Serre Weights: The Partially Ramified Case

by

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A Dissertation Submitted to the Faculty of the

Department of Mathematics

In Partial Fulfillment of the Requirements
For the Degree of

Doctor of Philosophy

In the Graduate College

The University of Arizona

2 0 1 2

THE UNIVERSITY OF ARIZONA
GRADUATE COLLEGE

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Serre weights: the partially ramified case
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Signed: _____ Ryan Smith _____

Dedication

My thesis is dedicated to Laurel and Malcolm, friends far away but always close in thought.

Acknowledgments

I would foremost like to thank my thesis advisor David Savitt for recommending this problem, and for his considerable patience and help as I worked on it.

I would also like to thank Victor Piercey, Jordan Schettler, Geillan Aly, and Mei Yin for their friendship and support. Finally, I would like to thank Christian Collberg, whose assistance helped make this document possible.

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Abstract

We study the possible weights of an irreducible 2-dimensional modular mod p representation of $\mathrm{Gal}(\overline{F}/F)$, where F is a totally real field in which p is allowed to ramify, and the representation is tamely ramified at primes above p. We describe a set of possible weights and completely determine the weights in some cases when $e = 2$, $f = 2$.

Chapter 1
Introduction

The connection between modular forms and Galois representations is one of the most important and far reaching discoveries in modern number theory. Modular forms play a key role in the proof of Fermat's last theorem, the converse direction of the Herbrand-Ribet theorem, and the Langlands program.

1.1 Background

Definition 1.1.1. Let $S_k(N, \chi)$ denote the space of cusp forms on $\Gamma_1(N)$ with character χ.

Let $f \in S_k(N, \chi)$ be a eigenform of weight k, level N, character χ, and Fourier coefficients a_ℓ. Let $p \nmid N$ be a prime, and let $\bar{\epsilon}$ be the reduction mod p of the p-adic cyclotomic character. Let $G_\mathbb{Q} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of $\mathbb{Q}$. For each prime ℓ let Frob_ℓ denote a Frobenius element for ℓ. The work of many people culminated in the following deep theorem:

Theorem 1. *(Eichler-Shimura, Deligne) Let f be as above. Then there is a continuous, semi-simple Galois representation*

$$\overline{\rho}_f : G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{F}}_\mathrm{p})$$

which is unramified at all primes $\ell \nmid pN$, and the characteristic polynomial of $\overline{\rho}_f(\mathrm{Frob}_\ell)$ is $x^2 - \overline{a_\ell}x + \overline{\chi}(\ell)\overline{\epsilon}^{k-1}(\ell)$.

By the Chebotarev Density theorem, this implies that the determinant of complex conjugation is $\overline{\chi}(c)\overline{\epsilon}^{k-1}(c) = (-1)^k(-1)^{k-1} = -1$. In 1973, Serre proposed a converse to this statement.

Conjecture 1.1.2. (Serre) Let $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_{\mathrm{p}})$ be a continuous, irreducible, odd representation (one in which the determinant of complex conjugation is -1). Then $\overline{\rho} \cong \overline{\rho}_f$ for some modular form f.

Serre also proposed a stronger version of his conjecture in [Ser87] which explicitly specified the minimal weight and prime to p level of the associated modular form. Fix a choice of embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_{\mathrm{p}}$ and let I_p denote the inertia subgroup of $G_{\mathbb{Q}_p}$. Let $\omega : I_p \to \mu_{p-1}(\overline{\mathbb{Q}}_{\mathrm{p}}) \to \mathbb{F}_p^{\times} \hookrightarrow \overline{\mathbb{F}}_{\mathrm{p}}{}^{\times}$ denote the mod p cyclotomic character, and let $\omega_2 : I_p \to \mathbb{F}_{p^2}^{\times} \hookrightarrow \overline{\mathbb{F}}_{\mathrm{p}}{}^{\times}$ be the niveau 2 fundamental character. If $\overline{\rho}\,|_{G_{\mathbb{Q}}}$ is semi-simple[1] then either

- $\overline{\rho}\,|_{I_p} \sim \begin{pmatrix} \omega^a & 0 \\ 0 & \omega^b \end{pmatrix}$ for unique $0 \leq a \leq b \leq p-2$, or
- $\overline{\rho}\,|_{I_p} \sim \begin{pmatrix} \omega_2^{a+bp} & 0 \\ 0 & \omega_2^{b+ap} \end{pmatrix}$ for unique $0 \leq a < b \leq p-1$.

In both cases Serre conjectured that the the minimal weight was $k(\overline{\rho}) := 1 + pa + b$.

Long before Serre's conjecture was known it was known in almost all cases that if $\overline{\rho}$ is modular, then it is modular of the conjectured weight and level ([Edi92], [Rib90], ...). There has been a great deal of work on Serre's conjecture in recent years, culminating in a proof of the full conjecture in the work of Khare-Wintenberger in [KW09a] and [KW09b], and Kisin [Kis09]. The proof of Serre's conjecture involves modularity lifting theorems. One measure of the strength of Serre's Conjecture is that it directly implies Fermat's Last Theorem (i.e., without resort to the Shimura-Taniyama Conjecture that all elliptic curves over $\mathbb{Q}$ are modular [BCDT01]).

1.2 Generalizations of Serre's conjecture

It is natural to ask if Serre's conjecture can be generalized to representations of other Galois groups, such as 2-dimensional mod p representations of G_F for a totally real

[1] The strong form of Serre's conjecture also includes a combinatorial formula for $k(\overline{\rho})$ if not semi-simple.

field F. As with the classical case, generalizations of Serre's conjecture contain two related but separate problems: the existence of modular lifts and the possible weights and levels of modular lifts. Although modularity lifting theorems are known for totally real fields, there is no base case for the inductive arguments used in [KW09a] and it seems that a proof of the modularity of representations will not be available in the near future. The weight part of generalizations of Serre's conjecture have also received significant interest, and many partial results are known.

In the seminal paper [BDJ10], Buzzard, Diamond, and Jarvis proposed a generalization of Serre's conjecture for a totally real field F. Naively one might attempt to replace modular forms with Hilbert modular forms in the formulation of Serre's conjecture, but the conjecture would be false since not all continuous, irreducible, totally odd representations

$$\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_{\mathrm{p}})$$

arise from Hilbert modular forms of level prime to p. They define modularity in terms of Shimura varieties, which generalize modular curves. In this context a weight is no longer an integer, so the authors introduced a generalization of a weight called a Serre weight.

Let F be a totally real field, with ring of integers $\mathcal{O}_F$ and residue fields $k_{\mathfrak{p}}$ for the places $\mathfrak{p}$ of F lying above p.

Definition 1.2.1. A *Serre weight* σ is an isomorphism class of irreducible $\overline{\mathbb{F}}_{\mathrm{p}}$-representations of $\mathrm{GL}_2(\mathcal{O}_F/p)$. Since

$$\mathrm{GL}_2(\mathcal{O}_F/p) \cong \prod_{\mathfrak{p}|p} \mathrm{GL}_2(\mathcal{O}_F/\mathfrak{p}^{e_{\mathfrak{p}}})$$

any irreducible $\overline{\mathbb{F}}_{\mathrm{p}}$-representation factors through $\prod_{\mathfrak{p}|p} \mathrm{GL}_2(k_{\mathfrak{p}})$. Therefore Serre weights have the form $\sigma = \bigotimes_{\mathfrak{p}|p} \sigma_{\mathfrak{p}}$ where $\sigma_{\mathfrak{p}}$ is an irreducible representation of $\mathrm{GL}_2(k_{\mathfrak{p}})$.

Buzzard, Diamond, and Jarvis say that $\overline{\rho}$ is modular if $\overline{\rho}$ occurs as a sub-quotient

of the étale cohomology of certain Shimura varieties. Subsequent work in [Gee11b], [GS11b], [BLGG11] and others defines modularity in terms of algebraic modular forms. We will use this definition for our work.

As in the classical case, the Serre weights of a modular representation contain information about the structure of the representation. However, unlike the classical case, a representation will generally be modular of multiple weights and the weights will not be well ordered. Provided that p is unramified in F, Buzzard, Diamond, and Jarvis go on to define an explicit set of Serre weights $W^?(\overline{\rho})$ that they conjecture is equal to $W(\overline{\rho})$, the set of Serre weights for which $\overline{\rho}$ is modular.

Schein [Sch08] and Gee [Gee11a] have generalized the description of $W^?(\overline{\rho})$ in [BDJ10] by removing the unramified hypothesis, and recent work in [BLGG11] significantly improved our understanding by showing that $W^?(\overline{\rho}) \subset W(\overline{\rho})$ for modular representations. The containment is known to be equality in the special cases where the places $\mathfrak{p}$ of F lying above p are either unramified [GLS12] or totally ramified[2] [GLS11]. It is currently unknown if equality holds in the general case.

In the tamely ramified case Schein conjectured an explicit set of local Serre weights $W^?_{\mathfrak{p}}(\overline{\rho})$ for each place of F lying above p in [Sch08]. We will now recall the set of weights defined by Schein, in the case when $\overline{\rho}\,|_{G_{F_{\mathfrak{p}}}}$ is irreducible for all places $\mathfrak{p}$ of F above p. His original definition was stated abstractly in terms of certain combinatorial data which he then described concretely. Fix an embedding of $\mathbb{F}_{p^s} \hookrightarrow \overline{\mathbb{F}}_{\mathrm{p}}$.

Definition 1.2.2. A fundamental system of niveau s is a set of characters $\{\psi_1, \ldots, \psi_s\}$ from the inertia subgroup $I_{F_{\mathfrak{p}}}$ such that $\psi_i : I_{F_{\mathfrak{p}}} \to \mathbb{F}_{p^s}^{\times} \hookrightarrow \overline{\mathbb{F}}_{\mathrm{p}}{}^{\times}$ are induced by local class field theory, do not factor through $\mathbb{F}_{p^d}^{\times}$ for any $d < s$, and $\psi_i^p = \psi_{i+1}$ for all i.

Let $\{\omega_\tau\}_\tau$ be a fundamental system of niveau s and let $\{\eta_{\tilde{\tau}}, \eta_{\tilde{\tau}'}\}_\tau$ be a fundamental system of niveau $2s$ with the property that $\eta_{\tilde{\tau}}\eta_{\tilde{\tau}'} = \omega_\tau$ for all embeddings $\tau : k_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{F}}_{\mathrm{p}}$.

[2]Earlier work on the totally ramified case in [GS11a] was limited to the tamely ramified case.

Definition 1.2.3. Suppose that $\overline{\rho}|_{I_{\mathfrak{p}}}$ is irreducible and has niveau $2s$. Then $W^?_{\mathfrak{p}}(\overline{\rho})$ consists precisely of those Serre weights at $\mathfrak{p}$

$$\sigma_{\mathfrak{p}} = \bigotimes_{\tau\in I}(\det{}^{\omega_\tau}\operatorname{Sym}^{k_\tau-2} k_{\mathfrak{p}}^2)\otimes_{k_{\mathfrak{p}},\tau}\overline{\mathbb{F}}_p$$

such that for each $\tau\in I$ there exists a labeling $\{\tilde{\tau},\tilde{\tau}'\}$ of its two lifts to $k'_{\mathfrak{p}}$ and an integer $0\leq\delta_\tau\leq e-1$ such that

$$\overline{\rho}|_{I_{\mathfrak{p}}}\sim\prod_{\tau\in I}\omega_\tau^{\omega_\tau}\begin{pmatrix}\prod_{\tau\in I}\eta_{\tilde{\tau}}^{k_\tau-1+\delta_\tau}\eta_{\tilde{\tau}'}^{e-1-\delta_\tau} & 0\\ 0 & \prod_{\tau\in I}\eta_{\tilde{\tau}}^{e-1-\delta_\tau}\eta_{\tilde{\tau}'}^{k_\tau-1+\delta_\tau}\end{pmatrix}$$

where e is the ramification index of $\mathfrak{p}$ in F.

Schein also defines a set of weights $W^?_{\mathfrak{p}}(\overline{\rho})$ when $\overline{\rho}|_{G_{F_{\mathfrak{p}}}}$ is reducible and decomposable, but we will not need to recall it here.

Conjecture 1.2.4. (Schein's Conjecture) If $\overline{\rho}$ is tamely ramified at all of the places lying above p then $W(\overline{\rho})=\otimes_{\mathfrak{p}|p}W^?_{\mathfrak{p}}(\overline{\rho})$.

1.3 Overview

In this work we use the methods of [GS11b] and [GS11a] to address certain cases where F is "partially ramified" at p, namely when the inertial and ramification indices are both 2 and $\overline{\rho}|_{G_{F_{\mathfrak{p}}}}$. Unlike the totally ramified case, we are not able to prove that $W^?(\overline{\rho})=W(\overline{\rho})$ in general because the information obtained from our arguments does not fully determine the weights when $f>1$. However, when $e=2$ and $f=2$ it is often possible to prove that the actual weights are the conjectured weights. The outline of this thesis is as follows.

The purpose of Chapter 2 is to classify all of the inertial types of parallel potentially Barsotti-Tate lifts of a local representation $\overline{\rho}$, and does so using purely local arguments. The existence of these lifts is related to questions involving finite flat group schemes. Potentially Barsotti-Tate Galois representations and their reductions

mod p can be described by certain semilinear algebra objects called (respectively) strongly divisible modules with descent data and Breuil modules with descent data. In section 2.2 we explicitly describe the Breuil modules associated to $\overline{\rho}|_{G_{F_\mathfrak{p}}}$ in the cases when $\overline{\rho}\,|_{G_{F_\mathfrak{p}}}$ is reducible and irreducible, respectively. In section 2.3 we explicitly construct strongly divisible modules with descent data and coefficients with certain properties which lift the Breuil modules of section 2.2. The existence of these strongly divisible modules allows us to conclude that $\overline{\rho}|_{G_{F_\mathfrak{p}}}$ has potentially Barsotti-Tate lifts ρ of certain inertial Galois types.

In Chapter 3 we prove global results, beginning with initial definitions and notation. In particular we define what it means for a global representation $\overline{\rho}$ to be modular of weight σ, and deduce that if $\overline{\rho}$ is modular then $\overline{\rho}$ must be modular of one or more weights which lie in a set determined by the inertial type of a potentially Barsotti-Tate lift of $\overline{\rho}\,|_{G_{F_\mathfrak{p}}}$. To strengthen this statement, in section 3.3 we use Buzzard's weight cycling argument to show that $\overline{\rho}$ must be modular of at least two weights. Finally, in section 3.4 we specialize to the case $e = 2$, $f = 2$. In this restrictive setting, the combination of weight cycling and combinatorics is sufficient to show that $W^?(\overline{\rho}) = W(\overline{\rho})$ in many cases.

Chapter 2
Local results

We begin with some definitions. Let K be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_K$ containing the maximal ideal $\mathfrak{p}$. Let $\rho : G_K \to \mathrm{GL}_2(E)$ be a p-adic representation and let $I_\mathfrak{p}$ denote the inertia subgroup of G_K. If k is any field $W(k)$ will always denote the Witt vectors of k.

Definition 2.0.1. Let $\tau : I_\mathfrak{p} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\mathrm{p})$ be a continuous representation with open kernel. We say that a representation $\rho : G_K \to \mathrm{GL}_2(E)$ has *inertial type* τ if the associated Weil-Deligne representation WD_ρ satisfies $\mathrm{WD}_\rho|_{I_\mathfrak{p}} \cong \tau$.

We are only concerned with tame inertial types, i.e. types τ that are trivial on wild inertia. Since tame inertia is abelian, a tame type τ will always decompose as a direct sum of characters $\tau \cong \chi_1 \oplus \chi_2$.

Definition 2.0.2. A representation $\rho : G_K \to \mathrm{GL}_2(E)$ is *Barsotti-Tate* if ρ is isomorphic to a representation arising from the Tate module of a p-divisible group defined over $\mathcal{O}_K$.

Barsotti-Tate representations are crystalline [Fon82], and the property of being Barsotti-Tate is equivalent to being crystalline with Hodge-Tate weights contained in $\{0, 1\}$ [Bre00].

Definition 2.0.3. A representation $\rho : G_K \to \mathrm{GL}_2(E)$ is *potentially* Barsotti-Tate (respectively crystalline or semi-simple) if there is a finite extension L/K such that $\rho|_{G_L}$ is Barsotti-Tate (respectively crystalline or semi-simple).

Definition 2.0.4. Let τ be an inertial type. We say that a representation $\rho : G_K \to \mathrm{GL}_2(E)$ is *parallel potentially Barsotti-Tate* of type τ if ρ is potentially Barsotti-Tate,

has determinant a finite order character of order prime to p times the cyclotomic character, and the corresponding Weil-Deligne representation, when restricted to $I_{\mathfrak{p}}$, is isomorphic to τ.

2.1 Breuil modules with coefficients

We begin by summarizing some properties of mod p representations and potentially semistable representations from sections 3 and 4 respectively of [Sav05]. Let K/L be a tamely ramified Galois extension of p-adic fields, and let K_0 be the maximal unramified subextension in K. Fix a uniformiser $\pi \in K$ and let $E(u)$ denote the minimal polynomial of π over K_0. Let e' denote the absolute ramification index of K. Let $k = \mathcal{O}_K/(\pi)$ denote the residue field of K. The p-adic field E, with ring of integers $\mathcal{O}_E$ and residue field k_E, will be the coefficient field for the representations.

The tools of p-adic Hodge theory make it possible to analyze p-adic and mod-p Galois representations by studying certain semi-linear objects. There is an anti-equivalence of categories between the category of Galois stable $\mathcal{O}_L$-lattices contained in representations ρ such that $\rho|_{G_L}$ is semi-stable and ρ has Hodge-Tate weights in $\{0, 1\}$, and the category of strongly divisible modules with E-coefficients and descent data (see section 4 of [Sav05]). When we speak of taking the reduction mod p of a Galois stable lattice in ρ we shall always mean the semi-simplification of the reduction mod $\mathfrak{m}_E$. Reducing such a lattice mod p corresponds to taking the special fiber of the associated strongly divisible module, an object we call a Breuil module with k_E-coefficients and descent data.

Let $\overline{\rho}$ be the reduction of ρ mod $\mathfrak{m}_E$. In order to show that $\overline{\rho}$ has potentially Barsotti-Tate lifts we first review some results on strongly divisible modules (resp. Breuil modules) with coefficients and descent data. This will allow us to explicitly describe a Breuil module $\overline{\mathcal{N}}$ with coefficients and descent data associated to $\overline{\rho}$, which will occupy all of section 2.2. We can then write down a strongly divisible module $\mathcal{N}$

whose special fiber is $\overline{\mathcal{N}}$. From this strongly divisible module we pass back to a Galois stable lattice, which gives the desired potentially Barsotti-Tate lift by inverting p.

Definition 2.1.1. The category of Breuil modules with k_E-coefficients and descent data from K to L, denoted $\mathrm{BrMod}_{\mathrm{dd},L}$ consists of quadruples $(\mathcal{M}, \mathrm{Fil}^1 \mathcal{M}, \phi_1, \{\hat{g}\})$ where:

- $\mathcal{M}$ is a finitely generated $(k \otimes_{\mathbb{F}_p} k_E)[u]/(u^{e'p})$-module, free over $k[u]/u^{e'p}$.
- $\mathrm{Fil}^1 \mathcal{M}$ is a $(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{e'p}$-submodule of $\mathcal{M}$ containing $u^{e'}\mathcal{M}$.
- $\phi_1 : \mathrm{Fil}^1 \mathcal{M} \to \mathcal{M}$ is a k_E-linear and ϕ-semilinear map (where $\phi : k[u]/u^{e'p} \to k[u]/u^{e'p}$ is the p-th power map) with image generating $\mathcal{M}$ as a $(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{e'p}$-module.
- $\hat{g} : \mathcal{M} \to \mathcal{M}$ are additive bijections for each $g \in \mathrm{Gal}(K/L)$, preserving $\mathrm{Fil}^1 \mathcal{M}$, commuting with the ϕ-, and k_E-actions, and satisfying $\hat{g_1} \circ \hat{g_2} = \widehat{g_1 \circ g_2}$ for all $g_1, g_2 \in \mathrm{Gal}(K/L)$, and $\hat{1}$ is the identity. Furthermore, if $a \in k \otimes_{\mathbb{F}_p} k_E$, $m \in \mathcal{M}$ then $g(au^i m) = g(a)((g(\pi)/\pi)^i \otimes 1)u^i g(m)$.

The category $\mathrm{BrMod}_{\mathrm{dd},L}$ is equivalent to the category of finite flat group schemes over $\mathcal{O}_K$ together with a k_E-action and descent data on the generic fiber from K to L. For proof of this equivalence see [Sav05], section 4 "Objects killed by p".

As defined in [Sav05] the functor $D^K_{st,2}$ is an equivalence of categories between the category of G_L-representations with E-coefficients which are semistable when restricted to G_K and have Hodge-Tate weights in $\{0, 1\}$, and the category of weakly admissible filtered (ϕ, N)-modules D with descent data and E-coefficients such that $\mathrm{Fil}^0(K \otimes_{K_0} D) = K \otimes_{K_0} D$ and $\mathrm{Fil}^2(K \otimes_{K_0} D) = 0$.

Let S_K be the p-adic completion of the divided power envelope of $W(k)[[u]]$ with respect to $\ker(\theta)$, where $\theta : W(k)[[u]] \twoheadrightarrow \mathcal{O}_K$ is given by $u \mapsto \pi$. It can be realized as

$$S_K = \left\{ \sum_{j=0}^{\infty} r_j \frac{u^j}{\lfloor j/e' \rfloor !}, \text{ where } r_j \in W(k), r_j \to 0 \text{ as } j \to \infty \right\}.$$

Define $S_{K,\mathcal{O}_E} = \mathcal{O}_E \otimes_{\mathbb{Z}_p} S_K$. Let $\mathrm{Fil}^1 S_{K,\mathcal{O}_E}$ be the closure of the ideal generated by $E(u)^j/j!$ for $j \geq 1$. There is a unique continuous Frobenius-semilinear map $\phi : S_{K,\mathcal{O}_E} \to S_{K,\mathcal{O}_E}$ defined by $\phi(u) = u^p$ and $\phi(u^{je'}/j!) = u^{je'p}/j!$. This map has $\phi(\mathrm{Fil}^1 S_{K,\mathcal{O}_E}) \subset pS_{K,\mathcal{O}_E}$, so we can define $\phi_1 = \left(\phi|_{\mathrm{Fil}^1 S_{K,\mathcal{O}_E}}\right)/p$. We can give S_K a $\mathrm{Gal}(K/L)$-action by defining $g(u) = (g(\pi)/\pi)u$ and letting $\mathrm{Gal}(L/K)$ act on $W(k)$ as usual, then extending by p-adic continuity.

Definition 2.1.2. A *strongly divisible $\mathcal{O}_E$-module with tame descent data from K to L* is a finitely generated $S_{K,\mathcal{O}_E}$ module $\mathcal{N}$, together with a $S_{K,\mathcal{O}_E}$-submodule $\mathrm{Fil}^1 \mathcal{N}$, maps $N, \phi : \mathcal{N} \to \mathcal{N}$, and additive bijections $\hat{g} : \mathcal{N} \to \mathcal{N}$ for each $g \in \mathrm{Gal}(K/L)$ satisfying the following conditions:

1. $\mathrm{Fil}^1 \mathcal{N}$ contains $(\mathrm{Fil}^1 S_{K,\mathcal{O}_E})\mathcal{N}$,
2. $\mathrm{Fil}^1 \mathcal{N} \cap I\mathcal{N} = I \,\mathrm{Fil}^1 \mathcal{N}$ for all ideals I in $\mathcal{O}_E$,
3. $\phi(sx) = \phi(s)\phi(x)$ for $s \in S_{K,\mathcal{O}_E}$ and $x \in \mathcal{N}$,
4. $\phi(\mathrm{Fil}^1 \mathcal{N})$ is contained in $p\mathcal{N}$ and generates it as an $S_{K,\mathcal{O}_E}$-module,
5. $N(sx) = N(s)x + sN(x)$ for $s \in S_{K,\mathcal{O}_E}$ and $x \in \mathcal{N}$,
6. $N\phi = p\phi N$,
7. $E(u)N(\mathrm{Fil}^1 \mathcal{N}) \subset \mathrm{Fil}^1 \mathcal{N}$,
8. $N(\mathcal{N}) \subset J\mathcal{N}$ where J is the ideal $\sum_{j=1}^{\infty} \frac{u^j}{\lfloor j/e' \rfloor !} S_{K,\mathcal{O}_E}$,
9. $\hat{g}(sx) = \hat{g}(s)\hat{g}(x)$ for $s \in S_{K,\mathcal{O}_E}$, $x \in \mathcal{N}$, and $g \in \mathrm{Gal}(K/L)$,
10. $\hat{g}_1 \circ \hat{g}_2 = \widehat{g_1 \circ g_2}$ for all $g_1, g_2 \in \mathrm{Gal}(K/L)$,
11. $\hat{g}(\mathrm{Fil}^1 \mathcal{N}) \subset \mathrm{Fil}^1 \mathcal{N}$ for all $g \in \mathrm{Gal}(K/L)$, and
12. ϕ and N both commute with $\hat{g}$ for all $g \in \mathrm{Gal}(K/L)$.

Set $S = S_{K,\mathcal{O}_E}$. By Theorem 4.13 of [Sav05] the functor $T^L_{st,2}$ is an equivalence of categories between the category of strongly divisible modules with $\mathcal{O}_E$-coefficients and descent data in $S \otimes_{W(k)} D^K_{st,2}(\rho)$, and the category of Galois stable $\mathcal{O}_E$-lattices in ρ. This equivalence is compatible with reduction mod $\mathfrak{m}_E$, so that applying $T^L_{st,2}$ to the object $(k \otimes_{\mathbb{F}_p} k_E)[u]/(u^{e'p}) \otimes_{S/\mathfrak{m}_E S} (\mathcal{N}/\mathfrak{m}_E\mathcal{N})$ of $\mathrm{BrMod}_{dd,L}$ yields a reduction mod p of ρ (see Section 4 of [Sav05],"Objects killed by p").

In [GS11b] the authors establish a general result relating the inertial type τ of a potentially Barsotti-Tate representation $\rho : G_F \to \mathrm{GL}_n(E)$ and the descent data on strongly divisible modules $\mathcal{N}$ corresponding to lattices in ρ.

Proposition 2.1.3 ([GS11b], Proposition 5.1)**.** *We have $\tau = \chi_1 \oplus \cdots \oplus \chi_n$ if and only if there is an S-basis $v_1, \ldots, v_n$ of $\mathcal{N}$ such that the descent data acts on $\mathcal{N}$ via $\hat{g} \cdot v_i = (1 \otimes \chi_i(g))v_i$ for all $g \in I(K/L)$.*

Corollary 2.1.4 ([GS11b], Corollary 5.2)**.** *Let $\mathcal{N}$ denote the Breuil module with descent data corresponding to the π_E-torsion in the Barsotti-Tate group corresponding to $\overline{\rho}\,|_{G_K}$. If $\tau = \chi_1 \oplus \ldots \chi_n$ then $\mathcal{N}$ has a $(k \otimes k_E)[u]/(u^{e'p})$-basis $\overline{v}_1, \ldots, \overline{v}_n$ such that $\hat{g}(\overline{v}_i) = (1 \otimes \overline{\chi}_i(g))\overline{v}_i$ for all $g \in I(K/L)$. (Here e' is the absolute ramification index of K).*

2.2 Computations with Breuil modules

Let $F_{\mathfrak{p}}$ be a finite extension of $\mathbb{Q}_p$, with $p > 2$. Fix a uniformiser $\pi_{\mathfrak{p}}$ of $F_{\mathfrak{p}}$ and let $q = p^f$ denote the cardinality of the residue field of $F_{\mathfrak{p}}$. Let $e > 1$ be the absolute ramification index of $F_{\mathfrak{p}}$. Let $\overline{\rho} : G_{F_{\mathfrak{p}}} \to \mathrm{GL}_2(E)$ be the reduction mod p of a potentially potentially Barsotti-Tate representation ρ with coefficients in a p-adic field E, with ring of integers $\mathcal{O}_E$ and residue field k_E.

Suppose that ρ has tame principal series type, then the associated Weil-Deligne representation factors through tame inertia and has the form $\chi_1 \oplus \chi_2$ where $\chi_1 = \chi_1^q$

and $\chi_2 = \chi_2^q$. Since inertial types are compatible with twisting, we can assume that ρ has type $1 \oplus \chi$ for some character $\chi : I_{\mathfrak{p}} \to \mathcal{O}_E^\times$ of order $q-1$. Since $\chi^q = \chi$ we know that $\overline{\chi} \neq 1$ where $\overline{\chi}$ is the reduction of χ mod p.

We make several definitions which depend on whether or not $\overline{\rho}$ is irreducible:

- If $\overline{\rho}$ is reducible, set $L = F_{\mathfrak{p}}$ and let K be the splitting field of $u^{p^f-1} - \pi_{\mathfrak{p}}$ over L.
- If $\overline{\rho}$ is irreducible, let L to be the unique unramified quadratic extension of $F_{\mathfrak{p}}$ and let K be the splitting field of $u^{p^{2f}-1} - \pi_{\mathfrak{p}}$ over L.

These fields are chosen so that $\tau(\rho)|_{I_K}$ will be reducible and have trivial inertial type mod p. We fix a choice of uniformiser $\varpi \in K$ with $\varpi^{e(K/L)} = \pi_{\mathfrak{p}}$.

In order to apply the results of [Sav05] on potentially Barsotti-Tate representations we require that the coefficient field k_E contain $k := \mathcal{O}_K/\varpi$; without loss of generality we can enlarge E in order to assume that k embeds into k_E. Fixing a choice of embedding $\sigma_0 : k \hookrightarrow k_E$ allows us to recursively define the other embeddings via $\sigma_{i+1}^p = \sigma_i$. In the reducible case the character $\bar{\eta} : \mathrm{Gal}(K/F_{\mathfrak{p}}) \to k^\times$ given by $g \mapsto g(\varpi)/\varpi \pmod{\varpi}$ can be used to define a system of fundamental characters of niveau f via $\omega_i = (\sigma_i \circ \bar{\eta})|_{I_{\mathfrak{p}}}$ for $i \in \{0, 1, \ldots, f-1\}$. In the irreducible case $\bar{\eta}$ gives rise to a system of fundamental characters of niveau $2f$ denoted $\eta_0, \ldots, \eta_{2f-1}$, and $\omega_i = \eta_i \eta_{i+f}$. Since the $\omega_0, \ldots, \omega_{f-1}$ form a fundamental system of niveau f, and by definition χ extends to a character of the Weil group $W_{F_{\mathfrak{p}}}$ we can write

$$\overline{\chi} = \prod_{i=0}^{f-1} \omega_i^{c_i}$$

where each $c_i \in [0, p-1]$ is uniquely determined since $\overline{\chi} \neq 1$ and $\sigma_{i+1}^p = \sigma_i$. Let $\bar{\epsilon}$ be the mod p cyclotomic character. Then we have $\bar{\epsilon} = \prod_{i=0}^{f-1} \omega_i^e$.

We recall from [Sav08] that if $\mathcal{M}$ is any $(k \otimes_{\mathbb{F}_p} k_E)$-module then $\mathcal{M}$ decomposes as a direct sum $\mathcal{M} = \oplus_{i=0}^{d-1} \mathcal{M}_i$, where $d = [k : \mathbb{F}_p]$ and $\mathcal{M}_i$ is the k_E-submodule on

which multiplication by $x \otimes 1$ is the same as multiplication by $1 \otimes \sigma_i(x)$. There is a collection of idempotents $e_i \in k \otimes_{\mathbb{F}_p} k_E$ such that $\mathcal{M}_i = e_i\mathcal{M}$ and $\phi(e_i) = e_{i+1}$.

Suppose that M is a rank 1 Breuil submodule of a rank 2 Breuil module with descent data from L to K which corresponds to a representation with inertial type $1 \oplus \overline{\chi}$. For any $g \in \mathrm{Gal}(K/L)$ and $v \in M$ the descent data acts via $\hat{g}(e_i v) = (1 \otimes \chi_i(g))(e_i v)$ for some character χ_i. This defines a set of characters $\{\chi_0, \chi_1, \ldots, \chi_{d-1}\}$. By corollary 2.1.4, since $\overline{\rho}$ has inertial type $1 \oplus \chi$ for each i the descent data corresponding to $g \in \mathrm{Gal}(K/L)$ must either act on trivially on e_i or via $\hat{g}(e_i) = \overline{\chi}(g)e_i$. Therefore $\chi_i \in \{1, \overline{\chi}\}$ for all i.

Finally, in both of the following sections if $c_0, \ldots, c_{f-1}$ are integers in the range $[0, p-1]$ we let $\lambda = c_0 + p^{f-1}c_1 + \cdots + pc_{f-1}$ and let $[p^i\lambda]$ denote an integer in the range $[0, p^f - 1)$ which is congruent to $p^i\lambda \pmod{p^f - 1}$, so $[p^i\lambda] = c_i + p^{f-1}c_{i+1} + \cdots + pc_{i-1}$.

2.2.1 The Reducible Case

Assume now that $\bar{\rho}$ is reducible, so there is a basis in which it can be written $\overline{\rho} = \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}$ for characters ψ_1 and ψ_2 which we will explicitly calculate. As explained in section 2.1, because $\overline{\rho}$ is the reduction mod p of a representation whose restriction to G_K is Barsotti-Tate, the p-torsion of the action of $G_{F_\mathfrak{p}}$ on an $\mathcal{O}_E$-lattice in ρ defines a finite flat group scheme $\mathcal{H}$ with generic fiber descent data from K to $F_\mathfrak{p}$. Let $\mathcal{G}$ be the subgroup scheme of $\mathcal{H}$ corresponding to ψ_1. By ([Sav08], Theorem 3.5) the Breuil module $\mathcal{M}$ with decent data to $F_\mathfrak{p}$ corresponding to $\mathcal{G}$ can be written

- $\mathcal{M} = ((k \otimes_{\mathbb{F}_p} k_E)[u]/u^{e(p^f-1)p}) \cdot v$ for some $v \in \mathcal{M}$,
- $e_i(\mathrm{Fil}^1 \mathcal{M}) = u^{r_i}\mathcal{M}_i$,
- $\phi_1(u^{r_i}e_i v) = (1 \otimes \gamma_i)e_{i+1}v$ for some $\gamma_i \in k_E^\times$,
- $\hat{g}(\sum_{i=0}^{f-1} e_i v) = \sum_{i=0}^{f-1}(\overline{\eta}(g)^{k_i} \otimes 1)e_i v$ for all $g \in \mathrm{Gal}(K/F_\mathfrak{p})$,

where the k_i and r_i are integers with $k_i \in [0, p^f - 1)$, $r_i \in [0, e(p^f - 1)]$. These coefficients must also satisfy $k_{i+1} \equiv p(k_i + r_i) \pmod{p^f - 1}$. Let $\chi_i = \sigma_i \circ \bar{\eta}^{k_i}$. Then, as remarked earlier for each i either $\chi_i = \overline{\chi}$ or $\chi_i = 1$. Let $J \subseteq \{0, \ldots, f-1\}$ be the set of i for which $\chi_i = \overline{\chi}$. If $i \in J$ then clearly

$$k_i \equiv p^i c_0 + p^{i-1} c_1 + \cdots + p^{i+1} c_{f-1} \equiv p^i (c_0 + p^{f-1} c_1 + \cdots + p c_{f-1}) \mod p^f - 1$$

. Recall that we defined $\lambda = c_0 + p^{f-1} c_1 + \cdots + p c_{f-1}$, so if $i \in J$ then $k_i \equiv p^i \lambda \pmod{p^f - 1}$, and $i \notin J$ is equivalent to $k_i \equiv 0 \pmod{p^f - 1}$.

The value of k_i is determined by whether or not $i \in J$, so the relation $k_{i+1} \equiv p(k_i + r_i) \pmod{p^f - 1}$ can be used to find restrict the possible values for the r_i. The values for the r_i coefficients are constrained by J,

$$r_i \equiv \begin{cases} 0 \pmod{p^f - 1} & i, i+1 \in J \\ 0 \pmod{p^f - 1} & i, i+1 \notin J \\ -p^i \lambda \pmod{p^f - 1} & i \in J, i+1 \notin J \\ p^i \lambda \pmod{p^f - 1} & i \notin J, i+1 \in J. \end{cases}$$

Definition 2.2.1. We say that J *flips* at i if $i \in J$ and $i + 1 \notin J$, or $i \notin J$ and $i + 1 \in J$.

If there is no flip at i then we can write $r_i = (p^f - 1)x_i$ for some $x_i \in [0, e]$. Recall that $[p^i \lambda]$ is the unique value in $[0, p^f - 1)$ which is congruent to $p^i \lambda \mod p^f - 1$, so if $i \in J$ and $i + 1 \notin J$ then $r_i = (p^f - 1)x_i - [p^i \lambda]$. If $i \notin J$ and $i + 1 \in J$ then $r_i = (p^f - 1)x_i + [p^i \lambda]$.

Define

$$s_0 := \frac{r_0 p^f + r_1 p^{f-1} + \cdots + r_{f-1} p}{p^f - 1}.$$

We are in the situation of Corollary 4.3 of [GS11b], so the Galois representation $T^K_{\mathrm{st},2}(\mathcal{M})$ coming from the Breuil module $\mathcal{M}$ is equal to $(\sigma_0 \circ \overline{\eta}^{k_0 + s_0})$ times an unramified character. Therefore $\psi_1|_{I_\mathfrak{p}} = \eta_0^{k_0 + s_0}$. We can view s_0 as $p^f x_0 + p^{f-1} x_1 + \cdots + p x_{f-1}$ plus a linear expression in the c_i's. Write $c_i(s_0)$ for the coefficient of c_i in s_0. For each

flip with $i \in J$ and $i+1 \notin J$ the coefficient of c_0 in $r_i p^{f-i}$ is $-p^f$, whereas if $i \notin J$, $i+1 \in J$ the coefficient of c_0 in $r_i p^{f-i}$ is p^f. The net number of flips from 0 to f are 0, so $c_i(s_0) = 0$ for all i. Let $c_i(k_0)$ be the coefficient of c_i in k_0,

$$c_i(k_0) = p^{f-i} c_i(k_i) = \begin{cases} p^{f-i} & i \in J \\ 0 & i \notin J. \end{cases}$$

When $i \notin J$ the coefficient of c_i in k_0 is 0, so

$$k_0 + s_0 = \sum_{i=0}^{f-1} c_i \cdot c_i(k_0) + \sum_{i=0}^{f-1} p^{f-i} x_i = \sum_{i \in J} p^{f-i} c_i + \sum_{i=0}^{f-1} p^{f-i} x_i$$

and $\psi_1|_{I_\mathfrak{p}}$ has the form

$$\psi_1|_{I_\mathfrak{p}} = \eta_0^{k_0+s_0} = \prod_{i \in J} \omega_0^{p^{f-i} c_i} \prod_{i=0}^{f-1} \omega_i^{x_i} = \prod_{i \in J} \omega_i^{c_i} \prod_{i=0}^{f-1} \omega_i^{x_i} \tag{2.2.1}$$

It remains to determine which values of x_i are valid. All of the r_i lie in the range $[0, e(p^f - 1)]$, which limits the possible values of x_i when there is a flip. By definition we have $0 < [p^i \lambda] < p^f - 1$. Observe that if there is no flip at i then $r_i = (p^f - 1)x_i$ and there is no restriction on x_i beyond the obvious requirement that $x_i \in [0, e]$. If $i \in J$ and $i+1 \notin J$ then $r_i = (p^f - 1)x_i - [p^i \lambda]$. Since $[p^i \lambda] > 0$ this precludes $x_i = 0$. If $i \notin J$ and $i+1 \in J$ then $r_i = (p^f - 1)x_i + [p^i \lambda]$ and the excluded case is $x_i = e$. This motivates the following definition:

Definition 2.2.2. A set of values $\{x_i \mid 0 \leq i < f\}$ is an allowable list (relative to a set $J \subseteq \{0, \ldots, f-1\}$) if for each i:

- if $i, i+1 \in J$ or $i, i+1 \notin J$ then $x_i \in [0, e]$,
- if $i \in J$ and $i+1 \notin J$ then $x_i \in [1, e]$,
- if $i \notin J$ and $i+1 \in J$ then $x_i \in [0, e-1]$.

As usual subscripts are interpreted mod f.

Remark 2.2.3. If $e \geq p-1$ then ψ_1 can be any niveau f character.

Proof. Setting $J = \varnothing$ assures that there are no flips and the allowable range of each x_i is $[0, e] \supseteq [0, p-1]$. Then $\sum_{i=0}^{f-1} p^{f-i} x_i$ can assume any value in $[0, p^f - 1)$. □

Theorem 2.2.4. *Let $\overline{\rho} : G_{F_\mathfrak{p}} \to \mathrm{GL}_2(k_E)$ be the reduction mod $\mathfrak{m}_E$ of a parallel potentially Barsotti-Tate representation ρ of type $1 \oplus \chi$. Suppose that $\overline{\chi} = \prod_{i=0}^{f-1} \omega_i^{c_i}$, where $\omega_0, \ldots, \omega_{f-1}$ are a fundamental system of niveau f and $c_i \in [0, p-1]$ for all i. Suppose that $\overline{\rho}|_{I_\mathfrak{p}} = \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}$ is reducible. Then are two possibilities:*

1. *If $e \geq p-1$ then ψ_1 and ψ_2 can be any inertial characters such that $\psi_1\psi_2 = \overline{\epsilon}\,\overline{\chi}$.*
2. *If $e < p-1$ then $\psi_1\psi_2 = \overline{\epsilon}\,\overline{\chi}$ and*

$$\psi_1|_{I_\mathfrak{p}} = \prod_{i \in J} \omega_i^{c_i} \prod_{i=0}^{f-1} \omega_i^{x_i}$$

where $J \subset \{0, \ldots, f-1\}$ and the x_i are an allowable list.

Remark 2.2.5. Taking $*$ to be zero it is easy enough to see that all of the allowable possibilities occur.

2.2.2 The Irreducible Case

We shall now treat the case where $\overline{\rho}$ is irreducible. Recall that we defined L to be the unramified quadratic extension of $F_\mathfrak{p}$ and let K be the splitting field of $u^{p^{2f}-1} - \pi_\mathfrak{p}$ over L. In this case, $\bar{\rho}|_{G_L} = \begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix}$ with $\psi_2 = \psi_1^q$ for a character ψ_1. As before we examine the possibilities for $\psi_1|_{I_\mathfrak{p}}$ via Breuil modules with descent data. There will be many Breuil modules with descent data and coefficients corresponding to $\overline{\rho}$, the primary difficulty in this section will be to find Breuil modules with a specific structure so that we can construct a lift in section 2.3. We first prove an analogous version of equation (2.2.1) for $\psi_1|_{I_\mathfrak{p}}$ when $\overline{\rho}$ is irreducible, but in order to construct a lift we require $i \in J \iff i+f \notin J$. When $e \geq p-1$ this is straightforward, but when

$e < p-1$ the situation is significantly more difficult and occupies the remainder of the section.

Let $\mathcal{H}$ be the finite flat group scheme with generic fiber descent data from K to L corresponding to $\overline{\rho}|_{G_L}$, and let $\mathcal{G}$ be the finite flat subgroup scheme corresponding to ψ_1. Let $\mathcal{M}$ be the Breuil modules with k_E-coefficients and descent data from K to L corresponding to $\mathcal{G}$. By Theorem 3.5 of [GS11a] $\mathcal{M}$ has the form

- $\mathcal{M} = ((k \otimes_{\mathbb{F}_p} k_E)[u]/u^{e(p^{2f}-1)p}) \cdot v$ for some $v \in \mathcal{M}$,
- $(\mathrm{Fil}^1 \mathcal{M})_i = u^{r_i}\mathcal{M}_i$,
- $\phi_1(u^{r_i}e_i v) = (1 \otimes \gamma_i)e_{i+1}v$ for some $\gamma_i \in k_E^\times$,
- $\hat{g}(\sum_{i=0}^{2f-1} e_i v) = \sum_{i=0}^{2f-1}(\overline{\eta}(g)^{k_i} \otimes 1)e_i v$ for all $g \in \mathrm{Gal}(K/L)$,

where the k_i and r_i are integers with $k_i \in [0, p^{2f}-1)$, $r_i \in [0, e(p^{2f}-1)]$. Note that the allowable ranges of k_i and r_i are greater than they were in the reducible case and the character $\overline{\eta} : \mathrm{Gal}(K/F_{\mathfrak{p}}) \to k^\times$ now has niveau $2f$. Let $\eta_i = (\sigma_i \circ \overline{\eta})|_{I_{\mathfrak{p}}}$ for $0 \le i < 2f$ be a system of fundamental characters of niveau $2f$; note that $\eta_i^p = \eta_{i-1}$. Then $\omega_i = \eta_i \eta_{i+f}$ for $0 \le i < f$ is a system of fundamental characters of niveau f. As before we can uniquely express $\overline{\chi}$ in terms of a fundamental system of characters of niveau f,

$$\bar{\chi} = \prod_{i=0}^{f-1} \omega_i^{c_i}$$

where each $c_i \in [0, p-1]$.

Much like the reducible case, the coefficients must satisfy $k_{i+1} \equiv p(k_i + r_i) \pmod{p^{2f}-1}$. Let $\chi_i = \sigma_i \circ \bar{\eta}^{k_i}$. Then for each i either $\chi_i = \overline{\chi}$ or $\chi_i = 1$. Let $J \subseteq \{0, \ldots, 2f-1\}$ be the set of i for which $\chi_i = \bar{\chi}$. Since $\bar{\eta}$ has niveau $2f$ but $\overline{\chi}$ only has niveau f, if $i \in J$ we must have

$$k_i \equiv (p^f + 1)[p^i \lambda] \pmod{p^{2f}-1}$$

if $\chi_i = \overline{\chi}$ (and 0 otherwise).

As before we can write $r_i = (p^{2f} - 1)x_i$ if $i, i+1 \in J$ or $i, i+1 \notin J$. If $i \in J$, $i+1 \notin J$ then $r_i = (p^{2f} - 1)x_i - (p^f + 1)[p^i\lambda]$; if $i \notin J$ and $i+1 \in J$ then the sign is reversed. This forces the same restrictions on when x_i can be 0 or e as in the reducible case, and for the same reasons:

Definition 2.2.6. A set of values $\{x_i \mid 0 \leq i < 2f\}$ is an allowable list (relative to a set $J \subseteq \{0, \ldots, 2f-1\}$) if for each i:

- if $i, i+1 \in J$ or $i, i+1 \notin J$ then $x_i \in [0, e]$,
- if $i \in J$ and $i+1 \notin J$ then $x_i \in [1, e]$,
- if $i \notin J$ and $i+1 \in J$ then $x_i \in [0, e-1]$.

As usual subscripts are interpreted mod f.

As with the reducible case, we have $\psi_1 = \eta_0^{k_0+s_0}$ where

$$s_0 := \frac{r_0 p^{2f} + r_1 p^{2f-1} + \cdots + r_{2f-1} p}{p^{2f} - 1}.$$

We regard s_0 as a linear combination of the x_i and c_i. If $i < f$ then each time there is a flip from $i \in J$ to $i+1 \notin J$ the $p^{2f-i} r_i$ term contributes $-p^{2f}(p^f + 1)c_0$ to the coefficient of c_0 in s_0, with the opposite sign for a flip from $i \notin J$ to $i+1 \in J$. When $i \geq f$ the $p^{2f-i} r_i$ term contributes $-p^f(p^f + 1)c_0$. If the net number of flips from 0 to f are 0, then the coefficient of c_0 in s_0 is 0. If the net flips from 0 to f is 1, then the net flips from f to $2f$ are -1, and the coefficient of c_0 in s_0 is $-p^f$. If the net flips from 0 to f are -1 then the total coefficient of c_0 in s_0 is p^f. The coefficient of c_0 in k_0 is $p^f + 1$ if $0 \in J$ and 0 otherwise. This yields four possibilities:

- If $0 \in J, f \in J$ then the coefficient of c_0 is $p^f + 1$.
- If $0 \in J, f \notin J$ the coefficient is 1.

- If $0 \notin J, f \in J$ the coefficient is p^f.
- If $0 \notin J, f \notin f$ the coefficient is 0.

The only contributions come from the terms in J, so by cyclic symmetry we have

$$k_0 + s_0 = \sum_{i \in J} p^{2f-i} c_{\pi(i)} + \sum_{i=0}^{2f-1} p^{2f-i} x_i$$

where $\pi : \mathbb{Z}/(2f)\mathbb{Z} \to \mathbb{Z}/f\mathbb{Z}$ is the natural projection map. This furnishes an explicit description of the character,

$$\psi_1|_{I_\mathfrak{p}} = \prod_{i \in J} \eta_i^{c_{\pi(i)}} \prod_{i=0}^{2f-1} \eta_i^{x_i}$$

Proposition 2.2.7. *If $e \geq p-1$, then as J varies over the subsets of $\{0, \ldots, 2f-1\}$ with $i \in J \iff i+f \notin J$ and the $x_0, \ldots, x_{2f-1}$ vary over the allowable lists for J, the character ψ_1 ranges over all inertial characters for which $\psi_1^{p^f+1} = \overline{\epsilon}\,\overline{\chi}$.*

Proof. Let $J = \{0, \ldots, f-1\}$ and let $x_{i+f} = e - x_i$. Let $X = p^{f-1}x_0 + p^{f-2}x_1 + \cdots + x_{f-1}$ then

$$\psi_1 = \prod_{i=0}^{f-1} \eta_i^{c_i} \prod_{i=0}^{2f-1} \eta_i^{x_i} = \prod_{i=0}^{f-1} \eta_i^{c_i} \prod_{i=0}^{f-1} \eta_i^{x_i} \eta_{i+f}^{e-x_i} = \left(\prod_{i=0}^{f-1} \eta_i^{c_i} \right) \eta_{2f-1}^{e(p^f-1)/(p-1)} \eta_{f-1}^{(1-p^f)X}$$

which depends only on $X \pmod{p^f+1}$. If $e \geq p$ then as the x_i range over $[0, e]$ for $0 \leq i < f-1$, and x_{f-1} ranges over $[1, e]$, X ranges over an interval which includes $[1, p^f]$. Repeating the process with $J = \{f, \ldots, 2f-1\}$ allows X to range over an interval which contains 0. If $e = p-1$ then X ranges over an interval which contains $[1, p^f - 1]$, and we can repeat the process with the compliment of J. □

Assume for the rest of the section that $e < p-1$. Suppose that J has the property that $i \in J \iff i+f \notin J$. The determinant condition $\psi_1^{p^f+1} = \overline{\epsilon}\,\overline{\chi}$ implies that

$$\prod_{i \in J} \omega_i^{c_{\pi(i)}} \prod_{i=0}^{f-1} \omega_i^{x_i + x_{i+f}} = \bar{\chi} \prod_{i=0}^{f-1} \omega_i^e$$

which reduces to

$$\prod_{i=0}^{f-1} \omega_i^{x_i+x_{i+f}-e} = 1.$$

This is equivalent to $\sum_{i=0}^{f-1} p^{f-i}(x_i + x_{i+f} - e) \equiv 0 \pmod{p^f - 1}$, however since $e < p-1$ and each $x_i \in [0, e]$ we have $x_i + x_{i+f} - e \in [-e, e] \subsetneq [-(p-1), p-1]$. The congruence must actually be equality since $\left|\sum_{i=0}^{f-1} p^{f-i}(x_i + x_{i+f} - e)\right| < p^f - 1$. Therefore $x_{i+f} = e - x_i$. Although it is possible that J might not satisfy the property $i \in J \iff i + f \notin J$, we can replace it with a set which does.

Proposition 2.2.8. *There is a set $J' \subset \{0, 1, \ldots, 2f-1\}$ and an allowable system of coefficients $x'_0, \ldots, x'_{2f-1}$ such that $i \in J' \iff i + f \notin J'$ and*

$$\psi_1|_{I_L} = \prod_{i \in J'} \eta_i^{c_{\pi(i)}} \prod_{i=0}^{2f-1} \eta_i^{x'_i}.$$

Proof. Define the sets $S = \{i \mid i < f, i \in J, i + f \in J\}$ and $T = \{i \mid i < f, i \notin J, i + f \notin J\}$. We say that $i \in J$ is a defect if $i \in S$ or $i \in T$. Since J is finite it suffices to produce a set J' and an allowable set of coefficients $x'_0, \ldots, x'_{2f-1}$ such that J' has fewer defects than J and $\psi_1|_{I_L} = \prod_{i \in J'} \eta_i^{c_{\pi(i)}} \prod_{i=0}^{2f-1} \eta_i^{x'_i}$.

Allowing for the possibility that J may have defects, the determinant condition $\psi^{q+1} = \overline{\epsilon}\,\overline{\chi}$ can be written

$$\prod_{i=0}^{f-1} \omega_i^{x_i+x_{i+f}-e} \prod_{i \in S} \omega_i^{c_i} \prod_{i \in T} \omega_i^{-c_i} = 1$$

Define $y_0, y_1, \ldots, y_{f-1}$ to be the exponents,

$$y_i = \begin{cases} x_i + x_{i+f} - e + c_i & i \in S \\ x_i + x_{i+f} - e - c_i & i \in T \\ x_i + x_{i+f} - e & i \notin S \cup T \end{cases}$$

Since each $y_i \in [-(2p-3), (2p-3)]$, we can write $y_i = a_i p - a_{i+1}$ for some $a_i \in \{0, \pm 1\}$ (the subscripts are taken mod f).

If $[j', j] \subseteq T$ then $[j', j] \not\subset J$ and $[j' + f, j + f] \not\subset J$. Given such an interval $[j', j] \subseteq T$ of maximal length, we define $J' := J \cup [j', j]$ and construct new coefficients x'_i. The details for the case where $[j', j] \subset J$ and $[j' + f, j + f] \subset J$ are almost identical (with the obvious change that $J' := J \setminus [j', j]$), so we treat the case of adding an interval from T to J.

Consider the case $J = \varnothing$. It is impossible that $x_i = 0$ for all i, since this would imply that ψ_1 is the trivial character, but ψ_1 has niveau 2. For the same reason not all of the x_i can be equal to e. Since there is no i for which $i \in J$ and $i + f \in J$, all of the a_i are 0 or -1. We claim that it is possible to choose j such that $x_{j-1} + a_j \in [0, e - 1]$. Assume by contradiction that there were no such j. Then for all j either $(x_{j-1}, a_j) = (e, 0)$, or $(x_{j-1}, a_j) = (0, -1)$. If $(x_{j-1}, a_j) = (e, 0)$ then we also have $(x_{j+f-1}, a_{j+f}) = (e, 0)$ since $a_j = a_{j+f} = 0$. But then $y_{j-1} = e - c_{j-1} > -(p - 1)$, so $a_{j-1} \neq -1$. However, if $a_j = 0$ then $(x_{j-2}, a_{j-1}) = (e, 0)$ as well. By induction $x_i = e$ for all i, which is a contradiction.

Choose j as in the claim. Define $J' = \{j, \ldots, j + f - 1\}$ and let

- $x'_i = x_i - c_i - pa_i + a_{i+1}$ for $i \in [j, j + f - 1)$,
- $x'_{j+f-1} = x_{j+f-1} - c_{j-1} - pa_{j-1}$,
- $x'_{j-1} = x_{j-1} + a_j$,
- $x'_i = x_i$ for all other i.

Allowability is automatic for the interior of the interval since there are no flips for these i and by construction $x'_i \in [0, e]$. By the choice of j we have $x'_{j-1} \in [0, e - 1]$, which implies $x'_{j+f-1} \in [1, e]$ (these are equivalent statements).

The case $J = \{0, 1, \ldots, 2f - 1\}$ is essentially the same and presents no new difficulties.

Now assume that there is a non-defective entry in J. Working with subscripts modulo $2f$ as usual, if T is nonempty let $[j', j] \subseteq T$ be a maximal interval (which is

to say $j'-1 \notin T$ and $j+1 \notin T$). Define $J' = J \cup \{j', \ldots, j\}$, and define

- $x'_i = x_i - c_i - pa_i + a_{i+1}$ for $i \in [j', j)$,
- $x'_j = x_j - c_j - pa_j$,
- $x'_{j'-1} = x_{j'-1} + a_{j'}$,
- $x'_i = x_i$ for all other i.

The characters ψ and ψ' agree by construction, what remains is to show that the new set of coefficients is allowable. First observe that for $i \in [j', j)$, allowability is not a concern since

$$x'_i = x_i - c_i - y_i = e - x_{i+f} \in [0, e]$$

and there are no restrictions on the boundary values since J' has no flip at i. The difficulty is to show the allowability of x'_j and $x'_{j'-1}$.

Note that if $a_i = 1$ then $i \in S$, and $a_i = -1$ implies $i \in T$.

We first show that x'_j is allowable in J'. The interval $[j', j]$ is maximal, so either $j+1 \in J$ or $j+f+1 \in J$. By definition $x'_j = e - x_{j+f} - a_{j+1}$, there are several ways in which it might not be allowable:

- If $x'_j = -1$ then $a_{j+1} = 1$ and $x_{j+f} = e$. But this would be impossible since $j+f \notin J$ and $j+f+1 \in J$ would necessitate $x_{j+f} \in [0, e-1]$.
- If $x'_j = 0$ and is not allowable, then $j+1 \notin J$ since adding j to J must create a flip. However, since $j+1 \notin T$ we must have $j+f+1 \in J$ and $x_{j+f} \in [0, e-1]$, which would contradict $a_{j+1} = 0$.
- If $x'_j = e$ then it is always allowable since the allowable range in J' will either be $[1, e]$ or $[0, e]$.
- If $x'_j = e+1$ then $a_{j+1} = -1$ which would contradict the maximality of $[j', j]$.

Allowability of $x'_{j'-1}$ is straightforward. Replacing $[j', j]$ by $[j'+f, j+f]$ if necessary, we can assume without loss of generality that $j'-1 \in J$. Then $x_{j'-1} \in [1, e]$ and $x'_{j'-1} \in [0, e]$ which is allowable in J'. □

Since we can always replace J with a set J' such that $i \in J' \iff i+f \notin J'$ we have proved the following theorem.

Theorem 2.2.9. *Let $\overline{\rho} : G_{F_\mathfrak{p}} \to \mathrm{GL}_2(k_E)$ be an irreducible representation which is the reduction mod p of a potentially Barsotti-Tate representation with coefficients in $\mathcal{O}_E$ and inertial type $1 \oplus \chi$. Then $\overline{\rho}|_{G_L} = \begin{pmatrix} \psi & 0 \\ 0 & \psi^{p^f} \end{pmatrix}$ and there are two possibilities:*

1. *If $e \geq p-1$ then ψ can be any character such that $\psi^{p^f+1} = \overline{\epsilon}\,\overline{\chi}$.*
2. *If $e < p-1$ then $\psi^{p^f+1} = \overline{\epsilon}\,\overline{\chi}$ and there exists a $J \subset \{0, \ldots, 2f-1\}$ with $i \in J$ if and only if $i+f \notin J$ such that*

$$\psi|_{I_\mathfrak{p}} = \prod_{i \in J} \eta_i^{c_{\pi(i)}} \prod_{i=0}^{2f-1} \eta_i^{x_i}$$

for an allowable list $x_0, \ldots, x_{2f-1}$, with $x_i + x_{i+f} = e$.

2.3 Local lifts

We retain the notation of section 2.2. Let $\overline{\mathcal{N}}$ be the rank 2 Breuil module over $(k \otimes k_E)[u]/(u^{e'p})$ with descent data from K to $F_\mathfrak{p}$ with generators $\overline{v}$ and $\overline{w}$ defined as follows:

For the reducible case let $J \subset \{0, \ldots, f-1\}$ (respectively $J \subset \{0, \ldots, 2f-1\}$ for the irreducible case) and set $\chi_i = \overline{\chi}$ if $i \in J$ and $\chi_i = 1$ otherwise. Choose an allowable list $\{x_0, \ldots, x_{f-1}\}$ (respectively $\{x_0, \ldots, x_{2f-1}\}$) for the set J and $\overline{\chi}$. This is equivalent to choosing $r_i \in [0, e']$ such that $\chi_{i+1} = \chi_i \eta_i^{r_i}$. In the reducible case we define $\chi_{i+f} = \{1, \overline{\chi}\} \setminus \{\chi_i\}$ as a pure formality; J is still a subset of $\{0, \ldots, f-1\}$ and the coefficients for x_i and r_i should still be read mod f. In the reducible case this notation is merely for convenience.

Set $r_i' = e - r_i$ and $\chi_i' = \chi_{i+f}$. With this notation, $\overline{\mathcal{N}}$ is defined as follows:

- $\mathrm{Fil}^1 \overline{\mathcal{N}}$ is generated by $\sum_{i=0}^{f-1} u^{r_i} e_i \overline{v}$ and $\sum_{i=0}^{f-1} u^{r_i'} e_i \overline{w}$ (respectively $\sum_{i=0}^{2f-1} u^{r_i} e_i \overline{v}$ and $\sum_{i=0}^{2f-1} u^{r_i'} e_i \overline{w}$).
- $\phi_1(u^{r_i} e_i \overline{v}) = (1 \otimes \gamma_i) e_{i+1} \overline{v}$ and $\phi_1(u^{r_i'} e_i \overline{w}) = (1 \otimes \gamma_i') e_{i+1} \overline{w}$.
- $\hat{g}(e_i \overline{v}) = (1 \otimes \chi_i(g))(e_i \overline{v})$ and $\hat{g}(e_i \overline{w}) = (1 \otimes \chi_{i+f})(e_i \overline{w})$

where $\gamma_i, \gamma_i' \in k_E^\times$. In the irreducible case let $\varphi \in \mathrm{Gal}(K/F_{\mathfrak{p}})$ be the element fixing ϖ and acting non-trivially on L, in the reducible case set $\varphi = 1$. When $\overline{\rho}$ is irreducible we impose the extra conditions:

- $i \in J \iff i + f \notin J$
- $r_i = r_{i+f}'$
- $\gamma_i = \gamma_{i+f}'$
- $\hat{\varphi}(\overline{v}) = \overline{w}$ and $\hat{\varphi}(\overline{w}) = \overline{v}$.

We will construct a strongly divisible module $\mathcal{N}$ with $\mathcal{O}_E$ coefficients and descent data from K to $F_{\mathfrak{p}}$ such that $\overline{\mathcal{N}}$ is the reduction mod $\mathfrak{m}_E$ of $\mathcal{N}$. The construction of $\mathcal{N}$ is almost identical to the construction given in section 6 of [GS11b], with the only differences coming from the relatively minor differences in the structure of the respective Breuil modules. Since there are few changes, the argument is reproduced here mostly for completeness.

First we recall our standard notation from section 2.2.

- If $\overline{\rho}$ is reducible, let $L = F_{\mathfrak{p}}$ and let K be the splitting field of $u^{p^f-1} - \pi_{\mathfrak{p}}$ over $F_{\mathfrak{p}}$. Let $\varpi \in K$ be an element such that $\varpi^{p^f-1} = \pi_{\mathfrak{p}}$.
- If $\overline{\rho}$ is irreducible, let L be the unramified quadratic extension of $F_{\mathfrak{p}}$ and let K be the splitting field of $u^{p^{2f}-1} - \pi_{\mathfrak{p}}$ over $F_{\mathfrak{p}}$. Let $\varpi \in K$ be an element such that $\varpi^{p^{2f}-1} = \pi_{\mathfrak{p}}$.

Let $E_0(u)$ be an Eisenstein polynomial for $\pi_{\mathfrak{p}}$ over $K_0 = W(k)[\frac{1}{p}]$, and set $E(u) = E_0(u^{p^f-1})$ or $E(u) = E_0(u^{p^{2f}-1})$ for the respective cases. Let e' be the absolute ramification index of K. We can write $E(u) = u^{e'} + pF(u)$ where $F(u)$ is polynomial in u^{p^f-1} (respectively $u^{p^{2f}-1}$) over $W(k)$ whose constant term is not divisible by p.

Lemma 2.3.1. *Let I be any ideal in $\mathcal{O}_E$, then $\operatorname{Fil}^1 \mathcal{N} \cap I\mathcal{N} = I \operatorname{Fil}^1 \mathcal{N}$.*

Proof. It suffices to check that $e_i \operatorname{Fil}^1 \mathcal{N} \cap e_i I\mathcal{N} = e_i I \operatorname{Fil}^1 \mathcal{N}$ for each i. There are two cases, in the first case we have $\chi_i = \chi_{i+1}$ and the lemma is established as part of the proof of Theorem 6.1 of [GS11b]. If $\chi_i \neq \chi_{i+1}$ the claim is follows from the proof of Theorem 6.5 of [GS11a]. In general this is a nontrivial step in the construction of strongly divisible modules with coefficients and descent data. □

Theorem 2.3.2. *There is a strongly divisible module $\mathcal{N}$ of inertial type $1 \oplus \chi$ with $\mathcal{O}_E$ coefficients and descent data from K to $F_{\mathfrak{p}}$ lifting $\overline{\mathcal{N}}$.*

Proof. Let $S = S_{K,\mathcal{O}_E}$ and define $\mathcal{N}$ to be the free S-module generated by v, w. Let $\tilde{e}_i \in W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ be idempotents lifting $e_i \in k \otimes_{\mathbb{F}_p} k_E$, and let $\tilde{\sigma}_i : K_0 \to E$ be a lift of σ_i. The map $\phi_1 : \mathcal{N} \to \mathcal{N}$ will depend on whether $\chi_i = \chi_{i+1}$, so $\operatorname{Fil}^1 \mathcal{N}$ necessarily depends on J. Let $\operatorname{Fil}^1 \mathcal{N}$ be the sum of $(\operatorname{Fil}^1 S)\mathcal{N}$ and submodules $\mathcal{M}_i \subset e_i\mathcal{N}$ defined as follows.

If $\chi_i = \chi_{i+1}$, let $g_i h_i$ be a factorization of $\tilde{\sigma}_i(E_0(u))$ in $\mathcal{O}_E[u]$ into monic polynomials. Since $E/\mathbb{Q}_p$ is Galois, the polynomial $\tilde{\sigma}_i(E_0(u))$ splits completely and we can choose g_i, h_i so that $\deg(g_i) = r_i/(p^f - 1)$ and $\deg(h_i) = r_i'/(p^f - 1)$ (respectively $r_i/(p^{2f} - 1)$ and $r_i'/(p^{2f} - 1)$ for the irreducible case). Define $G_i = g_i(u^{p^f-1})$ and $H_i = h_i(u^{p^f-1})$ (resp. $g_i(u^{p^{2f}-1})$ and $h_i(u^{p^{2f}-1})$); let $\mathcal{M}_i$ be the submodule generated by $(1 \otimes G_i(u))\tilde{e}_i v$ and $(1 \otimes H_i(u))\tilde{e}_i w$.

If $\chi_i \neq \chi_{i+1}$, let $y_i z_i$ be a factorization of $-p\tilde{\sigma}_i(F(u))$ in $\mathcal{O}_E[u]$ such that $y_i \in \mathfrak{m}_E$ and $z_i \in \mathfrak{m}_E[u]$, such a factorization is possible because p ramifies in E. Define $\mathcal{M}_i$ to be the S-module generated by $\tilde{e}_i(u_i^{r_i} v + (1 \otimes y_i)w)$ and $\tilde{e}_i((1 \otimes z_i)v + u^{r_i'} w)$.

We define descent data from K to $F_{\mathfrak{p}}$ on $\mathcal{N}$ by first defining the action of $g \in \mathrm{Gal}(K/L)$. Let $\tilde{\chi}_i$ be the Teichmüller lift of χ_i, define $\hat{g}(e_i v) = (1 \otimes \tilde{\chi}_i(g))e_i v$, and $\hat{g}(e_i w) = (1 \otimes \tilde{\chi}_{i+f}(g))e_i w$ so that either $\hat{g}(e_i v) = e_i v$ and $\hat{g}(e_i w) = (1 \otimes \tilde{\chi}(g))e_i w$ or the reverse depending on whether $i \in J$ or not. In the irreducible case we also define $\hat{\varphi}(v) = w$ and $\hat{\varphi}(w) = v$. Since $e_i \mapsto e_{i+f}$ under $\hat{\varphi}$, the descent data satisfies $\hat{\varphi}^{-1}\hat{g}\hat{\varphi} = \widehat{g^{p^f}} = \hat{g}$. Therefore the descent data extends to $\mathrm{Gal}(K/F_{\mathfrak{p}})$.

The definition of a strongly divisible module includes a map $\phi : \mathcal{N} \to \mathcal{N}$ which is ϕ semilinear on S such that $\phi_1 = \frac{1}{p}\phi|_{\mathrm{Fil}^1 \mathcal{N}}$ is well defined and equal to $\phi_1 : \overline{\mathcal{N}} \to \overline{\mathcal{N}}$ on the generic fiber of $\mathcal{N}$. If $\chi_i = \chi_{i+1}$ then we require that

$$\phi_1((1 \otimes G_i(u))e_i v) = \tilde{\gamma}_i e_{i+1} v$$
$$\phi_1((1 \otimes H_i(u))e_i w) = \tilde{\gamma_i}' e_{i+1} w$$

and if $\chi_i \neq \chi_{i+1}$ we must have

$$\phi_1(e_i(u_i^{r_i} v + (1 \otimes y_i)w)) = \tilde{\gamma}_i e_{i+1} v$$
$$\phi_1(e_i((1 \otimes z_i)v + u^{r_i'} w)) = \tilde{\gamma_i}' e_{i+1} w$$

where $\tilde{\gamma}_i$ and $\tilde{\gamma_i}'$ are lifts of γ_i and γ_i' to $\mathcal{O}_E^\times$. In the reducible case we require that $\tilde{\gamma}_{i+f} = \tilde{\gamma}_i$, in the irreducible case we require $\tilde{\gamma}'_{i+f} = \tilde{\gamma}_i$.

Recall that we defined $c = p\phi(E(u))^{-1} \in S^\times$, so if $\chi_i = \chi_{i+1}$ we can define

$$\phi(e_i v) = c^{-1}\phi(1 \otimes H_i(u))\tilde{\gamma}_i e_{i+1} v$$

$$\phi(e_i w) = c^{-1}\phi(1 \otimes G_i(u))\tilde{\gamma}'_i e_{i+1} w$$

If $\chi_i \neq \chi_{i+1}$ we define

$$\phi(e_i v) = c^{-1} e_{i+1}(u^{pr_i'}\tilde{\gamma}_i v - \phi(y_i)\tilde{\gamma}'_i w)$$
$$\phi(e_i w) = c^{-1} e_{i+1}(u^{pr_i}\tilde{\gamma}'_i w - \phi(z_i)\tilde{\gamma}'_i w)$$

Extending this map additively and ϕ-semilinearly to the rest of $\mathcal{N}$ defines a map $\phi_1 : \mathrm{Fil}^1 \mathcal{N} \to \mathcal{N}$ which lifts ϕ_1 of $\overline{\mathcal{N}}$. By construction $\mathcal{N}$ is a lift of $\overline{\mathcal{N}}$. The same argument employed in theorem 6.1 of [GS11b], to check that the strongly divisible module

$\mathcal{N}$ constructed there satisfies the remainder of the axioms for a strongly divisible module laid out in Definition 4.1 of [Sav05], applies to equally well to $(\mathcal{N}, \mathrm{Fil}^1 \mathcal{N}, \phi)$. In particular, they show that given $(\mathcal{N}, \mathrm{Fil}^1 \mathcal{N}, \phi)$ there is a unique monodromy operator $N : \mathcal{N} \to \mathcal{N}$ satisfying the remaining axioms of a strongly divisible module with coefficients and descent data. By lemma 2.3.1, $\mathrm{Fil}^1 \mathcal{N} \cap I\mathcal{N} = I \,\mathrm{Fil}^1 \mathcal{N}$ for any ideal $I \subset \mathcal{O}_E$.

We require proposition 2.1.3 to compute the type of the representation associated to $\mathcal{N}$. Recall that if ρ is a potentially Barsotti-Tate representation $G_F \to \mathrm{GL}_n(E)$ that becomes Barsotti-Tate over K then $D := D^K_{\mathrm{st},2}(\rho)$ is a weakly admissible (ϕ, N)-module (we assume that $K_0 \subset E$). Let $\mathcal{N}$ be a strongly divisible module with descent data over S contained in $S[1/p] \otimes_{K_0 \otimes_{\mathbb{Q}_p} E} D$ and let τ be the inertial type of ρ. By proposition 2.1.3 $\tau = \chi_1 \oplus \cdots \oplus \chi_n$ if and only if there is an S-basis $v_1, \ldots, v_n$ of $\mathcal{N}$ such that the descent data acts on $\mathcal{N}$ via $\hat{g} \cdot v_i = (1 \otimes \chi_i(g))v_i$ for all $g \in \mathrm{Gal}(K/L)$.

The representation associated to $\mathcal{N}$ is $\rho' := T^{F_{\mathfrak{p}}}_{\mathrm{st},2}(\mathcal{N})[1/p]$, which is potentially Barsotti-Tate. By applying proposition 2.1.3 to the basis $v' = \sum_{i \notin J} e_i v + \sum_{i \in J} e_i w$ and $w' = \sum_{i \in J} e_i v + \sum_{i \notin J} e_i w$ we deduce that ρ' has type $1 \oplus \chi$. Although $\det(\rho')$ may not be equal to a finite order character of order prime to p times the cyclotomic character, we can twist ρ' by an unramified character with trivial reduction to ensure that the lift is parallel potentially Barsotti-Tate.

The final step is to check that the labeled pairs of Hodge-Tate weights of ρ' are $\{0, 1\}$, but this follows from the same argument used at the end of theorem 6.1 of [GS11b].

□

Chapter 3

Global results

3.1 Notation

Let F be a totally real number field with a unique place $\mathfrak{p}$ lying above an odd prime p. Fix a uniformiser $\pi_{\mathfrak{p}}$ of $F_{\mathfrak{p}}$ and let $q = p^f$ denote the cardinality of the residue field k of $F_{\mathfrak{p}}$. Let $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_{\mathrm{p}})$ be an irreducible representation. In order to use modularity lifting theorems we assume throughout that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible. In the special case of $p = 5$ we need the additional hypotheses that the projective image of $\overline{\rho}$ is isomorphic to $\mathrm{PGL}(\mathbb{F}_5)$ and $[F(\zeta_p) : F] = 4$.

We recall from [GS11a] what it means for $\overline{\rho}$ to be modular of a given weight. Let D be a definite quaternion algebra with center F which is ramified at all infinite places of F and at a set of finite places Σ which does not contain $\mathfrak{p}$. Fix a maximal order $\mathcal{O}_D$ of D and for each place $v \notin \Sigma$ fix an isomorphism $(\mathcal{O}_D)_v \xrightarrow{\sim} M_2(\mathcal{O}_{F_v})$. For any finite place v let π_v denote a uniformiser of F_v.

Let $U = \prod_v U_v \subset (D \otimes_F \mathbb{A}_F^f)^\times$ be a compact open subgroup, with each $U_v \subset (\mathcal{O}_D)_v^\times$. We assume that $U_v = (\mathcal{O}_D)_v^\times$ for all $v \in \Sigma$.

Take A a topological $\mathbb{Z}_p$-algebra. Fix a continuous representation $\sigma : U_{\mathfrak{p}} \to \mathrm{Aut}(W_\sigma)$ with W_σ a finite free A-module. We view σ as a representation of U by letting U_v act trivially for $v \nmid p$. Fix a character $\psi : F^\times \setminus (\mathbb{A}_F^f)^\times \to A^\times$ such that for any finite place v of F, $\sigma|_{U_v \cap \mathcal{O}_{F_v}^\times}$ is multiplication by ψ^{-1}. We can view W_σ as a $U(\mathbb{A}_F^f)^\times$-module by letting $(\mathbb{A}_F^f)^\times$ act via ψ^{-1}.

Let $S_{\sigma,\psi}(U, A)$ denote the set of continuous functions

$$f : D^\times \setminus (D \otimes_F \mathbb{A}_F^f)^\times \to W_\sigma \tag{3.1.1}$$

such that for all $g \in (D \otimes_F \mathbb{A}_F^f)^\times$ we have

$$f(gu) = \sigma^{-1}(u)f(g) \text{ for all } u \in U,$$

$$f(gz) = \psi(z)f(g) \text{ for all } z \in (\mathbb{A}_F^f)^\times.$$

Choose U sufficiently small in the sense of ([BLGG11], section 2), so that the functor $W_\sigma \mapsto S_{\sigma,\psi}$ is exact in W_σ.

Let S be a set of finite places containing Σ, $\mathfrak{p}$, and the primes v for which $U_v \neq (\mathcal{O}_D)_v^\times$. Let $\mathbb{T}_{S,A}^{\mathrm{univ}} = A[T_v, S_v]_{v \notin S}$ be the polynomial ring in the formal variables T_v, S_v. Define a left action of $(D \otimes_F A_F^f)^\times$ on $S_{\sigma,\psi}(U, A)$ by $(gf)(z) = f(zg)$. We can view $S_{\sigma,\psi}(U, A)$ a $\mathbb{T}_{S,A}^{\mathrm{univ}}$-module by letting S_v act via the double coset $U \left(\begin{smallmatrix} \pi_v & 0 \\ 0 & \pi_v \end{smallmatrix}\right) U$ and T_v via $U \left(\begin{smallmatrix} \pi_v & 0 \\ 0 & 1 \end{smallmatrix}\right) U$. These are independent of the choices of π_v. We will write $T_{\sigma,\psi}(U, A)$ or $T_{\sigma,\psi}(U)$ for the image of $\mathbb{T}_{S,A}^{\mathrm{univ}}$ in $\operatorname{End} S_{\sigma,\psi}(U, A)$.

Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}_{S,A}^{\mathrm{univ}}$. We say that $\mathfrak{m}$ is in the support of (σ, ψ) if $S_{\sigma,\psi}(U, A)_\mathfrak{m} \neq 0$. Now let $\mathcal{O}$ be the ring of integers in $\overline{\mathbb{Q}}_p$, with residue field $\mathbb{F} = \overline{\mathbb{F}}_p$, and suppose that $A = \mathcal{O}$ in the above discussion, and that σ has open kernel. Consider a maximal ideal $\mathfrak{m} \subset \mathbb{T}_{S,\mathcal{O}}^{\mathrm{univ}}$ which is induced by a maximal ideal of $\mathbb{T}_{\sigma,\psi}(U, \mathcal{O})$. Then there is a semi-simple Galois representation $\overline{\rho}_\mathfrak{m} : G_F \to \mathrm{GL}_2(\mathbb{F})$ associated to $\mathfrak{m}$ which is characterized up to conjugacy by the property that if $v \notin S$ then $\overline{\rho}_\mathfrak{m}|_{G_{F_v}}$ is unramified, and if Frob_v is an arithmetic Frobenius at v then the trace of $\overline{\rho}_\mathfrak{m}(\mathrm{Frob}_v)$ is the image of T_v in $\mathbb{F}$.

We can now define what it means for a representation to be modular of some weight.

Definition 3.1.1. We say that $\overline{\rho}$ is modular of weight σ if for some D, S, U, ψ, and $\mathfrak{m}$ as above, with $U_\mathfrak{p} = \mathrm{GL}_2(\mathcal{O}_{F_\mathfrak{p}})$, we have $S_{\sigma,\psi}(U, \mathbb{F})_\mathfrak{m} \neq 0$ and $\overline{\rho}_\mathfrak{m} \cong \overline{\rho}$.

Assume from now on that $\overline{\rho}$ is modular of some weight, and fix D, S, U, ψ and $\mathfrak{m}$ as in the definition. Write $W(\overline{\rho})$ for the set of weights of $\overline{\rho}$.

The local representation $\sigma_{\mathfrak{p}}$ has a concrete description. Fix an embedding $\tau_0 : k_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{F}}_{\mathrm{p}}$, and let $\tau_i = \tau_0 \circ \mathrm{Frob}_{k_{\mathfrak{p}}}^{-i}$ for $0 \leq i < f$, where $\mathrm{Frob}_{k_{\mathfrak{p}}}$ is an arithmetic Frobenius on $k_{\mathfrak{p}}$. It is a standard fact that the isomorphism classes of irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(k_{\mathfrak{p}})$ are given by

$$\sigma_{(m_0,\ldots,m_{f-1}),(n_0,\ldots,n_{f-1})} := \bigotimes_{i=0}^{f-1} \left(\mathrm{Sym}^{m_i}(k_{\mathfrak{p}}{}^2) \otimes \det{}^{n_i}\right) \otimes_{k_{\mathfrak{p}},\tau_i} \overline{\mathbb{F}}_p \tag{3.1.2}$$

where $0 \leq m_i \leq p-1$, $0 \leq n_i \leq p-1$ and not all of the m_i are $p-1$ ([BL94], Proposition 1). Note that this differs slightly from the notation used in [BDJ10] and [Sch08].

3.2 Inertial local Langlands correspondence

In order to relate the set of Serre weights $W(\overline{\rho})$ of a modular representation and the set of inertial types τ such that $\overline{\rho}$ has a parallel potentially Barsotti-Tate lift of type τ we review some properties of the inertial local Langlands correspondence from Henniart's appendix to [BM02] and [EGH11].

Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$ and residue field k. Fix a reciprocity map $\tau_K : W_K \to K^\times$ given by sending a geometric Frobenius to a uniformiser of K, and write $\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$.

Definition 3.2.1. Let $\mathcal{A}_K(n)$ be the set of of isomorphism classes of smooth irreducible $\overline{\mathbb{Q}}_{\mathrm{p}}$-representations of $\mathrm{GL}_n(K)$. Let $\mathcal{G}_K(n)$ be the set of of isomorphism classes of n-dimensional Frobenius-semisimple[1] $\overline{\mathbb{Q}}_{\mathrm{p}}$-representations of the Weil-Deligne group of K.

Local Langlands correspondence: There is a canonical bijection

$$\mathrm{rec}_K : \mathcal{A}_K(n) \to \mathcal{G}_K(n).$$

[1] A representation of the Weil group (or more generally the Weil-Deligne group) of a non-archimedean local field is *Frobenius-semisimple* if Frobenius acts semisimply.

Let $\sigma \in \mathcal{G}_K(n)$ be an n-dimensional Frobenius-semisimple representation and let $\mathrm{rec}_K^{-1}(\sigma)$ be the corresponding representation of $\mathrm{GL}_n(K)$. If there are quasi-characters[2] $\chi_1, \ldots, \chi_n$ of $K^\times$ such that $\sigma = \chi_1 \circ \tau_K \oplus \cdots \oplus \chi_n \circ \tau_K$ then $\mathrm{rec}_K^{-1}(\sigma)$ is the principal series $\pi(\chi_1, \ldots, \chi_n)$. This correspondence is compatible with twists; if $\chi : K^\times \to \overline{\mathbb{Q}}_{\mathrm{p}}{}^\times$ then

$$\mathrm{rec}_K^{-1}((\chi \circ \tau_K) \otimes \sigma) = (\chi \circ \det) \otimes \mathrm{rec}_K^{-1}(\sigma).$$

Let $B_n(k)$ be the Borel subgroup of upper triangular matrices in $\mathrm{GL}_n(k)$. When restricted to the inertia subgroup I_K the (inertial) local Langlands correspondence take the following form:

Proposition 3.2.2 ([EGH11], Proposition 2.4.1 (ii))**.** *Suppose that π is an irreducible admissible smooth representation of $\mathrm{GL}_n(K)$ over $\overline{\mathbb{Q}}_{\mathrm{p}}$. If $\pi|_{\mathrm{GL}_n(\mathcal{O}_K)}$ contains the principal series representation*

$$\mathrm{Ind}_{B_n(k)}^{\mathrm{GL}_n(k)}(\chi_1 \otimes \cdots \otimes \chi_n)$$

for some distinct characters $\chi_i : k^\times \to \overline{\mathbb{Q}}_{\mathrm{p}}{}^\times$, then

$$\mathrm{rec}_K(\pi)|_{I_K} \cong \bigoplus_{i=1}^{n} \chi_i \circ \mathrm{Art}_K^{-1}$$

and $N(\pi) = 0$ for the monodromy operator on π.

By definition if ρ has inertial type $\chi_1 \oplus \chi_2$ then the action of the associated Weil-Deligne representation on inertia is $\chi_1 \oplus \chi_2$. Under the inertial local Langlands correspondence such ρ correspond to the principal series $\pi(\chi_1, \chi_2) := \mathrm{Ind}_{B(\overline{\mathbb{Q}}_{\mathrm{p}})}^{\mathrm{GL}_2(\overline{\mathbb{Q}}_{\mathrm{p}})}(\chi_1 \otimes \chi_2)$.

Let $\mathrm{JH}(\overline{V})$ denote the set of Jordan-Hölder factors of $\overline{V}$. By Lemma 3.4 of [GS11a], $\overline{\rho}$ is modular of some weight $\sigma \in \mathrm{JH}(\mathrm{LL}(\tau))$ if and only if $\overline{\rho}$ has a lift ρ such that $\rho|_{G_{F_{\mathfrak{p}}}}$ is parallel potentially Barsotti-Tate of type τ.

[2] In this context a quasi-character is a continuous homomorphism $K^\times \to \overline{\mathbb{Q}}_{\mathrm{p}}$.

3.3 Weight cycling

In this section suppose that $\overline{\rho}$ is modular and has a non-ordinary lift of type τ.

In the previous sections we explicitly described the inertial types τ of the form $1 \oplus \chi$ for which $\overline{\rho}|_{G_{F_\mathfrak{p}}}$ has a parallel potentially Barsotti-Tate lift. We will use a weight cycling argument to show that certain weights entail other weights, which strengthens our conclusion to show that $\overline{\rho}$ is modular of at least two potential weights.

The weights of a modular representation are limited–but not determined by–lifts of tame types, so we begin with a digression to talk about Jordan-Hölder factors of certain mod p representations. If V is a p-adic representation, then $\overline{V}$ will denote the semi-simplification of the reduction mod p of a lattice in V. Let $B(k)$ be the Borel subgroup of upper triangular matrices in $\mathrm{GL}_2(k)$. Given a pair of characters $\chi_1, \chi_2 : \mathbb{F}_{p^f} \to \overline{\mathbb{F}}_p$ we write $\chi_1 \otimes \chi_2 : B(\overline{\mathbb{Q}}_\mathrm{p}) \to \overline{\mathbb{F}}_p$ for the inflation of $\chi_1 \oplus \chi_2$ viewed as a representation of the torus. Let S denote the set of embeddings of $\mathbb{F}_{p^f} \hookrightarrow \overline{\mathbb{F}}_\mathrm{p}$ and let $\tilde{\tau}$ denote the Teichmüller lift of τ. Recall that $\sigma_{\vec{m}_J, \vec{n}_J}$ was defined in (3.1.2).

Proposition 3.3.1. *([Dia04] Proposition 1.1). Let* $V = \mathrm{Ind}_{B(\overline{\mathbb{Q}}_\mathrm{p})}^{\mathrm{GL}_2(\overline{\mathbb{Q}}_\mathrm{p})}(1 \otimes \prod_\tau \tilde{\tau}^{c_\tau})$ *with* $0 \leq c_\tau \leq p-1$ *for each* $\tau \in S$. *Then* $\overline{V} \sim \bigoplus_{J \subset S} \sigma_J$ *where* $\sigma_J = \sigma_{\vec{m}_J, \vec{n}_J}$ *with* $\vec{m}_J$ *and* $\vec{n}_J$ *defined as follows:*

$$m_{J,\tau} = \begin{cases} 0 & \text{if } \tau \in J \\ c_\tau + \delta_J(\tau) & \text{if } \tau \notin J \end{cases}$$

$$n_{J,\tau} = \begin{cases} c_\tau + \delta_J(\tau) - 1 & \text{if } \tau \in J \\ p - 1 - c_\tau - \delta_J(\tau) & \text{if } \tau \notin J \end{cases}$$

where $\delta_J(\tau)$ *is the characteristic function of* $J^{(p)} = \{\tau \circ \mathrm{Frob} \,|\tau \in J\}$. *Moreover the non-zero* σ_J *are inequivalent.*[3]

Definition 3.3.2. We say that a representation $\rho : G_{F_\mathfrak{p}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\mathrm{p})$ is *ordinary* if $\rho|_{I_\mathfrak{p}}$ is an extension of a finite order character by a finite order character times the cyclotomic character.

[3]The convention here is for $\mathrm{Sym}^{-1} = 0$.

Definition 3.3.3. Let τ be a tame inertial type, define $\sigma(\tau) := \overline{\mathrm{Ind}_{B(\overline{\mathbb{Q}}_p)}^{G(\overline{\mathbb{Q}}_p)}(\tau)}$.

The Jordan-Hölder factors of $\sigma(\tau)$ follow from proposition 3.3.1 and twisting.

Weight cycling arguments date back to Buzzard and state that if $\overline{\rho}$ is modular, then it must be modular of at least two weights. Previous weight cycling arguments tended to be context specific, but Emerton, Gee, and Herzig recently proved a much more general weight cycling result for n-dimensional representations. In their notation,

Proposition 3.3.4 ([EGH11], Proposition 2.3.1)**.** *Suppose that V is a weight and π is a smooth $G(F)$-representation. Suppose that $\mu \in X_*(T)_-$ and let $t = \mu(\varpi)$. Then we have the following commutative diagram, where the map on the right is induced by Frobenius reciprocity and the map on the top is injective.*

$$\begin{array}{ccc} (V \otimes_{\overline{\mathbb{F}}_p} \pi)^K & \hookrightarrow & \left(\mathrm{Ind}_{K \cap {}^tK}^K V^{N_{-\mu}(k)} \otimes \overline{\mathbb{F}}_p\right)^K \\ & \searrow^{\overline{T}_{\mu,\varpi}} & \downarrow \\ & & (V \otimes_{\overline{\mathbb{F}}_p} \pi)^K \end{array}$$

Specializing their result (and notation) to the context of this work we have the following.

Corollary 3.3.5. *Suppose that $\overline{\rho}$ is modular of weight σ and possess a non-ordinary parallel potentially Barsotti-Tate lift, then $\overline{\rho}$ is modular of at least two weights in $\sigma(\tau)$.*

Proof. By proposition 3.3.4 we have the following commutative diagram, where the map on the top is injective, the map on the right is surjective, but the map $\mathbb{T}$ fails to be injective because $\overline{\rho}$ has a non-ordinary lift.

$$\begin{array}{ccc} S(U,\sigma)_{\mathfrak{m}} & \hookrightarrow & S(U,\sigma(\tau))_{\mathfrak{m}} \\ & \searrow^{\mathbb{T}} & \downarrow \\ & & S(U,\sigma)_{\mathfrak{m}} \end{array}$$

From this we can conclude that $S(U,\sigma)_{\mathfrak{m}} \neq S(U,\sigma(\tau))_{\mathfrak{m}}$, so $S(U,\sigma')_{\mathfrak{m}} \hookrightarrow S(U,\sigma(\tau))_{\mathfrak{m}}$ for some $\sigma' \neq \sigma$. Hence $\overline{\rho}$ is also modular of weight σ'. □

3.4 The case $e = 2$, $f = 2$

For small e, f it is possible to give a more explicit description of the set of weights. For this section we impose the additional assumption that $[F : \mathbb{Q}] = 4$ and there is unique place $\mathfrak{p}$ of F lying above p.

Let $\overline{\rho}$ be an irreducible modular representation of G_F and let $e = f = 2$ for the ramification and inertial indices of $\mathfrak{p}$. We assume that the inertial type of $\overline{\rho}\,|G_{F_{\mathfrak{p}}}$ is $\tau = 1 \oplus \omega_0^{c_0}\omega_1^{c_1}$. We require that $\overline{\rho}|_{G_{F_{\mathfrak{p}}}}$ is generic in the sense that $c_i \in [2, p-3]$ for all i, and $p \geq 5$. Under these assumptions we able able to compute $W(\overline{\rho})$ in some cases.

Definition 3.4.1. Let $\chi_1, \chi_2 : I_{\mathfrak{p}} \to \overline{\mathbb{F}}_{\mathrm{p}}{}^{\times}$, define $I(\chi_1,\chi_2) := \mathrm{Ind}_{B(\overline{\mathbb{Q}}_{\mathrm{p}})}^{\mathrm{GL}_2(\overline{\mathbb{Q}}_{\mathrm{p}})}(\tilde{\chi_1} \otimes \tilde{\chi_2})$ where $\tilde{\chi_i}$ is the Teichmüller lift of χ_i.

Lemma 3.4.2. *The Jordan-Hölder factors of* $\overline{I(1,\omega_0^{c_0}\omega_1^{c_1})}$ *are*

$$\mathrm{JH}(\overline{I(1,\omega_0^{c_0}\omega_1^{c_1})}) = \left\{\sigma_{(0,0),(c_0,c_1)}, \sigma_{(c_0,c_1),(p-1-c_0,p-1-c_1)}, \sigma_{(0,c_1+1),(c_0-1,p-2-c_1)}, \sigma_{(c_0+1,0),(p-2-c_0,c_1-1)}\right\}$$

This is a special case of proposition 3.3.1.

Corollary 3.4.3. *More generally, the Jordan-Hölder factors of* $\overline{I(\omega_0^a\omega_1^b, \omega_0^c\omega_1^d)}$ *are*[4]*:*

$$\left\{\sigma_{(a,b),(c-a,d-b)}, \sigma_{(c,d),(p-1-c+a,p-1-d+b)}, \sigma_{(c_1,b),(p-2-c+a,d-b-1)}, \sigma_{(a,d+1),(c-a-1,p-2-d+b)}\right\}$$

where the subscripts for the twists are taken mod $p^2 - 1$.

Each weight in $\overline{I(1,\omega_0^{c_0}\omega_1^{c_1})}$ appears as a Jordan-Hölder factor for 4 types, however two of those types are $1 \oplus \omega_0^{c_0}\omega_1^{c_1}$ and $\omega_0^{c_0}\omega_1^{c_1} \oplus 1$, so by symmetry it suffices to check whether $\overline{\rho}$ has a lift of one specific type τ' per Jordan-Hölder factor. If any of the σ

[4] One can regard $\sigma_{(a,b),(c,d)}$ as $\det^{a+bp} \otimes \mathrm{Sym}^c(\mathbb{F}_{p^2}) \otimes (\mathrm{Sym}^d(\mathbb{F}_{p^2}) \circ \mathrm{Frob})$.

appear in the Jordan-Hölder factors of a type τ' for which $\overline{\rho}$ has no such lift then $\overline{\rho}$ cannot be modular of weight σ. For these τ' it is always possible to algebraically find suitable choices of J and x_i, however the x_i may not be allowable.

- If $\overline{\rho}|_{G_{F_\mathfrak{p}}}$ does not have a lift of type $\omega_0^{c_0}\omega_1^{p-1} \oplus \omega_0^{p-1}\omega_1^{c_1}$ then it is impossible to be modular of weight $\sigma_{(0,0),(c_0,c_1)}$.
- If $\overline{\rho}|_{G_{F_\mathfrak{p}}}$ does not have a lift of type $\omega_0^{c_0} \oplus \omega_1^{c_1}$ then it is impossible to be modular of weight $\sigma_{(c_0,c_1),(p-1-c_0,p-1-c_1)}$.
- If $\overline{\rho}|_{G_{F_\mathfrak{p}}}$ does not have a lift of type $\omega_0^{c_0-1}\omega_1^{p-1} \oplus \omega_1^{c_1+1}$ then it is impossible to be modular of weight $\sigma_{(0,c_1+1),(c_0-1,p-2-c_1)}$.
- If $\overline{\rho}|_{G_{F_\mathfrak{p}}}$ does not have a lift of type $\omega_0^{c_0+1} \oplus \omega_0^{p-1}\omega_1^{c_1-1}$ then it is impossible to be modular of weight $\sigma_{(c_0+1,0),(p-2-c_0,c_1-1)}$.

Since the relevant types are not of the form $1 \oplus \overline{\chi}$ we must revisit allowability in a more general context.

Lemma 3.4.4. *Under the hypotheses of this section, if $\overline{\rho}$ is modular of type $\omega_0^{d_0}\omega_1^{d_1} \oplus \omega_0^{c_0}\omega_1^{c_1}$ then it can be written*

$$\overline{\rho}|_{I_\mathfrak{p}} = \begin{pmatrix} \phi & 0 \\ 0 & \phi^{p^2} \end{pmatrix}$$

for

$$\phi = \left(\prod_{i\in J} \eta_i^{c_i}\right)\left(\prod_{i\notin J} \eta_i^{d_i}\right)\eta_0^{x_0}\eta_1^{x_1}\eta_2^{2-x_0}\eta_3^{2-x_1}$$

for a set $J \subset \{0,\dots,3\}$ such that $i \in J \iff i+2 \notin J$ and allowable $x_i \in [0,2]$.

Proof. This follows from proposition 2.2.8 and twisting. □

If $\overline{\rho}$ has inertial type $\omega_0^{d_0}\omega_1^{d_1} \oplus \omega_0^{c_0}\omega_1^{c_1}$ let J be the set of i for which $\overline{\chi}_i = \omega_0^{c_0}\omega_1^{c_1}$.

In the associated Breuil module we have

$$k_i = \begin{cases} c_0 + pc_1 & i \in J, i = 0, 2 \\ pc_0 + c_1 & i \in J, i = 1, 3 \\ d_0 + pd_1 & i \notin J, i = 0, 2 \\ pd_0 + d_1 & i \notin J, i = 1, 3. \end{cases}$$

Recall that $r_i \equiv p^{-1}k_{i+1} - k_i \pmod{p^2 - 1}$. We define $\lambda = d_0 - c_0 + p(d_1 - c_1)$ and write $[p\lambda] = d_1 - c_1 + p(d_0 - c_0)$. We have

$$r_i \equiv \begin{cases} \lambda & i \in J, i + 1 \notin J, i = 0, 2 \\ p\lambda & i \in J, i + 1 \notin J, i = 1, 3 \\ -\lambda & i \notin J, i + 1 \in J, i = 0, 2 \\ -p\lambda & i \notin J, i + 1 \in J, i = 1, 3 \\ 0 & \text{No flip at } i \end{cases}$$

The allowable range of x_0 is therefore

$$x_0 \in \begin{cases} [0, 1] & 0 \in J, 1 \notin J, \lambda > 0 \\ [1, 2] & 0 \in J, 1 \notin J, \lambda < 0 \\ [1, 2] & 0 \notin J, 1 \in J, \lambda > 0 \\ [0, 1] & 0 \notin J, 1 \in J, \lambda < 0 \\ [0, 2] & \text{No flip at } 0 \end{cases}$$

and the allowable range of x_1 is

$$x_1 \in \begin{cases} [0, 1] & 1 \in J, 2 \notin J, [p\lambda] > 0 \\ [1, 2] & 1 \in J, 2 \notin J, [p\lambda] < 0 \\ [1, 2] & 1 \notin J, 2 \in J, [p\lambda] > 0 \\ [0, 1] & 1 \notin J, 2 \in J, [p\lambda] < 0 \\ [0, 2] & \text{No flip at } 1 \end{cases}$$

Whether $\overline{\rho}$ has a lift of each of these specific types depends on the x_i and J. To determine if $\overline{\rho}$ has a lift of type τ' it suffices to determine if the associated J' and x_i' are allowable. If $\overline{\rho}$ has a lift of type $\omega_0^{a_0}\omega_1^{a_1} \oplus \omega_0^{b_0}\omega_0^{b_1}$ then we can write

$$\left(\prod_{i \in J} \eta_i^{c_i}\right)\left(\prod_{i \in J'} \eta_i^{-b_i}\right)\left(\prod_{i \notin J'} \eta_i^{-a_i}\right)\eta_0^{x_0 - x_0'}\eta_1^{x_1 - x_1'}\eta_2^{x_0' - x_0}\eta_3^{x_1' - x_1} = 1.$$

Remark 3.4.5. For some combinations of J and x_i the associated representations have lifts of all four types. In these case we can only assert that $\overline{\rho}$ is modular of two of the four specified weights. We have stayed within the case $c_i \in [2, p-3]$ since otherwise $\overline{\rho}$ has more lifts and some of the weights are not regular (in the sense of [Gee11b]). This case appears elusive for nearly all J and x_i.

Our arguments for the $e = 2$, $f = 2$ case rely on combinatorics to show that lifts of certain types are impossible in order to rule out weights. Since this is primarily accomplished through certain boundary restrictions on when the x_i may be 0 or e, these methods do not apply to most representations when $e > 2$. Weight cycling and eliminating weights through combinatorics are also unlikely to be successful when $f > 2$ since the number of Jordan-Hölder factors of $\overline{\mathrm{Ind}(1 \otimes \chi)}$ grows exponentially, so new approaches are required.

References

[BCDT01] Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor. On the modularity of elliptic curves over $\mathbb{Q}$. *J. of the AMS*, 14:843–939, 2001.

[BDJ10] Kevin Buzzard, Fred Diamond, and Frazer Jarvis. On Serre's conjecture for mod ℓ Galois representations over totally real fields. *Duke Math Journal*, 55(1):105–161, 2010.

[BL94] L. Barthel and R. Livné. Irreducible modular representations of GL_2 of a local field. *Duke Mathematical Journal*, 75(2):261–292, 1994.

[BLGG11] Thomas Barnet-Lamb, Toby Gee, and David Geraghty. Serre weights for rank two unitary groups. *Preprint*, 2011.

[BM02] Christophe Breuil and Ariane Mézard. Multiplicités modulaires et représentations de $\mathrm{GL}_2(\mathbb{Z}_p)$ et $\mathrm{Gal}(\mathbb{Q}_p)$ en $\ell = p$. *Duke Mathematical Journal*, 115:205–310, 2002.

[Bre00] Christophe Breuil. Groupes p-divisibles, groupes finis et modules filtr es. *Ann. of Math*, 152(2):489–549, 2000.

[Dia04] Fred Diamond. A correspondence between representations of local Galois groups and Lie-type groups. In *Proceedings of the LMS Durham Symposium on L-functions and Galois Representations*, 2004.

[Edi92] Bas Edixhoven. The weight in serre's conjectures on modular forms. *Inventiones Mathematicae*, 109:563–594, 1992.

[EGH11] Matthew Emerton, Toby Gee, and Florian Herzig. Weight cycling and Serre type conjectures for unitary groups. *Preprint*, 2011.

[Fon82] Jean-Marc Fontaine. Sur certains types de représentations p-adiques du groupe de galois d'un corps local; construction d'un anneau de Barsotti-Tate. *The Annals of Mathematics*, 115(3):529–577, May 1982.

[Gee11a] Toby Gee. Automorphic lifts of prescribed types. *Mathematische Annalen.*, 350(1):107–144, 2011.

[Gee11b] Toby Gee. On the weights of mod p Hilbert modular forms. *Inventiones mathematicae*, 184(1):1–46, 2011.

[GLS11] Toby Gee, Tong Liu, and David Savitt. Crystalline extensions and the weight part of Serre's conjecture. *Preprint*, 2011.

[GLS12] Toby Gee, Tong Liu, and David Savitt. The buzzard-diamond-jarvis conjecture for unitary groups. *Preprint*, 2012.

[GS11a] Toby Gee and David Savitt. Serre weights for mod p Hilbert modular forms: the totally ramified case. *J. Reine Angew. Math*, 2011.

[GS11b] Toby Gee and David Savitt. Serre weights for quaternion algebras. *Compositio Mathematica*, 147(Issue 04):1059–1086, 2011.

[Kis09] Mark Kisin. Modularity of 2-adic barsotti-tate representations. *Inventiones Mathematicae*, 178:587–634, 2009.

[KW09a] Chandrashekhar Khare and Jean-Pierre Wintenberger. Serre's modularity conjecture (I). *Inventiones Mathematicae*, 178:485–504, 2009. 10.1007/s00222-009-0205-7.

[KW09b] Chandrashekhar Khare and Jean-Pierre Wintenberger. Serre's modularity conjecture (II). *Inventiones Mathematicae*, 178:505–586, 2009. 10.1007/s00222-009-0206-6.

[Rib90] Kenneth A. Ribet. On modular representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ arising from modular forms. *Inventiones Mathematicae*, 100(2):431–476, 1990.

[Sav05] David Savitt. On a Conjecture of Conrad, Diamond, and Taylor. *Duke Mathematical Journal*, 1:141–197, 2005.

[Sav08] David Savitt. Breuil modules for Raynaud schemes. *Journal of Number Theory*, 128:2939–2950, 2008.

[Sch08] Michael M. Schein. Weights in Serre's conjecture for Hilbert modular forms: The ramified case. *Israel Journal of Math*, 166:369–391, 2008.

[Ser87] Jean-Pierre Serre. Sur les représentations modulaires de degré 2 de $Gal(\overline{\mathbf{Q}}/\mathbf{Q})$. *Duke Mathematical Journal*, 54(1):179–230, 1987.