an integral for large enough p. Then by the dominated convergence theorem, we have showed that $a_{j,T}$ converges strongly to a_j as $T \uparrow T_C$. For $\theta \in \Sigma_{A,T}$ define

$$\hat{s} := \frac{1}{(2M_T+1)} \frac{1}{2} \sum_{\theta' \in \Sigma_{P,T}} \frac{1}{\sin[\frac{1}{2}(\theta - \theta')]}.$$

The operator $\hat{s}_T$ in (3.76) is the image of $\hat{s}$ under the isometric embeddings given in (3.69) and (3.71). Hence $\sup_T \|\hat{s}_T\|_{op}$ is bounded, where $\|\cdot\|_{op}$ is the operator norm. We show that the operator $\hat{s}_T$ converges strongly to the operator h: $l^2(\Gamma_P^*) \to l^2(\Gamma_A^*)$ given by

$$hf(p) = \frac{1}{2L} \sum_{p' \in \Gamma_P^*} \frac{f(p')}{p - p'} \tag{3.77}$$

as $T \uparrow T_C$. We recognize h as the discrete Hilbert transform. From [Lo94], we have the well-known inequality

$$\left(\sum_{n \in \mathbb{Z}} \left| \sum_{\substack{m \in \mathbb{Z} \\ m \neq n}} \frac{a_m}{n - m} \right|^2 \right)^{\frac{1}{2}} \leq \pi \left(\sum_{m \in \mathbb{Z}} |a_m|^2 \right)^{\frac{1}{2}},$$

where a_m is a real valued function in $l^2(\mathbb{Z})$. Using this inequality, we have that

$$\big(\sum_{p \in \Gamma_A^*} |hf(p)|^2 \big)^{\frac{1}{2}} \leq C \|f\|_2$$

for $f \in l^2(\Gamma_P^*)$, where $\|\cdot\|_2$ is the $l^2(\Gamma_P^*)$ norm and C is a positive constant. Now, we prove strong convergence of $\hat{s}_T$. Define

$$\hat{s}_{T,L} f(p) = \frac{1}{4L} \sum_{p' \in \Gamma_P^*} \frac{\mu\, \chi_T^A(p) \chi_T^P(p')}{\sin \frac{1}{2}(p\mu - p'\mu)} f(p'). \tag{3.78}$$

We have

$$\|(\hat{s}_T - \hat{s}_{T,L}) f(p)\|_2 \leq \left\| \sum_{p' \in \Gamma_P^*} \frac{\mu\, \chi_T^A(p) \chi_T^P(p')}{\sin \frac{1}{2}(p\mu - p'\mu)} f(p') \right\|_2 \left\| \frac{1}{4L_T} - \frac{1}{4L} \right\|_2 \tag{3.79}$$

which converges to zero as $T \uparrow T_C$ since $\hat{s}_T$ is uniformly bounded and $2L_T > L$. We calculate the difference of the kernels of $\hat{s}_{T,L}$ in (3.78) and h in (3.77). We follow the analysis as given in [Pal06]. Since $\sin\theta \geq \theta - \frac{\theta^2}{\pi}$ for $0 < \theta < \pi$, we have

$$0 \leq \frac{1}{\sin\theta} - \frac{1}{\theta} \leq \frac{1}{(\pi - \theta)} \quad \text{for} \quad 0 < \theta < \pi \quad (\text{see} \quad [\text{Pal06}]),$$

which implies

$$\left|\frac{1}{\sin\theta}-\frac{1}{\theta}\right|\le\frac{1}{\pi-|\theta|}\quad\text{for}\quad 0<|\theta|<\pi.$$

Hence

$$\frac{\mu}{2}\left|\frac{1}{\sin\frac{1}{2}(\mu p-\mu p')}-\frac{1}{\frac{1}{2}(\mu p-\mu p')}\right|\le\frac{\mu}{2\pi-\mu|p-p'|}\tag{3.80}$$

for $0<\mu|p-p'|<2\pi$. Let χ_T^A denote the characteristic function of the set $\Gamma^*_{A,T}$. We want to show that $\hat{s}_T-\chi_T^A h$ converges strongly to zero as $T\uparrow T_C$ which implies that $\hat{s}_T-h$ converges strongly to zero. Since $\hat{s}_T-\chi_T^A h$ is uniformly bounded, it suffices to prove strong convergence on a dense subset of $l^2(\Gamma^*_P)$. Suppose that $f\in l^2(\Gamma^*_P)$ is in the dense set of compactly supported functions. Then there exists an $L'>0$ such that f has finite support contained in the set

$$\Gamma^*_{P,T,L'}:=\left\{p\in\frac{\pi}{L}\mu L'\{-M_T,...,M_T\}\right\}.$$

Choose T such that $\mu L'<1$. Then for $p'\in\Gamma^*_{P,T,L'}$, we have $\mu|p'|<\pi$ which implies that $\mu|p-p'|<2\pi$ for $\Gamma^*_{A,T}\ni p=\frac{\pi}{L}k$, where $k=-M_T+\frac{1}{2},...,M_T-\frac{1}{2}$. Define $p_{M_T+1/2}:=\frac{\pi}{L}(M_T+\frac{1}{2})$ and let $\hat{\Gamma}^*_{A,T}$ denote $\Gamma^*_{A,T}$ with $p_{M_T+1/2}$ omitted. Using (3.80), we have the following estimate

$$\|(\hat{s}_{T,L}-\chi_T^A h)f\|_2^2\le\frac{\mu^2}{4L^2}\sum_{p\in\hat{\Gamma}^*_{A,T}}\sum_{p'\in\Gamma^*_{P,T,L'}}\frac{\|f\|_2^2}{(2\pi-\mu|p-p'|)^2}+\tag{3.81}$$

$$+\frac{\mu^2}{4L^2}\sum_{p'\in\Gamma^*_{P,T,L'}}\|f\|_2^2\left|\frac{1}{\sin\frac{1}{2}(\mu p_{M_T+1/2}-\mu p')}-\frac{1}{\frac{1}{2}(\mu p_{M_T+1/2}-\mu p')}\right|^2,\tag{3.82}$$

where $\|\cdot\|_2$ is the $l^2(\Gamma^*_P)$ norm. We notice that the last term in the inequality above converges to zero as μ approaches zero. Define

$$\Sigma_{P,\mu,T}:=\left\{\theta'\in\frac{2\pi}{2M_T+1}\frac{L_T}{L}\mu L'\big\{-M_T,...,M_T\big\}\right\}$$

and

$$\Sigma_{A,T,L_T}:=\left\{\theta\in\frac{2\pi}{2M_T+1}\frac{L_T}{L}\big\{-M_T+\tfrac{1}{2},...,M_T-\tfrac{1}{2}\big\}\right\}.$$

Substitute $\theta = \mu p$ and $\theta' = \mu p'$ into the double sum in (3.81). Then it becomes

$$\frac{\mu^2}{4L^2} \sum_{\theta \in \Sigma_{A,T,L_T}} \sum_{\theta' \in \Sigma_{P,\mu,T}} \frac{\|f\|_2^2}{(2\pi - |\theta - \theta'|)^2}. \tag{3.83}$$

which tends to zero as $T \uparrow T_C$ since $\Sigma_{P,\mu,T} \to \{0\}$ and $\frac{\|f\|_2^2}{(2\pi - |\theta-\theta'|)^2}$ is in $l^1(\mathbb{Z})$. Thus,

$$\|(\hat{s}_T - \chi_T h)f\|_2 \leq \|(\hat{s}_T - \hat{s}_{T,L})f\|_2 + \|(\hat{s}_{T,L} - \chi_T h)f\|_2$$

which converges to zero as $T \uparrow T_C$. The strong convergence of $\hat{s}_T$ and $a_{j,T}$ imply that $X_{T,1}^{\alpha,\beta}$ converges strongly to $X_1^{\alpha,\beta}$: $l^2(\Gamma_P^*) \to l^2(\Gamma_A^*)$ as $T \uparrow T_C$. □

Define $X_{T,2} := BD^{-1}$, which is a map

$$X_{T,2} : l^2(\Sigma_{A,T}) \to l^2(\Sigma_{A,T}),$$

where BD^{-1} is conjectured to be given by (3.34). Conjugate the operator $X_{T,2}$ by O_T^A given in (3.71) such that we have the map

$$\hat{X}_{T,2} := O_T^A X_{T,2} O_T^{*A} : l^2(\Gamma_A^*) \to l^2(\Gamma_A^*).$$

Here

$$\hat{X}_{T,2} f(p) = \sum_{p' \in \Gamma_A^*} \chi_T^A(p) X_{T,2}(p,p') \chi_T^A(p') f(p'),$$

where

$$X_{T,2}(p,p') = \frac{e^{\frac{1}{2}v(p)}}{\sqrt{2L_T\mu^{-1}\sinh\gamma_T(p)}} \frac{e^{\frac{1}{2}v(p')}}{\sqrt{2L_T\mu^{-1}\sinh\gamma_T(p')}} \times \\ \times \frac{\sin\frac{1}{2}(p\mu - p'\mu)}{\sinh[\frac{1}{2}(\gamma_T(p) + \sinh\gamma_T(p'))]},$$

and where $\chi_T^A(p)$ and $\chi_T^A(p')$ denote the characteristic functions of the set $\Gamma_{A,T}^*$. Introduce the operator,

$$X_2 : l^2(\Gamma_A^*) \to l^2(\Gamma_A^*),$$

where

$$X_2 f(p) = \sum_{p' \in \Gamma_A^*} X_2(p,p') f(p')$$

and

$$X_2(p,p') = e^{\frac{1}{2}\tilde{v}(p)} \frac{1}{\sqrt{2L\omega(p)}} \frac{1}{\sqrt{2L\omega(p')}} \left(\frac{p-p'}{\omega(p)+\omega(p')} \right) e^{\frac{1}{2}\tilde{v}(p')}.$$

Again we multiply the kernels $X_{T,2}(p,p')$ and $X_2(p,p')$ with exponential factors which have smoothing properties: We define for $\alpha, \beta > 0$

$$X_{T,2}^{\alpha,\beta}(p,p') = e^{-\alpha\mu^{-1}\gamma_T(p)} \chi_T^A(p) X_{T,2}(p,p') \chi_T^A(p') e^{-\beta\mu^{-1}\gamma_T(p')}$$

and

$$X_2^{\alpha,\beta}(p,p') = e^{-\alpha\omega(p)} X_2(p,p') e^{-\beta\omega(p')}.$$

The pointwise limit

$$\lim_{T\uparrow T_C} X_{T,2}^{\alpha,\beta}(p,p') = X_2^{\alpha,\beta}(p,p')$$

is proved in Lemma 3.5. Including the exponential factors will make the pointwise convergence into Hilbert Schmidt class convergence:

Lemma 3.7. *Suppose that $\alpha, \beta > 0$. Then*

$$\lim_{T\uparrow T_C} \sum_{p\in\Gamma_A^*} \sum_{p'\in\Gamma_A^*} |X_{T,2}^{\alpha,\beta}(p,p') - X_2^{\alpha,\beta}(p,p')|^2 = 0.$$

Proof. We will use the dominated convergence theorem to prove the limit. Since

$$\left| \frac{p-p'}{\omega(p)+\omega(p')} \right| \leq 2,$$

and $e^{\frac{1}{2}\tilde{v}(p)}$ is uniformly bounded (see 3.75), we have

$$|X_2^{\alpha,\beta}(p,p')| \leq \frac{C}{2L} \frac{e^{-\alpha\omega(p)-\beta\omega(p')}}{\sqrt{\omega(p)}\sqrt{\omega(p')}}$$

for some positive constant C. Now for some T_0 such that $0 < T_0 < T_C$, we will show that there exists a constant C_1 such that for all $T \in [T_0, T_C]$, we have

$$|X_{T,2}(p,p')| \leq \frac{C_1}{2L} \frac{1}{\sqrt{\omega(p)}\sqrt{\omega(p')}}.$$

We have for some constant $C_2 > 0$,

$$\begin{aligned} &\left|\frac{\sin(\frac{1}{2}(p\mu - p'\mu))}{\sinh(\frac{1}{2}(\gamma_T(p)+\gamma_T(p')))}\right| \\ &= \left|\frac{\mu^{-1}\sin(\frac{1}{2}(p\mu - p'\mu))}{\mu^{-1}(\sinh\frac{\gamma_T(p)}{2}\cosh\frac{\gamma_T(p')}{2} + \sinh\frac{\gamma_T(p')}{2}\cosh\frac{\gamma_T(p)}{2})}\right| \\ &\leq C_2\left|\frac{p-p'}{\omega(p)+\omega(p')}\right| \\ &\leq 2C_2 \end{aligned}$$

restricted to $\hat{\Gamma}^*_{A,T}$. By the same lemma and using (3.75) we have for some constant $C_3 > 0$,

$$\left|\frac{e^{\frac{1}{2}v(p\mu)}e^{\frac{1}{2}v(p'\mu)}}{\sqrt{2L_T\mu^{-1}\sinh\gamma_T(p)}\sqrt{2L_T\mu^{-1}\sinh\gamma_T(p')}}\right| \leq \frac{C_3}{2L}\frac{1}{\sqrt{\omega(p)}}\frac{1}{\sqrt{\omega(p')}}$$

restricted to $\hat{\Gamma}^*_{A,T}$. Using the fact that $\gamma_T(p)$ and $\sin(p)$ are periodic functions, a simple calculation shows that for $p = p_{M_T+1/2}$, we have that $|X_{T,2}(p_{M_T+1/2}, p')|$ is bounded by $\frac{C_4}{2L}\frac{1}{\sqrt{\omega(q)}\sqrt{\omega(p')}}$ for some $q \in \hat{\Gamma}^*_{A,T}$ and constant $C_4 > 0$. This implies the existence of a constant K such that for all $T \in [T_0, T_C]$ we have

$$|X^{\alpha,\beta}_{T,2} - X^{\alpha,\beta}_2|^2 \leq \frac{K}{(2L)^2}\frac{1}{\omega(p)}\frac{1}{\omega(p')}e^{-2\alpha\omega(p)-2\beta\omega(p')},$$

and it is clear that the expression to the right of the inequality sign is in $l^1(\Gamma^*_A)$. Hence by the dominated convergence theorem, the lemma is proved. □

The operator

$$D^{-1} : l^2(\Sigma_{A,T}) \to l^2(\Sigma_{P,T})$$

with exponential smoothing factors and working in the isometric embeddings O^P_T and O^A_T of (3.69) and (3.71), converges strongly by the same method as for $D^{-\tau}$. The operator D^{-1} is conjectured to be given by (3.20). Similarly, the operator

$$D^{-1}C : l^2(\Sigma_{P,T}) \to l^2(\Sigma_{P,T})$$

with exponential smoothing factors and working in the isometric embeddings O^P_T of (3.69) converges in Hilbert-Schmidt norm by the same argument as for BD^{-1}. The operator $D^{-1}C$ is conjectured to be given by (3.36). We give their limits in the next section.

3.6. Pfaffian Formulas for the Scaling Functions

Before we provide the Pfaffian formulas for the Scaling functions, we recall some properties of the Hilbert Schmidt class operators that are central in our calculations. The proofs of these properties can be found in [Si77] or [RS80]. Recall that a bounded operator A on a Hilbert space is called Hilbert-Schmidt if and only if $\operatorname{tr}(A^*A) < \infty$ and it is called trace class if and only if $\operatorname{tr}|A| < \infty$. A bounded operator A on a Hilbert space is a trace class operator if and only if $A = BC$, where B and C are Hilbert-Schmidt class operators. We will need the following lemma.

Lemma 3.8. *Suppose that A_n is a sequence of bounded operators that converges strongly to a bounded operator A on a Hilbert space and suppose that B_n is a sequence of Hilbert Schmidt class operators that converges in Schmidt norm to B. Then A_nB_n converges in Schmidt norm to AB.*

On page 333 and 334 of [Pal06], Palmer provides Pfaffian formulas for vacuum expectations of products in spin representations in the infinite-volume limit under the pure state defined by plus boundary conditions:

Theorem 3.9. *[Pal06] Suppose that g_j is an element in a spin representation of the orthogonal group for $j = 1, ..., n$ and the matrix of $T(g_j)$ with respect to the Q polarization of W is given by*

$$T(g_j) = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix}.$$

If D_j is invertible for $j = 1, ..., n$, then

$$\langle g_n g_{n-1} \cdots g_1 \rangle_Q = \prod_{j=1}^{n} \langle g_j \rangle_Q \operatorname{Pf} \begin{pmatrix} 1 & -\mathcal{UB} \\ \mathcal{LC} & 1 \end{pmatrix}$$

where

$$\mathcal{B} = \begin{pmatrix} B_n D_n^{-1} & .. & .. & 0 \\ .. & .. & & .. \\ .. & & .. & .. \\ 0 & .. & .. & B_1 D_1^{-1} \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} D_n^{-1} C_n & .. & .. & 0 \\ .. & .. & & .. \\ .. & & .. & .. \\ 0 & .. & .. & D_1^{-1} C_1 \end{pmatrix} \tag{3.84}$$

and $\mathcal{L}$ and $\mathcal{U}$ are the strictly lower triangular and strictly upper triangular matrices given by

$$\mathcal{L} = \begin{pmatrix} 0 & 0 & .. & .. & & & 0 \\ 1 & 0 & & & & & \\ D_{n-1}^{-1} & 1 & & & & & \\ D_{n-1,n-2}^{-1} & D_{n-2}^{-1} & & .. & & & .. \\ .. & & .. & & & & .. \\ .. & & & & 1 & 0 & 0 \\ D_{n-1,2}^{-1} & .. & .. & & D_2^{-1} & 1 & 0 \end{pmatrix} \tag{3.85}$$

and

$$\mathcal{U} = \begin{pmatrix} 0 & 1 & D_{n-1}^{-\tau} & D_{n-1,n-2}^{-\tau} & .. & .. & D_{n-1,2}^{-\tau} \\ 0 & 0 & 1 & D_{n-2}^{-\tau} & & & .. \\ .. & & & & .. & & .. \\ .. & & & .. & & & .. \\ .. & & .. & & & 1 & D_2^{-\tau} \\ .. & & & & & 0 & 1 \\ 0 & .. & .. & & & 0 & 0 \end{pmatrix} \tag{3.86}$$

with $D_{i,j} = D_i D_{i-1} \cdots D_j$ for $i > j$.

We are interested in obtaining Pfaffian formulas for the scaling functions on the periodic lattice wrapped around a cylinder which is finite in the horizontal direction and infinite in the vertical direction. In [Pal06], the space $\mathrm{Alt}^2(W)$ is identified with the space of skew-symmetric maps $R : W \to W$ via the bilinear form $(\cdot,\cdot)$ on W. Palmer defined

$$\mathcal{R} := \tfrac{1}{2} \sum_k R w_k^* \wedge w_k \in \mathrm{Alt}^2(W),$$

where $\{w_k\}$ is a basis for W with dual basis $\{w_k^*\}$ with respect to the bilinear form. If

$$R w_k^* = \sum r_{j,k} w_j$$

with $r_{j,k} = (R w_k^*, w_j^*)$, then

$$\mathcal{R} = \tfrac{1}{2} \sum_{j,k} r_{j,k} w_j \wedge w_k.$$

On the periodic lattice, the analog of $\mathcal{R}$ is defined in Appendix B,

$$\mathcal{R} = \sum_{l,m=1}^{M} [(\tfrac{1}{2} a_{lm} e_l^+ \wedge e_m^+) + (b_{lm} e_l^+ \wedge \alpha_m^-) + (\tfrac{1}{2} c_{lm} \alpha_l^- \wedge \alpha_m^-)],$$

where

$$a = BD^{-1}, \quad b = D^{-\tau} \quad \text{and} \quad c = D^{-1}C,$$

and where B, C, D are the matrix elements of the induced rotation $T(g)$ associated with g. Here $\{e_l^+\}$ is a basis for W_+^A and $\{\alpha_l^-\}$ is a basis for W_-^P. Using this definition and the formalism given in Appendix B, we can prove Theorem 3.9 by a modification of the proof given in [Pal06]. From [Pal06] it is given that

$$\det \begin{pmatrix} 1 & -c\mathcal{UB} \\ c\mathcal{LC} & 1 \end{pmatrix} = \det(1 + c^2\mathcal{UBLC})$$

for a constant c. It is well-known (see [Si77]) that $A \mapsto \det(1 + A)$ is continuous in the trace norm for A. Since the Pfaffian is the square root of the determinant of a skew symmetric matrix, it follows that the square root of

$$c \mapsto \det(1 + c^2\mathcal{UBLC})$$

is holomorphic if the Pfaffian

$$\mathrm{Pf} \begin{pmatrix} I & -\mathcal{UB} \\ \mathcal{LC} & I \end{pmatrix}$$

is continuous in the Schmidt norm in $\mathcal{UB}$ and $\mathcal{LC}$. We control the scaling limit of the multi-spin correlation functions on the cylinder for temperatures below T_C in terms of the Bugrij-Lisovyy conjecture for the spin matrix elements on the finite, periodic lattice by proving the following theorem.

Theorem 3.10. *For $k = 1, ..., n$ and $n \leq 2M + 1$, let*

$$a_k := (x_k, y_k) \in \Gamma^x \times \Gamma^y$$

with $y_k < y_{k+1}$ for all $k \in \{1, ..., n\}$ and where Γ^x and Γ^y are defined in (3.44) and (3.46). Let $[a_k]$ denote the point which is closest to a_k, where $[a_k] \in \Gamma_T^x \times \Gamma_T^y$, and where Γ_T^x and Γ_T^y are defined in (3.43) and (3.45). Define $\mu^{-1}[a_k] = (\mu^{-1}[x_k], \mu^{-1}[y_k])$, where $\mu = 2(\mathcal{K} - \mathcal{K}^*)$ is the inverse correlation length at temperature T. Define $\bar{\Delta}_j := \frac{y_j - y_{j-1}}{2} > 0$. Then

$$\tau^{[-]}(a) := \lim_{T \uparrow T_C} \frac{\langle \sigma(\mu^{-1}[a_n]) \cdots \sigma(\mu^{-1}[a_1]) \rangle_{M_T}}{\langle 0_A, \sigma 0_P \rangle_{M_T}^n} = \mathrm{Pf} \begin{pmatrix} I & -\mathcal{UB} \\ \mathcal{LC} & I \end{pmatrix},$$

where

$$\mathcal{B} = \begin{pmatrix} B_n D_n^{-1} & .. & .. & 0 \\ .. & .. & & .. \\ .. & & .. & .. \\ 0 & .. & .. & B_1 D_1^{-1} \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} D_n^{-1} C_n & .. & .. & 0 \\ .. & .. & & .. \\ .. & & .. & .. \\ 0 & .. & .. & D_1^{-1} C_1 \end{pmatrix} \tag{3.87}$$

and $\mathcal{L}$ and $\mathcal{U}$ are the strictly lower triangular and strictly upper triangular matrices given by

$$\mathcal{L} = \begin{pmatrix} 0 & 0 & .. & .. & & & 0 \\ 1 & 0 & & & & & \\ D_{n-1}^{-1} & 1 & & & & & \\ D_{n-1,n-2}^{-1} & D_{n-2}^{-1} & & .. & & & .. \\ .. & & .. & & & & .. \\ .. & & & & 1 & 0 & 0 \\ D_{n-1,2}^{-1} & .. & .. & & D_2^{-1} & 1 & 0 \end{pmatrix} \tag{3.88}$$

and

$$\mathcal{U} = \begin{pmatrix} 0 & 1 & D_{n-1}^{-\tau} & D_{n-1,n-2}^{-\tau} & .. & .. & D_{n-1,2}^{-\tau} \\ 0 & 0 & 1 & D_{n-2}^{-\tau} & & & .. \\ .. & & & & .. & & .. \\ .. & & & .. & & & .. \\ .. & & .. & & & 1 & D_2^{-\tau} \\ .. & & & & & 0 & 1 \\ 0 & .. & .. & & & 0 & 0 \end{pmatrix} \tag{3.89}$$

with $D_{i,j} = D_i D_{i-1} \cdots D_j$ for $i > j$. The operators $D_j^{-\tau}$ and $D_j^{-1} C_j$ act on $l^2(\Gamma_P^*)$ and the operators D_j^{-1} and $B_j D_j^{-1}$ act on $l^2(\Gamma_A^*)$, where and Γ_P^* and Γ_A^* are defined in (3.47) and (3.48). In terms of the Bugrij-Lisovyy conjecture, they are given by

$$D_j^{-\tau} f(p) \propto \sum_{p' \in \Gamma_P^*} \frac{e^{\frac{1}{2}\tilde{v}(p)} e^{-\frac{1}{2}\tilde{v}(p')}}{2L\sqrt{\omega(p)}\sqrt{\omega(p')}} \chi_{j1}^{\bar{\Delta}_{j+1}, \bar{\Delta}_j}(p, p') f(p') \quad \textit{for} \quad p \in \Gamma_A^*,$$

$$B_j D_j^{-1} f(p) \propto \sum_{p' \in \Gamma_A^*} \frac{e^{\frac{1}{2}\tilde{v}(p)} e^{\frac{1}{2}\tilde{v}(p')}}{2L\sqrt{\omega(p)}\sqrt{\omega(p')}} \chi_{j2}^{\bar{\Delta}_{j+1}, \bar{\Delta}_{j+1}}(p, p') f(p') \quad \textit{for} \quad p \in \Gamma_A^*,$$

$$D_j^{-1} C_j f(p) \propto \sum_{p' \in \Gamma_P^*} \frac{e^{-\frac{1}{2}\tilde{v}(p)} e^{-\frac{1}{2}\tilde{v}(p')}}{2L\sqrt{\omega(p)}\sqrt{\omega(p')}} \chi_{j2}^{\bar{\Delta}_j, \bar{\Delta}_j}(p, p') f(p') \quad \textit{for} \quad p \in \Gamma_P^*,$$

$$D_j^{-1}f(p) \propto \sum_{p'\in\Gamma_A^*} \frac{e^{-\frac{1}{2}\tilde{v}(p)}e^{+\frac{1}{2}\tilde{v}(p')}}{2L\sqrt{\omega(p)}\sqrt{\omega(p')}}\chi_{j1}^{\bar{\Delta}_j,\bar{\Delta}_{j+1}}(p,p')f(p') \quad \textit{for} \quad p\in\Gamma_P^*,$$

where

$$\chi_{j1}^{\alpha,\beta}(p,p') = e^{-\alpha\omega(p)}\frac{\omega(p)+\omega(p')}{p-p'}e^{ix_j(p-p')}e^{-\beta\omega(p')}$$

and

$$\chi_{j2}^{\alpha,\beta}(p,p') = e^{-\alpha\omega(p)}\frac{p-p'}{\omega(p)+\omega(p')}e^{ix_j(p-p')}e^{-\beta\omega(p')}.$$

Proof. Rewrite

$$\langle\sigma([a_n])\cdots\sigma([a_1])\rangle_{M_T} = \langle\sigma_n\cdots\sigma_1\rangle_{M_T},$$

where

$$\sigma_j := V^{\bar{\Delta}_{j+1}}\sigma([x_j])V^{\bar{\Delta}_j}.$$

We have

$$\begin{aligned}
D(\sigma_j)^{-1} &= e^{-\bar{\Delta}_j\gamma(\theta)}D([x_j])^{-1}e^{-\bar{\Delta}_{j+1}\gamma(\theta')} \quad \text{for} \quad \theta'\in\Sigma_{A,T}, \quad \theta\in\Sigma_{P,T},\\
D(\sigma_j)^{-\tau} &= e^{-\bar{\Delta}_{j+1}\gamma(\theta)}D([x_j])^{-\tau}e^{-\bar{\Delta}_j\gamma(\theta')} \quad \text{for} \quad \theta'\in\Sigma_{P,T}, \quad \theta\in\Sigma_{A,T},\\
B(\sigma_j)D(\sigma_j)^{-1} &= e^{-\bar{\Delta}_{j+1}\gamma(\theta)}B([x_j])D([x_j])^{-1}e^{-\bar{\Delta}_{j+1}\gamma(\theta')} \quad \text{for} \quad \theta',\theta\in\Sigma_{A,T},\\
D(\sigma_j)^{-1}C(\sigma_j) &= e^{-\bar{\Delta}_j\gamma(\theta)}D([x_j])^{-1}C([x_j])e^{-\bar{\Delta}_j\gamma(\theta')} \quad \text{for} \quad \theta',\theta\in\Sigma_{P,T}.
\end{aligned}$$

Making the substitutions $[x_j]\mapsto\mu^{-1}[x_j]$ and $[y_j]\mapsto\mu^{-1}[y_j]$ and using the isometric embeddings O_T^P and O_T^A of (3.69) and (3.71) , we obtain the following Pfaffian formula

$$\frac{\langle\sigma(\mu^{-1}[a_n])\cdots\sigma(\mu^{-1}[a_1])\rangle_{M_T}}{\langle 0_A,\sigma 0_P\rangle_{M_T}^n} = \mathrm{Pf}\begin{pmatrix} I & -\mathcal{U}_T\mathcal{B}_T\\ \mathcal{L}_T\mathcal{C}_T & I\end{pmatrix},$$

where we used Theorem 3.9. From Lemma 3.7, we have that $\mathcal{B}_T$ converges to $\mathcal{B}$ in Schmidt norm and $\mathcal{C}_T$ converges to $\mathcal{C}$ in Schmidt norm as $T\uparrow T_C$. Lemma 3.6 shows that $\mathcal{U}_T$ converges strongly to $\mathcal{U}$ and $\mathcal{L}_T$ converges strongly to $\mathcal{L}$ as $T\uparrow T_C$. Then it follows from Lemma 3.8 that $\mathcal{U}_T\mathcal{B}_T\to\mathcal{U}\mathcal{B}$ and $\mathcal{L}_T\mathcal{C}_T\to\mathcal{L}\mathcal{C}$ in Schmidt norm as $T\uparrow T_C$. Since the Pfaffian is continuous in Schmidt norm in $\mathcal{U}\mathcal{B}$ and $\mathcal{L}\mathcal{C}$, the theorem follows. □

4. The one-point Green function

4.1. The Dirac Operator

Lisovyy [Lis05] calculated the Green function for the Dirac operator on the 1-punctured cylinder in the continuum. He used the representations of the one-point Green function to determine a projection onto the space of local solutions to the Dirac equation for a single branch point. In this section we determine the Green function for the Dirac operator on the 1-punctured cylinder with the special monodromies, $\lambda_v = \pm\frac{1}{2}$. We conjecture that the average value of the Green function with monodromy, $\lambda_v = \frac{1}{2}$, and the one with monodromy, $\lambda_v = -\frac{1}{2}$, is the appropriate Green function for the Ising model. The representations of these Green functions determine a projection onto the localization subspace for a single branch point which connects to the scaling limit calculations given in Chapter 5. Let $a = (a_1, a_2)$ be a point on the cylinder. We replace the cylinder with the strip,

$$C = \{(x_1, x_2) \in \mathbb{R}^2 : -L \leq x_1 \leq L)\},$$

where the left and right edges are identified. Let b be the branch cut with vertex a as shown in figure 4.1.

The Dirac operator $\mathcal{D}$ on $\mathbb{R}^2$ can be written as

$$\mathcal{D} := \begin{pmatrix} 0 & 2\partial z \\ 2\partial \overline{z} & 0 \end{pmatrix},$$

where the complex derivatives ∂z and $\partial \overline{z}$ are

$$\partial z = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right) \quad \text{and} \quad \partial \overline{z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$$

for

$$z = x_1 + ix_2, \quad \text{and} \quad \overline{z} = x_1 - ix_2.$$

We refer to $(mI - \mathcal{D})\psi = 0$ as the Dirac equation, where m is a mass parameter. We consider solutions to the Dirac equation on $C\backslash b$ that have continuations across b away

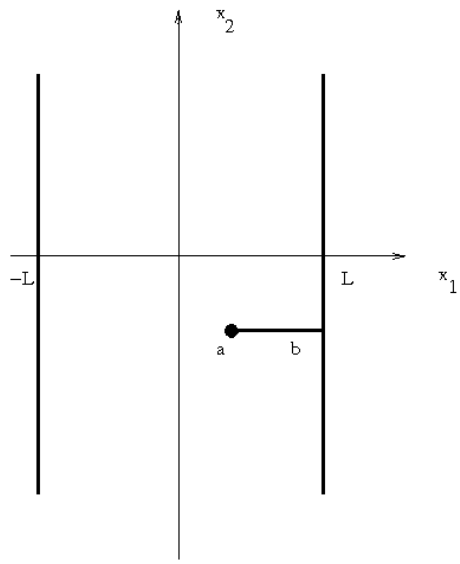

FIGURE 4.1. The figure shows the strip C with branch cut b and vertex a.

from the point a. The continuations across b, differ by the factor $e^{2\pi i\lambda_1}$ for $\lambda_1 = \pm\frac{1}{2}$. The continuations across the boundary differ by the factor $e^{2\pi i\lambda_0}$ where $\lambda_0 = 0$ for $x_2 < \Im a$, and by the factor $e^{2\pi i\lambda_1}$, where $\lambda_1 = \pm\frac{1}{2}$, for $x_2 > \Im a$. In other words, the solutions we are interested in have periodic boundary conditions for $x_2 < \Im a$ and anti-periodic boundary conditions for $x_2 > \Im a$.

4.2. Green Function for the Dirac Operator on the Cylinder with no Branch-Points.

In this section we calculate the Green function for the Dirac operator on the cylinder with no branch points. We follow the analysis as given in [Pal06] and [Lis05]. The domain of the Dirac operator consists in this case of functions that are periodic in x_1 and which is L^2 for x_2 near infinity. Using the two-dimensional Fourier transform,

$$\widehat{\psi}(p_1, p_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dx_2 \int_{-L}^{L} dx_1 \, \psi(x_1, x_2) e^{-i(x_1 p_1 + x_2 p_2)},$$

with $p_2 \in \mathbb{R}$ and $p_1 \in \Gamma_P^* = \{p \in \frac{\pi}{L}\mathbb{Z}\}$, the Dirac operator $mI - \mathcal{D}$ can be transformed into the matrix-valued multiplication operator,

$$(mI - \mathcal{D}) = \begin{pmatrix} m & -i\bar{p} \\ -ip & m \end{pmatrix},$$

where $p = p_1 + ip_2$ and $\bar{p} = p_1 - ip_2$. The inverse of the Dirac operator is given by the matrix multiplication operator,

$$(mI - \mathcal{D})^{-1} = \frac{1}{m^2 + |p|^2} \begin{pmatrix} m & i\bar{p} \\ ip & m \end{pmatrix}.$$

Using the inverse Fourier transform given by

$$\psi(x_1, x_2) = \frac{\pi}{L} \sum_{p_1 \in \Gamma_P^*} \int_{-\infty}^{\infty} dp_2 \, \widehat{\psi}(p_1, p_2) e^{i(x_1 p_1 + x_2 p_2)},$$

we obtain the following formula for the Green function, $G_0(x_1, x_2)$:

$$G_0(x_1, x_2) = \frac{1}{\pi(4L)} \sum_{p_1 \in \Gamma_P^*} \int_{-\infty}^{\infty} dp_2 \, \frac{1}{m^2 + |p|^2} \begin{pmatrix} m & i\bar{p} \\ ip & m \end{pmatrix} e^{i(x_1 p_1 + x_2 p_2)}.$$

Suppose first that $x_2 > 0$. The integral in the p_2 variable has a simple pole at $p_2 = i\sqrt{p_1^2 + m^2} := i\omega_m(p_1)$, where $\omega_m(p_1) > 0$. Let $p_\pm = \omega_m(p_1) \pm p_1$. Then by a residue calculation, we obtain

$$G_0(x_1, x_2) = \frac{1}{4L} \sum_{p_1 \in \Gamma_P^*} \begin{pmatrix} m & ip_+ \\ -ip_- & m \end{pmatrix} \frac{e^{\frac{i}{2}(xp_+ - \bar{x}p_-)}}{\sqrt{p_1^2 + m^2}}. \tag{4.1}$$

Define the angle θ_n through the relation,

$$\sinh \theta_n = \frac{\pi}{Lm} n, \quad n \in \mathbb{Z}.$$

Define

$$u := \frac{ip_-}{m} = ie^{-\theta_n} \tag{4.2}$$

which implies that

$$u^{-1} = -\frac{ip_+}{m} = -ie^{\theta_n}.$$

It follows that

$$e^{\frac{i}{2}(xp_+ - \bar{x}p_-)} = e^{-\frac{m}{2}(\bar{x}u + xu^{-1})} = e^{mx_1 i \sinh \theta_n - mx_2 \cosh \theta_n}.$$

Thus, for $x_2 > 0$, we have

$$G_0(x_1, x_2) = \frac{1}{4L} \sum_{n \in \mathbb{Z}} \begin{pmatrix} 1 & ie^{\theta_n} \\ -ie^{-\theta_n} & 1 \end{pmatrix} \frac{e^{mx_1 i \sinh \theta_n - mx_2 \cosh \theta_n}}{\cosh \theta_n}. \tag{4.3}$$

For $x_2 < 0$, we have

$$G_0(x_1, x_2) = \frac{1}{4L} \sum_{n \in \mathbb{Z}} \begin{pmatrix} 1 & -ie^{-\theta_n} \\ ie^{\theta_n} & 1 \end{pmatrix} \frac{e^{mx_1 i \sinh\theta_n + mx_2 \cosh\theta_n}}{\cosh\theta_n}. \tag{4.4}$$

The two series above converges for $-L \leq x_1 \leq L$. We rewrite the Green function as

$$G(x_1, x_2) := G_0(x_1, x_2)J, \quad \text{with} \quad J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

4.3. Canonical Basis on the Cylinder with one Branch Point.

In this section we follow [Lis05] and provide formulas for the canonical basis of solutions to the Dirac equation on the 1-punctured cylinder. These solutions have continuations to the lower half plane that is periodic in x_1 and continuations to the upper half plane that is anti-periodic in x_1. For each θ_n, the vector-valued functions

$$x \to e(x, \theta_n) := e^{-mx_1 i \sinh\theta_n - mx_2 \cosh\theta_n} \begin{pmatrix} 1 \\ -ie^{\theta_n} \end{pmatrix}$$

and

$$x \to e(x, \theta_n + \pi i) := e^{mx_1 i \sinh\theta_n + mx_2 \cosh\theta_n} \begin{pmatrix} 1 \\ ie^{\theta_n} \end{pmatrix}$$

are solutions to the Dirac equation. From [Lis05] we have, after a slight modification, that the elements of the canonical basis on the cylinder with one branch point that satisfy the Dirac equation, are given by:

$$\psi_{x_2<0}(x_1, x_2) = A \sum_{n \in \mathbb{Z}} \frac{G(\theta_n) e(x, \theta_n + \pi i)}{m2L \cosh\theta_n} \tag{4.5}$$

$$\psi_{x_2>0}(x_1, x_2) = A \sum_{n \in \mathbb{Z} - \frac{1}{2}} \frac{H(\theta_n) e(x, \theta_n)}{m2L \cosh\theta_n}, \tag{4.6}$$

where A is a normalization constant,

$$G(\theta) := -i \exp\Big(-\frac{\theta}{2} + \frac{1}{2}\eta(\theta) \Big), \quad H(\theta) := \exp\Big(-\frac{\theta}{2} - \frac{1}{2}\eta(\theta) \Big) \tag{4.7}$$

and

$$\eta(\theta) = \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} sech(\theta' - \theta) g(\theta') \tag{4.8}$$

for

$$g(\theta) = \ln\left(\frac{(1 - e^{-m2L\cosh\theta})^2}{(1 + e^{-m2L\cosh\theta})^2}\right). \tag{4.9}$$

4.4. The Green Function for the Dirac Operator on the Cylinder with one Branch Point.

In this section we give the formulas for the Green function for the Dirac operator on the cylinder with one branch point $a = (a_1, a_2)$. The domain $D^{a,\lambda}$ of the Dirac operator that we are interested in, consists of functions ψ that are square integrable at $|x_2| \to \infty$ and that have monodromies $e^{2\pi i\lambda_v}(v = 0, 1)$, where $\lambda_0 = 0$ and $\lambda_1 = \pm\frac{1}{2}$. Boundary condition at a limit the singular behavior at a and make the following equation for ψ well-posed

$$(mI - \mathcal{D})\psi = \phi,$$

where ϕ is a smooth function away from the branch point and where both ϕ and ψ have periodic boundary conditions in the lower half plane ($x_2 < a_2$) and anti-periodic boundary conditions in the upper half plane ($x_2 > a_2$). The function ψ is given by

$$\psi(z) = \int_{C\backslash b} G^{a,\lambda}(z, z')J\phi(z')idz' \wedge d\overline{z'}.$$

The kernel $G^{a,\lambda}(z, z')$ of $(mI - \mathcal{D})^{-1}$ is the Green function and can be expressed through the elements of the canonical basis given in (4.5) and (4.6). In order to determine the Green function, we will need the function to satisfy the following requirements [Lis05]: The rows of the Green function must be square integrable functions as $|x_2| \to \infty$, that satisfies the Dirac equation for all $z' \in C\backslash(b \cup \{z\})$, with monodromy $e^{2\pi i\lambda_v}$, $(v = 0, 1)$, where $\lambda_0 = 0$ and $\lambda_1 = \pm\frac{1}{2}$. Furthermore,

$$G^{a,\lambda}(z, z') - G(z, z')$$

must be smooth for z' in a neighborhood of z, where $G(z, z')$ is the Green function on the cylinder with no branch points. Following the method given in [Lis05], we obtain the following representations of the Green function, $G^{a,\lambda}(z, z')$:

For $x_2, x_2' < a_2$, we have

$$G^{a,-\frac{1}{2}}(z,z') = -i\sum_{n\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \frac{e^{-\theta_n}G(-\theta_l;\frac{1}{2})G(\theta_n;\frac{1}{2})}{m(2L)^2\cosh(\theta_l)\cosh(\theta_n)} \times \frac{e^{im(x_1-a_1)\sinh\theta_n+m(x_2-a_2)\cosh\theta_n-mi(x_1'-a_1)\sinh\theta_l+m(x_2'-a_2)\cosh\theta_l}}{e^{-\theta_n}+e^{\theta_l}} \times \begin{pmatrix} 1 & ie^{-\theta_l} \\ ie^{\theta_n} & -e^{\theta_n-\theta_l} \end{pmatrix} + G(z-z';0). \tag{4.10}$$

For $x_2, x_2' > a_2$, we have

$$G^{a,-\frac{1}{2}}(z,z') = -i\sum_{n\in\mathbb{Z}-\frac{1}{2}}\sum_{l\in\mathbb{Z}+\frac{1}{2}} \frac{e^{\theta_n}H(\theta_l;\frac{1}{2})H(-\theta_n;\frac{1}{2})}{m(2L)^2\cosh(\theta_l)\cosh(\theta_n)} \times \frac{e^{im(x_1-a_1)\sinh\theta_n-m(x_2-a_2)\cosh\theta_n-mi(x_1'-a_1)\sinh\theta_l-m(x_2'-a_2)\cosh\theta_l}}{e^{\theta_n}+e^{-\theta_l}} \times \begin{pmatrix} 1 & -ie^{\theta_l} \\ -ie^{-\theta_n} & -e^{\theta_l-\theta_n} \end{pmatrix} + G(z-z';-\tfrac{1}{2}). \tag{4.11}$$

For $x_2 > a_2 > x_2'$, we have

$$G^{a,-\frac{1}{2}}(z,z') = -i\sum_{n\in\mathbb{Z}-\frac{1}{2}}\sum_{l\in\mathbb{Z}} \frac{e^{\theta_n}G(-\theta_l;\frac{1}{2})H(-\theta_n;\frac{1}{2})}{m(2L)^2\cosh(\theta_l)\cosh(\theta_n)} \times \frac{e^{im(x_1-a_1)\sinh\theta_n-m(x_2-a_2)\cosh\theta_n-mi(x_1'-a_1)\sinh\theta_l+m(x_2'-a_2)\cosh\theta_l}}{e^{\theta_n}-e^{\theta_l}} \times \begin{pmatrix} 1 & ie^{-\theta_l} \\ -ie^{-\theta_n} & e^{-\theta_l-\theta_n} \end{pmatrix} \tag{4.12}$$

For $x_2 < a_2 < x_2'$, we have

$$G^{a,-\frac{1}{2}}(z,z') = -i\sum_{n\in\mathbb{Z}}\sum_{l\in\mathbb{Z}+\frac{1}{2}} \frac{e^{-\theta_n}G(\theta_n;\frac{1}{2})H(\theta_l;\frac{1}{2})}{m(2L)^2\cosh(\theta_l)\cosh(\theta_n)} \times \frac{e^{im(x_1-a_1)\sinh\theta_n+m(x_2-a_2)\cosh\theta_n-mi(x_1'-a_1)\sinh\theta_l-m(x_2'-a_2)\cosh\theta_l}}{e^{-\theta_n}-e^{-\theta_l}} \times \begin{pmatrix} 1 & -ie^{\theta_l} \\ ie^{\theta_n} & e^{\theta_l+\theta_n} \end{pmatrix}. \tag{4.13}$$

For $x_2, x_2' < a_2$, we have

$$G^{a,\frac{1}{2}}(z,z') = i\sum_{n\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \frac{e^{\theta_l}G(-\theta_l;\frac{1}{2})G(\theta_n;\frac{1}{2})}{m(2L)^2\cosh(\theta_l)\cosh(\theta_n)} \times \frac{e^{im(x_1-a_1)\sinh\theta_n+m(x_2-a_2)\cosh\theta_n-mi(x_1'-a_1)\sinh\theta_l+m(x_2'-a_2)\cosh\theta_l}}{e^{-\theta_n}+e^{\theta_l}} \times \begin{pmatrix} 1 & ie^{-\theta_l} \\ ie^{\theta_n} & -e^{\theta_n-\theta_l} \end{pmatrix} + G(z-z';0). \tag{4.14}$$

For $x_2, x_2' > a_2$, we have

$$G^{a,\frac{1}{2}}(z,z') = i \sum_{n\in\mathbb{Z}-\frac{1}{2}} \sum_{l\in\mathbb{Z}+\frac{1}{2}} \frac{e^{-\theta_l} H(\theta_l;\frac{1}{2}) H(-\theta_n;\frac{1}{2})}{m(2L)^2 \cosh(\theta_l)\cosh(\theta_n)}$$
$$\times \frac{e^{im(x_1-a_1)\sinh\theta_n - m(x_2-a_2)\cosh\theta_n - mi(x_1'-a_1)\sinh\theta_l - m(x_2'-a_2)\cosh\theta_l}}{e^{\theta_n}+e^{-\theta_l}}$$
$$\times \begin{pmatrix} 1 & -ie^{\theta_l} \\ -ie^{-\theta_n} & -e^{\theta_l-\theta_n} \end{pmatrix} + G(z-z';\tfrac{1}{2}). \tag{4.15}$$

For $x_2 > a_2 > x_2'$, we have

$$G^{a,\frac{1}{2}}(z,z') = -i \sum_{n\in\mathbb{Z}-\frac{1}{2}} \sum_{l\in\mathbb{Z}} \frac{e^{\theta_l} G(-\theta_l;\frac{1}{2}) H(-\theta_n;\frac{1}{2})}{m(2L)^2 \cosh(\theta_l)\cosh(\theta_n)}$$
$$\times \frac{e^{im(x_1-a_1)\sinh\theta_n - m(x_2-a_2)\cosh\theta_n - mi(x_1'-a_1)\sinh\theta_l + m(x_2'-a_2)\cosh\theta_l}}{e^{\theta_n}-e^{\theta_l}}$$
$$\times \begin{pmatrix} 1 & ie^{-\theta_l} \\ -ie^{-\theta_n} & -e^{-\theta_n-\theta_l} \end{pmatrix}. \tag{4.16}$$

For $x_2 < a_2 < x_2'$, we have

$$G^{a,\frac{1}{2}}(z,z') = -i \sum_{n\in\mathbb{Z}} \sum_{l\in\mathbb{Z}+\frac{1}{2}} \frac{e^{-\theta_l} G(\theta_n;\frac{1}{2}) H(\theta_l;\frac{1}{2})}{m(2L)^2 \cosh(\theta_l)\cosh(\theta_n)}$$
$$\times \frac{e^{im(x_1-a_1)\sinh\theta_n + m(x_2-a_2)\cosh\theta_n - mi(x_1'-a_1)\sinh\theta_l - m(x_2'-a_2)\cosh\theta_l}}{e^{-\theta_n}-e^{-\theta_l}}$$
$$\times \begin{pmatrix} 1 & -ie^{\theta_l} \\ ie^{\theta_n} & e^{\theta_l+\theta_n} \end{pmatrix}, \tag{4.17}$$

where the functions $H(\theta)$ and $G(\theta)$ are given in (4.7).
Here $G(z-z';0)$ is the Green function on the cylinder with no branch points and with periodic boundary conditions while $G(z-z';\pm\frac{1}{2})$ is the Green function on the cylinder with no branch points and with anti-periodic boundary conditions.

4.5. Projection Operators

In this section we describe the analysis as given in [Lis05], used to determine the projection onto the space of local solutions to the Dirac equation for a single branch point. Recall that

$$C = \{(x_1,x_2) \in \mathbb{R}^2 : -L \le x_1 \le L)\},$$

where the left and right edges are identified. Consider the circle

$$\mathbb{S}^1{}_{x_2^0} = \{(x_1, x_2) \in C : x_2 = x_2^0\}$$

and the fractional Sobolev space $H_\lambda^{\frac{1}{2}}(\mathbb{S}^1{}_{x_2^0})$, where $H_\lambda^{\frac{1}{2}}(\mathbb{S}^1{}_{x_2^0})$ is the space of $\mathbb{C}$-valued functions g on $\mathbb{S}^1_{x_2^0}$ that satisfies $g(x + 2L) = e^{2\pi i\lambda} g(x)$. For a description of $H^{\frac{1}{2}}$, see [LL97]. Here $\lambda = 0$ if g is periodic and $\lambda = \pm\frac{1}{2}$ if g is anti-periodic. For $g \in H_\lambda^{\frac{1}{2}}(\mathbb{S}^1{}_{x_2^0})$, we define the Fourier series

$$g(x_1) = \frac{\pi}{L} \sum_{n \in \mathbb{Z}+\lambda} \hat{g}(\theta_n) e^{imx_1 \sinh\theta_n} \quad \text{with} \quad \sinh\theta_n = \frac{\pi}{mL} n, \quad n \in \mathbb{Z} + \lambda.$$

Introduce the operators

$$Q_-(\theta) = \frac{1}{2\cosh\theta} \begin{pmatrix} e^{-\theta} & -i \\ i & e^{\theta} \end{pmatrix},$$

$$Q_+(\theta) = \frac{1}{2\cosh\theta} \begin{pmatrix} e^{\theta} & i \\ -i & e^{-\theta} \end{pmatrix},$$

which acts on $H_\lambda^{\frac{1}{2}}(\mathbb{S}^1{}_{x_2^0})$ such that

$$Q_\pm g(x_1) = \frac{\pi}{L} \sum_{n \in \mathbb{Z}+\lambda} Q_\pm(\theta_n) \hat{g}(\theta_n) e^{imx_1 \sinh\theta_n}.$$

These operators are projection operators, i.e they satisfies the properties,

$Q_+ + Q_- = I, \quad Q_+^2 = Q_+ \quad \text{and} \quad Q_-^2 = Q_-.$

Define the splitting $H_\lambda^{\frac{1}{2}}(\mathbb{S}^1{}_{x_2^0}) = Q_+ H_\lambda^{\frac{1}{2}}(\mathbb{S}^1{}_{x_2^0}) \oplus Q_- H_\lambda^{\frac{1}{2}}(\mathbb{S}^1{}_{x_2^0})$ and the polarization

$$\begin{pmatrix} g_+(\theta_n) \\ g_-(\theta_n) \end{pmatrix} = \begin{pmatrix} e^{\frac{\theta_n}{2}} & ie^{-\frac{\theta_n}{2}} \\ -ie^{-\frac{\theta_n}{2}} & -e^{\frac{\theta_n}{2}} \end{pmatrix} \begin{pmatrix} \hat{g}_1(\theta_n) \\ \hat{g}_2(\theta_n) \end{pmatrix}.$$

It follows that

$$\sum_{n \in \mathbb{Z}+\lambda} |Q_\pm(\theta_n)\hat{g}(\theta_n)|^2 \cosh\theta_n = \frac{1}{2} \sum_{n \in \mathbb{Z}+\lambda} |g_\pm(\theta_n)|^2. \tag{4.18}$$

Thus, we can write

$$g(x_1) = \frac{\pi}{L} \sum_{n \in \mathbb{Z}+\lambda} \frac{e^{imx_1 \sinh\theta_n}}{2\cosh\theta_n} \begin{pmatrix} e^{\frac{\theta_n}{2}} & ie^{-\frac{\theta_n}{2}} \\ -ie^{-\frac{\theta_n}{2}} & -e^{\frac{\theta_n}{2}} \end{pmatrix} \begin{pmatrix} g_+(\theta_n) \\ g_-(\theta_n) \end{pmatrix}.$$

We now rewrite the Dirac equation,

$$\begin{pmatrix} 0 & 2\partial \\ 2\bar{\partial} & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} - \begin{pmatrix} m\psi_1 \\ m\psi_2 \end{pmatrix} = 0.$$

Solving for $\partial_2 \psi$, we obtain

$$\partial_2 \psi = \begin{pmatrix} i\partial_1 & -im \\ im & -i\partial_1 \end{pmatrix} \psi.$$

By solving for ψ in this equation with the initial condition, $\psi(x_1, x_2^0) = g(x_1, x_2^0)$ for $g \in Q_+ H_\lambda^{\frac{1}{2}}(L_{x_2^0})$, we obtain

$$\psi_{x_2 > x_2^0}(x_1, x_2) = \frac{\pi}{L} \sum_{n \in \mathbb{Z}+\lambda} \frac{e^{imx_1 \sinh\theta_n - m(x_2 - x_2^0)\cosh\theta_n}}{2\cosh\theta_n} \begin{pmatrix} e^{\frac{\theta_n}{2}} & ie^{-\frac{\theta_n}{2}} \\ -ie^{-\frac{\theta_n}{2}} & -e^{\frac{\theta_n}{2}} \end{pmatrix} \begin{pmatrix} g_+(\theta_n) \\ 0 \end{pmatrix}. \tag{4.19}$$

This shows that the elements of $Q_+ H_\lambda^{\frac{1}{2}}(L_{x_2^0})$ represent the boundary values of solutions to the Dirac equation in the upper half-strip ($x_2 > x_2^0$) with monodromy λ. A similar calculation shows that the elements of $Q_- H_\lambda^{\frac{1}{2}}(L_{x_2^0})$ represent the boundary values of solutions to the Dirac equation in the lower half-strip ($x_2 < x_2^0$) with monodromy λ. Let $\Delta^{\mathcal{L}}$ and $\Delta^{\mathcal{U}}$ denote two positive real numbers, and let $a = (a_1, a_2) \in C$ denote a branch point. Define the horizontal strip

$$S_\Delta(a) := \{(x_1, x_2) : a_2 - \Delta^{\mathcal{L}} < x_2 < a_2 + \Delta^{\mathcal{U}}\}.$$

When a, $\Delta^{\mathcal{L}}$ and $\Delta^{\mathcal{U}}$ are understood, we write $S := S_\Delta(a)$. Denote the lower and the upper boundary of S by $\mathcal{L}$ and $\mathcal{U}$ respectively, where

$$\mathcal{U} = \{x : x_2 = a_2 + \Delta^{\mathcal{U}}\} \quad \text{and} \quad \mathcal{L} = \{x : x_2 = a_2 - \Delta^{\mathcal{L}}\}.$$

Here we assume that the upper boundary of S is negatively oriented. Recall that the Green function $G(z - z'; 0)$ on the cylinder without branch point is defined as

$$G(z - z'; 0) = G_0(z - z')J,$$

where

$$J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

We have the following proposition, slightly modified from [Lis05]:

Proposition 4.1. *[Lis05] Let $Q : H_0^{\frac{1}{2}}(\mathbb{S}^1{}_{x_2^0}) \to H_0^1(C\backslash\mathbb{S}^1{}_{x_2^0})$ be defined by*

$$(Q(g))(z) = \int_{\mathbb{S}^1{}_{x_2^0}} G(z-z';0)g(x_1')\,dx_1' \tag{4.20}$$

for $g \in H_0^{\frac{1}{2}}(\mathbb{S}^1{}_{x_2^0})$. Then the boundary values on $\mathbb{S}^1_{x_2^0}$ of restrictions of $(Qg)(z)$ to the upper and lower half-strip are equal to Q_+g and $-Q_-g$ respectively.

We now follow [Lis05] and [Pal06] and define the subspaces

$$\begin{aligned} X(\partial S) &:= Q_+H_0^{\frac{1}{2}}(\mathcal{L}) \oplus Q_-H_{\pm 1/2}^{\frac{1}{2}}(\mathcal{U}),\\ Y(\partial S) &:= Q_+H_{\pm 1/2}^{\frac{1}{2}}(\mathcal{U}) \oplus Q_-H_0^{\frac{1}{2}}(\mathcal{L}). \end{aligned} \tag{4.21}$$

Let $Z : X(\partial S) \to Y(\partial S)$ be a continuous linear map and define

$$W = \{x + Zx : x \in X(\partial S)\},$$

which is a subspace of $H^{\frac{1}{2}}(\partial S)$ with monodromies $\lambda = 0$ and $\lambda = \pm\frac{1}{2}$. Denote the space of boundary values of functions that solves $(mI - \mathcal{D})g = 0$ on S by W_{int} which is a subspace of W. If $g \in W$, we write $g^{\mathcal{L}}$ and $g^{\mathcal{U}}$ for the restriction of g to the lower and upper boundary respectively. We write $g_\pm = Q_\pm g$, where $Q_\pm$ is the projection onto g in the x_1 variable. For an element in $g \in W$, we use the notation

$$g_{\mathcal{L}\cup\mathcal{U}} = \begin{pmatrix} g_+^{\mathcal{L}} \\ g_-^{\mathcal{U}} \end{pmatrix} \oplus \begin{pmatrix} g_+^{\mathcal{U}} \\ g_-^{\mathcal{L}} \end{pmatrix}$$

for the restriction of g to the boundary of the strip S. First, assume that there is no branch points in the strip S, i.e both $g^{\mathcal{L}}$ and $g^{\mathcal{U}}$ have periodic boundary conditions. Define the map

$$\tilde{Q}(g)(z) := \int_{\mathcal{L}\cup\mathcal{U}} G(z-z';0)g(x_1')\,dx_1'$$

which satisfies the Dirac equation in S (see[Lis05]). Using the Fourier representations of g and the free Green function, we obtain

$$\begin{aligned} \tilde{Q}(g)(z) = {} & \frac{\pi}{L}\sum_{n\in\mathbb{Z}} \frac{e^{m(x_2-x_2^{\mathcal{U}})\cosh\theta_n + ix_1\sinh\theta_n}}{2\cosh\theta_n} \begin{pmatrix} e^{\frac{\theta_n}{2}} & ie^{-\frac{\theta_n}{2}} \\ -ie^{-\frac{\theta_n}{2}} & -e^{\frac{\theta_n}{2}} \end{pmatrix} \begin{pmatrix} 0 \\ g_-^{\mathcal{U}}(\theta_n) \end{pmatrix} \\ & + \frac{\pi}{L}\sum_{n\in\mathbb{Z}} \frac{e^{-m(x_2-x_2^{\mathcal{L}})\cosh\theta_n + ix_1\sinh\theta_n}}{2\cosh\theta_n} \begin{pmatrix} e^{\frac{\theta_n}{2}} & ie^{-\frac{\theta_n}{2}} \\ -ie^{-\frac{\theta_n}{2}} & -e^{\frac{\theta_n}{2}} \end{pmatrix} \begin{pmatrix} g_+^{\mathcal{L}}(\theta_n) \\ 0 \end{pmatrix}. \end{aligned}$$

By restricting g to the boundaries, $\mathcal{L}$ and $\mathcal{U}$, we find that $\tilde{Q}$ induces a map on W. This map is given by (see [Lis05]):

$$\tilde{Q} : \begin{pmatrix} g_+^{\mathcal{L}} \\ g_-^{\mathcal{U}} \end{pmatrix} \oplus \begin{pmatrix} g_+^{\mathcal{U}} \\ g_-^{\mathcal{L}} \end{pmatrix} \to \begin{pmatrix} g_+^{\mathcal{L}} \\ g_-^{\mathcal{U}} \end{pmatrix} \oplus \begin{pmatrix} 0 & \hat{w} \\ \hat{w} & 0 \end{pmatrix} \begin{pmatrix} g_+^{\mathcal{L}} \\ g_-^{\mathcal{U}} \end{pmatrix},$$

where

$$(\hat{w}g)(\theta_n) = e^{-m(x_2^{\mathcal{U}} - x_2^{\mathcal{L}})\cosh\theta_n} g(\theta_n)$$

in Fourier representation. We propose that the Green function we are interested in is the average value of the Green function with monodromy, $\lambda = \frac{1}{2}$, and the Green function with monodromy, $\lambda = -\frac{1}{2}$. Define

$$\tilde{G}^{a,\frac{1}{2}} := \tfrac{1}{2}(G^{a,\frac{1}{2}} + G^{a,-\frac{1}{2}}).$$

Now assume that the strip contains a branch point a. Generalizing the previous example we have the following theorem, slightly modified from [Lis05]:

Theorem 4.2. *[Lis05] Suppose that g is a function on (∂S) that satisfies $g^{\mathcal{L}} \in H_0^{\frac{1}{2}}(\mathcal{L})$ and $g^{\mathcal{U}} \in H_{\pm 1/2}^{\frac{1}{2}}(\mathcal{U})$. Let $a \in S = \{(x_1, x_2) : a_2 - \Delta^{\mathcal{L}} < x_2 < a_2 + \Delta^{\mathcal{U}}\}$. Then the map*

$$Pr(a)g(z) = \int_{\mathcal{L}\cup\mathcal{U}} \tilde{G}^{a,\lambda}_{\cdot,1}(z,z')g_1(z')\,dz' + \tilde{G}^{a,\lambda}_{\cdot,2}(z,z')g_2(z')\,d\bar{z}'$$

defines a projection onto the space of functions, $f \in D^{a,\lambda}$, that solve $(mI - \mathcal{D})f = 0$ on S. In the decomposition given in (4.21), the action of $Pr(a)$ is given by

$$Pr(a) : \begin{pmatrix} g_+^{\mathcal{L}} \\ g_-^{\mathcal{U}} \end{pmatrix} \oplus \begin{pmatrix} g_+^{\mathcal{U}} \\ g_-^{\mathcal{L}} \end{pmatrix} \to \begin{pmatrix} g_+^{\mathcal{L}} \\ g_-^{\mathcal{U}} \end{pmatrix} \oplus Z(a) \begin{pmatrix} g_+^{\mathcal{L}} \\ g_-^{\mathcal{U}} \end{pmatrix},$$

where

$$Z(a) = \begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ \hat{\gamma} & \hat{\delta} \end{pmatrix}$$

is a matrix of a map from $X(\partial S)$ to $Y(\partial S)$.
Thus, $\hat{\alpha} : Q_+ H_0^{\frac{1}{2}}(\mathcal{L}) \to Q_+ H_{\pm 1/2}^{\frac{1}{2}}(\mathcal{U})$, $\hat{\beta} : Q_- H_{\pm 1/2}^{\frac{1}{2}}(\mathcal{U}) \to Q_+ H_{\pm 1/2}^{\frac{1}{2}}(\mathcal{U})$, $\hat{\gamma} : Q_+ H_0^{\frac{1}{2}}(\mathcal{L}) \to Q_- H_0^{\frac{1}{2}}(\mathcal{L})$, $\hat{\delta} : Q_- H_{\pm 1/2}^{\frac{1}{2}}(\mathcal{U}) \to Q_- H_0^{\frac{1}{2}}(\mathcal{L})$. The maps, $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ and $\hat{\delta}$ are given by

$$
\begin{aligned}
&(\hat{\alpha}g)(\theta_n) \\
&= -i\frac{1}{2L}\sum_{l\in\mathbb{Z}}\frac{(e^{\theta_n}+e^{\theta_l})}{(e^{\theta_n}-e^{\theta_l})}\frac{e^{-m(X_2^{\mathcal{U}}-a_2)\cosh\theta_n+m(X_2^{\mathcal{L}}-a_2)\cosh\theta_l}e^{-ima_1(\sinh\theta_n-\sinh\theta_l)}}{\cosh\theta_l} \\
&\times e^{-\frac{1}{2}(\eta(\theta_n)-\eta(\theta_l))}g(\theta_l), \qquad n\in\mathbb{Z}+\tfrac{1}{2},
\end{aligned}
\tag{4.22}
$$

$$
\begin{aligned}
&(\hat{\beta}g)(\theta_n) \\
&= -\frac{1}{2L}\sum_{l\in\mathbb{Z}+\frac{1}{2}}\frac{(1-e^{\theta_l}e^{\theta_n})}{(1+e^{\theta_n}e^{\theta_l})}\frac{e^{-m(X_2^{\mathcal{U}}-a_2)(\cosh\theta_n+\cosh\theta_l)}e^{-ima_1(\sinh\theta_n-\sinh\theta_l)}}{\cosh\theta_l} \\
&\times e^{-\frac{1}{2}(\eta(\theta_n)+\eta(\theta_l))}g(\theta_l), \qquad n\in\mathbb{Z}+\tfrac{1}{2},
\end{aligned}
\tag{4.23}
$$

$$
\begin{aligned}
&(\hat{\gamma}g)(\theta_n) \\
&= \frac{1}{2L}\sum_{l\in\mathbb{Z}}\frac{(1-e^{\theta_n}e^{\theta_l})}{(1+e^{\theta_n}e^{\theta_l})}\frac{e^{m(X_2^{\mathcal{L}}-a_2)(\cosh\theta_n+\cosh\theta_l)}e^{-ima_1(\sinh\theta_n-\sinh\theta_l)}}{\cosh\theta_l} \\
&\times e^{\frac{1}{2}(\eta(\theta_n)+\eta(\theta_l))}g(\theta_l), \qquad n\in\mathbb{Z}.
\end{aligned}
\tag{4.24}
$$

$$
\begin{aligned}
&(\hat{\delta}g)(\theta_n) \\
&= -i\frac{1}{2L}\sum_{l\in\mathbb{Z}+\frac{1}{2}}\frac{(e^{\theta_n}+e^{\theta_l})}{(e^{\theta_n}-e^{\theta_l})}\frac{e^{m(X_2^{\mathcal{L}}-a_2)\cosh\theta_n-m(X_2^{\mathcal{U}}-a_2)\cosh\theta_l}e^{-ima_1(\sinh\theta_n-\sinh\theta_l)}}{\cosh\theta_l} \\
&\times e^{+\frac{1}{2}(\eta(\theta_n)-\eta(\theta_l))}g(\theta_l), \qquad n\in\mathbb{Z},
\end{aligned}
\tag{4.25}
$$

where

$$
\eta(\theta) = \int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\,\mathrm{sech}(\theta'-\theta)h(\theta')
\tag{4.26}
$$

for

$$
h(\theta) = \ln\left(\frac{(1-e^{-m2L\cosh\theta})^2}{(1+e^{-m2L\cosh\theta})^2}\right) \quad \textit{and} \quad \theta\in\mathbb{R}.
\tag{4.27}
$$

4.6. Connections with the Scaling Limit Calculations

We can use the image of the projection $Pr(a)$ above to think about a connection between the scaling limit of the operators, $D^{-\tau}$, BD^{-1}, $D^{-1}C$ and D^{-1} and the operators,

$\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ and $\hat{\delta}$ given in Theorem 4.2. Palmer [Pal06] found informally a similar connection for the Ising model for the infinite-volume case in the pure state defined by + boundary conditions. His method goes as follows: Let $a = (a_1, a_2) \in C$. A pair of functions, $(F^{\mathcal{L}}, F^{\mathcal{U}})$, is in W_{int} given that $F^{\mathcal{L}}$ is transfered to $F^{\mathcal{U}}$ in the following way. We transfer $F^{\mathcal{L}}$ from $x_2 = a_2 - \Delta^{\mathcal{U}}$ to $x_2 = a_2$ by multiplying the function with the free Dirac propagator,

$$\begin{pmatrix} e^{-\Delta^{\mathcal{L}} \cosh\theta_n} & 0 \\ 0 & e^{\Delta^{\mathcal{L}} \cosh\theta_l} \end{pmatrix} \quad \text{for} \quad l, n \in \mathbb{Z}.$$

Then multiply this result by $-\operatorname{sgn}(x_1 - x)$ and transfer the result via the free Dirac propagator,

$$\begin{pmatrix} e^{-\Delta^{\mathcal{U}} \cosh\theta_n} & 0 \\ 0 & e^{\Delta^{\mathcal{U}} \cosh\theta_l} \end{pmatrix},$$

to $x_2 = a_2 + \Delta^{\mathcal{U}}$, where $l, n \in \mathbb{Z} + \frac{1}{2}$. We obtain $F^{\mathcal{U}}$ if $(F^{\mathcal{L}}, F^{\mathcal{U}})$ is in W_{int}. The matrix of multiplication by $-\operatorname{sgn}(x_1 - x)$ is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

relative to the splitting, $W = W_+^P \oplus W_-^P$. Here the splitting is chosen to be

$$F_{\mathcal{L}} := f_+^{\mathcal{L}} + g_-^{\mathcal{L}} \quad \text{and} \quad F_{\mathcal{U}} := f_-^{\mathcal{U}} + g_+^{\mathcal{U}}.$$

Referring to this method, we expect the following:

$$\begin{aligned} \hat{\alpha} &= e^{-\Delta^{\mathcal{U}} \cosh\theta_n} D^{-\tau} e^{-\Delta^{\mathcal{L}} \cosh\theta_l} \quad \text{for} \quad l \in \mathbb{Z}, \quad n \in \mathbb{Z} + \tfrac{1}{2}, \\ \hat{\beta} &= e^{-\Delta^{\mathcal{U}} \cosh\theta_n} B D^{-1} e^{-\Delta^{\mathcal{U}} \cosh\theta_l} \quad \text{for} \quad l, n \in \mathbb{Z} + \tfrac{1}{2}, \\ \hat{\gamma} &= e^{-\Delta^{\mathcal{L}} \cosh\theta_n} D^{-1} C e^{-\Delta^{\mathcal{L}} \cosh\theta_l} \quad \text{for} \quad l, n \in \mathbb{Z}, \\ \hat{\delta} &= e^{-\Delta^{\mathcal{L}} \cosh\theta_n} D^{-1} e^{-\Delta^{\mathcal{U}} \cosh\theta_l} \quad \text{for} \quad l \in \mathbb{Z} + \tfrac{1}{2}, \quad n \in \mathbb{Z}, \end{aligned} \tag{4.28}$$

where $\Delta^{\mathcal{U}} = {x_2}^{\mathcal{U}} - a_2$ and $\Delta^{\mathcal{L}} = a_2 - {x_2}^{\mathcal{L}}$. We confirm (4.28), up to a similarity transform, by comparing the scaling limit calculations of $D^{-\tau}$, D^{-1}, BD^{-1} and $D^{-1}C$ with $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ and $\hat{\delta}$ found in Theorem 4.2. Recall that the proposed scaling limit calculations of $D^{-\tau}$, BD^{-1}, D^{-1}, and $D^{-1}C$ are proportional to

$$D^{-\tau} f(p) \propto \sum_{p' \in \Gamma_P^*} \frac{e^{\frac{1}{2}\tilde{v}(p)} e^{-\frac{1}{2}\tilde{v}(p')}}{2L\sqrt{\omega(p)}\sqrt{\omega(p')}} \frac{\omega(p) + \omega(p')}{p - p'} f(p') \quad \text{for} \quad p \in \Gamma_A^*, \quad p' \in \Gamma_P^*,$$

$$BD^{-1}f(p) \propto \sum_{p' \in \Gamma_A^*} \frac{e^{\frac{1}{2}\tilde{v}(p)} e^{\frac{1}{2}\tilde{v}(p')}}{2L\sqrt{\omega(p)}\sqrt{\omega(p')}} \frac{p-p'}{\omega(p)+\omega(p')} f(p') \quad \text{for} \quad p, p' \in \Gamma_A^*,$$

$$D^{-1}Cf(p) \propto \sum_{p' \in \Gamma_P^*} \frac{e^{-\frac{1}{2}\tilde{v}(p)} e^{-\frac{1}{2}\tilde{v}(p')}}{2L\sqrt{\omega(p)}\sqrt{\omega(p')}} \frac{p-p'}{\omega(p)+\omega(p')} f(p') \quad \text{for} \quad p, p' \in \Gamma_P^*,$$

$$D^{-1}f(p) \propto \sum_{p' \in \Gamma_A^*} \frac{e^{-\frac{1}{2}\tilde{v}(p)} e^{+\frac{1}{2}\tilde{v}(p')}}{2L\sqrt{\omega(p)}\sqrt{\omega(p')}} \frac{\omega(p)+\omega(p')}{p-p'} f(p') \quad \text{for} \quad p \in \Gamma_P^*, \quad p' \in \Gamma_A^*.$$

Let

$$r = ie^{-\theta_n} = \frac{ip_-}{m} \quad \text{and} \quad s = ie^{-\theta_l} = \frac{ip'_-}{m}$$

be the substitutions as introduced in (4.2). Recall that we defined

$$p_\pm = \omega_m(p) \pm p,$$

where $p \in \Gamma_P^*$ or $p \in \Gamma_A^*$. Then we have

$$p = \frac{m}{2}(ir - (ir)^{-1}) \quad \text{and} \quad \omega(p) = -\frac{m}{2}(ir + (ir)^{-1}).$$

A short calculation give for $m = 1$,

$$\frac{\omega(p)+\omega(p')}{p-p'} = \left(\frac{s+r}{s-r}\right) = \left(\frac{e^{\theta_n}+e^{\theta_l}}{e^{\theta_n}-e^{\theta_l}}\right),$$

where $\omega(p) = \sqrt{1+p^2}$. For the calculation of $\hat{\beta}$ we made the substitution, $\theta_n \to -\theta_n$. If we make this substitution in the calculation of the factor $\frac{p-p'}{\omega(p)+\omega(p')}$ in BD^{-1} we obtain

$$\frac{p-p'}{\omega(p)+\omega(p')} = \left(\frac{1-e^{\theta_n}e^{\theta_l}}{1+e^{\theta_n}e^{\theta_l}}\right).$$

Similarly, for the calculation of $\hat{\gamma}$ we made the substitution, $\theta_l \to -\theta_l$ which in the calculation of the factor $\frac{p-p'}{\omega(p)+\omega(p')}$ in $D^{-1}C$ give

$$\frac{p-p'}{\omega(p)+\omega(p')} = -\left(\frac{1-e^{\theta_n}e^{\theta_l}}{1+e^{\theta_n}e^{\theta_l}}\right).$$

For the calculation of $\hat{\delta}$ we made the substitutions, $\theta_l \to -\theta_l$ and $\theta_n \to -\theta_n$ which in the calculation of the factor $\frac{\omega(p)+\omega(p')}{p-p'}$ in D^{-1} give

$$\frac{\omega(p)+\omega(p')}{p-p'} = -\left(\frac{e^{\theta_n}+e^{\theta_l}}{e^{\theta_n}-e^{\theta_l}}\right).$$

We have

$$\sinh\theta_n = \frac{\pi n}{Lm} = \frac{p_n}{m} \quad \text{for} \quad n \in \mathbb{Z} \quad \text{or} \quad n \in \mathbb{Z}+\tfrac{1}{2}$$

so

$$\cosh\theta_n = \frac{1}{m}\sqrt{m^2+p_n^2} \quad \text{and} \quad d\theta_n = \frac{1}{\omega_m(p_n)}dp_n.$$

Then

$$\begin{aligned} g(\theta') &= \ln\left(\frac{1-e^{-m2L\cosh\theta'}}{1+e^{-m2L\cosh\theta'}}\right)^2 \\ &= -2\ln\big(\coth(\omega_m(p)L)\big) \end{aligned}$$

Thus,

$$\begin{aligned} \eta(\theta) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\cosh(\theta'-\theta)}g(\theta')\,d\theta' \\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{m^2\ln\big(\coth(\omega_m(p)L)\big)}{\sqrt{m^2+p^2}\sqrt{m^2+p'^2}-pp'}\frac{dp'}{\omega_m(p)} \\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\ln\big(\coth(\omega_m(p)L)\big)(\omega_m(p)\omega_m(p')+pp')}{m^2+p^2+p'^2}\frac{dp}{\omega_m(p)} \\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\ln\big(\coth(\omega_m(p)L)\big)\omega_m(p')}{m^2+p^2+p'^2}\,dp \\ &= -\tilde{v}(p'), \end{aligned}$$

where $\tilde{v}(p)$ is introduced in (3.54) with $m=1$. Here

$$\int_{-\infty}^{\infty}\frac{\ln\big(\coth(\omega_m(p)L)\big)pp'}{(m^2+p^2+p'^2)\omega_m(p)}\,dp = 0$$

since the integrand is odd. We now notice that the representations for $D^{-\tau}$, BD^{-1}, $D^{-1}C$ and D^{-1} differ from the corresponding operators, $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ and $\hat{\delta}$ by a constant and by a ‘similarity transform’ with $\sqrt{\omega(p)}$.

In the infinite-volume limit $M \to \infty$ the function $\eta(\theta)$ converges to 0.

In [Pal06] the conjugation (in (q,p) coordinates) defined by

$$* : \{x_k, y_k\} \to \{-\bar{x}_k, \bar{y}_l\}$$

is a symmetry on the lattice. This property is used in [Pal06] to find an appropriate Green function. We conjecture that this conjugation is also the key element in determining the appropriate Green function for our case. Define

$$C := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$* \begin{pmatrix} x_k \\ y_k \end{pmatrix} := \overline{\left(C \begin{pmatrix} x_k \\ y_k \end{pmatrix} \right)} = \begin{pmatrix} -\bar{x}_k \\ \bar{y}_k \end{pmatrix}.$$

We want to show that the conjugation, $*$, commutes with the action of the induced rotation $T(V)$ for the transfer matrix on W, i.e $*T(V)* = T(V)$. The induced rotation for the transfer matrix on the lattice can be written

$$T(V) = T_1 z + T_0 + T_{-1} z^{-1},$$

where

$$T_1 = -\frac{s_1}{2} \begin{pmatrix} s_2^* & i(1-c_2^*) \\ i(c_2^*+1) & s_2^* \end{pmatrix},$$

$$T_{-1} = -\frac{s_1}{2} \begin{pmatrix} s_2^* & -i(c_2^*+1) \\ i(c_2^*-1) & s_2^* \end{pmatrix}$$

and

$$T_0 = c_1 \begin{pmatrix} c_2^* & -is_2^* \\ is_2^* & c_2^* \end{pmatrix}.$$

We have

$$\begin{aligned} *T_1 * z &= \left[\overline{-\frac{s_1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s_2^* & i(1-c_2^*) \\ i(1+c_2^*) & s_2^* \end{pmatrix} \begin{pmatrix} -z^{-1} & 0 \\ 0 & z^{-1} \end{pmatrix}} \right] \\ &= \left[\overline{-\frac{s_1}{2} \begin{pmatrix} s_2^* & -i(1-c_2^*) \\ -i(1+c_2^*) & s_2^* \end{pmatrix} z^{-1}} \right] \\ &= -\frac{s_1}{2} \begin{pmatrix} s_2^* & i(1-c_2^*) \\ i(1+c_2^*) & s_2^* \end{pmatrix} z \end{aligned}$$

and

$$*T_{-1} * z^{-1} = \left[\overline{-\frac{s_1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} s_2^* & -i(1+c_2^*) \\ i(c_2^*-1) & s_2^* \end{pmatrix}\begin{pmatrix} -z & 0 \\ 0 & z \end{pmatrix}}\right]$$
$$= \left[\overline{-\frac{s_1}{2}\begin{pmatrix} s_2^* & i(1+c_2^*) \\ -i(c_2^*-1) & s_2^* \end{pmatrix} z}\right]$$
$$= -\frac{s_1}{2}\begin{pmatrix} s_2^* & -i(1+c_2^*) \\ i(c_2^*-1) & s_2^* \end{pmatrix} z^{-1}$$

and

$$*T_0* = \left[\overline{c_1\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_2^* & -is_2^* \\ is_2^* & c_2^* \end{pmatrix}\begin{pmatrix} c_2^* & -is_2^* \\ is_2^* & c_2^* \end{pmatrix}}\right]$$
$$= c_1\begin{pmatrix} c_2^* & -is_2^*) \\ is_2^* & c_2^* \end{pmatrix}.$$

So the calculation above shows that the conjugation

$$* : \{x_k, x_k\} \to \{-\bar{x_k}, \bar{y_k}\}$$

commutes with the action of the induced rotation $T(V)$ on W.

4.7. Nullvector

We are interested in finding a vector which is in the intersection of the positive subspace of the polarization with anti-periodic boundary conditions and the negative subspace of the polarization with periodic boundary conditions. The induced rotation associated with the transfer matrix can be written as a finite difference operator on the lattice that scales to the Euclidean Dirac equation in the continuum limit. Lisovyy [Lis05] used the continuum version of this vector to compute the Green function for the Dirac operator on the 1-punctured cylinder. We show that we can exhibit the 'new' elements V_+ and V_- in the Bugrij-Lisovyy formula as part of a holomorphic factorization of the periodic and anti-periodic summability kernels on the spectral curve.

Recall that the two cycles $\mathcal{M}_\pm$ of the spectral curve $\mathcal{M}$ associated with the induced rotation for the transfer matrix are given by

$$\mathcal{M}_\pm = \{(z, \lambda) = (e^{i\theta}, e^{\mp\gamma(\theta)})\}.$$

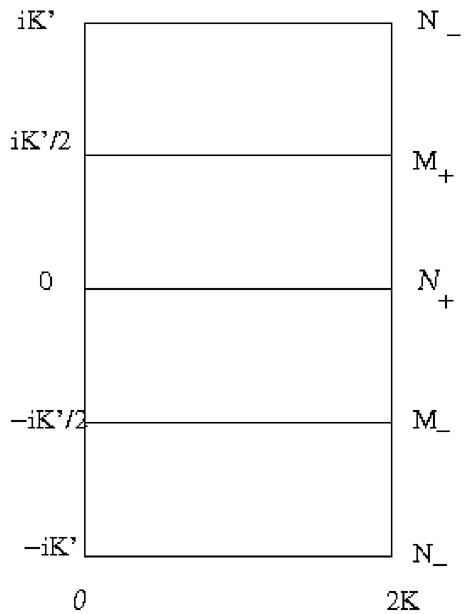

FIGURE 4.2. The figure shows the location of the cycles $\mathcal{M}_+$, $\mathcal{M}_-$, $\mathcal{N}_+$ and $\mathcal{N}_-$ in the periodic parallelogram in the uniformization parameter u.

Recall from Section C.1 that in the uniformization parameter u, the cycles $\mathcal{M}_\pm$ are located at the following positions in the periodic parallelogram

$$\mathcal{M}_\pm = \{u : 0 < \Re u < 2K, \Im u = \pm\tfrac{K'}{2}\}.$$

Introduce the cycles

$$\mathcal{N}_+ = \{u|0 < \Re u < 2K, \Im u = 0\}$$
$$\mathcal{N}_- = \{u|0 < \Re u < 2K, \Im u = \pm K'\}.$$

From (C.8), it follows that

$$\lambda(u \pm iK') = \lambda(u)^{-1},$$

$$z(u \pm iK') = z(u)^{-1},$$

so λ and z, defined on $\mathcal{N}_+$ and $\mathcal{M}_+$, both have inverses on $\mathcal{N}_-$ and $\mathcal{M}_-$ respectively. A simple calculation shows that a substitution of a with $a + \frac{K'}{2}$ in the elliptic parametrization of the Boltzmann weights, interchanges z and λ in the spectral curve (C.1), and sends s_1 to $-s_2$, s_2 to $-s_1$ and c_1c_2 to $-c_1c_2$. The calculation goes as follows: Define

a' by the relation

$$ia' = \frac{iK'}{2} + ia.$$

Then

$$\begin{aligned} z(u,a) &= k\,\mathrm{sn}(u - ia)\,\mathrm{sn}(u + ia) \\ &= k\,\mathrm{sn}(u + \tfrac{iK'}{2} - ia')\,\mathrm{sn}(u + \tfrac{iK'}{2} - iK' + ia') \\ &= \mathrm{sn}(u + \tfrac{iK'}{2} - ia')/\,\mathrm{sn}(u + \tfrac{iK'}{2} + ia') \\ &= \lambda(u + \tfrac{iK'}{2}, a') := \lambda' \end{aligned} \tag{4.29}$$

and

$$s_1' := -i\,\mathrm{sn}(2ia') = -i\,\mathrm{sn}(2ia + iK') = -ik^{-1}\,\mathrm{ns}(2ia) = -s_2.$$

The other calculations are similar. In particular, we notice that

$$z' := z(u + \tfrac{iK'}{2}, a') = \lambda(u, a).$$

In addition, it can be checked that z and λ are both $2K$, $2iK'$ periodic. Interchanging λ and z through the substitution, $a \mapsto a + \frac{iK'}{2}$, the spectral curve

$$s_1 \frac{z + z^{-1}}{2} + s_2 \frac{\lambda + \lambda^{-1}}{2} = c_1 c_2 \tag{4.30}$$

becomes

$$s_1 \frac{\lambda' + \lambda'^{-1}}{2} + s_2 \frac{z' + z'^{-1}}{2} = c_1 c_2. \tag{4.31}$$

The spectral curves given in (4.30) and (4.31) are both meromorphic functions of u. Recall that the set of the $2M+1$ roots of unity, i.e $z^{2M+1} = 1$, is denoted by Σ_P, and the set of the $2M+1$ roots of -1, i.e $z^{2M+1} = -1$, is denoted by Σ_A. We here use the short-hand notation, $p \in \Sigma_P$ for $z_P \in \Sigma_P$ and $\lambda_p := \lambda(p) = e^{\gamma(p)}$. Using (3.30) and the spectral curve in (4.31), we obtain

$$\begin{aligned} \prod_{p \in \Sigma_P} (c_1 c_2 - s_1 \cos p - s_2 \tfrac{z' + z'^{-1}}{2}) = \prod_{p \in \Sigma_p} s_1 (\tfrac{\lambda' + \lambda'^{-1}}{2}) - s_1 \cos p \\ = (\tfrac{s_1}{2})^{2M+1} \prod_{p \in \Sigma_p} ((\lambda' - e^{ip})(1 - \lambda'^{-1} e^{-ip})) \\ = (\tfrac{s_1}{2})^{2M+1} (\lambda'^{2M+1} - 1)^2 \lambda'^{-(2M+1)} \end{aligned} \tag{4.32}$$

and similarly

$$\prod_{p\in\Sigma_A}(c_1c_2 - s_1\cos p - s_2\tfrac{z'+z'^{-1}}{2}) = (\tfrac{s_1}{2})^{2M+1}(\lambda'^{2M+1}+1)^2\lambda'^{-(2M+1)}. \tag{4.33}$$

Then from (4.29), (4.32) and (4.33) we obtain

$$\begin{aligned}&\left(\frac{z(u)^{2M+1}+1}{z(u)^{2M+1}-1}\right)^2\\ =&\left(\frac{\lambda'^{2M+1}+1}{\lambda'^{2M+1}-1}\right)^2\\ =&\frac{\prod_{p\in\Sigma_A}(c_1c_2 - s_1\cos p - s_2\frac{z'+z'^{-1}}{2})}{\prod_{p\in\Sigma_P}(c_1c_2 - s_1\cos p - s_2\frac{z'+z'^{-1}}{2})}.\end{aligned} \tag{4.34}$$

Now we rewrite the factor in (4.34) using the spectral curve (4.30). It becomes

$$\begin{aligned}&\frac{\prod_{p\in\Sigma_A}(\lambda_p+\lambda_p^{-1}-(z'+z'^{-1}))}{\prod_{p\in\Sigma_P}(\lambda_p+\lambda_p^{-1}-(z'+z'^{-1}))}\\ =&\frac{\prod_{p\in\Sigma_A}(\lambda_p-z')(1-\lambda_p^{-1}z'^{-1})}{\prod_{p\in\Sigma_P}(\lambda_p-z')(1-\lambda_p^{-1}z'^{-1})}\\ =&\frac{\prod_{p\in\Sigma_A}(\lambda_p-\lambda(u))(1-\lambda_p^{-1}\lambda^{-1}(u))}{\prod_{p\in\Sigma_P}(\lambda_p-\lambda(u))(1-\lambda_p^{-1}\lambda^{-1}(u))}\\ =&\frac{\prod_{p\in\Sigma_A}(-1)(\lambda(u)-\lambda_p)(\lambda(u)-\lambda_p^{-1})\lambda^{-1}(u)}{\prod_{p\in\Sigma_P}(-1)(\lambda(u)-\lambda_p)(\lambda(u)-\lambda_p^{-1})\lambda^{-1}(u)}.\end{aligned}$$

Since $\lambda(p) = \lambda(-p)$ for $p \neq 0, \pi$, the right hand side of the equation above can be written

$$\frac{(\lambda(u)-\lambda_\pi)(\lambda(u)-\lambda_\pi^{-1})}{(\lambda(u)-\lambda_0)(\lambda(u)-\lambda_0^{-1})}\frac{\prod_{p>0\in\Sigma_A}(\lambda(u)-\lambda_p)^2(\lambda(u)-\lambda_p^{-1})^2}{\prod_{p>0\in\Sigma_P}(\lambda(u)-\lambda_p)^2(\lambda(u)-\lambda_p^{-1})^2}.$$

Now define

$$\tilde{V}_+(u) := \sqrt{\frac{(\lambda(u)-\lambda_\pi)}{(\lambda(u)-\lambda_0)}\frac{\prod_{p>0\in\Sigma_A}(\lambda(u)-\lambda_p)}{\prod_{p>0\in\Sigma_P}(\lambda(u)-\lambda_p)}} \tag{4.35}$$

$$\tilde{V}_-(u) := \sqrt{\frac{(\lambda(u)-\lambda_0^{-1})}{(\lambda(u)-\lambda_\pi^{-1})}\frac{\prod_{p>0\in\Sigma_P}(\lambda(u)-\lambda_p^{-1})}{\prod_{p>0\in\Sigma_A}(\lambda(u)-\lambda_p^{-1})}}. \tag{4.36}$$

The square roots are here chosen to have positive real parts. The function $\tilde{V}_+(u)$ is analytic on the part of the spectral curve where $|\lambda| \leq 1$ which occurs for

$-\epsilon \leq \Im u \leq K'$ and $-K' \leq \Im u \leq -K' + \epsilon$ for some $\epsilon > 0$. The function $\tilde{V}_{-}(u)$ is analytic where $|\lambda| \geq 1$ which occurs for $-K' \leq \Im u \leq \epsilon$ and $K' - \epsilon \leq \Im u \leq K'$. Thus, we have showed that near $\mathcal{N}_{+}$, we have

$$\frac{\tilde{V}_{+}(u)}{z(u)^{2M+1}+1} = -\frac{\tilde{V}_{-}(u)}{z(u)^{2M+1}-1}. \tag{4.37}$$

Near the curves

$$\mathcal{N}_{-} = \{u | 0 < \Re u < 2K, \Im u = \pm K'\},$$

we have

$$\frac{\tilde{V}_{+}(u)}{z(u)^{2M+1}+1} = \frac{\tilde{V}_{-}(u)}{z(u)^{2M+1}-1}. \tag{4.38}$$

We observe that on the curve $\mathcal{M}_{+}$, the function $\tilde{V}_{+}(u)$ is a multiple of $V_{+}(u)$ in (3.25) from the Bugrij-Lisovyy formula while $\tilde{V}_{-}(u)$ is a multiple of $V(u)$ in (3.26) on the curve $\mathcal{M}_{-}$. The eigenvectors of $T(V)$ corresponding to $e^{-\gamma(z)}$ and $e^{\gamma(z)}$ can be written in the form,

$$\begin{pmatrix} w(z) \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} w(z) \\ 1 \end{pmatrix} \tag{4.39}$$

respectively, where

$$w(z) = iz^{-1}\frac{\mathcal{A}_1(z)\mathcal{A}_2(z)}{\mathcal{A}_1(z^{-1})\mathcal{A}_2(z^{-1})} = i\sqrt{\frac{(\alpha_1 - z)(\alpha_2 - z)}{\alpha_1\alpha_2(z-\alpha_1^{-1})(z-\alpha_2^{-1})}}$$

and $\mathcal{A}_j = \sqrt{\alpha_j - z}$ for $j = 1, 2$. Recall that $\mathcal{A}_j$ and $w(z)$ are normalized so that $\mathcal{A}_j(1) > 0$ and $w(1) = i$. We now write $w(z)$ in terms of the uniformization parameter u. From page 67 of [Pal06] we have

$$\begin{aligned}
z - \alpha_1 &= (1-\alpha_1\alpha_2)\frac{1-kx}{x+\alpha_2}, \\
z - \alpha_1^{-1} &= \alpha_1^{-1}(\alpha_1 - \alpha_2)\frac{1-k^{-1}x}{x+\alpha_2}, \\
z - \alpha_2 &= \frac{1-\alpha_2^2}{x+\alpha_2}, \\
z - \alpha_2^{-1} &= \alpha_2^{-1}(\alpha_2^2 - 1)\frac{x}{x+\alpha_2}, \\
k &= \frac{\alpha_2 - \alpha_1}{\alpha_1\alpha_2 - 1},
\end{aligned}$$

where

$$x = \frac{1-\alpha_2 z}{z-\alpha_2}.$$

It follows that in the x variables, $w(z)$ can be written,

$$w = i\sqrt{\tfrac{(\alpha_1\alpha_2-1)}{\alpha_1-\alpha_2}\left(\tfrac{1-kx}{(1-k^{-1}x)x}\right)}. \tag{4.40}$$

Now substitute

$$x = k\,\mathrm{sn}^2(u)$$

into (4.40). We obtain

$$\sqrt{1-kx} = \sqrt{1-k^2\,\mathrm{sn}^2(u)} = \mathrm{dn}(u),$$

$$\sqrt{1-k^{-1}x} = \sqrt{1-\mathrm{sn}^2(u)} = \mathrm{cn}(u),$$

where

$$\frac{\alpha_1\alpha_2-1}{\alpha_1-\alpha_2} = -\frac{1}{k}.$$

Now using the addition formulas (C.4), (C.6), (C.7) and the translations found on page 72 of [Pal06]:

$$\mathrm{sn}(\tfrac{iK'}{2}) = ik^{-\frac{1}{2}},$$
$$\mathrm{cn}(\tfrac{iK'}{2}) = (1+k)^{\frac{1}{2}}k^{-\frac{1}{2}},$$
$$\mathrm{dn}(\tfrac{iK'}{2}) = (1+k)^{\frac{1}{2}},$$

in addition to the fact that

$$\mathrm{dn}(K) = k' \quad \mathrm{cn}(K) = 0, \quad \mathrm{sn}(K) = 1 \quad \text{(see page 499 of [WW62])},$$

it can be checked that we have

$$\frac{\mathrm{dn}(K+\frac{iK'}{2})}{k\,\mathrm{sn}(K+\frac{iK'}{2})\,\mathrm{cn}(K+\frac{iK'}{2})} = i.$$

Since $w(1) = i$ and $z(K+\frac{iK'}{2}) = 1$, it follows that we must choose the square root such that

$$w(z(x(u))) = \frac{\mathrm{dn}(u)}{k\,\mathrm{sn}(u)\,\mathrm{cn}(u)}.$$

This function is a meromorphic function of u on the spectral curve.

The elliptic functions $\mathrm{dn}(u)$, $\mathrm{cn}(u)$ and $\mathrm{sn}(u)$ all have a pole located at $u = iK'$. We consider a multiple of the eigenvectors given in (4.39) of the induced rotation $T(V)$ associated with the transfer matrix. In the u parametrization, the eigenvectors corresponding to $e^{-\gamma(z)}$ are then given by

$$e_+(u) = \begin{cases} \begin{pmatrix} \mathrm{dn}(u) \\ -k\,\mathrm{sn}(u)\,\mathrm{cn}(u) \end{pmatrix} & \text{if } -\epsilon \leq \Im u \leq K'; \\ \\ \begin{pmatrix} -\,\mathrm{dn}(u) \\ k\,\mathrm{sn}(u)\,\mathrm{cn}(u) \end{pmatrix} & \text{if } -K' \leq \Im u \leq -K' + \epsilon, \end{cases} \tag{4.41}$$

and the eigenvectors corresponding to $e^{\gamma(z)}$ are given by

$$e_-(u) = \begin{cases} \begin{pmatrix} -\,\mathrm{dn}(u) \\ k\,\mathrm{sn}(u)\,\mathrm{cn}(u) \end{pmatrix} & \text{if } -K' \leq \Im u \leq \epsilon; \\ \\ \begin{pmatrix} \mathrm{dn}(u) \\ -k\,\mathrm{sn}(u)\,\mathrm{cn}(u) \end{pmatrix} & \text{if } K' - \epsilon \leq \Im u \leq K'. \end{cases} \tag{4.42}$$

Since $\begin{pmatrix} \mathrm{dn}(u) \\ k\,\mathrm{sn}(u)\,\mathrm{cn}(u) \end{pmatrix}$ is $2iK'$ anti-periodic, the functions $e_+(u)$ and $e_-(u)$ are meromorhically continuous in a neighborhood around $\mathcal{N}_-$. In particular, we notice that

$$\begin{aligned} e_+(u) &= -e_-(u) \quad \text{on} \quad \mathcal{N}_+, \\ e_+(u) &= e_-(u) \quad \text{on} \quad \mathcal{N}_-. \end{aligned}$$

In order to take care of the sign difference on the cycle, $\mathcal{N}_+$ in (4.37), we multiply the expressions in (4.37) and (4.38) by the vector $e_+(u)$ on the left side of the equations and with the vector $e_-(u)$ on the right side of the equations. We have proved the following:

Proposition 4.3. *Near the curves,*

$$\mathcal{N}_- = \big\{u : 0 < \Re u < 2K, \Im u = \pm K'\big\} \quad \textit{and} \quad \mathcal{N}_+ = \big\{u : 0 < \Re u < 2K, \Im u = 0\big\},$$

we have the identity

$$\frac{\tilde{V}_+(u)}{z(u)^{2M+1} + 1} e_+(u) = \frac{\tilde{V}_-(u)}{z(u)^{2M+1} - 1} e_-(u), \tag{4.43}$$

where $\tilde{V}_+(u)$, $\tilde{V}_-(u)$, $e_+(u)$ and $e_-(u)$ are given in (4.35), (4.36), (4.41) and (4.42).

For $l = 0, ..., 2M+1$, consider the following sum over the anti-periodic spectral points,

$$\sum_{z_A \in \Sigma_A} \tilde{V}_+(z_A) e_+(z_A) z_A^{-l} \tag{4.44}$$

which is an element in W_+^A. This sum can be represented in terms of a contour integral

$$\frac{(2M+1)}{2\pi i} \int_{\partial_{A(\mathcal{M}_+)}} \tilde{V}_+(z(u)) e_+(u) \frac{z(u)^{2M-l}}{z(u)^{2M+1}+1} \frac{dz}{du} du,$$

where the integration is over the boundary of an annulus, $\partial_{A(\mathcal{M}_+)}$, containing the circle $\mathcal{M}_+$. Since $\tilde{V}_+$ is analytic on the part of the spectral curve where $|\lambda| \leq 1$ and $\tilde{V}_-$ is analytic on the part of the spectral curve where $|\lambda| \geq 1$, we can deform the integral from the boundary of the annulus about $\mathcal{M}_+$, through the curves $\mathcal{N}_\pm$, to the boundary of an annulus about $\mathcal{M}_-$. The pole contributions that come from the factor

$$\frac{\tilde{V}_-(z)}{z^{2M+1} - 1}$$

involve periodic spectral points z_P. This give us a way of going from a sum over the anti-periodic spectrum to a sum over the periodic spectrum given by

$$\sum_{z_P \in \Sigma_P} \tilde{V}_-(z_P) e_-(z_P) z_P^{-l} \tag{4.45}$$

which is an element in W_-^P.
However, the meromorphic functions $e_\pm(u) z(u)^{-1}$ have poles off the cycles $\mathcal{M}_\pm$ which obscure the significants of this calculation. A related factorization in the scaling limit was used by Lisovyy [Lis05] to compute a relevant Green function.

A. First Appendix

A.1. Grassmann Algebra and Fock Representations of the Clifford Algebra

In this section we introduce the Fock representations of the Clifford algebra. We follow [Pal06] closely and refer the reader to this work for more detail. Assume W is a finite even-dimensional complex vector space with a distinguished nondegenerate complex bilinear form $(\cdot,\cdot)$. It can be shown by a modification of the Gram Schmidt process, that there exists a basis for W which is orthonormal with respect to the nondegenerate complex bilinear form. A subspace V of W is isotropic if $(x, y) = 0$ for all $x, y \in V$. We are interested in a decomposing of the space W into two isotropic subspaces $W_\pm$,

$$W = W_+ \oplus W_-.$$

Such a splitting is called an isotropic splitting or a polarization. Our interest in the polarization of W is the reason for our assumption that W is even-dimensional. We parametrize each isotropic splitting by an operator Q defined by

$$Qx = \begin{cases} x & \text{if } x \in W_+; \\ -x & \text{if } x \in W_-. \end{cases} \tag{A.1}$$

Define $Q_\pm := \frac{1}{2}(I \pm Q)$. Since $Q^2 = I$, we observe that $Q_\pm$ are the projections onto the ± 1 eigenspaces for Q. We notice that $Q_+ + Q_- = I$ and $Q_+Q_- = 0$. If we define $W_\pm := Q_\pm W$, the space W is the direct sum

$$W = W_+ \oplus W_-.$$

There is a simple argument to show that the ± 1 eigenspaces of an operator Q with $Q^2 = I$ are isotropic if and only if Q is skew symmetric with respect to the complex bilinear form $(\cdot,\cdot)$: Suppose that $Q = -Q^\tau$. Then for $x, y \in W_+$, we have

$$\begin{aligned}(x, y) = (Qx, Qy) &= (Q^\tau Qx, y) \\ &= (-x, y),\end{aligned}$$

which implies $(x, y) = 0$. Similar argument applies for $x, y \in W_-$. To prove the other direction, suppose that $(x, y) = 0$ for $x, y \in W_+$. Then $(x, y) = (-x, y) = (-Q^\tau Q x, y)$ which implies that Q is skew symmetric. An operator Q that is skew symmetric and satisfies $Q^2 = I$ is called a polarization. Let S_k denote the group of permutations on the set $\{1, 2, ..., k\}$. The linear operator defined by

$$W^{\otimes k} \ni w \mapsto \mathrm{alt}(w) := \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) w_{\sigma_1} \otimes ... \otimes w_{\sigma_k}$$

is a projection from $W^{\otimes k}$ onto $\mathrm{Alt}^k(W)$, where $\mathrm{Alt}^k(W)$ is the space of alternating k tensors over W (see [Pal06]). The wedge product of $v \in \mathrm{Alt}^k(W)$ and $w \in \mathrm{Alt}^l(W)$ is here defined by

$$v \wedge w := \frac{\sqrt{(k+l)!}}{\sqrt{k!}\sqrt{l!}} \mathrm{alt}(v \otimes w) \in \mathrm{Alt}^{k+l}(W)$$

and it follows that for $v_i \in W$ and $i = 1, ..., k$,

$$v_1 \wedge v_2 \wedge ... \wedge v_k = \frac{1}{\sqrt{k!}} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) v_{\sigma_1} \otimes v_{\sigma_2} \otimes \cdots \otimes v_{\sigma_k}$$

(see[Pal06] and [Sp65]). The Clifford algebra of W is the associative algebra with multiplicative unit e, generated by the elements $x \in W$ that satisfy the Clifford relations,

$$xy + yx = (x, y)e \quad \text{for} \quad x, y \in W. \tag{A.2}$$

For each polarization Q of the isotropic splitting of the space W, there is a Fock representation F_Q of the Clifford algebra $\mathrm{Cliff}(W)$. This representation acts on the alternating tensor algebra,

$$\mathrm{Alt}(W_+) := \bigoplus_{k=0}^{n} \mathrm{Alt}^k(W_+),$$

where $\mathrm{Alt}^0(W_+) = \mathbb{C}$ and $n = \dim(W_+)$. The Fock representation is defined as

$$W \ni x \mapsto F_Q(x) := c(x_+) + a(x_-)$$

for the splitting $x = x_+ + x_- \in W_+ + W_-$. Here W_- is identified with the dual W_+^* via the nondegenerate complex bilinear form $W_+ \ni x_+ \mapsto (x_+, x_-)$ for $x_- \in W_-$. The creation operator $c(x_+)$ associated with $x_+ \in W_+$ acts on $\mathrm{Alt}^k(W_+)$ in the following way:

$$\mathrm{Alt}^k(W_+) \ni v \mapsto c(x_+)v = x_+ \wedge v \in \mathrm{Alt}^{k+1}(W_+).$$

The annihilation operator $a(x_-)$ associated with $x_- \in W_-$ is defined as $a(x_-) := c^\tau(x_-)$, where $c^\tau(x_-)$ is the transpose of $c(x_-)$ with respect to the complex bilinear form $(\cdot,\cdot)$. It is given by

$$a(x_-)v = \sum_{j=1}^{k} (-1)^{j-1}(x_-, v_j) v_1 \wedge ... \wedge \hat{v}_j \wedge ... \wedge v_k$$

for $v = v_1 \wedge ... \wedge v_j \wedge ... \wedge v_k \in \mathrm{Alt}^k(W_+)$, and where $\hat{v}_j$ signifies that the factor v_j is omitted from v. It can be checked that the creation and annihilation operators satisfy the anticommutation relations,

$$\begin{aligned} &c(x_+)c(y_+) + c(y_+)c(x_+) = 0, \\ &a(x_-)a(y_-) + a(y_-)a(x_-) = 0, \\ &a(x_-)c(y_+) + c(y_+)a(x_-) = (x_-, y_+)I \end{aligned} \tag{A.3}$$

for $x_\pm, y_\pm \in W_\pm$.

Since $W_\pm$ are isotropic subspaces and by using the anticommutation relations given in (A.3), it is not hard to see that F_Q satisfies the generator relations for the Clifford algebra,

$$F_Q(x)F_Q(y) + F_Q(y)F_Q(x) = (x, y)I \quad \text{for} \quad x, y \in W.$$

Suppose that W is an even dimensional complex Hilbert space with a Hermitian symmetric inner product $\langle u, v\rangle$ defined by $\langle u, v\rangle = (\bar{u}, v)$, where $u \mapsto \bar{u}$ is a conjugation of u, and W_+ and W_- are orthogonal with respect to the Hermitian inner products. Then we call

$$W = W_+ \oplus W_-$$

a Hermitian polarization. In this case, we define the Fock representation of the Clifford relations for W associated with the polarization $W = W_+ \oplus W_-$ by

$$W \ni x \mapsto F_Q(x) := c(x_+) + a(\bar{x}_-).$$

Here

$$a(x) := c^*(x) \quad \text{for} \quad x \in W_+,$$

where $c^*(x)$ is the adjoint of $c(x)$ with respect to the Hermitian inner product on $\mathrm{Alt}(W_+)$. Now F_Q satisfies the relation

$$F_Q(x)F_Q(y) + F_Q(y)F_Q(x) = (x, y)I \quad \text{for} \quad x, y \in W.$$

When the polarization is understood, we will drop the subscript Q and write $F := F_Q$. We write

$$0 := 1 \oplus 0 \oplus ... \oplus 0$$

for the vacuum vector in $\mathrm{Alt}(W_+)$. The vacuum vector is defined to be the unique vector that is annihilated by all the elements in W_- in the F_Q representation of the Clifford algebra. We are in particular interested in a subgroup $\mathcal{G}$ of the Clifford algebra $\mathrm{Cliff}(W)$. This group is called the Clifford group, and is defined to be the group of invertible elements g in the Clifford algebra $\mathrm{Cliff}(W)$ that satisfy

$$gvg^{-1} = T(g)v \quad \text{for} \quad v \in W \subseteq \mathrm{Cliff}(W), \tag{A.4}$$

for some linear map $T(g)$ on W. It follows from this equation that T is complex orthogonal, i.e.

$$(T(g)v, T(g), w) = (v, w) \quad \text{for} \quad v, w \in W \subseteq \mathrm{Cliff}(W).$$

For $X \in \mathrm{Cliff}(W)$, we define the vacuum expectation of X in the Q Fock representation to be given by

$$\langle X \rangle_Q = \langle 0, F_Q(X)0 \rangle.$$

We use this definition for the calculation of the spin matrix elements in the infinite-volume limit under the pure state defined by plus boundary conditions.

B. Second Appendix

B.1. Berezin Integral Representation for the Matrix Elements

In this section we introduce a representation of the creation and annihilation operators which is an analog of the holomorphic representations as given in Faddeev and Slavnov [FS80]. We will use this representation to write the matrix elements for the Fock representation of an element g in the Clifford group as Pfaffians of a skew symmetric matrix whose entries are given in terms of the matrix elements of the induced rotation associated with g.

Assume W is a finite even-dimensional complex vector space with a Hermitian inner product $\langle \cdot, \cdot \rangle$ and a distinguished nondegenerate complex bilinear form $(\cdot, \cdot)$ defined by

$$(u, v) = \langle \bar{u}, v \rangle \quad \text{for} \quad u, v \in W,$$

where $u \mapsto \bar{u}$ is a conjugation. We consider a Hermitian polarization,

$$W = W_+ \oplus W_-,$$

where $W_\pm$ are isotropic subspaces of W as defined in Appendix A. Let $\{e_k^+\}$ denote an orthonormal basis for W_+ with corresponding dual basis $\{e_k^-\}$ for W_- with respect to the complex bilinear form $(\cdot, \cdot)$, i.e $(e_k^-, e_l^+) = \delta_{kl}$. Suppose that W has dimension $2M$ and define

$$e_I^\pm := e_{I_1}^\pm \wedge ... \wedge e_{I_k}^\pm \quad \text{for} \quad 1 \leq I_1 < ... < I_k \leq M. \tag{B.1}$$

The set $\{e_I^\pm\}$ is then an orthonormal basis for $\mathrm{Alt}(W_\pm)$, where we define $e_\emptyset^+ = 1$. Let $\mathcal{P}$ denote the collection of subsets of $\{1, ..., M\}$. For an element J in $\mathcal{P}$, we write

$$J = \{J_1, J_2, ..., J_k\} \quad \text{with} \quad J_1 < J_2 < .... < J_k.$$

We write $\#J = k$ for the number of elements in J. If R is a $2M \times 2M$ matrix, we let $R_{I,J}$ denote the $(\#I + \#J) \times (\#I + \#J)$ submatrix of R made from the rows and columns of R indexed by I and J respectively. An element in $\mathrm{Alt}(W_+)$ is given by

$$G(e^+) := \sum_{I \in \mathcal{P}} G_I e_I^+,$$

where the map, $\mathcal{P} \ni I \to G_I \in \mathbb{C}$. For $1 \leq i \leq M$, the creation operator c acts on $\mathrm{Alt}^k(W_+)$ as

$$c(e_i^+)v := e_i^+ \wedge v \in \mathrm{Alt}^{k+1}(W_+)$$

for $e_i^+ \in W_+$. For $1 \leq i \leq M$, the annihilation operator $a(e_i^+) := \frac{\partial}{\partial e_i^+}$, is analogous to a 'derivative', and is the linear map

$$\frac{\partial}{\partial e_i^+} : \mathrm{Alt}(W_+) \to \mathrm{Alt}(W_+)$$

defined as follows: If the monomial, $X := e_{i_1}^+ \wedge e_{i_2}^+ \wedge .... \wedge e_{i_n}^+$ contains exactly one factor e_i^+ then

$$\frac{\partial}{\partial e_i^+}(e_{i_1}^+ \wedge e_{i_2}^+ \wedge ... \wedge ... \wedge e_{i_n}^+) = \pm e_{i_1}^+ \wedge ... \hat{e}_i^+ ... \wedge ... \wedge e_{i_n}^+,$$

where $\hat{e}_i^+$ signifies that the factor e_i^+ is omitted from X, and the plus or minus sign is determined by number of interchanges the operator $\frac{\partial}{\partial e_i^+}$ has to make from the left before it contracts the factor e_i^+. An even number of interchanges give a positive sign and an odd number give a minus sign. For example

$$\frac{\partial}{\partial e_2^+} e_3^+ \wedge e_2^+ \wedge e_5^+ = -e_3^+ \wedge e_5^+.$$

If the monomial does not contain any factor e_i^+, then $\frac{\partial}{\partial e_i^+} X = 0$. If X contains more than one factor of e_i^+, then $X = 0$. The operator $\frac{\partial}{\partial e_i^+}$ acts by the 'signed Leibniz rule' so in this case we have $\frac{\partial}{\partial e_i^+} X = 0$. The operators c and a satisfy the commutation relations

$$\begin{aligned} a(e_j)c(e_i) + c(e_i)a(e_j) = \delta_{ij}, \\ c(e_i)c(e_j) + c(e_j)c(e_i) = 0, \\ a(e_i)a(e_j) + a(e_j)a(e_i) = 0 \end{aligned} \tag{B.2}$$

for $e_i, e_j \in W_+$. We define Berezin integrals as linear functionals in the following way,

$$\int e^+ de^+ = 1, \quad \int e^- de^- = 1, \quad \int de^+ = 0, \quad \int de^- = 0,$$

where we assume that de^- and de^+ anticommute with each other as well as with e^- and e^+. An element in the Grassmann algebra $\mathrm{Alt}(W)$ is given by

$$G(e^+,e^-) = G_{00}+G_{01}e_1^-+G_{10}e_1^++G_{11}e_1^-\wedge e_1^++...+G_{1,...,M,M,...,1}e_1^-\wedge..\wedge e_M^-\wedge e_M^+\wedge...\wedge e_1^+,$$

where G_{00}, G_{01}, G_{10} ,..., and $G_{1,...,M,M,...,1}$ are complex numbers (see [FS80]). The integral of $G(e^+,e^-)$ is then defined as

$$\int G(e^+,e^-)\prod_{k=1}^{M} de_k^+ de_k^- = G_{1....M,M...1}.$$

The inner product of G and H on $\mathrm{Alt}(W_+)$ is given by (see [FS80], p. 53)

$$\langle G,H\rangle = \int \bar{G}(e^+)H(e^+)e^{-\sum_{k=1}^{M} e_k^+\wedge e_k^-}\prod_{k=1}^{M} de_k^+ de_k^-, \tag{B.3}$$

where

$$\bar{G}(e^+) = \sum_{I\in\mathcal{P}} \bar{G}_I e_I^-$$

and the conjugation is defined as

$$\overline{G_I e_M^+\wedge\cdot\cdot\wedge e_1^+} = \bar{G}_I e_1^-\wedge\cdot\cdot\wedge e_M^-.$$

It can be checked that the inner product (B.3) makes c and a conjugates of each other on $\mathrm{Alt}(W_+)$. It is here understood that $e^{\sum e_k^+\wedge e_k^-}$ is the power series in the exterior algebra,

$$\sum_{j=0}^{M}\left(\frac{\sum e_k^+\wedge e_k^-}{j!}\right)^j.$$

We are in particular interested in two Hermitian polarizations,

$$W = W_+^A\oplus W_-^A \quad\text{and}\quad W = W_+^P\oplus W_-^P.$$

Here $W_\pm^P$ and $W_\pm^A$ are isotropic subspaces of W defined by

$$W_\pm^A = Q_\pm^A W \quad\text{and}\quad W_\pm^P = Q_\pm^P W,$$

where

$$Q_\pm^A := \tfrac{1}{2}(I\pm Q^A),\quad Q_\pm^P := \tfrac{1}{2}(I\pm Q^P),$$

and where Q^A and Q^P are polarizations as defined in (A.1). Let F^P and F^A denote the Fock representations associated with the Clifford algebra Cliff(W) acting on Alt(W_+^P) and Alt(W_+^A) respectively. We consider a map

$$g : \text{Alt}(W_+^P) \to \text{Alt}(W_+^A),$$

which satisfies the intertwining relation

$$gF^P(x) = F^A(T(g)x)g$$

for $x \in W$, and where $T := T(g)$ is the induced rotation associated with g. Let $\{e_k^+\}$ denote an orthonormal basis for W_+^A with corresponding dual basis $\{e_k^-\}$ for W_-^A with respect to the complex bilinear form $(\cdot,\cdot)$. Similarly, let $\{f_k^+\}$ denote an orthonormal basis for W_+^P with corresponding dual basis $\{f_k^-\}$ for W_-^P. The sets $\{e_I^+\}$ and $\{f_I^+\}$ as defined in (B.1) are then orthonormal bases for Alt(W_+^A) and Alt(W_+^P) respectively. We have

$$gf_J^+ = \sum_{I \in \mathcal{P}} g_{I,J} e_I^+,$$

where the matrix elements of the operator g in the bases $\{e_I^+\}$ and $\{f_I^+\}$ are given by

$$g_{I,J} = \langle e_I^+, gf_J^+ \rangle$$

with kernel $g(e^+, f^-)$ defined as

$$g(e^+, f^-) = \sum_{I,J \in \mathcal{P}} g_{I,J} e_I^+ \wedge f_J^-.$$

Introduce the anticommuting 'dummy' variables $\alpha_i^\pm \in W_\pm^P$ which anticommute with $e_i^\pm$ as well. The action of the operator g on $G(\alpha^+) \in \text{Alt}(W_+^P)$ is given by
(see [FS80], p. 55)

$$(gG)(e^+) = \int g(e^+, \alpha^-) G(\alpha^+) e^{-\sum_{k=1}^M \alpha_k^+ \wedge \alpha_k^-} \prod_{k=1}^M d\alpha_k^+ d\alpha_k^-.$$

We write

$$T(g) := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{B.4}$$

for the matrix of the induced rotation associated with g, where T is a map $W_+^P \oplus W_-^P \to W_+^A \oplus W_-^A$. Since $T(g)$ is a complex, orthogonal matrix, we have the relation

$$T(g)^\tau T(g) = I,$$

where

$$T^\tau = \begin{pmatrix} D^\tau & B^\tau \\ C^\tau & A^\tau \end{pmatrix}.$$

This relation implies the following identities:

$$\begin{array}{cc} D^\tau A + B^\tau C^\tau = I, & D^\tau B + B^\tau D = 0 \\ C^\tau A + A^\tau C = 0, & C^\tau B + A^\tau D = I \end{array} \tag{B.5}$$

when D is invertible. We will show that g can be written as an exponential, whose argument is a quadratic form. In other words, we will show that the kernel $g(e^+, \alpha^-)$ can be written as

$$g(e^+, \alpha^-) = \langle 0_A, g0_P \rangle e^{\mathcal{R}}, \tag{B.6}$$

where $\langle 0_A, g0_P \rangle$ is the one point function, 0_A and 0_P are the vacuum states in $\mathrm{Alt}(W_+^A)$ and $\mathrm{Alt}(W_+^P)$, and the quadratic form $\mathcal{R}$ is defined in the following way: Introduce the $2M \times 2M$ skew symmetric matrix

$$R := \begin{pmatrix} a & b \\ -b^\tau & c \end{pmatrix}, \tag{B.7}$$

where a, b, c are maps

$$a : W_-^A \to W_+^A, \quad b : W_+^P \to W_+^A, \quad c : W_+^P \to W_-^P, \tag{B.8}$$

and a, c are skew symmetric with respect to the bilinear form $(\cdot, \cdot)$. Define

$$\mathcal{R} := \sum_{m=1}^{M} [\tfrac{1}{2}(ae_m^- \wedge e_m^+) - \tfrac{1}{2}(b^\tau e_m^- \wedge e_m^+) + \tfrac{1}{2}(b\alpha_m^+ \wedge \alpha_m^-) + \tfrac{1}{2}(c\alpha_m^+ \wedge \alpha_m^-)]. \tag{B.9}$$

It can be checked that $\mathcal{R}$ does not depend on the choice of bases. Writing

$$ae_m^- = \sum_l a_{lm} e_l^+, \quad \text{where} \quad a_{lm} = (ae_m^-, e_l^-),$$

$$b\alpha_m^+ = \sum_l b_{lm} e_l^+, \quad \text{where} \quad b_{lm} = (b\alpha_m^+, e_l^-),$$

$$c\alpha_m^+ = \sum_l c_{lm} \alpha_l^-, \quad \text{where} \quad c_{lm} = (c\alpha_m^+, \alpha_l^+),$$

we have

$$\mathcal{R} = \sum_{l,m=1}^{M} [(\tfrac{1}{2} a_{lm} e_l^+ \wedge e_m^+) + (b_{lm} e_l^+ \wedge \alpha_m^-) + (\tfrac{1}{2} c_{lm} \alpha_l^- \wedge \alpha_m^-)].$$

If D is invertible, we will show that the choices

$$a = BD^{-1}, \quad b = D^{-\tau} \quad \text{and} \quad c = D^{-1}C,$$

where B, C, D are the matrix elements of the induced rotation $T(g)$ associated with g, is a choice that makes (B.6) work. We also show that

$$g_{I,J} = \langle 0_A, g0_P \rangle \,\mathrm{Pf}(R_{I,J}),$$

where $\mathrm{Pf}(R_{I,J})$ is the Pfaffian of $R_{I,J}$. Here $R_{I,J}$ is the $(\#I + \#J) \times (\#I + \#J)$ matrix

$$R_{I,J} = \begin{pmatrix} BD^{-1}_{I\times I} & D^{-\tau}_{I\times J} \\ -D^{-1}_{J\times I} & D^{-1}C_{J\times J} \end{pmatrix}$$

with matrix elements,

$$(R_{I,J})_{\alpha,\beta} = \begin{cases} BD^{-1}_{I_\alpha, I_\beta} & \text{for } 1 \le \alpha < \beta \le \#I; \\ D^{-\tau}_{I_\alpha, J_\beta} & \text{for } 1 \le \alpha \le \#I \quad \text{and} \quad 1 \le \beta \le \#J; \\ -D^{-1}_{J_\beta, I_\alpha} & \text{for } 1 \le \alpha \le \#I \quad \text{and} \quad 1 \le \beta \le \#J; \\ D^{-1}C_{J_\alpha, J_\beta} & \text{for } 1 \le \alpha < \beta \le \#J. \end{cases}$$

Recall (see [Pal06]) that the Pfaffian of a $2M \times 2M$ skew symmetric matrix R with matrix elements $R_{j,k}$ is defined in the following way. Let $\{e_j\}$ denote the standard basis of $\mathbb{C}^{2n}$. The Pfaffian, $\mathrm{Pr}(R)$ of R, is defined by

$$\frac{1}{2^M M!} \left(\sum_{j,k=1}^{2M} R_{j,k} e_j \wedge e_k \right)^M = \mathrm{Pf}(R) e_1 \wedge \ldots \wedge e_{2M}. \tag{B.10}$$

We start by proving that $e^{\mathcal{R}}$ is the kernel of an element of the Clifford group. We prove the following.

Lemma B.1. *Let* $\{e_i^{\pm}\}$ *and* $\{\alpha_j^{\pm}\}$ *denote orthonormal bases for* $W_{\pm}^A$ *and* $W_{\pm}^P$ *respectively and define*

$$e_I^+ := e_{i_1}^+ \wedge ... \wedge e_{i_k}^+ \quad \textit{for} \quad 1 \leq i_1 < ... < i_k \leq \dim(W_+^A) := M$$

and

$$\alpha_J^- := \alpha_{j_1}^- \wedge ... \wedge \alpha_{j_k}^- \quad \textit{for} \quad 1 \leq j_1 < ... < j_k \leq \dim(W_+^P) := M.$$

Let $\tilde{g}(e^+, \alpha^-) := e^{\mathcal{R}}$*, where*

$$\mathcal{R} = \sum_{l,m} [(\tfrac{1}{2} a_{lm} e_l^+ \wedge e_m^+) + (b_{lm} e_l^+ \wedge \alpha_m^-) + (\tfrac{1}{2} c_{lm} \alpha_l^- \wedge \alpha_m^-)]$$

and $a = BD^{-1}$*,* $b = D^{-\tau}$ *and* $c = D^{-1}C$*, where*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is the matrix of a complex orthogonal map, $W_+^P \oplus W_-^P \to W_+^A \oplus W_-^A$*. Then*

$$\tilde{g}G(e^+) = \int \tilde{g}(e^+, \alpha^-) G(\alpha^+) e^{-\sum_{k=1}^M \alpha_k^+ \wedge \alpha_k^-} \prod_{k=1}^M d\alpha_k^+ d\alpha_k^- \tag{B.11}$$

defines a linear map which satisfies the intertwining relation,:

$$\tilde{g} F^P(x) G = F^A(Tx) \tilde{g} G \tag{B.12}$$

for $G \in \mathrm{Alt}(W_+^P)$ *and* $x \in W$*.*

Proof. For $\alpha_i^- \in W_-^P$, we have

$$F^P(\alpha_i^-) = a(\overline{\alpha_i^-}) = \frac{\partial}{\partial \alpha_i^+}.$$

It follows that for $G \in \mathrm{Alt}(W_+^P)$, we have

$$(\tilde{g} F^P(\alpha_i^-) G)(e^+) = \int \tilde{g}(e^+, \alpha^-) \frac{\partial}{\partial \alpha_i^+} G(\alpha^+) e^{-\sum_{k=1}^M \alpha_k^+ \wedge \alpha_k^-} \prod_{k=1}^M d\alpha_k^+ d\alpha_k^-.$$

Since $\frac{\partial}{\partial \alpha_i^+} g(e^+, \alpha^-) = 0$, the integral above can be written

$$\int \frac{\partial}{\partial \alpha_i^+} \Big(\tilde{g}(e^+, \alpha^-) G(\alpha^+) \Big) e^{-\sum_{k=1}^M \alpha_k^+ \wedge \alpha_k^-} \prod_{k=1}^M d\alpha_k^+ d\alpha_k^-. \tag{B.13}$$

By the 'signed Leibniz rule', we have

$$\frac{\partial}{\partial \alpha_i^+} e^{-\sum_{k=1}^M \alpha_k^+ \wedge \alpha_k^-} = -\alpha_i^- e^{-\sum_{k=1}^M \alpha_k^+ \wedge \alpha_k^-},$$

and since the volume element is in the cokernel of $\frac{\partial}{\partial \alpha_i^+}$, we have

$$\int \frac{\partial}{\partial \alpha_i^+} G(\alpha^+) \prod_{k=1}^M d\alpha_k^+ d\alpha_k^- = 0.$$

It follows that the integral in (B.13) can be written

$$(\tilde{g} F^P(\alpha_i^-) G)(e^+) = \int \alpha_i^- \tilde{g}(e^+, \alpha^-) G(\alpha^+) e^{-\sum_{k=1}^M \alpha_k^+ \wedge \alpha_k^-} \prod_{k=1}^M d\alpha_k^+ d\alpha_k^-. \tag{B.14}$$

We have for $v \in \mathrm{Alt}(W_+^P)$

$$\begin{aligned} F^A(T(\alpha_i^-))v &= [c(B\alpha_i^-) + a(\overline{D\alpha_i^-})]v \\ &= \sum_{k=1}^M \left[B_{ki} e_k^+ + D_{ki} \frac{\partial}{\partial e_k^+} \right] v. \end{aligned}$$

It follows that

$$\begin{aligned} &(F^A(T(\alpha_i^-))\tilde{g}G)(e^+) \\ &= \int \left(\sum_k B_{ki} e_k^+ + D_{ki} \frac{\partial}{\partial e_k^+} \right) \tilde{g}(e^+, \alpha^-) G(\alpha^+) e^{-\sum_{k=1}^M \alpha_k^+ \wedge \alpha_k^-} \prod_{k=1}^M d\alpha_k^+ d\alpha_k^-. \end{aligned} \tag{B.15}$$

Here

$$\begin{aligned} \frac{\partial}{\partial e_k^+} \tilde{g}(e^+, \alpha^-) &= \left[\tfrac{1}{2} \sum_{l,m} a_{lm} \delta_{lk} e_m^+ - \tfrac{1}{2} \sum_{l,m} a_{lm} \delta_{km} e_l^+ + \sum_{l,m} b_{lm} \delta_{lk} \alpha_m^- \right] \tilde{g}(e^+, \alpha^-) \\ &= \left[\tfrac{1}{2} \sum_m a_{km} e_m^+ - \tfrac{1}{2} \sum_l a_{lk} e_l^+ + \sum_m b_{km} \alpha_m^- \right] \tilde{g}(e^+, \alpha^-) \\ &= \left[\sum_l a_{kl} e_l^+ + \sum_l b_{kl} \alpha_l^- \right] \tilde{g}(e^+, \alpha^-), \end{aligned} \tag{B.16}$$

where the last equation follows from the fact that the matrix, a, is skew symmetric. Combining the first part of the integrand in (B.15) with (B.16), using $a = BD^{-1}$ and

$b = D^{-\tau}$, we obtain

$$\begin{aligned}&\sum_k [B_{ki}e_k^+ + D_{ki}(\sum_l a_{kl}e_l^+ + \sum_l b_{kl}\alpha_l^-)] \\ &= \sum_k B_{ki}e_k^+ - \sum_{k,l} D_{ik}^\tau((BD^{-1})_{kl})^\tau e_l^+ + \sum_{k,l} D_{ik}^\tau D_{kl}^{-\tau}\alpha_l^-. \\ &= \alpha_i^-,\end{aligned}$$

since

$$\sum_{k,l} D_{ik}^\tau((BD^{-1})_{kl})^\tau e_l^+ = \sum_l (D^\tau D^{-\tau}B^\tau)_{il}e_l^+ = \sum_l B_{li}e_l^+.$$

Thus, we have showed that $\tilde{g}F^P(\alpha_i^-)G = F^A(T\alpha_i^-)\tilde{g}G$ for $\alpha_i \in \mathrm{Alt}(W_-^P)$ and $G \in \mathrm{Alt}(W_+^P)$.

In a similar fashion, we have for $\alpha_i^+, G \in \mathrm{Alt}(W_+^P)$

$$(\tilde{g}F^P(\alpha_i^+)G)(e^+) = \int \tilde{g}(e^+,\alpha^-)\alpha_i^+G(\alpha^+)e^{-\sum_{k=1}^M \alpha_k^+\wedge\alpha_k^-}\prod_{k=1}^M d\alpha_k^+d\alpha_k^- \tag{B.17}$$

Using the fact that

$$\frac{\partial}{\partial\alpha_i^-}e^{-\sum_{k=1}^M \alpha_k^+\wedge\alpha_k^-} = \alpha_i^+e^{-\sum_{k=1}^M \alpha_k^+\wedge\alpha_k^-}$$

and that the volume element is again in the cokernel of $\frac{\partial}{\partial\alpha_i^-}$, we see that the integral in (B.17) can be written as

$$-\int \frac{\partial}{\partial\alpha_i^-}[\tilde{g}(e^+,\alpha^-)G(\alpha^+)]e^{-\sum_k \alpha_k^+\wedge\alpha_k^-}\prod_{k=1}^M d\alpha_k^+d\alpha_k^-.$$

The integral above can be written

$$(\tilde{g}F^P(\alpha_i^+G))(e^+) = -\int \frac{\partial}{\partial\alpha_i^-}[\tilde{g}(e^+,\alpha^-)]G(\alpha^+)e^{-\sum_k \alpha_k^+\wedge\alpha_k^-}\prod_{k=1}^M d\alpha_k^+d\alpha_k^-,$$

where

$$\begin{aligned}-\frac{\partial}{\partial\alpha_i^-}\tilde{g}(e^+,\alpha^-) &= \left(\sum_l D_{li}^{-\tau}e_l^+ - \sum_l (D^{-1}C)_{il}\alpha_l^-\right)\tilde{g}(e^+,\alpha^-) \\ &= (D^{-\tau} + D^{-1}C)\alpha_i^+\tilde{g}(e^+,\alpha^-).\end{aligned}$$

Now we have for $v \in \mathrm{Alt}(W_+^P)$

$$\begin{aligned} F^A(T(\alpha_i^+))v &= [c(A\alpha_i^+) + a(\overline{C\alpha_i^+})]v \\ &= \sum_{k=1}^{M} \left[A_{ki} e_k^+ + C_{ki} \frac{\partial}{\partial e_k^+} \right] v. \end{aligned}$$

It follows that

$$\begin{aligned} &(F^A(T(\alpha_i^+))\tilde{g}G)(e^+) \\ &= \int \left[\sum_{k=1}^{M} A_{ki} e_k^+ + C_{ki} \frac{\partial}{\partial e_k^+} \right] \tilde{g}(e^+, \alpha^-) G(\alpha^+) e^{-\sum_{k=1}^{M} \alpha_k^+ \wedge \alpha_k^-} \prod_{k=1}^{M} d\alpha_k^+ d\alpha_k^-. \end{aligned}$$

Now using (B.16) with $a = BD^{-1}$, $b = D^{-\tau}$, we obtain

$$\begin{aligned} &\left[\sum_k A_{ki} e_k^+ + C_{ki} \frac{\partial}{\partial e_k^+} \right] \tilde{g}(e^+, \alpha^-) \\ &= \left(\sum_k A_{ki} e_k^+ + \sum_{k,l} C_{ki} (BD^{-1})_{kl} e_l^+ + \sum_{k,l} C_{ki} D_{kl}^{-\tau} \alpha_l^- \right) \tilde{g}(e^+, \alpha^-) \\ &= (A + D^{-\tau} B^\tau C + D^{-1} C) \alpha_i^+ \tilde{g}(e^+, \alpha^-) \\ &= (D^{-\tau} + D^{-1} C) \alpha_i^+ \tilde{g}(e^+, \alpha^-), \end{aligned}$$

where we in the last equation used (B.5). Thus, we have showed that

$$\tilde{g} F^P(\alpha_i^+) G = F^A(T(\alpha_i^+)) \tilde{g} G$$

for $\alpha_i^+, G \in \mathrm{Alt}(W_+^P)$, and the lemma is proved. □

Since $\tilde{g}$ defined in (B.11) satisfies the relation in (B.12), the range of $\tilde{g}$ is invariant under the action of the Fock representation $F^A(x)$ for all $x \in W$. The Fock representation is irreducible [Pal06], so the range of $\tilde{g}$ is either trivial or all of $\mathrm{Alt}(W_+^A)$. Since $\tilde{g}0_P$ is nonzero, the range of $\tilde{g}$ must be $\mathrm{Alt}(W_+^A)$. Thus $\tilde{g}$ is invertible, so that $\tilde{g} \in \mathcal{G}$.
We can use this fact and Lemma B.1 to prove the following theorem.

Theorem B.2. *Suppose that g satisfies the intertwining relation*

$$gF^P(x)v = F^A(Tx)gv \quad \textit{for} \quad x \in W \quad \textit{and} \quad v \in \mathrm{Alt}(W_+^P),$$

where

$$T(g) := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is the matrix of its induced rotation, $T : W_+^P \oplus W_-^P \to W_+^A \oplus W_-^A$*. Suppose that the one-point function* $\langle 0_A, g\, 0_P \rangle$ *is nonzero. Let* $\{e_i^+\}$ *and* $\{\alpha_j^-\}$ *denote orthonormal bases for* W_+^A *and* W_-^P *respectively and define*

$$e_I^+ := e_{I_1}^+ \wedge ... \wedge e_{I_k}^+ \quad \textit{for} \quad 1 \leq I_1 < ... < I_k \leq \dim(W_+^A) := M$$

and

$$\alpha_J^- := \alpha_{J_1}^- \wedge ... \wedge \alpha_{J_k}^- \quad \textit{for} \quad 1 \leq J_1 < ... < J_k \leq \dim(W_+^P) := M.$$

Then the kernel $g(e^+, \alpha^-)$ *of* g *can be written as*

$$g(e^+, \alpha^-) = \langle 0_A, g\, 0_P \rangle \sum_{I,J \in \mathcal{P}} \mathrm{Pf}(R_{I,J}) e_I^+ \wedge \alpha_J^-,$$

where

$$R_{I,J} = \begin{pmatrix} BD^{-1}_{I\times I} & D^{-\tau}_{I\times J} \\ -D^{-1}_{J\times I} & D^{-1}C_{J\times J} \end{pmatrix},$$

and where 0_A *and* 0_P *are the vacuum states in* $\mathrm{Alt}(W_+^A)$ *and* $\mathrm{Alt}(W_+^P)$ *respectively. The sum is over all such* I *and* J *with* $\#I + \#J$ *even.*

Proof. Since g satisfies the intertwining relation

$$gF^P(x) = F^A(Tx)g \quad \text{for} \quad x \in W,$$

the previous lemma implies that there is a nonzero constant λ such that

$$g(e^+, \alpha^-) = \lambda\, e^{\mathcal{R}}, \tag{B.18}$$

where

$$\mathcal{R} = \sum_{l,m} [(\tfrac{1}{2} a_{lm} e_l^+ \wedge e_m^+) + (b_{lm} e_l^+ \wedge \alpha_m^-) + (\tfrac{1}{2} c_{lm} \alpha_l^- \wedge \alpha_m^-)]$$

for $a = BD^{-1}$, $b = D^{-\tau}$ and $c = D^{-1}C$. We determine the constant λ by showing that $\langle 0_A, \tilde{g} 0_P \rangle = 1$ such that

$$\lambda = \langle 0_A, g0_P \rangle.$$

Since the first two sums in $e^{\mathcal{R}^\tau}$ consists of a sum of products of annihilation operators, we have

$$0_A e^{\mathcal{R}^\tau} = 0_A e^{\frac{1}{2}\sum_{m,l=1}^{M} c_{lm}\alpha_l^- \wedge \alpha_m^-}.$$

It follows that

$$\langle 0_A, \tilde{g}0_P\rangle = \int e^{\frac{1}{2}\sum_{m,l=1}^{M} c_{lm}\alpha_l^- \wedge \alpha_m^-} e^{-\sum_{k=1}^{M}\alpha_k^+ \wedge \alpha_k^-} \prod_{k=1}^{M} d\alpha_k^+ \alpha_k^-. \qquad \text{(B.19)}$$

Taking into account that the matrix c is skew symmetric, the Taylor series expansion of

$$\exp\left[\tfrac{1}{2}\sum_{m,l=1}^{M} c_{lm}\alpha_l^- \wedge \alpha_m^-\right] \qquad \text{(B.20)}$$

is given by,

$$1 + \sum_{\substack{m,l=1\\ m>l}}^{M} c_{lm}\alpha_l^- \wedge \alpha_m^- + ... + \text{Pf}(c)\alpha_1^- \wedge \alpha_2^- \wedge ... \wedge \alpha_M^-,$$

when M is even. When M is odd the Taylor series expansion of the expression in (B.20) is given by

$$1 + \sum_{\substack{m,l=1\\ m>l}}^{M} c_{lm}\alpha_l^- \wedge \alpha_m^- + ... + \frac{1}{(\frac{M-1}{2})!}\left(\tfrac{1}{2}\sum_{m,l=1}^{M} c_{lm}\alpha_l^- \wedge \alpha_m^-\right)^{\frac{M-1}{2}}.$$

The Taylor series expansion of

$$\exp\left[-\sum_{k=1}^{M}\alpha_k^+ \wedge \alpha_k^-\right]$$

is given by,

$$1 - \sum_{k=1}^{M}\alpha_k^+ \wedge \alpha_k^- + \sum_{\substack{k_1,k_2=1\\ k_2>k_1}}^{M}\alpha_{k_1}^+ \wedge \alpha_{k_1}^- \wedge \alpha_{k_2}^+ \wedge \alpha_{k_2}^- +$$

$$-\sum_{\substack{k_1,k_2,k_3=1\\ k_3>k_2>k_1}}^{M}\alpha_{k_1}^+ \wedge \alpha_{k_1}^- \wedge \alpha_{k_2}^+ \wedge \alpha_{k_2}^- \wedge \alpha_{k_3}^+ \wedge \alpha_{k_3}^- + ... + (-1)^M \alpha_1^+ \wedge \alpha_1^- \wedge ... \wedge \alpha_M^+ \wedge \alpha_M^-.$$

Multiply the two Taylor series expansion above and notice that only the term $(-1)^M \alpha_1^+ \wedge \alpha_1^- \wedge ... \wedge \alpha_M^+ \wedge \alpha_M^-$ gives a nonzero contribution under integration. We obtain,

$$\langle 0_A, \tilde{g} 0_P \rangle = \int e^{\sum_{m,l=1}^{M} c_{lm} \alpha_l^- \wedge \alpha_m^-} e^{-\sum_{k=1}^{M} \alpha_k^+ \wedge \alpha_k^-} \prod_{k=1}^{M} d\alpha_k^+ \alpha_k^- = 1.$$

Thus, we have

$$g(e^+, \alpha^-) = \langle 0_A, g 0_P \rangle e^{\mathcal{R}}. \tag{B.21}$$

It is well-known (see [Pal06]) that

$$e^{\mathcal{R}} = \sum_{I,J \in \mathcal{P}} \mathrm{Pf}(R_{I,J}) e_I^+ \wedge \alpha_J^-, \tag{B.22}$$

where R is given in (B.7). This can be shown by using the Taylor series expansion for $e^{\mathcal{R}}$ and the definition of $\mathrm{Pf}(R)$. Here $\#I + \#J$ must be even to contribute to the sum. Combining (B.21) and (B.22), we obtain

$$g(e^+, \alpha^-) = \langle 0_A, g 0_P \rangle \sum_{I,J \in \mathcal{P}} \mathrm{Pf}(R_{I,J}) e_I^+ \wedge \alpha_J^-,$$

where

$$R_{I,J} = \begin{pmatrix} BD^{-1}_{I\times I} & D^{-\tau}_{I\times J} \\ -D^{-1}_{J\times I} & D^{-1}C_{J\times J} \end{pmatrix}.$$

□

C. Third Appendix

C.1. Introduction to Elliptic Functions

The set of pairs (λ, z) such that $\det(\lambda - T_z(V)) = 0$ is an elliptic curve $\mathcal{M}$ which is important for the spectral analysis of the transfer matrix. In particular, the map $\mathcal{M} \ni (\lambda, z) \mapsto z \in \mathbb{P}^1$ is a two fold covering, and there are two cycles $\mathcal{M}_\pm$ on $\mathcal{M}$ which cover the circle $\mathbb{S}^1 = \{z : |z| = 1\}$ and are relevant for spectral theory. On the cycle $\mathcal{M}_+$ we have $\lambda < 1$, and on the cycle $\mathcal{M}_-$, we have $\lambda > 1$. Just which points $z_j \in \mathbb{S}^1$ are relevant for the spectral analysis depend on the boundary conditions for the model. For spin periodic boundary conditions on the lattice, the $(2M+1)^{th}$ roots of unity, $z^{2M+1} = 1$, are relevant as are the $(2M+1)^{th}$ roots of -1, $z^{2M+1} = -1$. In the infinite-volume limit all the points $z \in \mathbb{S}^1$ are relevant. An elliptic substitution gives a uniformization of the whole complex curve $\mathcal{M}$ [Pal06]. A uniformization is an isomorphism from $\mathbb{C}$ modulo a lattice to $\mathcal{M}$. We recall facts about the Jacobian elliptic functions, $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$ and $\mathrm{dn}(u,k)$, where u is the uniformization parameter and k is the modulus. These functions play a central role in the calculation of the spin matrix elements in the infinite-volume limit in the pure state defined by plus boundary conditions. They are also a key element in the Pfaffian formalism of the spin matrix elements on the finite, periodic lattice as we will discover in Section 3.3. We follow the introduction of the Jacobian elliptic functions as given in [WW62] and [Pal06] and refer to these books for more details.

The spectral curve $\mathcal{M}$ associated with the induced rotation $T(V)$ for the transfer matrix is given by the set (z, λ) such that

$$s_1 \frac{z + z^{-1}}{2} + s_2 \frac{\lambda + \lambda^{-1}}{2} = c_1 c_2. \tag{C.1}$$

The spectral curve is topologically a torus and the two fold covering, $(\lambda, z) \mapsto z$, is ramified at $z = \alpha_1^\pm, \alpha_2^\pm$, (see[Pal06]) where

$$\alpha_1 = (c_1^* - s_1^*)(c_2 + s_2) \quad \text{and} \quad \alpha_2 = (c_1^* + s_1^*)(c_2 + s_2).$$

The roots α_1 and α_2 were introduced in Section 2.3 in connection with the Boltzmann weights. We are interested in the two cycles on $\mathcal{M}$ given by,

$$\mathcal{M}_\pm = \{(z,\lambda) = (e^{i\theta}, e^{\mp\gamma(\theta)})\}$$

with parameter $\theta \in [-\pi,\pi)$. Here we have introduced the notation $\gamma(\theta) := \gamma(e^{i\theta})$. Recall that the function $\gamma(z) > 0$ is defined as the positive root of

$$\operatorname{ch}\gamma(z) = c_1 c_2^* - s_1 s_2^* \frac{z+z^{-1}}{2}.$$

We use the shorthand notation $\operatorname{sn} u$, $\operatorname{cn} u$ and $\operatorname{dn} u$, for $\operatorname{sn}(u,k)$, $\operatorname{cn}(u,k)$ and $\operatorname{dn}(u,k)$ since the modulus k is fixed at $k = s_1^* s_2^*$ in our calculations. The functions $\operatorname{sn}(u)$, $\operatorname{cn}(u)$ and $\operatorname{dn}(u)$ are doubly periodic, meromorphic functions of u and they satisfy the equations

$$\begin{aligned} \operatorname{sn}^2(u,k) + \operatorname{cn}^2(u,k) &= 1, \\ \operatorname{dn}^2(u,k) + k^2 \operatorname{sn}^2(u,k) &= 1, \end{aligned} \tag{C.2}$$

$$\frac{d}{du}\operatorname{sn}(u,k) = \operatorname{cn}(u,k)\operatorname{dn}(u,k) \quad \text{and} \quad \operatorname{sn}(0) = 0. \tag{C.3}$$

We use the standard notation (see [WW62], [Pal06])

$$\operatorname{ns}(u) := \frac{1}{\operatorname{sn}(u)}, \quad \operatorname{cs}(u) := \frac{\operatorname{cn}(u)}{\operatorname{sn}(u)}$$

and in general

$$\operatorname{nx}(u) := \frac{1}{\operatorname{xn}(u)} \quad \text{and} \quad \operatorname{x}\operatorname{y}(u) := \frac{\operatorname{xn}(u)}{\operatorname{yn}(u)},$$

where x and y are one of either c, d or s. We give here some of the properties of the Jacobian elliptic functions that we will use later. From [WW62], we have the following addition formulas:

$$\operatorname{sn}(u+v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}, \tag{C.4}$$

$$\operatorname{sn}(u-v) = \frac{\operatorname{sn}^2(u) - \operatorname{sn}^2(v)}{\operatorname{sn}(u)\operatorname{cn}(v)\operatorname{dn}(v) + \operatorname{sn}(v)\operatorname{cn}(u)\operatorname{dn}(u)}, \tag{C.5}$$

$$\operatorname{cn}(u+v) = \frac{\operatorname{cn} u \operatorname{cn} v - \operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}, \tag{C.6}$$

$$\operatorname{dn}(u+v) = \frac{\operatorname{dn} u \operatorname{dn} v - k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}. \tag{C.7}$$

These formulas will be involved in finding a product formula for the spin matrix elements on the finite periodic lattice. If we translate the Jacobian elliptic functions by iK' (see page 503 of [WW62]), we obtain:

$$\begin{aligned} \mathrm{sn}(u+iK') &= k^{-1}\,\mathrm{ns}(u), \\ \mathrm{cn}(u+iK') &= -ik^{-1}\,\mathrm{ds}(u), \\ \mathrm{dn}(u+iK') &= -i\,\mathrm{cs}(u). \end{aligned} \tag{C.8}$$

On page 67 of [Pal06] the following fractional transformation in z-plane,

$$x = \frac{1-\alpha_2 z}{z-\alpha_2} \tag{C.9}$$

is introduced. The inverse of this transformation is given by

$$z(x) = \frac{1+\alpha_2 x}{x+\alpha_2}.$$

The fractional linear transformation in (C.9) maps the ramification points in the following way:

$$\alpha_2^{-1} < \alpha_1^{-1} < \alpha_1 < \alpha_2 \to 0 < k < k^{-1} < \infty \quad (\text{see} \quad [\text{Pal06}]).$$

Introduce the elliptic integrals (see page 501 of [WW62]),

$$K = \int_0^1 (1-t^2)^{-\frac{1}{2}}(1-k^2t^2)^{-\frac{1}{2}}\,dt, \tag{C.10}$$

$$K' = \int_0^1 (1-t^2)^{-\frac{1}{2}}(1-k'^2t^2)^{-\frac{1}{2}}\,dt, \tag{C.11}$$

for which the complementary modulus k' is defined by $k^2 + k'^2 = 1$. In [Pal06], the elliptic substitution $x = k\,\mathrm{sn}^2(u)$, leads to a uniformization of the complex curve $\mathcal{M}$:

Theorem C.1. *[Pal06] The map,*

$$[0, 2K] \times i[-K', K] \ni u \mapsto (z(u,a), \lambda(u,a))$$

with

$$z(u,a) = k\,\mathrm{sn}(u+ia)\,\mathrm{sn}(u-ia)$$

and

$$\lambda(u,a) = \frac{\operatorname{sn}(u-ia)}{\operatorname{sn}(u+ia)}$$

is a uniformization of the spectral curve

$$s_1 \frac{z+z^{-1}}{2} + s_2 \frac{\lambda+\lambda^{-1}}{2} = c_1 c_2,$$

where $k = \frac{1}{s_1 s_2}$, *and* $0 < 2a < K'$ *is defined by*

$$s_1 = -i \operatorname{sn}(2ia).$$

In the uniformization parameter u*, the cycles* $\mathcal{M}_\pm$ *are located at*

$$\mathcal{M}_\pm = \left\{ u : 0 < \Re u < 2K, \Im u = \pm \frac{K'}{2} \right\}.$$

C.2. Spin Matrix Elements in the Infinite-Volume Limit in the Pure State defined by Plus Boundary Conditions

In this chapter we calculate the matrix representation of the spin operator σ below the critical temperature in the infinite-volume limit in the pure state defined by plus boundary conditions. Some features of this calculation give insight into what happens on the finite periodic lattice. This representation can be expressed as the Pfaffian of a skew symmetric matrix whose entries are given as Jacobian elliptic functions. The Pfaffian of this matrix can subsequently be written as a product of the Jacobian elliptic functions. The spin matrix elements in the infinite-volume limit are well-known in the physics literature. However, we are not aware of any mathematical proofs of those formulas. The calculations in this chapter address this issue.

We refer the reader to [Pal06] for details regarding a generalization of the finite dimensional Fock representation to infinite dimensions. We start with some definitions that can be found in [Pal06]. Let W denote the complex Hilbert space $L^2(\mathbb{S}^1, \mathbb{C}^2)$ with an inner product $\langle \cdot, \cdot \rangle$ that is conjugate linear in the first slot and with a distinguished, nondegenerate bilinear form defined by $(u, v) = \langle \bar{u}, v \rangle$ for $u, v \in W$. Here $v \mapsto \bar{v}$ is a conjugation of v. We consider the Hermitian polarization

$$W = W_+ \oplus W_-,$$

where W_+ is the spectral subspace associated with the induced rotation for the transfer matrix in the infinite-volume limit for the interval $(0, 1)$, and W_- is the spectral subspace associated with the induced rotation for the transfer matrix for the interval $(1, \infty)$. In [Pal06] there is a map that identifies $W_\pm$ with the Hilbert space $L^2(0, 2K)$, where K is the elliptic integral given in (C.10) with modulus

$$k = \frac{1}{\sinh(\frac{2J_1}{k_B T})} \frac{1}{\sinh(\frac{2J_2}{k_B T})}.$$

The elements in $L^2(0, 2K) \oplus L^2(0, 2K)$ are written $\begin{bmatrix} f_+ \\ f_- \end{bmatrix}$ with conjugation $*$ defined by

$$\begin{bmatrix} f_+ \\ f_- \end{bmatrix}^* = \begin{bmatrix} \bar{f}_- \\ \bar{f}_+ \end{bmatrix}.$$

The Hermitian inner product on $L^2(0, 2K) \oplus L^2(0, 2K)$ is given by

$$\langle f, g \rangle = \int_0^{2K} (\bar{f}_+(u) g_+(u) + \bar{f}_-(u) g_-(u))\, du \quad \text{(see[Pal06])} \tag{C.12}$$

and the complex bilinear form is

$$(f, g) = \int_0^{2K} (f_-(u) g_+(u) + f_+(u) g_-(u))\, du \quad \text{(see[Pal06])}. \tag{C.13}$$

Define the $2K$ anti-periodic exponentials

$$e_l(u) := \frac{1}{\sqrt{2K}} \exp\left(\frac{il\pi u}{K}\right), \quad \text{for} \quad l \in \mathbb{Z} + \tfrac{1}{2} \quad \text{and} \quad u \in (0, 2K).$$

The set $\{e_l\}$ is an orthonormal basis for W_+. The set $\{e_{-l}\}$ is an orthonormal basis for W_- and is dual to $\{e_l\}$ with respect to the bilinear form $(\cdot, \cdot)$ given in (C.13). The Fock space is now given by the Hilbert space

$$\mathrm{Alt}(W_+) := \mathbb{C} \oplus \sum_{k=1}^{\infty} \oplus \, \mathrm{Alt}^k(W_+),$$

where $\mathrm{Alt}^k(W_+)$ is the space of alternating k tensors over W_+. For $x \in W_+$ and $w \in \mathrm{Alt}(W_+)$, define the creation operator

$$a^*(x)w = x \wedge w.$$

The annihilation operator is defined as $a(x) := a^*(x)^*$, where $a^*(x)^*$ is the adjoint of $a^*(x)$ for $x \in W_+$. The Fock representation associated with the Clifford algebra for the Hermitian polarization $W_+ \oplus W_-$ is

$$F(x) = a^*(x_+) + a(\bar{x}_-) \tag{C.14}$$

for the splitting $x = x_+ + x_- \in W_+ \oplus W_-$ (see [Pal06]). It is proved in [Pal06] that the spin operator σ in the infinite-volume limit in the pure state defined by plus boundary conditions acting on $\mathrm{Alt}(W_+)$ is an element in the Clifford group. Thus, there is a complex, orthogonal transformation $s := T(\sigma)$, acting on W such that

$$\sigma w \sigma^{-1} = sw \quad \text{for all} \quad w \in W. \tag{C.15}$$

We write the matrix of the induced rotation s associated with σ relative to the splitting $W = W_+ \oplus W_-$ as

$$s := \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

It is shown in [Pal06] that B and C are Hilbert-Schmidt class operators, and that A and D are invertible operators. For $l \in \mathbb{Z} + \frac{1}{2}$ define

$$v_l := \frac{1 - q^{2l}}{q^{2l} + 1}, \quad w_l := \frac{2q^l}{q^{2l} + 1},$$

where

$$q = \exp\left(\tfrac{-\pi K'}{K}\right)$$

and K' is the elliptic integral given in (C.11) for which the complementary modulus k' is defined by $k^2 + k'^2 = 1$. In terms of the bases $\{e_l\}$ and $\{e_{-l}\}$, an element in $L^2(0, 2K) \oplus L^2(0, 2K)$ is given by $\sum_{l=1}^{\infty}(x_l e_l + y_l e_{-l})$, where x_l and y_l are complex functions. Then (see [Pal06] and [WW62]), we have

$$s \begin{pmatrix} x_l \\ y_l \end{pmatrix} = \begin{pmatrix} v_l & w_l \\ w_l & v_l \end{pmatrix} \begin{pmatrix} x_l \\ y_l \end{pmatrix}. \tag{C.16}$$

(Note that on page 78 of [Pal06] the formula should be $\mathrm{ds}(u \pm iK') = \mp ik\,\mathrm{cn}(u)$ and the eigenvalue for B should be the negative of the one that appears on page 86 in [Pal06]). From this it follows that

$$BD^{-1}e_{-l} = \frac{2q^l}{1+q^{2l}}\frac{1+q^{2l}}{1-q^{2l}}e_{-l} = \frac{2q^l}{1-q^{2l}}e_{-l}. \tag{C.17}$$

We will prove the following:

Theorem C.2. *For $T < T_C$ in the pure state defined by + boundary conditions, the action of the spin operator σ on $f \in \mathrm{Alt}^{m'}[L^2(0,2K)]$ is given by*

$$(\sigma f)(u_1, u_2, ..., u_m) = \langle\sigma\rangle \times$$
$$\times \left(\frac{-i}{\pi}\right)^{\frac{m+m'}{2}} k^{\left(\frac{m+m'}{2}\right)^2} \int_0^{2K} ... \int_0^{2K} \left[\prod_{i<j}^{m+m'} \mathrm{sn}(v_i - v_j) f(u_{m+1}, ..., u_{m+m'})\right] \prod_{n=m+1}^{m+m'} du_n,$$

where

$$v_j = \begin{cases} -u_j + \frac{iK'}{2} & \text{for } 1 \le j \le m; \\ -u_j - \frac{iK'}{2} & \text{for } (m+1) \le j \le (m+m') \end{cases}$$

and $u_j \in (0, 2K)$ for $1 \le j \le (m+m')$ and $m + m' = even$.

Before we prove the theorem, we give an example on how to compute the formula for the matrix elements $\langle e_1 \wedge e_2, \sigma e_3 \wedge e_4\rangle$. By applying a generalized version of Wick's theorem (see page 295 of [Pal06]), we have

$$\langle e_1 \wedge e_2, \sigma e_3 \wedge e_4\rangle = \langle\sigma\rangle\,\mathrm{Pf}(r),$$

where $\mathrm{Pf}(r)$ is the Pfaffian of the 4×4 skew symmetric matrix r with i, j matrix element above the diagonal given by,

$$r_{i,j} = (Q_- e_i, (I - RQ_-)e_j) \quad \text{for} \quad i < j \tag{C.18}$$

and where

$$R = (s - I)(Q_- s + Q_+)^{-1}.$$

A short calculation gives

$$(I - RQ_-) = \begin{pmatrix} 1 & -BD^{-1} \\ 0 & D^{-1} \end{pmatrix}. \tag{C.19}$$

The Pfaffian $\mathrm{Pf}(r)$ can be written in terms of the following reduction formula (see page 295 of [Pal06]),

$$\mathrm{Pf}(r) = \sum_{l=2}^{2n} (-1)^l r_{1,l}\, \mathrm{Pf}(r^{1,l}), \tag{C.20}$$

where $r^{1,l}$ denotes the $(2n-2)\times(2n-2)$ matrix with the 1^{th} and l^{th} rows and columns removed. If r is the 2×2 matrix

$$r = \begin{pmatrix} 0 & r_{i,j} \\ -r_{i,j} & 0 \end{pmatrix}$$

we have, for example, that

$$\langle e_i e_j \sigma \rangle = \langle \sigma \rangle \, \mathrm{Pf}(r) = \langle \sigma \rangle r_{i,j} \quad \text{for} \quad i < j.$$

Thus,

$$r_{i,j} = \frac{\langle e_i e_j \sigma \rangle}{\langle \sigma \rangle} \quad \text{for} \quad i < j.$$

The matrix elements $r_{i,j}$ in (C.18) can then be calculated from formulas for $\langle e_i e_j \sigma \rangle$, $\langle \sigma e_i e_j \rangle$ and $\langle e_i \sigma e_j \rangle$. We show below that in terms of the uniformization parameter v_i, the matrix r can be written

$$r = \frac{ik2K}{\pi} \begin{pmatrix} 0 & -\operatorname{sn}(v_2 - v_1) & -\operatorname{sn}(v_2 - v_3) & -\operatorname{sn}(v_2 - v_4) \\ \operatorname{sn}(v_2 - v_1) & 0 & -\operatorname{sn}(v_1 - v_3) & -\operatorname{sn}(v_1 - v_4) \\ \operatorname{sn}(v_2 - v_3) & \operatorname{sn}(v_1 - v_3) & 0 & -\operatorname{sn}(v_3 - v_4) \\ \operatorname{sn}(v_2 - v_4) & \operatorname{sn}(v_1 - v_4) & \operatorname{sn}(v_3 - v_4) & 0 \end{pmatrix}.$$

By interchanging one column and one row, we obtain that $\mathrm{Pf}(r) = \mathrm{Pf}(r')$, where

$$r' = \frac{ik2K}{\pi} \begin{pmatrix} 0 & -\operatorname{sn}(v_1 - v_2) & -\operatorname{sn}(v_1 - v_3) & -\operatorname{sn}(v_1 - v_4) \\ \operatorname{sn}(v_1 - v_2) & 0 & -\operatorname{sn}(v_2 - v_3) & -\operatorname{sn}(v_2 - v_4) \\ \operatorname{sn}(v_1 - v_3) & \operatorname{sn}(v_2 - v_3) & 0 & -\operatorname{sn}(v_3 - v_4) \\ \operatorname{sn}(v_1 - v_4) & \operatorname{sn}(v_2 - v_4) & \operatorname{sn}(v_3 - v_4) & 0 \end{pmatrix}.$$

Now define the 4×4 diagonal matrix E with matrix elements along the diagonal given by $k^{\frac{1}{4}}\sqrt{\frac{i2K}{\pi}}$. Let s be the 4×4 skew symmetric matrix with matrix elements above the diagonal given by

$$s_{i,j} = -\sqrt{k}\, \operatorname{sn}(v_i - v_j) \quad \text{for} \quad 1 \le i < j \le 4.$$

Then we have

$$r' = EsE^{\tau}.$$

It is shown in [Pal06] that we have

$$\mathrm{Pf}(EsE^{\tau}) = \det(E)\,\mathrm{Pf}(s). \tag{C.21}$$

We will need the following lemma found on page 87 of [Pal06].

Lemma C.3. *[Pal06] Let r be the $2n \times 2n$ skew symmetric matrix with i, j matrix element $r_{i,j} = -\sqrt{k}\,\mathrm{sn}(u_i - u_j)$ for $i, j = 1, 2, ..., 2n$.*

Then

$$\mathrm{Pf}(r) = \prod_{i<j}^{2n} r_{i,j}.$$

By using Equation (C.21) and Lemma C.3, we obtain

$$\mathrm{Pf}(r) = \mathrm{Pf}(r') = -\left(\tfrac{2K}{\pi}\right)^2 k^4 \prod_{i<j}^{4} \mathrm{sn}(v_i - v_j).$$

Now we prove Theorem C.2.

Proof. We have for $f \in \mathrm{Alt}^{m'}[L^2(0, 2K)]$ and $m + m' = \text{even}$,

$$\begin{aligned}
&(\sigma f)(u_1, u_2, ..., u_m)\\
&= (2K)^{-\frac{m+m'}{2}} \int_0^{2K} ... \int_0^{2K} \Bigg[f(u_{m+1}, ..., u_{m+m'}) \times\\
&\times \sum_{k_1,...k_m,l_1,...,l_{m'}} \langle e_{k_1} \wedge ... \wedge e_{k_m}, \sigma e_{l_1} \wedge ... \wedge e_{l_{m'}} \rangle \prod_{i=1}^{m} e^{\frac{i\pi k_i u_i}{K}} \prod_{j=1}^{m'} e^{-\frac{i\pi l_j u_{j+m}}{K}} \Bigg] \prod_{n=m+1}^{m+m'} du_n
\end{aligned}$$

We will first calculate the matrix elements $\langle e_{k_1} \wedge e_{k_2} \wedge ... \wedge e_{k_m}, \sigma e_{l_1} \wedge e_{l_2} \wedge ... \wedge e_{l_{m'}} \rangle$. Using (C.15) and (C.16) one finds

$$\begin{aligned}
\sigma F\left(\begin{bmatrix} e_l \\ 0 \end{bmatrix} \right) \sigma^{-1} &= F\left(s \begin{bmatrix} e_l \\ 0 \end{bmatrix} \right)\\
&= F\left(\begin{bmatrix} v_l e_l \\ w_l e_l \end{bmatrix} \right).
\end{aligned} \tag{C.22}$$

Then using the definition of a Fock representation given in (C.14), we have

$$
\begin{aligned}
&\langle e_{k_1} \wedge e_{k_2} \wedge ... \wedge e_{k_m}, \sigma e_{l_1} \wedge e_{l_2} \wedge ... \wedge e_{l_{m'}} \rangle \\
&= \langle 0, a(e_{k_m}) a(e_{k_{m-1}}) ... a(e_{k_1}) \sigma a^*(e_{l_1}) a^*(e_{l_2}) ... a^*(e_{l_{m'}}) 0 \rangle \\
&= \left\langle 0, F \begin{bmatrix} 0 \\ e_{-k_m} \end{bmatrix} ... F \begin{bmatrix} 0 \\ e_{-k_1} \end{bmatrix} \sigma F \begin{bmatrix} e_{l_1} \\ 0 \end{bmatrix} ... F \begin{bmatrix} e_{l_{m'}} \\ 0 \end{bmatrix} 0 \right\rangle \\
&= \left\langle 0, F \begin{bmatrix} 0 \\ e_{-k_m} \end{bmatrix} .... F \begin{bmatrix} 0 \\ e_{-k_1} \end{bmatrix} F \left(s \begin{bmatrix} e_{l_1} \\ 0 \end{bmatrix} \right) .... F \left(s \begin{bmatrix} e_{l_{m'}} \\ 0 \end{bmatrix} \right) \sigma 0 \right\rangle \\
&= \left\langle 0, F \begin{bmatrix} 0 \\ e_{-k_m} \end{bmatrix} .... F \begin{bmatrix} 0 \\ e_{-k_1} \end{bmatrix} F \begin{bmatrix} v_{l_1} e_{l_1} \\ w_{l_1} e_{l_1} \end{bmatrix} .... F \begin{bmatrix} v_{l_{m'}} e_{l_{m'}} \\ w_{l_{m'}} e_{l_{m'}} \end{bmatrix} \sigma 0 \right\rangle \\
&= \left\langle \begin{bmatrix} 0 \\ e_{-k_m} \end{bmatrix} .... \begin{bmatrix} 0 \\ e_{-k_1} \end{bmatrix} \begin{bmatrix} v_{l_1} e_{l_1} \\ w_{l_1} e_{l_1} \end{bmatrix} .... \begin{bmatrix} v_{l_{m'}} e_{l_{m'}} \\ w_{l_{m'}} e_{l_{m'}} \end{bmatrix} \sigma \right\rangle \\
&:= \langle b_1 b_2 ... b_m b_{m+1} ... b_{m+m'} \sigma \rangle,
\end{aligned}
$$

where we in the third equation moved the spin operator σ to the right by using $(C.22)$. By applying a generalized version of Wick's theorem, we have

$$\langle b_1 b_2 ... b_m b_{m+1} ... b_{m+m'} \sigma \rangle = \langle \sigma \rangle \, \mathrm{Pf}(r), \tag{C.23}$$

where $\mathrm{Pf}(r)$ is the Pfaffian of the $(m + m') \times (m + m')$ skew symmetric matrix r with i, j matrix element above the diagonal given by,

$$r_{i,j} = (Q_- b_i, (I - RQ_-) b_j) \quad \text{for} \quad i < j \tag{C.24}$$

and where

$$(I - RQ_-) = \begin{pmatrix} 1 & -BD^{-1} \\ 0 & D^{-1} \end{pmatrix}. \tag{C.25}$$

Using (C.17), (C.24) and (C.25), we obtain for $1 \leq i < j \leq m$,

$$
\begin{aligned}
r_{k_i, k_j} &= \frac{\langle b_i b_j \sigma \rangle}{\langle \sigma \rangle} \\
&= \left(Q_- \begin{bmatrix} 0 \\ e_{-k_{m+1-i}} \end{bmatrix}, (I - RQ_-) \begin{bmatrix} 0 \\ e_{-k_{m+1-j}} \end{bmatrix} \right) \\
&= -\left[\left(\frac{2 q^{k_{m+1-j}}}{1 - q^{2k_{m+1-j}}} \right) \right] \delta_{k_{m+1-i} - k_{m+1-j}}.
\end{aligned} \tag{C.26}
$$

Here we used the bilinear form given in (C.13) and the fact that $(e_{-k_{m+1-i}}, e_{-k_{m+1-j}}) = \delta_{k_{m+1-i}-k_{m+1-j}}$. Using the fact that $BD^{-1}e_l = \frac{2q^l}{q^{2l}-1}e_l$, we have for $1 \leq i \leq m$ and $1 \leq j \leq m'$,

$$\begin{aligned} r_{k_i,m+l_j} &= \frac{\langle b_i \sigma b_{j+m} \rangle}{\langle \sigma \rangle} \\ &= \left(Q_- \begin{bmatrix} 0 \\ e_{-k_{m+1-i}} \end{bmatrix}, (I - RQ_-) \begin{bmatrix} v_{l_j} e_{l_j} \\ w_{l_j} e_{l_j} \end{bmatrix} \right) \\ &= \left[v_{l_j} + \left(\frac{2q^{l_j}}{1-q^{2l_j}} \right) w_{l_j} \right] \delta_{k_{m+1-i} l_j} \\ &= \left(\frac{1+q^{2l_j}}{1-q^{2l_j}} \right) \delta_{k_{m+1-i} l_j}, \end{aligned} \tag{C.27}$$

since $(e_{-k_{m+1-i}}, e_{l_j}) = \delta_{k_{m+1-i} l_j}$. For $1 \leq i < j \leq m'$, we have

$$\begin{aligned} r_{m+l_i,m+l_j} &= \frac{\langle \sigma b_{i+m} b_{j+m} \rangle}{\langle \sigma \rangle} \\ &= \left(Q_- \begin{bmatrix} v_{l_i} e_{l_i} \\ w_{l_i} e_{l_i} \end{bmatrix}, (I - RQ_-) \begin{bmatrix} v_{l_j} e_{l_j} \\ w_{l_j} e_{l_j} \end{bmatrix} \right) \\ &= w_{li} \left[v_{l_j} + \left(\frac{2q^{l_j}}{1-q^{2l_j}} \right) w_{l_j} \right] \delta_{-l_i l_j} \\ &= \left(\frac{2q^{l_j}}{1-q^{2l_j}} \right) \delta_{-l_i l_j}. \end{aligned} \tag{C.28}$$

We are now ready to calculate

$$\sum_{k_1,\ldots k_m,l_1,\ldots,l_{m'}} \mathrm{Pf}(r) \prod_{i=1}^{m} e^{\frac{i\pi k_i u_i}{K}} \prod_{j=1}^{m'} e^{-\frac{i\pi l_j u_{j+m}}{K}}$$

where the matrix elements for r are given by (C.26), (C.27) and (C.28). In order to perform this calculation, we will need the Fourier series for $\mathrm{sn}(u)$ which is given by (see [Pal06] and [WW62])

$$\mathrm{sn}(u) = \frac{\pi}{ikK} \sum_{l \in \mathbb{Z}+\frac{1}{2}} \frac{q^l}{1-q^{2l}} \exp\left(\frac{il\pi u}{K} \right). \tag{C.29}$$

The Pfaffian $\mathrm{Pf}(r)$ is a linear function of the vector which is the direct sum of the i^{th} row and i^{th} column of r [Pal06]. In the following calculations we make the substitutions

$$\begin{aligned} v_j: &\quad = -u_j + \tfrac{iK'}{2} \quad \text{for} \quad 1 \leq j \leq m, \\ v_{j+m}: &\quad = -u_{j+m} - \tfrac{iK'}{2} \quad \text{for} \quad 1 \leq j \leq m'. \end{aligned} \tag{C.30}$$

Using (C.26), (C.29) and the substitution in (C.30), one finds for $1 \leq i < j \leq m$,

$$\begin{aligned}
&\sum_{k_{m+1-i},k_{m+1-j}\in\mathbb{Z}+\frac{1}{2}} r_{k_i,k_j} e^{\frac{k_{m+1-i}\pi i u_{m+1-i}}{K}} e^{\frac{k_{m+1-j}\pi i u_{m+1-j}}{K}} \\
&= -2 \sum_{k_{m+1-j}\in\mathbb{Z}+\frac{1}{2}} \left(\frac{q^{k_{m+1-j}}}{1-q^{2k_{m+1-j}}}\right) e^{\frac{k_{m+1-j}\pi i}{K}[u_{m+1-j}-u_{m+1-i}]} \\
&= -\frac{ik2K}{\pi}\,\mathrm{sn}(v_{m+1-i}-v_{m+1-j}). \qquad \text{(C.31)}
\end{aligned}$$

Using (C.27), (C.29) and the substitution in (C.30), one finds for $1 \leq i \leq m$ and $1 \leq j \leq m'$,

$$\begin{aligned}
&\sum_{k_{m+1-i},l_j\in\mathbb{Z}+\frac{1}{2}} r_{k_i,m+l_j} e^{\frac{k_{m+1-i}\pi i u_{m+1-i}}{K}} e^{\frac{-l_j\pi i u_{j+m}}{K}} \\
&= \sum_{k_{m+1-i},l_j\in\mathbb{Z}+\frac{1}{2}} \left(\frac{1+q^{2l_j}}{1-q^{2l_j}}\right) e^{\frac{k_{m+1-i}\pi i u_{m+1-i}}{K}} e^{\frac{-l_j\pi i u_{j+m}}{K}} \delta_{k_{m+1-i} l_j} \\
&= \sum_{l_j\in\mathbb{Z}+\frac{1}{2}} \frac{q^{l_j}}{1-q^{2l_j}} e^{\frac{l_j\pi i}{K}[-u_{j+m}+u_{m+1-i}+iK']} + \frac{q^{l_j}}{1-q^{2l_j}} e^{\frac{l_j\pi i}{K}[-u_{j+m}+u_{m+1-i}-iK']} \\
&= -\frac{ik2K}{\pi}\,\mathrm{sn}(v_{m+1-i}-v_{j+m}). \qquad \text{(C.32)}
\end{aligned}$$

In the last equation, we used the fact that $\mathrm{sn}(u+2iK') = \mathrm{sn}(u)$. Similarly, from (C.28), (C.29) and the substitution in (C.30), we obtain for $1 \leq i < j \leq m'$,

$$\begin{aligned}
\sum_{l_i,l_j\in\mathbb{Z}+\frac{1}{2}} r_{m+l_i,m+l_j} e^{\frac{-l_i\pi i u_{i+m}}{K}} e^{\frac{-l_j\pi i u_{j+m}}{K}} &= 2 \sum_{l_j\in\mathbb{Z}+\frac{1}{2}} \left(\frac{q^{l_j}}{1-q^{2l_j}}\right) e^{\frac{l_j\pi i}{K}[u_{i+m}-u_{j+m}]} \\
&= -\frac{ik2K}{\pi}\,\mathrm{sn}(v_{i+m}-v_{j+m}). \qquad \text{(C.33)}
\end{aligned}$$

The calculations in (C.31), (C.32) and (C.33) imply that

$$\sum_{k_1,\ldots k_m,l_1,\ldots,l_{m'}} \mathrm{Pf}(r) \prod_{i=1}^{m} e^{\frac{i\pi k_i u_i}{K}} \prod_{j=1}^{m'} e^{-\frac{i\pi l_j u_{j+m}}{K}} = \mathrm{Pf}(r'),$$

where r' is the skew symmetric matrix with matrix elements above the diagonal given by

$$\begin{aligned} r'_{i,j} &= -\frac{ik2K}{\pi}\,\mathrm{sn}(v_{m+1-i}-v_{m+1-j}) \quad \text{for} \quad 1 \le i < j \le m, \\ r'_{i,j+m} &= -\frac{ik2K}{\pi}\,\mathrm{sn}(v_{m+1-i}-v_{m+j}) \quad \text{for} \quad 1 \le i \le m \quad \text{and} \quad 1 \le j \le m', \\ r'_{i+m,j+m} &= -\frac{ik2K}{\pi}\,\mathrm{sn}(v_{m+i}-v_{m+j}) \quad \text{for} \quad 1 \le i < j \le m'. \end{aligned}$$

It can be checked that by changing the order of the columns and rows an even number of times, we have

$$\mathrm{Pf}(r') = \mathrm{Pf}(r''),$$

where r'' is the skew symmetric matrix with matrix elements above the diagonal given by

$$r''_{i,j} = -\frac{ik2K}{\pi}\,\mathrm{sn}(v_i - v_j) \quad \text{for all} \quad 1 \le i < j \le (m+m').$$

Define E to be the diagonal matrix with matrix elements

$$e_{j,j} = (i2K\pi^{-1})^{\frac{1}{2}} k^{\frac{1}{4}} \quad \text{for} \quad 1 \le j \le (m+m').$$

Let s be the skew symmetric matrix with matrix elements above the diagonal given by

$$s_{i,j} = -\sqrt{k}\,\mathrm{sn}(v_i - v_j) \quad \text{for} \quad 1 \le i < j \le (m+m').$$

Then we have

$$r'' = EsE^{\tau}.$$

By using Equation (C.21) and Lemma C.3, we obtain

$$\begin{aligned} &\sum_{k_1,\ldots k_m,l_1,\ldots,l_{m'}} \mathrm{Pf}(r) \prod_{i=1}^{m} e^{\frac{i\pi k_i u_i}{K}} \prod_{j=1}^{m'} e^{-\frac{i\pi l_j u_{j+m}}{K}} \\ &= \left(\frac{-i2K}{\pi}\right)^{\frac{m+m'}{2}} k^{\left(\frac{m+m'}{2}\right)^2} \prod_{i<j}^{m+m'} \mathrm{sn}(v_i - v_j), \end{aligned}$$

where

$$v_j = \begin{cases} -u_j + \frac{iK'}{2} & \text{for } 1 \le j \le m; \\ -u_j - \frac{iK'}{2} & \text{for } (m+1) \le j \le (m+m') \end{cases}$$

and $u_j \in (0, 2K)$ for $1 \le j \le (m+m')$ and $m+m' =$ even. The theorem follows. □

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