On A-expansions of Drinfeld Modular Forms

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Dedication

For my mom and dad.

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Abstract

In this dissertation, we introduce the notion of Drinfeld modular forms with A-expansions, where instead of the usual Fourier expansion in t^n (t being the uniformizer at infinity), parametrized by $n \in \mathbb{N}$, we look at expansions in t_a, parametrized by $a \in A = \mathbb{F}_q[T]$. We construct an infinite family of such eigenforms. Drinfeld modular forms with A-expansions have many desirable properties that allow us to explicitly compute the Hecke action. The applications of our results include: (i) various congruences between Drinfeld eigenforms; (ii) interesting relations between the usual Fourier expansions and A-expansions, and resulting recursive relations for special families of forms with A-expansions; (iii) the computation of the eigensystems of Drinfeld modular forms with A-expansions; (iv) many examples of failure of multiplicity one result, as well as a restrictive multiplicity one result for Drinfeld modular forms with A-expansions; (v) the proof of diagonalizability of the Hecke action in 'non-trivial' cases; (vi) examples of eigenforms that can be represented as 'non-trivial' products of eigenforms; (vii) an extension of a result of Böckle and Pink concerning the Hecke properties of the space of cuspidal modulo double cuspidal forms for $\Gamma_1(T)$ to the groups $\mathrm{GL}_2(\mathbb{F}_q[T])$ and $\Gamma_0(T)$.

Chapter 1
Introduction

Due to the fundamental work of many leading mathematicians: Eisenstein, Jacobi, Dedekind, Ramanujan, Hilbert, Hecke, Siegel, Weil, Eichler, Shimura, Langlands, Deligne, Serre, Manin, Mazur, Ribet, Hida, Zagier, Wiles, etc.; we know that classical modular forms are ultimately related to arithmetic properties of number fields. Congruence properties of arithmetic functions, representability of integers by quadratic forms, 2-dimensional Galois representations, properties of zeta functions and L-functions, are all connected with classical modular forms.

A classical modular form for Γ (take $\Gamma_0(N)$ or $\Gamma_1(N)$ for concreteness) is a $\mathbb{C}$-valued holomorphic function on the Poincaré upper half-plane $\mathcal{H}$ that satisfies certain transformation properties with respect to Γ and which extends analytically to the cusps (the points that have to be added to compactify $\Gamma\backslash\mathcal{H}$). The last condition implies that any classical modular form f_{cl} can be expanded into a Fourier expansion (since $\Gamma_0(N)$ and $\Gamma_1(N)$ both contain translation by 1, we can use the parameter $q = e^{2\pi i z}$)

$$f_{\mathrm{cl}} = \sum_{n=0}^{\infty} a_n q^n, \qquad\qquad q = e^{2\pi i z}.$$

Modular forms that vanish at the cusps, which necessitates $a_0 = 0$, are called cuspidal. We have a family of Hecke operators $\{T_p\}_{p\in\mathrm{Spec}(\mathbb{Z})}$, which acts on modular forms and can be used to single out modular forms with Fourier coefficients that have arithmetic significance. Indeed, if f_{cl} is a newform (an eigenform for all the Hecke operators with $a_1 = 1$), then its coefficients lie in the ring of integers of a number field $F_{f_{\mathrm{cl}}}$. Shimura (weight 2), Deligne (general weight bigger than 2) and Deligne-Serre (weight 1) have shown that such eigenforms can be interpreted in cohomological terms, and given any prime $\ell \nmid N$ one can attach a 2-dimensional Galois representation $\varrho_{f_{\mathrm{cl}}}$ of the absolute

Galois group of $\mathbb{Q}$ with values in $F_{f_{\rm cl}} \otimes \mathbb{Q}_\ell$. The properties of $\varrho_{f_{\rm cl}}$ are ultimately related with the Fourier coefficients of $f_{\rm cl}$. For example, one can show that $\varrho_{f_{\rm cl}}$ is unramified outside of $N\ell$ and the characteristic polynomial of Frobenius at any unramified prime p is closely linked with a_p and the weight of the modular form $f_{\rm cl}$. Several important facts make the theory extremely useful: the Hecke action of T_p on a q-expansion is easily computable; eigenforms exist in abundance; by looking at the Fourier coefficients of a cuspidal modular form one can see whether it is an eigenform or not; both modular forms and Hecke operators have geometric interpretations in terms of moduli spaces of elliptic curves with additional properties.

Drinfeld [Dr1] studied the analog of the moduli space of elliptic curves for a global function field in characteristic $p > 0$. His work plays an important role in the theory and earned him the Fields medal for his contribution to the understanding the Langlands conjectures in dimension 2 for function fields. For that he used complex valued automorphic forms. In this thesis, we will concentrate on Drinfeld modular forms which take values in a field of characteristic p and were first defined by David Goss in his Harvard thesis [Go1]. Drinfeld modular forms are rigid analytic functions on the Drinfeld upper half-plane (an analog of the Poincaré upper half-plane) that satisfy certain transformation properties with respect to a congruence subgroup of $\mathrm{GL}_2(\mathbb{F}_q[T])$ and that extend analytically to the cusps. Goss defined the analog of the Hecke algebra that is spanned by the set $\{\mathrm{T}_\mathfrak{p}\}_{\mathfrak{p}\in\mathrm{Spec}(\mathbb{F}_q[T])}$ of Hecke operators. He also showed that there is an analog of the classical Fourier expansion, namely every Drinfeld modular form can be written as a power series

$$f_{\rm Drin} = \sum_{n=0}^{\infty} a_n t^n,$$

for an appropriate function $t = t(z)$. However there are several important differences that to this day make the theory of Drinfeld modular forms less understood in comparison to the classical theory. Arguably the most significant differences are: the apparent disconnect between the Fourier coefficients and the Hecke operators and eigenvalues (the first is indexed by the natural numbers, while the latter is indexed by elements of $A = \mathbb{F}_q[T]$); the lack of transparent arithmetic significance of the Fourier coefficients of an eigenform due to that disconnect; lack of diagonalizability of the Hecke action (for a space that is not diagonalizable, see Example 2.4.15); and the lack of multiplicity one property (if we assume that the notion of multiplicity one remains the same as in the classical case).

Note that these are not independent of each other and progress on one of them will ultimately lead to progress on the others. A great amount of work on Drinfeld modular forms has been done since Goss' original work. There has been some progress on understanding the points above due to Goss, Gekeler, Armana, Bosser, Cornellisen, Gardeyn, Lopez, Pellarin, Pink, Teitelbaum and many others. And in a recent paper, Böckle [Bö1] has shown how to attach 1-dimensional Galois representations to cuspidal eigenforms by using a new geometric theory of crystals [BöPi]. Therefore computations require sophisticated techniques (the theory of crystals), even though the representations are always abelian (unlike the 2-dimensional representations of the classical case). The work of Böckle gives an important theoretical tool for the study of Drinfeld eigenforms. Nevertheless the theory of Drinfeld modular forms and the Hecke action on them remains much more mysterious than the classical theory of modular forms.

The present dissertation aims to address the points presented above by developing another tool: Drinfeld modular forms with A-expansions. Examples of such forms have been known to Goss in the form of Eisenstein series, which until recently appeared to be the only examples. In a recent paper [Lo1], Bartolome Lopez showed two additional examples of Drinfeld modular forms with A-expansions. In this dissertation, we prove that there are infinitely many examples of cuspidal Drinfeld eigenforms with A-expansions. This is achieved by Theorem 3.1.10. Drinfeld modular forms with A-expansions are important because their properties with respect to the Hecke action are very similar to the properties of classical modular forms.

The action of $T_{\mathfrak{p}}$ on a Drinfeld modular form with an A-expansion is easily computable (Theorem 3.1.5), and the forms with A-expansions satisfy the so called 'multiplicity one property'. Classically multiplicity one refers to the property that a newform is uniquely determined up to a multiplicative constant by its set of eigenvalues $\{\lambda_p\}_{p\in\mathrm{Spec}(\mathbb{Z})}$. The situation is different in the case of Drinfeld modular forms. Goss found two cuspidal eigenforms with different weights, but with the same eigenvalues. Our Theorem 3.1.10 produces a variety of such examples. Gekeler asked if a cuspidal Drinfeld eigenform is uniquely determined by its eigenvalues $\{\lambda_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathrm{Spec}(\mathbb{F}_q[T])}$ and weight k. Corollary 3.1.7 in Chapter 3 shows that this is indeed the case for Drinfeld modular forms for $\mathrm{GL}_2(\mathbb{F}_q[T])$ with the additional assumption that the forms possess A-expansions. This is the restrictive multiplicity one property of Drinfeld modular forms with A-expansions. It is not known if this remains true in general for $\mathrm{GL}_2(\mathbb{F}_q[T])$ if we do not assume the existence of an A-expansion.

Böckle has shown ([Bö1, Example 15.4]) that the answer to Gekeler's question is negative for the congruence subgroup $\Gamma_1(T)$ even if we require that the form is a double cuspidal newform (i.e., that the form vanishes to order two at the cusps and is not induced from a bigger congruence subgroup).

Another question suggested by the classical theory is the question of diagonalizability of the Hecke action, i.e., the existence of a basis of our space of modular forms that consists of eigenforms for all the Hecke operators. For $\mathrm{SL}_2(\mathbb{Z})$ this always happens for cuspidal forms. As we mentioned before, in the Drinfeld setting there are known counterexamples to diagonalizability (see Example 2.4.15). It is not known which spaces are diagonalizable for general weight even for the group $\mathrm{GL}_2(\mathbb{F}_q[T])$. By using A-expansions, we prove a general result in that direction (see Corollary 3.1.23).

Böckle has shown that for $\Gamma_1(T)$ the space of cuspidal Drinfeld modular forms modulo the space of double cuspidal Drinfeld modular forms is always diagonalizable by eigenforms with very simple eigenvalues. We show that this space is spanned by Drinfeld modular forms with A-expansions and this allows us to reprove his result

(Example 4.3.3) and extend it to $\mathrm{GL}_2(A)$ (Corollary 3.1.11) and to $\Gamma_0(T)$ (Theorem 4.3.4).

We show interesting relations in A-expansions in Theorem 3.1.13 and Theorem 3.1.22. Another result is the proof of a new set of congruences between Drinfeld eigenforms, Theorem 3.1.24, that is easy to see through the A-expansions. Congruences between classical modular forms are a stepping stone for the development of mod p and p-adic modular forms and Galois representations, and therefore the notion of A-expansions may be useful when dealing with this aspect of the theory for Drinfeld modular forms.

It is the hope of the author that what we are seeing is merely a special case of a more general theory of non-standard expansions of Drinfeld modular forms, which would interact with recent works of Böckle and Pellarin. Generalizations in several different directions might be possible: more general A-expansions with characters and other modifications, product expansions, extensions to higher genus generalizations of $A = \mathbb{F}_q[T]$ and higher level congruence subgroups.

Throughout this thesis the following notation will be in place:

- We use the standard notation $:=$ to mean that the object on the left is defined to be the object on the right.
- p will be a non-zero prime in $\mathbb{Z}$, $q := p^e$, $\mathbb{F}_q$ will denote the field with q elements.
- $A := \mathbb{F}_q[T]$ will be the polynomial ring over $\mathbb{F}_q$, A_+ will denote the set of monic polynomials in T.
- $\mathfrak{p}$ will denote a prime ideal in A and $\wp$ will be its unique monic generator.
- deg will denote degree with respect to T. If we compute degree with respect to a different variable, we will put a subscript to denote this, for example $\deg_X$.
- $K := \mathrm{Frac}(A)$ is the fraction field of A.
- ∞ is the place of K that does not come from a prime ideal of A. The place ∞ has the uniformizer $\frac{1}{T}$.
- $|\cdot|$ will denote the absolute value coming from ∞, which is normalized so that $|a| = q^{\deg(a)}$ for all $a \in A$.
- K_∞ is the completion of K at ∞, A_∞ is its valuation ring, π_∞ a fixed generator of the unique maximal ideal of A_∞. We will choose $\pi_\infty := \frac{1}{T}$. Concretely, we have $K_\infty = \mathbb{F}_q((\frac{1}{T}))$, $A_\infty = \mathbb{F}_q[[\frac{1}{T}]]$.
- $\mathbb{C}_\infty$ is the completion of an algebraic closure of K_∞ with respect to the absolute value induced from ∞.
- $K_\mathfrak{p}$ will be the completion of K at $\mathfrak{p}$, $A_\mathfrak{p}$ its valuation ring.
- L will generally denote a finite extension of K, the integral closure of A in L will be denoted by A_L.
- $\mathbb{A}_K^*$ will denote the ideles of K.

Chapter 2
Drinfeld Modular Forms

2.1 Drinfeld modules over affine A-schemes

In this section, we review some necessary background on Drinfeld modules (see [Tha], [Go4], and [Bö1]).

Let R be a commutative A-algebra via $\iota : A \to R$. That is, let $\operatorname{Spec} R$ be an affine A-scheme via $s : \operatorname{Spec} R \to \operatorname{Spec} A$, with $s^\sharp = \iota$. We will denote by $\mathbb{G}_{a,R}$ the additive group scheme over R. Let φ be the Frobenius endomorphism $X \to X^q$. The ring $\operatorname{End}_{\mathbb{F}_q}(\mathbb{G}_{a,R})$ of $\mathbb{F}_q$-linear endomorphisms of $\mathbb{G}_{a,R}$ can be explicitly described (see [Tha, Section 2.2]): $\operatorname{End}_{\mathbb{F}_q}(\mathbb{G}_{a,R}) = R\{\varphi\}$, the polynomial ring in φ over R with commutation relation $\varphi r = r^q \varphi$ for all $r \in R$.

Given any $\phi = x_0\varphi^0 + x_1\varphi + \cdots + x_n\varphi^n \in R\{\varphi\}$ we define

$$\partial\,\phi := x_0,$$

$$\operatorname{rk}\phi := n.$$

Definition 2.1.1. A *Drinfeld module* of rank $r > 0$ over R is an $\mathbb{F}_q$-linear ring homomorphism

$$\phi : A \to R\{\varphi\} : a \mapsto \phi_a$$

such that for every non-zero $a \in A$ we have:

$$\partial\,\phi_a = \iota(a)$$

$$\operatorname{rk}\phi_a = r\deg(a).$$

Example 2.1.2. Let $R = K$. Define the Drinfeld module ρ by

$$\rho_T := T\varphi^0 + \varphi.$$

This is a rank 1 Drinfeld module called the *Carlitz module*, which will play a central role in this dissertation. Let $a \in A$ be non-zero. We define $l_i(a)$ to be the φ^i-th coefficient of ρ_a, i.e., $l_i(a)$ is defined by

$$\rho_a = l_0(a)\varphi^0 + l_1(a)\varphi + \cdots + l_{\deg(a)}(a)\varphi^{\deg(a)}.$$

By using the relations coming from $\rho_a\rho_T = \rho_T\rho_a$ we can compute:

$$l_i(a) = \frac{l_{i-1}(a)^q - l_{i-1}(a)}{T^{q^i} - T}.$$

The *characteristic* of the Drinfeld module ϕ is the kernel of the map ι. In this dissertation, we will consider only Drinfeld modules of *generic characteristic*, i.e., Drinfeld modules such that ι is injective. If ϕ, ϕ' are two Drinfeld modules, then $\mu \in R\{\varphi\}$ is a *morphism* between ϕ and ϕ' if and only if $\mu\phi_a = \phi'_a\mu$ for all $a \in A$. A non-zero morphism μ is called an *isogeny.*

The ring of all endomorphisms $\phi \to \phi$ will be denoted by $\mathrm{End}(\phi)$. Given an isogeny $\mu \in \mathrm{End}(\phi)$ we denote its kernel by $\phi[\mu]$. For example, if $R = \overline{k}$ is an algebraically closed field, then $\phi[\mu] = \{z \in R : \mu(z) = 0\}$. An important isogeny ϕ_a is the *multiplication by a*. We denote its kernel by $\phi[a]$.

Definition 2.1.3. Given an ideal $\mathfrak{n} \subset A$ we define $\phi[\mathfrak{n}]$ by

$$\phi[\mathfrak{n}] := \bigcap_{a \in \mathfrak{n}} \phi[a].$$

We call $\phi[\mathfrak{n}]$ the set of $\mathfrak{n}$-*torsion points* of ϕ. Since A is a PID we see that if $\mathfrak{n} = (b)$, then

$$\phi[\mathfrak{n}] = \phi[b].$$

Theorem 2.1.4. (See [Tha, Section 2.3].) *Assume that ϕ is of generic characteristic. If $R = \overline{k}$ is algebraically closed field, then we have an isomorphism as A-modules*

$$\phi[\mathfrak{p}^n] \cong (A/\mathfrak{p}^n)^r. \quad \square$$

Remark 2.1.5. One can define the notion of Drinfeld modules of rank r over a general Noetherian A-scheme S and prove that $\phi[\mathfrak{n}]$ is a finite flat, étale group scheme of rank $|A/\mathfrak{n}|^r$ over S away from $\mathfrak{n}$. For more on this see [Bö1, Section 1.2] and the references therein.

2.2 Lattices and Drinfeld modules over $\mathbb{C}_\infty$

Let $\Lambda \subset \mathbb{C}_\infty$ be a *lattice* (a discrete projective A-module) of rank r. Two lattices Λ, Λ' will be called *homothetic* if and only if there exists $c \in \mathbb{C}_\infty^*$ such that $\Lambda = c\Lambda'$. We define the exponential function of Λ by the formula

$$e_\Lambda(z) := z \prod_{\lambda\in\Lambda}{}' \left(1 - \frac{z}{\lambda}\right) \in \mathbb{C}_\infty, \qquad\qquad z \in \mathbb{C}_\infty.$$

Here we have used the standard convention that a primed product (or sum) is a product (or sum) that omits zero. The discreteness of Λ shows that the infinite product converges for all z and non-archimedean analysis shows that $e_\Lambda(z)$ has kernel Λ, surjects onto $\mathbb{C}_\infty$, and is an entire function on $\mathbb{C}_\infty$. Approximating Λ by sublattices with finitely many elements shows that e_Λ is $\mathbb{F}_q$-linear and has a power series expansion

$$e_\Lambda(z) = z + \alpha_1 z^q + \cdots = \sum_{j=0}^{\infty} \alpha_j z^{q^j}.$$

If Λ' and Λ are homothetic, so $\Lambda' = c\Lambda$ for some $c \in \mathbb{C}_\infty^*$, then it is immediate that $ce_\Lambda(z) = e_{\Lambda'}(cz)$.

Given a lattice Λ, we let

$$S_{\Lambda,n} := \sum_{\lambda\in\Lambda} \frac{1}{(z+\lambda)^n}.$$

By logarithmic differentiation,

$$S_{\Lambda,1} = \frac{e'_\Lambda(z)}{e_\Lambda(z)} = \frac{1}{e_\Lambda(z)}.$$

We will denote $S_{\Lambda,1}$ by t_Λ, i.e., define

$$t_\Lambda := S_{\Lambda,1}.$$

David Goss observed [Go1, Theorem 2.2.3] that the following useful relations hold between the $S_{\Lambda,n}$'s and t_Λ.

Theorem 2.2.1. *Given n, there exists a monic polynomial $G_{\Lambda,n}(X)$ of degree n such that*

$$S_{\Lambda,n} = G_{\Lambda,n}(t_\Lambda).$$

In addition, we have the formulas

$$\begin{aligned} G_{\Lambda,n}(X) &= X^n, && 1 \le n \le q; \\ G_{\Lambda,pn}(X) &= (G_{\Lambda,n}(X))^p, && \forall n \in \mathbb{Z}_{\ge 1}; \\ G_{\Lambda,n}(X) &= X(G_{\Lambda,n-1}(X) + \cdots + \alpha_i G_{\Lambda,n-q^i}(X) + \cdots), && n - q^i \ge 0; \end{aligned}$$

where the α_i's are the coefficients of the power series expansion for e_Λ. □

In [Ge3, (3.8)] Gekeler uses generating functions to obtain an explicit formula for the coefficients of $G_{\Lambda,n}$:

Theorem 2.2.2. *We have*

$$G_{\Lambda,n}(X) = \sum_{0 \le j \le n-1} \left(\sum_{\underline{i}} \binom{j}{\underline{i}} \alpha^{\underline{i}} \right) X^{j+1},$$

where $\underline{i} = (i_0, \ldots, i_s)$ runs over the set of $(s+1)$-tuples ($s \ge 0$, arbitrary) of non-negative integers satisfying

$$i_0 + i_1 + \cdots + i_s = j \quad \textit{and} \quad i_0 + i_1 q + \cdots + i_s q^s = n - 1,$$

and $\alpha^{\underline{i}} := \alpha_0^{i_0} \cdots \alpha_s^{i_s}$, and $\binom{j}{\underline{i}}$ denotes the multinomial coefficient $\frac{j!}{i_0! \cdots i_s!}$. □

The reader should note that there is no assumption that $i_\nu \le (q-1)$, for $0 \le \nu \le s$.

Remark 2.2.3. A closer look at the formula in the theorem shows that for $n > 1$

$$\mathrm{ord}_X(G_{\Lambda,n}(X)) = 2 \iff n = q^l + 1$$

for some $l \ge 0$. Indeed, if $n = q^l + 1$, then $j = 1$ is the lowest term we can take in the formula, because $n - 1 = q^l$ and $j = i_l = 1 = \mathrm{ord}_X(G_{\Lambda,n}(X)) - 1$. On the other hand, if $\mathrm{ord}_X(G_{\Lambda,n}(X)) = 2$, then according to the theorem above we must have

$i_l = 1 = \mathrm{ord}_X(G_{\Lambda,n}(X)) - 1$, for some $l \geq 0$, and therefore $n - 1 = q^l$. We conclude that in the notation of the theorem

$$G_{\Lambda,q^l+1}(X) = \alpha_l X^2 + \mathcal{O}(X^3).$$

Given a lattice Λ, one can construct a Drinfeld module ϕ_*^Λ. The rank of ϕ_*^Λ is equal to the rank of the lattice Λ. The construction of ϕ_*^Λ is quite explicit. Let $a \in A$ and define

$$\phi_a^\Lambda(z) := az \prod_{\lambda \in \frac{1}{a}\Lambda \setminus \Lambda}' \left(1 - \frac{z}{e_\Lambda(\lambda)}\right).$$

Using non-archimedean analysis over $\mathbb{C}_\infty$, we find that

$$e_\Lambda(az) = \phi_a^\Lambda(e_\Lambda(z))$$

and ϕ_a^Λ determines an element in $\mathbb{C}_\infty\{\varphi\}$.

We have $\phi_a^\Lambda = c\phi_a^{c\Lambda}c^{-1}$, i.e., the Drinfeld modules of two homothetic lattices are isomorphic.

Conversely, given a Drinfeld module ϕ one can use the functional equation $e(az) = \phi_a(e(z))$ to solve for the coefficients of $e(z)$ and take the corresponding lattice Λ_ϕ to be its kernel.

By these considerations, Drinfeld (see [Tha, Section 2.4]) proved that

Theorem 2.2.4. *The natural correspondence described above gives a bijection between isomorphism classes of Drinfeld modules of rank r over $\mathbb{C}_\infty$ and homothety classes of rank r lattices inside $\mathbb{C}_\infty$.* □

Example 2.2.5 (Carlitz module)**.** Since there exists a unique class of homothetic lattices of rank 1, we see that there is a unique isomorphism class of Drinfeld modules of rank 1 over $\mathbb{C}_\infty$. The representative of this class such that $\phi_T = T\varphi^0 + \varphi$ corresponds to the lattice $\widetilde{\pi}A$, where $\widetilde{\pi}$ is an irrational in $\mathbb{C}_\infty$. As we have already remarked, the Drinfeld module corresponding to $\widetilde{\pi}A$ is called the *Carlitz module*. It was first studied by Carlitz long before Drinfeld defined the notion of Drinfeld modules. Carlitz was able to compute the coefficients of $e_{\widetilde{\pi}A}(z)$ (see [Tha, Section 2.5]), which we now recall.

For $i \geq 0$, let

$$[i] := T^{q^i} - T = \text{product of all monic primes of degree dividing } i,$$

$$D_i := [i][i-1]^q \cdots [1]^{q^{i-1}} = \text{product of all monics of degree } i,$$

$$L_i := [i][i-1] \cdots [1] = \text{the l.c.m. of all monics of degree } i,$$

so that $D_0 = L_0 = 1$. With this notation, we have

$$e_{\widetilde{\pi}A}(z) = \sum_{j=0}^{\infty} \frac{z^{q^j}}{D_j}.$$

From now on, we fix the following notation ($a \in A_+$):

$$\rho := \phi_*^{\widetilde{\pi}A},$$

$$t := t(z) = t_{\widetilde{\pi}A}(\widetilde{\pi}z) = \frac{1}{e_{\widetilde{\pi}A}(\widetilde{\pi}z)} = \frac{1}{\widetilde{\pi}e_A(z)} = \frac{1}{\widetilde{\pi}} \sum_{\lambda \in A} \frac{1}{z+\lambda},$$

$$e(z) := e_{\widetilde{\pi}A}(z),$$

$$\psi_a(X) := \rho_a(X^{-1})X^{q^{\deg(a)}} = 1 + \cdots ,$$

$$t_a = t(az) := \frac{t^{q^{\deg(a)}}}{\psi_a(t)} = t^{q^{\deg(a)}} + \cdots .$$

We call ψ_a the a^{th} *inverse cyclotomic polynomial*, in contrast with $\rho_a(X)$: the a^{th} *cyclotomic polynomial* for the Carlitz module.

We denote by G_n the multiple of $G_{\widetilde{\pi}A,n}$ which has 1 as its first non-zero coefficient. We will need the following lemma and its corollaries in Theorem 3.1.24.

Lemma 2.2.6. *Let n be a positive integer. Let $d > \log_q(n)$. If $\mathfrak{p}$ is any prime of degree d, then the polynomial $G_{\widetilde{\pi}A,n}(X)$ has coefficients with denominators not divisible by $\mathfrak{p}$.*

Proof. By Theorem 2.2.2 and the explicit description of the coefficients of $e(z)$, we have

$$G_{\widetilde{\pi}A,n}(X) = \sum_{0 \leq j \leq n-1} \left(\sum_{\underline{i}} \binom{j}{\underline{i}} \alpha^{\underline{i}} \right) X^{j+1},$$

where $\underline{i} = (i_0, \ldots, i_s)$ runs over the set of $(s+1)$-tuples ($s \geq 0$, arbitrary) satisfying

$$i_0 + i_1 + \cdots + i_s = j \quad \text{and} \quad i_0 + i_1 q + \cdots + i_s q^s = n - 1,$$

and

$$\alpha^{\underline{i}} = \frac{1}{D_0^{i_0} D_1^{i_1} \cdots D_s^{i_s}} = \frac{1}{D_1^{i_1} \cdots D_s^{i_s}}.$$

Because D_l = product of all monic polynomials of degree l, we see that $\lfloor \log_q(n) \rfloor$ is the biggest degree that a prime dividing the denominators of the coefficients of $G_{\tilde{\pi}A,n}(X)$ can have. □

Remark 2.2.7. Since we are inverting the coefficient of the lowest term of $G_{\tilde{\pi}A,n}$ in the definition of G_n, the theorem above does not remain true if we replace $G_{\tilde{\pi}A,n}$ with G_n for general n. From Theorem 2.2.2, we see that the nature of the lowest degree coefficient of $G_{\tilde{\pi}A,n}$ depends in a complicated manner on the q-adic properties of $n-1$ and the vanishing of certain multinomial coefficients modulo p.

We give conditions based on the formula in Theorem 2.2.2 that ensure that the coefficients of G_n have denominators that are no worse than the denominators of $G_{\tilde{\pi}A,n}$ in the next corollaries.

Corollary 2.2.8. *Let $n > 0$ be an integer. Let $0 \leq j \leq n-1$. Let $\mathfrak{S}_n$ be the set of $(s+1)$-tuples $\underline{i} = (i_0, \ldots, i_s)$, $s \geq 0$ arbitrary, that satisfy*

$$i_0 + i_1 + \cdots + i_s = \operatorname{ord}_X(G_{\tilde{\pi}A,n}), \qquad i_0 + i_1 q + \cdots + i_s q^s = n-1,$$
$$\text{and} \quad \binom{\operatorname{ord}_X(G_{\tilde{\pi}A,n})}{\underline{i}} \not\equiv 0 \bmod p.$$

If $|\mathfrak{S}_n| = 1$, then for any prime $\mathfrak{p}$ of degree d with $d > \log_q(n)$, the polynomial $G_n(X)$ has coefficients with denominators not divisible by $\mathfrak{p}$.

Proof. According to Theorem 2.2.2, we have

$$G_n(X) = \frac{D_0^{i_0} \cdots D_s^{i_s}}{\binom{\operatorname{ord}_X(G_{\tilde{\pi}A,n})}{\underline{i}}} G_{\tilde{\pi}A,n}(X).$$

Now the result follows from Lemma 2.2.6. □

Corollary 2.2.9. *Let n be an integer such that $1 \le n \le q^2$. If $\mathfrak{p}$ is any prime of degree d with $d > \log_q(n)$, then the polynomial $G_n(X)$ has coefficients with denominators not divisible by $\mathfrak{p}$.*

Proof. If n is between 1 and q, or is a p^{th}-power, then we have $G_n(X) = X^n$. So $G_n(X) = G_{\widetilde{\pi}A,n}(X)$, hence the result is Lemma 2.2.6.

Therefore, assume that $q < n < q^2$. Let n_0, n_1 be the q-adic digits of $n-1$, i.e.,

$$n - 1 = n_0 + n_1 q, \qquad\qquad 0 \le n_0, n_1 \le q - 1.$$

Then the set $\{(i_0, i_1) : i_0 + i_1 q = n - 1\}$ consists of the tuples $(n_0 + lq, n_1 - l)$ for $0 \le l \le n_1$. The equality

$$n_0 + lq + n_1 - l = n_0 + l'q + n_1 - l'$$

implies that $l = l'$. Therefore, the set $\mathfrak{S}_n$ of tuples (defined in the previous corollary) consists of only one element and the proof is complete by Corollary 2.2.8. □

Corollary 2.2.10. *Let $n > 1$ be a positive integer such that $n = q^l + 1$. If $\mathfrak{p}$ is any prime of degree d with $d > \log_q(n)$, then the polynomial $G_n(X)$ has coefficients with denominators not divisible by $\mathfrak{p}$.*

Proof. Assume that $n = q^l + 1$. Then according to Remark 2.2.3 and the explicit description of the coefficients of $e(z)$, we have

$$G_n(X) = D_l \cdot G_{\widetilde{\pi}A,n}(X).$$

Therefore the result follows from Lemma 2.2.6. □

Corollary 2.2.11. *Let n be a positive integer such that $n = q^l - 1$. If $\mathfrak{p}$ is any prime of degree d with $d > \log_q(n)$, then the polynomial $G_n(X)$ has coefficients with denominators not divisible by $\mathfrak{p}$.*

Proof. The condition $n = q^l - 1$ allows us to use a result of Gekeler [Ge3, (3.10)] about computing $G_{\Lambda,n}$ for $\mathbb{F}_q$-lattices Λ. As a special case of this result with $\Lambda = \widetilde{\pi}A$, we have

$$G_{\widetilde{\pi}A,n}(X) = \sum_{i<l} \frac{(-1)^i}{L_i} X^{q^l - q^i}.$$

Therefore

$$G_{q^l-1}(X) = (-1)^{l-1} L_{l-1} G_{\widetilde{\pi}A,q^l-1}(X)$$

and because L_{l-1} is the l.c.m. of monics of degree $l-1$, Lemma 2.2.6 completes the proof. □

Remark 2.2.12. As we have already remarked, determining the coefficients of $G_{\widetilde{\pi}A,n}(X)$, consequently of $G_n(X)$, appears difficult in general. In each of the corollaries above, we were basically using the fact that numerator of the lowest degree coefficient of $G_{\widetilde{\pi}A,n}(X)$ is not divisible by primes $\mathfrak{p}$ of degree $> \log_q(n)$. Based on numerical evidence, we conjecture that this is always the case for $q = 2$. More precisely we conjecture that the following is true:

If $q = p = 2$, then

$$\operatorname{ord}_X G_{\widetilde{\pi}A,n}(X) = 2^{\sum_i \nu_i} \quad \text{where} \quad (n-1) = \nu_0 + \nu_1 2 + \cdots + \nu_r 2^r,$$

with $0 \le \nu_i \le 1$ (i.e., the ν_i's are the 2-adic digits of $n-1$). Keeping the same notation, we also conjecture that

$$G_n(X) = ([1]^{e_1}[2]^{e_2} \cdots [r]^{e_r}) \cdot G_{\widetilde{\pi}A,n}(X)$$

where e_i are non-negative integers. If the last statement is true, then it follows that for any n the polynomial $G_n(X)$ has coefficients with denominators not divisible by $\mathfrak{p}$ of degree $> \log_2(n)$.

For prime q in general we cannot hope for such a result about the divisibility of the coefficients of $G_n(X)$. However, it appears that the order of vanishing of $G_{\widetilde{\pi}A,n}(X)$ is again given by the formula

$$\mathrm{ord}_X(G_{\widetilde{\pi}A,n}(X)) = \prod_{0\le j<p} (p-j)^{\mu_j},$$

where μ_j are numbers depending on the p-adic digits of $n-1$. For example, if $p=3$, then computations suggest that

$$\mathrm{ord}_X(G_{\widetilde{\pi}A,n}(X)) = 2^{\mu_2}3^{\mu_3},$$

where

$$\mu_2 = (\text{number of times 1 occurs among the 3-adic digits of } (n-1)) - \delta,$$

$$\mu_3 = (\text{number of times 2 occurs among the 3-adic digits of } (n-1)) + \delta$$

with $\delta = 1$ if 1 occurs more than once as a 3-adic digit of $(n-1)$, and $\delta = 0$ otherwise. It is interesting to note that a similar formula occurs in the context of multizeta recursions [La1, Section 5].

When $q = p = 2$ we have two infinite families (in addition to the families from the previous corollaries) of values for n for which we can prove the conjecture.

Theorem 2.2.13. *Let $q = p = 2$.*

- *If $l \ge 2$ is an integer, then*

$$\mathrm{ord}_X(G_{\widetilde{\pi}A,2^l+2}(X)) = 4, \qquad \text{and} \qquad G_{2^l+2}(X) = D_{l-1}^2 \cdot G_{\widetilde{\pi}A,2^l+2}(X).$$

- *If $l \ge 3$ is an integer, then*

$$\mathrm{ord}_X(G_{\widetilde{\pi}A,2^l+2^2}(X)) = 8, \qquad \text{and} \qquad G_{2^l+2^2}(X) = D_{l-2}^4 \cdot G_{\widetilde{\pi}A,2^l+2^2}(X).$$

In particular, if $n = 2^l+2$ (with $l \ge 2$) or $n = 2^l+2^2$ (with $l \ge 3$), then the polynomial $G_n(X)$ has coefficients with denominators not divisible by $\mathfrak{p}$, for $\mathfrak{p}$ of degree $> \log_2(n)$.

Proof. Assume that n is of the form above. The theorem follows by checking the the finitely many possibilities for $j \leq$ the respective $\mathrm{ord}_X(G_{\tilde{\pi}A,n}(X))$ in the formula from Theorem 2.2.2. $\square$

Let $\mathfrak{p} \in \mathrm{Spec}(A)$. Then by Theorem 2.1.4, we know that $\rho[\mathfrak{p}]$ is isomorphic to $(A/\mathfrak{p})$. Therefore $\rho[\mathfrak{p}]$ is an $\mathbb{F}_q$-lattice of rank $d = \deg(\mathfrak{p})$, which we denote by $\Lambda_{\mathfrak{p}}$.

For future reference, we record how to compute the coefficients of $e_{\Lambda_{\mathfrak{p}}}(z)$.

By definition,

$$e_{\Lambda_{\mathfrak{p}}}(z) = z \prod_{\lambda \in \Lambda_{\mathfrak{p}}}{}' \left(1 - \frac{z}{\lambda}\right) = z + \cdots$$

is a polynomial of degree $q^{\deg(\mathfrak{p})}$ with the same zeros as $\rho_{\wp}(z) = \wp z + \cdots$. Thus

$$e_{\Lambda_{\mathfrak{p}}}(z) = \frac{1}{\wp} \rho_{\wp}(z).$$

Since $\rho_{\wp}$ can be computed by using the formulas in Theorem 2.2.1, this shows that $e_{\Lambda_{\mathfrak{p}}}$ and $G_{\Lambda_{\mathfrak{p}},n}$ can be explicitly computed.

Example 2.2.14. Let $q = 5$ and $\mathfrak{p} = (T^2 + T + 1)$. Then using the formulas from Example 2.2.5, we compute

$$\rho_{T^2+T+1} = T^2 + T + 1 + (T^5 + T + 1)\varphi + \varphi^2,$$

and therefore

$$e_{\Lambda_{\mathfrak{p}}}(z) = 1 + (T^3 - T^2 + 1)z^5 + \frac{1}{T^2 + T + 1} z^{25}.$$

2.3 The Drinfeld Upper Half-Plane

In this section, we consider the problem of classifying isomorphism classes of rank 2 Drinfeld modules over $\mathbb{C}_\infty$, or, which is the same, classifying homothety classes of rank 2 lattices in $\mathbb{C}_\infty$. Let $\Lambda' = Aw_1 \oplus Aw_2$ be such a lattice. Remaining in the same equivalence class, we can rescale Λ' to get $\Lambda_z := Az \oplus A$, where $z \in \mathbb{C}_\infty \backslash K_\infty$.

Definition 2.3.1. The set $\Omega := \mathbb{C}_\infty \backslash K_\infty$ is called the *Drinfeld upper half-plane.* It is an analog of the Poincaré upper half-plane and was first considered by Drinfeld in [Dr1].

The set Ω comes with a natural left action of $\mathrm{GL}_2(K_\infty)$ by linear fractional transformations. That is, if

$$\gamma = \begin{bmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{bmatrix} \in \mathrm{GL}_2(K_\infty),$$

then we define

$$\gamma z := \frac{a_\gamma z + b_\gamma}{c_\gamma z + d_\gamma}.$$

Note that two points $z, z' \in \Omega$ correspond to the same homothety class of rank 2 lattices if and only if there exists $\gamma \in \mathrm{GL}_2(A)$ such that

$$\frac{a_\gamma z + b_\gamma}{c_\gamma z + d_\gamma} = z'.$$

Therefore, the set of homothety classes of rank 2 Drinfeld modules over $\mathbb{C}_\infty$ is in bijection with the set of orbits $\mathrm{GL}_2(A)\backslash\Omega$. In turn the set $\mathrm{GL}_2(A)\backslash\Omega$ is in bijection with the $\mathbb{C}_\infty$-valued points of a certain rigid analytic space, which is a quotient of the rigid analytic space with $\mathbb{C}_\infty$-values points bijecting with Ω. To develop the theory further, we need to study the Bruhat-Tits tree for $\mathrm{GL}_2(A)$. Our main reference is [Bö1, Chapter 3].

Definition 2.3.2. By an A_∞-lattice in $V_\infty := K_\infty^2$ we will mean a free rank 2 A_∞-submodule of V_∞. Two A_∞-lattices Λ_1, Λ_2 will be called *equivalent* if there is $c \in K_\infty^*$ such that $c\Lambda_1 = \Lambda_2$. The equivalence class of Λ will be denoted by $[\Lambda]$. We define the Bruhat-Tits tree, $\mathcal{T}$, to be the tree with

- $X(\mathcal{T})$:= set of vertices of $\mathcal{T}$ equal to the set of equivalence classes of A_∞-lattices in V_∞;
- $Y(\mathcal{T})$:= set of edges of $\mathcal{T}$ equal to the set of pairs of vertices $\{v_1, v_2\}$ such that there exist A_∞-lattices Λ_1, Λ_2 with $\Lambda_2 \subset \Lambda_1$, $\Lambda_1/\Lambda_2 \cong A_\infty/\pi_\infty A_\infty$, $v_1 = [\Lambda_1]$, and $v_2 = [\Lambda_2]$.

The set of all vertices and edges will be called the *simplicies* of $\mathcal{T}$. We will usually write $\sigma \in \mathcal{T}$ for a simplex σ.

One can show [Se1, Section II.1.1] that $\mathcal{T}$ is a simply connected, $(q+1)$-regular tree, i.e., we have $(q+1)$ edges originating from every vertex. It does not come with a natural orientation on the edges, but we will always make a choice of an orientation. Once we fix an orientation, we will write $-e = \{v_2, v_1\}$ if $e = \{v_1, v_2\}$ for any two vertices v_1, v_2. The *geometric realization* of $\mathcal{T}$ is defined to be the graph in $\mathbb{R}^2$ where for each $e = \{v_1, v_2\} \in Y(\mathcal{T})$ we assign an interval of length one, i_e. We identify the ends of i_e with $i_{v_1,v'}$ for every $\{v_1, v'\} \in Y(\mathcal{T})$ (there are $q+1$ of these edges). We denote the geometric realization of $\mathcal{T}$ by $|\mathcal{T}|$. A point in $|\mathcal{T}|$ is called *rational* if it has rational coordinates.

The group $\mathrm{GL}_2(K_\infty)$ acts in a natural way on $\mathcal{T}$ via its action on A_∞-lattices:

$$\gamma * \Lambda := \Lambda\gamma^{-1}.$$

This action is transitive on vertices and edges. The importance of the Bruhat-Tits tree stems from the following result (see [DeHu, Chapter 3]):

Theorem 2.3.3. *There exists a $GL_2(K_\infty)$-equivariant map $r : \Omega \to |\mathcal{T}|$, which surjects onto the rational points of $|\mathcal{T}|$. We call this the* reduction map. □

Define $v_0 := [A_\infty \oplus A_\infty]$, $v_1 := [\pi_\infty A_\infty \oplus A_\infty]$. We call v_0 the *standard vertex*, and the oriented edge $e_0 := (v_0, v_1)$ the *standard oriented edge.* Further define

$$\mathfrak{U}_{v_0} := \left\{ z \in \Omega : \frac{1}{q} < |z| < q, |z-\beta| > \frac{1}{q}, \forall \beta \in \mathbb{F}_q \right\},$$
$$\mathfrak{U}_{e_0} := \{ z \in \Omega : 1 < |z| < q \}.$$

Theorem 2.3.4. *The cover $\mathfrak{U} := \{\gamma\mathfrak{U}_{v_0}, \gamma\mathfrak{U}_{e_0} : \gamma \in GL_2(A_\infty)\}$ of Ω is the set of $\mathbb{C}_\infty$-valued points of an admissible affinoid cover of a rigid analytic space $\mathcal{M}_2^{rig}$ defined over K_∞.* □

For a nonzero ideal $\mathfrak{n} \subset A$, define the *principal congruence subgroup of level* $\mathfrak{n}$ to be

$$\Gamma(\mathfrak{n}) := \left\{ \gamma \in \mathrm{GL}_2(A) : \gamma \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod \mathfrak{n} \right\}.$$

We will also be interested in other congruence subgroups, such as

$$\Gamma_0(\mathfrak{n}) := \left\{ \gamma \in \mathrm{GL}_2(A) : \gamma \equiv \begin{bmatrix} a_\gamma & b_\gamma \\ 0 & d_\gamma \end{bmatrix} \bmod \mathfrak{n} \right\},$$
$$\Gamma_1(\mathfrak{n}) := \left\{ \gamma \in \mathrm{GL}_2(A) : \gamma \equiv \begin{bmatrix} 1 & b_\gamma \\ 0 & 1 \end{bmatrix} \bmod \mathfrak{n} \right\}.$$

In this notation, $\mathrm{GL}_2(A) = \Gamma(1)$ and $\Gamma(\mathfrak{n}) \subset \Gamma_1(\mathfrak{n}) \subset \Gamma_0(\mathfrak{n}) \subset \mathrm{GL}_2(A)$. For the rest of the section, let Γ be one of the above matrix groups for some fixed $\mathfrak{n}$. The group Γ acts on $\mathcal{T}$ via the action of $\mathrm{GL}_2(A_\infty)$ on A_∞-lattices, and we can construct the quotient graph $\Gamma \backslash \mathcal{T}$.

Definition 2.3.5. A *half-line* in $\mathcal{T}$ is an infinite sequence of adjacent vertices. Two half-lines are called *equivalent* if they differ by a finite set of vertices. An *end* of $\mathcal{T}$ is an equivalence class of half-lines.

Any end $[s]$ has a unique representative that starts at v_0 and has a sequence of representative A_∞-lattices $\Lambda_i := lA_\infty + \pi_\infty^i A_\infty$ for some $l \in A_\infty \oplus A_\infty$ (see [Bö1, p. 33]). The vector $l := (l_0, l_1)$ is defined up to multiplication by A_∞^*. Define a left action of $\gamma \in \mathrm{GL}_2(A_\infty)$ on $(l_0 : l_1) \in \mathbb{P}^1(K_\infty)$ by viewing $(l_0 : l_1)$ as a row vector and multiplying by γ^{-1} on the right.

Definition 2.3.6. The map which sends the end $[s]$ to the line $l \in \mathbb{P}^1(K_\infty)$ induces an $\mathrm{GL}_2(K_\infty)$-equivariant bijection between the ends of $\mathcal{T}$ and $\mathbb{P}^1(K_\infty)$. An end $[s]$ is called *rational* if it corresponds to an element of $\mathbb{P}^1(K)$ under this bijection. The elements of $\Gamma \backslash \mathbb{P}^1(K)$ are called the *cusps* of $\Gamma \backslash \mathcal{T}$.

We set $[\infty] := (1 : 0)$ and we call this the *cusp at infinity*. For each cusp $[s]$ we choose $\gamma_s \in \mathrm{GL}_2(A)$ such that $[s] = \gamma_s[\infty]$. The group $\gamma_s^{-1}\Gamma_{[\infty]}\gamma_s$ contains a maximal subgroup of transformations of the form $z \mapsto z+b$ with $b \in A$, and the set of such b is an ideal I_s of A. Define

$$t_{[s]}(z) := \frac{1}{\widetilde{\pi} e_{I_s}(z)} = \frac{1}{\widetilde{\pi}} \sum_{\lambda \in I_s} \frac{1}{z+\lambda}.$$

For example, when $[s] = [\infty]$ we choose $\gamma_s = \mathtt{Id}$. Then

$$\mathrm{GL}_2(A)_\infty := \left\{ \begin{bmatrix} a_\gamma & b_\gamma \\ 0 & d_\gamma \end{bmatrix} \in \mathrm{GL}_2(A) : b_\gamma \in A \right\},$$

and

$$t_{[\infty]}(z) := \frac{1}{\widetilde{\pi} e_A(z)} = \frac{1}{\widetilde{\pi}} \sum_{\lambda \in A} \frac{1}{z+\lambda}.$$

The following theorem describes the geometry behind $\Gamma\backslash\mathcal{T}$ and $\Gamma\backslash\Omega$:

Theorem 2.3.7. *The graph $\Gamma\backslash\mathcal{T}$ consists of a finite graph and a finite set of ends $[v_{i,s}]_{i\geq 0}$ that is in one-to-one correspondence with the set of cusps $[s] \in \Gamma\backslash\mathbb{P}^1(K)$. The space $\Gamma\backslash\Omega$ has an affinoid cover given by*

$$\mathfrak{U}_\Gamma := \{\Gamma_\sigma\backslash\mathfrak{U}_\sigma : \sigma \in \mathcal{T}\}.$$

In addition, for every cusp $[s] \in \Gamma\backslash\mathbb{P}^1(K)$, define

$$\mathfrak{U}_s := \bigcup_{i\geq 0} r^{-1}(v_{i,s}).$$

Then

$$\Gamma_s\backslash\mathfrak{U}_s \cong \text{ punctured disk.}$$

under the map $z \to t_{[s]}(z)$. □

Therefore, we can compactify the space $\Gamma\backslash\Omega$ by adding a finite number of points, which we appropriately call *cusps*. According to the results above, the cusps correspond to $\Gamma\backslash\mathbb{P}^1(K)$. This compactification will be denoted by $\Gamma\backslash\overline{\Omega}$.

Example 2.3.8 (The case of $\mathrm{GL}_2(A)$)**.** We define the notion of *imaginary part* on Ω by

$$|z|_i := \inf_{x\in K_\infty} |z-x|.$$

We will think of $[\infty]$, the unique cusp of $\mathrm{GL}_2(A)$, as a point that we have added to Ω which is the limit of all sequences $\{z_n\} \in \Omega$ with $|z_n|_i \to \infty$. This is justified by the fact that if we define $\Omega_c := \{z \in \Omega : |z|_i > c\}$, then according to Theorem 2.3.7 for some rational $c_0 > 1$ we have

$$\Omega_{c_0} = \mathfrak{U}_\infty.$$

And $t = t(z) = t_{[\infty]}(z)$ induces a bijection between Ω_{c_0} and a punctured disk.

Example 2.3.9 (The case of $\Gamma_0(T)$)**.** The group $\Gamma_0(T)$ has two cusps $[\infty]$ and $[0] = (0:1)$. We choose $\gamma_0 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then the stabilizers at ∞ and 0 respectively are given by

$$\Gamma_0(T)_\infty = \left\{ \begin{bmatrix} a_\gamma & b_\gamma \\ 0 & d_\gamma \end{bmatrix} \in \Gamma_0(T) : b_\gamma \in A \right\}$$
$$\gamma_0^{-1}\Gamma_0(T)_\infty\gamma_0 = \left\{ \begin{bmatrix} a_\gamma & b_\gamma \\ 0 & d_\gamma \end{bmatrix} \in \Gamma_0(T) : b_\gamma \in TA \right\}.$$

We have one uniformizer at the infinite cusp

$$t = t_{[\infty]}(z) = \frac{1}{\tilde{\pi} e_A(z)} = \frac{1}{\tilde{\pi}} \sum_{\lambda \in A} \frac{1}{z+\lambda}.$$

and a different uniformizer at the cusp at zero

$$s := t_{[0]}(z) = \frac{1}{\tilde{\pi} e_{TA}(z)} = \frac{1}{\tilde{\pi}} \sum_{\lambda \in A} \frac{1}{z+T\lambda}.$$

We note that

$$Ts(Tz) = \frac{1}{\tilde{\pi}} \sum_{\lambda \in A} \frac{T}{Tz+T\lambda} = t(z).$$

2.4 Drinfeld Modular Forms for $\mathrm{GL}_2(A)$

In this section, we define the central objects of this dissertation: Drinfeld modular forms. These are analogs of classical modular forms in the function field setting.

They were first defined by David Goss in his Harvard thesis. Our main references are [Go1], [Go2], [Ge2], [Ge3].

Let $\gamma \in \mathrm{GL}_2(K_\infty)$ and k, m be non-negative integers. For any function $f : \Omega \to \mathbb{C}_\infty$ we define the *slash operator* $|_{[\gamma]_{k,m}}$ by

$$f\ |_{[\gamma]_{k,m}} := (\det \gamma)^m (c_\gamma z + d_\gamma)^{-k} f(\gamma z).$$

It is checked that

$$f\ |_{[\gamma_1 \gamma_2]_{k,m}} = \left(f\ |_{[\gamma_1]_{k,m}}\right)\ |_{[\gamma_2]_{k,m}}\ .$$

As k, m will usually be understood from the context, we will simplify the notation and write $f|_{[\gamma]}$ for the slash operator.

Definition 2.4.1. Let k, m be two non-negative integers. A *Drinfeld modular form* for $\mathrm{GL}_2(A)$ of weight k and type m is a rigid analytic function $f : \Omega \to \mathbb{C}_\infty$ such that

$$\begin{aligned} &(1) \quad f|_{[\gamma]} = f \quad \forall \gamma \in \mathrm{GL}_2(A), \\ &(2) \quad f \quad \text{is holomorphic at infinity.} \end{aligned}$$

The space of all Drinfeld modular forms for $\mathrm{GL}_2(A)$ of weight k and type m will be denoted by $M_{k,m}(\mathrm{GL}_2(A))$.

The 'holomorphic at infinity' condition requires some explanation. Let $f \in M_{k,m}(\mathrm{GL}_2(A))$. Recall the imaginary distance $|z|_i$ and the neighborhoods of infinity Ω_c defined in the previous section. Because of condition (1), we see that f is invariant under $r_a : z \mapsto z + a,\ a \in A$. Therefore the function f descends to a function on the set $\langle r_a : a \in A\rangle \backslash \Omega_c$ for some $c > 1$. But this set is homeomorphic to a punctured disk with uniformizing parameter $t = t(z) = t_{[\infty]}(z)$. The condition 'holomorphic at infinity' means that f is holomorphic on this punctured disk, i.e., there exists a positive real number c such that for $|z|_i > c$ we have

$$f(z) = \sum_{n=0}^{\infty} a_n(f) t^n(z) = \sum_{n=0}^{\infty} a_n t^n.$$

We call this the *Fourier expansion*, or *t-expansion*, of f. Since Ω is a connected rigid analytic space, the t-expansion of a Drinfeld modular form f determines f uniquely.

Remark 2.4.2. From the definition of $|_{[\gamma]}$ it is clear that m should be thought of a non-negative integer taken modulo $|\det \mathrm{GL}_2(A)| = q-1$.

Definition 2.4.3. We define the space $S_{k,m}(\mathrm{GL}_2(A))$ of *cuspidal Drinfeld modular forms* of weight k and type m; and the space $S^2_{k,m}(\mathrm{GL}_2(A))$ of *double cuspidal Drinfeld modular forms* of weight k and type m as follows:

$$S_{k,m}(\mathrm{GL}_2(A)) := \{f \in M_{k,m}(\mathrm{GL}_2(A)) : a_0(f) = 0\},$$
$$S^2_{k,m}(\mathrm{GL}_2(A)) := \{f \in M_{k,m}(\mathrm{GL}_2(A)) : a_0(f) = a_1(f) = 0\}.$$

Let $\theta \in \mathbb{F}_q^*$. Because $z \mapsto \theta z$ is a transformation induced from a matrix in $\mathrm{GL}_2(A)$ and since $t(\theta z) = \theta^{-1} t(z)$, we see that for $f \in M_{k,m}(\mathrm{GL}_2(A))$ and $n \not\equiv m \bmod (q-1)$ we have $a_n(f) = 0$. Indeed, given $\theta \in \mathbb{F}_q^*$, we have

$$\sum_{n=0}^{\infty} \theta^{-n} a_n(f) t^n(z) = f(\theta z) = \theta^m f(z) = \theta^m \sum_{n=0}^{\infty} a_n(f) t^n(z),$$

hence if $a_n(f) \neq 0$, then by uniqueness of the t-expansions $\theta^{m-n} = 1$. Therefore,

$$f = \sum_{j=0}^{\infty} a_{m+j(q-1)} t^{m+j(q-1)}.$$

This shows that for $m \neq 1$ we have $S_{k,m}(\mathrm{GL}_2(A)) = S^2_{k,m}(\mathrm{GL}_2(A))$.

In addition, using the functional equation for trivial action of a diagonal matrix with diagonal entries $\theta \in \mathbb{F}_q^*$, we see that

$$f(z) = f\left(\frac{\theta z}{\theta}\right) = \theta^{-2m} \theta^k f(z),$$

hence $M_{k,m}(\mathrm{GL}_2(A)) = 0$ when $k \not\equiv 2m \bmod (q-1)$.

Remark 2.4.4. Classically, cuspidal modular forms of weight 2 for a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ correspond to holomorphic differentials on the Riemann surface that represents the compactified modular curve for Γ over the complex numbers (see [DiSh, Chapter 2]). David Goss [Go2, Theorem 1.80] observed that, unlike the classical case, holomorphic differentials on the compactified Drinfeld modular curve, $\Gamma\backslash\overline{\Omega}$, correspond to double cuspidal Drinfeld modular forms for a congruence subgroup $\Gamma \subset \mathrm{GL}_2(A)$. For example, for $\mathrm{GL}_2(A)$ this just boils down to

$$dt = -\widetilde{\pi} \cdot t^2 dz,$$

in contrast with the classical case for $\mathrm{SL}_2(\mathbb{Z})$:

$$dq = 2\pi i \cdot q dz.$$

Teitelbaum [Te1] has shown that cuspidal Drinfeld modular forms correspond to functions of a special kind, the Γ-invariant harmonic cocycles, on the Bruhat-Tits tree $\mathcal{T}$. For more information about interpreting Drinfeld modular forms as sections on $\Gamma\backslash\overline{\Omega}$ and as harmonic cocycles on $\Gamma\backslash\mathcal{T}$ see [Go1], [Co1], [Bö1, Sections 5, 6] and Section 3.5.

Example 2.4.5 (Eisenstein series)**.** Let k be a fixed positive integer. Consider the series

$$E_k(z) := \frac{1}{\widetilde{\pi}^k} \sideset{}{'}\sum_{(a,b)\in A^2} \frac{1}{(az+b)^k}.$$

The double sum converges absolutely for any $z \in \Omega$ and defines a rigid analytic function on Ω. Using the absolute convergence of the double sum, we get

$$E_k\left(\begin{bmatrix} 1 & b_\gamma \\ 0 & 1 \end{bmatrix} z\right) = E_k(z + b_\gamma) = E_k(z)$$

and

$$\begin{aligned} E_k\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} z\right) &= \frac{1}{\widetilde{\pi}^k} \sideset{}{'}\sum_{(a,b)\in A^2} \frac{1}{(a\frac{1}{z} + b)^k} \\ &= \frac{z^k}{\widetilde{\pi}^k} \sideset{}{'}\sum_{(a,b)\in A^2} \frac{1}{(bz+a)^k} \\ &= z^k E_k(z). \end{aligned}$$

These two types of matrices generate $\mathrm{PGL}_2(A)$, and therefore $E_k(z)$ satisfies condition (1) of the definition of Drinfeld modular forms for weight k and type 0. The function E_k is not identically zero on Ω if and only if $k \equiv 0 \bmod (q-1)$.

One can see ([Go2, Corollary 2]) that $E_k(z)$ is holomorphic at infinity with constant term $\widetilde{\pi}^k \sum'_{\lambda \in A} \frac{1}{\lambda}$. Define g_k to be the multiple of E_k that has constant term 1. We have

$$g_k = 1 + \mathcal{O}(t) \in M_{k,0}(\mathrm{GL}_2(A)),$$

where the notation $\mathcal{O}$ is the standard order notation. We caution the reader that our notation is different from the notation in [Ge3, (6.8)], where Gekeler uses g_{k_0} to denote $g_{q^{k_0}-1}$ in our notation.

Example 2.4.6 (Drinfeld modular forms that are coefficients of Drinfeld modules)**.** Let $z \in \Omega$, and let Λ_z be the rank 2 lattice $\Lambda_z := zA \oplus A$. Consider

$$\phi_a^{\Lambda_z} = a\varphi^0 + l_1(a)\varphi + \cdots + l_{2\deg(a)}(a)\varphi^{2\deg(a)}.$$

By using the $\mathrm{GL}_2(A)$ action on Ω and properties of Drinfeld modules, one can show (see [Ge3, (5.10)]) that

$$l_i(a) \in M_{q^i-1,0}(\mathrm{GL}_2(A)).$$

As a special case of the construction, we define $g_\dagger(z)$ and $\varDelta_\dagger(z)$ by

$$\phi_T^{\Lambda_z} = T\varphi^0 + g_\dagger(z)\varphi + \varDelta_\dagger(z)\varphi^2.$$

It follows from the considerations above that

$$g_\dagger(z) \in M_{q-1,0}(\mathrm{GL}_2(A)), \qquad \varDelta_\dagger(z) \in M_{q^2-1,0}(\mathrm{GL}_2(A)).$$

Since for a rank 2 Drinfeld module the coefficient of φ^2 is non-zero, we see that $\varDelta_\dagger$ is non-zero for any $z \in \Omega$. One can also show that $\varDelta_\dagger$ can be expressed as a linear combination of Eisenstein series (see [Ge3, (2.10)]), and this can be used to show that $\varDelta_\dagger \in S^2_{q^2-1,0}(\mathrm{GL}_2(A))$. One can also see that $\varDelta_\dagger$ must be cuspidal from the moduli space interpretation: $\Delta_\dagger$ vanishes at the cusp at infinity, which corresponds to the degeneration to rank one. We will renormalize $\Delta_\dagger$ to have 1 as a first non-zero coefficient and denote this form by Δ.

It turns out that $g_\dagger$ and g_{q-1} (the normalized Eisenstein series of weight $q-1$) are both non-cuspidal and differ by a constant multiple, since they generate the one-dimensional space $M_{q-1,0}(\mathrm{GL}_2(A))$.

Throughout this dissertation, we fix the following notation:

$$g := g_{q-1} = 1 + \mathcal{O}(t) \in M_{q-1,0}(\mathrm{GL}_2(A))$$

and

$$\Delta = t^{q-1} + \mathcal{O}(t^q) \in S^2_{q^2-1,0}(\mathrm{GL}_2(A)).$$

Example 2.4.7 (The form h)**.** Gekeler studied the form h, which is a $(q-1)$st root of Δ and a well-defined Drinfeld modular form [Ge3, (5.13)] of weight $q+1$ and type 1. We have

$$h = t + \mathcal{O}(t^2) \in S_{q+1,1}(\mathrm{GL}_2(A)).$$

The notation h will remain in effect throughout the rest of the dissertation.

Remark 2.4.8. We note that our normalizations of h and Δ differ from the standard reference [Ge3]. In [Ge3], h and Δ denote the negatives of the forms in our notation.

The proof of the following result can be found in [Ge3, (5.12)] and [Co1, Section 4].

Theorem 2.4.9. *The space*

$$M(GL_2(A)) := \oplus_{k,m} M_{k,m}(GL_2(A))$$

is generated as an algebra over $\mathbb{C}_\infty$ by the forms h and g. In addition, the spaces $M_{k,m}(GL_2(A))$, $S_{k,m}(GL_2(A))$, $S^2_{k,m}(GL_2(A))$ are finite-dimensional $\mathbb{C}_\infty$-vector spaces.

Their dimensions are given by the following formulas when $k \equiv 2m \bmod (q-1)$*:*

$$\begin{aligned}
\dim_{\mathbb{C}_\infty} M_{k,m}(GL_2(A)) &= 1 + \left\lfloor \frac{k - m(q+1)}{q^2-1} \right\rfloor, \\
\dim_{\mathbb{C}_\infty} S_{k,m}(GL_2(A)) &= \dim_{\mathbb{C}_\infty} M_{k,m}(GL_2(A)) && \text{if } m \neq 0, \\
&= \dim_{\mathbb{C}_\infty} M_{k,m}(GL_2(A)) - 1 && \text{if } m = 0, \\
\dim_{\mathbb{C}_\infty} S^2_{k,m}(GL_2(A)) &= \dim_{\mathbb{C}_\infty} S_{k,m}(GL_2(A)) && \text{if } m \neq 1, \\
&= \dim_{\mathbb{C}_\infty} S_{k,m}(GL_2(A)) - 1 && \text{if } m = 1.
\end{aligned}$$

In all other cases, the dimensions of these spaces are 0. □

Example 2.4.10. Let $q = 3$. The first five non-zero spaces $M_{k,0}(\mathrm{GL}_2(A))$ are generated by powers of g and h as follows:

$$\begin{aligned}
M_{2,0}(\mathrm{GL}_2(A)) &= \langle g \rangle, \\
M_{4,0}(\mathrm{GL}_2(A)) &= \langle g^2 \rangle, \\
M_{6,0}(\mathrm{GL}_2(A)) &= \langle g^3 \rangle, \\
M_{8,0}(\mathrm{GL}_2(A)) &= \langle g^4, h^2 \rangle, \\
M_{10,0}(\mathrm{GL}_2(A)) &= \langle g^5, gh^2 \rangle.
\end{aligned}$$

Remark 2.4.11. Let q be arbitrary. Since both g^j and $g_{j(q-1)}$ generate the one-dimensional space $M_{j(q-1),0}(\mathrm{GL}_2(A))$, and because of the normalizations that we have chosen, we get $g^j = g_{j(q-1)}$ for $1 \leq j \leq q$. If $j > q$ it may not be true that $g^j = g_{j(q-1)}$: for instance,

$$g^{q+1} = g_{q^2-1} + [1]h^{q-1}.$$

From [Ge3, Proposition 6.9], we have the following formula[1]:

$$g_{q^{k_0}-1} = g_{q^{k_0-1}-1} g^{q^{k_0-1}} + [k_0 - 1] g_{q^{k_0-2}-1} h^{(q-1)q^{k_0-2}}, \qquad k_0 \geq 2.$$

We will derive a similar recursive formula in Section 3.3 for another family of Drinfeld modular forms.

[1] The formula looks different in [Ge3], because of the different normalizations for h and g_k that we have chosen.

2.4.1 The Hecke Algebra

Let $\mathfrak{n}$ be a non-zero ideal in A. We can define the *Hecke operator* $\mathrm{T}_{\mathfrak{n}}$ at $\mathfrak{n}$ by the same formalism as in the classical case [DiSh, Chapter 5], i.e., by looking at double-coset operators $\mathrm{GL}_2(A)g\mathrm{GL}_2(A)$ for

$$g \in \mathfrak{D} = \{\gamma \in M_2(A) : \det \gamma \in A_+\}.$$

Instead, we proceed by defining $\mathrm{T}_{\mathfrak{p}}$ for $\mathfrak{p} \in \mathrm{Spec}(A)$ first, and then define $\mathrm{T}_{\mathfrak{n}}$ in terms of the $\mathrm{T}_{\mathfrak{p}}$'s.

Let $\mathfrak{p} \in \mathrm{Spec}(A)$. Consider the double-coset space $\mathrm{GL}_2(A) \begin{bmatrix} 1 & 0 \\ 0 & \wp \end{bmatrix} \mathrm{GL}_2(A)$. The group $\mathrm{GL}_2(A)$ acts on this double-coset space on the left. Let $\{\gamma_i\}_i$ be a finite system of left coset representatives of this action such that $\det \gamma_i \in A_+$. We define the *Hecke operator* $\mathrm{T}_{\mathfrak{p}}$ to be

$$\mathrm{T}_{\mathfrak{p}} f := \wp \sum_i f \mid_{[\gamma_i]} .$$

Here we have used the normalization that is used in [Ge3, Section 7]. The operator $\mathrm{T}_{\mathfrak{p}}$ is a well-defined operator on $M_{k,m}(\mathrm{GL}_2(A))$ which preserves both the space $S_{k,m}(\mathrm{GL}_2(A))$ and the space $S^2_{k,m}(\mathrm{GL}_2(A))$. Just like the classical case, we have

$$\mathrm{T}_{\mathfrak{p}} \mathrm{T}_{\mathfrak{p}'} = \mathrm{T}_{\mathfrak{p}'} \mathrm{T}_{\mathfrak{p}}, \qquad \text{for } \mathfrak{p} + \mathfrak{p}' = A.$$

We can compute that

$$\mathrm{T}_{\mathfrak{p}^n} = \mathrm{T}_{\mathfrak{p}^{n-1}} \mathrm{T}_{\mathfrak{p}} + |A/\mathfrak{p}| \mathrm{T}_{\mathfrak{p}^{n-2}} \mathrm{T}_{\mathfrak{p}^2},$$

which is precisely what happens in the classical case. However, in contrast with the classical case, $|A/\mathfrak{p}|$ is zero in characteristic p. Therefore unlike the classical case we have total multiplicativity

$$\mathrm{T}_{\mathfrak{p}^n} = \mathrm{T}_{\mathfrak{p}^{n-1}} \mathrm{T}_{\mathfrak{p}} .$$

Since the Hecke operators commute with each other and because we have total multiplicativity, we can define $\mathrm{T}_{\mathfrak{n}}$ as follows

$$\mathrm{T}_{\mathfrak{n}} := \mathrm{T}^{n_1}_{\mathfrak{p}_1} \cdots \mathrm{T}^{n_s}_{\mathfrak{p}_s}, \qquad \text{where } \mathfrak{n} = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_s^{n_s}.$$

Definition 2.4.12. The *Hecke algebra*, $\mathbb{T}$, is defined to be the commutative sub-$\mathbb{C}_\infty$-algebra of $\mathrm{End}_{\mathbb{C}_\infty}(M_{k,m}(\mathrm{GL}_2(A)))$ generated by the $\mathrm{T}_\mathfrak{p}$'s.

We will sometimes abuse notation and write $\mathbb{T}$ for the sub-$\mathbb{C}_\infty$-algebra of the algebra $\mathrm{End}_{\mathbb{C}_\infty}(S_{k,m}(\mathrm{GL}_2(A)))$ or $\mathrm{End}_{\mathbb{C}_\infty}(S^2_{k,m}(\mathrm{GL}_2(A)))$, but the meaning of $\mathbb{T}$ will be clear from the context.

By choosing representatives for the left cosets in the formula for $\mathrm{T}_\mathfrak{p}$, one can show that

$$\mathrm{T}_\mathfrak{p}\, f = \wp^k f(\wp z) + \sum_{\beta \in S_\mathfrak{p}} f\left(\frac{z+\beta}{\wp}\right),$$

where $S_\mathfrak{p} = \{\beta \in A : \deg(\beta) < \deg(\wp)\}$.

Goss computed the action of $\mathrm{T}_\mathfrak{p}$ on a t-expansion in [Go1, Section 1.8]. The following formula can be found in [Ge3, Section 7], and it recasts Goss' result in our notation:

$$\mathrm{T}_\mathfrak{p}\left(\sum_{n=0}^{\infty} a_n t^n\right) = \wp^k \sum_{n=0}^{\infty} a_n t_\wp^n + \sum_{n=0}^{\infty} a_n G_{\Lambda_\mathfrak{p},n}(\wp t).$$

Recall that

$$t_\wp = \frac{t^{q^{\deg(\wp)}}}{\psi_\wp(t)} = t^{q^{\deg(\wp)}} + \mathcal{O}(t^{q^{\deg(\wp)}+1}).$$

If we define $\omega_{n,\wp} = \mathrm{ord}_X\, G_{\Lambda_\mathfrak{p},n}$, then (see [Ge3, Corollary 3.9]) $\omega_{n,\wp} \geq n/q^{\deg(\wp)} + 1$. Therefore computing the effect of $\mathrm{T}_\mathfrak{p}$ with $\deg(\mathfrak{p}) = d$ requires the knowledge of a number of coefficients exponential in d, which is often not practical.

Remark 2.4.13. The formula for the effect of $\mathrm{T}_\mathfrak{p}$ on a t-expansion shows that $\mathbb{T}$ preserves both $S_{k,m}(\mathrm{GL}_2(A))$ and $S^2_{k,m}(\mathrm{GL}_2(A))$. Indeed, this follows from the formula above and the fact that $G_1(X) = X$ and $X^2 \mid G_n(X)$ for $n > 1$. It is known that the analogous result fails if we consider triple cuspidal forms because it is not true that $X^3 \mid G_n(X)$ for $n > 2$. Triple cuspidality is not preserved. For example, if $q = 3$, then for $h^4 \in S^2_{16,0}(\mathrm{GL}_2(A))$, we can compute

$$\mathrm{T}_T(h^4) = \mathrm{T}_T(t^4 + \cdots) = Tt^2 + \cdots = Th^2g^4 + T^4h^4.$$

Definition 2.4.14. A Drinfeld modular form f will be called a *simultaneous eigenform* for $\mathbb{T}$, or simply an *eigenform*, if there exist $\lambda_{\mathfrak{p}}$'s in $\mathbb{C}_\infty$ such that

$$\mathrm{T}_{\mathfrak{p}}\, f = \lambda_{\mathfrak{p}} f, \qquad \forall \mathfrak{p} \in \mathrm{Spec}(A).$$

For such an f the values $\{\lambda_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathrm{Spec}(A)}$ will be called the *eigensystem* of f.

In stark contrast with the case of classical modular forms, it is difficult to determine if a Drinfeld modular form is an eigenform. Indeed, despite the appearance of an easier set of Hecke operators (because of the complete multiplicativity among them), the crucial Petersson inner product (see [DiSh, Chapter 5]) is missing in the case of Drinfeld modular forms. Therefore there are no theoretical tools that ensure the existence of a basis of eigenforms for the Hecke operators if the space of Drinfeld modular forms in question has a large dimension. We have examples of spaces that are not diagonalizable:

Example 2.4.15 (A space that is not diagonalizable)**.** Let $q = 2$. Since $q - 1 = 1$, there is only one type, $m = 0$. Consider the space

$$S_{9,0}(\mathrm{GL}_2(A)) = \langle hg^6, h^2g^3, h^3\rangle.$$

Computing with the basis that we have chosen, we get

$$\mathrm{CharPol}_{\mathrm{T}_T}(X) = (X + T)(X^2 + T^8 + T^7 + T^5).$$

This is an inseparable polynomial and therefore there cannot exist a basis of eigenforms for the Hecke operator T_T.

Even if it is clear that a Drinfeld modular form is an eigenform (for instance, for dimensional reasons), it may not be clear what the eigensystem is. A result of Goss, [Go1, Theorem 2.2.3], shows cases in which we can compute the eigensystem:

Theorem 2.4.16. *Let $f \in S_{k,0}(GL_2(A))$ be an eigenform with eigensystem $\{\lambda_\mathfrak{p}\}_{\mathfrak{p}\in\mathrm{Spec}(A)}$ and t-expansion*

$$f = \sum_{j=1}^{\infty} a_{j(q-1)} t^{j(q-1)}.$$

If $a_{q-1} \neq 0$ and $a_{j(q-1)} = 0$ when $j \not\equiv 0, 1 \bmod q$, then

$$\lambda_\mathfrak{p} = \wp^{q-1}. \quad \square$$

The previous theorem was used by Goss to show that Δ and $g^q\Delta$ are eigenforms with the same eigensystem $\{\lambda_\mathfrak{p} = \wp^{q-1}\}_{\mathfrak{p}\in\mathrm{Spec}(A)}$ but with different weights. It is non-trivial to see that the forms Δ and $g^q\Delta$ satisfy the hypothesis of Theorem 2.4.16. We will use A-expansions to prove that Δ, $g^q\Delta$, and other forms, such as $g^{q(q-1)}\Delta$, satisfy the hypothesis of Theorem 2.4.16.

In addition to the eigenforms just mentioned, one can show (see [Ge3, (7.2) and (7.6)]) that:

$$\mathrm{T}_\mathfrak{p}\, g_k = \wp^k g_k, \qquad \mathrm{T}_\mathfrak{p}\, h = \wp h.$$

We will return to the subject of eigenforms and their eigensystems in Chapter 3, where we will show how to recover the above eigensystems and compute many more by using properties of A-expansions.

Classically the action of the Hecke algebra on modular forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$ can be simultaneously diagonalized for any k. This is no longer the case for Drinfeld modular forms as Example 2.4.15 above shows. Exactly which spaces of Drinfeld modular forms are diagonalizable with respect to $\mathbb{T}$ is unknown. Most of the results in this direction have to do with trivial cases, when the spaces in question are of dimension one. For non-trivial examples see [Ar1, Theorem 7.7] and [LiMe, Section 7]. We confirm the results from [Ar1] and reprove the diagonalizability in several new non-trivial cases, but with extra assumptions, in Theorem 3.1.23.

2.4.2 Drinfeld Quasi-Modular Forms

In this subsection, we recall some results due to Pellarin ([Pe2], [Pe3]) since we will need them in Chapter 3.

We start by defining the *false Eisenstein series* from [Ge3, Section 8] by the following conditionally convergent series:

$$E(z) := \sum_{a \in A_+} a t_a = \frac{1}{\widetilde{\pi}} \sum_{a \in A_+} \left(\sum_{b \in A} \frac{a}{az+b} \right).$$

The function E is not a Drinfeld modular form, but it satisfies the following functional equation

$$E(\gamma z) = (\det \gamma)^{-1}(c_\gamma z + d_\gamma)^2 E(z) - c_\gamma \widetilde{\pi}^{-1} (\det \gamma)^{-1}(c_\gamma z + d_\gamma).$$

Therefore E should be thought of as an analog of the Eisenstein series of weight 2 in the classical case.

Following Pellarin, define the *space of Drinfeld quasi-modular forms* of weight k, type m and depth l to be

$$M_{k,m}^{\leq l} := M_{k,m}(\mathrm{GL}_2(A)) \oplus M_{k-2,m-1}(\mathrm{GL}_2(A))E \oplus \cdots \oplus M_{k-2l,m-l}(\mathrm{GL}_2(A))E^l.$$

Because E has a t-expansion, one can consider the space $M_{k,m}^{\leq l}$ as a subset of $\mathbb{C}_\infty[[t]]$. Introduce a new variable u, and consider $\mathbb{C}_\infty[[t,u]]$ equipped with the 'partial Frobenius'

$$\widetilde{\varphi} : \mathbb{C}_\infty[[t,u]] \to \mathbb{C}_\infty[[t,u]], \quad \sum_{i=0}^{\infty} c_i u^i \mapsto \sum_{i=0}^{\infty} c_i^q u^i.$$

The ring $\mathbb{M}$ of *deformations of Drinfeld quasi-modular forms* is the subring of $\mathbb{C}_\infty[[t,u]]$ generated by $g, h, \mathbb{E}, \widetilde{\varphi}\mathbb{E}$. In fact, Pellarin shows that $\mathbb{M}$ is contained in $\mathbb{C}_\infty[u][[t]]$. The function $\mathbb{E}$ is originally defined in terms of Anderson's functions (see [Pe2, Section 2.1]), but Pellarin has shown in [Pe3, Corollary 5] that

$$\mathbb{E}(t,u) = \sum_{a \in A_+} a(u) t_a,$$

and we take this as the definition of $\mathbb{E}$.

According to [Pe2, Proposition 9], $\mathbb{E}$ satisfies the $\widetilde{\varphi}$-difference equation

$$\widetilde{\varphi}^2\,\mathbb{E} = \frac{1}{u - T^{q^2}}(g^q\,\widetilde{\varphi}\,\mathbb{E} - h^{q-1}\mathbb{E}),$$

which we will use in Chapter 3.

Looking back at the definition of $\mathbb{E}$ and the false Eisenstein series E, it is easy to see that

$$E = \mathbb{E}(t,u)\mid_{u=T}.$$

This is no coincidence. The elements of $\mathbb{M}$ are called deformations of Drinfeld quasi-modular forms since for any $\mathbb{F} \in \mathbb{M}$, the t-expansion $\mathbb{F}_{|_{u=T}}$ is a Drinfeld quasi-modular form. Pellarin has profitably used the space $\mathbb{M}$ to prove results about the order of vanishing of Drinfeld quasi-modular forms, and the reader should consult the references that we have quoted for more information.

2.5 Drinfeld Modular Forms of level T

Throughout this section, we let Γ be either $\Gamma_1(T)$ or $\Gamma_0(T)$. The group Γ has two cusps, $[\infty]$ and $[0]$. Recall that in Example 2.3.9, we fixed a matrix γ_0 that switches them, as well as uniformizers at the cusps: $t = t_{[\infty]}$ and $s = t_{[0]}$.

Definition 2.5.1. A rigid analytic function $f : \Omega \to \mathbb{C}_\infty$ is called a *Drinfeld modular form* of weight k and type m for Γ if it satisfies

(1) $f|_{[\gamma]} = f,\ \forall \gamma \in \Gamma$,

(2) f is holomorphic at the cusps.

We denote the space of such functions by $M_{k,m}(\Gamma)$.

The last condition means that there exists a positive real number $c > 1$ such that for z with $|z|_i > c$ (i.e, a neighborhood of infinity) we have

$$f(z) = \sum_{n=0}^{\infty} a_n(f) t^n(z) = \sum_{n=0}^{\infty} a_n t^n,$$
$$f\,|_{[\gamma_0]}\,(z) = \sum_{n=0}^{\infty} b_n(f) s^n(z) = \sum_{n=0}^{\infty} b_n s^n.$$

The first expansion is called the expansion of f at $[\infty]$ with respect to t, while the second is called the expansion of f at $[0]$ with respect to s.

Definition 2.5.2. If $f \in M_{k,m}(\Gamma)$ with $a_0(f) = b_0(f) = 0$, then f is called *cuspidal.* If, in addition, $a_1(f) = b_1(f) = 0$, then f is called *double cuspidal.* Similarly to the case for $\mathrm{GL}_2(A)$ we denote the spaces of such forms by $S_{k,m}(\Gamma)$ and $S^2_{k,m}(\Gamma)$, respectively.

Remark 2.5.3. As in the case $\Gamma = \mathrm{GL}_2(A)$, the type m should be thought of as a non-negative integer modulo $|\det \Gamma|$ and we will implicitly take $m < |\det \Gamma|$. In particular, since $|\det \Gamma_1(T)| = 1$, we see that the spaces $M_{k,m}(\Gamma_1(T))$ are the same for any fixed k.

There are two natural maps from $M_{k,m}(\mathrm{GL}_2(A))$ to $M_{k,m}(\Gamma)$ which respect cuspidality and double cuspidality (this is shown in the course of the proof of Theorem 2.5.4 below):

$$\iota : M_{k,m}(\mathrm{GL}_2(A)) \to M_{k,m}(\Gamma) : f(z) \mapsto f(z),$$
$$\iota_T : M_{k,m}(\mathrm{GL}_2(A)) \to M_{k,m}(\Gamma) : f(z) \mapsto F(z) = f(Tz).$$

Regarding these maps, we have the following result about the expansions at $[\infty]$ and $[0]$ for f and F:

Theorem 2.5.4. *Let $f(z) \in M_{k,m}(GL_2(A))$. Let*

$$f(z) = \sum_{n\geq 0} a_n t(z)^n, \qquad F(z) = \sum_{n\geq 0} b_n t(z)^n,$$

be the expansions at the cusp infinity. Then

$$f(z) = f\mid_{[\gamma_0]}(z) = \sum_{n\geq 0} b_n T^n s(z)^n,$$

and

$$F\mid_{[\gamma_0]}(z) = T^{-k}\sum_{n\geq 0} a_n T^n s(z)^n.$$

are the expansions at the cusp at 0*.*

Because of several notational differences from [Co2, Proposition 1.3], we include a proof of this result.

Proof. Assume that

$$f(z) = \sum_{n\geq 0} a_n t^n, \qquad\qquad F(z) = \sum_{n\geq 0} b_n s^n.$$

We note the following matrix equation

$$\begin{bmatrix} T & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & T \end{bmatrix}.$$

By definition $F(z) = f(Tz) = T^{-m} f\mid_{\left[\begin{smallmatrix} T & 0 \\ 0 & 1 \end{smallmatrix}\right]}$ and therefore

$$\begin{aligned} F\mid_{[\gamma_0]} &= T^{-m} f\mid_{\left[\begin{smallmatrix} T & 0 \\ 0 & 1 \end{smallmatrix}\right][\gamma_0]} \\ &= T^{-m} f\mid_{[\gamma_0]\left[\begin{smallmatrix} 1 & 0 \\ 0 & T \end{smallmatrix}\right]} \\ &= T^{-m} f\mid_{\left[\begin{smallmatrix} 1 & 0 \\ 0 & T \end{smallmatrix}\right]} \\ &= T^{-k} f\left(\frac{z}{T}\right) \\ &= T^{-k}\sum_{n\geq 0} a_n t^n(z/T) \\ &= T^{-k}\sum_{n\geq 0} a_n T^n s^n(z). \end{aligned}$$

Similarly,

$$\begin{aligned} f\mid_{[\gamma_0]}(z) &= f(z) \\ &= F\left(\frac{z}{T}\right) \\ &= \sum_{n\geq 0} b_n t^n(z/T) \\ &= \sum_{n\geq 0} b_n T^n s^n(z). \end{aligned}$$

□

The spaces $M_{k,m}(\Gamma)$ are finite-dimensional and we have the following results (due to Cornelissen [Co2, Theorem 1.7], Gekeler [Ge2, Proposition 6.4], and Teitelbaum [Te1, Lemma 15]) regarding their dimensions:

Theorem 2.5.5. *We have*

- $\dim_{\mathbb{C}_\infty} M_{k_0(q-1),0}(\Gamma_0(T)) = k_0 + 1$;
- $\dim_{\mathbb{C}_\infty} S_{k_0(q-1),0}(\Gamma_0(T)) = \dim_{\mathbb{C}_\infty} S^2_{k_0(q-1),0}(\Gamma_0(T)) = k_0 - 1$;
- $M_{k_0(q-1),0}(\Gamma_0(T)) = \langle g^i G^j : i + j = k_0\rangle$;
- $\dim_{\mathbb{C}_\infty} S_{k,k-1}(\Gamma_1(T)) = \dim_{\mathbb{C}_\infty} S_{k,k-1}(\Gamma_0(T)) = k + 1$. □

Using the previous two theorems, we can explicitly determine the spaces $M_{k,0}(\Gamma_0(T))$, $S_{k,0}(\Gamma_0(T)) = S^2_{k,0}(\Gamma_0(T))$, for fixed k.

Example 2.5.6. Let $q = 3$. We will describe eigenbases for several spaces of low weight.

We have $M_{2,0}(\Gamma_0(T)) = \langle g - G, G\rangle$, $S_{2,0}(\Gamma_0(T)) = 0$.

When $k = 4$, define $\phi = g^2 - (T^2 + 1)gG + T^2G^2$. Then $M_{4,0}(\Gamma_0(T)) = \langle \phi, g^2 - G^2, G^2\rangle$, $S_{4,0}(\Gamma_0(T)) = \langle\phi\rangle$.

When $k = 6$, define

$$\phi_1 = g^3 - (T + 1)^2 g^2 G - T(T + 1)^2 gG^2 + T^3G^3,$$

$$\phi_2 = g^3 - (T - 1)^2 g^2 G + T(T - 1)^2 gG^2 - T^3G^3.$$

Then $M_{6,0}(\Gamma_0(T)) = \langle \phi_1, \phi_2, g^3 - G^3, G^3 \rangle$, $S_{6,0}(\Gamma_0(T)) = \langle \phi_1, \phi_2 \rangle$.

When $k = 8$, define

$$\psi_1 = g^3 G - (T^2+1) g^2 G^2 + T^2 g G^3,$$
$$\psi_2 = (T^2+1) g^4 - (T^4 + T^2 + 1) g^3 G + T^4 g G^3,$$
$$\psi_3 = g^4 - T^2 g^3 G - g G^3 + T^2 G^4,$$
$$\psi_4 = g^4 - g G^3,$$
$$\psi_5 = g^4 - T^2 g^3 G - g G^3 + T^8 G^4.$$

Then $M_{8,0}(\Gamma_0(T)) = \langle \psi_1, \psi_2, \psi_3, \psi_4, \psi_5 \rangle$, $S_{8,0}(\Gamma_0(T)) = \langle \psi_1, \psi_2, \psi_3 \rangle$.

2.5.1 Full Hecke Algebra and Hecke Algebra away from T

Let $\mathfrak{p} \in \operatorname{Spec}(A)$ be relatively prime to (T). The Hecke operator $\mathrm{T}_{\mathfrak{p}}$ on $M_{k,m}(\Gamma)$ is defined in the same manner as before. Recall:

$$\mathrm{T}_{\mathfrak{p}} f = \wp^k f(\wp z) + \sum_{\beta \in S_{\mathfrak{p}}} f\left(\frac{z+\beta}{\wp}\right), \qquad S_{\mathfrak{p}} = \{\beta \in A, \deg(\beta) < \deg(\wp)\}.$$

For the ideal (T), we define the operator

$$\mathrm{U_T} f := \sum_{\beta \in \mathbb{F}_q} f\left(\frac{z+\beta}{T}\right).$$

One can compute the effect of $\mathrm{U_T}$ on an expansion at infinity:

$$\mathrm{U_T}\left(\sum_{n=0}^{\infty} a_n t^n\right) = \sum_{n=0}^{\infty} a_n G_{\Lambda_T, n}(Tt).$$

The two kinds of operators above commute with one another and among themselves, and therefore for any non-zero ideal $\mathfrak{n} \subset A$ we can define

$$\mathrm{T}_{\mathfrak{n}} := \mathrm{U_T}^{\operatorname{val}_T(\mathfrak{n})} \mathrm{T}_{\mathfrak{p}_1}^{n_1} \cdots \mathrm{T}_{\mathfrak{p}_s}^{n_s}, \qquad \text{where } \mathfrak{n} T^{-\operatorname{val}_T \mathfrak{n}} = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_s^{n_s}.$$

Definition 2.5.7. We call the sub-$\mathbb{C}_\infty$-algebra $\mathbb{T}_T$ of $\operatorname{End}_{\mathbb{C}_\infty} M_{k,m}(\Gamma)$ generated by $\mathrm{U_T}$ and all the $\mathrm{T}_{\mathfrak{p}}$ with $\mathfrak{p} + (T) = A$ the *full Hecke algebra at level T*. The sub-$\mathbb{C}_\infty$-algebra $\mathbb{T}'_T$ of $\operatorname{End}_{\mathbb{C}_\infty} M_{k,m}(\Gamma)$ generated by $\mathrm{T}_{\mathfrak{p}}$ with $\mathfrak{p} + (T) = A$ will be called the *Hecke algebra away from level T*.

Both $\mathbb{T}_T$ and $\mathbb{T}'_T$ preserve the spaces $S_{k,m}(\Gamma)$ and $S^2_{k,m}(\Gamma)$.

Definition 2.5.8. If $f \in M_{k,m}(\Gamma)$ is an eigenform for $\mathbb{T}_T$, we will call f a *simultaneous eigenform*, or simply an eigenform. If f is an eigenform for $\mathbb{T}'_T$, we say that f is an *eigenform away from the level.*

We will use the notation $\lambda_T(f)$ for the eigenvalue of $\mathrm{U_T}$, even though $\mathrm{U_T}$ is not the same as T_T. We hope that the context will make it clear which operator we are using.

Using the definition of $\mathbb{T}$ in terms of double-cosets, one can show (see [DiSh, Chapter 5] for the proof in the classical case, which carries over to the Drinfeld setting) the following result.

Theorem 2.5.9. *Let $f \in M_{k,m}(GL_2(A))$ be an eigenform for $\mathbb{T}$. Then $f, F \in M_{k,m}(\Gamma)$ are eigenforms for $\mathbb{T}'_T$ with the same eigenvalues as f.* □

Remark 2.5.10. We can also determine how $\mathrm{U_T}$ acts on f and F. By the definition of $\mathrm{U_T}$ for any $\phi \in M_{k,m}(\Gamma)$, we have

$$\mathrm{U_T}\,\phi(z) = \mathrm{T}_T\,\phi(z) - T^k\phi(Tz).$$

Therefore if $\mathrm{T}_T\, f = \lambda_T f$, then

$$\mathrm{U_T}\, f = \lambda_T f - T^k F.$$

Since f and F are linearly independent (as can be seen by looking at their t-expansions) we see that f is never an eigenform for $\mathrm{U_T}$. On the other hand,

$$\mathrm{U_T}\, F = \sum_{\beta \in \mathbb{F}_q} f\left(T \cdot \frac{z+\beta}{T}\right) = \sum_{\beta \in \mathbb{F}_q} f(z) = 0.$$

Example 2.5.11. We can prove that the bases from Example 2.5.6 for $k = 2, 4$ are eigenbases for $\mathbb{T}_T$. We can show that:

$g - G \in M_{2,0}(\Gamma_0(T))$ has eigensystem $\{\lambda_T = T^2, \lambda_{\mathfrak{p}} = \wp(T)^2\}_{\mathfrak{p} \in \operatorname{Spec} A \setminus T}$,

$G \in M_{2,0}(\Gamma_0(T))$ has eigensystem $\{\lambda_T = 0, \lambda_{\mathfrak{p}} = \wp(T)^2\}_{\mathfrak{p} \in \operatorname{Spec} A \setminus T}$,

$g^2 - G^2 \in M_{4,0}(\Gamma_0(T))$ has eigensystem $\{\lambda_T = T^4, \lambda_{\mathfrak{p}} = \wp(T)^4\}_{\mathfrak{p} \in \operatorname{Spec} A \setminus T}$

$G^2 \in M_{4,0}(\Gamma_0(T))$ has eigensystem $\{\lambda_T = 0, \lambda_{\mathfrak{p}} = \wp(T)^2\}_{\mathfrak{p} \in \operatorname{Spec} A \setminus T}$,

$\phi \in S_{4,0}(\Gamma_0(T))$ has eigensystem $\{\lambda_T = -T^2, \lambda_{\mathfrak{p}} = \wp(T) \cdot \wp(-T)\}_{\mathfrak{p} \in \operatorname{Spec} A \setminus T}$,

$g^3 - G^3 \in M_{6,0}(\Gamma_0(T))$ has eigensystem $\{\lambda_T = T^6, \lambda_{\mathfrak{p}} = \wp(T)^6\}_{\mathfrak{p} \in \operatorname{Spec} A \setminus T}$,

$G^3 \in M_{6,0}(\Gamma_0(T))$ has eigensystem $\{\lambda_T = 0, \lambda_{\mathfrak{p}} = \wp(T)^6\}_{\mathfrak{p} \in \operatorname{Spec} A \setminus T}$,

$\psi_2 \in S_{8,0}(\Gamma_0(T))$ has eigensystem $\{\lambda_T = T^2, \lambda_{\mathfrak{p}} = \wp(T)^2\}_{\mathfrak{p} \in \operatorname{Spec} A \setminus T}$,

$\psi_3 \in S_{8,0}(\Gamma_0(T))$ has eigensystem $\{\lambda_T = 0, \lambda_{\mathfrak{p}} = \wp(T)^2\}_{\mathfrak{p} \in \operatorname{Spec} A \setminus T}$,

$\psi_4 \in M_{8,0}(\Gamma_0(T))$ has eigensystem $\{\lambda_T = T^8, \lambda_{\mathfrak{p}} = \wp(T)^8\}_{\mathfrak{p} \in \operatorname{Spec} A \setminus T}$,

$\psi_5 \in M_{8,0}(\Gamma_0(T))$ has eigensystem $\{\lambda_T = 0, \lambda_{\mathfrak{p}} = \wp(T)^8\}_{\mathfrak{p} \in \operatorname{Spec} A \setminus T}$.

The result for ϕ is [Bö1, Example 15.4], while the eigensystems can be proved by using the theory of A-expansions from Chapter 3 and Theorem 4.3.1. The reader will notice that we have not written down eigensystems for ϕ_1, ϕ_2, ψ_1. For these three forms we have conjectural eigensystems which have been verified for $\mathfrak{p}$ of degree less than or equal to 4. They are

$\phi_1 \in S_{6,0}(\Gamma_0(T))$ has eigensystem $\{\lambda_T = -T^3, \lambda_{\mathfrak{p}} = \wp(T) \cdot \wp(-T^2)\}_{\mathfrak{p} \in \operatorname{Spec} A \setminus T}$,

$\phi_2 \in S_{6,0}(\Gamma_0(T))$ has eigensystem $\{\lambda_T = T^3, \lambda_{\mathfrak{p}} = \wp(T) \cdot \wp(T^2)\}_{\mathfrak{p} \in \operatorname{Spec} A \setminus T}$,

$\psi_1 \in S_{8,0}(\Gamma_0(T))$ has eigensystem $\{\lambda_T = -T^4, \lambda_{\mathfrak{p}} = \wp(T) \cdot (-\wp(T))\wp(T^3)\}_{\mathfrak{p} \in \operatorname{Spec} A \setminus T}$.

Chapter 3
A-expansions

In this chapter, we introduce the main results of this dissertation. We define the concept of Drinfeld modular forms with A-expansions, and we show that there are infinitely many cuspidal Drinfeld modular forms with A-expansions. The utility of Drinfeld modular forms with A-expansions stems from the fact that the action of the Hecke algebra on them can be easily computed. All the examples that we obtain turn out to be eigenforms with simple eigensystems. Once we introduce our main result (Theorem 3.1.10) we give various applications, including a multiplicity one result (Corollary 3.1.7), results on the diagonalizability of the Hecke action (Corollary 3.1.11 and Corollary 3.1.23), non-trivial congruences between simultaneous eigenforms (Theorem 3.1.24) and other relations among Drinfeld modular forms (Corollary 3.1.13 and Section 3.3).

3.1 A-expansions and Their Properties

In Section 2.4.1 we defined the Hecke algebra $\mathbb{T}$ that acts on the space of Drinfeld modular forms for $\mathrm{GL}_2(A)$. We noted that, in contrast to the classical case, the action of $\mathrm{T}_{\mathfrak{p}}$ on the t-expansion of a Drinfeld modular form is difficult to compute. Classically the action of the Hecke algebra on the Fourier expansion of a modular form is well-understood. One obvious difference is that in the case of classical modular forms both the Hecke operators and the Fourier expansion are indexed by the positive integers, while in the case of Drinfeld modular forms the t-expansion is indexed by the positive integers, while the Hecke operators are indexed by the set of monic polynomials A_+. In this section, we develop the theory of A-expansions, which addresses this discrepancy and recaptures the computability of the Hecke action in a manner similar to the classical case. However, the formalism does not seem to apply to all Drinfeld modular forms (see the examples in Section 3.6.1).

Recall that in Chapter 2 we defined G_n to be a multiple of the n^{th} Goss polynomial for $\widetilde{\pi}A$, normalized so that its first non-zero coefficient equals 1. We also defined t_a to be the function $t(az)$. For $a \in A_+$, we have

$$t_a = \frac{t^{q^{\deg(a)}}}{\psi_a(t)} = \frac{t^{q^{\deg(a)}}}{1 + \cdots},$$

and therefore the function t_a can be expanded into a t-expansion for every $a \in A_+$.

Definition 3.1.1. A modular form $f \in M_{k,m}(\mathrm{GL}_2(A))$ is said to have an A-*expansion* if there exists a positive integer n and coefficients $c_0(f), c_a(f) \in \mathbb{C}_\infty$ such that

$$f = c_0(f) + \sum_{a \in A_+} c_a(f) G_n(t_a).$$

Here the equality above is meant as an equality in $\mathbb{C}_\infty[[t]]$ between the t-expansion of f on the left side and the expression on the right side. We will call the integer n an A-*exponent* of f and the number $c_a = c_a(f)$ the a^{th} *coefficient* of f. Clearly, if $f \in S_{k,m}(\mathrm{GL}_2(A))$, then $c_0(f) = 0$.

Remark 3.1.2. If we consider the action of scalar matrices on f and on the right hand side it is easy to see that $n \equiv m \bmod (q-1)$. We will show that if f is a simultaneous eigenform, then the A-exponent n of f is unique. This will follow from our multiplicity one result for forms with A-expansions (Corollary 3.1.7). For a general Drinfeld modular form f with A-expansion we do not know, but we strongly suspect, that the A-exponent n is unique.

Example 3.1.3 (Eisenstein series again)**.** Recall that we defined g_k to be a multiple of

$$\frac{1}{\widetilde{\pi}^k} \sideset{}{'}\sum_{(a,b) \in A^2} \frac{1}{(az+b)^k} = \sideset{}{'}\sum_{b \in A} \frac{1}{(\widetilde{\pi}b)^k} - \sum_{a \in A_+} \sum_{b \in A} \frac{1}{(\widetilde{\pi}az + \widetilde{\pi}b)^k}$$

The second sum is precisely $\sum G_k(t_a)$ and therefore, if we define δ_k^{-1} to be equal to the first sum, we have

$$g_k = 1 - \delta_k \sum_{a\in A_+} G_k(t_a).$$

This is our first family of examples of Drinfeld modular forms with A-expansions. It consists of non-cuspidal eigenforms and it was known to Goss ([Go1, Section 1.7], [Go2, Section 2]) and Gekeler ([Ge3, (5.9)]).

The constant δ_k can be explicitly computed. For instance (see [Ge3, (6.4)]), if $k_0 \leq q$ we have

$$g_{k_0(q-1)} = 1 + (-1)^{k_0}[1]^{k_0} \sum_{a\in A_+} G_{k_0(q-1)}(t_a).$$

Until the work of Bartolome Lopez [Lo1] in 2011, the family $\{g_k\}$ gave the only examples of Drinfeld modular forms with A-expansions. Lopez showed that there are two additional examples, the forms h and $\varDelta$. He proved that

$$h = \sum_{a\in A_+} a^q t_a, \qquad \varDelta = \sum_{a\in A_+} a^{q(q-1)} t_a^{q-1}.$$

We will see, Theorem 3.1.10 below, that these are just two examples in a whole family of infinitely many Drinfeld modular forms that possess A-expansions. All of the new examples are cuspidal or double cuspidal eigenforms and we will be able to explicitly compute their eigensystems. Before we do that we look at some properties of A-expansions.

Theorem 3.1.4 (Uniqueness of an A-expansion)**.**

$$c_0 + \sum_{a\in A_+} c_a G_n(t_a) = c_0' + \sum_{a\in A_+} c_a' G_n(t_a) \implies c_a = c_a' \quad \forall a \in A_+ \cup \{0\}.$$

I learned the following proof from Lopez [Lo2], who attributes the idea of the proof to Gekeler.

Proof. Assume that

$$\sum_{a\in A_+} c_a G_n(t_a) = 0. \qquad (*)$$

Then we will show that $c_a = 0$ for all $a \in A_+$.

Let S be a finite subset of A_+, and let a_S be a maximal element of S with respect to degree. The function $G_n(t_{a_S}(z))$ has a pole at $1/a_S$, while if $a \in S$, $a \neq a_S$, the functions $G_n(t_a(z))$ are holomorphic at $1/a_S$. Therefore,

$$\sum_{a \in S} c_a G_n(t_a) = 0 \Longrightarrow c_{a_S} = 0.$$

Repeating the argument with $S' = S - \{a_S\}$ shows that

$$\sum_{a \in S} c_a G_n(t_a) = 0 \Longrightarrow c_a = 0,\ \forall a \in S.$$

We use the standard notation $A_{d+} := \{a \in A_+ : \deg(a) = d\}$.

Let U_d be the $\mathbb{C}_\infty$-vector space generated by the set $\{G_n(t_a(z)) : a \in A_{d+}\}$. By what we have just shown, U_d has dimension $|A_{d+}| = q^d$ over $\mathbb{C}_\infty$. Consider the set $V_d := \{\mathrm{ord}_t(g) : g \in U_d\}$. We have $|V_d| \leq q^d$. If two non-zero elements g_1, g_2 have the same ord_t, then you can take a linear combination of g_1 and g_2 that will have ord_t which is strictly bigger than $\mathrm{ord}_t(g_1) = \mathrm{ord}_t(g_2)$ and replace g_1 with that linear combination. Applying this procedure inductively to a basis of U_d shows that $|V_d| = \dim_{\mathbb{C}_\infty} U_d = q^d$.

For each $i \in V_d$, choose $s_{d,i} \in U_d$ such that $\mathrm{ord}_t(s_{d,i}) = i$. Because the i's are distinct, any relation

$$0 = \sum_{i \in V_d} \alpha_i s_{d,i}, \qquad \alpha_i \in \mathbb{C}_\infty,$$

is impossible unless $\alpha_i = 0$ for all $i \in V_d$. Indeed, if $i_0 = \min\{i \in V_d : \alpha_i \neq 0\}$, then s_{d,i_0} is equal to a linear combination of terms with $\mathrm{ord}_t > i_0$, which contradicts the existence of i_0. Therefore the $s_{d,i}$, for $i \in V_d$, are linearly independent, and we conclude that the set $\{s_{d,i} : i \in V_d\}$ is a basis for U_d.

Let $d \neq d'$. Since $A_{d+} \cup A_{d'+}$ is finite of size $q^d + q^{d'}$ we see that the set $\{G_n(t_a(z)) : a \in A_{d+} \cup A_{d'+}\}$ is linearly independent over $\mathbb{C}_\infty$. Let $U_{d,d'}$ be the $\mathbb{C}_\infty$-vector space generated by this set. By the same reasoning as for V_d, the set $V_{d,d'} := \{\mathrm{ord}_t(g) :$

$g \in U_{d,d'}\} = V_d \cup V_{d'}$ has size $q^d + q^{d'}$. It follows that if $i \in V_d$ and $i' \in V_{d'}$, then $i \neq i'$ (otherwise $V_{d,d'}$ will have size less than $q^d + q^{d'}$). We conclude that $\operatorname{ord}_t(s_{d,i}) = \operatorname{ord}_t(s_{d',i'})$ if and only if $d = d'$ and $i = i'$.

Let $a_{d,1}, a_{d,2}, \ldots, a_{d,q^d}$ be the elements in A_{d+}. We express $G_n(t_{a_{d,j}}(z))$ in the basis $\{s_{d,i} : i \in V_d\}$:

$$G_n(t_{a_{d,j}}(z)) = \sum_{i=1}^{q^d} \alpha_{d,i,j} s_{d,i}.$$

Note that the change of basis matrix $(\alpha_{d,i,j})_{i,j}$ is invertible. Substituting in equation $(*)$, we get

$$\begin{aligned} 0 &= \sum_{d=0}^{\infty} \sum_{a \in A_{d+}} c_a G_n(t_a) \\ &= \sum_{d=0}^{\infty} \sum_{j=1}^{q^d} c_{a_{d,j}} G_n(t_{a_{d,j}}) \\ &= \sum_{d=0}^{\infty} \sum_{j=1}^{q^d} c_{a_{d,j}} \sum_{i=1}^{q^d} \alpha_{d,i,j} s_{d,i} \\ &= \sum_{d=0}^{\infty} \sum_{i \in V_d} \beta_{d,i} s_{d,i}, \end{aligned}$$

where

$$\beta_{d,i} = \sum_{j=1}^{q^d} c_{a_{d,j}} \alpha_{d,i,j}.$$

Because $\operatorname{ord}_t s_{d,i} = \operatorname{ord}_t s_{d',i'}$ if and only if $d = d', i = i'$ it follows that $\beta_{d,i} = 0$. Since the matrix $(\alpha_{d,i,j})_{i,j}$ is invertible we conclude that $c_{a_{d,j}} = 0$. □

As mentioned before, Drinfeld modular forms with A-expansions have the property that the action of any Hecke operator $\mathrm{T}_{\mathfrak{p}}$ on them can be explicitly computed. We have the following result.

Theorem 3.1.5. *Suppose that $f \in S_{k,m}(GL_2(A))$ is an eigenform for $\mathrm{T}_{\mathfrak{p}}$ with eigenvalue $\lambda_{\mathfrak{p}}$ and that f has an A-expansion with exponent n. Then $\lambda_{\mathfrak{p}}(f) = \wp^n$ and $c_{\mathfrak{p}}(f) = \wp^{k-n} c_1(f)$.*

Proof. Since f and $\wp$ are fixed, we let $c_a = c_a(f)$ and

$$S_{\mathfrak{p}} = \{\beta \in A : \deg(\beta) < \deg(\wp)\}.$$

We compute the Hecke action

$$\begin{aligned}
\mathrm{T}_{\mathfrak{p}} f &= \wp^k \sum_{a \in A_+} c_a G_n(t_{\wp a}) + \sum_{\beta \in S_{\mathfrak{p}}} \sum_{a \in A_+} c_a G_n\left(t_a\left(\frac{z+\beta}{\wp}\right)\right) \\
&= \wp^k \sum_{a \in A_+} c_a G_n(t_{\wp a}) + \frac{1}{\widetilde{\pi}^n} \sum_{\beta \in S_{\mathfrak{p}}} \sum_{a \in A_+} \sum_{b \in A} \frac{c_a \wp^n}{(az + a\beta + b\wp)^n} \\
&= \wp^k \sum_{a \in A_+} c_a G_n(t_{\wp a}) + \frac{1}{\widetilde{\pi}^n} \sum_{a \in A_+} \sum_{b \in A} c_a \wp^n \sum_{\beta \in S_{\mathfrak{p}}} \frac{1}{(az + a\beta + b\wp)^n}.
\end{aligned}$$

If $(a, \wp) = 1$, then the map $A \times S_{\mathfrak{p}} \to A$, which sends (b, β) to $a\beta + b\wp$, is surjective and injective. The inner double sum is absolutely convergent, therefore by rearranging

$$\sum_{b \in A} \sum_{\beta \in S_{\mathfrak{p}}} \frac{1}{(az + a\beta + b\wp)^n} = \sum_{b \in A} \frac{1}{(az + b)^n} = G_n(t_a).$$

If $(a, \wp) = \wp$, then the map $A \times S_p \to A$, which sends (b, β) to $a\beta + b\wp$, is surjective, but every output has a number of preimages which is divisible by q. Hence

$$\sum_{b \in A} \sum_{\beta \in S_{\mathfrak{p}}} \frac{1}{(az + a\beta + b\wp)^n} = 0.$$

It follows that

$$\mathrm{T}_{\mathfrak{p}} f = \wp^k \sum_{a \in A_+} c_a G_n(t_{\wp a}) + \wp^n \sum_{a \in A_+, (a,\wp)=1} c_a G_n(t_a).$$

Noting that $\mathrm{T}_{\mathfrak{p}} f = \lambda_{\mathfrak{p}} f$ and comparing coefficients in the A-expansions, we see that if there exists $a \in A_+$ such that $(a, \wp) = 1$ and $c_a \neq 0$, then $\lambda_{\mathfrak{p}} = \wp^n$. But if all the c_a for $(a, \wp) = 1$ are zero, then again looking at the A-expansions on both sides we see that f cannot be an eigenform for $\mathrm{T}_{\mathfrak{p}}$. Indeed, by the computation above

$$f = \sum_{a \in A_+} c_{\wp a} G_n(t_{\wp a}) \Longrightarrow \lambda_{\mathfrak{p}} \sum_{a \in A_+} c_{\wp a} G_n(t_{\wp a}) = \wp^k \sum_{a \in A_+} c_{\wp a} G_n(t_{\wp^2 a}),$$

which contradicts the uniqueness of the A-expansion.

By comparing $\wp^{\text{th}}$ coefficients on both sides, we get

$$c_\wp = \frac{\wp^k}{\lambda_\mathfrak{p}} c_1 = \wp^{k-n} c_1.$$

□

Corollary 3.1.6. *Assume that $f \in S_{k,m}(GL_2(A))$ is a modular form that possesses an A-expansion with exponent n. Let $a = \prod_{i=1}^{\nu} \wp_i^{e_i}$ for distinct monic primes $\wp_i$. If f is an eigenform for $T_{\mathfrak{p}_1}, \dots, T_{\mathfrak{p}_\nu}$, then $c_a(f) = a^{k-n} c_1(f)$.*

Proof. This follows by induction on the factorization of a. □

In the case of classical modular forms, a system of eigenvalues $\{\lambda_\ell : \ell \in \mathrm{Spec}(\mathbb{Z})\}$ determines a cuspidal eigenform $f_{\mathrm{cl}} \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ uniquely up to a multiplicative constant. This is known as the *multiplicity one* property of cusp forms for $\mathrm{SL}_2(\mathbb{Z})$. Together with the theory of Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$, this shows that any eigenform for $\mathrm{SL}_2(\mathbb{Z})$, not necessarily cuspidal, is determined up to a multiplicative constant by its eigensystem. As we have already remarked, this is not true for Drinfeld modular forms for $\mathrm{GL}_2(A)$. The forms g, $g^q\Delta$, Δ are counterexamples to the classical notion of multiplicity one, and indeed Theorem 3.1.10 below will provide infinitely many counterexamples. Gekeler asked if a Drinfeld eigenform for $\mathrm{GL}_2(A)$ is determined up to a multiplicative constant by its eigenvalues and its weight. We do not know if the answer to Gekeler's question is positive or negative when f is an eigenform, or even a cuspidal eigenform, for $\mathrm{GL}_2(A)$. The situation is much more favorable if we assume that the eigenform has an A-expansion. The following result shows that a cuspidal Drinfeld eigenform with A-expansion is uniquely determined by its eigensystem $\{\lambda_\mathfrak{p}\}$ and its weight k (as predicted by a positive answer to Gekeler's question). In addition, the eigensystem is of a very simple kind.

Corollary 3.1.7 (Multiplicity One for modular forms with A-expansions.). *If $f \in S_{k,m}(GL_2(A))$ is an eigenform that possesses an A-expansion with exponent n, then*

$$f = \sum_{a \in A_+} a^{k-n} G_n(t_a).$$

Therefore f is determined uniquely by its weight k and the eigenvalues $\{\lambda_{\mathfrak{p}} = \wp^n\}$.

We can obtain the following partial result if we assume that f is an eigenform for all $\mathfrak{p}$ with $\deg(\mathfrak{p}) \leq d$.

Corollary 3.1.8. *Let $f \in S_{k,m}(GL_2(A))$ be a modular form with A-expansion of exponent n. If f is an eigenform for $\{\mathrm{T}_{\mathfrak{p}}\}_{\deg(\mathfrak{p}) \leq d}$, then*

$$f = \sum_{a \in A_+, \deg(a) \leq d} a^{k-n} G_n(t_a) + O(t^{\omega_{n,\wp} q^{d+1}}).$$

Here $\omega_{n,\wp} = \mathrm{ord}_X\, G_{\Lambda_{\mathfrak{p}},n}(X)$.

Cécile Armana has shown [Ar1, Theorem 7.7] that if M is either $S_{k,0}(\mathrm{GL}_2(A))$ with $k < (q+1)^2(q-1)$, or $S^2_{k,1}(\mathrm{GL}_2(A))$ with $k < q^2(q+1)$, then an eigenform $f \in M$ for all primes of degree 1 is uniquely determined up to a multiplicative constant. If we assume that in addition f has an A-expansion, then using Corollary 3.1.8 we can recover a special case of her result in the case $S_{k,1}(\mathrm{GL}_2(A))$ and extend it to $S_{k,m}(\mathrm{GL}_2(A))$ with $m > 1$:

Corollary 3.1.9. *Let M be one of the following spaces:*

1. *$S_{k,1}(GL_2(A))$ with $k < q^{d+1}(q+1)$*
2. *$S_{k,m}(GL_2(A))$ with $m > 1$, $k < mq^{d+1}(q+1)$.*

If $f \in M$ is an eigenform for the operators $\{\mathrm{T}_{\mathfrak{p}}\}_{\deg(\mathfrak{p}) \leq d}$ and f has an A-expansion, then f is uniquely determined by its eigenvalues up to a multiplicative constant.

Proof. Indeed, f is determined by the formula in Corollary 3.1.8 up to at least $t^{q^{d+1}}$. In all the cases of M above any form is determined by the first i coefficients of its t-expansion (see [Ge3, (5.17)]), for some $i < q^{d+1}$. □

The following theorem constructs an infinite family of cuspidal eigenforms with A-expansions:

Theorem 3.1.10 (Existence of cusp forms with A-expansions.)**.** *Let k, n be two positive integers such that $k - 2n$ is a positive multiple of $(q-1)$ and $n \leq p^{\mathrm{val}_\mathfrak{p}(k-n)}$. Then*

$$f_{k,n} := \sum_{a \in A_+} a^{k-n} G_n(t_a)$$

is an element of $S_{k,m}(GL_2(A))$, where $m \equiv n \bmod (q-1)$.

Before we give a proof, we would like to present some consequences of Theorem 3.1.10. We will give computational examples in Section 3.6.

As mentioned before the proof, the forms $f_{k,n}$ are eigenforms. This follows from the properties of A-expansions that we have developed (see Theorem 3.1.5 and the corollary that follows it). Indeed, we see that

$$T_\mathfrak{p} f_{k,n} = \wp^n f_{k,n},$$

and therefore $f_{k,n}$ is an eigenform with eigensystem $\{\lambda_\mathfrak{p} = \wp^n\}_{\mathfrak{p} \in \mathrm{Spec}(A)}$.

As an important special case of Theorem 3.1.10, we have

Corollary 3.1.11. *Let $s \geq 0$. Then*

$$f_s := f_{q+1+s(q-1),1} = \sum_{a \in A_+} a^{q+s(q-1)} t_a$$

is an element of $S_{q+1+s(q-1),1}(GL_2(A)) \setminus S^2_{q+1+s(q-1),1}(GL_2(A))$.

Since the weights $k = q+1+s(q-1), s \geq 0$, are precisely the weights for which $S_{k,1}(\mathrm{GL}_2(A)) \neq 0$, and because $S_{k,m}(\mathrm{GL}_2(A)) = S^2_{k,m}(\mathrm{GL}_2(A))$ for $m \neq 1$, this shows that the space

$$S_{k,m}(\mathrm{GL}_2(A))/S^2_{k,m}(\mathrm{GL}_2(A))$$

is diagonalizable, and the eigenforms for $S_{k,1}(\mathrm{GL}_2(A))$ whose images form a coset eigenbasis (when the space is non-trivial) have eigenvalues $\lambda_{\mathfrak{p}} = \wp$. Thus given any cuspidal modular form which is not double cuspidal, we can add a double cuspidal modular form to make the sum (which is again cuspidal, but not double cuspidal) an eigenform with eigensystem $\{\lambda_{\mathfrak{p}} = \wp\}_{\mathfrak{p}\in\mathrm{Spec}(A)}$. In [Bö1, Example 15.7] Böckle shows[1] that the same result holds for $\Gamma_1(T)$. We will see in Chapter 4 that using A-expansions we can reprove Böckle's result and extend it to other congruence subgroups as hinted by [Bö1, Remark 12.9 and Example 15.7].

Definition 3.1.12. We call the family $\{f_s\}_{s\geq 0}$ the *special family.* Since $(q-1)$ divides $(q^\nu - q)$ we can define a subfamily $\{F_\nu \in S_{q^\nu+1,1}(\mathrm{GL}_2(A))\}_{\nu\in\mathbb{Z}_{>0}}$ by the formula

$$F_\nu := \sum_{a\in A_+} a^{q^\nu} t_a.$$

We will prove (see Section 3.3) that the subfamily $\{F_\nu\}$ can be obtained by a recursive procedure from $F_1 = h$ and $F_2 = hg^q$.

In [Ge1] Gekeler proved[2] that h has a product expansion that is indexed by the monic polynomials. If we use the notation ψ_a for the a^{th} inverse cyclotomic polynomial, we can write the product expansion as

$$h = t \prod_{a\in A_+} \psi_a(t)^{q^2-1}.$$

[1]The reader should be aware that Böckle uses a different normalization for $\mathrm{T}_{\mathfrak{p}}$ and with this normalization he shows that the eigenvalues are all equal to 1, which corresponds to $\lambda_{\mathfrak{p}} = \wp$ in our notation.

[2]Actually he derived a product expansion for Δ. The result for h follows immediately from that.

Corollary 3.1.13. *If $1 \leq j \leq q$, then*

$$h^j = \sum_{a \in A_+} a^{qj} t_a^j = t^j \prod_{a \in A_+} \psi_a(t)^{(q^2-1)j}.$$

In particular,

$$h = \sum_{a \in A_+} a^q t_a, \qquad \Delta = h^{q-1} = \sum_{a \in A_+} a^{q(q-1)} t_a^{q-1}.$$

Proof. We know that h^j as well as the claimed A-expansion for it are in the one-dimensional space $S_{j(q+1),j}(\mathrm{GL}_2(A))$ by Theorem 3.1.10. Comparing the first non-zero coefficient of the t-expansions on both sides, the claimed equality follows. □

Remark 3.1.14. We want to remark that while the relations

$$h^j = t^j \prod_{a \in A_+} \psi_a(t)^{(q^2-1)j}$$

are immediate from the product formula for h, the equations that follow from the theorem

$$\left(\sum_{a \in A_+} a^q t_a \right)^j = \sum_{a \in A_+} a^{qj} t_a^j, \qquad 1 \leq j \leq q$$

are non-trivial and imply relations between the coefficients of the t-coefficients on both sides.

Remark 3.1.15. The relations between products indexed by A_+ and A-expansions are very peculiar and reminiscent of results of Borcherds for classical eigenforms (see [Ono, Chapter 4]). It would be interesting to see if this is merely a coincidence or if a theory similar to Borcherds' exists for Drinfeld modular forms.

Corollary 3.1.13 above should remind the reader of Remark 2.4.11 from the previous chapter. The examples from that remark can be restated as saying

$$\left(1 - [1] \sum_{a \in A_+} t_a^{q-1} \right)^j = 1 + (-1)^j [1]^j \sum_{a \in A_+} G_{j(q-1)}(t_a).$$

Looking at computational evidence for $q = 2, 3, 4, 5$, we can conjecture that this a more general phenomenon:

Conjecture 3.1.16. *Suppose that (k, n) and (k', n') are two pairs of integers that satisfy the hypothesis of Theorem* 3.1.10*. If $G_n(X) \cdot G_{n'}(X) = G_{n+n'}(X)$, then*

$$\left(\sum_{a \in A_+} a^{k-n} G_n(t_a)\right) \cdot \left(\sum_{a \in A_+} a^{k'-n'} G_{n'}(t_a)\right) = \sum_{a \in A_+} a^{k+k'-(n+n')} G_{n+n'}(t_a).$$

Remark 3.1.17. Notice that in general the left hand side above equals

$$\sum_{a \in A_+} a^{k+k'-(n+n')} G_n(t_a) G_{n'}(t_a) + \sum_{a,b \in A_+, a \neq b} a^{k-n} b^{k'-n'} G_n(t_a) G_{n'}(t_a),$$

and therefore the conjecture implies enormous cancellation in the second sum whenever $G_n(X) \cdot G_{n'}(X) = G_{n+n'}(X)$.

Remark 3.1.18. Observe that if (k, n) and (k', n') both satisfy the hypothesis of Theorem 3.1.10, then so does the pair $(k+k', n+n')$. Therefore, all the A-expansions in Conjecture 3.1.16 are Drinfeld modular forms. Any proof of the conjecture would immediately imply Corollary 3.1.13, because by Theorem 2.2.1

$$G_i(X) \cdot G_j(X) = X^i X^j = X^{i+j} = G_{i+j}(X) \qquad \text{for } i + j \leq q.$$

Example 3.1.19. Since all the A-expansions in Conjecture 3.1.16 are Drinfeld modular forms (by Theorem 3.1.10) and because a Drinfeld modular form of type k and weight m is uniquely determined by the first i coefficients in its t-expansion, with $i \leq \frac{k}{q+1} + 1$, we can verify the conjecture case by case. We present several examples for various q.

Let $q = 3$. Then $G_7(X) \cdot G_8(X) = G_{15}(X)$ and by the procedure above we can verify that

$$\left(\sum_{a \in A_+} a^9 G_7(t_a)\right) \cdot \left(\sum_{a \in A_+} a^{18} G_8(t_a)\right) = \sum_{a \in A_+} a^{27} G_{15}(t_a).$$

Let $q = 4$. Then $G_7(X) \cdot G_4(X) = G_{11}(X)$ and we have

$$\left(\sum_{a\in A_+} a^{16}G_7(t_a)\right) \cdot \left(\sum_{a\in A_+} a^{16}G_4(t_a)\right) = \sum_{a\in A_+} a^{32}G_{11}(t_a).$$

As we have noted the examples above entail an enormous amount of cancellation.

Remark 3.1.20. The condition $G_n(X) \cdot G_{n'}(X) = G_{n+n'}(X)$ is necessary even if the pairs $(k, n), (k', n'), (k+k', n+n')$ all satisfy the hypothesis of Theorem 3.1.10, as the example (with $q = 3$)

$$\left(\sum_{a\in A_+} a^{18}G_4(t_a)\right) \cdot \left(\sum_{a\in A_+} a^{18}G_6(t_a)\right) \neq \sum_{a\in A_+} a^{36}G_{10}(t_a).$$

shows.

Remark 3.1.21. Conjecture 3.1.16 and Corollary 3.1.7 give examples of eigenforms that can be represented as products of eigenforms. Classically this rarely happens and such products have been explicitly determined (see [Gh1] and [Jo1]). In contrast to the classical case, in the case of Drinfeld modular forms we can have high order vanishing at the cusps. In the case of Drinfeld modular forms, one 'trivial' way of obtaining infinitely many such products is to take p^{th} powers of known eigenforms (for example, $h, h^p, h^{p^2}, \ldots$). Our results show 'non-trivial' examples of such eigenproducts, such as

$$h^j = \sum_{a\in A_+} a^{qj}t_a^j, \qquad 1 \le j \le q,$$

$$hg^j = \sum_{a\in A_+} a^{qj}t_a, \qquad 1 \le j \le q.$$

Additional examples have been shown in 3.1.19.

In Chapter 2 we promised the reader that we will show that Δ and $g^q\Delta$ satisfy the hypothesis of Theorem 2.4.16. Recall that Theorem 2.4.16, due to Goss, allows for the computation of the eigenvalues of cuspidal eigenforms with special t-expansions. The next theorem shows that there are infinitely many examples of cuspidal eigenforms that satisfy the hypothesis of that theorem.

Theorem 3.1.22. *Let k be a positive multiple of $(q-1)$ and $(q-1) \leq p^{\mathrm{val}_p(k-q+1)}$, so that by Theorem* 3.1.10

$$f_{k,q-1} = \sum_{a \in A_+} a^{k-(q-1)} t_a^{q-1}$$

is an element of $S_{k,0}(GL_2(A))$. Then

$$f_{k,q-1} = \sum_{j=1}^{\infty} a_{j(q-1)} t^{j(q-1)}$$

with $a_{q-1} \neq 0$ and $a_{j(q-1)} = 0$ when $j \not\equiv 0, 1 \bmod q$.

Note that $f_{k,q-1}$ is an eigenform by Corollary 3.1.5. The condition on the t-expansion, together with the cuspidality of f, is the hypothesis of Theorem 2.4.16.

Proof. Looking at the A-expansions, we get

$$\sum_{a \in A_+} a^{k-(q-1)} t_a^{q-1} = t^{q-1} + \mathcal{O}(t^{q(q-1)}),$$

and therefore $a_{q-1} = 1$. Next, we look at t_a with $a \in A_+$. To simplify notation, put $d = \deg(a)$. By definition

$$t_a = \frac{t^{q^d}}{\psi_a(t)} = \frac{1}{\rho_a(t^{-1})} = \frac{1}{at^{-1} + \cdots + l_{d-1}(a)t^{-q^{d-1}} + t^{-q^d}}.$$

And inverting the power series, we get

$$t_a = t^{q^d} \left(1 - (l_{d-1}(a)t^{q^{d-1}(q-1)} + \cdots + at^{q^d-1}) + \cdots \right)$$

Looking at the powers of t that occur with non-zero coefficients, we see that they all look like $i = q^d + q(q-1)i_0$ for some positive integer i_0. Hence t_a^{q-1} has a series expansion of the required form. $\square$

Even though this result[3] does not give us eigensystems that we could not have computed with our results on A-expansions, it shows how properties of the t-expansion of a Drinfeld modular form come from its A-expansion.

[3] Together with Theorem 2.4.16.

As promised, we see that

$$\varDelta = \sum_{a \in A_+} a^{q(q-1)} t_a^{q-1}$$

and

$$g^q \varDelta = \sum_{a \in A_+} a^{2q(q-1)} t_a^{q-1}$$

have t-expansions that satisfy the hypothesis of Theorem 2.4.16. These are only two of infinitely many eigenforms that satisfy the hypothesis of Theorem 2.4.16.

We turn to the question of the diagonalizability of the action of $\mathbb{T}$. One problem when working with the Hecke algebra $\mathbb{T}$ is that, unlike the classical case, we are not guaranteed that any of the Hecke operators will be diagonalizable. In fact, there are examples (see Example 2.4.15) of spaces of Drinfeld modular forms for which no basis of simultaneous eigenforms exists. By using Theorem 3.1.10, we can show that $S_{k,m}(\mathrm{GL}_2(A))$ is diagonalizable and determine the eigenforms and eigensystems in several non-trivial cases:

Corollary 3.1.23. *If $q+1 \leq d \leq 2q+1$, then $S_{q+1+d(q-1),1}(GL_2(A))$ is two-dimensional and has a basis of simultaneous eigenforms*

$$S_{q+1+d(q-1),1}(GL_2(A)) = \langle f_{q+1+d(q-1),1}, f_{q+1+d(q-1),1+(d-q)(q-1)} \rangle.$$

Proof. Corollary 3.1.11 shows that $f_{q+1+d(q-1),1}$ is indeed a modular form. On the other hand, $f_{q+1+d(q-1),1+(d-q)(q-1)}$ is also a modular form by Theorem 3.1.10, since

$$k-n = q+1+d(q-1)-1-d(q-1)+q(q-1) = q^2 \geq q^2-1 = 1+(d-q)(q-1).$$

□

Another important topic classically is that of congruences between modular forms. Essentially, the recasting of the classical definition was given in [Ge3, Section 12], and several results have appeared that seem to mirror the classical situation (see [Ge3, Section 12] and [Vi1]). It turns out that we can use Theorem 3.1.10 to obtain a new result regarding congruences between Drinfeld eigenforms.

Theorem 3.1.24. *Let k, n be two positive integers that satisfy the hypothesis of Theorem 3.1.10. Assume further that n satisfies at least one of the following conditions:*

1. *We have $1 \leq n \leq q^2$, or $n = q^r - 1$, or $n = q^r + 1$ for some $r \geq 1$.*
2. *The set of $(s+1)$-tuples $\underline{i} = (i_0, \dots, i_s)$, $s \geq 0$ arbitrary that satisfy*

$$i_0 + i_1 + \ldots + i_s = \mathrm{ord}_X(G_{\widetilde{\pi}A,n}), \qquad i_0 + i_1 q + \ldots + i_s q^s = n - 1,$$

$$\text{and} \quad \binom{\mathrm{ord}_X(G_{\widetilde{\pi}A,n})}{\underline{i}} \not\equiv 0 \bmod p,$$

consists of only one element.

For any integer $l \geq 0$, define

$$F_{k,n,l} := \sum_{a \in A_+} a^{(k-n)q^l} G_n(t_a) \in S_{(k-n)q^l+n,n}(GL_2(A)).$$

Let $\nu_0 = \mathrm{val}_p(k-n)$, and let ν be any non-negative integer.

Then if $\mathfrak{p}$ is any prime of degree d with $d > \log_q(n)$, the congruence

$$F_{k,n,d+\nu} \equiv F_{k,n,\nu} \bmod \mathfrak{p}^{q^\nu p^{\nu_0}}$$

holds.

Note that the weights of the forms in the congruence are $(k-n)q^{d+\nu} + n$ and $(k-n)q^\nu + n$ respectively.

Proof. The first part of the result (the fact that $F_{k,n,l}$ is actually a Drinfeld eigenform) is a direct consequence of Theorem 3.1.10. We want to point out that we cannot replace q with p in the definition of $F_{k,n,l}$ and still use Theorem 3.1.10 because of the condition $k - 2n \equiv 0 \bmod (q-1)$.

Let $\mathfrak{p}$ be a prime of degree d. By a well-know analog of Fermat's Little Theorem for function fields, or equivalently since $(A/\mathfrak{p})^*$ is a finite group of size q^d-1, we know $\mathfrak{p} \mid (a^{q^d}-a)$ for all $a \in A$. Therefore, we have

$$\mathfrak{p}^{q^\nu p^{\nu_0}} \mid \left(a^{(k-n)q^{d+\nu}} - a^{(k-n)q^\nu}\right)$$

for all $a \in A$. Because of the A-expansions on both sides, the congruence

$$F_{k,n,d+\nu} \equiv F_{k,n,\nu} \bmod \mathfrak{p}^{q^\nu p^{\nu_0}}$$

will follow if we can prove that $\mathfrak{p}$ does not divide the denominators of the coefficients of $G_n(X)$ (note that if $a \in A_+$, then t_a has no denominators in its t-expansion). Since we are taking $d > \log_q(n)$, in each case we are considering this follows from one of the Corollaries 2.2.8, 2.2.9, 2.2.10, or 2.2.11. □

It appears (see Remark 2.2.12) that when $q = p = 2$ the theorem is true with no assumptions on n. We want to point out that the assumptions on n are needed because of our choice to use $G_n(X)$ rather than $G_{\tilde{\pi}A,n}(X)$ (the difference being that G_n is normalized to have its first non-zero coefficient equal to 1, while $G_{\tilde{\pi}A,n}$ is normalized to have its top coefficient equal to 1). If we use $G_{\tilde{\pi}A,n}(X)$ in our definition of A-expansions, then we get the theorem with no assumptions on n:

Theorem 3.1.25. *Let k, n be two positive integers that satisfy the hypothesis of Theorem 3.1.10. For any integer $l \geq 0$, define*

$$F^\star_{k,n,l} := \sum_{a \in A_+} a^{(k-n)q^l} G_{\tilde{\pi}A,n}(t_a) \in S_{(k-n)q^l+n,n}(GL_2(A)).$$

Let $\nu_0 = \mathrm{val}_p(k-n)$, and let ν be any non-negative integer.

Then if $\mathfrak{p}$ is any prime of degree d with $d > \log_q(n)$ we have the congruence

$$F^\star_{k,n,d+\nu} \equiv F^\star_{k,n,\nu} \bmod \mathfrak{p}^{q^\nu p^{\nu_0}}.$$

Note that the weights of the forms in the congruence are $(k-n)q^{d+\nu}+n$ and $(k-n)q^{\nu}+n$ respectively.

Remark 3.1.26. Note that we actually have

$$[d]^{q^{\nu}p^{\nu_0}} \mid \left(a^{(k-n)q^{d+\nu}} - a^{(k-n)q^{\nu}}\right), \qquad \forall a \in A.$$

Therefore, if $[d]$ (recall that $[d] = T^{q^d} - T$ is the product of all monic primes of degree dividing d) is relatively prime to the denominators of the coefficients of $G_n(X)$, then we obtain the stronger congruence

$$F_{k,n,d+\nu} \equiv F_{k,n,\nu} \bmod [d]^{q^{\nu}p^{\nu_0}}.$$

Remark 3.1.27. The proof above is deceptively simple, however this is because the A-expansions have packaged the t-expansions on both sides in a special way. In Section 3.3 we will show how to express some of the forms constructed in Theorem 3.1.10 in terms of h and g. It is unclear how to prove the result of the previous theorem without observing the A-expansions, i.e., by just looking at the t-expansions or at the expressions in terms of h and g.

Remark 3.1.28. One should note that Theorem 3.1.24 gives congruences in two directions: for varying d and fixed ν, and for fixed d and varying ν. We will present examples of both of these below.

Some of the results before the present work, particularly $g_{q^d-1} \equiv 1 \bmod [d]$ from [Ge3, (6.12)], were also proven by using the A-expansions of Eisenstein series. It is interesting to see if there are other congruences that come from A-expansions of forms that are not eigenforms.

We end this section with several examples of congruences obtained from Theorem 3.1.24.

Example 3.1.29. We present several examples with increasing d and fixed $\nu = 0$.

We have

$$F_{q+1,1,d} = F_{d+1} = \sum_{a\in A_+} a^{q\cdot q^d} t_a,$$

so $h = F_{q+1,1,0}$, $hg^q = F_{q+1,1,1}$, etc.

$$h \equiv hg^q = F_2 \bmod [1]^q,$$
$$h \equiv hg^{q^2+q} - [1]^{q^2} h^{q(q-1)+1} = F_3 \bmod [2]^q,$$
$$h \equiv hg^{q^3+q^2+q} - [2]^{q^2} h^{q(q-1)+1} g^{q^3} - [1]^{q^3} h^{q^2(q-1)+1} g^q = F_4 \bmod [3]^q,$$

Another family for which we obtain congruences is

$$F_{q(q-1)+1,1,d} = \sum_{a\in A_+} a^{q(q-1)q^d} t_a^{q-1}, \qquad d \geq 0,$$

where $\Delta = F_{q(q-1)+1,1,0}$. We have the congruences

$$\Delta \equiv \Delta g^{q^2-q} = F_{q(q-1)+1,1,1} \bmod [1]^q,$$
$$\Delta \equiv \Delta g^{q^3-q} + [1]^{q^2} \Delta^{q+1} g^{q^3-q^2-2q} + [1]^{(q-1)q^2} \Delta^{q^2-q+1} = F_{q(q-1)+1,1,2} \bmod [2]^q.$$

Since $G_1(X) = X$ and $G_{q-1}(X) = X^{q-1}$, we are in the situation described in Remark 3.1.26. We note that we cannot improve the congruence to mod$[d+1]$ (i.e.,to mod $\mathfrak{p}$ with $\mathfrak{p}$ of degree $d+1$) because

$$h \not\equiv hg^q \bmod [2], \qquad h \not\equiv hg^{q^2+q} - [1]^{q^2} h^{q(q-1)+1} \bmod [3].$$

Example 3.1.30. Let us fix $d = 1$ and let ν vary.

Notice that $F_{q+1,1,1+\nu} = F_{2+\nu}$. Then we have

$$\begin{aligned} F_5 &= (hg^{q^4+q^3+q^2+q} - [3]^{q^2} h^{q(q-1)+1} g^{q^4+q^3} - [2]^{q^3} h^{q^2(q-1)+1} g^{q^4+q} \\ &\quad - [1]^{q^4} h^{q^3(q-1)+1} g^{q^2+q} + [1]^{q^4}[3]^{q^2} h^{(q^3+q)(q-1)+1}) \\ &\equiv (hg^{q^3+q^2+q} - [2]^{q^2} h^{q(q-1)+1} g^{q^3} - [1]^{q^3} h^{q^2(q-1)+1} g^q) \bmod [1]^{q^2\cdot q} \\ &= F_4 \end{aligned}$$

We can also see $F_6 \equiv F_5 \bmod [1]^{q^3\cdot q}$, $F_7 \equiv F_6 \bmod [1]^{q^4\cdot q}$, ...

3.2 The Proof of the Main Theorem

Throughout this subsection, we will assume that k and n are positive integers such that $k \geq 2n$, $k - 2n \equiv 0 \mod (q-1)$ and $n \leq p^{\mathrm{val}_p(k-n)}$. We use the standard notation $A_{<d} := \{a \in A : \deg(a) < d\}$, $A_{<d+} := A_{<d} \cap A_+$, and $A^2_{<d} := \{(a,b) : a, b \in A_{<d}\}$.

We note that $n \leq p^{\mathrm{val}_p(k-n)}$ if and only if $(T-1)^n \mid (T^{k-n} - 1)$. Let

$$F(T) = \sum_{i=0}^{k-2n} \xi_i T^i$$

be defined by $T^{k-n} - 1 = (T-1)^n F(T)$. If $k = 2n$, then $n = k - n$ is a p^{th}-power and $F(T) = 1$. In general, by setting $T = 0$ we see that $\xi_0 = (-1)^{n+1}$.

Lemma 3.2.1 (Key Lemma). *Given $r > 0$, there exists a positive integer d_r such that for all $d \geq d_r$*

$$\sum_{a \in A_{<d}} a^j = 0, \qquad \forall j,\ 1 \leq j \leq r.$$

Proof. Define

$$S_{r,d} := \sum_{a \in A_{<d}} a^r,$$

and let $v(r)$ be the sum of the p-adic digits of r.

If $(q-1) \nmid r$, then $S_{r,d} = 0$. This follows since

$$A_{<d} - \{0\} = \{\theta a_+ : \theta \in \mathbb{F}_q^*, a_+ \in A_{<d+}\}$$

and summing over $\mathbb{F}_q^*$ first, we get 0.

The result for $r \equiv 0 \mod (q-1)$ is due to Lee (see [Tha, Section 5.6]). □

Remark 3.2.2. If $q = p^e$, then it is a result due to Lee that $S_{r,d} = 0$ whenever $v(r) < de(p-1)$. A complete vanishing criterion was given by Carlitz. However, Carlitz simply asserts the result without proving it. It turns that the proof is not trivial and was only achieved by Jeffrey Sheats in the late 1990s. For more on this, see [Tha, Sections 5.6-5.8] and the references therein.

Remark 3.2.3. The previous lemma also follows easily from the vanishing of the Carlitz zeta function at negative 'even' integers, which was first proved by Goss (see [Go4, Sections 8.8, 8.13]).

Lemma 3.2.4. *If* $d \geq d_{k-2n}$*, then*

$$\sum_{(u,v)\in A^2_{<d}}{}' \frac{(vz)^{k-n} - u^{k-n}}{(vz-u)^n} = \begin{cases} 0 & k-2n \neq 0, \\ -1 & k-2n = 0. \end{cases}$$

Here the prime on the summation means that we are taking pairs $(u, v) \neq (0, 0)$*.*

Proof. We break the sum into three parts.

When $u = 0$ we have $v \neq 0$. By Key Lemma 3.2.1,

$$\sum_{v\in A_{<d}}{}' v^{k-2n} z^{k-2n} = 0 \qquad \text{for } k-2n \neq 0.$$

The case $k - 2n = 0$ gives -1 for the sum, since we are summing over non-zero v. Therefore

$$\sum_{v\in A_{<d}}{}' v^{k-2n} z^{k-2n} = \begin{cases} 0 & k-2n \neq 0, \\ -1 & k-2n = 0. \end{cases}$$

When $v = 0$ we have $u \neq 0$. By the same argument as for the previous sum,

$$\sum_{u\in A_{<d}}{}' u^{k-2n} = \begin{cases} 0 & k-2n \neq 0, \\ -1 & k-2n = 0. \end{cases}$$

If $v \neq 0, u \neq 0$, then

$$\begin{aligned} \sum_{u\in A_{<d}}{}' \sum_{v\in A_{<d}}{}' \frac{(vz)^{k-n} - u^{k-n}}{(vz-u)^n} &= \sum_{u\in A_{<d}}{}' \sum_{v\in A_{<d}}{}' u^{k-2n} F\left(\frac{vz}{u}\right) \\ &= \sum_{u\in A_{<d}}{}' \sum_{v\in A_{<d}}{}' \sum_{i=0}^{k-2n} \xi_i (vz)^i u^{k-2n-i}. \end{aligned}$$

Summing over v and using Key Lemma 3.2.1 we see that only the term $i = 0$ remains. But if $k - 2n \neq 0$, then for $i = 0$ we can sum over u and get 0. Therefore, if $k - 2n \neq 0$

$$\sum_{u\in A_{<d}}{}' \sum_{v\in A_{<d}}{}' \frac{(vz)^{k-n} - u^{k-n}}{(vz-u)^n} = 0.$$

On the other hand, if $k-2n=0$, we get

$$\sum_{u\in A_{<d}}{}' \sum_{v\in A_{<d}}{}' \frac{(vz)^{k-n}-u^{k-n}}{(vz-u)^n} = \sum_{u\in A_{<d}}{}' \sum_{v\in A_{<d}}{}' \xi_0 (vz)^0 u^0 = \xi_0 = (-1)^{n+1}.$$

Combining these proves the lemma. $\square$

Lemma 3.2.5. *Let $k-2n>0$. If $d \geq d_{k-2n}$, then for any $a, b \in T^d A$ (not both zero) we have*

$$\sum_{(u,v)\in A^2_{<d}} \frac{(a+u)^{k-n}}{((a+u)z+b+v)^n} = \sum_{(u,v)\in A^2_{<d}}{}' \frac{(bu-av)^{k-n}}{(az+b)^{k-n}((a+u)z+b+v)^n}.$$

Note that the left sum does not have the condition that $(u,v)\neq(0,0)$.

Proof. Assume that $(u,v)\neq(0,0)$. Then

$$\frac{(bu-av)^{k-n}}{(az+b)^{k-n}((a+u)z+b+v)^n} - \frac{(a+u)^{k-n}}{((a+u)z+b+v)^n}$$

is equal to

$$\frac{((bu-av)-(a+u)(az+b))^n \sum_{i=0}^{k-2n} \xi_i (bu-av)^i((a+u)(az+b))^{k-2n-i}}{(az+b)^{k-n}((a+u)z+b+v)^n}.$$

Here we have used the identity $X^{k-n}-Y^{k-n} = Y^{k-2n}(X-Y)^n F(X/Y)$. Since $(bu-av)-(a+u)(az+b) = -a((a+u)z+b+v)$, the last expression reduces to

$$\frac{(-a)^n \sum_{i=0}^{k-2n} \xi_i (bu-av)^i((a+u)(az+b))^{k-2n-i}}{(az+b)^{k-n}}.$$

For $1\leq i\leq k-2n$ we consider

$$\sum_{(u,v)\in A^2_{<d}}{}' (bu-av)^i(a+u)^{k-2n-i}.$$

Expanding $(bu-av)^i$ by the binomials theorem and summing over v, we see that by Key Lemma 3.2.1 only the term $(bu)^i(a+u)^{k-2n-i}$ survives. Thus

$$\sum_{(u,v)\in A^2_{<d}}{}' (bu-av)^i(a+u)^{k-2n-i} = \sum_{u\in A_{<d}}{}' (bu)^i(a+u)^{k-2n-i}.$$

Expanding $(a+u)^{k-2n-i}$ by the binomial theorem and summing over u we get 0 by Key Lemma 3.2.1.

For $i=0$ we have

$$\begin{aligned}\sum_{(u,v)\in A_{<d}^2}{}' (a+u)^{k-2n} &= \sum_{u\neq 0}(a+u)^{k-2n}\sum_{v\in A_{<d}} 1 + a^{k-2n}\sum_{v\neq 0} 1\\ &= -a^{k-2n}\end{aligned}$$

Therefore

$$\sum_{(u,v)\in A_{<d}^2}{}' \frac{(-a)^n \sum_{i=0}^{k-2n}\xi_i (bu-av)^i((a+u)(az+b))^{k-2n-i}}{(az+b)^{k-n}}$$

equals

$$\frac{(-1)^{n+1}a^{k-n}}{(az+b)^n}\xi_0.$$

But $\xi_0 = (-1)^{n+1}$ (we are using $k\neq 2n$ here) and the lemma follows. □

Proof of Theorem 3.1.10. In order to reduce notation, we impose the following conventions: in what follows we will assume that $a\in T^dA$, $b\in T^dA$, $u\in A_{<d}$ and $v\in A_{<d}$.

Define

$$\phi_{k,n}(z) := \sum_{(u,v)}{}' \frac{u^{k-n}}{(uz+v)^n} + \sum_{(a,b)}{}' \sum_{(u,v)}{}' \frac{(bu-av)^{k-n}}{(az+b)^{k-n}((a+u)z+b+v)^n}.$$

Note that the first sum is finite, while the second sum converges, since each term is bounded by $\frac{1}{\min\{|a|,|b|\}^n}$ in absolute value. We compute

$$\phi_{k,n}\left(\frac{-1}{z}\right) = \sum_{(u,v)}{}' \frac{u^{k-n}z^n}{(vz-u)^n} + \sum_{(a,b)}{}' \sum_{(u,v)}{}' \frac{z^k(bu-av)^{k-n}}{(bz-a)^{k-n}((b+v)z-(a+u))^n}$$

which by Lemma 3.2.4 equals

$$\sum_{(u,v)}{}' \frac{(vz)^{k-n}z^n}{(vz-u)^n} + \sum_{(a,b)}{}' \sum_{(u,v)}{}' \frac{z^k(bu-av)^{k-n}}{(bz-a)^{k-n}((b+v)z-(a+u))^n}.$$

And by replacing u with $-u$ and a with $-a$, this equals

$$z^k \sum_{(u,v)}{}' \frac{v^{k-n}}{(vz+u)^n} + z^k \sum_{(a,b)}{}' \sum_{(u,v)}{}' \frac{(av-bu)^{k-n}}{(bz+a)^{k-n}((b+v)z+(a+u))^n}$$
$$= z^k \phi_{k,n}(z).$$

Therefore we have the correct functional equation with respect to $z \mapsto -1/z$.

It remains to show that $\phi_{k,n}$ has an A-expansion. By Lemma 3.2.5

$$\sum_{(a,b)}{}' \sum_{(u,v)}{}' \frac{(bu-av)^{k-n}}{(bz-a)^{k-n}((b+v)z-(a+u))^n} = \sum_{(a,b)}{}' \sum_{(u,v)} \frac{(a+u)^{k-n}}{((a+u)z+b+v)^n}.$$

Thus the sum defining $\phi_{k,n}$ is equal to

$$\sum_{(u,v)}{}' \frac{u^{k-n}}{(uz+v)^n} + \sum_{b}{}' \sum_{(u,v)} \frac{(bu)^{k-n}}{b^{k-n}(uz+b+v)^n}$$
$$+ \sum_{a}{}' \sum_{b} \sum_{(u,v)} \frac{(a+u)^{k-n}}{(az+b)^{k-n}((a+u)z+b+v)^n},$$

which, after multiplying by $1/\widetilde{\pi}^{k-n}$, becomes

$$\sum_{u \in A_{<d}} u^{k-n} G_n(t_u) + \sum_{a \in T^d A} a^{k-n} G_n(t_a).$$

Finally, notice that $G_n(t_{\theta a}) = \theta^{-n} G_n(t_a)$ and hence the expression above is precisely

$$-\sum_{a \in A_+} a^{k-n} G_n(t_a).$$

This shows that $\phi_{k,n}$ is invariant under translations by A (i.e., invariant under $z \mapsto z+a$ for all $a \in A$) and that

$$\frac{-1}{\widetilde{\pi}^{k-n}} \phi_{k,n} = f_{k,n} = \sum_{a \in A_+} a^{k-n} G_n(t_a) \in S_{k,n}(\mathrm{GL}_2(A)). \qquad \square$$

Remark 3.2.6. We want to briefly mention two cases outside of Theorem 3.1.10 which are also of some interest.

First, if $k = n$, we can make the same definition for $\phi_{k,n}$, i.e.,

$$\phi_{k,n}(z) = \sum_{(u,v)}{}' \frac{1}{(uz+v)^n} + \sum_{(a,b)}{}' \sum_{(u,v)}{}' \frac{1}{((a+u)z+b+v)^n}.$$

Then $\phi_{k,n}$ has the correct functional equation under $z \mapsto -1/z$, but Lemma 3.2.5 is no longer true, since

$$\sum_{(a,b)}{}' \sum_{(u,v)}{}' \frac{1}{((a+u)z+b+v)^n} \neq \sum_{(a,b)} \sum_{(u,v)}{}' \frac{1}{((a+u)z+b+v)^n}.$$

Therefore $\phi_{k,n}$ does not have a t-expansion. To fix this we add the $(u,v) = (0,0)$ term to the double sum. The resulting expression is essentially the non-normalized Eisenstein series E_n:

$$\phi_{k,n} + \sum_{(a,b)}{}' \frac{1}{(az+b)^n} = \widetilde{\pi}^n E_n.$$

The second case is $k = 2n$. As $n \leq p^{\mathrm{val}_p(k-n)}$, we see that in this case $k-n = n = p^\nu$ for some non-negative integer ν. We define $\phi_{k,n}$ as in the proof. Using Lemma 3.2.4, we get

$$\phi_{k,n}\left(\frac{-1}{z}\right) = z^k \phi_{k,n} + z^n.$$

Lemma 3.2.5 is not true, but it is replaced by the equation

$$-\sum_{(u,v)\in A^2_{<d}} \frac{(a+u)^{k-n}}{((a+u)z+b+v)^n} = \sum_{(u,v)\in A^2_{<d}}{}' \frac{(bu-av)^{k-n}}{(az+b)^{k-n}((a+u)z+b+v)^n}.$$

Therefore if we define

$$\phi^*_{k,n} := \sum_{(u,v)}{}' \frac{u^{k-n}}{(uz+v)^n} - \sum_{(a,b)}{}' \sum_{(u,v)}{}' \frac{(bu-av)^{k-n}}{(az+b)^{k-n}((a+u)z+b+v)^n},$$

we have

$$\phi^*_{k,n}\left(\frac{-1}{z}\right) = z^k \phi^*_{k,n} + z^n$$

and

$$\frac{-1}{\widetilde{\pi}^{k-n}} \phi^*_{k,n} = f^*_{k,n} = \sum_{a\in A_+} a^{k-n} G_n(t_a) = \sum_{a\in A_+} a^{p^\nu} G_{p^\nu}(t_a).$$

The first equation looks like the functional equation of a quasi-modular form and the second equation shows that this is E^{p^ν}, the p^ν-th power of the false Eisenstein series.

3.3 Recursion for F_ν

In this section, we prove a recursive formula for F_ν that is similar to Gekeler's recursive formula [Ge3, (6.9)] for g_k:

$$g_{q^{k_0}-1} = [k_0-1]g_{q^{k_0-2}-1}\Delta^{q^{k_0-2}} + g_{q^{k_0-1}-1}g^{q^{k_0-1}}, \qquad (k_0 \geq 2).$$

The proof uses results of Pellarin ([Pe2], [Pe3]) that we recorded in Section 2.4.2.

Recall that we have defined

$$F_\nu = \sum_{a\in A_+} a^{q^\nu} t_a \in S_{q^\nu+1,1}(\mathrm{GL}_2(A)).$$

Theorem 3.3.1. *We have $F_1 = h$, $F_2 = hg^q$ and the recursive formula for $\nu \geq 2$*

$$F_\nu = \frac{g^q}{h^{q-1}}F_{\nu-1}^q - \frac{[\nu-2]^{q^2}}{h^{q-1}}F_{\nu-2}^{q^2}.$$

Proof. In Chapter 2, we defined

$$\mathbb{E}(z,u) = \sum_{a\in A_+} a(u)t_a \in \mathbb{C}_\infty[[t,u]],$$

where u is a new variable independent of t and T. Let $\widetilde{\varphi}$ be the partial Frobenius (the map that fixes u and acts on the elements of $\mathbb{C}_\infty[[t]]$ by $x \to x^q$). The space $\mathbb{C}_\infty[[t,u]]$ also has the usual Frobenius, φ, which acts as $x \to x^q$ on every element of $\mathbb{C}_\infty[[t,u]]$. By definition of φ and $\widetilde{\varphi}$, we have

$$(\varphi\circ\widetilde{\varphi}^{-1})^\nu \mathbb{E}(z,u)_{|u=T} = F_\nu.$$

Pellarin has shown ([Pe2, Proposition 9]) that $\mathbb{E}$ satisfies the $\widetilde{\varphi}$-difference equation

$$\widetilde{\varphi}^2\,\mathbb{E} = \frac{1}{u-T^{q^2}}(-h^{q-1}\mathbb{E} + g^q\,\widetilde{\varphi}\,\mathbb{E}),$$

which we rewrite as

$$\mathbb{E} = \frac{g^q}{h^{q-1}}\,\widetilde{\varphi}\,\mathbb{E} - \frac{(u-T^{q^2})}{h^{q-1}}\,\widetilde{\varphi}^2\,\mathbb{E}.$$

Applying $(\varphi \circ \widetilde{\varphi}^{-1})^\nu$ to both sides and plugging in $u = T$, we get the recursion

$$F_\nu = \frac{g^q}{h^{q-1}} F_{\nu-1}^q - \frac{T^{q^n} - T^{q^2}}{h^{q-1}} F_{\nu-2}^{q^2}.$$

□

Remark 3.3.2. The same idea, but using the difference equation for

$$\mathbb{E}^2 = \sum_{a \in A_+} a(u)^2 t_a^2,$$

can be used to compute recursive relations between

$$\Phi_{\nu,2} = \sum_{a \in A_+} a^{2q^\nu} t_a^2,$$

for different ν's. However, one should note that, rather than having $\mathbb{E}^2$ satisfy an equation in $\mathbb{E}, \widetilde{\varphi}\,\mathbb{E}^2, \widetilde{\varphi}^2\,\mathbb{E}^2$, we have:

$$\mathbb{E}^2 = \frac{g^{2q}}{h^{2(q-1)}}\,\widetilde{\varphi}\,\mathbb{E}^2 + \frac{(u - T^{q^2})^2}{h^{2(q-1)}}\,\widetilde{\varphi}^2\,\mathbb{E}^2 - 2\frac{g^q(u - T^{q^2})}{h^{2(q-1)}}\,\widetilde{\varphi}\,\mathbb{E} \cdot \widetilde{\varphi}^2\,\mathbb{E}.$$

Computational evidence suggests that we also have recursive relations between

$$\Phi_{\nu,j} = \sum_{a \in A_+} a^{jq^\nu} t_a^j$$

for $j \leq q$. It was also observed computationally that

$$\mathbb{E}^j = \sum_{a \in A_+} a(u)^j t_a^j, \qquad j \leq q.$$

If we can prove the equation for $\mathbb{E}^j$, then the difference equation for $\mathbb{E}^j$ will allow us to prove the recursive relations among the $\Phi_{\nu,j}$. We hope to return to this in future work.

Example 3.3.3. Using the recursion, one easily computes:

$$
\begin{aligned}
F_3 &= \sum_{a \in A_+} a^{q^3} t_a = hg^{q^2+q} - [1]^{q^2} h^{q(q-1)+1}, \\
F_4 &= \sum_{a \in A_+} a^{q^4} t_a = hg^{q^3+q^2+q} - [2]^{q^2} h^{q(q-1)+1} g^{q^3} - [1]^{q^3} h^{q^2(q-1)+1} g^{q}, \\
F_5 &= \sum_{a \in A_+} a^{q^5} t_a = hg^{q^4+q^3+q^2+q} - [3]^{q^2} h^{q(q-1)+1} g^{q^4+q^3}, \\
&\qquad - [2]^{q^3} h^{q^2(q-1)+1} g^{q^4+q} - [1]^{q^4} h^{q^3(q-1)+1} g^{q^2+q}, \\
&\qquad + [1]^{q^4} [3]^{q^2} h^{(q^3+q)(q-1)+1}.
\end{aligned}
$$

Write

$$F_\nu = h\Delta^{a_\nu} g^{b_\nu} \phi_\nu(j),$$

where j is the Drinfeld j-invariant

$$j(z) := \frac{g(z)^{q+1}}{\Delta(z)},$$

and the exponents a_ν, b_ν are determined by $q^\nu - q = a_\nu(q^2-1) + b_\nu(q-1)$ with $a_\nu \geq 0$, $0 \leq b_\nu \leq q$.

For the remainder of this section, we investigate the properties of the sequences $\{a_\nu\}$, $\{b_\nu\}$ and $\{\phi_\nu\}$. To that end, note that plugging into the recursion we get

$$F_\nu = h(\Delta^{qa_{\nu-1}} g^{q(b_{\nu-1}+1)} \phi_{\nu-1}^{q} - [\nu-2]^{q^2} \Delta^{q^2 a_{\nu-2}+q} g^{q^2 b_{\nu-2}} \phi_{\nu-2}^{q^2}).$$

Thus we have $a_1 = 0, a_2 = 0, a_3 = q = 0 \cdot q^2 + q$ and

$$a_\nu = \max\{qa_{\nu-1}, q^2 a_{\nu-2} + q\}.$$

By induction one proves that

Theorem 3.3.4.

$$a_{odd} = q^2 a_{odd-2} + q, \qquad a_{even} = q a_{even-1}.$$

and

$$b_{odd} = 0, \qquad b_{even} = q.$$

Proof. All we need to prove is the recursion for the a_ν's. We induct on ν.

First, let ν be even. Then

$$\begin{aligned} a_\nu &= \max\{qa_{\nu-1}, q^2a_{\nu-2}+q\} \\ &= \max\{q(q^2a_{\nu-3}+q), q^2qa_{\nu-3}+q\} \\ &= \max\{q^3a_{\nu-3}+q^2, q^3a_{\nu-3}+q\} \\ &= qa_{\nu-1}. \end{aligned}$$

Second, let ν be odd. Then

$$\begin{aligned} a_\nu &= \max\{qa_{\nu-1}, q^2a_{\nu-2}+q\} \\ &= \max\{q^2a_{\nu-2}, q^2a_{\nu-2}+q\} \\ &= q^2a_{\nu-2}+q. \end{aligned}$$

□

Next, we look at the properties of ϕ_ν.

One can easily compute the first several ϕ_ν's:

$$\begin{aligned} \phi_1 &= 1, \\ \phi_2 &= 1, \\ \phi_3 &= j^q - [1]^{q^2}, \\ \phi_4 &= j^{q^2} - [2]^{q^2} j^{q^2-q} - [1]^{q^3}, \\ \phi_5 &= j^{q^3+q} - [3]^{q^2} j^{q^3} - [2]^{q^3} j^{q^3-q^2+q} - [1]^{q^4} j^q + [1]^{q^4}[3]^{q^2}. \end{aligned}$$

Using the recursion for the F_ν's, we get

$$\phi_\nu = \Delta^{qa_{\nu-1}-a_\nu} g^{qb_{\nu-1}+q-b_\nu} \phi_{\nu-1}^q - [k-2]^{q^2} \Delta^{q^2a_{\nu-2}+q-a_\nu} g^{q^2b_{\nu-2}-b_\nu} \phi_{\nu-2}^{q^2}.$$

This and the properties of the a_ν's and the b_ν's show that we have $\phi_1 = \phi_2 = 1$ and the recursion:

$$\begin{aligned} \nu \text{ odd}\ ,\ \phi_\nu &= j^q \phi_{\nu-1}^q - [\nu-2]^{q^2} \phi_{\nu-2}^{q^2}; \\ \nu \text{ even}\ ,\ \phi_\nu &= \phi_{\nu-1}^q - [\nu-2]^{q^2} j^{q^2-q} \phi_{\nu-2}^{q^2}. \end{aligned}$$

Putting $\phi_\nu = \psi_\nu^q$, this shows that $\psi_1 = \psi_2 = 1$ and we have the recursion:

$$\nu \text{ odd },\ \psi_\nu = j\psi_{\nu-1}^q - [\nu-2]^q \psi_{\nu-2}^{q^2};$$
$$\nu \text{ even },\ \psi_\nu = \psi_{\nu-1}^q - [\nu-2]^q j^{q-1} \psi_{\nu-2}^{q^2}.$$

The polynomials ψ_ν are not separable. Indeed, if ν is odd, then

$$\gcd(\psi_\nu, \psi_\nu') = (\psi_{\nu-2}).$$

And if ν is even, then

$$\gcd(\psi_\nu, \psi_\nu') = (\psi_{\nu-1}).$$

Thus ψ_1, ψ_2, ψ_3 are separable, and the rest of them are not.

Computing the Newton polygon of the ψ_ν suggests that

Conjecture 3.3.5. *The polynomial ψ_ν has $\nu - 2$ zeros with $|j| = q^{q^{\nu-1}}$ and the rest of its zeros satisfy $|j| \leq q^q$.*

3.4 Relations with Hasse Derivatives

In this section, we recall the theory of Hasse derivatives[4] (see [BoPe, Section 3]). We will recall when Hasse derivatives take Drinfeld modular forms to Drinfeld modular forms (with different weight and possibly different type). Hasse derivatives take A-expansions to A-expansions, and therefore they can be used to generate A-expansions from the special family $\{f_s\}$. It is unclear if this process generates all the forms $f_{k,n}$ from Theorem 3.1.10.

Definition 3.4.1. Let $f : \Omega \to \mathbb{C}_\infty$ be any rigid analytic function. Define the n^{th} *unaltered Hasse derivative*, $D_n f(z)$, by the formula

$$f(z+\varepsilon) = \sum_{n\geq 0} D_n f(z) \varepsilon^n,$$

[4]Some authors also call them divided derivatives.

where ε is taken small in absolute value. The n^{th} *Hasse derivative*, $\mathcal{D}_n f$, is defined by the formula

$$\mathcal{D}_n = \frac{(-1)^n}{\widetilde{\pi}^n} D_n.$$

One can show (see [BoPe, Section 3] and the references therein) that the $\mathcal{D}_n$'s form an iterative family of higher differentials, i.e.,

- they are $\mathbb{C}_\infty$-linear maps, with $\mathcal{D}_0$ a multiple of the identity;
- we have a Leibniz rule: $\mathcal{D}_i(fg) = \sum_{r=0}^{i} (\mathcal{D}_r f)(\mathcal{D}_{i-r} g)$;
- we have $\mathcal{D}_i \mathcal{D}_j = \mathcal{D}_j \mathcal{D}_i = \binom{i+j}{i} \mathcal{D}_{i+j}$.

It is an easy exercise involving the geometric series to show that

$$\mathcal{D}_n \left(\frac{1}{(\widetilde{\pi}az + \widetilde{\pi}b)^w} \right) = \binom{w+n-1}{n} \frac{a^n}{(\widetilde{\pi}az + \widetilde{\pi}b)^{n+w}}.$$

Therefore, using $w = 1$, we see that

$$\mathcal{D}_n \left(\sum_{a \in A_+} c_a t_a \right) = \sum_{a \in A_+} c_a a^n G_{\widetilde{\pi}A, n+1}(t_a) = \kappa_n \sum_{a \in A_+} c_a a^n G_{n+1}(t_a).$$

Here κ_n is the constant such that $G_{\widetilde{\pi}A,n+1}(X) = \kappa_n G_{n+1}(X)$. Bosser and Pellarin [BoPe, Lemma 3.4] have shown the following result concerning the action of $\mathcal{D}_n$ on Drinfeld modular forms:

Theorem 3.4.2. *If $f \in M_{k,m}(GL_2(A))$, then for $\gamma \in GL_2(A)$ we have*

$$\begin{aligned} \mathcal{D}_n f(\gamma z) &= (\det \gamma)^{-m-n} (cz+d)^{k+2n} \mathcal{D}_n f(z) \\ &\quad + (\det \gamma)^{-m-n} (cz+d)^{k+2n} \sum_{j=1}^{n} \binom{k+n-1}{j} \left(\frac{c}{cz+d} \right)^j \mathcal{D}_{n-j} f(z). \quad \square \end{aligned}$$

This shows that if

$$\binom{k+n-1}{j} \equiv 0 \bmod p, \qquad \forall j,\ 1 \le j \le n,$$

then $\mathcal{D}_n$ preserves modularity, i.e., $\mathcal{D}_n : M_{k,m}(\mathrm{GL}_2(A)) \to M_{k+2n,m+n}(\mathrm{GL}_2(A))$.

We want to use the Hasse derivatives $\mathcal{D}_n$ and the special family $\{f_s\}$ to produce other forms with A-expansions. We have the following result:

Theorem 3.4.3. *Let $q = p$. Given $p^{\nu-1} \leq n < p^{\nu}$ one can find $s_{\nu-1}$, where $0 \leq s_{\nu-1} < p$, depending only on n, so that*

$$\mathcal{D}_n : M_{p^\nu s_{\nu-1}+p^\nu(p-1)w-n+1,1}(GL_2(A)) \to M_{p^\nu s_{\nu-1}+p^\nu(p-1)w+n+1,1+n}(GL_2(A)),$$

for any non-negative integer w. Applying this to the appropriate f_s, we have

$$\mathcal{D}_n \left(\sum_{a \in A_+} a^{p^\nu s_{\nu-1}+p^\nu(p-1)w-n} t_a \right) = \kappa_n \sum_{a \in A_+} a^{p^\nu s_{\nu-1}+p^\nu(p-1)w} G_{n+1}(t_a).$$

Proof. Put $\sigma = k + n - 1$, where k for now is arbitrary positive integer. Let $\nu = \lfloor \log_p(n) \rfloor$ as above.

It is a standard exercise in using Lucas' Theorem, i.e.,

$$\binom{\sum_i \sigma_i p^i}{\sum_i j_i p^i} \equiv \binom{\sigma_0}{j_0}\binom{\sigma_1}{j_1} \cdots \binom{\sigma_{\nu-1}}{j_{\nu-1}} \bmod p \qquad \text{where } 0 \leq \sigma_i, j_i < p,$$

to show that

$$\left(\binom{\sigma}{j} \equiv 0 \bmod p, \quad \forall j, 1 \leq j \leq n \right) \iff \sigma = p^\nu u.$$

Assume now that f is a Drinfeld modular form of weight k, and use the result of Bosser and Pellarin (Theorem 3.4.2). It follows that if $\mathcal{D}_n f$ is a modular form, then we must have $k + n - 1 = p^\nu u$, where u is a positive integer.

We still have to show that $k = p + s(p-1) + 1$ for some s. We do this next.

Recall that we have fixed $n = n_0 + n_1 p + \cdots + n_{\nu-1}p^{\nu-1}$, so we can solve recursively for the p-adic digits of $s = s_0 + s_1 p + \cdots + s_{\nu-1}p^{\nu-1} + p^\nu w$ in

$$p^\nu u = p + (p-1)(s_0 + s_1 p + \cdots + s_{\nu-1}p^{\nu-1} + p^\nu w) + n_0 + n_1 p + \cdots + n_{\nu-1}p^{\nu-1}.$$

Note that $s_{\nu-1}$ is uniquely determined by n. On the other hand, w depends on n and u, and conversely u depends on w and n. Therefore, we think of w as an arbitrary non-negative integer and of u as a positive integer determined by a choice of w and n.

Once we have solved for the s_i's we get that

$$k = p^\nu s_{\nu-1} + p^\nu (p-1)w - n + 1,$$

where w is an arbitrary non-negative integer. Combining this with Theorem 3.4.2 gives the rest.

□

Remark 3.4.4. The proof of the previous theorem also shows that

$$\mathcal{D}_n\, g_k = 0$$

for any n for which $\mathcal{D}_n$ preserves modularity. Indeed, if we put $k = p^\nu u - n + 1$, then

$$\mathcal{D}_n \left(\frac{1}{(\widetilde{\pi} az + \widetilde{\pi} b)^k} \right) = \binom{k+n-1}{n} * = \binom{p^\nu u}{n} * = 0.$$

An argument similar to the proof of Theorem 3.4.3 shows:

Theorem 3.4.5. *Let $q = p^e$. Given $p^{\nu-1} \leq n < p^\nu$ we have:*

1. *if $\nu \leq e$, then*

$$\mathcal{D}_n : S_{q(n+1)+p^\nu(q-1)w+1-n,1}(GL_2(A)) \to M_{q(n+1)+p^\nu(q-1)w+1+n,1+n}(GL_2(A))$$

and

$$\mathcal{D}_n \left(\sum_{a \in A_+} a^{q(n+1)+p^\nu(q-1)w-n} t_a \right) = \kappa_n \sum_{a \in A_+} a^{q(n+1)+p^\nu(q-1)w} G_{n+1}(t_a).$$

2. *if $\nu > e$, then one can find $\sigma < p^\nu$ such that*

$$\mathcal{D}_n : S_{q\sigma+p^\nu(q-1)w+1-n,1}(GL_2(A)) \to M_{q\sigma+p^\nu(q-1)w+1+n,1+n}(GL_2(A))$$

and

$$\mathcal{D}_n \left(\sum_{a \in A_+} a^{q\sigma+p^\nu(q-1)w-n} t_a \right) = (-1)^n \sum_{a \in A_+} a^{q\sigma+p^\nu(q-1)w} G_{n+1}(t_a). \quad \square$$

The reader should notice that all the A-expansions that are produced from the results above can be found by Theorem 3.1.10. It is not known if we can prove Theorem 3.1.10 by just showing that the special family $\{f_s\}$ consists of Drinfeld modular forms and use Hasse derivatives to complete the proof. However, we know that we can prove important special cases of Theorem 3.1.10 by this procedure:

Corollary 3.4.6. *Assume that we know that the special family* $\{f_s\}$ *consists of Drinfeld modular forms. Let* $n < q^\nu$*. Then we have*

$$\sum_{a\in A_+} a^{q^\nu(n+1)} G_{n+1}(t_a) \in S_{(q^\nu+1)(n+1),n+1}(GL_2(A)).$$

Proof. If $j < q^\nu$, then

$$\binom{q^\nu(n+1)}{j} = \binom{q+1+(q^\nu-1)(n+1)-(q-1)+n-1}{j} \equiv 0 \bmod p.$$

Hence we can use $\mathcal{D}_n$ to preserve modularity and get

$$\mathcal{D}_n\left(\sum_{a\in A_+} a^{q+(q^\nu-1)(n+1)-(q-1)} t_a\right) = \kappa_n \sum_{a\in A_+} a^{q^\nu(n+1)} G_{n+1}(t_a).$$

□

3.5 The Residues of f_s on the Quotient Tree $\mathbf{GL}_2(A)\backslash\mathcal{T}$

In this section, we recall the theory of harmonic cocycles on $\mathrm{GL}_2(A)\backslash\mathcal{T}$ and its connections with cuspidal Drinfeld modular forms following [Te1] and [Te3]. Our goal is to compute the residues of the form f_s from Corollary 3.1.11 on the tree $\mathrm{GL}_2(A)\backslash\mathcal{T}$. This is achieved in Theorem 3.5.6. The reader should be aware that the normalizations from [Te3, Section 1], which we follow, differ from the normalizations in [Bö1, Chapter 5].

Recall that in Theorems 2.3.4 and 2.3.7 in Chapter 2, we defined a covering of Ω by affinoids that reflected the combinatorial structure of the Bruhat-Tits Tree $\mathcal{T}$ of

$\mathrm{GL}_2(K_\infty)$. We also described the structure of the quotient tree $\Gamma\backslash\mathcal{T}$, when Γ is a congruence subgroup. In his 1991 paper [Te1], Jeremy Teitelbaum showed that the space of cuspidal Drinfeld modular forms for Γ is isomorphic to a space of certain functions on the tree $\mathcal{T}$, called harmonic cocycles (see Definition 3.5.1 below). In this section, we will only look at the case $\Gamma = \mathrm{GL}_2(A)$, but the reader should be aware that for other congruence subgroups (such as $\Gamma_1(T)$ and $\Gamma(T)$), Böckle has described exactly what happens with double cuspidal forms under this isomorphism (see [Bö1, Theorem 5.19]). Until further notice, e will denote an oriented edge of $\mathcal{T}$, v a vertex of $\mathcal{T}$, and the notation $e \mapsto v$ will mean that e ends at v.

Again from Chapter 2, recall that we have the following affinoid subsets of Ω

$$\begin{aligned}\mathfrak{U}_{v_0} &= \left\{ z \in \Omega : \ \frac{1}{q} < |z| < q, |z-\beta| > \frac{1}{q}, \beta \in \mathbb{F}_q \right\}, \\ \mathfrak{U}_{e_0} &= \{ z \in \Omega : \ 1 < |z| < q \} .\end{aligned}$$

According to Theorem 2.3.7, the quotient graph $\mathrm{GL}_2(A)\backslash\mathcal{T}$ has one infinite end corresponding to the cusp at infinity. Therefore, $\mathrm{GL}_2(A)\backslash\mathcal{T}$ is a half line that consists of vertices $\{v_\nu\}_{\nu\geq 0}$ and oriented edges $\{e_\nu = (v_\nu, v_{\nu+1})\}_{\nu\geq 0}$ such that, under the reduction map e_0, corresponds to $\mathfrak{U}_{e_0}$ and v_0 corresponds to $\mathfrak{U}_{v_0}$. Recall that in Chapter 2 we also had an action of $\mathrm{GL}_2(A)$ on $\mathcal{T}$ that was induced from the natural action on lattices. Under this action we have

$$v_\nu = \begin{bmatrix} T^\nu & 0 \\ 0 & 1 \end{bmatrix} v_0$$

and consequently

$$e_\nu = \begin{bmatrix} T^\nu & 0 \\ 0 & 1 \end{bmatrix} e_0.$$

Definition 3.5.1. Let $\mathtt{X}$ and $\mathtt{Y}$ be two indeterminates. A *harmonic cocycle of weight k* and *type m* is a map w from the set of oriented edges of $\mathcal{T}$ to the space of $\mathbb{C}_\infty$-valued functions on the set of homogeneous polynomials in $\mathtt{X}$ and $\mathtt{Y}$ of degree $k-2$ such that w satisfies

$$w(-e) = -w(e), \qquad \sum_{e\mapsto v} w(e) = 0.$$

Any such map w is uniquely determined by its values on the monomials $\mathtt{X}^j\mathtt{Y}^{k-2-j}$ with $0 \leq j \leq k-2$. We will usually write w_e to mean $w(e)$. Notice that so far the type m is irrelevant.

We define a left action of $\mathrm{GL}_2(A)$ on the space of harmonic cocycles of weight k and type m by

$$\gamma * w_e(\mathtt{X}^j\mathtt{Y}^{k-2-j}) := (\det\gamma)^{1-m} w_e((a_\gamma\mathtt{X} + b_\gamma\mathtt{Y})^j(c_\gamma\mathtt{X} + d_\gamma\mathtt{Y})^{k-2-j}).$$

A harmonic cocycle is called $\mathrm{GL}_2(A)$-*invariant* if $\gamma * w_e = w_{\gamma * e}$ for all $\gamma \in \mathrm{GL}_2(A)$. The space of $\mathrm{GL}_2(A)$-invariant harmonic cocycles of type k and weight m is a finite-dimensional $\mathbb{C}_\infty$-vector space and will be denoted by $\mathrm{C}^{har}_{k,m}(\mathrm{GL}_2(A))$.

Remark 3.5.2. Teitelbaum has shown that every $\mathrm{GL}_2(A)$-invariant harmonic cocycle is *cuspidal*, i.e., it vanishes off of a finite subgraph of $\mathcal{T}$. This is actually one way of showing that $\mathrm{C}^{har}_{k,m}(\mathrm{GL}_2(A))$ is finite-dimensional. For congruence subgroups Γ with no ℓ-power torsion (for all primes ℓ, $\ell \neq p$), we have an even better result. For such Γ one can show that any Γ-invariant harmonic cocycle is determined by the its values on special kinds of edges called *sources* (see [Te1, page 505]). As the set of sources can be determined in many cases, such as $\Gamma_1(T)$, where the set consists of only one element, this allows for explicit computations with harmonic cocycles (see [LiMe, Sections 7, 8] for examples). Unfortunately $\mathrm{GL}_2(A)$ and $\Gamma_0(\mathfrak{n})$ have ℓ-power torsion when $q > 2$. But if $f \in S_{k,m}(\mathrm{GL}_2(A))$, then by inclusion also $f \in S_{k,m}(\Gamma_1(T))$. Therefore any such f is determined uniquely by its values on the source of $\Gamma_1(T)$, which is the edge e_0' corresponding to the annulus

$$\mathfrak{U}'_{e_0} = \left\{ z \in \Omega : \ \frac{1}{q} < |z| < 1 \right\}$$

(see [LiMe, Section 7]).

We need the theory of residues from rigid analytic geometry. Instead of giving the most general treatment, we present the ad hoc approach from [Te3, Section 1].

Let $f \in M_{k,m}(\mathrm{GL}_2(A))$. By the theory of rigid analytic functions on $\mathfrak{U}_{e_0}$ (see [FrvD, Section 2.3]), we know that if we restrict f to $\mathfrak{U}_{e_0}$, then we can expand f into a two-sided convergent series

$$f(z) = \sum_{n\in\mathbb{Z}} \xi_n z^n.$$

We define the *residue* of f on $\mathfrak{U}_{e_0}$ (or on e_0 for short) to be

$$\mathrm{Res}_{e_0}(f(z)dz) := \mathrm{Res}_{\mathfrak{U}_{e_0}}(f(z)dz) = \xi_{-1}.$$

If $e = \gamma * e_0$ is any other edge of $\mathcal{T}$, then we define the *residue* of f on e by expanding $f(z)dz$ into a series

$$\sum_{n\in\mathbb{Z}} \xi_n u^n du, \qquad \text{where } u = \gamma^{-1}(z),$$

and declaring

$$\mathrm{Res}_e(f(z)dz) = \xi_{-1}.$$

Definition 3.5.3. Let $f \in M_{k,m}(\mathrm{GL}_2(A))$. We define the *residue* $\mathrm{Res}(f)$ of f to be the harmonic cocycle of weight k and type m uniquely defined by

$$\mathrm{Res}(f)_e(\mathtt{X}^j\mathtt{Y}^{k-2-j}) := \mathrm{Res}_e(z^j f(z)dz), \qquad 0 \le j \le k-2.$$

Remark 3.5.4. It is a theorem that the definition actually produces a $\mathrm{GL}_2(A)$-invariant harmonic cocycle (see [Te3, page 198]).

Theorem 3.5.5. *The map $f \mapsto \mathrm{Res}(f)$ restricts to an isomorphism between the vector spaces $S_{k,m}(GL_2(A))$ and $C^{har}_{k,m}(GL_2(A))$.* □

This is a special case of the main result (Theorem 16) of [Te1], where Teitelbaum proves that this holds for any congruence subgroup Γ. The isomorphism between cuspidal modular forms and harmonic cocycles described in the theorem is used in the proof of the main result of [Böl], which we will recall in the next chapter. The reader should be aware that the homological interpretation used by Böckle is valid only when Γ has no ℓ-power torsion (ℓ different than p), therefore it is valid for $\Gamma_1(T)$ and $\Gamma(T)$, but not for $\mathrm{GL}_2(A)$ or $\Gamma_0(T)$.

Since the forms $f_{k,n}$ from Theorem 3.1.10 are explicitly given, we want to compute their residues on the tree $\mathrm{GL}_2(A)\backslash\mathcal{T}$. We only give the formulas for the special family f_s, since the formulas for general $f_{k,n}$ can be computed by the same method, but get very complicated. As we have already mentioned (see Remark 3.5.2), the harmonic cocycle associated to f_s is uniquely determined by its value on e_0'. One checks that

$$\mathrm{Res}(f)_{e_0}(\mathtt{X}^j\mathtt{Y}^{k-2-j}) = (-1)^{k-2-j}\,\mathrm{Res}(f)_{e_0'}(\mathtt{X}^{k-2-j}\mathtt{Y}^j).$$

Indeed, by using the procedure above with

$$e_0' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} e_0$$

shows that we can take $u = \frac{-1}{z}$ as the variable on the set corresponding to e_0'. We compute that if

$$z^j f(z)dz = z^j \sum_{n\in\mathbb{Z}} a_n z^n dz, \qquad\qquad a_n \in \mathbb{C}_\infty$$

then

$$u^j f(u)du = u^j \sum_{n\in\mathbb{Z}} (-1)^n a_n u^{-n-2} du.$$

Hence, $\mathrm{Res}_{e_0}(z^j f(z)dz) = a_{1-j}$ and $\mathrm{Res}_{e_0'}(u^j f(u)du) = (-1)^{j-1} a_{j-1}$.

Since we have this explicit relation between Res on e_0 and e_0', and because the values of a $\Gamma_1(T)$-equivariant harmonic cycle on e_0' determine it uniquely, it will suffice to find the values of $\mathrm{Res}(f)$ on e_0. To simplify the notation, let us define

$$w_s(j) := \mathrm{Res}_{e_0}(z^j f_s(z)dz).$$

We remind the reader of the following standard notation:

$$A_d := \{a \in A : \deg(a) = d\}, \quad A_{<d} := \{a \in A : \deg(a) < d\}$$

$$\text{and} \quad A_{\leq d} := \{a \in A : \deg(a) \leq d\}.$$

Theorem 3.5.6. *Let f_s be one of the forms from the special family of Corollary* 3.1.11. *Let $d \geq d_{q+s(q-1)}$. Recall that by Lemma* 3.2.1 *that this ensures that the l-power sum over $A_{<d}$ vanishes for $1 \leq l \leq q+s(q-1)$. Let $0 \leq j \leq (s+1)(q-1)$, and write $i := (s+1)(q-1)-j$. We have*

$$-\widetilde{\pi}^{q+s(q-1)} w_s(j) = \begin{cases} 0 & \textit{if } j = 0, \\ 0 & \textit{if } j \not\equiv 0 \bmod (q-1), \\ 1 + \sum_{\mu=1}^{d-1} \sum_{u \in A_\mu} \sum_{v \in A_{\leq \mu}} v^j u^i & \textit{otherwise.} \end{cases}$$

Proof. If $z \in \mathfrak{U}_{e_0}$, then $|z| = q^{\epsilon}$, with $0 < \epsilon < 1$. Notice that the infinite part of the sum defining f_s is absolutely convergent, hence holomorphic, and so does not contribute to the residue. Therefore

$$\mathrm{Res}_{e_0}(z^j f_s(z)dz) = \mathrm{Res}_{e_0}\left(z^j \sum_{(u,v)}{}' \frac{u^{q+s(q-1)}}{uz+v} dz \right) = \sum_{(u,v)}{}' \mathrm{Res}_{e_0}\left(z^j \frac{u^{q+s(q-1)}}{uz+v} dz \right).$$

We have several cases:

If $v = 0$, then $u \neq 0$ and we have

$$\mathrm{Res}_{e_0}\left(z^j \frac{u^{q+s(q-1)}}{uz+v} dz \right) = \mathrm{Res}_{e_0}\left(z^{j-1} u^{(s+1)(q-1)} dz \right).$$

Therefore the residue from such a term is zero unless $j = 0$, in which case it is $u^{(s+1)(q-1)}$.

If $u = 0$, then the residue is 0.

If $u \neq 0, v \neq 0$ and $|\frac{uz}{v}| < 1$, i.e., $\deg(v) > \deg(u)$, then we have

$$\frac{u^{q+s(q-1)}}{uz+v} = \frac{u^{q+s(q-1)}}{v} \frac{1}{1+\frac{uz}{v}},$$

and expanding in $\frac{uz}{v}$ shows that the residue is 0.

Finally, if $u \neq 0, v \neq 0$ and $|\frac{v}{uz}| < 1$, i.e., $\deg(v) \leq \deg(u)$, then we have

$$\frac{u^{q+s(q-1)}}{uz+v} = \frac{u^{(s+1)(q-1)}}{z} \frac{1}{1+\frac{v}{uz}} = \sum_{n=0}^{\infty} (-1)^n v^n u^{(s+1)(q-1)-n} z^{-(n+1)}$$

and therefore

$$\mathrm{Res}_{e_0}\left(z^j \frac{u^{q+s(q-1)}}{uz+v} dz \right) = (-1)^j v^j u^{(s+1)(q-1)-j}.$$

If $j = 0$, then

$$\begin{aligned}
\mathrm{Res}_{e_0}(z^0 f_s(z)dz) &= \sum_{u\in A_{<d}}{}' u^{(s+1)(q-1)} + \sum_{u\in A_{<d}}{}' u^{(s+1)(q-1)} \sum_{v\in A_{\leq \deg(u)}}{}' 1 \\
&= \sum_{u\in A_{<d}}{}' u^{(s+1)(q-1)} \sum_{v\in A_{\leq \deg(u)}} 1 \\
&= 0.
\end{aligned}$$

If $j > 0$, let $i = (s+1)(q-1) - j$. We have

$$\begin{aligned}
\mathrm{Res}_{e_0}(z^j f_s(z)dz) &= (-1)^j \sum_{u\in A_{<d}}{}' \sum_{v\in A_{\leq \deg(u)}}{}' v^j u^i \\
&= (-1)^j \left(\sum_{\theta\in\mathbb{F}_q^*} \sum_{\sigma\in\mathbb{F}_q^*} \theta^j \sigma^i + \sum_{\mu=1}^{d-1} \sum_{u\in A_\mu} \sum_{v\in A_{\leq\mu}} v^j u^i \right)
\end{aligned}$$

If $j \not\equiv 0 \bmod (q-1)$, then both sums are 0. Otherwise the first double sum is equal to 1 and we obtain the result. □

The constant $-\widetilde{\pi}^{q+s(q-1)}$ appears above because we are using the lattice $\widetilde{\pi}A$, rather than A, to define G_n.

Example 3.5.7. We compute the values of $\mathrm{Res}(f_s)_{e_0}$ for $0 \leq s \leq 7$ when $q = 3$. In order to simplify things we will use the following notation for the tuple of values:

$$\overrightarrow{\mathrm{Res}}(f_s) := -\widetilde{\pi}^{q+s(q-1)}(w_s(0), w_s(1), \ldots, w_s((s+1)(q-1))).$$

In this notation,

$$\begin{aligned}
\overrightarrow{\mathrm{Res}}(f_0) &= \overrightarrow{\mathrm{Res}}(h) = (0,0,1), \\
\overrightarrow{\mathrm{Res}}(f_1) &= \overrightarrow{\mathrm{Res}}(hg) = (0,0,1,0,1), \\
\overrightarrow{\mathrm{Res}}(f_2) &= \overrightarrow{\mathrm{Res}}(hg^2) = (0,0,1,0,1,0,1), \\
\overrightarrow{\mathrm{Res}}(f_3) &= \overrightarrow{\mathrm{Res}}(hg^3) = (0,0,1,0,1,0,1,0,1), \\
\overrightarrow{\mathrm{Res}}(f_4) &= (0,0,1,0,1,0,1,0,[1]^2+1,0,1), \\
\overrightarrow{\mathrm{Res}}(f_5) &= (0,0,1,0,1,0,1,0,[1]^2+1,0,1,0,1), \\
\overrightarrow{\mathrm{Res}}(f_6) &= (0,0,1,0,1,0,1,0,[1]^2+1,0,1,0,1,0,1), \\
\overrightarrow{\mathrm{Res}}(f_7) &= (0,0,1,0,1,0,1,0,[2][1]+1,0,1,0,1,0,[2][1]+1,0,1).
\end{aligned}$$

Remark 3.5.8. As mentioned before, we can also compute the residues of f_s for the tree $\Gamma_1(T)\backslash\mathcal{T}$, as presented in [LiMe, Section 7]. Harmonic cocycles that are $\Gamma_1(T)$-invariant are in some sense simpler to compute with. Indeed, as we noted in Remark 3.5.2, they are uniquely determined by the values on the edge e_0', and we have

$$\mathrm{Res}(f)_{e_0}(\mathtt{X}^j\mathtt{Y}^{k-2-j}) = (-1)^{k-2-j}\,\mathrm{Res}(f)_{e_0'}(\mathtt{X}^{k-2-j}\mathtt{Y}^j).$$

Therefore the theorem above shows that for j in the range $0 \le j \le (s+1)(q-1)$ and $i = (s+1)(q-1) - j$, we have

$$-\widetilde{\pi}^{q+s(q-1)}\,\mathrm{Res}_{e_0'}(f_s)(\mathtt{X}^j\mathtt{Y}^i) = \begin{cases} 0 & \text{if } j = (s+1)(q-1), \\ 0 & \text{if } j \not\equiv 0 \bmod (q-1), \\ 1 + \sum_{\mu=1}^{d-1}\sum_{u\in A_\mu}\sum_{v\in A_{\le\mu}} v^i u^j & \text{otherwise.} \end{cases}$$

Unfortunately, we have been unsuccessful in finding special properties of the residues of the family f_s (or the bigger family $f_{k,n}$), which will single them out among harmonic cocycles. It would be interesting to understand the Hecke action in terms of the residues as in [LiMe, Section 7]. Classically the residues are related to special values of L-functions (see [Da1, Chapters 5, 6]). We hope to return to these issues in future work.

3.6 Computational Examples

We want to end Chapter 3 with explicit computations based on Theorem 3.1.10.

3.6.1 $q = 2$

There is only one type, $m = 0$. The form g is of weight 1 and h is of weight 3. The pairs (k, n) that satisfy the hypothesis of Theorem 3.1.10 are

$(3,1), (4,1), (5,1), (6,1), (6,2), (7,1), (7,3), (8,1), (8,2), (9,1), (10,1), (10,2),$

$(11,1), (11,3), (12,1), (12,2), (12,4), (13,1), (13,5), (14,2), (14,6), (15,1), (15,3),$

$(15,7), (16,1), (16,2), (16,4), (17,1), (18,1), (18,2), (19,1), (19,3), \ldots$

Some of the resulting eigenforms are

$$\sum_{a\in A_+} a^2 t_a = h \in S_{3,0}(\mathrm{GL}_2(A)),$$
$$\sum_{a\in A_+} a^3 t_a = hg \in S_{4,0}(\mathrm{GL}_2(A)),$$
$$\sum_{a\in A_+} a^4 t_a = hg^2 \in S_{5,0}(\mathrm{GL}_2(A)),$$
$$\sum_{a\in A_+} a^5 t_a = hg^3 + [1]^2 h^2, \quad \sum_{a\in A_+} a^4 t_a^2 = h^2 \in S_{6,0}(\mathrm{GL}_2(A)),$$
$$\sum_{a\in A_+} a^6 t_a = hg^4 + [1]^2 h^2 g, \quad \sum_{a\in A_+} a^4 G_3(t_a) = h^2 g \in S_{7,0}(\mathrm{GL}_2(A)),$$
$$\sum_{a\in A_+} a^7 t_a = hg^5 + [1]^3 h^2 g^3, \quad \sum_{a\in A_+} a^6 t_a^2 = h^2 g^2 \in S_{8,0}(\mathrm{GL}_2(A)),$$
$$\sum_{a\in A_+} a^8 t_a = hg^6 + [1]^4 h^3 \in S_{9,0}(\mathrm{GL}_2(A)),$$
$$\sum_{a\in A_+} a^9 t_a = hg^7 + [2]^2 h^2 g^4 + [1]^4 h^3 g \in S_{10,0}(\mathrm{GL}_2(A)),$$
$$\sum_{a\in A_+} a^8 t_a^2 = h^2 g^4 \in S_{10,0}(\mathrm{GL}_2(A)).$$

The reader can see that this agrees with our recursion for

$$F_\nu = \sum_{a\in A_+} a^{q^\nu} t_a.$$

For dimension reasons, the spaces $S_{j,0}(\mathrm{GL}_2(A))$, $3 \le j \le 8$, are diagonalizable with respect to the Hecke action of $\mathbb{T}$. We have already remarked that $S_{9,0}(\mathrm{GL}_2(A))$ is not diagonalizable.

The space $S_{10,0}(\mathrm{GL}_2(A))$ is 3-dimensional and the examples above give two eigenforms: $hg^7+[2]^2h^2g^4+[1]^4h^3g$ with eigensystem $\{\lambda_\mathfrak{p} = \wp\}$, and h^2g^4 with eigensystem $\{\lambda_\mathfrak{p} = \wp^2\}$. The form

$$\phi = h^2 g^4 + [1] h^3 g$$

is double cuspidal and we have verified that

$$\mathrm{T}_\mathfrak{p}\,\phi = \wp^3\phi, \qquad \text{for all } \mathfrak{p} \text{ of degree} \le 4.$$

This already shows that the space $S_{10,0}(\mathrm{GL}_2(A))$ is diagonalizable for the action of $\mathbb{T}$ with eigenbasis $\{hg^7 + [2]^2h^2g^4 + [1]^4h^3g, h^2g^4, \phi\}$. The last statement follows from the commutativity of the operators $\{\mathrm{T}_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathrm{Spec}\,A}$ and the fact that the characteristic polynomial of $\mathrm{T}_{\mathfrak{p}}$ with $\deg(\mathfrak{p}) \le 4$ does not have repeated roots. However, it is much more difficult to show that $\lambda_{\mathfrak{p}}(\phi) = \wp^3$ for all $\mathfrak{p}$, which we expect from our computations for $\mathfrak{p}$ with $\deg(\mathfrak{p}) \le 4$. One thing that we can prove though is that if we assume that ϕ is an eigenform with eigensystem $\{\lambda_{\mathfrak{p}} = \wp^3\}$, then ϕ does not have an A-expansion. If it did, then by multiplicity one for forms with A-expansions, we would have

$$\phi = \sum_{a\in A_+} a^7 G_3(t_a),$$

but this is false, i.e., the t-expansions on both sides do not agree.

We also give several examples of spaces of higher weight.

If $k = 47$, then the space $S_{47,0}(\mathrm{GL}_2(A))$ has dimension equal to 14. Using Theorem 3.1.10, we get four cuspidal eigenforms with A-expansions:

$$\sum_{a\in A_+} a^{46} t_a,$$

$$\sum_{a\in A_+} a^{44} G_3(t_a) = \sum_{a\in A_+} a^{44}(t_a^2 + [1]t_a^3),$$

$$\sum_{a\in A_+} a^{40} G_7(t_a) = \sum_{a\in A_+} a^{40}(t_a^4 + [2]t_a^6 + [2][1]t_a^7),$$

$$\sum_{a\in A_+} a^{32} G_{15}(t_a) = \sum_{a\in A_+} a^{32}(t_a^8 + [3]t_a^{12} + [3][2]t_a^{14} + [3][2][1]t_a^{15}).$$

As the examples for small k show, it is not true that the number of forms produced by Theorem 3.1.10 necessarily grows with k. For example, if $k = 101$ the 33-dimensional space $S_{101,0}(\mathrm{GL}_2(A))$ only has three forms with A-expansions:

$$\sum_{a\in A_+} a^{100}t_a,$$
$$\sum_{a\in A_+} a^{96}G_4(t_a),$$
$$\sum_{a\in A_+} a^{64}G_{37}(t_a).$$

3.6.2 $q = 3$

There are two types $m = 0$ and $m = 1$.

We first look at type $m = 0$. The pairs (k, n) that satisfy the hypothesis of Theorem 3.1.10 are

$$(8,2),(14,2),(20,2),(22,4),(24,6),(26,2),(26,8),(32,2),(38,2),(40,4),(42,6),$$
$$(44,2),(44,8),(50,2),(56,2),(58,4),(60,6),(62,2),(62,8),(64,10),(66,12),\ldots$$

Therefore the first forms with A-expansions are

$$\sum_{a\in A_+} a^6t_a^2 = h^2 \in S_{8,0}(\mathrm{GL}_2(A)),$$
$$\sum_{a\in A_+} a^{12}t_a^2 = h^2g^3 \in S_{14,0}(\mathrm{GL}_2(A)),$$
$$\sum_{a\in A_+} a^{18}t_a^2 = h^2g^6 \in S_{20,0}(\mathrm{GL}_2(A)),$$
$$\sum_{a\in A_+} a^{18}G_4(t_a) = h^2g^7 - [1]h^4g^3 \in S_{22,0}(\mathrm{GL}_2(A)),$$
$$\sum_{a\in A_+} a^{18}t_a^6 = h^6 \in S_{24,0}(\mathrm{GL}_2(A)),$$
$$\sum_{a\in A_+} a^{24}t_a^2 = h^2g^9 - [1]^6h^6g, \ \sum_{a\in A_+} a^{18}G_8(t_a) = h^6g \in S_{26,0}(\mathrm{GL}_2(A)).$$

Notice that there are spaces that have no cuspidal eigenforms with A-expansions. One such space is $S_{10,0}(\mathrm{GL}_2(A))$, which is a one-dimensional space spanned by h^2g. We do not know of any way to compute the eigensystem of h^2g. However, from computations for $\mathfrak{p}$ of degree ≤ 4, we suspect that the eigensystem is $\{\lambda_\mathfrak{p} = \wp(T) \cdot \wp(-T^3 - T)\}$.

If we continue to compute the Hecke action, we can see that h^2g^2 is an eigenform, since $S^2_{12,0}(\mathrm{GL}_2(A))$ is one-dimensional. We have

$$\mathrm{T}_{\mathfrak{p}}\, h^2g^2 = \wp^4 h^2 g^2, \qquad \text{for all } \mathfrak{p} \text{ of degree} \leq 4.$$

However, h^2g^2 does not have an A-expansion. There are other examples of the same phenomenon.

So far we have only seen eigenforms with rational eigenvalues, but this is not always the case. The space $S_{16,0}(\mathrm{GL}_2(A))$ is the first space where we see eigenvalues that are quadratic over K. Let δ be an element such that $\delta^2 = [2]/[1]$. The form

$$\phi = h^2g^4 + [1](1+\delta)h^4$$

is an eigenform (at least for $\mathrm{T}_{\mathfrak{p}}$ with $\mathfrak{p}$ of degree ≤ 4). And for $\deg(\mathfrak{p}) \leq 4$ we have

$$\mathrm{T}_{\mathfrak{p}}\, \phi = \wp(T) \cdot \wp(T + [1]\delta)\phi.$$

There are more examples of quadratic eigenvalues for other weights.

Let us now look at type $m = 1$. The pairs (k, n) that satisfy the hypothesis of Theorem 3.1.10 are

$(4,1), (6,1), (8,1), (10,1), (12,1), (12,3), (14,1), (14,5), (16,1), (16,7), (18,1),$

$(18,3), (20,1), (22,1)(24,1), (24,3), (26,1), (28,1), (30,1), (30,3), (32,1),$

$(32,5), (34,1), (34,7), (36,1), (36,3), (36,9), \ldots$

Because the spaces $S_{2j,1}(\mathrm{GL}_2(A))$ are one or two-dimensional for $2 \leq j \leq 9$, we see from the list above that all these spaces are diagonalizable and have a basis of eigenvectors with A-expansions.

We also have examples:

$$\sum_{a \in A_+} a^{27} G_{25}(t_a) = h^9 g^8 - [1]h^{11}g^4 + [1]^2 h^{13} \in S_{52,1}(\mathrm{GL}_2(A)),$$

$$\sum_{a \in A_+} a^{81} G_{79}(t_a) = h^{27}g^{26} - [1]h^{29}g^{22} + [1]^2 h^{31} h^{18}$$
$$- [2]h^{33}g^{14} - [1][2]h^{35}g^{10} + [2]^2 h^{39} g^2 \in S_{160,1}(\mathrm{GL}_2(A)).$$

3.6.3 $q = 4$

There are three types, $m = 0, 1, 2$, and we have A-expansions in every type.

For type $m = 0$ the pairs (k, n) are

$$(15,3),(27,3),(30,6),(39,3),(51,3),(54,6),(57,9),(60,12),\ldots$$

For type $m = 1$ the pairs (k, n) are

$$(5,1),(8,1),(11,1),(14,1),(17,1),(20,1),(20,4),(23,1),(23,7),\ldots$$

For type $m = 2$ the pairs (k, n) are

$$(10,2),(13,5),(16,2),(22,2),(28,2),(34,2),(37,5),(40,2),(40,8),\ldots$$

3.6.4 Natural Questions

The previous examples have answered or given evidence for the answers to several natural questions, which we record below.

- *Do all cuspidal eigenforms have A-expansions?* The answer is clearly *No*. Indeed there are far fewer forms with A-expansions than to linear independent forms as q and k grow.

- *Does every cuspidal eigenform with eigensystem $\{\lambda_{\mathfrak{p}} = \wp^n\}$ possess an A-expansion?* We strongly suspect that the answer is *No*. The example that we have in mind is h^2g^2 for $q = 3$, $k = 12$, $m = 0$. The only reason that we cannot be completely certain is that we cannot show that the eigenform h^2g^2 has eigenvalues $\lambda_{\mathfrak{p}} = \wp^4$ for all $\mathfrak{p}$, but we have verified this for $\mathfrak{p}$ of degree ≤ 4.

- *Do all the cuspidal forms with A-expansions come from differentiating the special family of type 1 forms?* We strongly suspect that the answer is *Yes*, since this is true in all cases that we have computed.

Chapter 4
Galois Representations attached to cuspidal eigenforms

In this chapter, we summarize the theory of crystals over function fields and its connections with Drinfeld modular forms: see [BöPi] and [Bö1]. The association of Hecke characters to cuspidal and double cuspidal Drinfeld eigenforms is a recent development, and there are many open questions, suggested by the theory of classical modular forms, that remain to be answered. We will show that applications of A-expansions to Drinfeld modular forms for $\Gamma_1(T)$ and $\Gamma_0(T)$ will reprove results of Böckle regarding the diagonalizability of the quotient space of cuspidal Drinfeld modular forms modulo double cuspidal Drinfeld modular forms.

4.1 Hecke Characters and Compatible Systems of Galois Representations

We want to recall the definitions of Hecke characters and compatible systems of Galois representation. Our main references are [Bö2] and [Se2]. The reader is advised to consult the first reference for the most general definitions.

We begin by fixing an algebraic closure K^{alg} of K, and the separable closure K^{sep} of K inside K^{alg}. We let G_K denote the Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$.

Definition 4.1.1. A *set of Hodge-Tate weights* is a finite subset of embeddings $\Sigma \subset \mathrm{Emb}(K, K^{\mathrm{alg}})$, together with numbers $n_\sigma \in \mathbb{Z}[1/p]$ for $\sigma \in \Sigma$. A *Hodge-Tate character* is a homomorphism $\psi : K^* \to (K^{\mathrm{alg}})^*$ for which there exists a set of Hodge-Tate weights $(\Sigma, (n_\sigma))$ such that

$$\psi : a \mapsto \prod_{\sigma \in \Sigma} \sigma(a)^{n_\sigma}.$$

Recall that $\mathbb{A}_K^*$ denotes the set of ideles of K.

Definition 4.1.2. An *algebraic Hecke character*, or simply a *Hecke character*, of K is a continuous homomorphism

$$\chi : \mathbb{A}_K^* \to (K^{\text{alg}})^*$$

such that its restriction to K^* is a Hodge-Tate character. The set of Hodge-Tate weights for this Hodge-Tate character is called the *set of Hodge-Tate weights* for χ.

Let $L \subset K^{\text{alg}}$ be a finite extension of K. We use v to denote a place of L.

Definition 4.1.3. An *L-rational strictly compatible system* $\{\varrho_v\}$ of n-dimensional mod v representations of G_K with defect set $\mathcal{D} \subset \text{Spec}(A_L)$ and ramification set $\mathcal{S} \subset \text{Spec}(A)$ consists of the following data

- for each place v outside of $\mathcal{D}$, a continuous semisimple representation

$$\varrho_v : G_K \to \text{GL}_n(\mathbb{F}_v)$$

  which is unramified outside of a finite set that includes $\mathcal{S}$.

- for each place $\mathfrak{p}$ outside of $\mathcal{S}$ and for each v outside of $\mathcal{D}$, a monic polynomial $f_{\mathfrak{p}} \in L[z]$ of degree n such that

$$\text{CharPol}(\varrho_v(\varphi_{\mathfrak{p}})) \equiv f_{\mathfrak{p}} \bmod v.$$

Remark 4.1.4. If $\varrho : G_K \to \text{Aut}(V)$ is a v-adic or mod v Galois representation, then V has a composition series

$$0 = V_m \subset \cdots \subset V_1 \subset V_0 = V$$

of G_K-invariant subspaces with V_i/V_{i+1} irreducible. The representation defined by $V' = \oplus_{i=0}^{m-1} V_i/V_{i+1}$ is semi-simple and has characteristic polynomials at Frobenii that agree with those of ϱ for almost all primes. We call this the *semisimplification* of ϱ and denote it by ϱ^{ss}.

In [Bö2, Section 2.6] Böckle shows how, given a Hecke character χ with Hodge-Tate weights $(\Sigma, (n_\sigma)_{\sigma\in\Sigma})$ and a finite extension L that contains $\sigma(K)$ for all $\sigma \in \Sigma$ and $\chi(\mathbb{A}_K^*) \subset L$, one can construct an L-rational strictly compatible system of 1-dimensional representations of G_K. This strictly compatible system $\{\varrho_{\chi,v}\}$ has defect set and ramification set that are explicitly computable, and the corresponding family of characteristic polynomials is given by

$$f_{\mathfrak{p}}(z) = z - \chi((1, \ldots, \wp, \ldots, 1)).$$

The main result of [Bö2] states that this gives all mod v strictly compatible abelian systems. We state the theorem for 1-dimensional strictly compatible systems and refer the interested reader to [Bö2, Theorem 2.20] for the general case:

Theorem 4.1.5. *Let $\{\varrho_v\}$ be an L-rational strictly compatible abelian system of* 1-*dimensional mod v representations of G_K. There exists a unique Hecke character χ such that the mod v Galois representation of G_K attached to χ is isomorphic to the semisimplification ϱ_v^{ss} of ρ_v, for v away from the defect set of $\{\varrho_v\}$.* □

4.2 Connections with Drinfeld Cuspidal Forms

In this section, we summarize the process of attaching Galois representations to cuspidal Drinfeld eigenforms. The original reference for this is [Bö1]. In order to simplify the exposition, we follow [Bö3].

Let R be an $\mathbb{F}_q$-algebra, and let φ be the Frobenius $x \mapsto x^q$ for $x \in R$. We denote by R_φ the ring R viewed as an R-algebra via φ, i.e., we have $x \cdot y = \varphi(x)y = x^q y$ for all $x, y \in R$.

Definition 4.2.1. For an R-module M, we define the R-module $\varphi^*M := M \otimes_R R_\varphi$. A *$\tau$-module* (M, τ), or simply M, over R is a finitely generated projective R-module together with an injective R-module homomorphism

$$\tau : \varphi^*M \to M.$$

A *morphism* of τ-modules is an R-linear homomorphism that respects the action of τ.

Remark 4.2.2. Any R-linear homomorphism $\tau : \varphi^* M \to M$ can be regarded as a φ-semi-linear homomorphism $M \to M$, i.e., a map τ such that

$$\tau(rm) = \varphi(r)\tau(m), \qquad r \in R, m \in M.$$

The concept of τ-sheaf brings the notion of a τ-module into a geometric setting, i.e., affine τ-sheaves are τ-modules, but we drop the injectivity condition on τ. However, crystals (the technical objects that are 'classes' of τ-sheaves) will always have τ-sheaf representative with τ injective.

Let X be a Noetherian $\mathbb{F}_q$-scheme. We write φ_X, or simply φ, for the q-power Frobenius on X. We will write $X \times A$ for the scheme $X \times_{\mathbb{F}_q} \operatorname{Spec}(A)$.

Definition 4.2.3. A τ-sheaf on X over A is a pair $(\mathcal{F}, \tau)$ consisting of a coherent sheaf of $\mathcal{O}_{X\times A}$-modules together with an $\mathcal{O}_{X\times A}$-linear homomorphism

$$\tau : (\varphi \times \mathrm{id})^* \mathcal{F} \to \mathcal{F}.$$

We will often simply write $\mathcal{F}$ instead of $(\mathcal{F}, \tau)$. A *morphism* of τ-sheaves is an $\mathcal{O}_{X\times A}$-linear morphism that respects the action of τ. A τ-sheaf is called *locally free of rank* r if the underlying sheaf $\mathcal{F}$ is a locally free $\mathcal{O}_{X\times A}$-module of finite rank r. A τ-sheaf $\mathcal{F}$ is called *nilpotent* if and only if τ^n vanishes for some $n > 0$. A morphism of τ-sheaves is called *nil-isomorphism* if and only if both its kernel and cokernel are nilpotent.

The category $\mathbf{Coh}_\tau(X, A)$ of all τ-sheaves on X over A is abelian. For any $\mathbb{F}_q$-morphism $f : Y \to X$, we have a *base change* functor f^* that attaches to any τ-sheaf $\mathcal{F}$ on X a τ-sheaf on Y with underlying sheaf $(f \times \mathrm{id})^* \mathcal{F}$. In particular, we have a *restriction to a point* morphism for any point $i_x : x = \operatorname{Spec}(k_x) \to X$, which gives a localization $(\mathcal{F}_x, \tau_x)$. Another functor from $\mathbf{Coh}_\tau(X, A)$ to $\mathbf{Coh}_\tau(X, A)$ is the *change of coefficients* functor, which for any A-algebra A' and τ-sheaf on X over A, gives a τ-sheaf on X over A' by $\_ \otimes_A A'$.

For technical reasons, we need to work with 'classes' of τ-sheaves, i.e., we need to consider a new category $\mathbf{Crys}(X, A)$ of A-crystals on X. The category $\mathbf{Crys}(X, A)$ consists of the same objects as $\mathbf{Coh}_\tau(X, A)$, but with a different notion of morphism. Namely, any morphism $\mathcal{F} \to \mathcal{G}$ in $\mathbf{Crys}(X, A)$ is a diagram

$$\mathcal{F} \leftarrow \mathcal{H} \rightarrow \mathcal{G} \qquad\qquad \text{in } \mathbf{Coh}_\tau(X, A),$$

for some $\mathcal{H} \in \mathbf{Coh}_\tau(X, A)$, where the morphism $\mathcal{H} \to \mathcal{F}$ is a nil-isomorphism. This is the main subject of study of [BöPi]. We will use the concept of a crystal as a black box and refer the reader to this reference for more details.

The functors f^*, and $\_ \otimes_A A'$ pass to the new category of crystals. In fact, f^* is exact, but $\_ \otimes_A A'$ is not. We call a crystal *flat* if all the higher Tor-groups for all coefficient changes $\_ \otimes_A A'$ vanish.

It can be shown that every crystal has a τ-sheaf representative on which τ is injective. We will usually choose such a representative and use it when we talk about crystals. We want to attach Galois representations to crystals, and therefore we would like to work with locally free τ-sheaves of finite rank. In [Ga1], Gardeyn puts forward the following definitions that are useful in this context.

Definition 4.2.4. A τ-sheaf $\mathcal{F}$ on X is called *good* if it is locally free and, after base change to any $x \in X$, the endomorphism τ_x is injective.

A τ-sheaf $\mathcal{F}$ on X is called *generically good* if its base change to any generic point is good.

From now on, we assume that X is an irreducible curve with generic point η.

Definition 4.2.5. Let $\mathcal{F}_\eta$ be a good τ-sheaf on η. A *model* for $\mathcal{F}_\eta$ is a locally free generically good τ-sheaf on X with generic fiber agreeing with $\mathcal{F}_\eta$. A model is called *good* if it is good as a τ-sheaf. A model $\mathcal{H}$ for $\mathcal{F}_\eta$ is called *maximal* if, for every τ-sheaf $\mathcal{G}$, every morphism $f_\eta : \mathcal{G}_\eta \to \mathcal{F}_\eta$ of τ-sheaves on η extends uniquely to a morphism $f : \mathcal{G} \to \mathcal{H}$ of τ-sheaves on X.

Gardeyn has shown that every good τ-sheaf on η admits a maximal model that is unique up to a unique isomorphism. In addition, every good model is maximal. For any generically good τ-sheaf $\mathcal{F}$ on X, we write $\mathcal{F}^{\max}$ for the maximal model of $\mathcal{F}_\eta$. By the universal property of $\mathcal{F}^{\max}$, we have a canonical homomorphism $\mathcal{F} \to \mathcal{F}^{\max}$.

The following summarizes the results about crystals that are of interest to us:

Theorem 4.2.6.

- *Let $X = \mathrm{Spec}(F)$ for F a field, and let $\mathcal{F}$ be a flat A-crystal on X. Then the underlying sheaf of some representative of $\mathcal{F}$, with injective τ, is locally free.*

- *Let X be an irreducible curve. Let $\mathcal{F}$ be the representative of a flat A-crystal on X which is generically good and on which τ is injective. Let $\kappa : \mathcal{F} \to \mathcal{F}^{\max}$ be the canonical morphism. Then there exists a finite set $S \subset X$ such that the restriction of κ to $X \backslash S$ is an isomorphism and such that Ker(κ) is locally free on S. Therefore, the A-crystal that $\mathcal{F}$ represents has a locally free representative on a dense open set in X.* □

Let $\mathcal{F}$ be a flat A-crystal. The *rank* of $\mathcal{F}$ is defined to be the rank of $\mathcal{F}_\eta$ according to the first part of the theorem above.

A flat crystal has both an étale and an analytic interpretation. We begin by describing the étale one.

We continue to assume that X is an irreducible curve with generic point $\eta = \mathrm{Spec}(K)$. Let $\mathcal{F}$ be a generically good representative of a flat crystal of rank r. Fix a prime $\mathfrak{p} \subset A$. Let L be a finite separable extension of K; that is, let $u : \mathrm{Spec}\, L \to \mathrm{Spec}\, K$ be étale. To this data, and positive integer n, we attach an $\mathbb{F}_q$-vector space as follows:

$$(u : \mathrm{Spec}\, L \to \mathrm{Spec}\, K) \mapsto \Gamma(\mathrm{Spec}\, L, u^*(\mathcal{F}_\eta) \otimes_A A/\mathfrak{p}^n)^\tau,$$

where the notation M^τ means that we take the τ-invariant part of M. By [Ka1, Theorem 4.1], if τ is an isomorphism on $\mathcal{F}_\eta \otimes_A A/\mathfrak{p}$, this assignment defines a lisse étale sheaf of $\mathbb{F}_q$-vector spaces over $\eta = \operatorname{Spec} K$ of rank equal to $r \dim_{\mathbb{F}_q} A/\mathfrak{p}^n$. This étale sheaf is also an $A/\mathfrak{p}^n$-module and one can show that the construction also defines a lisse étale sheaf of $A/\mathfrak{p}^n$-modules of rank r. In turn, this gives a mod $\mathfrak{p}^n$ Galois representation

$$\varrho_{\mathcal{F},\mathfrak{p}^n} : G_K \to \mathrm{GL}_r(A/\mathfrak{p}^n).$$

Varying n, one gets an inverse limit of representations that produces a $\mathfrak{p}$-adic Galois representation

$$\varrho_{\mathcal{F}}^{\mathfrak{p}} : G_K \to \mathrm{GL}_r(A_{\mathfrak{p}}).$$

It turns out [Bö3, Corollary 3.7] that the construction above descends to flat A-crystals and the representations that we have constructed form a strictly compatible system.

Theorem 4.2.7. *Suppose that X is smooth, and let $\mathcal{F}$ be a flat crystal. The representations $(\varrho_{\mathcal{F}}^{\mathfrak{p}})_{\mathfrak{p}}$, where $\mathfrak{p}$ ranges over the maximal ideals of A at which $\mathcal{F}_\eta \otimes_A A/\mathfrak{p}$ is lisse, form a strictly compatible system of Galois representations.* □

This is the étale side of a flat A-crystal. Next, we describe the analytic interpretation of $\mathcal{F}$.

Let L_∞ be a complete valued field containing K, and denote by $L_\infty\langle u\rangle$ the Tate algebra over L_∞:

$$L_\infty\langle u\rangle = \left\{ \sum_{n=0}^{\infty} a_n u^n : a_n \in L_\infty, \lim_{n\to\infty} a_n = 0 \right\}.$$

As in Section 2.4.2, denote by $\widetilde{\varphi}$ the partial Frobenius, i.e., the map such that

$$\widetilde{\varphi}\left(\sum_{n=0}^{\infty} a_n u^n \right) = \sum_{n=0}^{\infty} a_n^q u^n.$$

Definition 4.2.8. An *analytic* τ-module on L_∞ is a pair (M_∞, τ_∞) consisting of a finitely generated $A \otimes_{\mathbb{F}_q[u]} L_\infty\langle u\rangle$-module and a linear map $\tau_\infty : \widetilde{\varphi}^* M \to M$, i.e., a $\widetilde{\varphi}$-semi-linear homomorphism $M \to M$.

One denotes the category of such objects by $\mathbf{Coh}_\tau^{\mathrm{an}}(L_\infty, A)$. As in the algebraic case, we have a related category $\mathbf{Crys}^{\mathrm{an}}$ with the same objects, but more morphisms and isomorphisms (see [Bö1, Chapter 8]).

Since injective affine τ-sheaves are τ-modules, we know that any injective τ-sheaf $\mathcal{F}$ on Spec(K) is a pair (M, τ) consisting of a finitely generated $K \otimes_{\mathbb{F}_q} A$-module M together with a $(\varphi \times \mathrm{id})$-semi-linear homomorphism $\tau : M \to M$. We define the analytic τ-module associated to $\mathcal{F}$ by

$$\mathcal{F}^{\mathrm{an}} = (M_\infty = M \otimes_{K[u]} L_\infty\langle u \rangle, \tau_\infty),$$

where τ_∞ is the induced endomorphism from the tensor product. We denote by $\mathbf{1}_{L_\infty, A}$ the analytic τ-sheaf $A \otimes_{\mathbb{F}_q[u]} L_\infty\langle u \rangle$ together with the map φ that fixes u and acts on other elements of the tensor product as $x \to x^q$, i.e., φ is essentially $\widetilde{\varphi}$.

Definition 4.2.9. The analytic τ-module $\mathcal{F}^{\mathrm{an}}$ is called the *analytification* of $\mathcal{F}$.

Definition 4.2.10. Let $\mathcal{F}$ be a flat A-crystal on X of rank r. Then $\mathcal{F}$ is *uniformizable* for L_∞, where $K \hookrightarrow L_\infty$, if $\mathrm{rank}_A(\mathrm{Hom}_{\mathbf{Crys}^{\mathrm{an}}}(\mathbf{1}^{\mathrm{an}}_{L_\infty, A}, \mathcal{F}^{\mathrm{an}})) = r$.

The crystals that will turn out to be connected to Drinfeld modular forms are of the following special kind:

Definition 4.2.11. A *Hecke crystal* on X is the following data:

- a flat A-crystal $\mathcal{F}$ on X which is uniformizable for some $K \hookrightarrow L_\infty$ and
- a commutative A-subalgebra $\mathbb{T} \subset \mathrm{End}_{\mathbf{Crys}}(\mathcal{F})$, called a *Hecke algebra*, generated by *Hecke operators* T_x, $x \in X$

such that, denoting by $\mathcal{W}$ the finite set $\mathfrak{p} \in \mathrm{Spec}\, A$ for which $\varrho_{\mathcal{F},\mathfrak{p}}$ is not lisse and $\mathcal{S} \subset X$ a suitable closed finite subset, one has for all $\mathfrak{p} \in \mathrm{Spec}\, A \backslash \mathcal{W}$, all $x \in X \backslash \mathcal{S}$ and all positive integers n:

$$\varphi_x^{-1} = \mathrm{T}_x \in \mathrm{End}_{\mathbf{et}}(\mathcal{F}^{\mathrm{et}}_{\mathfrak{p}^n}).$$

In addition, for all $\mathfrak{p} \in \operatorname{Spec} A \backslash \mathcal{W}$ and for all $x \in X \backslash \mathcal{S}$, the $\mathfrak{p}$-adic Galois representation $\varrho_{\mathcal{F}}^{\mathfrak{p}}$ is abelian and

$$\operatorname{CharPol}_{\varrho_{\mathcal{F}}^{\mathfrak{p}}(\varphi_x^{-1})} = \operatorname{CharPol}_{T_x} \in A[z].$$

After introducing all this notation, we are ready to state the main result of [Bö1], which shows how cuspidal Drinfeld modular forms are related to Hecke cystals.

Theorem 4.2.12. *Let $\mathfrak{n}$ be a non-trivial ideal of A and let $X_{\mathfrak{n}} = \operatorname{Spec} A \backslash \mathfrak{n}$. Let Γ be one of $\Gamma(\mathfrak{n})$, $\Gamma_1(\mathfrak{n})$.*

- *There exists a Hecke crystal $\mathcal{S}_{k,m}(\Gamma)$ on $X_{\mathfrak{n}}$ which is uniformizable by K_∞. The space*
$$(\mathcal{S}_{k,m}(\Gamma))^\tau \otimes_A \mathbb{C}_\infty$$
is dual, as a Hecke module, to the space of cuspidal Drinfeld modular forms $S_{k,m}(\Gamma)$.
- *There exists a Hecke crystal $\mathcal{S}^2_{k,m}(\Gamma)$ on $X_{\mathfrak{n}}$ which is uniformizable by K_∞. The space*
$$(\mathcal{S}^2_{k,m}(\Gamma))^\tau \otimes_A \mathbb{C}_\infty$$
is dual, as a Hecke module, to the space of double cuspidal Drinfeld modular forms $S^2_{k,m}(\Gamma)$. □

Remark 4.2.13. The reader should be warned that Böckle uses a slightly different Hecke algebra. If $\mathrm{T}_{\mathfrak{p}}^{\text{Bö}}$ denotes the Hecke operator at $\mathfrak{p}$ on modular forms according to Böckle, then we have

$$\mathrm{T}_{\mathfrak{p}}^{\text{Bö}} = \frac{1}{\wp}\,\mathrm{T}_{\mathfrak{p}}\,.$$

The theorem above is called the *Eichler-Shimura isomorphism for Drinfeld modular forms.* This theorem together with the étale realization of crystals that we have summarized shows that one can attach Galois representations to cuspidal and double cuspidal Drinfeld modular forms that are eigenforms away from the level $\mathfrak{n}$. We will see that, with the use of A-expansions, we will be able to recover several results of Böckle and Pink that produce Drinfeld eigenforms and their eigensystems by using the corresponding crystals.

4.3 Galois representations and A-expansions

Let Γ denote either $\Gamma_1(T)$ or $\Gamma_0(T)$. Recall that Γ has two cusps, $[\infty]$ and $[0]$, with uniformizers $t = t_{[\infty]}(z)$ and $s = t_{[0]}(z)$, respectively.

Definition 4.3.1. A Drinfeld modular form f for Γ is said to have an *A-expansion with exponent n at the cusp* $[\sigma]$ if its expansion at the cusp $[\sigma]$ equals

$$\sum_{a \in A_+} c_{a,\sigma}(f) G_n(t_{[\sigma]}(az)).$$

Remark 4.3.2. Notice that in our definition we fix a cusp $[\sigma]$ and talk about A-expansions, rather than consider the A-expansions at all the cusps.

We will see below that all the examples that we give will have A-expansions at both $[\infty]$ and $[0]$. However, there are computational examples for $\Gamma(T)$ that show that there are Drinfeld modular forms that only have an A-expansion at a single cusp.

Suppose that

$$f = \sum_{a \in A_+} c_a(f) G_n(t_a) \in S_{k,m}(\mathrm{GL}_2(A)).$$

Then by inclusion $f = \iota(f) \in S_{k,m}(\Gamma)$. And since $f \mid_{[\gamma_0]} = f$, we see that

$$f \mid_{[\gamma_0]} (z) = T^n \sum_{a \in A_+} c_a(f) G_n(s_{Ta}).$$

Recall that we have also defined $\iota_T f(z) = f(Tz)$. We have

$$F = \iota_T f = \sum_{a \in A_+} c_a(f) G_n(t_{aT}) \in S_{k,m}(\Gamma).$$

It follows from the proof of Theorem 2.5.4 that

$$F\ |_{[\gamma_0]} = \sum_{a\in A_+} c_a(f) G_n(s_a) \in S_{k,m}(\Gamma).$$

This proves that:

Theorem 4.3.1. *If $f \in S_{k,m}(GL_2(A))$ has an A-expansion with exponent n, then f and F (both of them viewed as modular forms for Γ) have A-expansions at both $[0]$ and $[\infty]$ with exponent n, which are given by the formulas above.*

Using the proof of Theorem 3.1.5, we can show that if $a^{k-n} = c_a(f) = c_{a,[\infty]}(f)$, then both f and F are eigenforms away from T with eigenvalues $\lambda_{\mathfrak{p}} = \wp^n$.

In addition to the maps ι and ι_T, there is also an inclusion map i from $S_{k,m}(\Gamma_0(T))$ to $S_{k,m}(\Gamma_1(T))$. This is often useful since we know that g and G generate $M_{k,0}(\Gamma_0(T))$, and one can determine which generators give cusp forms. Since $\det \Gamma_1(T) = 1$, we see that $S_{k,m}(\Gamma_1(T)) = S_{k,m'}(\Gamma_1(T))$ for any m, m', and the same result holds for $S^2_{k,m}(\Gamma_1(T))$. In the examples below we will take $m = 0$ when we talk about $\Gamma_1(T)$. Using this, together with the last part of Theorem 2.5.5, shows that

$$\dim_{\mathbb{C}_\infty} S_{k,m}(\Gamma_1(T)) = k + 1.$$

It is also known (see [Bö1, Proposition 5.18]) that

$$\dim_{\mathbb{C}_\infty} S^2_{k,m}(\Gamma_1(T)) = k - 1.$$

In the previous section, we learned that the cuspidal Drinfeld eigenforms have associated Galois representations. Eigenforms with A-expansions provide concrete examples.

In [Bö1, Chapter 15] Böckle[1] presents several examples from the point of view of crystals. We recall some of these below and show how they can be recovered by using A-expansions.

[1] These are examples due to Böckle and Pink.

Example 4.3.3 (Cuspidal forms that are not double cuspidal)**.** Example 15.7 in [Bö1] shows that the quotient space $S_{k,0}(\Gamma_1(T))/S^2_{k,0}(\Gamma_1(T))$, which is two-dimensional, is always diagonalizable with respect to the Hecke algebra away from T. And any eigenform in this space has eigenvalues[2] $\lambda_{\mathfrak{p}} = \wp$ for $\mathfrak{p} \neq T$. This is a result that we have already noted for $\mathrm{GL}_2(A)$ (see Corollary 3.1.11 in Chapter 3).

The use of A-expansions allows us to see that this also follows without looking at the respective crystals. Indeed, if $k \equiv 1 \bmod (q-1)$, then write $k = s(q-1)+1$. We have $f_s \in S_{k,1}(\mathrm{GL}_2(A))$ as in Corollary 3.1.11. Using ι and ι_T to induce f_s to $\Gamma_1(T)$, we have two linearly independent forms

$$\iota(f_s), \iota_T(f_s) \in S_{k,0}(\Gamma_1(T))/S^2_{k,0}(\Gamma_1(T)),$$

with the same eigensystems away from the level $\{\lambda_{\mathfrak{p}} = \wp\}_{\mathfrak{p} \neq (T)}$.

Again, the argument applies verbatim to $\Gamma_0(T)$, and since the quotient space $S_{k,1}(\Gamma_0(T))/S^2_{k,1}(\Gamma_0(T))$ is two-dimensional, we have:

Theorem 4.3.4. *The quotient space $S_{k,1}(\Gamma_0(T))/S^2_{k,1}(\Gamma_0(T))$ has a basis of eigenforms away from T. Each element of this basis has eigensystem $\{\lambda_{\mathfrak{p}} = \wp\}_{\mathfrak{p} \neq (T)}$.* □

Böckle observed in Remark 12.9 and Example 15.7 of [Bö1] that the quotient space may be generated by Poincaré series. Our results show that this space is generated by forms with A-expansions for $\Gamma = \mathrm{GL}_2(A), \Gamma_0(T)$ and $\Gamma_1(T)$. We do not know if this happens for general congruence subgroups. In the Drinfeld setting, a Poincaré series $P_{k,m}(z)$ is defined [Ge3, (5.11)] by:

$$P_{k,m}(z) = \sum_{\gamma \in H \backslash \mathrm{GL}_2(A)} \frac{(\det \gamma)^m t^m(\gamma z)}{(c_\gamma z + d_\gamma)^k},$$

where

$$H = \left\{ \gamma = \begin{bmatrix} * & * \\ 0 & 1 \end{bmatrix} : \gamma \in \mathrm{GL}_2(A) \right\}.$$

[2]With Böckle's normalization the eigenvalues are actually $\lambda_{\mathfrak{p}} = 1$.

In [Ge3, (5.11)] Gekeler observes that the map $\gamma \mapsto (c_\gamma, d_\gamma)$ produces a bijection between $H\backslash \mathrm{GL}_2(A)$ and the set of pairs $S = \{(c,d) : c, d \in A, (c,d) = 1\}$. One can also take $\det\gamma = 1$ in the definition of $P_{k,m}$. For $s \geq 0$

$$P_{q+1+s(q-1),1}(z) = \sum_{(c,d)\in S} \frac{t(\gamma_{c,d} z)}{(cz+d)^{q+1+s(q-1)}}.$$

Gekeler proved that

$$P_{q+1,1} = -h.$$

by showing $P_{q+1,1} \in S_{q+1,1}(\mathrm{GL}_2(A))$ (a one-dimensional space) and

$$P_{q+1,1} = -t + \mathcal{O}(t^2).$$

Using Gekeler's argument verbatim, we get

$$P_{q+1+s(q-1),1} = -t + \mathcal{O}(t^2).$$

Unfortunately, we were not able to compute the t-expansion of $P_{q+1+s(q-1),1}$ beyond this. The only thing that we can say is that (for $\mathrm{GL}_2(A)$) if the quotient space $S_{q+1+s(q-1),1}(\mathrm{GL}_2(A))/S^2_{q+1+s(q-1),1}(\mathrm{GL}_2(A))$ is generated by Poincaré series, then

$$f_s = -P_{q+1+s(q-1),1}.$$

Example 4.3.5 (Proposition 15.2 and 15.6 from [Bö1])**.** Let θ be a variable different from T. Put $X = \mathrm{Spec}(\mathbb{F}_q[\theta, \theta^{-1}])$.

Böckle shows that for any q we have

$$\begin{aligned} \mathcal{S}^2_{4,0}(\Gamma_1(T)) &\cong (\mathcal{O}_{X\times A}, (T/\theta+1)(\varphi \times \mathrm{id})) \\ &= (\mathcal{O}_{\mathrm{Spec}\,\mathbb{F}_q[\theta,\theta^{-1},T]}, (T/\theta+1)(\varphi\times\mathrm{id})). \end{aligned}$$

In addition, Böckle computes that any eigenform away from T in this space has eigenvalues[3] $\lambda_{\mathfrak{p}} = \wp(T)\cdot\wp(-T)$.

[3]Here we use the definition of Hecke operators that we have, rather than the one used by Böckle.

Looking at Example 2.5.6, we see that for $q = 3$ this agrees with the form

$$\phi = g^2 - (T^2 + 1)gG + T^2G^2.$$

This form does not have an A-expansion according to our definitions (if it did the eigensystem will be $\lambda_{\mathfrak{p}} = \wp^n$ for some n). It is not clear if there is a way to redefine the concept of A-expansions in a way that will allow us to encompass ϕ. However, the computational evidence suggests (see Example 2.5.6) that we always have eigenvalues $\lambda_{\mathfrak{p}} = \wp(T) \cdot \wp(\mathrm{Pol}_f(T))^{e_f}$, where $\mathrm{Pol}_f(X)$ is a polynomial with coefficients in a finite extension of K and e_f is a non-negative integer. That is, with Böckle's normalization of the Hecke operators we always have

$$\mathrm{T}_{\mathfrak{p}}^{\text{Bö}} f = \wp(\mathrm{Pol}_f(T))^{e_f} f,$$

for $\mathfrak{p} \neq (T)$. At the moment it is not clear whether this will hold in complete generality.

Remark 4.3.6. In Example 15.4 from [Bö1], Böckle proves that the space $\mathcal{S}^2_{5,0}(\Gamma_1(T))$ is two-dimensional and has two linearly independent forms with the same eigensystems. This shows that Gekeler's question about the eigensystem and weight determining a newform has a negative answer for $\Gamma_1(T)$. The question is open for general congruence subgroups, such as $\mathrm{GL}_2(A)$, $\Gamma_0(T)$, or $\Gamma(T)$.

References

[Ar1] Cécile Armana, *Drinfeld modular forms and the Hecke action*, J. Number Theory **131**, pp 1435–1460, 2011.

[BöPi] Gebhard Böckle, Richard Pink, *Cohomology Theory of Crystals over Function Fields*, EMS Tracts in Mathematics 9, 2011.

[Bö1] Gebhard Böckle, *An Eichler-Shimura isomorphism over function fields between Drinfeld modular forms and cohomology classes of crystals*, preprint, 2002.

[Bö2] Gebhard Böckle, *Algebraic Hecke Characters and Compatible Systems of Abelian Mod p Galois Representations over Global Fields* preprint, 2008.

[Bö3] Gebhard Böckle, *Hecke Characters Associated to Drinfeld Modular Forms* preprint, 2008.

[BoPe] Vincent Bosser, Frederico Pellarin, *Hyperdifferential Properties of Drinfeld Quasi-Modular Forms*, International Math. Research Notices, 2008.

[Co1] Gunther Cornelissen, *A Survey of Drinfeld Modular Forms*, in [GPR], pp 167 – 187.

[Co2] Gunther Cornelissen, *Geometric Properties of Modular Forms over Rational Function Fields*, Dissertation at the University of Gent, 1997.

[Da1] Henri Darmon, *Rational Points on Modular Elliptic Curves*, CBMS Regional Conference Series in Mathematics **101**, AMS, 2004.

[DeHu] P. Deligne, D. Husemöller, *Survey of Drinfeld Modules*, Contemporary Math. **67**, pp 25–91, 1987.

[DiSh] Fred Diamond, Jerry Shurman, *An Introduction to Modular Forms*, Graduate Text in Mathematics, Springer-Verlag, 2005.

[Dr1] Vladimir Drinfeld, *Elliptic Modules* (English translation), Math USSR, Sbornik Vol. 23, No. 4, 1974.

[FrvD] J. Fresnel and M. van der Put, *Rigid Analytic Geometry and Its Applications*, Progress in Math. 18, Birkhäuser, 1981.

[Ga1] F. Gardeyn, *t-motives and Galois Representations*, Dissertation, 2001.

[Ge1] Ernst-Ulrich. Gekeler, *A Product Expansion for the Discriminant Function of Drinfeld modules of Rank Two*, J. Number Theory **21**, pp 135-140, 1985.

[Ge2] Ernst-Ulrich. Gekeler, *Drinfeld Modular Curves*, Lecture Notes in Math. 1231, Springer-Verlag, 1986.

[Ge3] Ernst-Ulrich Gekeler, *On the coefficients of Drinfeld modular forms*, Invent. math. **93**, pp 667–700, 1988.

[GPR] E.-U. Gekeler, M. van der Put, M. Reversat, J. Van Geel (Eds), *Drinfeld modules, modular schemes and applications*, Proceedings of a workshop held in Alden-Biesen, World Scientific Publishing Co., 1997.

[Gh1] Eknath Ghate, *On products of eigenforms*, Acta Arithmetica **102**(1), pp 27–44, 2002.

[Go1] David Goss, *Modular forms for* $\mathbb{F}_r[T]$, J. Reine Angew. Math. **317** , pp 16–39, 1980.

[Go2] David Goss, π*-adic Eisenstein series for function fields*, Compositio Math. **41**, pp 3–38, 1980.

[Go3] David Goss et al. (Eds), *The Arithmetic of Function Fields*, Proceedings of a workshop held at the Ohio State University, OSU Math. Res. Inst. Publ. **2**, Walter de Gruyter & Co., 1992.

[Go4] David Goss, *Basic Structures of Function Field Arithmetic*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **35**, Springer-Verlag, 1996.

[Jo1] Matthew Johnson, *Hecke Eigenforms as Products of Eigenforms*, arXiv: 1110.6430v1.

[Ka1] Nicholas Katz, *p-adic Properties of Modular Schemes and Modular Varieties* in Modular Functions on One Variable III, LMN **350**, Springer-Verlag, pp 69–191, 1973.

[La1] José Alejandro Lara Rodríguez, *Relations between Multizeta Values in Characteristic p*, J. Number Theory **131**, pp 2081–2099, 2011.

[LiMe] Wen-Ching Winnie-Li, Yotsanan Meemark, *Hecke Operators on Drinfeld Cusp Forms*, J. Number Theory **128**, pp 1941–1965, 2008.

[Lo1] Bartolome Lopez, *A non-standard Fourier expansion for the Drinfeld discriminant function*, Archiv der Mathematik **95**, Num 2, 143–150, 2011.

[Lo2] Bartolome Lopez, private communication.

[Ono] Ken Ono, *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-series*, CBMS Number 102, American Math. Society, 2004.

[Pe1] Frederico Pellarin, *Values of certain L-series in positive characteristic*, arXiv 1107.4511v3.

[Pe2] Frederico Pellarin, *Estimating the Order of Vanishing at Infinity of Drinfeld Quasi-Modular Forms*, arXiv: 0907.4507v3.

[Pe3] Frederico Pellarin, *τ-recurrent Sequences and Modular Forms*, arXiv: 1105.5819v3.

[Se1] Jean-Pierre Serre, *Arbres, amalgames, SL_2*, Asterisque No. **46**, Societe Math. France, 1977.

[Se2] Jean-Pierre Serre, *Abelian l-adic representations and elliptic curves*, Research Notes in Math. **7**, A K Peters Ltd., 1998.

[Te1] Jeremy Teitelbaum, *The Poisson Kernel for Drinfeld Modular Curves*, J. Amer. Math. Soc., **4** (3), pp 491–511, 1991.

[Te2] Jeremy Teitelbaum, *Modular Symbol for* $\mathbb{F}_q(T)$, Duke Math. J. **68** (2), pp 271-295, 1992.

[Te3] Jeremy Teitelbaum, *Rigid Analytic Modular Forms: An Integral Transform Approach*, in [Go3], pp 189–207.

[Tha] Dinesh Thakur, *Function Field Arithmetic*, World Scientific Publishing, 2004.

[Vi1] Christelle Vincent, *Drinfeld Modular Forms Modulo* $\mathfrak{p}$, Proceedings of the AMS, Vol. **138** (12), pp 4217–4229, 2010.