FIFTH GRADERS' REPRESENTATIONS AND REASONING ON CONSTANT GROWTH FUNCTION PROBLEMS: CONNECTIONS BETWEEN PROBLEM REPRESENTATIONS, STUDENT WORK AND ABILITY TO GENERALIZE

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Kathleen M. Ross

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As members of the Dissertation Committee, we certify that we have read the dissertation

prepared by Kathleen M. Ross

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and recommend that it be accepted as fulfilling the dissertation requirement for the

Degree of Doctor of Philosophy.

____ Date: 12/2/11
Marcy Wood

____ Date: 12/2/11
Marta Civil

____ Date: 12/2/11
Erin Turner

Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copies of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

____ Date: 12/2/11
Dissertation Director: Marcy Wood

____ Date: 12/2/11
Dissertation Director: Marta Civil

STATEMENT BY AUTHOR

SIGNED: Kathleen M. Ross

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ABSTRACT

Student difficulties learning algebra are well documented. Many mathematics education researchers (e.g., Bednarz & Janvier, 1996; Davis, 1985, 1989; Vergnaud, 1988) argued that the difficulties students encounter in algebra arose when students were expected to shift suddenly from arithmetic to algebraic reasoning and that the solution to the problem was to integrate opportunities for elementary school students to simultaneously develop both arithmetic and algebraic reasoning. The process of generalization, or describing the overall pattern underlying a set of mathematical data, emerged as a focal point for extending beyond arithmetic reasoning to algebraic reasoning (Kaput, 1998; Mason, 1996). Given the critical importance for students to have opportunities to develop understanding of the fundamental algebraic concepts of variable and relationship, one could argue that providing opportunities to explore linear functions, the first function studied in depth in a formal algebra course, should be a priority for elementary students in grades 4-5. This study informs this debate by providing data about connections between different representations of constant growth functions and student algebraic reasoning in a context open to individual construction of representations and reasoning approaches. Participants included 9 fifth graders from the same elementary class. Data shows that students can generate representations which are effective reasoning tools for finding particular cases of the function and generalizing the function but that this depends on features of the problem representation, most importantly the representation of the additive constant. I identified four categories of algebraic reasoning on the task to find the tenth term and found that only students who used reasoning approaches with the additive constant separate and functional reasoning to find the variable component were able to generalize the function. These instances occurred on a story problem and two geometric pattern problems. None of the students used such a reasoning approach or were able to generalize on the numeric sequence problem which did not represent the additive constant separately. Implications for future research and for teaching for conceptual understanding of variable and relationship are discussed.

CHAPTER ONE

INTRODUCTION

It is May and fifth graders at Pueblo Elementary School are excited. The city built a new pool at a nearby park and started to fill it with water. The students wonder how long it will take. Their mathematics and science teacher, Ms. Walker, decides this would be a good “real world” situation to build a lesson around. Here is the problem she gives her fifth graders: “The pool is 4 feet deep. The water in the pool is now 5 inches deep. It is filling at a rate of 1.5 inches per day. Describe the pattern using words, pictures or objects. Find how deep the water will be after 4 days. Find how deep the water will be after 10 days. Write a rule to find how deep the water is after any number of days. Use your rule to find how long it will take to fill the pool.”

Walking around the classroom, Ms. Walker sees students using different reasoning tools to solve the tasks: some have drawn pictures, some are using number lines, others have written number sentences or organized their answers into a function table. They are using their work on finding the particular cases (i.e., day 4, day 10) to reason about a rule and using their rule to find out how long it will take to fill the pool.

In this vignette, students are engaged in a meaningful (to them) exploration of a linear function situation. It is an opportunity for them to develop understanding of the concepts underlying all functions, i.e., variable and relationship, based on the personal meaning they develop in working with familiar (to them) variable quantities (e.g., days, inches) on a constant growth function problem. The mathematical activity context is open to different reasoning approaches and students may choose not only their reasoning tools but also which units of the quantities they want to work with. They generate a variety of representations which support their conversations about how they solved the tasks. Generating their own reasoning tools and sharing their reasoning with their peers seem to support students in making shifts from arithmetic to algebraic reasoning. What is not yet known, and is the subject of this dissertation, is how students might leverage the mathematical information incorporated in constant growth function problems in order to engage in algebraic rather than arithmetic reasoning.

Problem Statement

Algebraic reasoning takes many forms, including: (1) generalizing and formalizing patterns (e.g., reasoning from particular cases to a general rule or property and expressing this general rule or property using algebraic notation); (2) syntactically-guided manipulation of formalisms (e.g., equation solving); and (3) working with functions, relations and joint variation which involve relationships between variable quantities (Kaput, 1998). Student difficulties with this reasoning are well documented but mathematics education researchers have proposed quite different causes and solutions.

Filoy & Rojano (1989) argued that these difficulties are rooted in limitations in student cognitive development, i.e., developmental constraints, and that students would gradually evolve in their mathematics reasoning to the point at which they are ready for abstract algebraic reasoning. The solution from the perspective of researchers who attribute difficulties to developmental constraints is to delay introduction of algebra until students are developmentally ready. This perspective would imply that earlier introduction of opportunities to engage in algebraic reasoning would be ineffective if not injurious to student cognitive development.

Other mathematics education researchers (e.g., Bednarz & Janvier, 1996; Davis, 1985, 1989; Vergnaud, 1988) argued that the difficulties students encounter in algebra arose when students were expected to shift suddenly from arithmetic to algebraic reasoning and that the solution to the problem was to integrate opportunities for elementary school students to simultaneously develop both arithmetic and algebraic reasoning.

From the perspective of those who espouse the integration of algebraic reasoning into the elementary school curriculum, this integration might also preclude difficulties such as student reliance on deeply engrained arithmetic reasoning approaches when presented with problems involving relationships between variable quantities (Bednarz & Janvier, 1996) and student development of misconceptions during arithmetic instruction, such as a letter symbol standing for “the answer” and the “=” symbol meaning “yields” rather than equivalence (Kieran, 1981).

During the 1990s, work progressed on the approach proposed by Davis (1985, 1989) and Vergnaud (1983, 1988), to integrate opportunities for elementary school students to engage in algebraic reasoning. The process of generalization, or describing the overall pattern underlying a set of mathematical data, emerged as a focal point for extending beyond arithmetic reasoning to algebraic reasoning (Kaput, 1998; Mason, 1996). This work culminated in 2000, when the National Council of Teachers of Mathematics incorporated a research-based algebra strand in its *Principles and Standards for School Mathematics* (2000).

More importantly, research results published since 2000 have documented that elementary school students can in fact engage in algebraic reasoning and that integration of opportunities to generalize do not interfere with, but rather enrich, students' development of arithmetic reasoning (e.g., Blanton, 2008; Blanton & Kaput, 2004; Carpenter & Franke, 2001; Schliemann, Carraher & Brizuela, 2007). This research in the field of *early algebra* is based on grade-level appropriate introduction of algebraic reasoning not early introduction of the content of a formal algebra course.

One vein of this research, the function approach, is based on the notion that functions are fundamental objects of algebra (Schwarz, 1990). Research studies with elementary school students, primarily grades 3-4, have demonstrated that students can develop understanding of the fundamental algebraic concepts of variable and relationship and engage in algebraic reasoning, specifically generalizing the relationship underlying constant growth functions (cf, Blanton, 2008; Moss, Beatty, Barkin & Shillolo, 2008; Schliemann, Carraher & Brizuela, 2007;Warren & Cooper, 2008).

Constant growth functions are a subset of linear functions appropriate for elementary school in which the rate of growth is a positive constant. For example, the cost of buying Girl Scout cookies is a constant growth function relating the total cost to the number of boxes. If each box costs the same amount, say $4, the cost goes up $4 for each additional box. The total cost is a function of the number of boxes, specifically, the total cost equals the number of boxes times the constant rate of growth, $4. If graphed as cost versus number of boxes, particular cases could be connected with a straight line which is why it is called a linear function. This is a rich and important domain for student development of algebraic reasoning grounded in development of the underlying concepts of variable and relationship.

Linear functions are particularly appropriate for older elementary school students poised at the transition to middle school mathematics and the opportunity to take the formal algebra course. Given the critical importance for students to have opportunities to develop understanding of the fundamental algebraic concepts of variable and relationship, one could argue that providing opportunities to explore linear functions, the first function studied in depth in a formal algebra course, should be a priority for elementary students in grades 4-5.

This is a very different approach than introducing algebra through early introduction of syntactically-guiding manipulation of equations solving for an unknown. One can only imagine what effect this different approach might have on student learning of equation solving based on understanding linear equations as representing particular cases of linear functions: perhaps many of the documented difficulties might be averted if students were operating on objects they understood. In addition, there may be benefits to subsequent learning assuming students are supported in making connections between concepts underlying linear functions and other functions they study (e.g., quadratic, exponential, trigonometric).

In its current state, early algebra research using a function approach has established the following with students as young as grades K-2, but primarily with students grades 3-4: (1) students can engage meaningfully in generalization activities designed to support students in exploring representations of constant growth functions

(Blanton, 2008; Moss, Beatty, Barkin & Shillolo, 2008; Schliemann, Carraher & Brizuela, 2007; Warren &Cooper, 2008); (2) students need scaffolding in using function tables as a reasoning tool for generalization of the relationship between two variable quantities (Blanton, 2008; Schliemann, Carraher & Brizuela, 2007); (3) students need scaffolding in recognizing and using the term number as the independent variable in geometric patterns and numeric sequences (Moss, Beatty, McNab & Eisenband, 2006; Moss, Beatty, Barkin & Shillolo, 2008; Warren &Cooper, 2008); and (4) students need scaffolding in the symbolization process, i.e., the use of letter-symbolic notation to express a generalization (Blanton, 2008; Moss, Beatty, Barkin & Shillolo, 2008; Schliemann, Carraher & Brizuela, 2007; Warren & Cooper, 2008).

Thus, these studies identify difficulties elementary students encounter using representations of the function used as algebraic reasoning tools, in particular, difficulties with using function tables and geometric pattern representations.

It is intriguing that the difficulties encountered by elementary students in grades 3-4 and interventions to scaffold students in overcoming these difficulties began with making sense of the function representations. While research has identified student difficulties associated with constant growth function representations, research is sparse on features of different representations of constant growth functions which elementary students might leverage to engage in algebraic reasoning. To investigate this would require a different mathematics activity context, one which is open to student generation of representations, i.e., reasoning tools, and student choice of reasoning approaches, much like the context described in my introductory vignette.

In summary, the problem addressed in this research study is the need for research with individual elementary school students to identify the connections between different representations of constant growth functions and student algebraic reasoning in a context open to individual construction of representations and reasoning approaches. Because research on student algebraic reasoning on constant growth patterns has been primarily conducted with students in grades 3 and 4 in the context of teaching interventions, more research is needed with the oldest elementary school students, i.e., fifth graders at the end of the final year of elementary school mathematics in a different context. These students are poised at the transition to middle school and will be challenged with problems relating two variable quantities. Thus, this study is intended to inform the mathematics education community on ways fifth graders themselves can reason algebraically and how they might leverage the mathematical information defining a constant growth function. Since recent research on elementary students solving constant growth functions employed story problems with familiar variable quantities, geometric pattern problems and numeric sequence problems, these three types of problem representations are explored in this study as well.

This study examines fifth graders' work to solve constant growth function problems that vary in how the function is represented (i.e., story problems with familiar variable quantities, geometric pattern problems and numeric sequence problems). It provides a different perspective on the "algebra problem," one that focuses on individual student reasoning approaches developed through interaction with the problem representation and generation of their own reasoning tools, i.e., representations.

Specifically, it will inform the discussion of the means by which linear functions are introduced as meaningful and accessible to students before the first course in algebra so as to support development of algebraic reasoning grounded in student understanding of the concepts underlying all functions: variable and relationship. This understanding is evidenced in this study when a student writes a general rule to find any term in the pattern. This is a particularly powerful representation of a function not only because it represents the functional relationship but also because algebraic formulas (i.e., letter-symbolic representations of the general rule) are used extensively in school algebra.

Overview of the Dissertation

In the subsequent chapters of this dissertation, I elaborate my theoretical framework and present my analysis of data before describing the implications of the findings for research and curriculum and instruction.

In Chapter 2, I add details to the brief review in this chapter of the research base in which this study is situated and establish the theoretical framework for this study of student representations and reasoning on constant growth function problems.

In Chapter 3, I describe my research process. I discuss the theoretical basis for my methods and then describe the setting for this study including the school and the students. This description is followed by details of the data I collected, my analysis procedures, and limitations of my methods.

Chapters 4 and 5 present my findings in two parts. First, in Chapter 4, I present findings on the connections between the problem representation of a constant growth patterns and the students' work on a task to find the tenth term in the pattern. Then, in Chapter 5, I extend these findings to the connections between the students' work on this task and ability to engage in algebraic reasoning on two tasks: a task to write a rule to find any term in the pattern and a supplementary task to find the 100th term in the pattern,

Finally, in Chapter 6, I look across the two findings chapters to answer my research questions. I discuss my findings, elaborate implications for teachers and researchers, and discuss limitations and future research directions.

CHAPTER TWO

LITERATURE REVIEW

This research project specifically examines elementary school student engagement in algebraic reasoning on constant growth function problems. Mathematics education researchers have demonstrated that elementary school students have the capacity to engage in algebraic reasoning about functions including constant growth functions. Their research, conducted in the context of elementary mathematics classroom teaching experiments, used different representations of functions, primarily geometric patterns (Blanton, 2008; Moss, Beatty, Barkin & Shillolo, 2008; Warren & Cooper, 2008), story problems (Blanton, 2008; Schliemann, Carraher & Brizuela, 2007) and integrated activities linking geometric pattern representations with numeric pattern representations (Moss, Beatty, McNab & Eisenband, 2006).

Although this important body of research work provides evidence that elementary school students have the capacity to reason algebraically, it also points to a number of challenges students face, particularly with the use of particular representations of the functions as algebraic reasoning tools. These considerations are discussed in this chapter in the first two sections on algebraic reasoning and representations, both focused on constant growth functions.

What is not known from prior research is how individual elementary school students might interact with constant growth function problems in a context open to student choice of representations and reasoning approaches. Of particular interest are: (1) identifying aspects of different problem representations of constant growth functions which students leverage in using algebraic reasoning approaches and (2) identifying obstacles students encounter in engaging in algebraic reasoning with different problem representations. Research is needed to understand how individual elementary school students use different representations of constant growth functions to generate their own representations and construct their own reasoning approaches.

In the sections that follow, I first provide an overview of constant growth functions using a plant growth example. I then present a framework for examining individual student interactions with different problem representations of constant growth functions and the connections between the problem representation, student work and algebraic reasoning. This framework is organized into three parts: algebraic reasoning about constant growth functions, representations of constant growth functions and mathematical activity contexts for algebraic reasoning, frequently referred to in the research literature as generalizing activities.

Research considerations related to mathematical activity contexts and student reasoning on constant growth functions are discussed in this final section of framework. The chapter concludes with research questions to explore student reasoning approaches across different problem representations of constant growth functions.

Overview of Constant Growth Functions

A *function* is a mathematical statement which describes how two or more quantities vary in relation to each other in a unique way, i.e., it defines the specific value of one quantity given a particular value of the other. A function can be represented in different ways: as a story problem, as a mapping of corresponding values such as a set of ordered pairs or a function table, as a Cartesian coordinate graph or as a rule defining the operations to perform to find the value of one variable given the other such as an algebraic formula. For example, consider a situation in which you plant a seed and a plant grows 3 inches per week after sprouting. One could represent this function with a table of values of the number of weeks in the first column (e.g., 1, 2, 3) and the corresponding heights in inches in the second column (e.g., 3, 6, 9), a graph of height in inches versus number of weeks or an algebraic formula using letters to represent the number of weeks and the height in inches such as $H = 3W$.

The plant growth situation is an example of a *constant growth function*, a linear function restricted to rates of change that are positive. When graphed, particular cases of a constant growth function (i.e., ordered pairs) can be connected by a straight line and the slope of the line is positive. The prior example of plant growth represents a constant growth situation in which the plant starts at a height of zero inches and grows 3 inches per week. Consider another situation in which you plant a seedling 4 inches tall and it grows at a constant rate of 3 inches per week. The plant's height is a function of (i.e., related to) the number of weeks since the seedling was transplanted, but in addition to the growth that varies by the number of weeks, the plant's height contains another component, the 4 inches tall it was when planted as a seedling. Specifically, the plant's height H after any number of weeks W can be found using the relationship rule $H = 3W + 4$. After 10 weeks, the plant would reach a height of 34 inches, the 30 inches the plant grew in 10 weeks plus the 4 inches starting height.

In the plant growth example, one could find the height after any number of weeks by multiplying a rate of change, 3 inches per week, times the number of weeks to find the height component that varies according to the number of weeks and then add the constant component, the 4 inches. More generally, any linear function can be represented in the form $y = mx + b$, where x is the independent variable quantity (e.g., number of weeks), y is the dependent variable quantity, m is a constant rate of change (e.g., inches per week) and b is a constant known as the *additive constant.*

In the remainder of this chapter, I articulate the theoretical framework for this research project, addressing algebraic reasoning, representation and mathematical activity context.

Algebraic Reasoning on Constant Growth Functions

In this first section, I define algebraic reasoning and describe the specific form of algebraic reasoning that is fundamental to the study of mathematics, generalization. This is followed by discussion of categories of algebraic reasoning and research on reasoning associated with constant growth functions.

Reasoning can be thought of in terms of detecting, expressing, analyzing, explaining and providing evidence in support of assertions. Algebraic reasoning is reasoning about algebra and takes different forms depending on the topic being studied. To reason algebraically about constant growth functions means to detect and analyze the relationship between two variable quantities and to express and explain this relationship. For example, in the plant growth situation, a student could reason algebraically that the plant is 4 inches tall to start and the height after any number of weeks would be that 4 inches plus the rate of growth (i.e., 3 inches per week) times the number of weeks. The student might express this relationship rule using written words (e.g., “to find how tall the plant is after any number of weeks, multiply 3 inches per week times the number of weeks and then add the 4 inches it was to begin with.”) or letter-symbolic notation (e.g., the algebraic formula $H = 3W + 4$).

Thus, this research project focuses on the first of two core aspects of algebra identified by Kaput (1998): systematic symbolization of generalizations of regularities and constraints. It looks at student reasoning to find particular cases and write a rule to find any case (i.e., generalize) on linear function problems, one of the topic strands identified by Kaput.

This research study does not focus on the second of two core aspects of algebra identified by Kaput: syntactically guided reasoning and actions on generalizations expressed in conventional symbol systems (e.g., equation solving). Research has documented connections between syntactically guided (procedural) equation solving and student development of misconceptions, such as thinking an equal sign means “yields” or “here comes the answer” rather than “is equivalent to” or “this side equals that side” (Kieran, 1981) or thinking a letter represents “an unknown” or “the answer” rather than a quantity or magnitude that can vary, a “variable” (Smith, 2003). This latter misconception is evidenced in research by students assigning values to what are intended to be variable quantities, such as assigning a value of 20 to “any term” in a task to write a rule to find any term in a function (Schliemann, Carraher & Brizuela, 2007; Smith, 2003).

In contrast to introducing elementary school students to algebra through activities to generate and solve linear equations in one unknown, the function approach to early algebra uses problems which represent relationships between variable quantities. The students engage in activities to detect and express the pattern or relationship between two variable quantities (i.e., generalize a function). Thus, the function approach presents opportunities for students to reason about variables, rather than unknowns; about the mathematical relationship rather than procedural rules; and about the big idea that algebra is useful for modeling real world situations, including those associated with the sciences, rather than viewing it as procedures to be learned but not necessarily understood.

In the next section I define what it means for elementary school students to engage in generalization on constant growth function problems. This is followed by the section on different categories of student reasoning on constant growth functions tasks to find particular cases, interwoven with a discussion of research findings related to algebraic reasoning including generalization.

Generalization. A mathematical generalization is a statement describing a general truth about a set of mathematical data. In early algebra *generalization* refers to making "a single statement to refer to multiple instances." (Kaput, Blanton & Moreno, 2008, p. 20). A generalization describing a constant growth function is thus a single statement which describes the general truth, i.e., the underlying functional relationship between two variable quantities. This is the way I use the term "generalization" in this study.

Ellis (2007) developed a taxonomy for classifying children's generalizations. She distinguished generalizing actions, mental activities that could be inferred from what students did and said as they solved problems, and the statements (i.e., generalizations) students generated which were grounded in their generalizing actions.

Generalizing actions relevant to this study include searching for and extending patterns in constant growth pattern tasks. Students, given a constant growth pattern, will search for patterns. For example, a student may detect that the terms increase by a constant amount. This could be inferred, for example, from how a student describes the pattern (e.g., “increasing by 3”). Or, a student may detect a pattern as a relationship between increases in the term numbers and term values. For example, a student may write “the pattern is that you add 6 to the terms and 2 to the term number.” These examples are both based on student reasoning about a relationship between terms. In contrast, a student may detect a pattern between the term and its term number and express this, for example, by saying “each term is a multiple of 3 plus 1.”

Extending actions are particularly relevant to this study in which students solve tasks to find particular cases of a constant growth function before solving the task to write a rule to find any term in the pattern. To generate new cases, students working with a constant growth pattern might repeat an existing pattern (e.g., adding 3 to the term value and 1 to term number) or operate upon an object (i.e., adding 6 to the term value and 2 to the term number, indicating that the student operated on the ratio of 3 to 1).. Ellis (2007) labels these two extending actions *continuing* and *operating*, respectively. Both are based on the student detecting a specific correspondence between the rates of change in the term values and term numbers.

It is important to remember that Ellis (2007) is referring to mental activities in categorizing generalizing actions. This study of individual student reasoning on constant growth patterns extends and builds upon Ellis' work by focusing on what younger students, fifth graders, do and say as they solve for particular cases of the function represented by the pattern. Specifically, the focus of this study is how students reason about constant growth patterns depending on how the mathematical information is represented in the problem, and how this may impact student generalization of the pattern.

In this study, the focus is on one specific kind of generalization, a statement of a rule to find one variable quantity in terms of another. This corresponds to Ellis' (2007) category *General Principle*, which includes rule and pattern generalizations. The first, *rule*, is a description of a general formula. In this study, it would be a description of how to find one variable quantity in terms of the other. The second, *pattern*, is a statement identifying a general pattern which might describe how one variable quantity changes but does not capture the relationship between the two variable quantities. Students who describe, for example, that the pattern is to add 3 to each term value would be using a pattern generalization. I expect that fifth graders in this study will express both types of generalization on the task to write a rule to find any term in the pattern.

Different researchers have used different approaches, sometimes referred to as generalizing activities, to support students in generalizing constant growth functions. For example, Blanton (2008) enunciates a framework for generalizing from particular cases: (1) explore a mathematical situation; (2) develop conjectures (i.e., mathematical statements about the situation that are true or false); (3) test conjectures; (4) if false, revise and test new conjecture; until (5) a conjecture is found to be true based on testing and becomes a generalization. Schliemann, Carraher and Brizuela (2007) used a similar approach, engaging students in activities to express and test conjectures of the rule relating corresponding values of two variable quantities represented with a function table. In this approach, students exploring a constant growth pattern may generate a table of particular cases to use as a reasoning tool to generate and test conjectures about the relationship between two variable quantities. The student is reasoning from particular cases to the general case.

In contrast to this research approach to supporting student algebraic reasoning, specifically generalization, through a process of reasoning from particular cases to the general case, there is another corpus of research on the function approach to early algebra which is based on the assumption that students generalize using their innate abilities to detect patterns (Moss, Beatty, Barkin & Shillolo, 2008; Moss, Beatty, McNab & Eisenband, 2006; Warren, 2005; Warren & Cooper, 2008). These researchers emphasize exploration activities with geometric patterns, such as constructing and deconstructing figures or drawing or constructing a particular figure. Through these activities, students may detect the underlying functional relationship between the two variable quantities and express a generalization. More details on these different approaches to generalization and challenges encountered by students in using different representations of constant growth functions to generalize are provided in later sections on mathematical activity contexts and representations, respectively.

Regardless of the generalizing activities employed in particular mathematics classroom teaching experiment research, elementary school students needed scaffolding to write a letter-symbolic formula for finding the value of one variable quantity in terms of the other. Research with children as young as first grade (Blanton, 2008) but primarily with children in grades 3-5 (Blanton, 2008; Davis, 1989; Moss, Beatty, Barkin & Shillolo, 2008; Schliemann, Carraher & Brizuela, 2007; Warren & Cooper, 2008) has demonstrated that elementary school children can use letter-symbolic notation to express a generalization, but that this required scaffolding to understand the use of a letter to represent a variable quantity.

One advantage of story problems in this regard is the presence of familiar measured quantities that students can use as quasi-variables (Schliemann, Carraher & Brizuela, 2007). That is, students can engage in generalizing about the relationship between the two variable quantities, first describing the relationship using words and then transitioning to letter-symbolic notation with scaffolding. While some mathematicians might argue that generalization necessarily involves symbolic expression and that these written or oral word representations are therefore quasi-generalizations at best, they are important indicators at the elementary school level that students are engaging in algebraic reasoning, i.e., considering the relationship between two variable quantities that underlies the constant growth function representation (Kaput, Blanton & Moreno, 2008).

In summary, recent research using constant growth function representations as algebraic reasoning tools has shown that elementary school students can generalize and express their generalizations about constant growth functions. There are different views on how to support students in engaging in algebraic reasoning about constant growth functions, but research studies using generalizing activities in elementary classroom teaching experiments has shown that elementary school children can generate generalizations about constant growth functions in a variety of forms, including oral words and gestures, written words, and algebraic (letter-symbolic) notation (Blanton, 2008; Moss, Beatty, Barkin & Shillolo, 2008; Moss, Beatty, McNab & Eisenband, 2006; Schliemann, Carraher and Brizuela, 2007; Warren & Cooper, 2008).

Reasoning approaches (y=mx). Vergnaud (1988) has done extensive research on elementary student reasoning approaches to solving constant growth functions of the form $y = mx$ (i.e., constant growth functions with an additive constant equal to zero). He analyzed reasoning approaches in terms of the relationships the student's reasoning approach was based on. If the additive constant is zero, the relationship is multiplicative. On tasks to find particular cases of these constant growth functions, Vergnaud found that students either reasoned about the relationship between the two variable quantities (functional reasoning) or reasoned about the patterns of the two variable quantities separately (scalar reasoning). This distinction is important in terms of ability to generalize as well as development of understanding of the fundamental concepts underlying functions, variable and relationship.

In the following sections, I present Vergnaud's (1988) definitions of these two categories of reasoning and discuss the connections between these two categories and student algebraic reasoning on constant growth functions. These connections relate to Vergnaud's framework of analysis: scalar and functional reasoning approaches differ depending upon what relationships the student takes into account in choosing particular operations to perform on constant growth function problems..

Scalar reasoning. *Scalar reasoning approaches* are those in which students choose procedures based on the relationship between quantities or magnitudes of the same kind (Vergnaud, 1988). For example, students used scalar reasoning approaches to fill in a function table by extending the patterns in each column separately (Schliemann, Carraher & Brizuela, 2007). If the first column represented the number of boxes of cookies and the second column represented the total cost at $3 per box, the student using a scalar reasoning approach would operate on the number of boxes (successively adding 1 box) to extend the first column and would operate on the cost in dollars (successively adding $3) to extend the second column.

Likewise, given a numeric or geometric pattern problem, a student using a scalar reasoning approach would operate on the term or figure number (successively adding 1) and would separately operate on the magnitude of the term or the number of objects in each figure (successively adding the magnitude or number of objects of change per term or figure number, respectively). For example, given the first three terms of a numeric pattern “3, 6, 9,” the student using a scalar reasoning approach could reason that the terms are increasing by 3 and add 3 successively to find the value of the 10^{th} term. Given the first three figures of a geometric pattern, the student could likewise find the number of objects being added successively and find the number of objects in the 10^{th} figure by adding 3 successively. The student is using a scalar reasoning approach because he or she is operating on quantities or magnitudes of the same kind.

A more efficient scalar approach to extending a constant growth function to a particular case might involve proportional reasoning (Vergnaud, 1988). A student who skips term numbers and adds a multiple of the successive change in the term values, for example, is still operating on quantities of the same kind. For example, if the rate of change is 3, the student might add 2 to each term number and 2 times 3, or 6, to each term value to extend the pattern, or any other ratio equivalent to 1:3. Little is known, however, about connections between this form of scalar reasoning about constant growth functions and generalization, other than reasoning from particular cases to write a function rule as discussed in the section on generalization.

Using multiplication to separately extend the two patterns (the term number and the term value) is also a form of scalar reasoning. For example, a student finding the 10th term could multiply the term number 1 times 10 and multiply the term value corresponding to the first term, 3, times 10. Here again, the student is using scalar reasoning because the student is operating on the two variable quantities separately, i.e., the student's reasoning approach is based on operating on magnitudes or quantities of the same kind. This form of scalar reasoning, while appropriate for the constant growth functions Vergnaud (1988) was investigating, is not appropriate for constant growth functions with an additive constant. This restriction is discussed in the section on reasoning about constant growth patterns with an additive constant.

Research using constant growth functions has shown that students can use scalar reasoning approaches effectively to extend constant growth functions to near terms of the function . In addition to the work of Vergnaud(1988) and Davis (1989), more recent research has also demonstrated the ease with which students can find near cases of a constant growth function (e.g., the cost of 4 boxes of cookies or 10 boxes of cookies or the number of objects in the fourth or tenth figure of a geometric pattern) using scalar reasoning approaches. In their research, Schliemann, Carraher and Brizuela (2007) found that elementary school children easily extended function tables past the first three (given) cases by extending each column separately. The students usually extended the value in the first column by 1's and easily found the change between successive values given in the second column and used this to extend that column through successive addition (e.g., if the first three values in the second column were "3, 6, 9," the student extended that column through successive addition of 3).

In this and other scalar reasoning approaches described above, the student's reasoning is based on detection of two separate patterns and operating on each separately. This is quite different than reasoning about the relationship between variable quantities, i.e., the relationship that defines the function.

Scalar reasoning approaches become cumbersome to use for far cases of the function, however, and do not support reasoning about the function rule. Schliemann, Carraher and Brizuela (2007) found that they needed to intervene after elementary school students used scalar reasoning approaches to fill in a table of particular cases of a function: they needed to shift the students from reasoning about separate relationships of two variable quantities to reasoning about one relationship between the two variable quantities. This category of reasoning is discussed next, again using Vergnaud's (1988) framework for analyzing reasoning approaches on constant growth functions of the form y=mx (i.e., constant growth functions with no additive constant).

Functional reasoning**.** In contrast to scalar approaches, Vergnaud (1988) identifies a *functional reasoning approach* based on the relationship between the values of the independent and dependent variables quantities. In the plant growth example, the student would use the relationship that the values in the second column are 3 times the corresponding values in the first column to find the plant height after 10 weeks as 3 inches per week times 10 weeks equals 30 inches. This approach necessarily involves procedures based on a relationship between quantities or magnitudes of a different kind (e.g., weeks and inches per week).

Unlike scalar reasoning approaches, in which the student reasons about two separate patterns for the two variable quantities, in a functional reasoning approach the student is reasoning about the underlying relationship that defines the function. It is not clear from the early algebra research how common this type of reasoning might be among students in contexts open to student choice of representations and reasoning approaches. But research does indicate that it is difficult to shift students from scalar reasoning to functional reasoning (Schliemann, Carraher & Brizuela, 2007; Smith, 2003)

In summary, students may use scalar or functional reasoning approaches on constant growth function problems.   Both scalar and functional reasoning approaches are important in the study of linear functions in algebra: the scalar approach supports understanding the concept of slope; the functional approach supports understanding the relationship between the two variables. However, to be able to write a rule to find any case in a pattern (i.e., generalize), the student needs to express the relationship between the two variables. This suggests there is a connection between functional reasoning and generalization but it is not clear whether functional reasoning emerges as students engage in generalization or whether functional reasoning emerges as students explore and discover algebraically useful patterns in constant growth function representations. What is known is that elementary school students can use scalar reasoning approaches effectively to extend constant growth functions to near terms of the function and students can use functional reasoning approaches to generalize. Little is known about shifts between these forms of reasoning at the individual level.

Reasoning approaches on constant growth functions with an additive constant (y=mx+b). Vergnaud's (1988) framework distinguishes forms of reasoning about problems of the form $y = mx$. Constant growth function problems can take the form $y = mx + b$ where b is an additive constant which is a quantity of the same kind as y, the dependent variable. Using the plant growth example, suppose the plant height is measured after a seedling, which is 1 inch tall, is transplanted into a larger container. Now, the height of the plant in inches depends not only on the 3 inches per week (the rate of change) times the number of weeks, but also on the starting amount (i.e., the height at time zero). After one week, the plant height is 4 inches (the 1 inch to start with plus the 3 inches it grew the first week).

For this composite constant growth function, Vergnaud's framework (1988) could be used to distinguish scalar or functional reasoning approaches to students finding the variable component, mx, recognizing that the student would then use a scalar reasoning approach to find $mx + b$ by adding the variable component mx to the fixed component b since they would be quantities of the same kind. Returning to the plant growth example, the student might find the variable component to be 30 inches by multiplying 3 inches per week times 10 weeks. In this step, the student is using functional reasoning since the quantities are of a different kind. The student would then add 30 inches to 1 inch, the starting amount to find a plant height of 31 inches after 10 weeks. In this step, the student is using scalar reasoning, adding two quantities of the same kind.

One can think of a linear function as comprised of a component *mx* that varies depending on the value of *x, the variable component,* and a component, the additive constant, that does not vary and therefore is a *fixed component.* In the plant growth example, the variable component is the amount the plant grows and the fixed component, the additive constant, is the 1 inch to start with.

Alternatively, the student may use a reasoning approach in which she or he treats the rate of change *m* as being the same kind of quantity as the additive constant (i.e., the same kind of quantity as the dependent variable): such reasoning approaches would be considered scalar. In our plant growth example, a student may treat 3 inches per week as 3 inches and use a scalar reasoning approach in which he or she adds 3 inches to 1 inch to find the height after one week, plus 3 more inches to find the height after two weeks and so on. The student is finding each case of the constant growth function, i.e., a height of 4 inches after 1 week, 7 inches after 2 weeks and so on. A different scalar reasoning approach might be to add 3 inches together a particular number of times to find the variable component, keeping the additive constant (i.e., the starting amount of 1 inch) separate. To find the height after 10 weeks using this approach, the student would sum ten 3’s to find the plant grew 30 inches in 10 weeks, then add the 1 inch to start with.

Both are scalar reasoning approaches but are different in the treatment of the additive constant. More importantly, they differ in whether the student is extending the pattern one case at a time (the first method) or finding a particular case (e.g., the 10^{th} case) directly. The second method is similar to the functional reasoning approach described in the previous paragraph in that the student is finding the 10^{th} term in the pattern directly from the value of the independent variable, 10 days. .

Research in early algebra has not systematically addressed the connections between student reasoning approaches and different representations of constant growth patterns of the form $y = mx + b$. Research is needed to examine student reasoning approaches in a context open to student choice of representations and individual construction of reasoning approaches on constant growth functions to identify aspects of different problem representations students may leverage in engaging in algebraic reasoning and the connections between representations students generate and ability to generalize the function.

Representation

In this section, I discuss representations using Goldin and Schteingold's (2001) definition of a representation: "A representation is typically a sign or a configuration of signs, characters, or objects. The important thing is that it can stand for (symbolize, depict, encode, or represent) something other than itself" (p. 3-4). I first discuss problem representations of constant growth functions as algebraic reasoning tools, then student generation of representations and their use as algebraic reasoning tools.

Problem representations. This research study seeks to inform the discussion of the means by which constant growth function problems can be introduced to elementary school students through a variety of problem representations, including those commonly encountered in elementary school curricula and research studies, stories (written words), geometric patterns (pictorial), and numeric sequences (numeric).

In mathematics, there are conventional systems of representation for functions including function tables, Cartesian coordinate graphs and algebraic (i.e., letter-symbolic) equations. Constant growth function research with elementary students has used both conventional function representations (e.g., function tables, algebraic formulas) and non-conventional representations (e.g., geometric patterns). This research has identified challenges students face using these representations and interventions that scaffold students in overcoming these challenges.

Elementary school students needed scaffolding to use function tables to reason about the relationship between two variable quantities (Schliemann, Carraher, & Brizuela, 2007; Smith, 2003). Otherwise, since students used scalar reasoning approaches to extend each of two patterns (one for each variable quantity), the researchers needed to intervene to shift student attention to the pattern across the rows representing corresponding values of the two variable quantities. This raises serious questions about the usefulness of tabular representations as algebraic reasoning tools on constant growth functions before students understand these representations as intended (i.e., function tables). As I note below, more research is needed on representations students might generate that would be more useful as algebraic reasoning tools, and connections between problem representations and student generation of useful (to them) algebraic reasoning tools.

Non-conventional representations of constant growth functions such as geometric patterns can also be problematic as algebraic reasoning tools for elementary school students. One difficulty common to both geometric and numeric patterns is recognizing the figure or term number as the independent variable (Smith, 2003). Both Moss and colleagues (2008) and Warren and Cooper (2008) incorporated the use of number cards to make the value of the independent variable (i.e., the position number) explicit in geometric pattern problems. Another obstacle to student use of geometric pattern representations of constant growth functions as an algebraic reasoning tool relates to student perception. Lee (1996) noted that students working with geometric patterns might encounter difficulty perceiving “algebraically useful patterns” (p. 95). Both Lee and Mason (1996) proposed that student engagement in algebraic reasoning about geometric patterns requires opportunities to develop their capacity to see multiple patterns (what Lee calls “perceptual agility” and Mason calls “multiple seeings”). For example, students working with a geometric pattern representing the constant growth function $f(x)=4x+2$ might perceive a figure as having two equal parts and that each part has one more block than two times the value of x. This might be a less useful pattern for generalization that detecting a pattern of four times the value of x plus two, since it would require reasoning about doubling the detected pattern (i.e., two times the value of x plus one).

Recent research using patterning activities (Moss, Beatty, Barkin & Shillolo, 2008; Warren & Cooper, 2008) has explored Mason’s recommendation that visualization and manipulation of figures (i.e., constructing and deconstructing figures) can scaffold elementary school students in developing understanding of the relationship between the number of objects in each figure and the figure number. More research is needed with individual students to understand the connection between features of geometric pattern representations and detection of algebraically useful patterns.

Student generation of representations as reasoning tools. In contrast to using problem representations (e.g., numeric sequence or geometric pattern), elementary students may invent their own representations to solve tasks. The may use these representations as algebraic reasoning tools on the task to write a rule to find any case in a constant growth function (i.e., the generalization task).

Although research using the generalized arithmetic approach has demonstrated that students can engage in algebraic reasoning through generation of their own representations and reasoning approaches (Blanton, 2008; Carpenter & Franke, 2001), student engagement in algebraic reasoning about constant growth functions using this approach has received scant attention. Blanton (2008) reports on young children's generation and use of picture representations as effective reasoning tools on a constant growth function problem about dogs. Researchers using geometric pattern representations report on effective use of object representations as algebraic reasoning tools (Moss, Beatty, Barkin & Shillolo, 2008; Warren & Cooper, 2008).

Research is lacking in general on older elementary school student generation of algebraic reasoning tools on constant growth function problems. Since representations are tools for communication as well, student's intuitive representations provide a glimpse into their reasoning and are thus useful from a methodological standpoint for understanding student cognition on constant growth functions.

Representations of generalizations. On constant growth functions, generalization takes the form of reasoning about the relationship between two variable quantities. Classroom research shows that elementary school children can generalize and symbolize constant growth functions using a variety of non-conventional symbols, including oral words and notations and, with scaffolding, use conventional letter-symbolic notation to symbolize the generalization (Blanton, 2008; Blanton & Kaput, 2004; Moss, Beatty, Barkin & Shillolo, 2008; Moss, Beatty, McNab & Eisenband, 2006; Schliemann, Carraher and Brizuela, 2007; Warren & Cooper, 2008). Although some mathematicians might argue that generalization necessarily involves symbolic expression, early algebra researchers have demonstrated that student generation of generalizations using oral word, gesture and written word representations are important for emergence of algebraic reasoning (Kaput, Blanton & Moreno, 2008). These generalizations also support students in transitioning from meaningful (to them) representations to letter-symbolic representations. Familiar variable quantities (e.g., days, pennies, inches) have been identified as important scaffolds in this regard since students can use them as quasi-variables in their oral or written word representations (Schliemann, Carraher & Brizuela, 2007).

More research is needed on the connections between constant growth function representations, student work finding particular cases and generalization. In the next section, I discuss the nature of mathematical activities that encourage students to generate representations to show and explain their work on constant growth problems. I begin by defining context in terms of the mathematical activities students are engaging in while solving constant growth functions.

Mathematical activity context

In this section, I discuss mathematical activity context (Boaler, 1993). I focus on the nature of mathematical activities which would allow exploration of connections between different representations of constant growth functions and individual student reasoning. The focus of this study is not on differences in story problem situations and my use of the term context does not refer to story problem situations. Rather, the focus is on differences in how students use the mathematical information about a constant growth function in different types of problems (i.e., story, geometric pattern, numeric sequence). To explore these differences, the mathematical activity context needs to be open to student choice of reasoning approaches.

Students construct personal meaning and develop understanding through interaction with a problem during mathematical activities, such as solving a sequence of tasks. The *mathematical activity context* includes the tasks students engage in solving as well as the way they are engaged (Boaler, 1993).

Much of the research in early algebra using a function approach has been conducted in the context of engaging students in specific generalizing activities and/or using specific representations of constant growth functions as reasoning tools. In these contexts, students may be guided or channeled into particular reasoning approaches using particular representations of the function, such as function tables (Blanton, 2008;

Schliemann, Carraher & Brizuela, 2007) or geometric pattern object representations (Moss and colleagues, 2006, 2008; Warren & Cooper, 2008).

This teaching intervention research varied in the degree to which students were encouraged to generate and use other representations as reasoning tools. Blanton (2008), for example, reports results of young elementary school students (i.e., K-3) drawing pictures that were effective algebraic reasoning tools for the students on a story problem about dogs. And, Moss and colleagues (2006, 2008) report on older elementary students constructing geometric figures with objects and how these activities seemed to support students in generalizing constant growth functions. However, research is sparse on what reasoning tools older elementary students might generate when solving constant growth pattern problems and possible connections to their ability to generalize.

The primary difference between these research studies is the nature of the generalizing activity. For example, Schliemann, Carraher and Brizuela (2007) and Blanton and Kaput (2005) believe students develop conceptual understanding of variable and relationship as they engage in activities to generalize from a set of particular cases of a constant growth function. Other researchers including Boaler (1993) believe students develop conceptual understanding as they construct meaningful (to them) reasoning approaches to solve tasks. The geometric patterning activity approach (e.g., Moss, Beatty, Barkin & Shillolo, 2008; Warren & Cooper, 2008) is more aligned with Boaler's (1993) perspective since its focus is on student discovery of patterns as they generate object or notational representations as reasoning tools. However, these researchers have concentrated on use of geometric pattern representations of constant growth functions.

Research is sparse on student generalization of numeric sequence problems, although Moss (2005) reported on scaffolding student algebraic reasoning on numeric sequence representations through integration with geometric patterning activities.

Summary and Research Questions

In summary, more research is needed on how elementary students engage in solving constant growth function problems in a mathematical activity context which encourages students to develop their own reasoning approaches and generate their own representations, allowing a glimpse into their cognition. Specifically, research is needed on connections between different types of constant growth function problems (i.e., story, geometric pattern, numeric sequence) and student algebraic reasoning, including generalization. Research is sparse on how individual students engage in generalization depending on how the mathematical information is represented in the problem.

Because this research study with individual students will be conducted in a context open to student choice of representations and reasoning approaches, with minimal intervention, I assume that students will generate a variety of representations in solving tasks to find particular cases of the function. I also assume that students will vary in their ability to generalize and to symbolize a generalization (i.e., they may or may not generalize, and if they do, they may express it using a variety of unique representations including written words, oral words and gestures, algebraic notations, or some unique combination of forms of representation).

Thus, the potential for contributions to the early algebra function approach research field include: (1) identifying connections between different problem representations of constant growth functions and student reasoning approaches to finding particular cases; (2) adding to the research on student generalization on constant growth functions by identifying connections between student reasoning approaches to finding particular cases and student generalization; and (3) identifying representations which students who are able to generalize use as reasoning tools.

In conclusion, the primary purpose of this research project is to identify connections between problem representations of constant growth functions, student representations and student algebraic reasoning. A secondary purpose is to examine the connections between student work finding particular cases of the function and student representations of their generalizations in those instances in which students are able to generalize.

The overarching research question is:

What are the relationships between problem representations of constant growth functions and students' algebraic reasoning?

Sub-question (1): How do students use the problem representation to find the 10^{th} case in the pattern?

Sub-question (2): How do students use their work on finding the 10^{th} case to generalize the pattern?

CHAPTER THREE

STUDY DESIGN AND METHODS

Overview

In this chapter, I first describe the study design for exploring the relationships between problem representations of linear growth functions and student algebraic reasoning. I then provide details about development of materials for this study (i.e., the constant growth function problems and the task sequence for each problem) before describing the site and participants, the data collection methods, and the analysis methods employed in this study.

Study Design

The purpose of this research study is to examine the connections between different representations of constant growth patterns and student reasoning approaches. Since the focus is on individual construction of reasoning approaches, clinical interview methods are necessary to "capture the distinctive nature of the child's thought" (Ginsberg, 1997, p. 58). Ginsberg notes that clinical interviews are useful for understanding the personal context of an individual student's thinking and for examining shifts in student's thinking, both of which are important in this research study. Ginsberg details two important considerations for conducting clinical interviews, establishing a relationship between interviewer and participant and positioning the participant as expert. To address these considerations, I incorporated mathematics class observations and a background interview prior to clinical interviews with participants.

To meet the goal of positioning the student as expert as well as the goal of establishing rapport before the task-based interviews, I knew it was important for the students to become familiar with me in the role of someone interested in student thinking. I incorporated into my research study design twice weekly classroom observations before the consenting process, during which students would become familiar with me in that role (e.g., asking students about their work and what they were thinking about mathematics problems but not suggesting reasoning approaches). I then presented a brief explanation of my research project to the whole class in terms of seeking their help on my research, to further serve the purpose of positioning the students as experts who could help me in my research on student mathematics problem solving.

To establish a relationship with individual participants based on trust and mutual respect and to develop comfort in talking together (Ginsberg, 1997), I developed a protocol for an initial background interview in which the student answered questions focused primarily on the student's prior schooling (e.g., how long at current school, prior schools attended) and interests in outside of school activities (e.g., games, sports, hobbies) and how these activities might involve mathematics (see Background interview protocol, Appendix A-1). I developed mathematical problems based on the data collected in these interviews, but reporting on those results is beyond the scope of this dissertation. To gauge the comfort level of the individual participants in reading mathematics problems aloud and expressing their thoughts, I included a segment in which the student would read aloud three of the constant growth pattern problems and state whether they thought particular tasks within each would be easy, medium or hard to solve.

Task-based interviews. A task-based interview is a form of clinical interview in which the student engages in solving content-specific tasks. This interview approach, grounded in Piaget’s methodology for understanding student strategies and thinking, presents students with particular tasks to be solved and then asks the student to explain the specific methods she or he just used in a concrete problem. The advantage of this approach is that it “gives the child the opportunity to think about something ’concrete’ and to introspect about mental activities transpiring in the immediate present” (Ginsberg, 1997, p. 123).

In this study, I used audio- and video-taped task-based interviews with individual students as a primary method of data collection because I needed to collect not only student work artifacts but also student oral word and gesture representations explaining their work on specific mathematical tasks. I also needed to be able to ask follow-up prompts to probe instances of emergent algebraic reasoning since I assumed that students would vary in their zones of proximal development (Vygotsky, 1978). I constructed a semi-structured interview protocol (see Appendix A-2) with sample follow-up prompts (see Appendix A-3) to accomplish two objectives: (1) consistency for systematic study of student reasoning approaches on tasks across problems (e.g., for each task, first read aloud then solve/show work then explain aloud); and (2) flexibility for probing reasoning or clarifying how the student used a representation (e.g., follow-up prompts). I expected that some students might talk aloud as they solved tasks, similar to interview situations in which education researchers use “think-aloud” protocols (Celedón-Pattichis, 1999; Schoenfeld, 1985), while others might say little as they work or during their explanations and therefore need follow-up prompts to fully explain their work on a task.

In summary, student-generated representations on tasks within each problem, including not only written notations but also oral word and gesture representations, comprise the main data source for the analysis in this dissertation of relationships between problem representations of constant growth functions and student engagement in algebraic reasoning. This consideration informed the research design to encourage student generation of notational representations--the problem as presented to the students had reminders to "show your work" and had space for students to show their work for each task. Further, the interview protocol included prompts to explain their work, thus encouraging oral word and gesture representations.

Constant growth function problems. In this section, I describe construction of constant growth problems for this study. In this dissertation, I use the term *problem* to refer to a representation of a constant growth function. Rather than referring to a single *task*, I refer to *tasks* the student solves for each problem. Thus, it is useful to think of each task-based interview as consisting of solving a set of similar tasks on different constant growth problems, following a semi-structured protocol for each task. The rationale for the set of tasks is described following discussion of the problems.

As described in the previous chapter, recent research demonstrating the capacity of elementary school students to reason algebraically about constant growth functions used story problems with familiar measured quantities (Blanton, 2008; Schliemann, Carraher & Brizuela, 2007), geometric pattern problems (Moss, Beatty, Barkin, & Shillolo, 2008; Warren & Cooper, 2008), and both geometric and numeric pattern problems (Moss, 2005). I first searched publicly available fourth grade items from the

National Assessment of Educational Progress (NAEP) for constant growth problems. I selected a numeric sequence item from the grade 4 mathematics NAEP main assessment but did not find a story problem with familiar measured quantities nor did I find a geometric pattern problem representing a constant growth function. Being familiar with *Investigations* (2008), a research-based curriculum developed by TERC with funding from the National Science Foundation, I decided to use two such representations of constant growth functions from the fourth grade unit on patterns, functions, and change. In the sections that follow, I describe problems used in this study, and then describe the rationale for the tasks students solved for each problem.

Penny jar story problem. For the story problem with familiar measured quantities, I selected the penny jar story problem from the fourth grade investigation unit on patterns, functions, and change (2008) because it: (1) represents a linear growth function of the form $y=mx+b$, where b is an additive constant; (2) involves familiar measured quantities for the variable quantities, i.e., days and pennies; and (3) can be represented in linguistically simplified form. Linguistic complexity is an important factor in the design of assessment items since it has been shown to have a disproportional effect on the performance of English language learners on mathematics assessments (Abedi & Lord, 2001). I knew based on the population of the school that I might be working with students who spoke another language as their first language. I did not want reading or understanding English to be an impediment to working on the problem. Therefore, in constructing the written word representation of the penny jar story problem, I used research-based considerations to reduce linguistic complexity such as using verbs in present tense and commonly used words and minimizing the length of prepositional phrases (Abedi & Lord, 2001), and to enhance reading one sentence at a time with understanding (Martiniello, 2008).

In the penny jar story problem, the first sentence introduces the story, “Marisa wants to save pennies.” The second sentence (on the next line) represents the additive constant, 4 pennies: “Her mother gives her a jar with 4 pennies in it to start.” The rate of change, m, is represented by the next sentence (again on a new line) “Marisa adds 3 pennies to the jar each day (Day 1, Day 2, Day 3, …).” This sentence clarified that Marisa adds pennies starting on day 1, so as to avoid student confusion regarding the day number when Marisa begins to add pennies. Thus, the function represented using a written word representation in the penny jar story problem is $f(x)=3x+4$. The penny jar story problem with familiar measured quantities is shown in Appendix 3-4, Model Problem A.

Geometric pattern problems. For the geometric pattern problem, I selected the tower problem from the same fourth grade investigation unit on patterns, functions and change (2008) because it: (1) represents a linear function of the form $f(x)=mx+b$, where b is the additive constant; (2) represents the figures using the familiar measured quantity, blocks. I included simple labels beneath each diagram of the first three figures, i.e., “figure 1,” “figure 2,” “figure 3.” In this geometric tower problem, the additive constant is represented as a bottom row of 4 blocks. The rate of change is represented as a row of 2 blocks, with one row in figure 1, two rows in figure 2 and three rows in figure 3. Thus, the function represented by this representation of the first three figures of a geometric pattern is f(x) =2x+4, where y is the number of blocks in any figure and x is the figure number. The geometric tower problem is shown in Appendix 3-4, Model Problem B.

I designed another geometric pattern problem with a different orientation. I wanted to explore the possibility that students would interpret geometric patterns differently depending on the way the mathematical information is represented in the pictures of the first three figures. Again, I used the variable quantities “blocks” and “figure number” for the dependent and independent variables y and x, respectively. The additive constant in this geometric pattern representation is a last column of 3 blocks in each figure; the rate of change is represented as a column of 5 blocks, with one column in figure 1, two columns in figure 2 and three columns in figure 3. Thus, the function represented by the geometric array problem, shown in Appendix 3-4, Model Problem G, is f(x)=5x+3.

Numeric sequence problem. The third form of representation of a constant growth function selected for this study was a numeric sequence representation from a publicly released NAEP mathematics items (grade 4 level). In a numeric sequence representation, only the first three values of the dependent variable quantity (i.e., the first three terms) are represented. The problem selected had the first three terms “4, 7, 10” followed by blanks and an ellipsis indicating that the sequence continued under the same rule. Thus, the function represented by the numeric sequence problem, shown in Appendix 3-4 Model Problem C, is f(x)=3x+1.

Validation problems. As mentioned previously, a search of the publicly released items of the fourth grade NAEP main mathematics assessment yielded few items representing constant growth functions other than one story problem and several numeric sequence problems. The story problem item did not meet the criteria of representing a constant growth function with familiar variable quantities: it related cricket chirps per minute to temperature in degrees. However, I included this open-response problem and one of the multiple choice numeric sequence items to validate student participant performance on age-appropriate constant growth function tasks (see Appendix A-4, model problem J, extending a function table three cases and model problem H, extending a numeric sequence two terms). Student work on these two problems was not a data source for the research analysis addressing connections between problem representations and student algebraic reasoning nor was it used to validate results of my findings.

Task sequence. Elementary school mathematics standards (NCTM, 2000) in the patterns, relations, and functions strand specify that students should be able to describe a pattern and find specific instances, and possibly write a general rule. The latter activity would presumably depend on the student's development of conceptual understanding of variable and relationship and ability to generalize. My design of the sequence of tasks for students to solve on each of the different constant growth pattern problems used in this study (i.e., model problems A, B, C, and G) was informed by both the NCTM standards and considerations for opening the mathematical activity context to student choice of reasoning approaches (Boaler, 1993). For reasons described earlier in the discussion of the penny jar story problem construction, I used Abedi and Lord's (2001) recommendations to reduce linguistic complexity in the written word representations of the tasks. Specifically, I used imperative sentences with verbs in present tense and minimized use and complexity of prepositional phrases.

Describe the pattern. The first task opens the mathematical activity context to student choice of representations: “Describe the pattern using words, pictures or objects.” This is the only task with the same wording across different problems (i.e., the story, geometric and numeric sequence problems). The representations students generate on this task would thus provide data on how each individual student initially interprets each constant growth pattern.

Find the fourth case in the pattern. The second task in the task sequence is modeled after a task commonly presented to students with numeric sequence and geometric sequence representations of a constant growth function. In these representations, the first three cases are represented and the task to find the fourth case is framed as finding the next case in the pattern. This task takes on a different meaning in story problems which do not include the first three cases, such as model problem A, the penny jar story problem. A student might find the fourth case by representing or finding the first three cases or by using the relationship between the two variable quantities. This task is designed to provide additional data on the relationship between the problem representation and student reasoning approaches and allow analysis of shifts in reasoning approaches.

Unlike the first task, this task is phrased differently in each problem:

- “Find the fourth term in the pattern.” (numeric sequence problem)
- “Find the number of blocks in figure 4 of the pattern.” (geometric problems)
- “Find the number of pennies in the jar after Marisa adds pennies on day 4.” (story

problem)

Find the tenth case in the pattern. The third task in the sequence asks students to find the 10th case in the pattern. This task can be solved by extending the pattern (i.e., finding all the intervening cases) or by using a reasoning approach based on the relationship between the two variable quantities. This task was designed to provide additional data on the relationship between the problem representation and student reasoning approaches and allow analysis of shifts in reasoning approaches. This task is phrased similarly to the task to find the fourth case in the pattern:

- "Find the tenth term in the pattern." (numeric sequence problem)
- "Find the number of blocks in figure 10 of the pattern." (geometric problems)
- "Find the number of pennies in the jar after Marisa adds pennies on day 10." (story problem)

Write a rule to find any case in the pattern. The fourth task in the task sequence is the generalization task. This is of particular interest in this study since being able to write a rule to find any case in the pattern, also known as the general rule, is indicative of algebraic reasoning. Since the generalization task follows student work on solving three other tasks, this task sequence provides data to analyze how students rely upon (or not) their work and representations on the prior tasks.

This task is phrased differently depending on the variable quantities shown italicized:

- "Write a rule to find any *term* in the pattern." (numeric sequence problem)
- "Write a rule to find the *number of blocks* in *any figure* of the pattern." (geometric

problems)

- “Write a rule to find the *number of pennies* in the jar after *any number of days* (story problem)

Find the 100th case in the pattern. I added this task to find a far case as an alternative task for students who did not understand the task to write a rule to find any case in the pattern (e.g., assigned a specific value to what was intended to be a variable quantity). Student work on this task provided data on the relationships underlying student reasoning approaches, i.e., what relationships the students were taking into account when they solved for the 100th case in the pattern, what Vergnaud (1988) terms *theorems-in-action.* This allows for analysis of student reasoning approaches when presented with a task for which scalar reasoning approaches would be cumbersome, at best, and on which students may therefore shift in their reasoning approaches. In contrast to the other tasks which were presented to the students in written word form, I presented this task to students orally as interview time allowed. Therefore, not all students were presented with this task and particular students were presented it on some but not all problems.

In summary, the task sequence I designed provides the data necessary to analyze connections between the problem representation, student work on solving tasks (i.e., student-generated representations and student-constructed reasoning approaches) and student algebraic reasoning, specifically generalization. By asking students to work through intermediate cases, I crafted the possibility that students would build representations and ideas they might draw upon and that I would be able to analyze how student reasoning changed as the number of the case increased. It allows for collection of data necessary to analyze student reasoning approaches on specific tasks and shifts in reasoning approaches across tasks within each problem. Finally, it provides data to analyze student ability to generalize in relation to the different problem representations of constant growth functions and in relation to the different representations and reasoning approaches students constructed on prior tasks.

In the next section, details on data collection and analysis methods follow descriptions of the study site and participants.

Methods

Site. The site was a public elementary school in an urban school district in south central Arizona. I selected this school because I had participated in pilot research studies at this school involving students and teachers the prior school year. The school district serves a predominantly Latino population (88%) of approximately 17,000 K-12 students, 33% of whom are English language learners (ELLs) and 86% of whom received free or reduced-price meals. The site was a Title I school serving 725 students in 2007-08. Half of these students were either classified as English language learners (33%) or had been reclassified proficient in English (16%) based on student scores on the Arizona ELL Assessment (AZELLA). This assessment purports to classify students into four categories: Initially Fluent in English, Basic (conversational), Intermediate, and Reclassified English Proficient (former English language learners now testing within the normal range of fluency). On Arizona's standardized test, the Arizona Instrument to Measure Standards (AIMS), administered in Spring 2008, 55% of tested students at this school site met proficiency on the AIMS mathematics assessment, 56%, the reading, and

73% the writing.

Participants. I selected a fifth grade classroom for the study that was a regular classroom (i.e., one teacher instructed the children in multiple subjects each day, including mathematics) whose teacher was open to hosting a research project which would involve observations of mathematics classes and pull-out of individual students during the school day for the three interviews. There were 25 students in this fifth grade classroom. Nine of these 25 students were consented and participated in the study and none withdrew during the data collection phase. The interviews with each of the nine participants occurred approximately once per week for 30 to 50 minutes, during the last month of the school year. These nine participants reflected a range of mathematics quarterly benchmark test scores. Three students were in the top third of their class, three in the middle third, and three in the lower third. The participants also represented a range of English language proficiency classifications based on the AZELLA: one student was currently classified Intermediate ELL after having been classified as Reclassified English proficient; one student was currently classified Basic ELL after having been classified Intermediate ELL; two students were classified Reclassified English Proficient; two students were classified Initially Fluent in English; and three students were classified non-ELL (i.e., they were never administered the AZELLA because they were presumed to be fluent in English based on that being the language used in the home). Thus, although this was a convenience sample, it did reflect a range of student proficiency in fifth-grade mathematics and English language.

Data collection. The research design for data collection addressed the following considerations: (1) positioning the student as the expert and the interviewer as a person interested in understanding the student's thinking; (2) allowing students opportunities to explore each problem representation to develop personal meaning and to construct reasoning approaches based on the understanding they develop through exploration of the pattern; (3) encouraging the student to generate representations including not only notational representations of their work on the tasks in each problem but also oral word and gesture representations explaining their reasoning approaches; and (4) collecting data on student representations including notational representations (i.e., student work artifacts) and oral word and gesture representations (i.e., videotapes and transcripts of interviews).

During mathematics classroom observations, I took field notes focused on student participation, mathematics content and problem solving activities. Student work samples were collected from each consented student participant for use in the background interview. I collected data on the English language proficiency classifications of the consented students from site administrators prior to interviewing the students. I also collected data on the mathematics proficiency demonstrated by consented students on quarterly benchmark tests from the students' fifth grade teacher at the conclusion of the study during an interview that I audiotaped and transcribed. Thus, I knew the English language proficiency classification of the students when I interviewed them, but not their level of mathematics proficiency as indicated by their performance of quarterly benchmark testing. This reduced the possibility that my analysis of data collected in the interviews would be influenced by knowledge of student performance on benchmark tests.

I interviewed each student participant individually three times during the last month of the 2009-2010 school year. Each interview lasted approximately 30-60 minutes. Interviews were conducted in English (the students' language of mathematics teaching and learning). Student interviews were video- and audio-taped by trained videographers, and transcribed by trained educational research transcribers. I also viewed the videotapes and modified the transcripts as necessary to fully capture the student's oral word and gesture representations. I also collected all student work artifacts from the interviews.

The three interviews (i.e., a background interview and two task-based interviews), which were spaced approximately one week apart, are next described in terms of data collected.

The first interview consisted of three parts: (1) background questions about schools attended, use of English and Spanish inside school, specifically in mathematics class, and outside of school (see Appendix 3-1 for background interview protocol and sample questions); (2) questions about a sample of their class work to see how the students described their mathematics work; and (3) questions about three model problems (see Appendix 3-4 for model problems A, B, and C). In this third part, I presented the three problems one at a time, randomizing the order for each student. The student participants read each problem aloud and were asked to identify any words someone in their mathematics class might not know. If a student identified any words, I asked the student what the word(s) meant. I then asked the students whether they thought the tasks would be easy, medium, or hard, but the students did not solve the tasks.

After the individual background interview, each participant engaged in solving constant growth function problems during two individual task-based interviews. Two task-based interviews were necessary to allow ample time for students to engage in developing personal meaning and understanding, to generate their own representations and to explain their reasoning approaches. During the task-based interviews, materials were available for the participants' use including grid paper and manipulatives (i.e., pennies and a jar, pattern blocks, and connecting cubes). I interviewed each participant in a meeting room adjacent to the school library during school hours, usually in the afternoon.

In the first task-based interview, student participants read aloud and solved each task in model problems A, B, and C (see Appendix 3-4). The problems were randomized and presented one at a time. I used the semi-structured task-based interview protocol (see Appendix 3-2) for each task within each problem. I used follow-up prompts to ask students to explain their solutions (see Appendix 3-3).

In the second task-based interview, student participants read aloud and solved each task in model problems G, H, and J (see Appendix 3-4) randomized and presented one at a time. I used the same semi-structured task-based interview protocol as in the first task-based interview for each task within each problem and the same follow-up prompts.

Analyzing data for connections. This is an exploratory study and data analysis followed a qualitative analysis approach based on category and theme development

(Bogdan & Biklen, 2003). I began with a coding scheme based on my theoretical framework (i.e., Vergnaud's categorization of reasoning approaches as scalar or functional, Goldin's definition of representation).

I conducted the analysis in multiple passes through the data, with continued category and theme development coincident with writing interpretations of the data. Examples of data analysis methods used in this study are shown in Table 3-1. I first organized student work artifacts and transcript and videotape data by mathematical problems and research questions. I conducted multiple passes through the data with ongoing category development, analysis of emerging themes and writing preliminary findings.

Table 3-1. Data Sources and Data Analysis Methods for Major Research Question: What are the relationships between different problem representations of constant growth functions and student algebraic reasoning?

RESEARCH SUBQUESTION	DATA	ANALYSIS	EXAMPLES
(1) How do students use the problem representation to find the tenth case in the pattern?	Videotapes, transcripts of task-based interviews; student work artifacts.	Categorization of reasoning approach depending on representation of the additive constant in the problem representation and the student's representation. Analysis of emerging themes	Additive constant separately represented in problem representation and student representation; Additive constant separate in problem representation but not in student representation. Connections between the problem representation and student-generated representations (e.g., pictures, tabular)
(2) How do students use the problem representation and their work on finding the 10th case in the pattern to generalize?	Videotapes, transcripts of task-based interviews; student work artifacts.	Categorization of student responses on task to write a rule depending on their work finding the 10th case. Analysis of emerging themes	Student writes/does not write a general rule after using a reasoning approach with the additive constant separate/not separate. Form of representation used by student to write a rule (e.g., written word in terms of the variable quantities, letter-symbolic equation). Connections between the student's work on the 10th case and the student's work on the task to write a rule.

Since the students' representations (i.e., written notations, objects, oral words and gestures) were the data source for analysis of the relationships between different problem representations of constant growth functions and student algebraic reasoning, I first focused on analyzing the connections between the problem representations and the representations the student generated in solving tasks on each problem. I continually developed categories related to representations and reasoning while also writing preliminary findings on connections between the problem representation of the constant growth function and the student representations and reasoning on the tasks to describe the pattern, find increasingly larger cases and write a general rule to find any case in the pattern.

As themes emerged on potential connections, I focused on category development in these areas and looked for counter-examples. I continued to refine my preliminary writing to focus on important themes that emerged related to student algebraic reasoning depending on the problem representation of a constant growth function. Since interpretation of the data is the major factor impacting the validity of findings, two mathematics education researchers were closely involved during this phase of data analysis, reading preliminary findings in which I presented possible interpretations of the data and engaging in discussions with me of themes I identified, limitations and interpretations of potential counter-examples.

Finally, a coherent argument emerged from the themes, leading to draft findings on connections between different problem representations of constant growth functions and student algebraic reasoning. The same two mathematics education researchers continued to provide comments based on the understanding of the data they had developed through their involvement in the data analysis and preliminary writing phase.

CHAPTER FOUR

CONNECTIONS BETWEEN PROBLEM REPRESENTATIONS AND STUDENT REASONING TO FIND THE TENTH TERM

This chapter explores connections between student reasoning approaches on the task to find the 10^{th} term in a constant growth pattern and four different representations of constant growth patterns, specifically a story problem, a numeric sequence problem and two geometric pattern problems. These four problems differed in how the mathematical information about the constant growth function was represented.

Although the focus of this chapter is on the task to find the 10^{th} term, in my analysis I traced connections to the student's work on the two prior tasks (i.e., to describe the pattern and to find the 4^{th} term). Where these connections were important for understanding the student's reasoning approach, I included the connections in the discussion of findings. I also identified instances in which students shifted in their reasoning approach from one task to another, particularly on the task to find the 10^{th} term after finding the 4^{th} term in the pattern. I interweave this theme of shifts in reasoning approaches throughout the discussion of findings related to connections between the problem representations of constant growth patterns and student reasoning approaches.

As described in Chapter Two, there is a difference between the constant growth functions in this study and those which Vergnaud (1988) analyzed (i.e., $f(x) = ax$ where a is a constant of proportionality and x is the independent variable). The constant growth functions solved by students in this dissertation study have both a variable component (i.e., mx, where m is a constant rate of change and x is the independent variable) and a fixed component known as the additive constant (i.e., $f(x) = mx + b$, where b is the additive constant). Reasoning about the variable component of a constant growth function involves reasoning about the relationship between two variable quantities and is thus algebraic in nature. Thus, this dissertation makes an important addition to Vergnaud's work because results indicated that attending to the additive constant seemed to be connected in important ways to algebraic reasoning about constant growth functions.

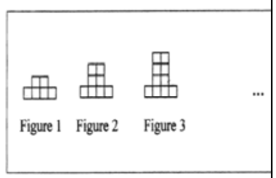
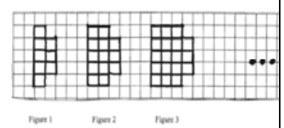
I begin with an overview of how each of the four problems used in this dissertation represented the mathematical information about a constant growth pattern. In this chapter I will use both "constant growth patterns" and "linear functions" interchangeably to refer to the mathematical problems. Following this overview, I present results of student reasoning approaches to finding the 10^{th} term in the pattern on the different problems, organized by the major connection that emerged between the different problems and algebraic reasoning, students' use of the additive constant in their reasoning approaches.

Overview of Constant Growth Pattern Problems

This background section provides an overview of the different problem representations and the tasks. The problems differed in both form of representation and mathematical information conveyed.

Problem representations. Each problem represented a constant growth pattern where the value of one variable (e.g., the number of blocks in a figure or the number of pennies in a jar) depended on the value of another variable (e.g., the figure number or the number of days pennies are added to the jar). Because the number of blocks or number of pennies in the jar depended on the figure number or number of days, these variables are known as the *dependent variable* and the *independent variable*, respectively. The four problems and their associated tasks (see Appendix A-4) are displayed in Table 4-1.

Table 4-1. Constant Growth Pattern Problems.

Penny Jar Story Problem (A)	Geometric Tower Problem (B)	Numeric Sequence Problem (C)	Geometric Array Problem (G)
PROBLEM REPRESENTATIONS			
Written Words Marisa wants to save pennies. Her mother gives her a jar with 4 pennies in it to start. Marisa adds 3 pennies to the jar each day (day one, day two, day three, ...)	Geometric Pictures 	Numeric List 4, 7, 10, ___, ___, ___, ...	Geometric Pictures 
TASK WORDING			
1. Describe the pattern using words, pictures, or objects.	1. Describe the pattern using words, pictures, or objects.	1. Describe the pattern using words, pictures, or objects.	1. Describe the pattern using words, pictures, or objects.
2. Find the number of pennies in the jar after Marisa adds pennies on day four. Show your work.	2. Find the number of blocks in Figure 4 of the pattern. Show your work.	2. Find the fourth term in the pattern. Show your work.	2. Find the number of blocks in Figure 4 of the pattern. Show your work.
3. Find the number of pennies in the jar after Marisa adds pennies on day ten. Show your work.	3. Find the number of blocks in Figure 10 of the pattern. Show your work.	3. Find the tenth term in the pattern. Show your work.	3. Find the number of blocks in Figure 10 of the pattern. Show your work.
4. Write a rule to find the number of pennies in the jar after any number of days.	4. Write a rule to find the number of blocks in any figure of the pattern.	4. Write a rule to find any term in the pattern.	4. Write a rule to find the number of blocks in any figure of the pattern.

In the penny jar story problem, a child named Marisa adds three pennies to the jar each day. The three pennies per day is known as the *rate of change*. In the geometric pattern problems, there are a certain number of blocks being added from one figure to the next. The rate of change is represented as a row of 2 blocks in the geometric tower pattern and a column of 5 blocks in the geometric array pattern. The numeric sequence problem represents an arithmetic sequence in which the rate of change is 3. The rate of change is not represented explicitly but can be detected by students as the (constant) difference between consecutive terms given a numeric list of the first three terms.

In this study, the constant growth patterns also have a component that does not vary, the *additive constant*. In the penny jar story problem, it is the 4 pennies Marisa's mother gave her to start. In the geometric tower pattern, it is a bottom row of 4 blocks and in the geometric array pattern, it is the last column of 3 blocks that is the same in each figure. The additive constant in the numeric sequence problem is equal to 1 but it is implicit (i.e., not shown). The value of the additive constant is embedded in the first three terms (i.e., the first term is 4, the sum of 3 and 1; the second term is 7, the sum of the first term plus the rate of change, 3).

In summary, each problem in this study represented a constant growth pattern with a variable component (the rate of change times the value of the independent variable) and a fixed component (the additive constant). Mathematically, the different problems represented the following constant growth patterns, where x stands for the independent variable quantity and $f(x)$ stands for the dependent variable quantity:

- penny jar story problem $f(x) = 3x+4$

- Numeric sequence $f(x) = 3x+1$
- geometric tower pattern $f(x) = 2x+4$
- geometric array pattern $f(x) = 5x+3$.

Mathematical information in different problem representations. This section, summarizes the different ways in which the mathematical information about a constant growth pattern was represented in the four types of problems students solved in this study:

- The numeric sequence problem represented explicitly only the first three values of the dependent variable;
- Both the geometric tower and the geometric array problems represented the first three values of the independent variable as figure numbers (figure 1, figure 2, figure 3) and the first three values of the dependent variable as three geometric figures comprised of individual blocks with the additive constant and the rate of change represented as rows or columns of blocks, respectively, as already described;
- The penny jar story problem represented explicitly the first three values of the independent variable as day numbers (i.e., day 1, day 2, day 3), the additive constant (4 pennies in the jar to start) and the rate of change (3 pennies each day).

Tasks. Each problem had 4 tasks for students to solve. The wording of the first task was the same, but the wording of the other tasks depended on the problem (shown in parentheses):

- Task 1 Describe the pattern using words, pictures, or objects;

- Task 2 Find the (number of pennies in the jar on day 4, number of blocks in figure 4, fourth term in the pattern);
- Task 3 Find the (number of pennies in the jar on day 10, number of blocks in figure 10, tenth term in the pattern);
- Task 4 Write a rule to find (the number of pennies in the jar after any number of days, the number of blocks in any figure in the pattern, any term in the pattern).

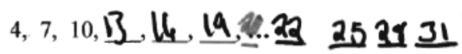
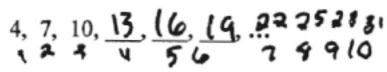
Student Reasoning Approaches to Finding the Tenth Term

I expected different reasoning approaches on the different constant growth pattern problems because the problems differed in how the mathematical information was represented as described in the previous section. This section on student reasoning approaches on the task to find the 10^{th} term in a constant growth pattern goes beyond examining connections between different representations of the pattern and student reasoning approaches to analyze what students leveraged when they engaged in algebraic reasoning. In a mathematical activity context open to student choice of representations and reasoning, I found that students generated a variety of notational representations based on their personal understanding of the problem and tasks. The findings in this chapter are grounded in these written representations as well as the oral words and gestures a student used to explain his or her work on finding the 10^{th} term.

The focus of this dissertation is on algebraic reasoning. I briefly illustrate typical examples of arithmetic reasoning on the numeric sequence problem then describe in greater depth the algebraic reasoning approaches students used to find the 10^{th} term in the pattern on this and other problems.

Arithmetic reasoning. I include two examples of arithmetic reasoning in order to demonstrate the contrast between arithmetic and algebraic reasoning. Some fifth graders in this study found the 10^{th} term in a constant growth pattern operating only on the term values. Two examples are shown in Figure 4-1.

Figure 4-1. Two Examples of Arithmetic Reasoning on the Numeric Sequence Problem.

Tomás 	"I have six here (Points to the list) then one, two, three, four. (Draws a new blank as he says each number) Nineteen, twenty-two, twenty-five, twenty-eight, thirty-one."
Hugo 	Hugo: "Mm, so it would be like this is one, this is two, and this is three and this is four? (Points to the numbers four, seven, ten and thirteen, respectively) Kathleen: Mhmm. Hugo: (Writes the numbers one through ten in the bottom of the box, representing term numbers) "Thirteen, sixteen, seventeen, eighteen, nineteen. (Talking to himself as he writes.) Twenty-five, twenty-eight, thirty-one."

Tomás began his work on finding the 10th term by counting the number of blanks in the problem representation and drawing in four more blanks. Hugo began by writing the numbers 1 through 10 as shown in Figure 4-1. Then they extended the term values using repeated addition of 3. Neither of these students was reasoning that the term value went up by 3 when the term number went up by 1, a form of algebraic reasoning about the relationship between two variable quantities. Rather, they were reasoning only about one quantity, the terms in a numeric list. These reasoning approaches are similar to those reported in prior research with elementary school students working with function tables, in which the students operated on the term numbers and term values as separate patterns (Schliemann, Carraher & Brizuela, 2007; Smith, 2003).

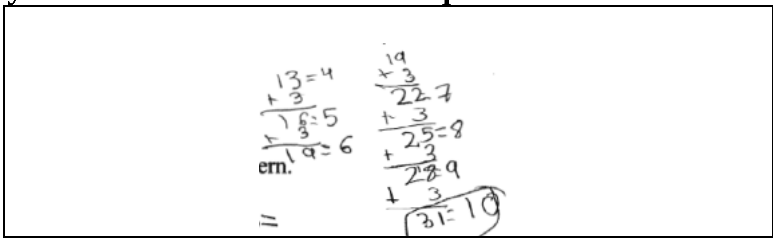
Students could also solve the penny jar story problem and the geometric pattern problems using arithmetic reasoning. On these problems, there were instances in which students generated a numeric list of the first three terms using the mathematical information in the problem, then solved for the 10^{th} term using arithmetic reasoning approaches similar to the two examples just described.

These arithmetic approaches were provided as a contrast for the focal discussion of this chapter, student algebraic reasoning approaches. In the next section on algebraic reasoning, I will give one more example in order to further emphasize the difference between arithmetic reasoning and scalar reasoning.

Algebraic reasoning. Algebraic reasoning about a constant growth function involves reasoning about an underlying relationship between two variable quantities, the term number and the term value. There are two categories of algebraic reasoning approaches: scalar and functional. Reasoning is categorized as scalar if it involves operating on quantities of the same kind such as pennies in the story problem or blocks in the geometric pattern problems. The other category, functional reasoning, involves operating on quantities of a different kind, such as a rate of pennies per day times the number of days. There were instances of both kinds of algebraic reasoning in this study. I begin with Taylor’s work on the numeric sequence problem as the algebraic reasoning example of a student continuing the pattern one case at a time.

Scalar reasoning. Taylor’s work finding the tenth term is provided in Figure 4-2.

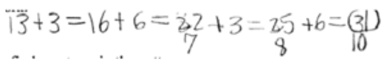
Figure 4-2. Taylor's Work on the Numeric Sequence Problem.



Notice the notations for the term numbers. Each time Taylor added 3 to the term values, he added 1 to the term numbers. Taylor's reasoning was algebraic since it was based on a relationship between the rates of change of the two variable quantities, specifically that the term value increased a certain amount, 3, for each increase of 1 in the term number. It was scalar because he was operating on magnitudes of the same kind. It is important to distinguish this algebraic reasoning approach from the arithmetic approaches used by students who wrote the term numbers first. It may be that reported difficulties generalizing constant growth pattern after scalar reasoning approaches may be due to the fact that students are actually reasoning arithmetically about a single pattern. Simply showing the term numbers should not be assumed to indicate algebraic reasoning.

A different and stronger example of algebraic reasoning is shown in Figure 4-3.

Figure 4-3. Juan's Work on the Numeric Sequence Problem

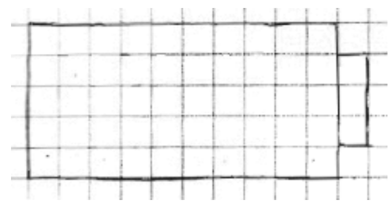
	"First I did thirteen plus three, which equals sixteen. Then plus six, which equals twenty-two and that's the seventh term. And so I did plus three again which equals twenty-five. Then I doubled it up again and I got thirty-one."

In contrast to the prior example in which Taylor extended the pattern one case at a time, Juan did not find every case between the 4^{th} and 10^{th} terms. As shown in Figure 4-3, Juan sometimes added 6 to the term value and 2 to the term number to find the 10^{th} term. In his explanation of his work, he described doubling the amount added to the terms from 3 to 6 and it is clear from his work that he made an appropriate change to the term numbers also (i.e., adding 2 or its equivalent, skipping term numbers). For example, when he added 6 to the value of the 8^{th} term, he added 2 to the term number, maintaining a ratio of 3 to 1 for the change in term values to the change in term numbers. Juan's reasoning was scalar since he was increasing term numbers and term values separately, thus operating on quantities of the same kind. Thus, it was a scalar form of algebraic reasoning about a constant growth function grounded in detection of a relationship between the rates of change of two variable quantities.

Juan's reasoning to extend the pattern is what Ellis (2007) calls an operating type of extending. Juan operated on the ratio of the change in the term value to the change in the term number. Juan was clearly reasoning about the relationship between two variable quantities as a ratio of their rates of change. This is a more powerful reasoning approach for finding far terms as will be shown in the next chapter.

Instances of students using scalar approaches to find the 10th term in a constant growth pattern were not limited to reasoning about two corresponding patterns (i.e., term numbers and term values). Hugo's work on the geometric array problem, shown in Figure 4-4, is quite different in terms of the quantities he was reasoning about.

Figure 4-4. Hugo's Work on the Geometric Array Problem.

Hugo: "Find the number of blocks in Figure 10 of the pattern…OK. It would be... let me see (Hugo draws the figure). Like that. **Kathleen**: OK, and how many blocks would that be? **Hugo**: Five, ten, fifteen, twenty, twenty-five, thirty, thirty-five, forty, forty five, fifty, fifty-three.	

Hugo drew figure 10 first as a rectangular area five blocks high by 10 blocks wide followed by a rectangular area one 3 blocks high by one block wide. Using this representation, Hugo counted the blocks in the first area by 5's. This was a scalar reasoning approach because he was reasoning about quantities of the same kind, blocks. What is different about his reasoning was that it was based on a relationship between the number of groups of 5 blocks and the figure number. He kept the additive constant (i.e., the 3 blocks that stays the same in each figure) separate and accounted for it in a final addition ("fifty, fifty-three").

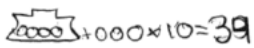
This was a very different scalar approach from Juan's. Hugo was reasoning about what I refer to as the *variable component*, the part of the pattern that depends on the value of the independent variable quantity, in this case the figure number. He was reasoning about how to find the variable component for a particular figure in the pattern directly from the figure number. In contrast, Juan was reasoning about the relationship between the rates of change of two corresponding patterns. Juan found values of intermediary terms as he extended both patterns. Both are forms of algebraic reasoning but, as we shall see in the next chapter, they led to different outcomes on the generalization task, suggesting that this distinction was significant in some important ways.

So far, we have seen three examples of types of scalar reasoning used by students to find the 10^{th} term in a constant growth pattern: two were based on extending the pattern (i.e., Taylor's continuing the pattern one case at a time, Juan's operating on the ratio to extend the pattern skipping cases and one was based on finding the term value directly (i.e., Hugo's counting the blocks in the variable component by 5's). They were scalar because the student was operating on quantities of the same kind.

There were also instances in which students reasoned about quantities of a different kind when they found the 10^{th} term as described next.

Functional reasoning. On the penny jar story problem, Taylor found the tenth term (i.e., the number of pennies on day 10) as shown in Figure 4-5.

Figure 4-5. Taylor's Work on the Penny Jar Story Problem.

	"I drew the jar with the four pennies again (referring to also beginning his work on the prior task with a similar picture) and then I did three times ten. *I did four plus three times ten equals thirty* [emphasis added]. Oh wait. I messed up (starts erasing). This is actually sixteen (changes his answer on task 2 from "12" to "16") and that's thirty-four (changes his answer on task 3 from "30" to "34.") Because you add this number (pointing to the picture of the jar with 4 pennies in it)."

Taylor began his work on finding the number of pennies on day 10 by drawing a picture of a jar with 4 pennies followed by an addition of 3 pennies times 10. He was thinking about the 10^{th} term (i.e., the number of pennies on day 10) as the sum of two parts: the 4 pennies in the jar to start with and the pennies Marisa added at a rate of 3 pennies each day. However, he actually multiplied 3 times 10 and wrote an answer of "30." It was during his explanation, also shown in Figure 4-5, that he realized he had made a mistake, erased his answer and wrote "34."

Taylor found the number of pennies Marisa added using multiplication. This was the variable component in the pattern, i.e., the part of the pattern that varied depending on the value of the independent variable, in this case, the number of days. Since he was operating on quantities of a different kind, 3 pennies each day times 10 days, this was functional reasoning. It was based on recognition of a direct relationship between the number of pennies Marisa added and the number of days. It is similar to Hugo's work on the geometric array in that both reasoned about the 10^{th} term as the sum of two parts: a fixed part and a variable part. It is different from Hugo's in how they found the value of the variable component. Hugo used a scalar approach, counting 10 columns of blocks by 5; Taylor used a functional approach, multiplying the rate of change times 10.

As we will see in the next chapter, this distinction-- how the student found the variable component--was less important than the fact that both solved for the 10th term directly from a value of 10 rather than finding intervening term values as Juan did in his solution to the numeric sequence problem. They did this by keeping the fixed value in the 10th term (e.g., the 4 pennies in the jar to start with, the 3 blocks at the end of each figure) separate and reasoning about how to find the other part of the 10th term, a variable component that depended on the independent variable quantity (e.g., the number of days, the figure number).

The four algebraic reasoning approaches presented differed in the quantities the students operated on in finding the 10th term and the relationship they were considering. On the numeric sequence problem, Juan and Taylor extended the pattern, operating on quantities of the same kind (i.e., term values and term numbers) which is scalar reasoning. On the geometric array problem, Hugo also operated on quantities of the same kind (i.e., number of blocks in a figure and figure numbers) and thus used scalar reasoning. But his work was based on a direct relationship between the term value (i.e., the number of blocks in figure 10) and the term (i.e., figure) number. On the penny jar story problem, Taylor also found the 10th term, the number of pennies on day 10, directly from 10, the number of days. But he used a functional approach to find the variable component, the pennies Marisa added. He multiplied the rate of 3 pennies each day times 10 days, quantities of a different kind. These differences were important to student success on the subsequent tasks to write a rule to find any term in the pattern and to find a far term, the 100^{th} term, the topic of the next chapter.

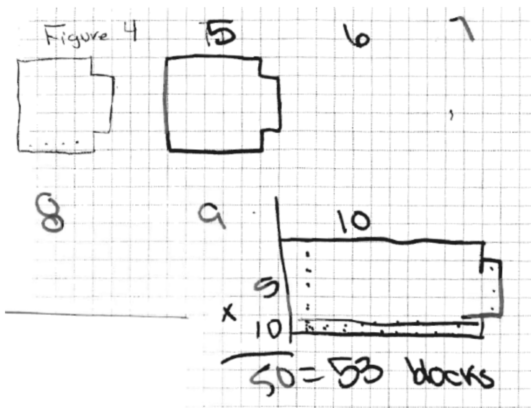
Before presenting findings on the connections between student algebraic reasoning and the four problems which represented a constant growth function differently (i.e., story, numeric sequence, geometric tower and geometric array), there is one additional reasoning approach to present which was used by students on the geometric array problem.

Emergent functional reasoning. Alejandra's work on the geometric array problem is shown in Figure 4-6.

Figure 4-6. Alejandra's Work on the Geometric Array Problem.

<table>
<tr>
<td>

Drawing of figure 10



</td>
<td>"I saw that in figure four there was four on the bottom and then three like always so I did three, and then I saw there was five on the bottom, then I did six, seven, eight, nine, and then ten. I did ten on the bottom, then the five up and then three over here."</td>
</tr>
</table>

Alejandra began solving for the number of blocks in figure 10 by writing figure numbers 5 through 10 as if she were going to draw each figure following her representation of figure 4 from the prior task. She then drew figure 10. As shown in the transcript segment, she was reasoning about the pattern as the sum of a fixed part (the 3 blocks that was the same in each figure) and a variable part. She used her drawing as a reasoning tool: "I did that because *I saw that this is ten (pointing to the bottom of figure 10, and then this is five (pointing to the left side of figure 10), so ten times five is fift*y [emphasis added], plus three equals fifty three." Her explanation indicated that she "saw" either a rectangular configuration of blocks or an array model of multiplication in her representation of the variable component (although she would not call it that) and multiplied the dimensions to find the number of blocks it contained.. Alejandra's work on the geometric array problem suggested that (1) she was multiplying the dimensions of the variable component and (2) she recognized that the number of blocks on the bottom (i.e., the length of the rectangle) was the same as the figure number. Other students who drew figure 10 on this problem also found the number of blocks in figure 10 using multiplication of 5 times 10.

Unless the student made explicit that he or she was multiplying a rate of change (i.e., 5 blocks per figure) times the figure number, one cannot be sure the student was using functional reasoning. I categorized Alejandra's reasoning as emergent functional since she did not make this explicit. As we shall see in the next chapter, this distinction was less important for generalization than the fact that the students reasoned about how to find the number of blocks in a figure directly from the figure number.

To summarize, students used one of four categories of algebraic reasoning to find the 10^{th} term in a constant growth pattern. The first category consisted of reasoning approaches in which the student extended the pattern to find the 10^{th} term (i.e., Juan and Taylor on the numeric sequence problem). The additive constant was implicit, i.e., embedded in the term values. These reasoning approaches were scalar since the student operated on quantities of the same kind (i.e., term values and term numbers). The student reasoned about the pattern as a relationship between the rates of change in the term values and term numbers. The other three categories consisted of reasoning approaches in which the student kept the additive constant separate and reasoned about a relationship between the variable component and the term number. They reasoned about how to find the 10^{th} term directly from the value of the independent variable quantity (i.e., the term number, 10). These three categories differed in whether the student used scalar, emergent functional or functional reasoning to find the variable component. These reasoning categories are shown in Table 4-2.

Table 4-2. Algebraic Reasoning on the Task to Find the Tenth Term.

	Additive Constant Separate	Additive Constant Not Separate
Scalar	*Geometric Array* Hugo	*Numeric Sequence* Juan (operating on ratio to extend pattern) Taylor (continuing to extend pattern)
Emergent Functional	*Geometric Array* Alejandra	_____
Functional	*Penny Jar Story* Taylor	------

The final section of this chapter is on connections between the different problems (story, geometric pattern, numeric sequence) and student algebraic reasoning approaches.

Connections Between Problem Representations and Student Algebraic Reasoning

The most significant connection was between explicit representation of the additive constant in the problem and student use of reasoning approaches with the additive constant separate. It was significant because students who kept the additive constant separate reasoned about the part of the function which varied (i.e., mx in $f(x) = mx + b$) and found the 10^{th} term directly from its term number, 10.

The numeric sequence problem did not show the additive constant; it displayed the first three term values whose values included the additive constant. On this problem, all the students extended the pattern to find the 10^{th} term. Depending on whether they reasoned about only one pattern (i.e., the term values) or reasoned about two related patterns (i.e., term values and term numbers), their reasoning was arithmetic or algebraic reasoning, respectively.

The story problem and geometric pattern problems did represent the additive constant explicitly, but not all students used reasoning approaches with it separate. Some students used the mathematical information in these problems to solve for the first three term values then used similar reasoning as described for the numeric sequence problem.

There were 11 instances in which students kept the additive constant separate and solved for the variable component of the 10^{th} term. These instances are important because the students found the 10^{th} term directly from its value, 10. The results by problem are shown in Table 4-3 and features that students leveraged in using such a reasoning approach are discussed next.

Table 4-3. Student Algebraic Reasoning with the Additive Constant Separate.

Penny Jar Story (n = 8)	4
Geometric Tower (n = 7)	1
Geometric Array (n = 8)	6
Total	11

Besides the explicit representation of the additive constant, there were other features of the problems which seemed to be connected to algebraic reasoning with the additive constant separate. One of these related to student use of picture representations. A picture representation of figure 10 was a particularly powerful reasoning tool for students on the geometric pattern problems. As already noted, the rectangular (array) configuration of blocks supported students in using multiplication to find the variable component and reasoning about how to find the 10^{th} term directly from 10, the term number.

One student used a picture representation on the geometric tower problem (in which the pattern grew vertically), corresponding to the one instance shown in Table 4-3 in which a student reasoned about the variable component on that problem. Four students used picture representations as reasoning tools on the geometric array problem (in which the pattern grew horizontally), including Alejandra and Hugo, whose work was described in this chapter, and these accounted for 4 of the 6 instances shown in Table 4-3.

In contrast, on the penny jar story problem, only Taylor generated a picture representation with the additive constant separate. The other three pictures generated by students on this problem modeled the actions described in the story in which a set of 3 pennies was added to the starting amount of 4 pennies in the jar each day. These pictures did not represent the additive constant separate and only led to students using a reasoning approach of counting all the pennies in their pictures (i.e., scalar with additive constant not separate). Thus, the connection between use of picture representations and reasoning approaches was not as definitive on the story problem as on the geometric pattern problems.

Finally, the evidence suggested that the value of 5 for the rate of change on the geometric array problem, coupled with the location of the additive constant, 3, as a last column of blocks, supported students in reasoning about the 10^{th} term directly from the figure number, 10. These aspects may account for the difference in reasoning approaches on the two geometric pattern problems. Students who began reasoning about this pattern by counting the number of blocks in the first three figures started on the left side of the figure. Thus, they were counting the variable component first on the geometric array problem and this led to detection of algebraically useful patterns relating the value of the variable component to the figure number. In contrast, all but one student counted the blocks in the first 3 figures of the geometric tower problem and used these term values as a starting point for finding the number of blocks in figure 4, then extended this pattern to find the number of blocks in figure 10.

In the next chapter, I extend and build upon the discussion of findings in this chapter as I discuss the connections between student work finding the 10^{th} term and student work on subsequent tasks to write a rule to find any term in the pattern (i.e., generalize the function) and to find a far term, the 100^{th} term. In this study, the fifth graders used their work on term 10 as reasoning tools on these subsequent tasks.

CHAPTER FIVE

CONNECTIONS BETWEEN STUDENT WORK FINDING THE TENTH TERM AND GENERALIZATION

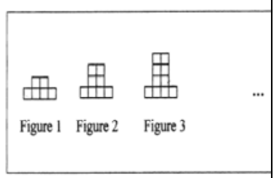
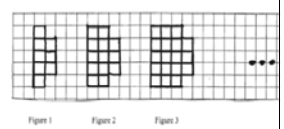
This chapter explores the connections between student work finding the 10^{th} term in a constant growth pattern and work on the subsequent tasks to write a rule to find any term in the pattern and to find the 100^{th} term in the pattern. It is organized by the three categories of algebraic reasoning approaches students used to find the 10^{th} term, the topic of Chapter 4. I found that students used their work on the 10^{th} term in important ways when they solved these two tasks, which I refer to as the generalization task and the 100^{th} term or far term task. I also found that these tasks were associated with beneficial shifts in reasoning for some students, depending on how they had reasoned about the 10^{th} term in the pattern. Shifts in reasoning and the effects on generalization are a topic of this chapter as well.

I begin with a brief overview of the generalization and far term tasks. This is followed by findings on student generalization and shifts in reasoning leading to generalization. The focus is on algebraic reasoning approaches supportive of generalization.

Overview of the Tasks

The four problems differed in task wording depending on the variable quantities. The wording of the tasks to find the 10^{th} term in the pattern and to write a rule to find any term in the pattern can be seen in Table 5-1.

Table 5-1. Task Wording by Problem.

Penny Jar Story Problem (A)	Geometric Tower Problem (B)	Numeric Sequence Problem (C)	Geometric Array Problem (G)
PROBLEM REPRESENTATIONS			
Written Words Marisa wants to save pennies. Her mother gives her a jar with 4 pennies in it to start. Marisa adds 3 pennies to the jar each day (day one, day two, day three, ...)	Geometric Pictures 	Numeric List 4, 7, 10, ___, ___, ___, ...	Geometric Pictures 
TASK WORDING			
3. Find the number of pennies in the jar after Marisa adds pennies on day ten. Show your work.	3. Find the number of blocks in Figure 10 of the pattern. Show your work.	3. Find the tenth term in the pattern. Show your work.	3. Find the number of blocks in Figure 10 of the pattern. Show your work.
4. Write a rule to find the number of pennies in the jar after any number of days.	4. Write a rule to find the number of blocks in any figure of the pattern.	4. Write a rule to find any term in the pattern.	4. Write a rule to find the number of blocks in any figure of the pattern.

Students seemed less sure of the meaning of the generalization task compared to the tasks to find particular terms which preceded it in each problem. Some students read the task and assigned a particular value to “any” term. For example, students might have said they would find term 5 or term 20. Other students expressed uncertainty about the meaning of the task or declined to solve it.

The task to find a far term in the pattern, specifically the 100th term, was designed to capture student thinking about the pattern or relationship between the two variable quantities after finding particular terms. It was presented orally as a prompt using wording such as "What if you wanted to find the 100th term?" Students did not receive this prompt on all problems due to interview time constraints.

Generalization by Reasoning Approaches on the Task to Find the Tenth Term

In Chapter 4, I discussed different categories of algebraic reasoning about constant growth functions exhibited by fifth graders in this study and summarized them in a chart which is shown again here.

Table 5-2. Algebraic Reasoning on the Task to Find the Tenth Term.

	Additive Constant Separate	Additive Constant Not Separate
Scalar	*Geometric Array* Hugo	*Numeric Sequence* Juan (operating on ratio to extend pattern) Taylor (continuing to extend pattern)
Emergent Functional	*Geometric Array* Alejandra	
Functional	*Penny Jar Story* Taylor	------

Shown in Table 5-2 are three categories of reasoning with the additive constant separate (scalar, emergent functional and functional) and one category of reasoning with the additive constant not separate (scalar). When students kept the additive constant separate, they reasoned about how to find the variable component, i.e., the part of the pattern that varied depending on the value of the independent variable quantity (e.g., term number, figure number). They reasoned about how to find the 10^{th} term directly from 10, the term number. The other category of reasoning (i.e., additive constant not separate) was associated with reasoning about the ratio of the rates of change in the term values and term numbers. Since the student operated on quantities of the same kind (i.e., term numbers and term values), this category involved scalar reasoning.

In this chapter, I begin discussion of student generalization with the categories of algebraic reasoning in which the student kept the additive constant separate. I then present results on the category in which the student did not. Students sometimes shifted in how they thought about the pattern, particularly on the tasks to write a rule to find any term in the pattern or to find the 100^{th} term. Significant shifts in reasoning are included in the discussion of student work on these tasks.

Stable Functional Reasoning. In 4 instances, students kept the additive constant separate and used functional reasoning to find the variable component of the 10^{th} term in the pattern, corresponding to the reasoning approach exemplified by Taylor on the penny jar story problem. In two of these instances, the students used their work on this task to write a rule to find any term in the pattern. In the other instance, the student used the same reasoning approach on the task to find the 100^{th} term and wrote a written word description of how he found it but did not write a general rule In this section, I describe how these students' work on the 10^{th} term supported them on the tasks to write a rule or find a far term.

I begin with Taylor's work on the penny jar story problem which was featured in chapter 4 as an example of a functional reasoning approach to finding the variable component. On the task to write a rule to find the total number of pennies after any number of days, what I refer to as the generalization task, he generated the letter-symbolic representation shown in Figure 5-1.

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Figure 5-1. Taylor's Generalization on the Penny Jar Story Problem.

Find the number of pennies in the jar after Marisa adds pennies on day 10. 	During his explanation of his work, Taylor realized he forgot to add 4, erased "30" and wrote "34" for his answer to find the number of pennies after 10 days.
Write a rule to find the number of pennies in the jar after any number of days. 	Taylor: You can draw the jar again. (Draws four pennies in a jar). And you can add four plus three times M equals X. (Writes "+ OOO x M = X" next to the jar with four pennies) Kathleen- Okay. So M is what? Taylor- Is the number of days and X is the answer.

As can be seen in Figure 5-1, Taylor began to write the letter "M" after the "+" sign, then erased it and put the three circle icons representing the rate of change (3 pennies each day) first, thus maintaining the structure of his prior representation. He then wrote "x M = X". He substituted the letter "M" for 10, the value of the independent variable (number of days) and verified orally that "M" stood for the number of days. He substituted the letter "X" for his answer (i.e., 34, the number of pennies on day 10) and described it as "the answer." Since the task was to write a rule to find the number of pennies after any number of days, "X" referred to the number of pennies after any number of days.

Taylor had reasoned about the 10th term in the penny jar story problem as the sum of two components, a fixed component (i.e., the 4 pennies in the jar to start with) and a variable component (i.e., the number of pennies Marisa added at a rate of 3 pennies each day). He had used a functional reasoning approach to finding the variable component since he multiplied two quantities of a different kind (i.e., a rate of 3 pennies each day times the number of days). This reasoning supported him in reasoning about how to find any term in the pattern from the value of the independent variable quantity. He did not shift in his reasoning on the task to write a rule.

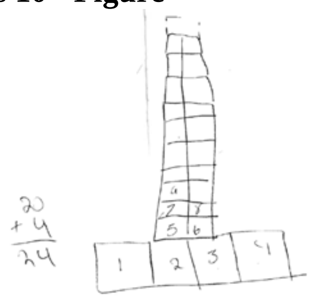
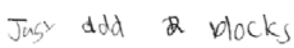
In this instance, Taylor used his representation from the prior task as a symbolization tool as well as a reasoning tool. He easily generated a letter-symbolic representation of the general rule by substituting letters to represent the two variable quantities in the problem. Thus, he was using his representation on the prior task as a template for his general rule. This example illustrates the importance of student-generated representations to support not only reasoning but also generalization and symbolization processes.

On the geometric array problem, Taylor generated a written word representation of the general rule after solving for figure 10. His work on the 10^{th} term had included a picture representation of part of figure 10 (he did not need to draw all of the figure before he solved the task) and arithmetic notations showing the multiplication of 5 times 10 and the final answer of 53. His explanation indicated he was multiplying the rate of change times the figure number to find the variable component, *mx*. This was a functional reasoning approach. His rule described these operations in terms of the variable quantities: “To find the number of blocks in any figure you can look at the figure number and multiply it by 5 and add 3.” Again, his written work on the prior task, specifically his arithmetic notations, was useful as a referent for writing a general rule. Again, Taylor's reasoning on the two tasks was stable; he did not shift.

Taylor was not unique in his consistent use of functional reasoning. Two other students also exhibited functional reasoning on both the 10^{th} term and the general rule. The work of these students was not elaborated in Chapter 4, but I will go into some detail here in order to illustrate interesting features of their functional reasoning. Alfredo generated a letter-symbolic representation of the general rule after using numeric sentences on the tasks to find particular cases on the penny jar story problem: "3x4=12+4=16" for day 4 and "3x10=30+4=34" for day 10. The algebraic formula he generated was: "*T*x3+4=*Z*," where *T* represented days and *Z* represented "all the pennies." The connection between his work on the prior tasks and his work on the task to write a rule was apparent in the parallel structure. His numeric sentences, although unconventional in the use of the equal sign, nevertheless functioned as an effective reasoning and symbolization tool for generating a letter-symbolic representation of the general rule. Like Taylor, his reasoning approach was stable; he did not shift on the subsequent tasks.

Tomás was the only student who solved for figure 10 on the geometric tower problem using functional reasoning to find the variable component. His work is shown in Figure 5-2.

Figure 5-2. Tomás' Work on the Geometric Tower Pattern Problem.

Tomás 10th Figure 	"Times ten, ten times two. Okay, there's twenty down. (Draws a narrow rectangle) And four on the bottom. (Draws bottom rectangle below the narrow rectangle and draws lines to make it a row of four blocks, then adds in lines to the first rectangle to make ten pairs of blocks). Okay, twenty plus four. Twenty-four. (writes 20 + 4 =24 using vertical notations)."
Tomás Rule to Find Any Figure 	
Tomás 100th Figure **Oral Words:** "It would be two hundred plus four equals two hundred and four." **Written Words:** "For figure 100 we multiply 2 and the total is 200 + 4 = 204." (Whispers as he writes): What I did was multiply 100 with 2 and gave me 200 then just add four."	**Kathleen** (After he writes his description of his work): Mhmm. So, now, a rule uh, is what your using, what you use to find the = **Tomas-** The answer. **Kathleen-** = two hundred and four. Yes, to find it. So what did you do? Can you write what you did to find that one?

Tomas talked aloud about his approach as he generated his picture representation of figure 10 without the numbers inside the blocks (i.e., the picture shown in Figure 5-2 has numbers that he added after he found the number of blocks using functional reasoning.

After using a functional reasoning approach, however, Tomás immediately wrote "just add 2 blocks" for the rule to find any figure. This was the same as his response on the first task, describe the pattern. The fact that Tomás generated the same written word representation on both tasks suggested that he interpreted the two tasks the same way. It did not necessarily indicate a shift in reasoning.

In fact, on the task to find the number of blocks in figure100, Tomás easily found the answer using the same (functional) reasoning approach he had used to find the number of blocks in figure 10: “It would be two hundred plus four equals two hundred and four.” When prompted if he could write how he found this answer for figure 100, Tomás wrote a description of his reasoning approach: “For figure 100 we multiply 2 and the total is 200 + 4 = 204.” When prompted what he multiplied by 2, Tomás indicated he multiplied 100 times 2, then rewrote his answer for the rule to make this explicit. It may be that Tomás simply did not understand the meaning of the generalization task or it may be that he could not think abstractly about his reasoning approach as a way to find any term in the pattern. What is important is the way he reasoned (i.e., functional) and his ability to generate a description of his reasoning approach. I would suggest that it would be fairly easy to scaffold him in expressing this as a rule in terms of the independent variable quantity, figure number, and that solving for additional cases might support him in thinking about it as a way to find any term in the pattern.

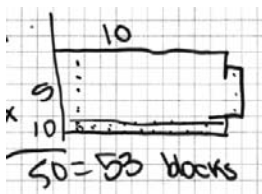
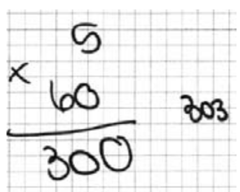
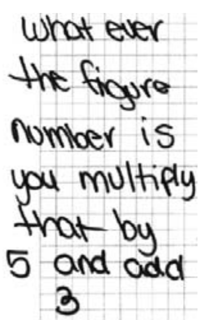
This instance illustrates why the task to find the 100^{th} term in a constant was useful for analyzing student reasoning. It provided evidence that Tomás’ functional reasoning approach was stable. It had become a reasoning tool for him which he could use to find any term in the pattern even though he did not express it in general terms.

To summarize, functional reasoning approaches were stable. The four instances in which students had used functional reasoning to find the 10^{th} term also used functional reasoning on a subsequent task. It was a reasoning tool for them and also a tool for generating a letter-symbolic representation (i.e., a symbolization) in two instances.

The next instances I discuss are those in which students found the 10^{th} term using reasoning approaches other than functional to finding the variable component. There were 3 instances in which student work on the tasks to write a rule or find a far term resulted in a shift to functional reasoning. The two categories in which these shifts occurred included emergent functional and scalar with the additive constant separate.

Emergent Functional Reasoning. In Chapter 4, I presented the work of Alejandra on the geometric array problem. Her work on this and subsequent tasks is shown in Figure 5-3.

Figure 5-3. Alejandra’s Generalization on the Geometric Array Problem.

Find the number of blocks in figure 10 of the pattern.  	
Write a rule to find the number of blocks in any figure of the pattern.  	“Let's say that it's sixty, you could just imagine like this (writes vertical multiplication), like sixty by five, because it still has to be five up. Three hundred...and then plus three, so three hundred and three.
 	**Kathleen**: OK. So the rule is that you... **Alejandra**: So you… (long pause). **Kathleen**: Where did the sixty come from? **Alejandra**: Um, *because on the bottom, like, let's say it's figure sixty* [emphasis added]... **Kathleen**: Oh, OK, the figure, which one you picked, the figure. What do we call that? A figure... **Alejandra**: (writes the rule). OK, I put whatever the figure number is, you multiply that, the figure's number, by five and add three.

Alejandra had also used multiplication to find the variable component of figure 10, leveraging the rectangular configuration of the variable component in her picture representation. She then interpreted the task to find the number of blocks in any figure of the pattern as an invitation to pick a figure number and solve for the number of blocks.

She chose a value of 60, a far term. She used the same reasoning approach to find the number of blocks in figure 60 as she had used on figure 10. The connection between her work on figure 10 and her work on figure 60 is apparent in Figure 5-3. Notice the parallel structure of the vertical multiplications and the increase of 3 to account for the additive constant.

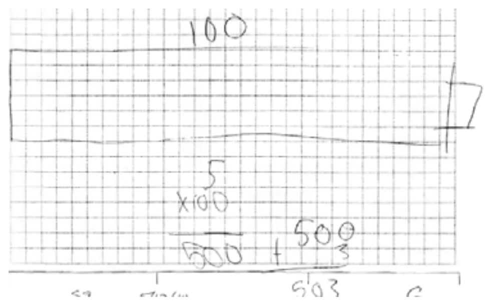
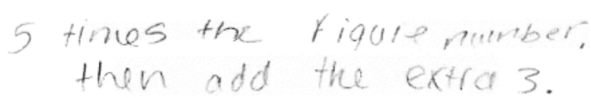
Assigning a value to what was intended in the task to be a variable quantity was not uncommon in this study. There were 4 other instances in which students did the same, but with lesser values of the term number such as 5 or 30 which did not support them in reasoning about the general rule. Alejandra's work on this, a far term task, seemed to support her in making sense of the task to write a rule to find any term. She generated the written word representation of the rule shown in Figure 5-3 after solving for the 60^{th} term.

Alejandra might have been able to generalize after finding the number of blocks in the 10^{th} figure but solving the task to find the number of blocks in the 60^{th} figure seemed to help her understand the meaning of the task to write a rule for *any* figure, i.e., a way to find the number of blocks in figure 10 or figure 60 or *any* figure.

Alejandra's representation of the general rule included the independent variable quantity (i.e., figure number). She needed scaffolding in using these words. It may be that the wording of the generalization task on the geometric pattern problems was problematic since it did not include the words "figure number." This was a minor obstacle to generalizing the function, however, compared to the major obstacle encountered by students who had used reasoning approaches with the additive constant separate, the topic of a later section.

Scalar reasoning with the additive constant separate. In Chapter 4, I also presented the work of Hugo on the task to find the number of blocks in figure 10 of the geometric array problem. Hugo had used a picture representation of figure 10 and counted the number of blocks in the variable component by 5's before adding 3, the additive constant. This was a scalar reasoning approach since he was operating on quantities of the same kind, blocks. After reading the task to write a rule, Hugo expressed the rule this way: "I kept on adding by five in every figure and then the extra three." Hugo was describing his reasoning approach, how he added 5 to extend the pattern of the variable component, since he references then adding 3. But this was not the reasoning approach he used to find the number of blocks in figure 100. His work is shown in Figure 5-4.

Figure 5-4. Hugo's Work on the Task to Find the Hundredth Term on the Geometric Array Problem.

	**Hugo:** "I would have to have a hundred blocks going this way (drawing a horizontal line almost the width of the paper) and then going down I would have to have five, and then this way a hundred again (drawing another horizontal line below…), and then three right here (drawing a small rectangle on the right side of the figure). **Kathleen:** And how many would that be? **Hugo:** "That would be a lot…That would be a…thousand. Oh, no, no, no. Let's see. Five times 100…Five hundred. **Kathleen:** And is that the number of blocks in figure 100? **Hugo:** Yeah… No. I have to add three more. (writes 500 + 3 = 503 vertically)
	**Kathleen:** So, would that always work? **Hugo:** Yeah. **Kathleen:** You do five times… **Hugo:** Times… **Kathleen:** Times what? **Hugo:** A hundred. **Kathleen:** OK, but let's… **Hugo:** Five times any kind of number that… **Kathleen:** Any kind of… **Hugo**: …the figure. **Kathleen:** The figure number. Oh. **Hugo:** (writes rule)

Hugo drew a representation of figure 100 (not to scale) and his oral words indicated that he connected the figure number and the number of "blocks going this way (horizontally)." Because his drawing was not to scale, a reasoning approach of counting blocks in the variable component by 5's, as he had done for figure 10, would not work. There was no indication that he even considered using that approach, however. Hugo leveraged the rectangular configuration of the variable component to use multiplication of 5 times 10 to find the variable component.

This is another example of how a student leveraged a picture representation of a figure to use multiplication to find the variable component on the geometric array problem. Hugo had been thinking about a relationship between successive figures in the pattern, specifically that the variable component increased by 5 blocks. Now he shifted to reasoning about how to find the variable component directly from the value of the figure number. Hugo had been reasoning about 5 as a rate of change so this was a shift to a functional approach to finding the variable component. His subsequent work on generating a general rule supports this interpretation.

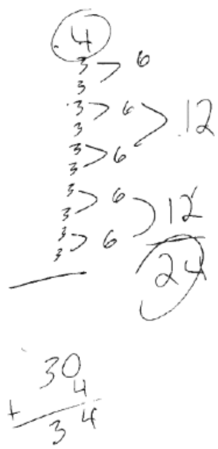
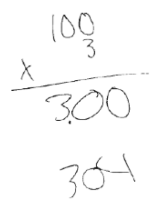
Hugo's shift from successive addition of 5 using counting by 5's to multiplication of 5 times the figure number was a significant and beneficial shift. Hugo described his work on finding the 100th term to write a rule to find the number of blocks in any figure. As shown in the transcript segment associated with his generation of the rule in Figure 5-4, he was thinking about multiplying 5 times the figure number, not a dimension of 5 times a dimension of 10. Since he had been reasoning about 5 as a rate of change, this provided further evidence that he had shifted to functional reasoning about any term as the product of a rate of change times the figure number plus the additive constant.

Hugo, like Alejandra, needed scaffolding of what to call the figure number, as shown in the second transcript segment in figure 5-4. The generalization task (i.e., "Write a rule to find the number of blocks in any figure in the pattern") did not include the words "figure number." Unlike Alejandra who seemed to make sense of the generalization task after finding the 100th term, Hugo needed additional scaffolding, beginning with the question: "Would that (his reasoning approach) always work?" This question seemed to help Hugo think of describing a general approach in terms of the figure number.

In contrast to Alejandra, whose written work on the figure 10 supported her shift to using multiplication to find the variable component on that task, it was Hugo's written work on the far term task (i.e., figure 100) which supported him in shifting from scalar reasoning to functional reasoning to find the variable component. This demonstrates again the importance of student written work as a reasoning tool on the generalization task. It also provides an example of how the task to find a far term supported a shift to functional reasoning. I would argue that this beneficial impact was related to the difficulty of finding a far term using an approach other than multiplication. Since only students who engaged in functional reasoning were able to generalize, this was a highly significant shift.

On the penny jar story problem, Hugo had used a chunking strategy to find the variable component (i.e., the number of pennies Marisa added in 10 days), as shown in Figure 5-5.

Figure 5-5. Hugo's Work on the Penny Jar Story Problem.

Day 10 	**Kathleen:** How many 3's did you add up? **Hugo:** Ten. **Kathleen**: And when you add ten three's what do you get? **Hugo:** I got (pause) twenty-four. (Pause) Thirty-three. **Kathleen:** Ten, ten three's? **Hugo:** Three, six, nine, twelve, fifteen, eighteen, twenty-one, twenty-four, twenty-seven, thirty." **Kathleen:** Thirty. Does that surprise you that when you added ten threes you got thirty? **Hugo:** Mhmm (indicating yes).
Day 100 	**Kathleen:** So what would you have to do if you were going to find day one hundred? She could save pennies for one hundred days. **Hugo:** I would divide it. No, times it. **Kathleen:** Oh, what would you times? **Hugo:** One hundred times three. **Kathleen:** Why did you use one hundred? **Hugo:** Because that's the days. **Kathleen:** Why did you use three? **Hugo:** Because that's the days that she's, like that's how much she saves each day.

Notice Hugo's chunking of 3's into 6's and 6's into 12's. Hugo accounted for the other 6, adding it to 24, and showed adding 4, the additive constant, to 30. His reasoning on the variable component was scalar since he was operating on quantities of the same kind, pennies. He did not understand the meaning of the generalization task, saying "So I just pick out any day?" So I prompted him to consider day 100. Hugo did not make a connection between adding ten 3's and multiplication as shown in the transcript segment in the top of Figure 5-5. His representation did not show the addition of 6 to 24 and he was confused about what he found when he combined the ten 3's. His response was to count out the multiples of 3, still a scalar approach. He even responded that he was surprised the answer was 30. However, this brief exchange about his work scaffolded him in making this connection as shown in the transcript and work in the bottom half of Figure 5-5.

Although Hugo was not able to find the 100th term on this problem without scaffolding, his consideration of this more difficult task and using his written work on the 10th term as a referent led to him using a functional approach to finding the variable component, multiplying the number of days times the rate of change, “how much she saved each day.” He benefited by making a connection between adding ten 3’s and multiplication of 10 times 3. His confusion on this task initially occurred because he did not show the addition of 6 to 24, once again illustrating the critical importance of student work on the task to find the 10th term as a reasoning tool on the more difficult tasks that followed.

In the next section, I discuss the third and final category of algebraic reasoning in which the student reasoned about term values (i.e., additive constant not separate). I found that students who used this reasoning approach shifted to reasoning approaches which would not work on the generalization and far term tasks. Therefore, I label these reasoning approaches as unstable to contrast the results with those of functional reasoning approaches which, once used by students, continued to be used in subsequent tasks.

Unstable Scalar Reasoning about Term Values. The third and final category of algebraic reasoning exhibited in this study consisted of instances in which the student operated on term numbers and term values (scalar reasoning with the additive constant not separate). It involved reasoning about a relationship between the rates of change in the term number and term values. It was quite different than the two prior categories in which students reasoned about how to find the term value directly from the term number. Instead, the student reasoned about extending two corresponding patterns based on a relationship between their rates of change. This was a scalar reasoning approach since the student operated on quantities of the same kind, term numbers and term values.

In Chapter 4, I presented the example of Juan who found particular term values on the numeric sequence problem by extending the term values and term numbers using successive additions. His work finding the 10^{th} term indicated that he was using a scalar form of algebraic reasoning. His work on this task and the task to write a rule to find any term in the pattern are shown in Figure 5-6.

Figure 5-6. Juan's Work on the Numeric Sequence Problem.

4th Term 4+3=7+3=10+3=(13)	
10th Term 13+3=16+6=22+3=25+6=(31) 7 8 10	
Write a rule to find any term in the pattern. Add 3 or any mutiple of three	"What I wrote was add three or any multiple of three because six is a multiple of three so if you do like a bigger multiple than six, that's a factor of three. Like, you would get the answer faster or something."

Juan found the 10th term in the numeric sequence problem by reasoning that he could add 3 to a term value to find the next term or add 6 to a term value and add 2 to the term number (or, alternatively, skip a term number). He actually alternated these operations when he found the 10th term, as shown in the middle row of Figure 5-6. His response to the task to write a rule to find any term in the pattern is also shown in Figure 5-6. He expressed his reasoning in terms of adding 3 or any multiple of 3. He did not describe the corresponding values to add to the term numbers (i.e., 1 or the same multiple of 1), however, which he correctly accounted for in his work on the 10th term.

Juan's reference to "a factor of three" in his explanation was unclear, especially after he had just identified 6 as a "multiple" of 3. It appeared that he might have shifted in his thinking, so I probed his reasoning further as shown in the following transcript segment, beginning with his explanation of his rule to find any term in the pattern. His work on finding the 4th term in the pattern was also shown in Figure 5-6 since I prompted Juan to look at it during the interaction (line 163).

> 162 **Juan** What I wrote was add three or any multiple of three because six is a multiple of three so if you do like a bigger multiple than six um, that's a factor of three, like, you would get the answer faster or something.
>
> 163 **Kathleen** Mhmm, mhmm. It's like a shortcut, huh? So, for term ten…well let's start back at the beginning. For term two…What's term two there (pointing to his work on the task to find the fourth term)?
>
> 164 **Juan** Seven.
>
> 165 **Kathleen** How many times did you add three to get to seven?
>
> 166 **Juan** Um, only twice. Because if, *probably they skipped the number one because if you add three and three it would equal six instead of seven* [emphasis added].

Juan shifted from reasoning about the second term as the first term, 4, plus 3 to reasoning about the second term as 1 more than "3 + 3." In response to my question in line 165, I had expected him to say he added 3 once (to the first term, "4") to find the second term, based on his original work on the task to find the 4th term in the pattern, shown in the top row of Figure 5-6. Thus, he had shifted from reasoning about the pattern as a sequence formed by successive addition of 3 to reasoning about the composition of a particular term. This was a rare occurrence in this study: students rarely shifted from reasoning about change between successive terms, as Juan had exhibited on the task to find the 10th term on this problem, to reasoning about the value of a pattern.

With scaffolding, Juan was able to express how to find the 10th term, writing “10 times 3 plus 1,” and to verify that this approach worked for lesser term numbers as well. But his subsequent work on the task to find the 100th term in the pattern suggested that his reasoning was not based on a solid understanding of the relationship between the term number and the term value, as shown in the following transcript segment.

205 **Kathleen** Could you do the same type of thing for term one hundred?

206 **Juan** Mm, I think probably.

207 **Kathleen** Mm, want to try it over here?

208 **Juan** (Writes "33 x 3 = 99 + 1")

209 **Kathleen** Oh, maybe you misunderstood my question, I was saying can you find term one hundred.

210 **Juan** Oh. (Pause)

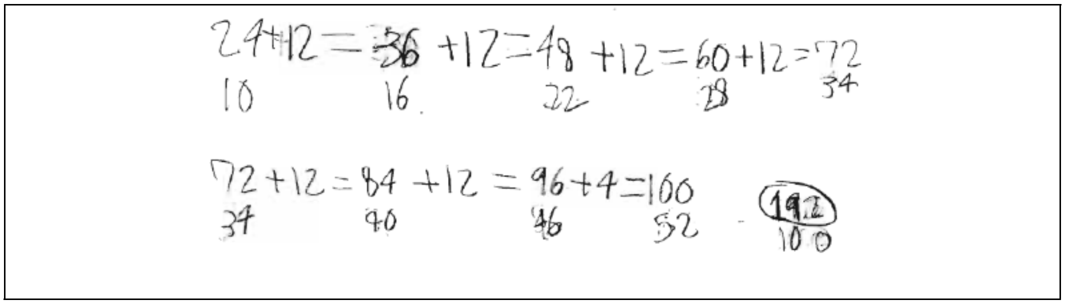
211 **Kathleen** But you didn't do term one hundred here. When you did thirty-three times three what term were you finding there?

212 **Juan** Um, term (long pause) eleven.

Juan reasoned backwards about a term whose value was 100 rather than reasoning about how to find the 100th term. This was not uncommon in this study since the task to find the 100th term was presented orally and the student may have thought I said “100” rather than “term 100.” What was significant was that Juan could not identify the term number in his representation, “33,” when prompted (lines 211-212), suggesting that his emergent reasoning about how to find a particular term from the term number was tenuous. Nevertheless, his shift in reasoning on this task was significant because he shifted to a reasoning approach associated with ability to generalize based on student recognition of a direct relationship between the two variable quantities. The rule Juan had originally written on this task (i.e., add 3 or any multiple of 3) would not have been feasible for finding a far term in a constant growth pattern.

Before concluding the chapter with a summary of findings on connections between student work on the 10^{th} term and student reasoning about finding a far term or writing a rule to find any term in a constant growth pattern, I present Juan's work on the task to find the number of blocks in figure 100 on the geometric tower problem, shown in Figure 5-7, to illustrate the difficulty of finding a far term using scalar reasoning based on a relationship between corresponding rates of change in term values and term numbers.

Figure 5-7. Juan's Work on Finding the Hundredth Term on the Geometric Tower Problem.



Juan began his reasoning approach with his answer to the task to find the 10th term, 24, and was successful extending the pattern to the 46th term using a tabular representation as a reasoning tool. As can be seen in Figure 5-7, Juan extended the term values and term numbers adding 12 and 6, respectively, on a problem in which the rate of change was 2, a scalar approach. His mistake occurred when he added 4, instead of 12, to 96, the value of the 46th term, while still adding 6 to the term number. Similar to his work on the numeric sequence problem, Juan was reasoning about extending pattern to a term *value* of 100 rather than to the 100th term. Juan was the only student in this study who attempted to extend the pattern to a far term using a scalar reasoning approach based on a relationship between the rates of change of the term numbers and term values. He extended the pattern using an operating on the ratio approach in contrast to the predominant scalar reasoning approach among fifth graders in this study, i.e., continuing. Using a continuing approach, i.e., finding each case of the pattern one at a time, would have been even less feasible for finding the 100th term.

In general, a scalar reasoning approach with the additive constant not separate did not support generalization or finding the 100th term in the pattern. This scaffolded exchange with Juan was included to demonstrate how a task to find a far term might engage students in reconsidering the pattern and identifying an algebraically useful pattern supportive of generalization.

Solving for a far term was problematic for other students who used scalar reasoning with the additive constant not separate. There were 8 more instances of this scenario. For example, on the penny jar story problem, one student assigned a value of 30 to the number of days and added his solution for day 10 (i.e., 34) three times. On the numeric sequence problem, the same student multiplied his 10^{th} term solution (i.e., 31), by 3. Alejandra used scalar reasoning and wrote that you could double (no specifics) on the numeric sequence problem and that you could multiply by 2 when you reach 55 on the geometric tower problem. Thus, scalar reasoning approaches with the additive constant not separate were problematic in terms of student generalization. In 4 instances, although the students made a connection between the repeated addition of 3 to the term values in their work and multiplication by 3, they forgot the additive constant.

In summary, student work on finding a far term (i.e., the 100^{th} term) provided more evidence of a disconnect between ability to generalize and reasoning approaches based on finding a particular term from prior terms (i.e., scalar reasoning with the additive constant not separate). More importantly, however, this task led to beneficial shifts in reasoning for students who had reasoned about the variable component, keeping the additive constant separate. Students might shift from scalar to functional reasoning about the variable component in order to solve a far term task and then be able to generalize.

Summary of Connections between Student Problem Representations and Student Algebraic Reasoning

Student generalization depended on the student's reasoning approach on the task preceding it. There were no instances of generalization after students used scalar reasoning and extending actions to find the 10th term, i.e., finding terms from preceding terms. Some students used their work on the 10th term task as a reasoning and generalization tool on the task to write a general rule (Taylor and Alfredo on the penny jar story problem); others used their work on a far term task as a reasoning tool (Alejandra and Hugo on the geometric array problem). These results are displayed in Table 5-3.

Table 5-3. Generalization by Algebraic Reasoning Category.

	Additive Constant Separate	Additive Constant Not Separate
Scalar	*Geometric Array* Hugo Shift to Functional on the 100th term then Written Word Generalization	*Numeric Sequence* Juan (operating on ratio to extend pattern) Did not generalize Taylor (continuing to extend pattern) Did not generalize
Emergent Functional	*Geometric Array* Alejandra Shift to Functional on the 60th term then Written Word Generalization	
Functional	*Penny Jar Story* Taylor and Alfredo Letter-symbolic Generalizations	------

Students who generated a rule to find any term in the pattern leveraged their work on the prior task to find the 10th term in one of two ways: (1) generating a written word description of their (functional) reasoning approach in terms of the variable quantities or (2) generating a letter-symbolic representation using their representation as a template.

These instances occurred after students had reasoned about how to find a particular term from the term number. This was not a sufficient condition, however. Factors associated with ability to generalize included a written representation of their reasoning approach on the prior task to use as a tool for generalization and understanding the meaning of the task to write a rule.

The task to find a far term in the pattern led to additional instances of student generalization but only in those instances in which the student had reasoned about how to find it from the term number (i.e., additive constant separate). It supported students in understanding the meaning of the generalization task. More importantly, it supported two students who had used scalar approaches to finding the variable component of the 10^{th} term in shifting to a functional approach and then generalizing using their work on the 100^{th} term as a reasoning tool. For this reason, I categorized scalar reasoning approaches with the additive constant separate as emergent functional reasoning about the constant growth function.

Pictures were particularly powerful reasoning tools which supported students in using multiplication of the rate of change times the term number on the geometric pattern problems. The distinction between students using product of measures or functional reasoning to find the variable component did not seem significant since students demonstrated that, even if they were leveraging the rectangular configuration of the variable component, they understood that the two dimensions corresponded to the rate of change and the term number, and were able to express a rule in terms of the independent variable quantity, the term number. For this reason, I either used evidence to show the student was making this connection, particularly on the generalization task, or argued that product of measures was an emergent functional reasoning approach based on evidence of shifts to functional reasoning.

In the next chapter, I synthesize the findings on (1) connections between the problem representation and student work on the 10th term and (2) connections between student work on the 10th term and ability to generalize to answer the main research question of this dissertation: What are the relationships between problem representations of constant growth functions and students' algebraic reasoning? I then discuss research and teaching implications of the findings reported in this study.

CHAPTER SIX

CONCLUSION

In Chapter 1, I introduced the “algebra problem” and situated this study in the context of early algebra opportunities for elementary school students to develop algebraic reasoning. From the perspective of many researchers in the field of early algebra, the problems students encounter in algebra are due to an expectation that students make an abrupt shift from arithmetic to algebraic reasoning; the solution is to provide opportunities for elementary school students to develop both algebraic and arithmetic reasoning. To shift from arithmetic reasoning to algebraic reasoning involves a process of reasoning from particular instances to reasoning about the overall pattern, i.e., generalization. Research has demonstrated that elementary school students can engage in generalization in a variety of mathematical contexts.

As I argued in Chapter 2, generalization is a human activity. Therefore, children have capabilities to draw upon in engaging in generalizing activities such as reasoning about how to find any term in a constant growth pattern (Kaput, Blanton & Moreno, 2008). Because functions are fundamental objects of algebra (Schwartz, 1990, older elementary school students need opportunities to explore grade level appropriate function problems. Through engagement in generalizing activities around constant growth patterns, students have the opportunity not only to develop understanding of the underlying concepts of variable and relationship but also to develop algebraic reasoning.

In Chapter 2, I situated this research study in the function approach to introducing algebraic reasoning in elementary school. I elaborated a theoretical framework for analyzing student representations and algebraic reasoning on constant growth pattern problems. This theoretical framework integrated Goldin and Shteingold's (2001) broad definition of a representation focused on its functionality (e.g., depicting or symbolizing something other than the signs, characters or objects themselves); Kaput, Blanton and Moreno's (2008) framework on generalization and symbolization processes; Ellis' (2007) taxonomy of generalizing actions and reflection generalizations; and Vergnaud's (1988) framework on reasoning approaches students use on linear functions without an additive constant. I used this theoretical framework to refine my questions about connections between problem representations and student algebraic reasoning on constant growth functions.

In Chapter 2, I also situated this research study in the methodological tradition of ethnomathematics (Bishop, 1988; Boaler, 1993). In the view of these researchers, individual students construct mathematical knowledge through participation in activities both in and out of school, and use this knowledge in future activities. This study is aligned with Boaler's argument that activity contexts for learning mathematics should be open to student choice of reasoning approaches grounded in the personal meaning they develop as they interact with representations of a mathematical situation to solve tasks. These considerations influenced the design of the mathematical activity context used in this study as described in Chapter 3.

In Chapter 3, I explained my research methods, outlining my data collection and analysis processes. I described the school, the student participants, and design of the mathematical activity context. I explained how I collected data during task-based interviews and analyzed student work to identify connections between different problem representations and student algebraic reasoning.

In the two findings chapters, Chapters 4 and 5, I illustrated how my theoretical framework and analytical tools provided insights into student algebraic reasoning. I presented findings on connections between student reasoning and features of the problem representations, focusing on what students leveraged when they engaged in algebraic reasoning. Chapter 4 focused on the connections between the problem representations of the constant growth functions and student reasoning on the task to find the 10^{th} term. Chapter 5 presented the findings on the connections between the student work on finding the 10^{th} term and student work on the tasks to write a rule to find any term in the pattern and to find the 100^{th} term. Interwoven throughout these chapters were findings related to student-initiated shifts in reasoning, including shifts from arithmetic to algebraic reasoning.

In this concluding chapter, I synthesize the findings from both chapters to address the main research question of this dissertation: What are the connections between different problem representations of constant growth functions and student algebraic reasoning? This leads to discussions of an addition to Vergnaud's (1988) framework for analyzing student algebraic reasoning on linear functions and a different way to support students in recognizing algebraically useful patterns and generalizing functions. Research recommendations and curricular and instructional implications are interwoven throughout these sections. The chapter concludes with reflections on the potential benefits from continuing the lines of research suggested in this chapter.

The primary purpose of this dissertation study was to add to the research base on student algebraic reasoning about constant growth functions, specifically focusing on the reasoning approaches fifth graders invented as they solved for particular cases (i.e., the 4th, 10th and 100th terms) and responded to a generalization task. I found that students generated a variety of representations on the task to find the 10th term and used these as reasoning tools on the subsequent tasks to write a rule to find any term and/or to find the 100th term. I also found instances in which students shifted in their reasoning approaches as the demands of the tasks increased, i.e., from finding a near term, the 10th term, to finding a far term, the 100th term.

Students were not able to generalize after using reasoning approaches in which they extended the pattern using scalar reasoning. Students who were able to write a rule reasoned about how to find the 10th or the 100th term directly from the term number, 10 or 100. These reasoning approaches were characterized by keeping the additive constant separate and finding the variable component (i.e., mx in the function $f(x) = mx + b$ where b is the additive constant). They used their work on finding these particular cases of the function to either generated a letter-symbolic representation by substituting a letter for the value of 10 or described the reasoning approach in written words in terms of the independent variable quantity (e.g., number of days, figure number or term number in the story problem, geometric pattern problem or numeric sequence problem, respectively).

The task to find a far term in the pattern supported some students in shifting from scalar approaches to finding the variable component to a functional approach. This required students to make a connection between their scalar approaches (e.g., counting blocks in chunks or summing 10 sets of 3 pennies in the geometric pattern or story problem, respectively) and multiplication of the rate of change times the term number. This task led to problematic shifts in reasoning, however, for students who used scalar reasoning approaches with the additive constant not separate.

A secondary purpose of this dissertation study was to examine the connections between student work finding particular cases of the function and student representations of their generalizations in those instances in which students were able to generalize. I found that student representations on the tasks to find the 10th term or the 100th term were critical as reasoning tools.

These key findings were presented in detail in the preceding findings chapters and are briefly summarized in the following section on connections between problem representations and algebraic reasoning.

Connections between Problem Representations and Algebraic Reasoning

I found that the students who were able to generalize used a reasoning approach in which they kept the additive constant separate and reasoned about the part of the pattern that was algebraic in nature, what I call the variable component. In a function of the form $f(x) = mx + b$, in which m is the rate of change and b is the additive constant, mx is the variable component.

There was a disconnect between the numeric sequence problem and generalization. None of the fifth graders in this study found the additive constant and used such a reasoning approach on the numeric sequence problem and none were able to generate a rule which would work to find any term in the pattern. I found that students could solve for the 10^{th} term in the numeric sequence problem using arithmetic reasoning about only the term values and their work indicated they thought of the term numbers like a counter. Students who reasoned algebraically about this constant growth function used scalar approaches to extend the pattern which did not support them in writing a rule to find any term nor in finding the 100^{th} term.

Only on problems with an explicit representation of the additive constant were there instances of student success on these tasks. The way the additive constant was represented impacted the number of instances. In the penny jar story problem, the additive constant was a starting amount and most students generated a tabular representation of the function similar to a function table, showing particular cases of the function. That is, they did not keep the additive constant separate; it was embedded in the term values. Their reasoning was similar to that used on the numeric sequence problem, i.e., scalar reasoning based on a relationship between the rates of change in the term values and term numbers. Just as on the numeric sequence problem, this reasoning approach, although algebraic, did not support them in finding a far term or writing a rule to find any term in this pattern (i.e., the number of pennies after any number of days).

In contrast, all but one student used a reasoning approach with the additive constant separate on the geometric array problem. This problem represented the additive constant as a final column of 3 blocks preceded by the variable component. Students seemed to leverage the rectangular configuration of the variable component to use functional reasoning to find its value and remembered to add the fixed part, the 3 blocks at the end. Student drawings of figure 10 or figure 100 supported them in using this algebraic reasoning approach. They found the number of blocks in a particular figure directly from the figure number and used their work as a reasoning tool on the tasks to write a rule to find any term or to find a far term (e.g., figure 100). This functional reasoning approach was stable; students who used such an approach did not shift to a different form of reasoning.

There was only one instance of a student using such a reasoning approach on the geometric tower problem. This problem represented the additive constant as a bottom row of 4 blocks and students who began reasoning about this pattern by counting blocks in the given figures did not notice the variable component as related to the figure number. They treated the additive constant similarly as they did on the penny jar story problem, combining it with the variable component and using scalar reasoning about term values. Again, this did not support students in reasoning about how to find any term in the pattern and most generalized the pattern in ways very similar to their generalizations on the numeric sequence problem. They described the rule simply in terms of the change in term values (e.g., “keep adding 2 blocks”) or proposed reasoning approaches which would not work to find a far term (e.g., doubling terms or multiplying the 10^{th} term by 10 to find the 100^{th} term).

Thus, the critical feature of constant growth pattern problems for student engagement in algebraic reasoning was explicit representation of the additive constant. Students who attended to the additive constant as a separate component used reasoning that was algebraic in nature to find the other (variable) component whose value depended on the value of the independent variable quantity. On the penny jar story problem, for example, algebraic reasoning was linked to approaches in which the student kept separate the 4 pennies Marisa's mother gave her to start (i.e., the additive constant), detected a relationship between the number of days and the number of pennies Marisa added at a rate of 3 pennies per day and found the total number of pennies on a particular day as the sum of the number of pennies Marisa added (i.e., the variable component) and the number of pennies her mother gave her to start.

Similarly, algebraic reasoning on the geometric pattern problems involved keeping the number of blocks that were constant across consecutive figures separate and reasoning about the relationship between the figure number and the other blocks in the figure (i.e., the variable component). In contrast, the numeric sequence problem did not represent the additive constant separately and none of the students used algebraic reasoning on this problem.

The findings on student algebraic reasoning laid the foundation for an addition to Vergnaud's (1988) framework for analyzing student algebraic reasoning which is discussed in the next section. I then turn to discussion of shifts in reasoning that occurred in a mathematical activity context in which students were free to shift to different reasoning approaches on different tasks within a problem.

Framework for Student Algebraic Reasoning on Linear Functions

I used Vergnaud's (1988) framework to analyze reasoning approaches students used to solve tasks. I found this framework useful for distinguishing scalar versus functional reasoning approaches to finding the variable component and identifying shifts in reasoning. In this section, I propose a framework for analyzing reasoning approaches on linear functions with an additive constant. This is an important addition to Vergnaud's framework since the results indicated that only when students attended to the additive constant as a separate component were they successful generalizing the function or finding the 100^{th} term in the pattern.

The framework based on the results of this study has three categories of algebraic reasoning approaches on linear functions with an additive constant. The first is a student reasoning approach with the additive constant not separate. Students reason about two corresponding patterns (i.e., the term numbers and the term values) based on a relationship that the ratio of the corresponding rates of change is constant. For example, a student might reason that if a plant which is currently 5 inches tall grows 3 inches per month, the student would extend this pattern through successive addition of 3 to the height and 1 to the month until he or she reached month 10. This is algebraic reasoning because the student is reasoning about a relationship between two corresponding patterns. Since the student is reasoning about quantities of the same kind (i.e., inches plus inches, months plus months), this is a scalar reasoning approach.

The other two categories correspond to reasoning approaches in which the student keeps the additive constant separate. If a student multiplies the rate of change times the value of the independent variable, it is a functional approach to finding the variable component. In the plant growth example, the student could find how tall the plant is after 10 months directly from the value of 10: he or she could multiply 3 inches per month times 10 months and add the starting height of 5 inches to this variable component.

In this study, I also found instances in which students kept the additive constant and used scalar reasoning to find the variable component, the third category. For example, the student might add ten 3's rather than multiply 10 times 3. This category is important because students sometimes shifted to a functional approach to finding the variable component. These shifts, discussed next, did not necessarily indicate a developmental leap (i.e., making a connection between their scalar reasoning approaches to finding the variable component and a functional reasoning approach). Students were free to choose different reasoning approaches on different tasks and it may be that the shifts indicated nothing more than a student selecting an approach based on a change in the task.

Student-initiated Shifts in Reasoning

There were instances in this study of fifth graders shifting from scalar to functional reasoning to solve for the variable component as the difficulty of the task increased (i.e., the term number increased). Since there were instances in which fifth graders in this study generalized after shifting from scalar to functional reasoning to find the variable component, this distinction between scalar and functional reasoning to find the variable component was found to be less critical for supporting generalization than keeping the additive constant separate. More research is needed on what students leverage when they make these beneficial shifts from scalar to functional reasoning to find the variable component. My findings suggest that increasing the term number led to beneficial shifts for students who had used reasoning approaches with the additive constant separate. In fact, the task structure of the mathematical activity context seemed to support beneficial shifts from scalar to functional reasoning since the difficulty of the tasks increased from finding the 4th term to finding the 10th term and then the 100th term. In contrast, scalar reasoning approaches with the additive constant not separate were unstable. Students who used such approaches (i.e., category 1) shifted from scalar reasoning approaches that worked to scalar reasoning approaches that would not work on the task to find the 100th term. Examples included describing strategies of doubling term values or multiplying the 10th term by 10 to find the 100th term or vague statements related to continuing to add the rate of change.

In the next section, I discuss curriculum and instruction implications in further depth in terms of classroom settings supportive of student engagement in algebraic reasoning, including generalization, on constant growth function problems.

Implications for Teaching and Learning

In this section I discuss implications for teaching and learning grounded in this dissertation research. I begin with a discussion of the mathematical activity context used in the task-based interviews, identifying the benefits. I then present a mathematical activity model for use in a classroom setting to support students in developing algebraic on constant growth problems. The recommendations in the activity model and the short section on preparation students need to understand numeric sequence representations in a way that supports development of algebraic reasoning are based on the research findings of this study. The latter section is included because reasoning about numeric pattern problems is an important topic with respect to the implementation of the Core Curriculum State Standards (2009), which includes study of numeric patterns at the 5th grade level.

Mathematical activity design considerations. The results of this study suggest that fifth graders may frequently shift between arithmetic and algebraic reasoning in a mathematical activity context in which they solve a sequence of increasingly difficult tasks in a constant growth pattern problem. Given research-based recommendations to provide opportunities for development of both algebraic and arithmetic reasoning in elementary school mathematics (see discussion in chapters 1 and 2), this is an important vein of research to explore in further depth and breadth than afforded by this study with its small sample size and limited problem set.

As noted earlier, students were free to use whatever reasoning approaches made sense to them based on the personal meaning they constructed as they engaged in solving the constant growth function problems. In most instances, students generated their own representations as reasoning tools. The activity context encouraged this through a statement to "shown your work" on each task. Students relied on the representations they generated on the task to find the 10th term as reasoning tools on the generalization task. Students who were successful writing a generalization that would be workable for finding any term in the pattern used their work on the 10th term in one of two ways: to describe their reasoning approach in terms of the independent variable quantity (e.g., "multiply 3 times the number of days and add the extra 4 pennies her mother gave her, "5 times the figure number then add the extra 3"); or as a template for a letter-symbolic representation of the rule, replacing the specific value of the independent variable quantity, 10, with a letter representing any value of the quantity. The reader is referred to chapter 5 for examples. These results suggest that student ability to generalize depends on both using a reasoning approach with the additive constant separate and also generating a written representation of their work on tasks to find particular terms prior to engaging in the task to generalize. This implies that students should be encouraged to generate representational reasoning tools that make sense to them as they solve for particular terms.

There were other findings related to tasks and student algebraic reasoning. As already described briefly, the wording of the task to write a rule to find any term in the pattern was problematic for most students in this study. This could simply be due to lack of experience solving such tasks. More importantly, there were instances in which students did not initially understand the generalization task on a particular problem, but understood it and wrote a rule that would work for far terms after solving for the 100th term. This result suggests that it would be advantageous for students to solve for a term considerably larger than the 10th term before being tasked to write a generalization. Due to the limited number of instances of students solving both the tasks to find the 100th term and the task to write a general rule, more research is needed on a sequence of tasks to support students understanding of the generalization task. Based on the results with the fifth graders in this study, the students made sense of the task as "write a rule to find any term in the pattern *like the 10th term or the 100th term* [emphasis added]." Therefore, I recommend including reference to particular terms they found in prior tasks in the wording of the generalization task to support students who may not be familiar with such a task.

Overall, classroom research is needed on student algebraic reasoning and ability to generalize constant growth function problems in a mathematical context which: (1) encourages students to show their work while also (2) leaving the choice of representations and reasoning approaches to individual students or groups of students who are tasked to find different ways to solve the task and (3) provides challenging tasks to find far terms in the pattern after finding the 10^{th} term. Of particular interest is how students might learn from interactions with other students who use reasoning approaches with the additive constant separate.

Mathematics classroom activity model for constant growth function problems. In this study, the fifth graders engaged in a mathematical activity context more typical of a classroom focused on group work and student thinking, an approach shown to support student learning with understanding (Boaler, 1993; Hiebert, et al., 1997). This has important implications for future research, since the evidence reported in this study supports the notion that fifth graders can generate a variety of representations and reasoning approaches which would provide rich resources for exploring functions, the fundamental object of algebra. The goal would be *early development of conceptual understanding* of the underlying concepts of variable and relationship versus *early procedural instruction in linear equation solving* before students conceptually understand these objects.

This proposed mathematical activity model for exploring the linear relationships underlying constant growth functions and developing algebraic reasoning about this relationship is based on the research findings of this dissertation.

1. Students should have ample opportunities to explore constant growth functions represented in both story problem representations and geometric pattern problem representations: the more time spent on this and the more open it is to student representation of the pattern using words, pictures or objects, the more this will support development of personal meaning leading to more mathematically interesting than the quick responses of some fifth graders in this study (e.g., "+3" or "adding 3"). Student work in small groups should include ample opportunities to share their representations with one another coupled with the teacher as expert leading a discussion of different features. Particular attention should be paid to helping all students make sense of constant growth patterns as having a fixed component (i.e., one that does not vary) and a variable component.
2. Teachers might next introduce the task to find the 10^{th} term. For example, in a whole class discussion, they might lead their students in exploring whether there could be more than one right answer (depending on the solution approach). The class could revisit the additive constant as an important component of the pattern, since in many instances in this study, students either claimed that it didn't matter or forgot about it after they found the variable component. If students work cooperatively, they might be tasked to find the 10^{th} term in the pattern using three different approaches (including at least one with the additive constant separate).

Students may construct and represent these reasoning approaches fairly quickly but may need scaffolding from the teacher to resolve issues arising from students finding different solutions. This would help students understand that functions have one particular term value associated with a particular term number.

3. Allowing time for students to generate different representations and use different reasoning approaches on the task to find the 10^{th} term would provide rich material for whole class discussion, especially if different reasoning approaches are compared and contrasted (e.g., additive constant not separate, additive constant separate and functional reasoning to find the variable component, additive constant separate and scalar reasoning to find the variable component). With representations as communication tools, students could, for example, analyze how the additive constant and variable component are represented and explain why certain approaches to finding the variable component work. Students may draw connections between the functional approach (multiplication) and scalar approaches such as chunking or counting by the chunk size.

4. When students are ready for tasks to find far terms, such as the 60^{th} term or the 100^{th} term, they can try using the different reasoning approaches used to find the 10^{th} term. By encouraging students to try at least two different reasoning approaches, one with the additive constant separate and one with it not separate, students may come to appreciate the usefulness of reasoning approaches with the additive constant separate. However, it is important for teachers to attend to student shifts to reasoning approaches that will not work and important for

students to discuss why the approaches did not work to avoid the risk of students forming misconceptions.

5. The task to write a general rule should follow student engagement in finding far terms in the pattern and should refer in the wording to these terms to support student understanding of this generalization task (e.g.,“write a rule to find any term in the pattern, such as the 10^{th} term, the 100^{th} term or even the 1000^{th} term”). It may be useful to have a whole class discussion of what it means to “write a rule” perhaps relating it to other rules they have used such as finding the area of a rectangle. Next, students may benefit from initially writing descriptions in their own words of how they found two particular terms. Finally, just as on tasks to find far terms in the pattern, it is important that the teacher be alert for problematic shifts in reasoning on this task and perhaps allow students more opportunities to find particular terms before engaging in finding far terms, since it is important to avoid development of misconceptions. What is more important than learning to write a rule is that students develop reasoning approaches based on the underlying relationship between the variable component and the independent variable quantity.

Preparatory activities for using numeric sequence problems in constant growth function activities. The fifth graders in this study easily found the fourth and tenth terms in this representation of a constant growth pattern. The tabular representations they generated to extend the pattern demonstrated that these nine fifth graders understood that one variable quantity was the term number and that the term number and term value co-varied. It was easy for the students to solve these tasks with scalar reasoning. Clearly, these tasks were not challenging for the fifth graders in this study.

Imagine, however, if the students constructed these objects and developed personal understanding of how these objects are constructed. For example, students who are studying the multiples of 5 could construct numeric sequences, not just of the multiples of 5 but also the multiples of 5 plus 1, the multiples of 5 plus 2. The multiples of 5 minus a constant would be an interesting way to explore the concept of negative numbers, as well. The objects the students construct could then be used as reasoning tools for not only defining the pattern but also generalizing the pattern. I believe the results of this research study argue strongly for development of mathematical activities with numeric sequences that support understanding the relationship implicitly represented (i.e., “hidden”), and would recommend activities to support students in recognizing fixed and variable components across constant growth pattern problems.

This is a very different approach that assumes that: (1) students can develop understanding of variable and relationship in a context encouraging multiple representations and reasoning approaches, such as might occur in a complex instruction environment with a group worth task (Cohen, 1994); and (2) constant growth problems and task sequences can be designed (and differentiated) to enhance the engagement of the mathematics class in algebraic reasoning, specifically reasoning approaches in which the additive constant is kept separate and the relationship between the variable component and the independent variable is easily discovered. The objectives of this approach are twofold. First, this approach supports students in developing conceptual understanding of variable and relationship. Second, it encourages students to look at patterns from a different perspective, as comprised of two components, one of which depends on the independent variable quantity.

With this conceptual grounding, students could engage in linear equation solving in a context that is meaningful, finding the specific value of the dependent variable for a particular value of the independent variable. This is a very different way to think about generalizing a linear function. Rather than using a mapping representation as a reasoning tool for generalization, the student would use an operational representation. In a mapping representation, particular cases of the constant growth function are displayed as corresponding values. Students then use this reasoning tool to detect a relationship between numbers, although research has shown they need scaffolding to focus on the relationship across rows (i.e., between corresponding values of the independent and dependent variable quantities). In contrast, operational representations might consist of a series of numeric equations generated by students to find the term values. They would generate similar representations of the rate of change times the value of the independent variable, plus the additive constant, to find the term values associated with each term number (e.g., 5 x *1* + 2 = 7 for term *1*, 5 x *2* + 2 = 12 for term *2*, 5 x *10* + 2 = 52 for term *10*). The relationship between the term number and term value is expressed mathematically, and may better support students in reasoning about the relationship between the two variable quantities.

Concluding Remarks

In this chapter I have identified a research agenda for a different approach to supporting older elementary students in shifting from arithmetic to algebraic reasoning on constant growth pattern problems. Unlike the teaching experiment research which has demonstrated student ability to engage in algebraic reasoning about these grade-level appropriate linear functions using particular representations such as function tables (Blanton, 2008; Schliemann, Carraher and Brizuela, 2007) or object representations of geometric patterns (Moss, Beatty, Barkin & Shillolo, 2008; Warren & Cooper, 2008). Given the lack of significant progress on improving algebra readiness in this country, it is vitally important to explore different approaches.

The results of this study suggest that students can reason algebraically about constant growth functions given opportunities to develop personal meaning and freedom to use reasoning approaches that make sense to them, including generation of their own representations. The results reinforce the idea that student generation of meaningful (to them) representations were critical for generalization since students relied on their written work on finding particular terms when solving the task to write a rule to find any term in the pattern.

I identified research implications, including those related to limitations of the study and recommended a research agenda to examine whether the same beneficial shifts from arithmetic to algebraic reasoning would occur in a classroom. I presented a framework for analyzing student reasoning approaches on linear functions of the form $f(x) = mx + b$ that is grounded in Vergnaud's (1988) framework on student reasoning approaches on linear functions of the form $f(x) = mx$ and the results of this study. I discussed future directions in research to test this framework as an analytical tool for understanding emergence of algebraic reasoning about constant growth functions. Finally, I discussed implications for the teaching and learning of linear functions, proposing a mathematical activity context model for explorations of linear functions based on this framework and the findings of this study.

Before concluding, I would like to remark on the potential of this research agenda to improve algebra readiness of students before and during middle school. This opportunity is linked to what courses students take in high school, which in turn determines whether students are prepared for college coursework and careers in science, technology, engineering and mathematics. Thus, providing opportunities for students to develop algebraic reasoning and conceptual understanding of variable and relationship is a social justice issue.

We are poised for dramatic changes in school mathematics curricula as the majority of States implement the Common Core State Standards. It would be a missed opportunity not to take advantage of this initiative to seriously consider changes in the way we teach mathematics. Currently, the curricula are being driven by assessments of fragmented knowledge even as we decry the weakness of our students in making connections between knowledge they acquire. Our curriculum is criticized for being “a mile wide and inch deep” but our assessments do not focus on making connections between big ideas so such important mathematical work is relegated to a lower priority. Thus, we have lost opportunities to support students in developing mathematical competency grounded in number sense and growth in conceptual understanding and development of algebraic reasoning.

My hope is we will turn away from this irrational approach to our curriculum and focus on engaging students in in-depth exploration of content using reasoning approaches that are meaningful to them. The results of this study offer optimism: the students engaged in serious ways solving a sequence of increasingly difficult tasks in constant growth pattern problems, generating and using a rich variety of representations as meaningful (to them) reasoning tools. The students initiated their own shifts in reasoning approaches, including significant shifts to functional reasoning supportive of generalization. This study demonstrating student mathematical competency grounded in the students' emergent understanding of variable and relationship is a call for action. We are doing these students a great disservice if we are not providing opportunities for them to engage in reasoning about problems rich in potential for development of conceptual understanding and algebraic reasoning.

APPENDIX A

PROTOCOLS

A-1 **Background Interview Sample Questions**

A-2 **Task-Based Interview Protocol**

A-3 **Task-Based Interview Sample Prompts and Questions**

A-4 **Model Problems**

APPENDIX A-1

Background Interview Sample Questions

<u>Opening Questions</u>

1. What games do you like playing? Why? Do you use math in the games? How?

2. What do you do in your free time? Do you use math in the activity? How?

3. In what ways do you use mathematics outside the classroom? [probe to find out when students use math, with whom, if they ever learn mathematics outside of the classroom, etc.]

4. Do you think mathematics is important to know? What are some things that mathematics is important for? Why else is mathematics important?

5. Tell me about where you have gone to school.
i. Elicit student's schooling history for elementary school
ii. Ask about language of mathematics instruction

6. What language or languages do you speak at home? What language do you usually speak with friends? (In the neighborhood? At school?)

7. What language(s) do you speak in your classroom? If you could choose, would your classes be in English, Spanish, or both?

8. What language(s) do you speak in mathematics class? When you work in small groups in mathematics class, what language do you speak?

9. What language do you like to speak more, Spanish or English?

<u>Math Work Sample Questions</u>

1. Can you tell me about the math problem and how you could solve it?

2. Did you find any part of this problem easy? Why or why not? What part of this problem is easy?

3. Did you find any part of this problem hard? Why or why not? What part of this problem is hard?

4. How do you think this problem is related to your everyday life? (helps you think about your life?)

5. Do you feel that problems like this help you understand mathematics better? Why? Do you like this kind of problem?

APPENDIX A-2

Task-Based Interview Protocol

Introduction

1. Hello, my name is **________** and I'll be interviewing you today and asking you to talk about how you solve some mathematics problems.

2. I'm doing this as part of a research project that looks at how students solve mathematics problems. I'm hoping that we can find ways to make learning mathematics better for all different types of students. So I'm glad you are able to participate.

3. I want to thank you for giving up your time to help me complete this research. I want you to know that I will not be reporting anything from this interview to your teacher or other people you know.

4. Is it all right with you that we videotape the interview? I will be the only one listening to the tape, mostly so I can have an accurate record of your thinking and see the mathematics problems you are talking about.

5. Thanks. Now here is the first mathematics problem for you to look at.

<u>Mathematics Problem Solving Questions</u> (see Appendix D.4 for sample prompts and questions)
[Repeat for each task]

Conclusion:

Thank you for helping me today. Do you have any questions? Comments?

APPENDIX A-3

Task-Based Interview Sample Prompts and Questions

1. Can you read the problem to yourself and underline the words someone in your mathematics class might not know?

   [Follow up prompt: tell me something about this (word/sentence) you underlined.]

2. Can you please read the problem aloud now?
3. What do you think the problem is asking you to do?
4. Please talk about your work while you solve the problem.

   [Possible scaffolding prompts:
   What do you think you need to know in order to do the problem?
   Do you see a pattern?
   Would it help to use objects to do the problem?
   Can you draw a picture to show what is going on in the problem?]

5. Do you think this problem is easy or hard? Why?
6. What part of this problem is easy?
7. What part of this problem is hard?
8. How do you think these problems are related to your everyday life?
9. Do you feel that problems like this help you understand mathematics better? Why?

APPENDIX A-4
MODEL PROBLEM A

Marisa wants to save pennies.
Her mother gives her a jar with 4 pennies in it to start.
Marisa adds 3 pennies to the jar each day (day one, day two, day three,…)

Tasks:

1. Describe the pattern using words, pictures, or objects.

2. Find the number of pennies in the jar after Marisa adds pennies on day four.

   Show your work.

3. Find the number of pennies in the jar after Marisa adds pennies on day ten.

   Show your work.

4. Write a rule to find the number of pennies in the jar after any number of days

APPENDIX A-4
MODEL PROBLEM B

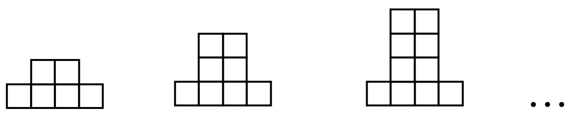


Figure 1 Figure 2 Figure 3

Tasks:

1. Describe the pattern using words, pictures, or objects.

2. Find the number of blocks in Figure 4 of the pattern.

   Show your work.

3. Find the number of blocks in Figure 10 of the pattern.

   Show your work.

4. Write a rule to find the number of blocks in any figure in the pattern.

APPENDIX A-4
MODEL PROBLEM C

4, 7, 10, ___, ___, ___, …

Tasks:

1. Describe the pattern using words, pictures, or objects.

2. Find the fourth term in the pattern.

   Show your work.

3. Find the tenth term in the pattern.

   Show your work.

4. Write a rule to find any term in the pattern.

REFERENCES

Abedi, J., & Lord, C. (2001). The language factor in mathematics tests. *Applied Measurement in Education, 14*, 219-234.

Bednarz, N., & Janvier, B. (1996). Emergence and development of algebra as a problem-solving tool: Continuities and discontinuities with arithmetic. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra. Perspectives for research and teaching* (pp. 115-136). Dordrecht, The Netherlands: Kluwer.Blanton, M., & Kaput, J. (2004). Elementary students' capacity for functional thinking. In . J. Holmes & A. B. Fugelstad (Eds.), *Proceedings of the 28th PME International Conference, 2,* 135-142.

Blanton, M. (2008). Algebra and the elementary classroom: Transforming thinking, transforming practice. Portsmouth, NH: Heinemann.

Blanton, M., & Kaput, J. (2004). Design principles for instructional contexts that support students' transition from arithmetic to algebraic reasoning: Elements of task and culture. In R. Nemirovsky, B. Warren, A. Rosebery, & J. Solomon (Eds.), *Everyday matters in science and mathematics* (pp. 211–234). Mahwah, NJ: Lawrence Erlbaum.

Boaler, J. (1993). Encouraging the transfer of 'school' mathematics to the 'real world' through the integration of process and content, context and culture. *Educational Studies in Mathematics*, 25(4), 341-373.

Bogdan, R. C., & Biklen, S. K. (2003). *Qualitative research for education: An introduction ot theories and methods* (Fourth ed.). Boston: Allyn and Bacon.

Carpenter, T., & Franke, M. (2001). Developing algebraic reasoning in the elementary school: Generalization and proof. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the Twelfth ICMI Study Conference: The future of the teaching and learning of algebra: Vol. 1* (pp. 155-162). Melbourne, Australia: University of Melbourne Press.

Celedon-Pattichis, S. (2003). Constructing meaning: Think-aloud protocols of ELLs on english and spanish word problems. *Educators for Urban Minorities,* 2 (2), 74-90.

Cohen, E. G. (1994). *Designing groupwork: Strategies for the heterogeneous classroom* (Second ed.). New York: Teachers College Press.

Davis, R. (1985). ICME-5 Report: Algebraic thinking in the early grades. *Journal of Mathematical Behavior,* 4, 195-208.

Davis, R. (1989). Theoretical considerations: Research studies in how humans think about algebra. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (Vol. 4, pp. 266-274). Reston, VA: The National Council of Teachers of Mathematics/Lawrence Erlbaum Associates.

Ellis, A. (2007). The influence of reasoning with emergent quantities on student generalizations. *Cognition and Instruction,* 25(4), 439-478.

Filoy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics*, 9(2), 19-25.

Ginsberg, H. (1997). *Entering the child's mind : the clinical interview in psychological research and practice*. Cambridge, MA: Cambridge University Press.

Goldin, G. & Shteingold, N. (2001). Systems of representations and the development of mathematical concepts. In A. A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics. The National Council of Teachers of Mathematics 2001 Yearbook* (pp. 1-23). Reston, VA: The National Council of Teachers of Mathematics.

Hiebert, J., Carpenter, T., Fennema, E., Fuson, K., Human, P., Murray, H., Olivier, A., & Wearne, D. (1997). *Making sense: Teaching and learning mathematics with understanding.* Portsmouth, NH: Heinemann.

Kaput, J. (1998). Transforming algebra from an engine of inequality to an engine of mathematical power by “algebrafying” the K-12 curriculum. In National Council of Teachers of Mathematics (Eds.), *The nature and role of algebra in the K-14 curriculum*. Washington, DC: National Academy Press.

Kaput, J., Blanton, M., & Moreno, L. (2008). Algebra from the symbolization point of view. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the Early Grades* (pp. 19-56). Mahwah, NJ: Lawrence Erlbaum Associates/Taylor & Francis Group.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12,* 317–326.

Lee, L. (1996). An initiation into algebraic culture through generalization activities. In N. Bednarz, C. Kieran, & L. Lee (Eds.), Approaches to algebra: Perspectives for research and teaching (pp. 65–86). Dordrecht: Kluwer.

Martiniello, M. (2008). Language and the Performance of English Language Learners in Math Word Problems. *Harvard Educational Review*, 78(2).

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), Approaches to algebra: Perspectives for research and teaching (pp. 65–86). Dordrecht: Kluwer.

Moss, J. (2005). *Integrating numeric and geometric patterns: A developmental approach to young students' learning of patterns and functions*. Paper presented at the Canadian mathematics education study group annual meeting, Ottowa, Canada.

Moss, J., Beatty, R., Barkin, S., & Shillolo, G. (2008). What is your theory? What is your rule? Fourth graders build an understanding of functions through patterns and generalizing problems. In C. E. Greenes & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics: NCTM Seventieth Yearbook* (pp. 155-68). Reston, VA: NCTM.

Moss, J., Beatty, R., McNab, S. L., & Eisenband, J. (2006, April). The potential of geometric sequences to foster young students' ability to generalize in mathematics. Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Schliemann, A., Carraher, D., & Brizuela, B. (2007). From quantities to ratio, functions, and algebraic notation. In A. D. Schliemann, D. W. Carraher, & B. M. Brizuela (Eds.), *Bringing out the algebraic character of arithmetic: From children's ideas to classroom practice* (pp. 85-104). Mahwah, NJ: Erlbaum.

Schoenfeld, A. (1995). Report of Working Group 1. In C. B. LaCampagne (Ed.), *The algebra initiative colloquium.* Vol. 2: Working group papers (pp. 11-18). Washington, DC: U. S. Department of Education, OERI.

Schwartz, J. (1990). Getting students to function in and with algebra. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology* (pp. 261-289). Washington, DC: Mathematics Associations of America.

Smith, E. (2003). Stasis and change: Integrating patterns, functions, and algebra throughout the K-12 curriculum. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 136-150). Reston, VA: National Council of Teachers of Mathematics.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics: Concepts and process* (pp. 189-199). New York: Academic Press.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number Concepts and Operations in the Middle Grades* (pp. 141-161). Reston, VA: NCTM.

Vygotsky, L. S. (1978). *Mind in society*. Cambridge, MA: Harvard University Press.

Warren, E. (2005). Young children's ability to generalize the pattern rule for growing patterns. In H. Chick and J. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (pp. 305–12). Melbourne, Victoria: University of Melbourne.

Warren, E., & Cooper, (2008). Patterns that support early algebraic thinking in the elementary school. In C. E. Greenes & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics: NCTM Seventieth Yearbook* (pp. 113-126). Reston, VA: NCTM.