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A COMPARISON OF TWO MULTIVARIATE CUMULATIVE SUM CONTROL CHART TECHNIQUES

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A COMPARISON OF TWO MULTIVARIATE CUMULATIVE SUM CONTROL CHART TECHNIQUES

by

Kathryn Schuler Korpela

A Thesis Submitted to the Faculty of the

DEPARTMENT OF SYSTEMS AND INDUSTRIAL ENGINEERING

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For the Degree of

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In the Graduate College

THE UNIVERSITY OF ARIZONA

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ABSTRACT

This study utilizes a Monte Carlo simulation to examine and compare the performance of two proposed multivariate cumulative sum control chart schemes for controlling the mean of a multinormal process. The study compares the performance of the proposed methods with multiple univariate Shewhart charts, a multivariate Shewhart chart and multiple univariate cumulative sum control charts.

The results indicate that one of the proposed multivariate cumulative sum control charts is superior to the other and that in many cases the superior method has certain advantages over the classical univariate and multivariate control chart techniques as well.

CHAPTER 1

INTRODUCTION

Control charts are statistical devices used primarily for the study and control of repetitive processes. At the basis of the theory of control charts is a differentiation of the causes of variation in quality. Statistical theory recognizes that the variation in the quality of a product belongs to two general categories: chance variations and assignable causes. Since chance variations behave in a random manner, future variations cannot be predicted from knowledge of past variations. However, variation produced by chance causes does follow statistical laws and knowledge of the behavior of chance variations is the foundation on which control chart analysis exists. If a group of data from a process is studied and it is found that its variation conforms to a statistical pattern that might reasonably be produced by chance causes, then it is assumed that no assignable causes are present and the process is said to be in control. Otherwise, it is concluded that one or more assignable causes are at work and the process is said to be out of control.

Hotelling is credited with recognizing that the quality of an item could depend on two or more correlated characteristics. Early attempts to monitor the quality of p characteristics employed the use of p univariate control charts. The most widely used of the univariate charts is the Shewhart chart which was developed in 1931. Numerous publications have dealt with the application of these charts and some modifications to the method, either in presentation or interpretation, have been proposed.

The concept of cumulative sum (cusum) control charts was introduced by Page [18] in 1954. This control chart method is based on Wald sequential schemes. Since its inception the cusum control chart has increased in popularity due to its well established advantages over Shewhart charts when there is a small to moderate shift from the process target value [6,12,21].

The shortcomings of the use of a series of univariate control charts running separately for each of the variables have been pointed out in publications which outline the appropriate multivariate procedures [2,3]. The univariate approach is generally not adequate because it ignores the correlations among the variables. On the other hand, multivariate techniques tend to shed more light on the process since they take into consideration the relationships and interdependence between the quality characteristics

In this study we use a Monte Carlo simulation to examine the performance of two proposed multivariate cumulative sum schemes for monitoring a process where each item it characterized by two quality characteristics. These schemes combine aspects of the cumulative sum scheme and various multivariate procedures in the hopes of improved performance over established control chart methods. These schemes are referred to as multivariate cusum #1 and multivariate cusum #2, and their simulated average run lengths will be compared to the theoretical average run lengths of multiple univariate Shewhart control charts, multiple univariate cusum charts and multivariate Shewhart charts.

The following chapter contains a review of control chart procedures and the proposed development of the two multivariate cusum control chart schemes. A discussion of the average run length as a measure of performance, the analytical procedures utilized to theoretically compute the average run lengths for the Shewhart and univariate cusum schemes and the details of the simulation study are discussed in Chapters 3 and 4. The results of the Monte Carlo simulation are presented and discussed in Chapter 5. Conclusions and directives for further study are presented in Chapter 6.

CHAPTER 2

CONTROL CHART METHODS

In this section we review the univariate and multivariate Shewhart and univariate cumulative sum (cusum) control charts. We also describe the proposed development of two multivariate cusum charts. For successive samples control chart techniques can be interpreted as repeated tests of signifigance of the form:

$$H_o: \quad \mu = \mu_o$$
$$\text{vs}$$
$$H_a: \quad \mu \neq \mu_o$$

where μ represents a process parameter whose true value is unknown and μ_o is the target value for the parameter [3,11]. In the multivariate case, μ would represent a vector of parameters.

In the study which follows it is assumed that the population standard deviation is a known constant. It should also be noted that while other more complex decision rules such as run tests may be used to determine if the process is in or out of control they are not considered here.

The Univariate Shewhart Chart

The univariate Shewhart control chart consists of examining samples of a fixed size at regular intervals of time.

A statistic (for our purposes, a sample mean) is calculated and plotted on the control chart. Corrective action is taken if the point falls outside of the control limits.

When there is only one normally distributed quality characteristic the univariate Shewhart control chart for the mean is of the form:

$$\text{UCL:} \quad \mu_o + Z_{\alpha/2}(\sigma_o/\text{SQRT}(n))$$

$$\text{CL:} \quad \mu_o$$

$$\text{LCL:} \quad \mu_o - Z_{\alpha/2}(\sigma_o/\text{SQRT}(n))$$

where μ_o and σ_o denote the standard values specified for the population mean and standard deviation of the distribution of the quality characteristic and $Z_{\alpha/2}$ is the upper $\alpha/2$ point of the standard normal distribution, that is, Prob.$(Z > Z_{\alpha/2}) = \alpha/2$.

In practice a random sample of size n parts is obtained and the mean of the quality characteristic measurements on those parts is computed. If the plotted value of $\bar{X}$ falls within the control limits, then the process is deemed in control, otherwise, one or more assignable causes are sought to explain the unusual variation.

The Multivariate Shewhart Chart

The multivariate analogue of the Shewhart chart is commonly referred to as the chi-square chart. A column vector $\underline{X}$ of p components $X_1, X_2, \ldots, X_p$ is said to have a p-variate nonsingular normal distribution if its probability distribution function (pdf) is of the form:

$$(2\pi)^{-p/2} \, |\underline{\sigma}|^{-1/2} \exp[-1/2 \, (\underline{X} - \underline{\mu})' \, \underline{\sigma}^{-1} (\underline{X} - \underline{\mu})]$$

$$-\infty < X_i < \infty \qquad (i = 1, 2, \ldots, p)$$

where $\underline{\mu}$ and $\underline{\sigma}$ are the parameters of the distribution, $\underline{\mu}$ is a column vector of elements $\mu_i (i = 1,2,\ldots,p)$ such that each μ_i is finite and the variance-covariance matrix $\underline{\sigma} = \{\sigma_{ij}\}$ is a positive definite symmetric matrix of order p. The notation $\underline{X} \sim N_p(\underline{\mu}, \underline{\sigma})$ indicates that the random vector $\underline{X}$ has a p-variate non-singular normal distribution with parameters $\underline{\mu}$ and $\underline{\sigma}$.

If we consider a process where the quality of p characteristics are to be monitored, then we essentially wish to perform a likelihood ratio test of the form:

$$H_o : \underline{\mu} = \underline{\mu}_o$$

$$\text{vs.}$$

$$H_a : \underline{\mu} \neq \underline{\mu}_o .$$

The test specifies that the null hypothesis is rejected if

$$n(\bar{\underline{X}} - \underline{\mu}_o)' \, \underline{\sigma}_o^{-1} \, (\bar{\underline{X}} - \underline{\mu}_o) > \chi^2_{p,\alpha}$$

where $\bar{\underline{X}}$ denotes the p by 1 vector of samples means and $\chi^2_{p,\alpha}$ is the upper α 100%-age point of the chi-square distribution with p degrees of freedom [4].

In practice the statistic is calculated and plotted on a control chart with an upper control limit of $\chi^2_{p,\alpha}$. If the point plots above the upper contol limit, the process mean is deemed out of control and the assignable causes of the variation are sought.

The Univariate Cusum Chart

A shortcoming of the Shewhart-type control chart is that although the results of previous samples are recorded on the chart, none of the previous results are used by the process inspection rule directly. The cumulative sum chart was developed by Page in 1954 [18] in an attempt to make direct use of this information. Prior to its development quality control engineers were trying to use the Shewhart-type chart intelligently by giving some weight to runs of results above or below the mean. A number of more intricate rules using runs of results outside of warning limits were also devised, but these rules are inefficient compared with control schemes based on cusum charts [13].

The cusum chart has several distinct advantages over the Shewhart chart. These advantages are:

1. It is at least equally effective at less expense.
2. It picks up a sudden and persistent change in the process average more quickly.
3. It locates the time of the change more sharply.
4. A change in quality can be seen much more easily by visual inspection.

Cusum schemes can be devised to detect only upward or downward shifts in the process mean or devised to detect shifts in both directions simultaneously. These schemes are referred to as one-sided and two-sided schemes respectively. Our discussion will be confined to one-sided schemes since a two-sided scheme is just a combination of two one-sided schemes.

In a one-sided univariate cusum scheme for detecting increases in the process mean, we consider a normally distributed quality characteristic X whose mean is μ and whose variance is σ^2. For each sample of n observations $X_1, X_2, \ldots, X_n$ collected at time t, calculate the sample mean, $\bar{X}_t$ and the cumulative sum

$$c_t = \sum_{i=1}^{t} (\bar{X}_i - \mu_o).$$

Note that at time t:

$$c_t \sim N(0, t\sigma^2/n)$$

and

$$Z = [c_t/\mathrm{SQRT}(\mathrm{VAR}(c_t))] \sim N(0,1).$$

Therefore, a variation of Page's original scheme may operate by plotting Z at time t on a control chart with an upper control limit of $Z_{\alpha/2}$. If the point plots above the UCL, then the process is deemed out of control.

In order to prevent the accumulation of a large backlog of results when the process is in control the cusum is zeroed out periodically. In practice when the cusum is being used to detect shifts above the mean it is zeroed out and restarted when it becomes negative. Likewise, if the chart is used to detect shifts below the mean the control limit can be taken to be $-Z_{\alpha/2}$ and the cusum c_t would be zeroed out whenever the value of c_t becomes positive.

Multivariate Cusum Charts

Cumulative sum charts have many advantages over the Shewhart chart and because of these advantages it is only natural to investigate whether the principles governing the univariate case can be extended to the multivariate case.

In the development of the two multivariate cusum charts we consider a vector of p quality characteristics $\underline{X}$ with probability distribution function $N_p(\underline{\mu}, \underline{\sigma})$ where $\underline{\sigma}$ is known. At a given time t, a sample of n multivariate observations on $\underline{X}$ (consisting of pn measurements) is taken and the vector of sample means is calculated as

$$\bar{\underline{X}}_t = (1/n) \sum_{i=1}^{n} \underline{X}_i .$$

The two proposed multivariate cumulative sum charts are to be referred to subsequently as multivariate cusum #1 and multivariate cusum #2. Multivariate cusum #1 was first proposed by Pignatiello and Kasunic in 1984 [19]. The development of the multivariate cusums is outlined in the following sections.

Multivariate Cusum #1

In the case of multivariate cusum #1 we calculate the multivariate cusum as

$$\underline{C}_t = \sum_{i=1}^{t} (\bar{\underline{X}}_i - \underline{\mu}_o) .$$

Since $\underline{C}_t$ may be written as

$$\underline{C}_t = \sum_{i=1}^{t} \bar{\underline{X}}_i - \sum_{i=1}^{t} \underline{\mu}_o = \sum_{i=1}^{t} \bar{\underline{X}}_i - t\underline{\mu}_o$$

$(1/t)\ \underline{C}_t$ is the mean accumulated vector difference between the sample average at time t and the standard value for the mean. We note that since $\underline{C}_t \sim N_p(\underline{0}, t\underline{\sigma}/n)$ the statistic

$$\chi^2 = [n\underline{C}_t{}' \ \underline{\sigma}^{-1} \ \underline{C}_t]/t$$

has a chi-square distribution with p degrees of freedom. This statistic represents the square of the distance of the accumulated sample average vector from its target. The multivariate cusum chart #1 operates by plotting the statistic χ^2 at time t on a control chart with an upper control limit of $\chi^2_{p,\alpha}$. If the point plots above the UCL then the process is deemed out of control.

Multivariate Cusum Chart #2

As an alternative to the multivariate cusum above one could consider

$$d_i = n(\bar{\underline{X}}_i - \underline{\mu}_o)' \ \underline{\sigma}^{-1} \ (\bar{\underline{X}}_i - \underline{\mu}_o)$$

which has a chi-square distribution with p degrees of freedom. The value d_i represents the square of the distance of the i^{th} sample mean from the target value of $\underline{\mu}_o$. The statistic

$$D_t = \sum_{i=1}^{t} d_i$$

which has a chi-square distribution with tp degrees of freedom could also be used as a multivariate cusum statistic on a control chart with an upper control limit of $\chi^2_{tp,\alpha}$. If the point plots above the UCL the process is deemed out of control.

It should be noted that as in the univariate case, the multivariate cusums need to be zeroed out periodically, but as the best method for zeroing out these cusums was not obvious this was studied as a design consideration.

The next sections contain a discussion of run length as a measure of control chart performance and a description of the average run length simulation used to compare the two multivariate cusum charts with each other and with the other methods discussed in this section.

CHAPTER 3

THE RUN LENGTH AS A MEASURE OF PERFORMANCE

In 1950, Aroian and Levene [5] considered the type of measures that one should use to assess the statistical properties of process inspection schemes. If there is an abrupt change in product quality we need to know the distribution of the amount produced by the process before the change is noticed by the inspection scheme we are using. This distribution could be used to do a detailed study of the costs involved in any particular case, but for general comparisons of inspection schemes, a more direct method of comparison is preferable.

If the process is operating at a constant rate and samples are taken at a constant rate, the average amount produced by the process before corrective action is demanded by the inspection scheme is proportional to the average number of samples taken before action is demanded by the inspection scheme. The average number of samples taken before action is demanded by the inspection scheme is commonly referred to as the average run length (ARL). The ARL(δ) is defined to be the average number of samples taken from the process to detect a shift in the process average from $\mu = \mu_o$ to $\mu = \mu_o + \delta$. If the mean remains at the control value, μ_o, then the average run length should be large so that the frequency of false alarms is low and if the mean shifts from the control value the procedure should signal quickly after the change to reduce the amount of poor quality product produced by the process before corrective action is taken.

The ARL may be calculated analytically for the Shewart and the univariate cusum schemes. However, the calculation of the ARL for cusum schemes is complicated by the fact that successive cusums are correlated.

Let us first consider the ARL for a univariate Shewhart scheme. The number of samples each of n items which are examined before action is demanded by the sampling scheme is commonly referred to as the run length (RL). The run length is a geometric variable with probability function:

$$P(RL = k) = P_a^{k-1}(1-P_a) \qquad k = 1, 2, \ldots$$

where P_a is the probability that a given sample falls between the control limits [11]. Thus,

$$E(RL) = 1/(1-P_a).$$

This analysis can be extended to the multivariate case by using a central χ^2 approximation to a noncentral χ^2 distribution with noncentrality parameter λ [1]. As in the case of the univariate Shewhart the expected run length is $1/(1-P_a)$ where P_a is the probability that a given sample falls between the control limits. When the process average shifts from μ_o to $\mu_o + \delta$ the noncentrality parameter λ is equal to δ^2 and the approximation of the run length is as follows:

$$\mathrm{Prob}[\chi^{2'} <= CL \mid \lambda, v] \sim \mathrm{Prob}[\chi^2_{v*} <= CL/(1 + b)]$$

where v represents the degrees of freedom for the noncentral distribution $\chi^{2'}$, CL represents the control limit, $a = v + \lambda$, $b = \lambda/(\lambda + v)$ and $v^* = a/(1 + b)$.

The classical method of studying the average run length of the univariate cusum control chart has been to regard the scheme as a sequence probability ratio tests . A different approach has been proposed by Brook and Evans [8] in which the set of all possible values the cusum can assume are discretized and then treated as a Markov chain. The transition probability matrix for this chain is obtained and then the properties of the transition matrix are used to determine the average run length for the scheme. This method may be used with any discrete distribution and also as an accurate approximation with any continuous distribution.

The Markov chain approach was used by the Department of Mathematical Sciences of the IBM Thomas J. Watson Research Center in the development of an APL software package, DARCS, for the design, analysis and implementation of cusum-Shewhart control schemes [24]. This software package was utilized to approximate the average run lengths for the univariate cusum schemes. A more detailed discussion of this method is beyond the scope of this paper, however, it is discussed at length in the references mentioned above.

CHAPTER 4

DISCUSSION OF THE SIMULATION STUDY

Almost all of the discussions on cusum schemes since Page's original article have dealt primarily with the topic of how one may best find an approximation of the ARL [8,12,14,16,22]. Both Reynolds [20] and Van Dobben de Bruyn [21] suggest that computer simulation methods are both easier and sufficient to estimate the ARL. Consequently, we used a Monte Carlo simulation approach in this study to compute estimates of the ARL's for the multivariate cusum methods and compare these results with the analytical results from the classical control chart procedures.

Simulation denotes a computer-based numerical technique for the experimental study of a stochastic or deterministic process over time. The designation Monte Carlo is appropriate for any numerical procedure utilizing random or pseudo-random numbers.

The control charts for the simulated process were designed to maintain control of the process mean at a nominal level denoted by H_o. A shift in the process mean was purposely induced away from the nominal level to an out of control level H_a and the run length was estimated to be the number of samples taken before this shift was detected by the various control chart techniques. The number of replications performed in the simulation was determined as a function of precision and the cost of computer time. There was a tradeoff between practicality and precision to be considered here so pilot runs were conducted. It was determined that 61 replications of 100 runs each would be satisfactory for our purposes. In other words the run length for the process was simulated 100 times and an average was calculated for these 100 runs. This process was repeated 61 times. Then, a point estimate for the mean of this distribution is just the average of the 61 averages and a 95% confidence interval for the mean can be calculated as:

$$\bar{X} \pm t_{60,0.975}\mathrm{SQRT}(s^2/n)$$

where $t_{60,0.975}$ is the upper critical point for the t distribution with 60 degrees of freedom.

We simulated a process where the sampled manufactured parts were monitored on the basis of two quality characteristics. Let $\underline{X}_n = (X_{1n}, X_{2n})'$ denote the 2 by 1 vector of quality characteristic measurements made on the n^{th} part. We assumed that the successive $\underline{X}_n$ are independent identically distributed bivariate normal random vectors with known covariance matrix $\underline{\sigma}$. Without loss of generality, we took the nominal level H_o for the process to be the zero vector. The variance-covariance matrix is of the form:

$$\underline{\sigma} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

Specifically, the simulation considers the case where $\sigma_1 = \sigma_2 = 1$ so that:

$$\underline{\sigma} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

Thus, under H_0 the conditions are that $\underline{X}$ are i.i.d. $N_2(\underline{0},\underline{\sigma})$.

The out of control conditions that we induced differed from the nominal conditions only in the mean values. For each of the process control detection schemes the ARL under H_0 was calibrated to be 200 or as in the case of the simulated average run lengths so that 200 was covered by a 95% confidence interval on the mean run length. We then evaluated the performance of the different control chart methods over the parameter space when the shift from the process nominal value was a sudden and persistent shift of moderate to large magnitude.

The theoretical run lengths for the Shewhart and univariate cusum charts were calculated using the methods described in the previous section. In the case of the univariate control charts it was assumed that in a process with two quality characteristics there would be a control chart monitoring each of the quality characteristics. Therefore, the average run lengths for these methods were based on the assumption that there were two control charts running simultaneously. It should also be noted that in the univariate charts any correlation between the quality characteristics was ignored.

The theoretical run lengths for the two univariate Shewhart charts running simultaneously were calibrated so that

$$E(RL) = 1/(1-P_a) = 200$$

or $P_a = 0.005$. The critical value of the χ^2 distribution was then calculated such that:

$$Pr(\text{One is out}) + Pr(\text{Second is out}) - Pr(\text{Both out}) = 0.005.$$

Thus, the critical value of the χ^2 distribution with two degrees of freedom is 3.02 and the average run length was then calculated to be:

$$ARL = 1/(Pr(\chi^2_1 > 3.02 \mid \mu = \mu_1) + Pr(\chi^2_2 > 3.02 \mid \mu = \mu_2))$$

where μ_1 and μ_2 represent the values that each of the quality characteristics' means have shifted to under the alternate hypothesis.

The univariate cusum charts were calibrated so that the average run length of each of the individual charts under the null hypothesis was 400. Therefore, under the null hypothesis the average run length of the two charts running simultaneously was 1/(1/400 + 1/400) or 200 [11]. The univariate cusums were optimized for three cases involving shifts to alternate hypotheses of 1, 2 and 3 using the DARCS program described previously. The DARCS program calibrated the univariate cusums to average run lengths of 400 each by setting the decision interval, h, to a value that corresponded to the control limit which gives a Type I error of 0.0025. They were then optimized by choosing a reference value, $k > 0$, such that the average run lengths for the various alternate hypotheses were optimized. The ARL of the two univariate cusum charts running simultaneously was then approximated as:

$$ARL = 1/(1/ARL_1 + 1/ARL_2)$$

where ARL_1 and ARL_2 represent the average run lengths of the individual cusum charts. It should be noted that preliminary studies indicated that the approximated theoretical cusum results agreed well with simulated data and these approximations were therefore used in the comparison of methods.

The method used for zeroing out the multivariate cusums was chosen based on the results of a preliminary study which indicated that the chosen method performed the best in that the average run lengths were the smallest for all values of μ. The cusums were zeroed out when the $[\text{cusum} - T(v + \delta^2/2)]$ was less than or equal to zero where cusum represents the appropriate statistic for each of the multivariate cusums, T represents the number of χ^2 statistics in the cusum, v represents the degrees of freedom and δ was used as a parameter of the experiment. In the case of multivariate cusum #1 T is always equal to 1 and in multivariate cusum #2 T is equal to t, the number of chi-square statistics in the cusum.

The simulation study considered various alternate hypotheses and correlation coefficients and the distance between the target value and a given alternate hypothesis was considered to be:

$$\text{Distance} = \text{SQRT}((\underline{\mu}_o - \underline{\mu}_a)' \; \underline{\sigma}^{-1} \; (\underline{\mu}_o - \underline{\mu}_a)).$$

The relationship between the actual shift in the means of the two quality characteristics that we studied and the true distance of the shift is shown in Table 1. It is interesting to note that in the case where the correlation coefficient equals 0.0 shifts of equal distances are represented by concentric circles about the mean as indicated in Figure 1. Therefore, a shift in the mean to an alternate hypothesis of (0.5,0.0) is equidistant to shifts with means of (0.0,0.5), (-0.5,0.0) and (0.0,-0.5). If the correlation coefficient is not equal to zero then shifts of equal distance are represented by concentric ellipses about the mean as shown in Figure 2. For example, when the correlation coefficient is equal to 0.5, a shift in the mean to an alternate hypothesis of (2,1) is equidistant to shifts with means of (1,2), (-1,1), (-2,-1), (-1,-2) and (1,-1). Multivariate techniques such as the multivariate Shewhart and multivariate cusums detect shifts of equal distance equally well regardless of the direction of the shift in the mean. Thus, without loss of generality we can consider shifts in only one direction in the mean of a single quality characteristic and effectively consider shifts in every direction in one or more of the quality characteristics.

The conditions which were considered in the simulation study are shown in Table 2. These conditions include an analysis

Table 1. Distance of the shift in the means of the quality characteristics for various correlation coefficients.

		ρ	
Shift	0.0	0.5	0.9
(0.0,0.0)	0.0000	0.0000	0.0000
(0.5,0.0)	0.5000	0.5774	1.1471
(1.0,0.0)	1.0000	1.1547	2.2942
(1.5,0.0)	1.5000	1.7321	3.4412
(2.0,0.0)	2.0000	2.3094	4.5883
(2.5,0.0)	2.5000	2.8868	5.7354
(3.0,0.0)	3.0000	3.4641	6.8825
(3.5,0.0)	3.5000	4.0415	8.0296
(4.0,0.0)	4.0000	4.6188	9.1766

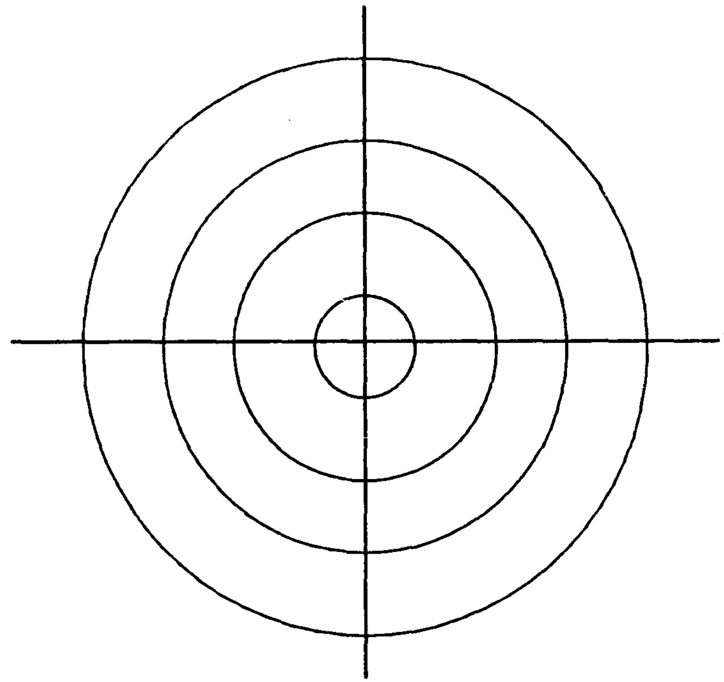

Figure 1: Concentric circles representing equal distances zero correlation.

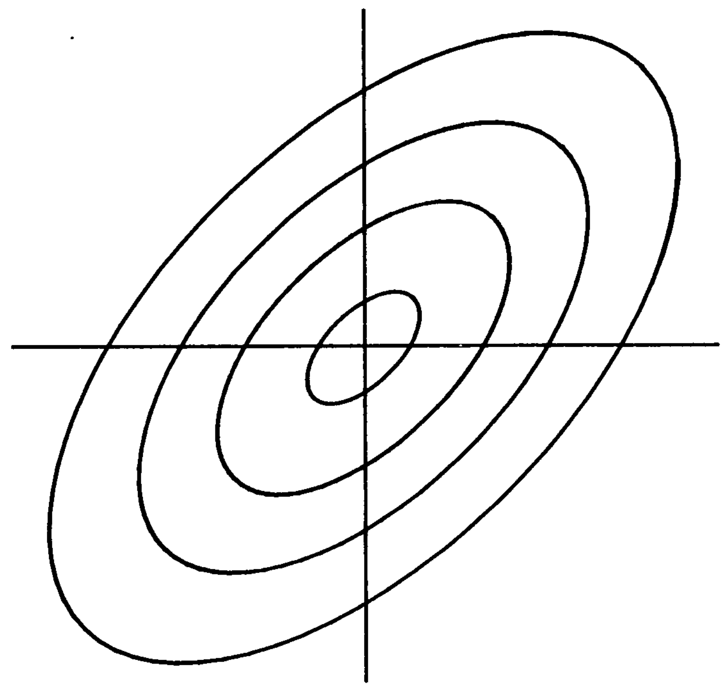

Figure 2: Concentric ellipses representing equal distances nonzero correlation.

Table 2. Conditions considered in the Monte Carlo simulation.

Size of Shift X_1	X_2	Correlation Coefficient	Delta
0.5	0.0	0.0	0
1.0	0.0	0.5	1
1.5	0.0	0.9	2
2.0	0.0		3
2.5	0.0		
3.0	0.0		
3.5	0.0		
4.0	0.0		

of the effect of various amounts of correlation between the quality characteristics, several values of δ in the zeroing out rule and a number of alternate hypotheses representing moderate to large shifts in the mean values. Various combinations of these conditions were considered in the simulation.

The program used in the simulation is written in Fortran/77 code and appears in Appendix A. Table 3 lists the primary functions of each of the program routines. Appendix A also contains a short program to convert the results of the program into a summary file containing the point estimate of the mean run length, a 95% confidence interval for the mean run length and its standard deviation.

The results of the simulation study are presented and discussed in the next section. This discussion includes both tabular and graphic presentation of the data as well as a statistical evaluation of the results.

Table 3. Description of the program routines.

Routine	Function
ARL	Input program parameters, generate random vectors, write results to output file.
MUSUM1	Evaluate data using multivariate cumulative sum #1 technique.
MUSUM2	Evaluate data using multivariate cumulative sum #2 technique.
TRPNRM	Generates unit normal deviate by Ahrens and Dieter composition method. Area under the normal curve is divided into 5 different areas [7].
UNIF	Generates uniform (0,1) random numbers [7].
DCHISQ	Computes the cumulative distribution function for a chi-square random variable in double precision [9].
DNML	Computes the cumulative distribution function for a normal random variable in double precision [9].

Chapter 5

RESULTS AND DISCUSSION

The purpose of this chapter is to present and discuss the results of the Monte Carlo simulation. The results of the Monte Carlo simulation are given in tabular form (see Tables 4 - 17) and graphical form (see Figures 3 - 15). The statistical evaluation of the significance of the results is also presented in this chapter (see Tables 18 - 20).

The actual simulated average run lengths and their respective standard deviations are shown in Tables 4 - 17. It should be noted that the values of α' shown in these tables are not the probability of type I error but rather a calibration constant used in the simulation to obtain a probability of type I error equal to 0.005 under the null hypothesis.

Figures 3 - 15 are graphs of the average run length vs. the actual shift in the means of the two quality characteristics. Each of the points plotted on the graphs represents the average number of samples taken to detect a shift in the mean values to the indicated alternate hypothesis. These graphs illustrate how each of the different control chart methods perform under the various conditions considered both on an individual method by method basis and when they are compared to one another. The graphical presentation of the data provides a quick indication of the relative efficiency of each of the methods under the specified conditions.

The theoretical run lengths for two univariate Shewhart charts running simultaneously are presented in Table 4 and graphed in Figure 3. The calibration point was purposely left off the graph in order to improve the resolution of the graph.

The results for the multivariate Shewhart chart are presented in Table 5 and graphed in Figure 4. The average run lengths were calculated for each of the three correlation coefficients and each of the three lines on the graph represent one of the coefficients. As the correlation coefficient increases in absolute value the multivariate Shewhart method responds more quickly to shifts in the mean.

The theoretical run lengths for the two univariate cumulative sum charts running simultaneously are presented in Table 6 and graphed in Figure 5. The univariate cusums were optimized for the three alternate hypotheses of (1,0), (2,0) and (3,0) using the DARCS program discussed in a previous section. The values of h and k for the optimized cusums are included in the table. It is interesting to note that the univariate cusum optimized for (1,0) has the best run length for a shift in the mean from (0,0) to (1,0), the cusum optimized for (2,0) has the best run length for a shift in the mean from (0,0) to (2,0) and so on. Each line on the graph represents the average run length

Table 4. Theoretical average run lengths for two univariate Shewhart charts running simultaneously.

Shift	Average Run Length
(0.0,0.0)	200.000
(0.5,0.0)	140.219
(1.0,0.0)	43.562
(1.5,0.0)	15.263
(2.0,0.0)	6.446
(2.5,0.0)	3.303
(3.0,0.0)	2.027
(3.5,0.0)	1.435
(4.0,0.0)	1.194

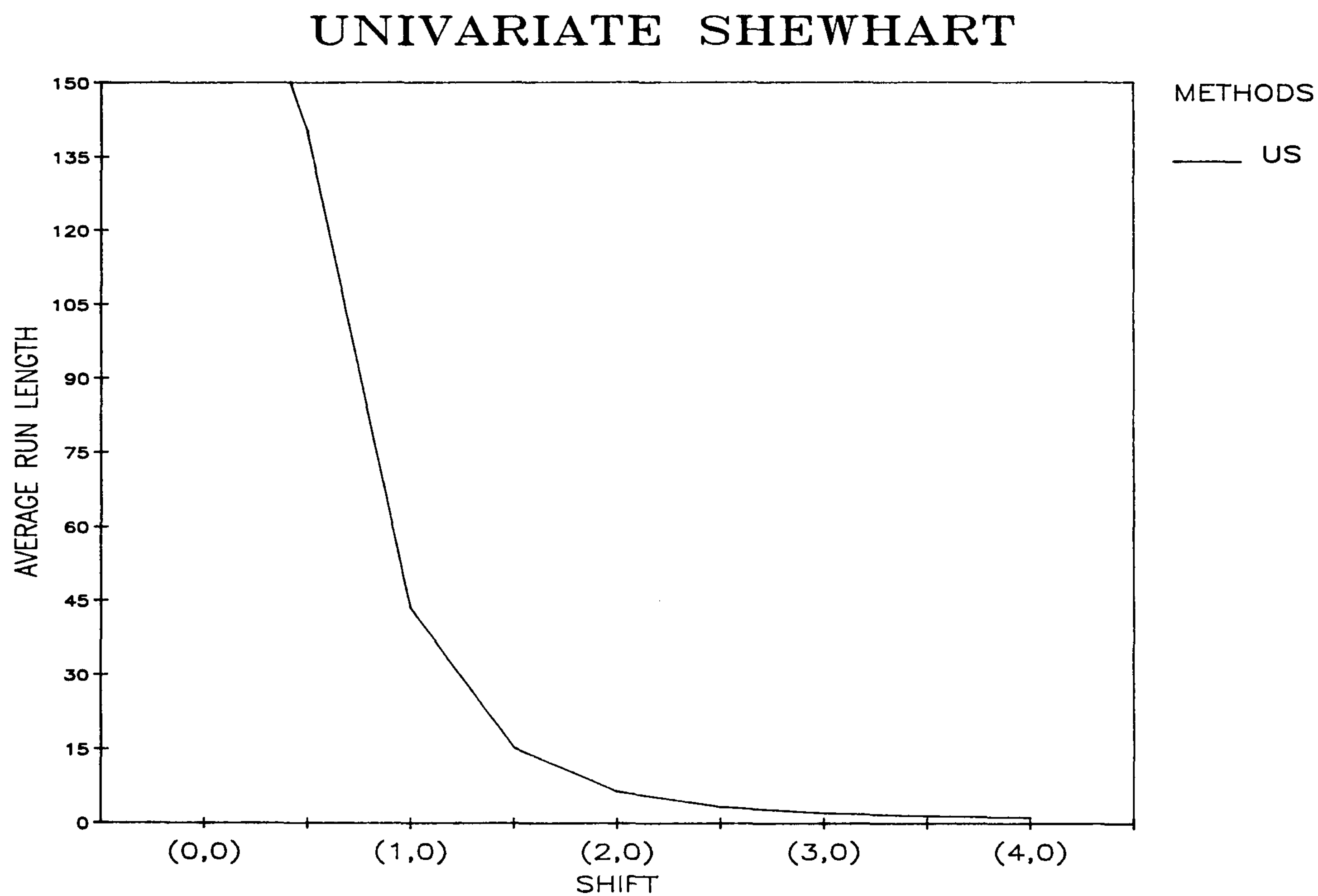

Figure 3: Theoretical average run lengths univariate Shewhart.

Table 5. Theoretical average run lengths for the multivariate Shewhart chart with various correlation coefficients.

		ρ	
Shift	0.0	0.5	0.9
(0.0,0.0)	200.000	200.000	200.000
(0.5,0.0)	114.567	98.502	30.652
(1.0,0.0)	41.091	30.198	4.784
(1.5,0.0)	15.940	10.799	1.602
(2.0,0.0)	7.161	4.692	1.065
(2.5,0.0)	3.716	2.471	1.002
(3.0,0.0)	2.230	1.580	1.000
(3.5,0.0)	1.546	1.207	1.000
(4.0,0.0)	1.224	1.061	1.000

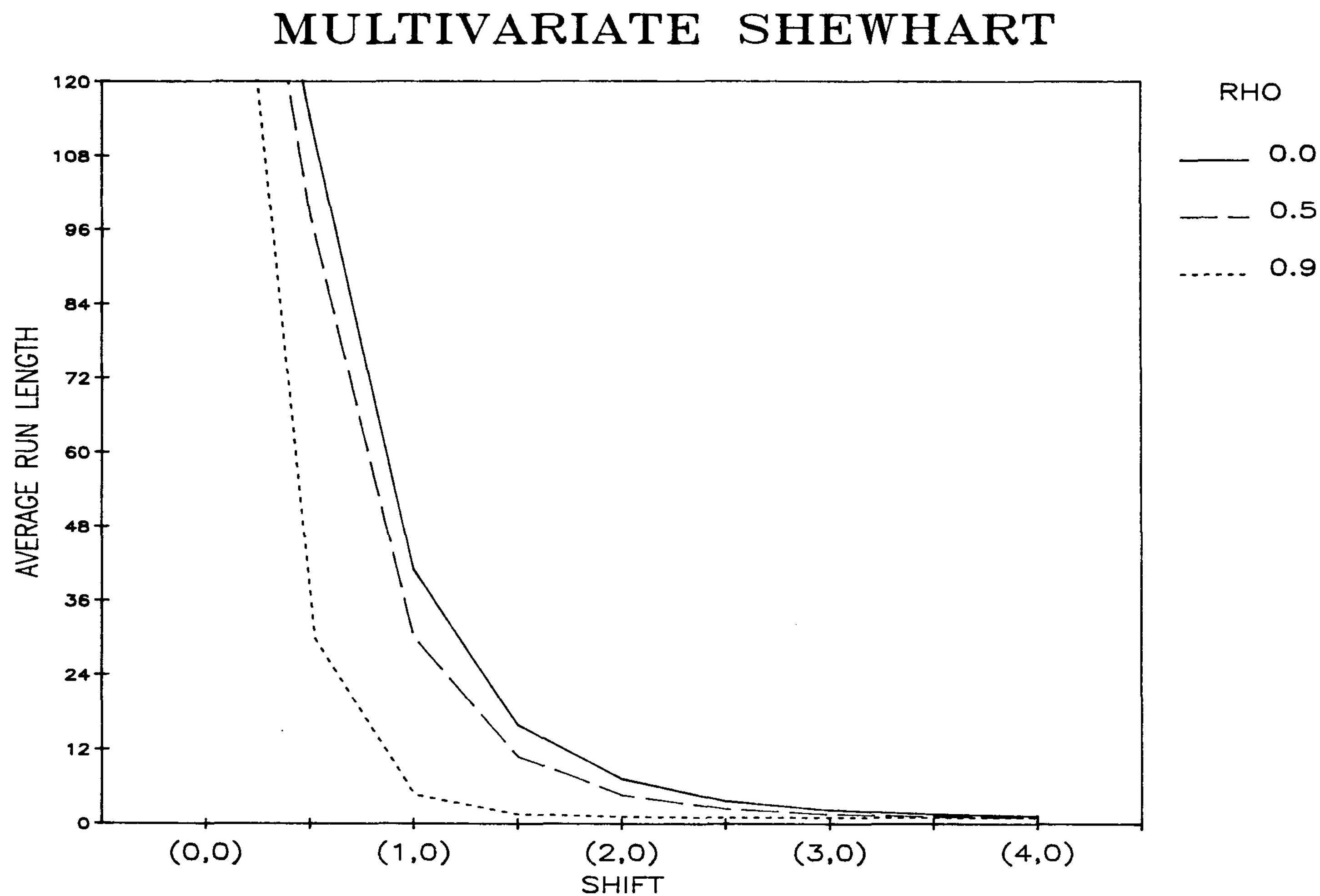

Figure 4: Theoretical ARLs multivariate Shewhart various correlation coefficients

Table 6. Theoretical average run lengths for two univariate cumulative sum charts running simultaneously optimized for various alternate hypotheses.

	H_a		
	(1,0) h=4.2 k=0.4	(2,0) h=2.22 k=1.00	(3,0) h=1.39 k=1.50
Shift			
(0.0,0.0)	200.000	200.000	200.000
(0.5,0.0)	26.866	44.207	63.441
(1.0,0.0)	8.611	11.179	17.591
(1.5,0.0)	4.841	4.841	6.396
(2.0,0.0)	3.470	2.978	3.273
(2.5,0.0)	2.682	2.089	2.089
(3.0,0.0)	2.287	1.693	1.594
(3.5,0.0)	1.990	1.395	1.296
(4.0,0.0)	1.792	1.196	1.097

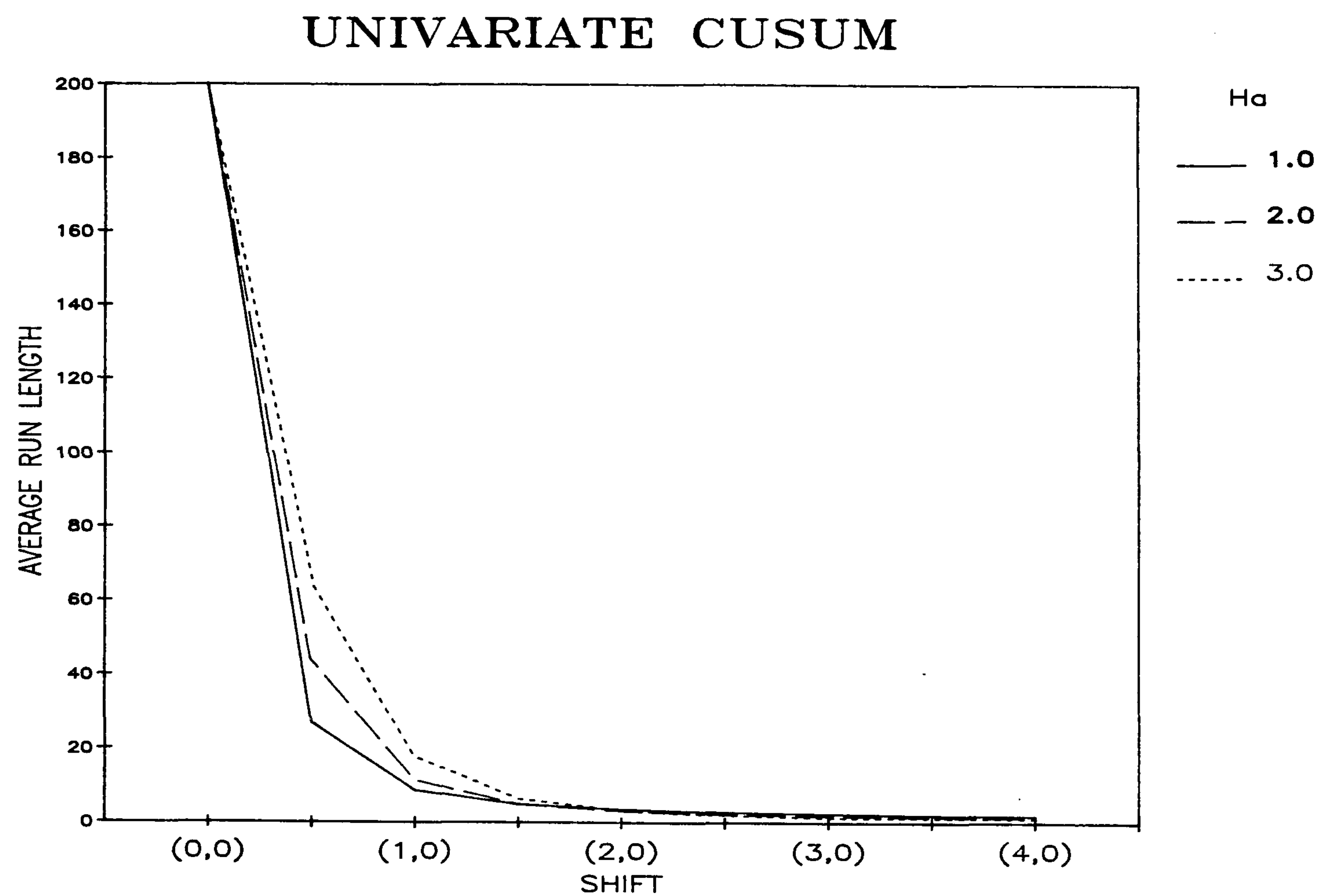

Figure 5: Theoretical ARLs for combination of two univariate cusums.

curve for the cusum optimized for a given alternate hypothesis.

The simulated run lengths for multivariate cusum #1 with $\rho = 0.0$ and various values of δ in the zeroing out rule are presented in Table 7. The table includes the values of α' which were used to calibrate the cusum. The standard deviations of the average run lengths are contained in Table 8. The average run length curves are presented in Figure 6 and each line on the graph represents a different value of δ. The best zeroing out rule appears to be when the value of δ is equal to 0.

The simulated run lengths for multivariate cusum #2 with $\rho = 0.0$ and various values of δ in the zeroing out rule are presented in Table 9. The table includes the values of α' which were used to calibrate the cusum. It should be noted that in the case where $\delta = 0.0$ we were unable to calibrate the cusum to 200. The largest average run length we were able to obtain under the null hypothesis for $\delta = 0.0$ was 87.235. The standard deviations of the average run lengths are contained in Table 10. The average run length curves are presented in Figure 7 and each line on the graph represents a different value of δ. There doesn't appear to be any particular best value of δ for this multivariate cusum method. It looks as though smaller values of δ are preferable for smaller shifts and larger values of δ are preferable for larger shifts. This behavior is similar to the behavior of the univariate cusum in that the multivariate cusum optimized for a specific alternate hypothesis by adjusting the

Table 7. Simulated average run lengths for multivariate cusum #1, $\rho = 0.0$, various deltas.

		δ		
Shift	0	1	2	3
(0.0,0.0)	203.885	206.871	203.836	202.950
(0.5,0.0)	30.983	35.027	46.970	82.226
(1.0,0.0)	9.671	10.585	12.386	21.002
(1.5,0.0)	4.937	5.296	5.765	7.759
(2.0,0.0)	3.149	3.306	3.487	4.042
(2.5,0.0)	2.248	2.319	2.424	2.578
(3.0,0.0)	1.739	1.832	1.844	1.892
(3.5,0.0)	1.430	1.477	1.504	1.472
(4.0,0.0)	1.223	1.259	1.260	1.247
α'	.0035	.0026	.00225	.0031

Table 8. Standard deviation of simulated average run lengths for multivariate cusum #1, ρ = 0.0, various deltas.

		δ		
Shift	0	1	2	3
(0.0,0.0)	24.1363	25.5702	30.0910	15.8033
(0.5,0.0)	2.3102	2.3909	3.6719	8.6942
(1.0,0.0)	0.6370	0.5558	0.9251	1.9540
(1.5,0.0)	0.2991	0.2929	0.2735	0.5483
(2.0,0.0)	0.1492	0.1361	0.1731	0.2614
(2.5,0.0)	0.0917	0.1217	0.1234	0.1356
(3.0,0.0)	0.0852	0.0825	0.0868	0.0847
(3.5,0.0)	0.0617	0.0618	0.0587	0.0681
(4.0,0.0)	0.0376	0.0462	0.0398	0.0456

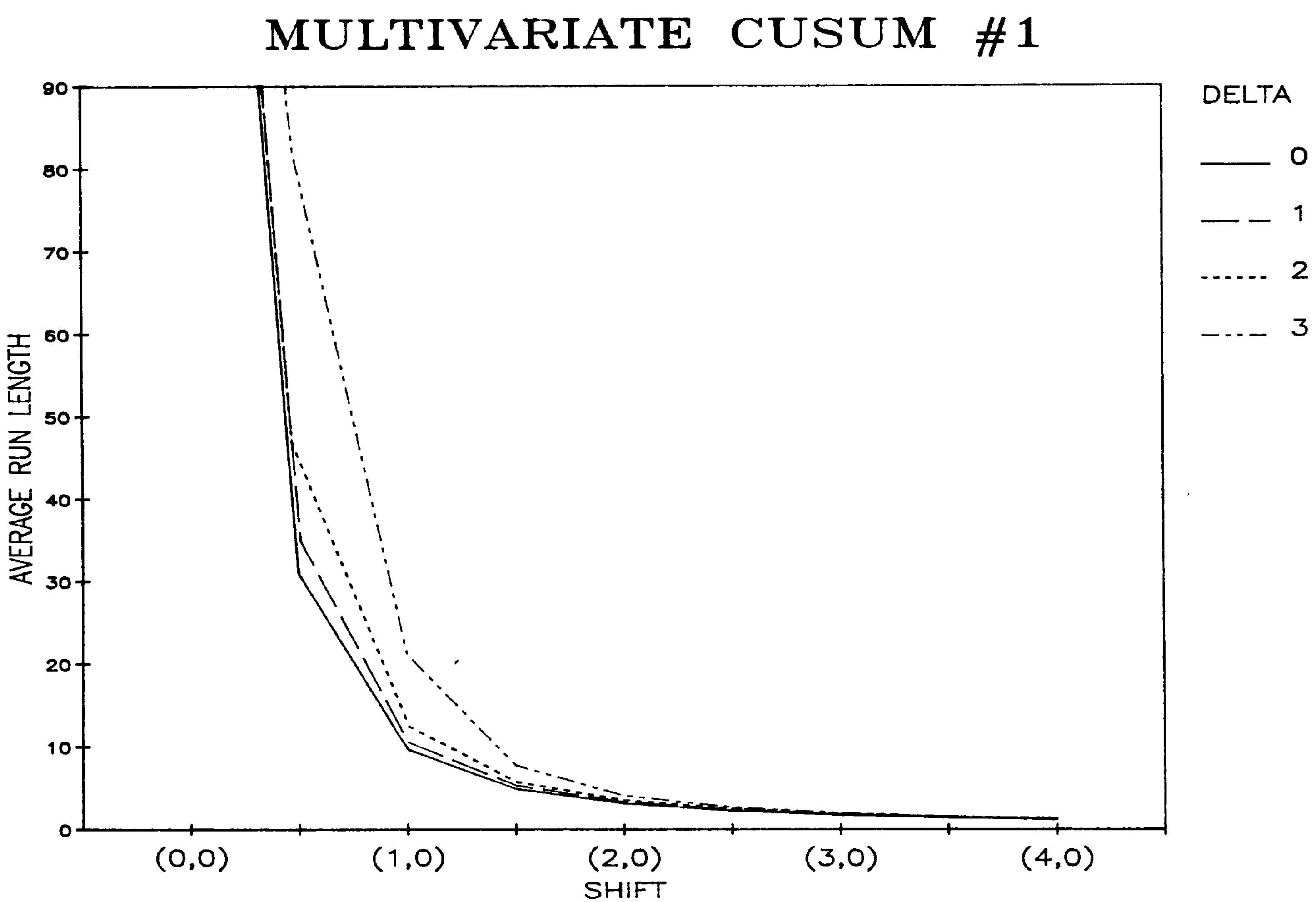

Figure 6: Average run length curve multivariate cusum #1 Rho = 0.0, various deltas

Table 9. Simulated average run lengths for multivariate cusum #2, ρ = 0.0, various deltas.

	δ			
Shift	0	1	2	3
(0.0,0.0)	87.235	202.247	199.883	202.503
(0.5,0.0)	54.232	90.261	108.495	109.303
(1.0,0.0)	28.419	25.781	29.764	36.395
(1.5,0.0)	12.382	9.742	9.866	12.447
(2.0,0.0)	5.770	4.808	4.633	5.409
(2.5,0.0)	3.440	2.936	2.816	2.938
(3.0,0.0)	2.310	2.028	1.970	1.947
(3.5,0.0)	1.710	1.539	1.513	1.485
(4.0,0.0)	1.386	1.266	1.246	1.242
α'	.001	.0026	.0031	.0039

Table 10. Standard deviation of simulated average run lengths for multivariate cusum #2, ρ = 0.0, various deltas.

		δ		
Shift	0	1	2	3
(0.0,0.0)	5.9003	18.7490	18.0402	16.9295
(0.5,0.0)	2.8760	8.7364	10.5084	10.2974
(1.0,0.0)	1.2949	1.9687	3.2316	4.1079
(1.5,0.0)	0.7576	0.6115	0.7062	1.1114
(2.0,0.0)	0.2789	0.3309	0.3482	0.4646
(2.5,0.0)	0.2084	0.2142	0.1746	0.2060
(3.0,0.0)	0.1095	0.0863	0.0948	0.1287
(3.5,0.0)	0.0875	0.0779	0.0761	0.0817
(4.0,0.0)	0.0573	0.0426	0.0500	0.0478

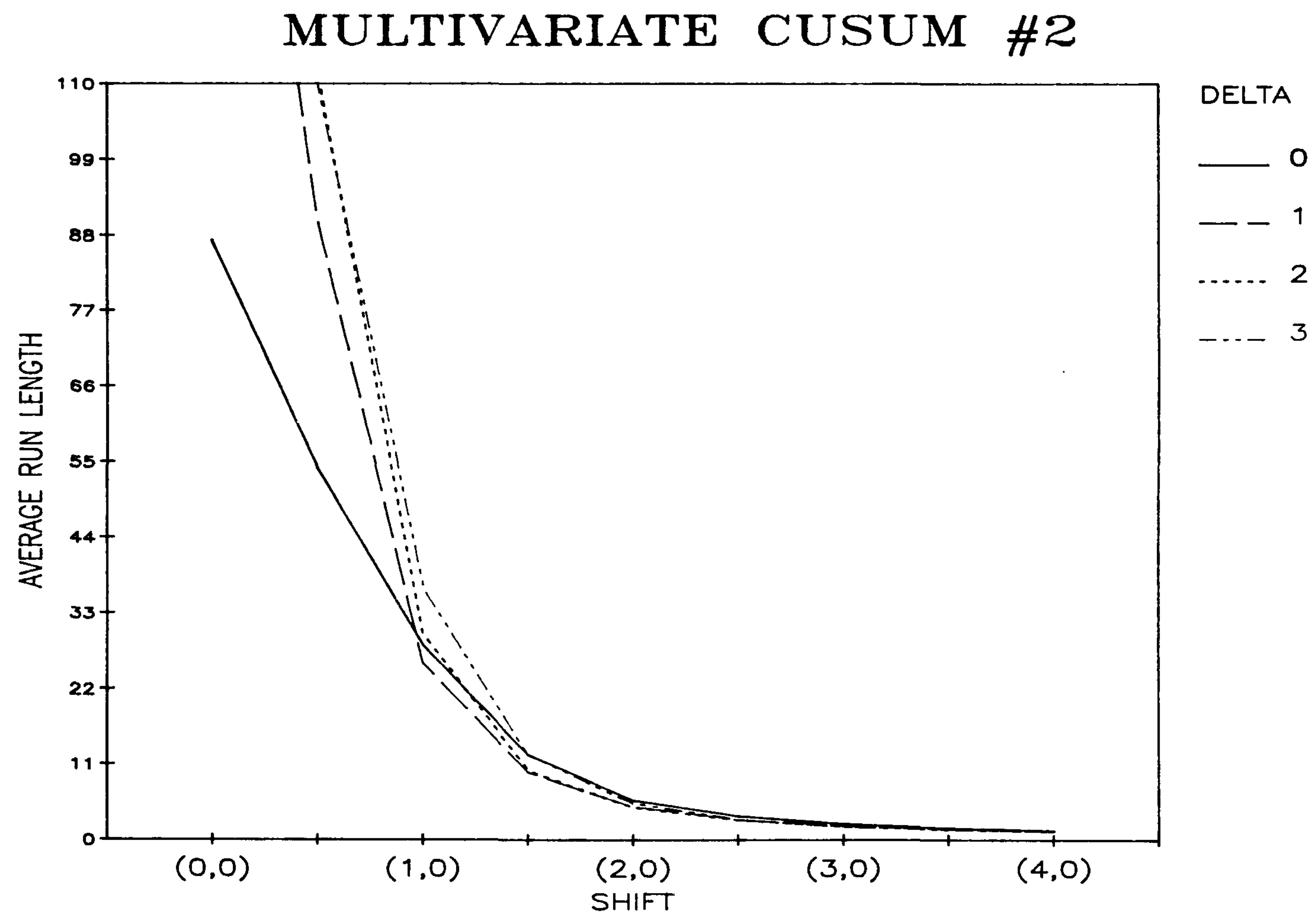

Figure 7: Average run length curve multivariate cusum #2 Rho = 0.0, various deltas

value of δ will achieve the smallest average run length for that particular shift in the mean.

The various control chart procedures are compared to each other in the case where $\rho = 0.0$ for each of the values of δ. These comparisons are shown in Figures 8 - 11. Each line on these figures represents the average run length curve for a given method. It should be noted that the univariate cusum on each of these charts represents the cusum optimized for an alternate hypothesis equal to $(\delta,0)$ except for the case where $\delta = 0.0$. This chart shows the univariate cusum optimized for an alternate hypothesis of (1,0). These figures indicate that regardless of the value of δ the two best methods appear to be the univariate cusum and multivariate cusum #1 especially for small to moderate shifts in the mean. The methods all appear to perform relatively well for large shifts in the mean, although it is difficult to distinguish between the lines for the various methods when the shift is large.

The simulated run lengths for multivariate cusum #1 with $\rho = 0.5$ and various values of δ in the zeroing out rule are presented in Table 11. The table includes the values of α' which were used to calibrate the cusum. The standard deviations of the average run lengths are contained in Table 12. The average run length curves are presented in Figure 12 and each line on the graph represents a different value of δ. The best zeroing out rule appears to be when the value of δ is equal to 0.

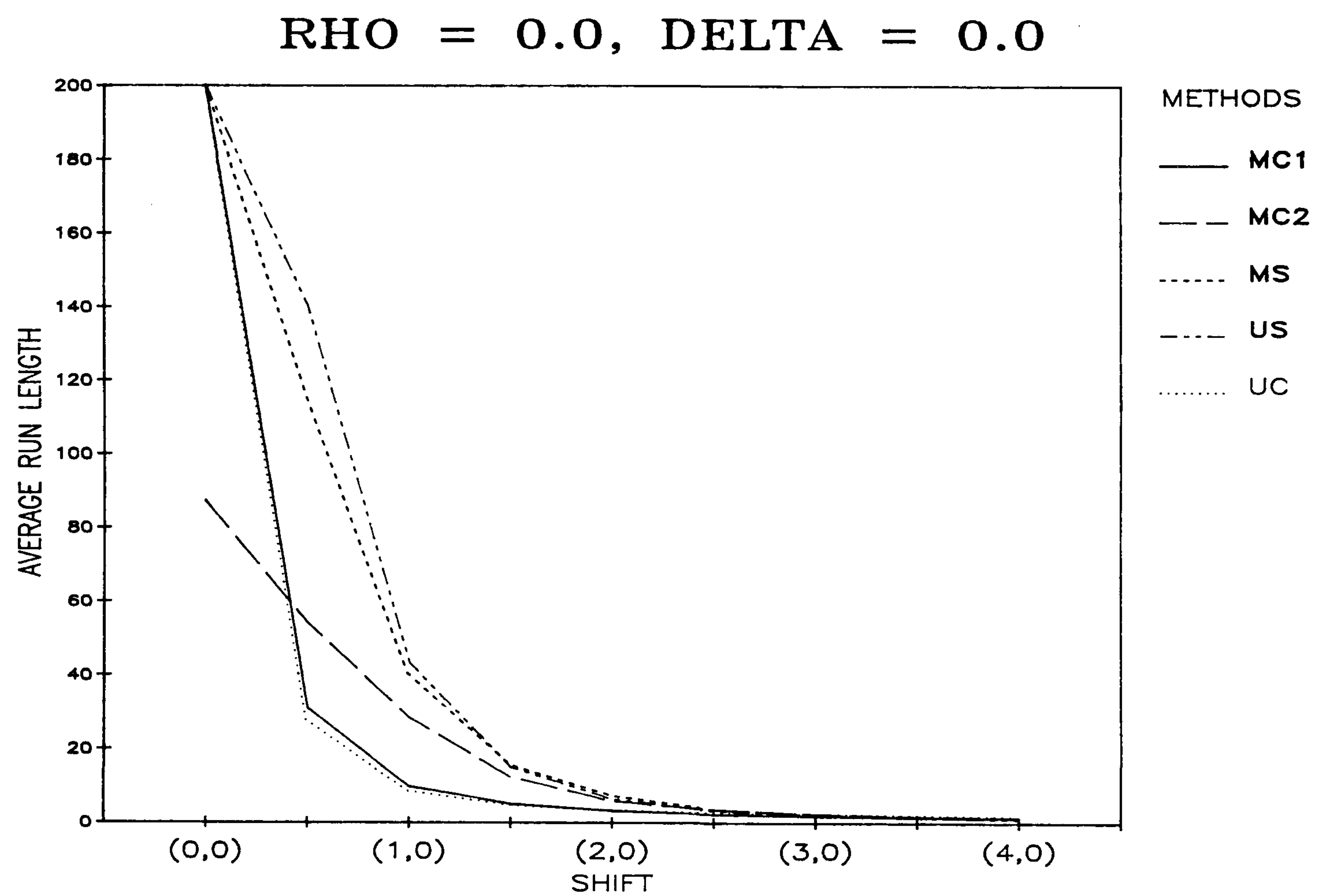

Figure 8: Average run length curve Rho = 0.0, Delta = 0.0.

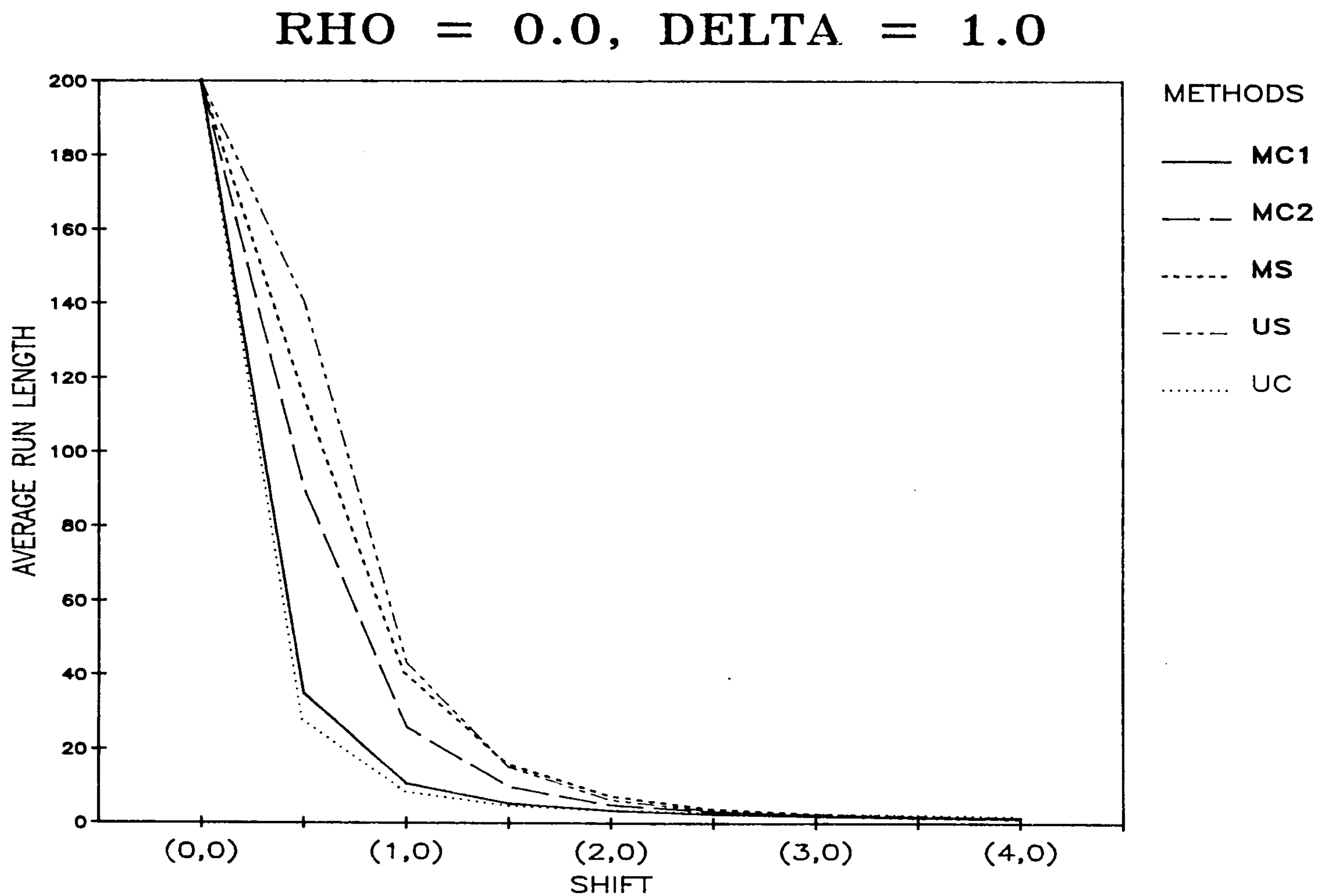

Figure 9: Average run length curve Rho = 0.0, Delta = 1.0.

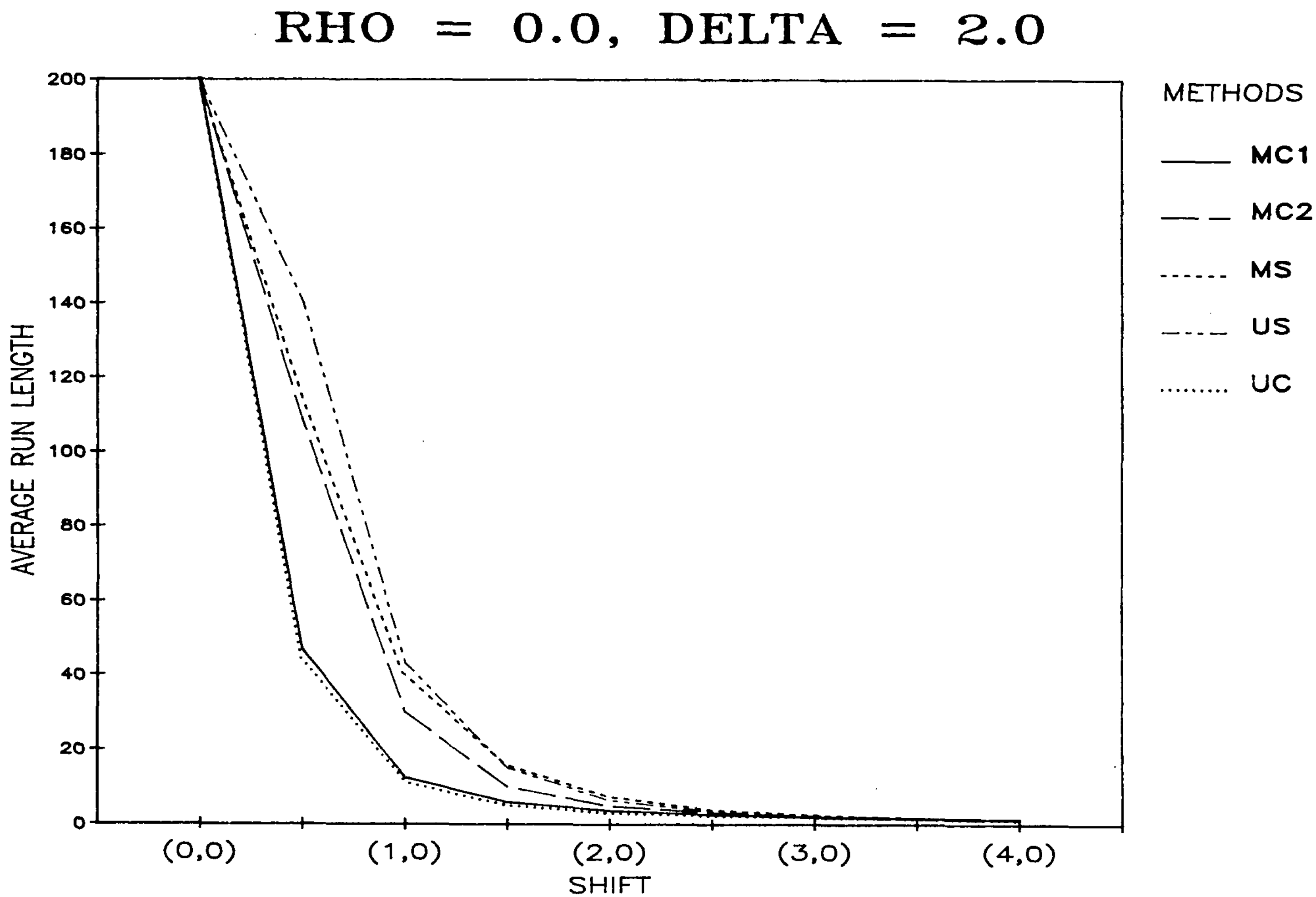

Figure 10: Average run length curve Rho = 0.0, Delta = 2.0.

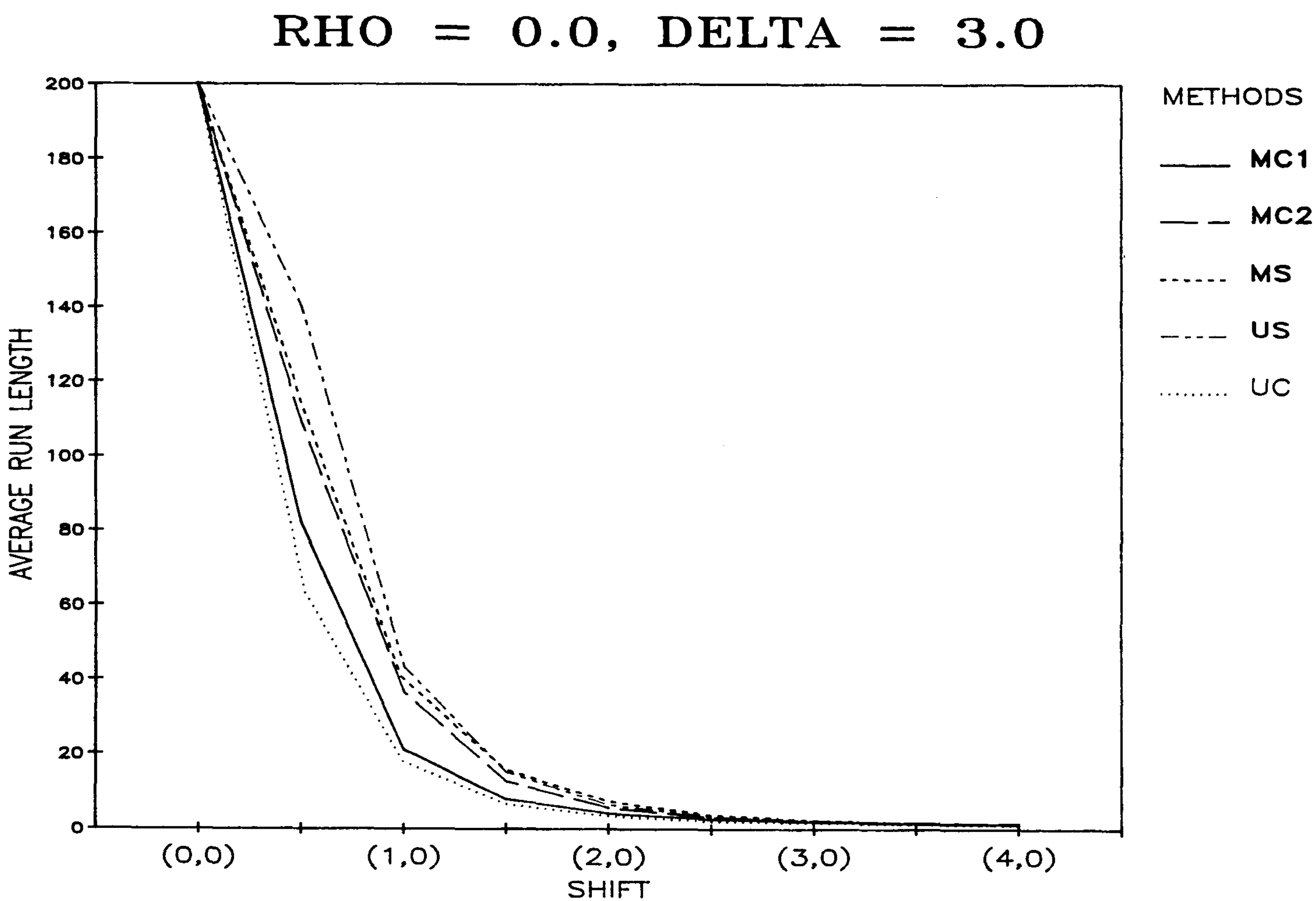

Figure 11: Average run length curve Rho = 0.0, Delta = 3.0.

Table 11. Simulated average run lengths for multivariate cusum #1, $\rho = 0.5$, various deltas.

			δ		
Shift	0	0.5	1.0	1.5	2.0
(0.0,0.0)	204.387	205.535	201.312	200.018	199.069
(0.5,0.0)	25.039	25.362	26.646	30.150	35.885
(1.0,0.0)	7.732	7.858	8.042	8.565	9.323
(1.5,0.0)	3.962	4.015	4.130	4.371	4.477
(2.0,0.0)	2.540	2.558	2.626	2.693	2.760
(2.5,0.0)	1.837	1.854	1.900	1.935	1.953
(3.0,0.0)	1.444	1.455	1.486	1.513	1.524
(3.5,0.0)	1.218	1.223	1.226	1.254	1.252
(4.0,0.0)	1.082	1.085	1.098	1.098	1.105
α'	.0034	.0032	.00278	.0023	.0023

Table 12. Standard deviations of simulated average run lengths for multivariate cusum #1, ρ = 0.5, various deltas.

	δ				
Shift	0	0.5	1.0	1.5	2.0
(0.0,0.0)	24.4110	22.4335	21.8826	20.2783	24.3482
(0.5,0.0)	1.6744	1.7780	1.9288	2.3688	3.2321
(1.0,0.0)	0.3714	0.4999	0.4506	0.4680	0.5110
(1.5,0.0)	0.2074	0.2137	0.1943	0.1922	0.2526
(2.0,0.0)	0.1303	0.1153	0.1083	0.1234	0.1536
(2.5,0.0)	0.0817	0.0758	0.0811	0.0882	0.0847
(3.0,0.0)	0.0613	0.0613	0.0639	0.0620	0.0639
(3.5,0.0)	0.0495	0.0457	0.0424	0.0548	0.0513
(4.0,0.0)	0.0268	0.0301	0.0296	0.0289	0.3610

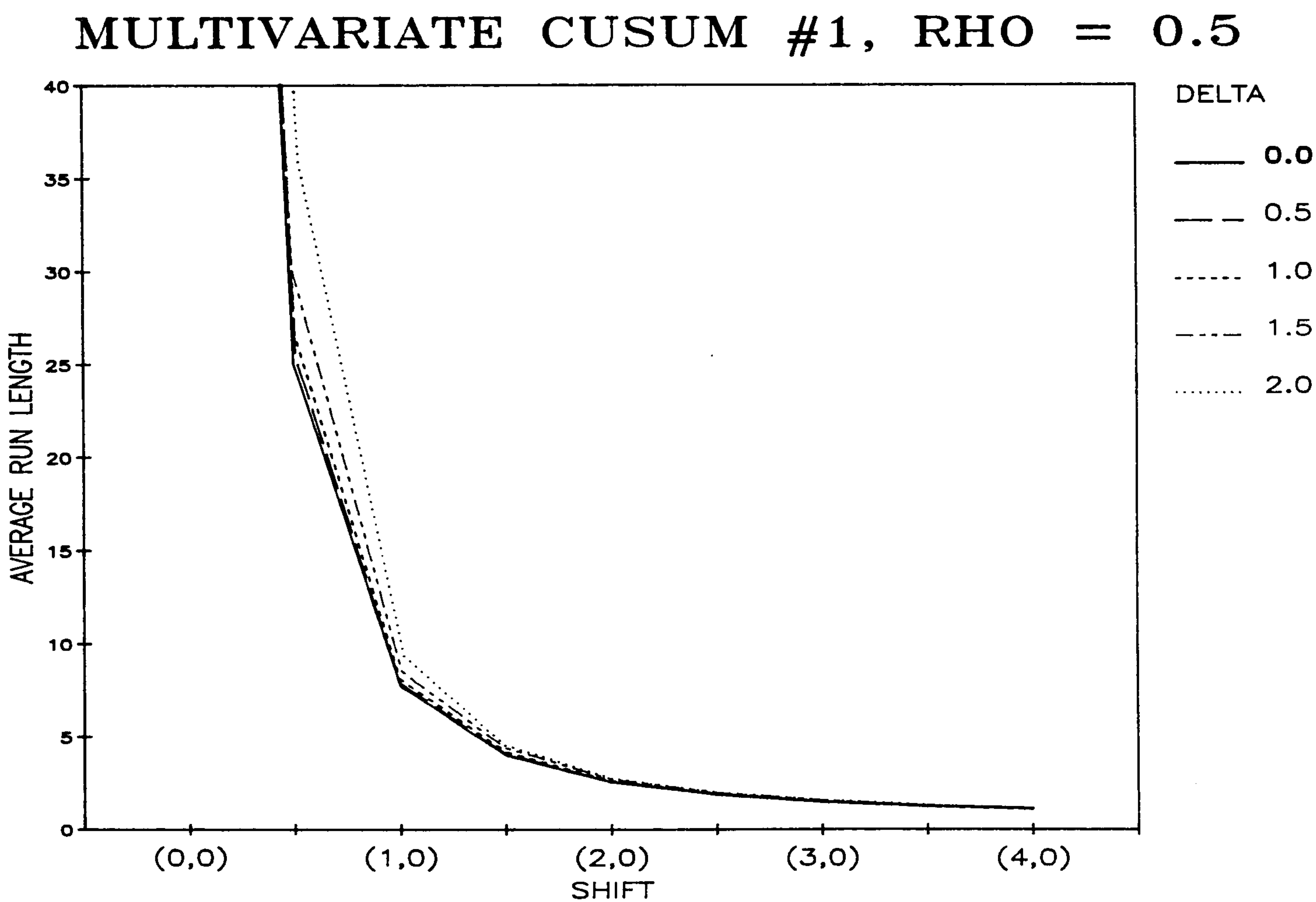

Figure 12: Average run length curve MC #1 Rho=0.5, various deltas.

The simulated run lengths for multivariate cusum #1 with ρ = 0.9 and various values of δ in the zeroing out rule are presented in Table 13. The table includes the values of α' which were used to calibrate the cusum. The standard deviations of the average run lengths are contained in Table 14. The average run length curves are presented in Figure 13 and each line on the graph represents a different value of δ. The best zeroing out rule appears to be either a value of δ equal to 0.0 or 0.5. Each of these values is slightly better in several cases, but for the purposes of comparison with multivariate cusum #2 δ = 0.0 is chosen.

The simulated average run lengths for the multivariate cusums with ρ = 0.5 and δ = 0.0 are shown in Table 15. It should be noted that we were unable to calibrate multivariate cusum #2 to an average run length of 200 under the null hypothesis. Figure 14 is a comparison of the various control chart procedures under these conditions. The univariate cusum used on this figure is optimized for an alternate hypothesis of (1,0). The best methods under these conditions still appear to be the univariate cusum and multivariate cusum #1 for small to moderate shifts in the mean while all the methods perform relatively well for large shifts in the mean.

The simulated average run lengths for the multivariate cusums with ρ = 0.9 and δ = 0.0 are presented in Table 16. As we have noted previously we were unable to calibrate multivariate

Table 13. Simulated average run lengths for multivariate cusum #1, $\rho = 0.9$ various deltas.

			δ		
Shift	0	0.5	1.0	1.5	2.0
(0.0,0.0)	202.060	202.652	203.023	199.975	203.419
(0.5,0.0)	7.816	7.757	8.147	8.705	9.558
(1.0,0.0)	2.532	2.566	2.658	2.752	2.783
(1.5,0.0)	1.456	1.467	1.499	1.536	1.532
(2.0,0.0)	1.086	1.090	1.101	1.107	1.124
(2.5,0.0)	1.007	1.006	1.007	1.009	1.009
(3.0,0.0)	1.001	1.000	1.001	1.001	1.001
(3.5,0.0)	1.000	1.000	1.000	1.000	1.000
(4.0,0.0)	1.000	1.000	1.000	1.000	1.000
α'	.0036	.0035	.00275	.0024	.0022

Table 14. Standard deviations of simulated average run lengths for multivariate cusum #1, $\rho = 0.9$, various deltas.

	δ				
Shift	0	0.5	1.0	1.5	2.0
(0.0,0.0)	48.7800	91.7124	22.1351	24.0526	18.7320
(0.5,0.0)	0.4464	0.4529	0.4187	0.5364	0.7027
(1.0,0.0)	0.1078	0.1028	0.1400	0.1330	0.1385
(1.5,0.0)	0.0563	0.0694	0.0669	0.0659	0.0641
(2.0,0.0)	0.0288	0.0246	0.0315	0.0282	0.0336
(2.5,0.0)	0.0092	0.0742	0.0095	0.0108	0.0098
(3.0,0.0)	0.0025	0.0000	0.0022	0.0022	0.0022
(3.5,0.0)	0.0000	0.0000	0.0000	0.0000	0.0000
(4.0,0.0)	0.0000	0.0000	0.0000	0.0000	0.0000

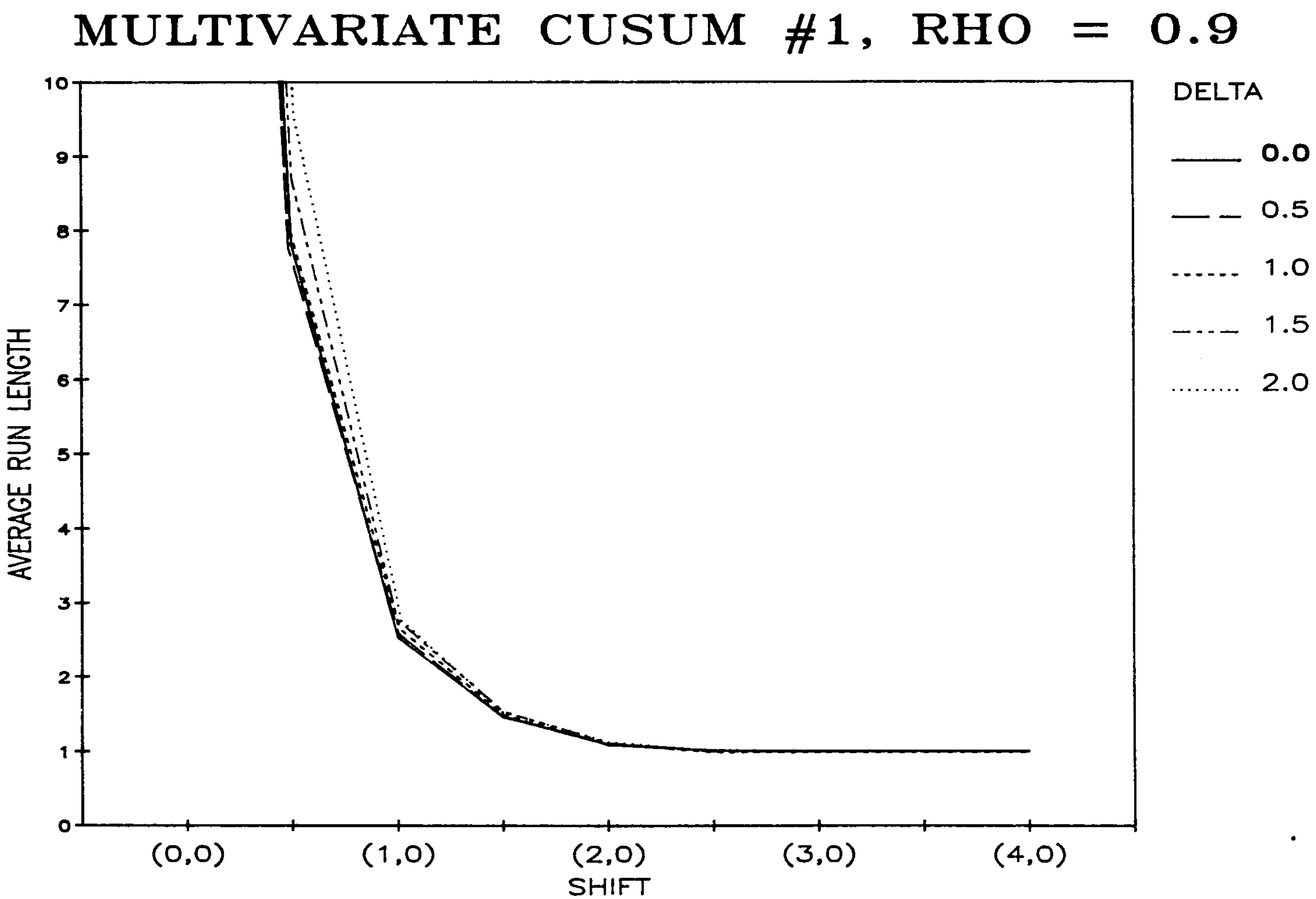

Figure 13: Average run length curve MC #1 Rho = 0.9, various deltas.

Table 15. Simulated average run lengths for multivariate cusums, $\rho = 0.5$, $\delta = 0.0$.

Shift	MC #1	MC #2
(0.0,0.0)	204.387	86.509
(0.5,0.0)	25.039	49.417
(1.0,0.0)	7.732	22.494
(1.5,0.0)	3.962	8.519
(2.0,0.0)	2.540	4.142
(2.5,0.0)	1.837	2.513
(3.0,0.0)	1.444	1.753
(3.5,0.0)	1.218	1.374
(4.0,0.0)	1.082	1.162
α'	.0034	.001

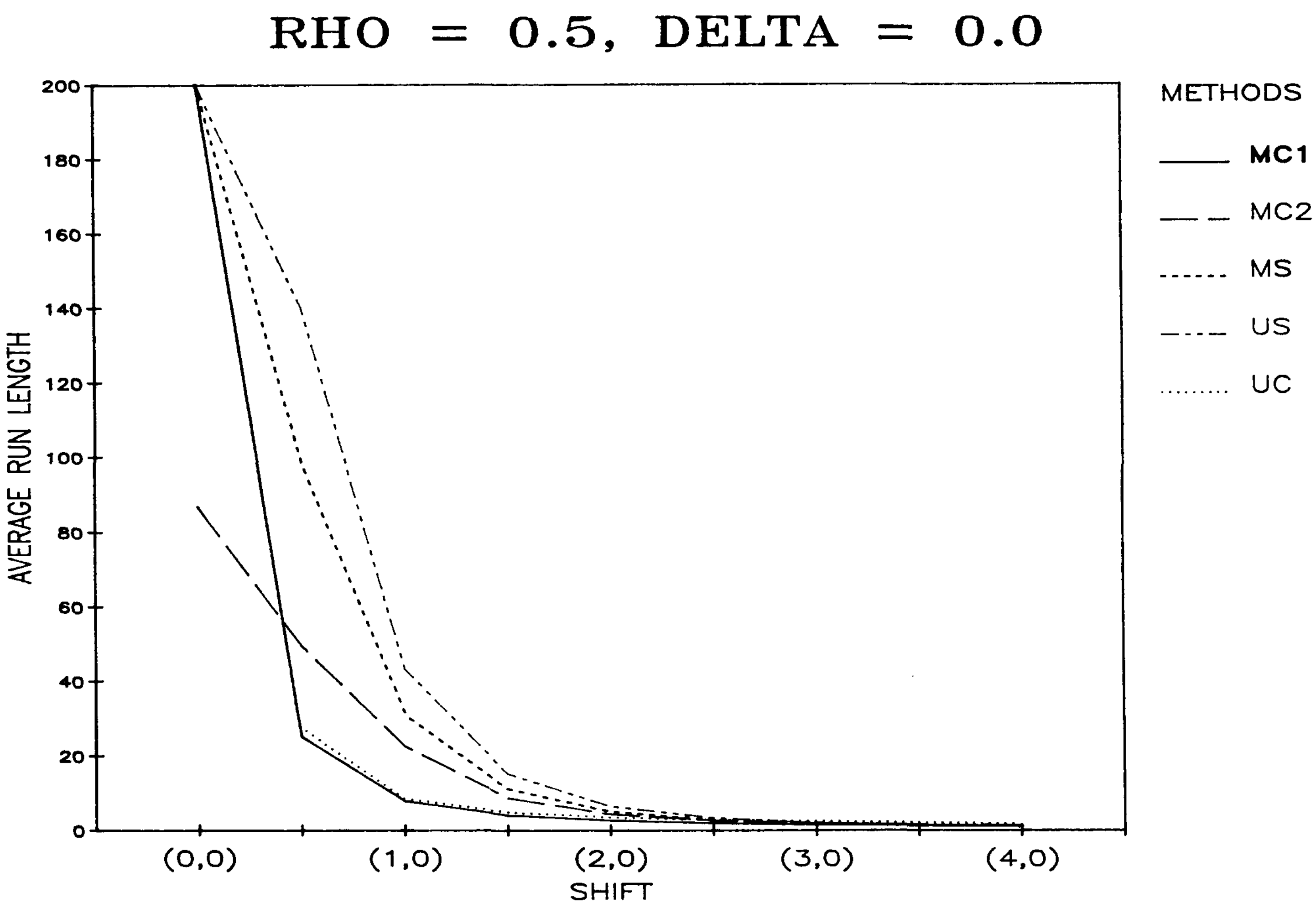

Figure 14: Average run length curve Rho = 0.5, Delta = 0.0.

Table 16. Simulated average run lengths for multivariate cusums, $\rho = 0.9$, $\delta = 0.0$.

Shift	MC #1	MC #2
(0.0,0.0)	202.060	89.125
(0.5,0.0)	7.816	22.905
(1.0,0.0)	2.532	4.165
(1.5,0.0)	1.456	1.791
(2.0,0.0)	1.086	1.161
(2.5,0.0)	1.007	1.018
(3.0,0.0)	1.001	1.000
(3.5,0.0)	1.000	1.000
(4.0,0.0)	1.000	1.000
α'	.0036	.001

cusum #2 to 200 under the null hypothesis. Table 17 contains the standard deviations of the average run lengths for multivariate cusum #2 with $\rho = 0.5$ and 0.9. Figure 15 is a comparison of the various control chart procedures under these conditions. The univariate cusum used on this figure is optimized for an alternate hypothesis of (1,0). The best methods under these conditions appear to be the multivariate cusums for small shifts in the mean, all the multiviariate methods for moderate shifts in the mean, and any of the methods for large shifts in the mean.

<u>Statistical Evaluation of the Results</u>

It is extremely difficult to determine any one best method for detecting a shift in the mean values of a process solely from studying the figures representing the average run length curves or from perusing the actual numerical results. Therefore, the differences between the methods were evaluated statistically for selected conditions. These selected conditions consisted of the shifts in the means from (0,0) to alternate hypotheses of (1,0), (2,0), (3,0) and (4,0) respectively for each of the correlation coefficients with $\delta = 0.0$.

Initially, the purpose of the statistical evaluation was to compare multivariate cusum #1 with multivariate cusum #2. This was achieved by first comparing the variances of their simulated average run lengths for equality using Bartlett's test for equality of variances [17]. Then, if the variances were

Table 17. Standard deviation of simulated average run lengths for multivariate cusum #2,variouscorrelation coeffiecients, δ = 0.0.

	ρ	
Shift	0.5	0.9
(0.0,0.0)	5.0906	5.6357
(0.5,0.0)	2.0898	1.5045
(1.0,0.0)	1.1998	0.2350
(1.5,0.0)	0.5094	0.0856
(2.0,0.0)	0.2518	0.0434
(2.5,0.0)	0.1387	0.0117
(3.0,0.0)	0.0920	0.0013
(3.5,0.0)	0.0561	0.0000
(4.0,0.0)	0.0466	0.0000

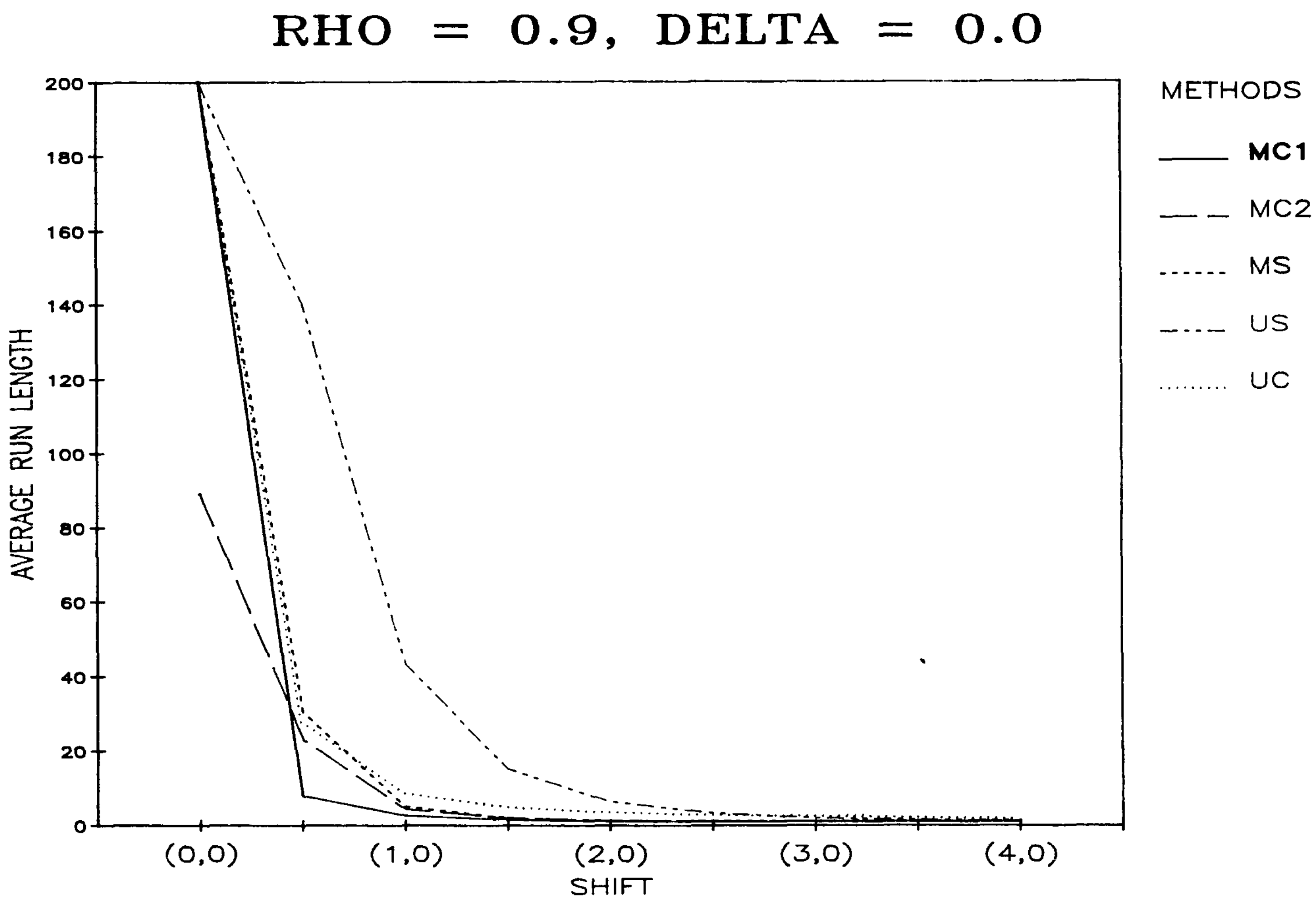

Figure 15: Average run length curve Rho = 0.9, Delta = 0.0.

equal a t-test on means with equal but unknown variances was performed. If the variances were unequal, then the Hsu solution [10] to the Behren's Fisher problem was used to test the equality of the means. A compilation of the numerical calculations from the tests are available on request.

The results of Bartlette's test for equality of variances indicated that, in all but one case, the variances of the multivariate cusums were not equal. The results of each of the following tests on means indicated that there was only one case where there was not a significant difference between the simulated average run lengths for multivariate cusum #1 and multivariate cusum #2. The case in which there was no significant difference occurred when the shift was equal to (4,0) and the correlation coefficient was 0.9. Therefore, multivariate cusum #1 performs as well or better than multivariate cusum #2 under all the conditions considered.

The simulation results are summarized in Tables 18 - 20. These tables show which methods performed the best for each of the shifts and correlation coefficients when $\delta = 0.0$ by ranking the methods from 1 to 7 according to their average run lengths, where 1 indicates the smallest average run length and 7 indicates the largest. It should be noted that any statistical comparisons considered the results for the univariate Shewhart, multivariate Shewhart and univariate Shewhart to be constants. The univariate cusums optimized for each of the three alternate hypothesis were considered to be different methods for the purposes of comparison.

A summary of the results for the case when the correlation coefficient and delta are equal to zero is shown in Table 18. It is interesting to note that the univariate cusum charts had the smallest average run lengths for all the shifts considered. Furthermore, the average run lengths for the univariate cusums are significantly smaller than those for multivariate cusum #1 in all cases. However, when the univariate cusum is optimized for a particular alternate hypothesis and the shift in the means is either substantially larger or smaller than the shift it is optimized for multivariate cusum #1 responds more quickly to the change. For instance, when a shift from (0,0) to (2,0) occurs the univariate cusum optimized for an alternate hypothesis of (2,0) responds the most rapidly, but multivariate cusum #1 responds more quickly than the univariate cusum optimized for either (1,0) or (3,0).

The results for the case when the correlation coefficient is equal to 0.5 and delta is equal to zero are detailed in Table 19. Under these conditions multivariate cusum #1 has significantly smaller average run lengths for all the alternate hypotheses except (4,0). In the case where the mean shifts from (0,0) to an alternate hypothesis of (4,0) the multivariate Shewhart has a significantly smaller average run length than multivariate cusum #1. Practically speaking, however, it is of interest to note that in the case where the multivariate Shewhart

Table 18. Ranking of control chart methods, $\rho = 0.0$, $\delta = 0.0$.

			Method				
Shift	US	MS	UC (1,0)	UC (2,0)	UC (3,0)	MC#1	MC#2
(1,0)	7	6	1*	3	4	2	5
(2,0)	6	7	4	1*	3	2	5
(3,0)	4	5	6	2	1*	3	7
(4,0)	2	5	7	3	1*	4	6

* Indicates that the univariate cusum methods have significantly better (P-value ~ 0.0000) average run lengths than multivariate cusum #1.

Table 19. Ranking of control chart methods, $\rho = 0.5$, $\delta = 0.0$.

Shift	US	MS	UC (1,0)	UC (2,0)	UC (3,0)	MC#1	MC#2
			Method				
(1,0)	7	6	2	3	4	1*	5
(2,0)	7	6	4	2	3	1*	5
(3,0)	6	2	7	3	4	1*	5
(4,0)	5	1**	7	6	3	2	4

* Indicates that multivariate cusum #1 has significantly better (P-value ~ 0.0000) average run lengths than other methods.

** Indicates that multivariate Shewhart method has significantly better (P-value ~ 0.0000) average run length than multivariate cusum #1.

is significantly better the actual average run lengths of the multivariate Shewhart and multivariate cusum #1 are very close.

A summary of the simulation results for the case when the correlation coefficient is equal to 0.9 and delta is equal to zero is shown in Table 20. Under these conditions multivariate cusum #1 has a significantly smaller average run length for a small shift, the multivariate Shewhart method is the best for moderate shifts and any of the multivariate methods would be equally effectively for a large shift.

The next section contains a summary of the overall findings of the simulation and directives for further study. The advantages of using various methods are discussed and possible improvements and design considerations for multivariate cusum #1 are suggested.

Table 20. Ranking of control chart methods, $\rho = 0.9$, $\delta = 0.0$.

Shift	US	MS	UC (1,0)	UC (2,0)	UC (3,0)	MC#1	MC#2
			Method				
(1,0)	7	3	4	5	6	1*	2
(2,0)	7	1**	6	4	5	2	3
(3,0)	6	1***	7	5	4	3	2
(4,0)****	3	1	5	4	2	1	1

* Indicates that multivariate cusum #1 has significantly better (P-value ~ 0.0000) average run lengths than other methods.

** Indicates that multivariate Shewhart method has significantly better (P-value ~ 0.0000) average run lengths than other methods.

*** Indicates that multivariate Shewhart method is significantly better (P-value ~ 0.025) than multivariate cusum #1.

**** All three multivariate methods equal in this case.

Chapter 6

SUMMARY

The purpose of this chapter is to summarize the results of the Monte Carlo simulation and discuss directives for further study. The results of the simulation indicate that there are definite advantages to using multivariate control chart methods rather than their univariate counterparts, particularly when the quality characteristics are correlated. The univariate control chart schemes fail to take into consideration any interrelationships between the quality characterictics, and therefore, fail to consider the distance of the shift with regard to the interdependence between the characteristics.

Although, the univariate cusums optimized for a particular shift had significantly smaller average run lengths than multivariate cusum #1 when $\rho = 0.0$, it is interesting to note that this was only the case when the univariate cusum was optimized for the particular shift in question. The successful use of the univariate cusum therefore, requires some specific foreknowledge as to the size of the shift to be detected in order to gain optimum results. Multivariate cusum #1 , on the other hand, will detect a shift in the means to any alternate hypothesis equidistant to the one in question with the same average run length. This is to say that any shift of a given magnitude will be detected equally well regardless of the direction of the shift. Multivariate cusum #1 will therefore, detect an infinite number of shifts in the means equally well while the univariate cusum is only optimized for a single shift.

When the correlation between the quality characteristics was moderate ($\rho = 0.5$) multivariate cusum #1 was superior to any of the other methods except the multivariate Shewhart in the case of a large shift. The multivariate Shewhart performs better in the case of a large shift because it doesn't have any backlogged results to slow it down.

In the case where the correlation between the quality characteristics was high multivariate cusum #1 was superior for small shifts in the mean values, while, the multivariate Shewhart chart excelled for larger shifts. Any of the multivariate methods would detect a shift in the means from (0,0) to (4,0) equally well.

Regardless of the conditions, multivariate cusum #1 is clearly superior to multivariate cusum #2. Even though multivariate cusum #2 had an initial headstart, since in many cases we were unable to calibrate it to 200 under the null hypthesis, multivariate cusum #1 still achieved uniformly smaller average run lengths.

Overall, the two best methods appear to be multivariate cusum #1 and the multivariate Shewhart chart unless there is specific knowledge about the shift to be detected and the variables are uncorrelated.

Directives for Further Study

Since the multivariate Shewhart control chart performs well for large shifts in the means when there is moderate correlation between the quality characteristics and for moderate to large shifts in the means when there is a high degree of correlation between the characterictics it would be of interest to investigate a combined multivariate cusum #1 - Shewhart scheme. Work of a similar nature has been conducted by Lucas [15] and suggests that this might combine the advantages of the swift detection of small to moderate shifts provided by the multivariate cusum with the equally swift detection of large shifts by the multivariate Shewhart.

In addition to combined schemes of this nature, it would also be of interest to investigate the possibility of implementing a fast initial response (FIR) feature for multivariate cusum #1 as discussed by Lucas and Crosier [16]. This involves not resetting the cusum to zero at startup or after an out-of-control signal. Instead the cusum is set to an initial head start value. If the process is in control the head start value should have little effect, however, if the process is out of control a signal should be given faster with the head start.

Additional topics of interest for further study include such design considerations as sample size, sampling interval, different underlying probability distributions for the quality characteristics, erratic shifts in the means, detecting changes in the variance-covariance matrix, the case where the variance-covariance matrix is unknown, considering a process with three or more quality characteristics and other methods of zeroing out the multivariate cusum. We suspect that these investigations will shed more light on the circumstances when multivariate cusum #1 is the appropriate method to use for process control.

APPENDIX A

FORTRAN CODE

```
      program arl
        real x(2)
        logical flagm1,flagm2
        double precision limit
        integer count,count1,count2,pmts,reps
        character*64 nfile
        common arlm1,arlm2,count,count1,count2,flagm1,flagm2,
     f        limit,reps,summ1,summ2
        write(*,*)'enter new output file name'
        read(*,1)nfile
1       format(a64)
        open(unit=1,file=nfile,status='new')
        write(*,*)'enter new data output file name'
        read(*,1)nfile
        write(*,*)'enter random number seed'
        read(*,*)ix
        write(1,*)' random number seed entered=',ix
        nruns=61
        write(*,*)'nruns=',nruns
        pmts=2
        write(*,*)'enter alpha limit'
        read(*,*)limit
        write(1,*)' limit =',limit
        write(*,*)'enter correlation'
        read(*,*)rho
        write(*,*)'correlation=',rho
        write(1,*)'rho=',rho
        a1=0.5*sqrt(1.0-rho**2)
        a=sqrt( .5 + a1)
        b=sqrt( .5 - a1)
        write(*,*)'enter delta for zeroing out rule'
        read(*,*)delta
        write(1,*)'delta=',delta
        dsq2=delta*delta/2.0
        open(unit=2,file=nfile,status='new')
        write(1,*)'arl data in file:',nfile
        write(*,*)'enter ha mean'
        read(*,*)ha
        dist=ha/sqrt(1.-rho*rho)
        write(1,*)'dist=',dist,'  ha=',ha
        write(*,*)'dist=',dist,'  ha=',ha
        write(*,*)'use cusum no. 1 or 2 ?'
        read(*,*)icnum
        write(1,*)'cusum no.=',icnum,' no. runs= ',nruns
        do 900 kk=1,nruns
          write(1,*)'starting random number seed for this
```

```
     f         run=',ix
           write(1,*)'ha = ',ha,' distance from zero is ',dist
           reps=100
           flagm1=.true.
           flagm2=.true.
           if(icnum.eq.1)flagm2=.false.
           if(icnum.eq.2)flagm1=.false.
c
c *** the null hypothesis is that the population mean vector is
c *** the zero vector.
c *** the simulation is repeated reps times for time units or
c***until a run length is determined for each of the three
c *** methods for a given n.
c
           do 70 count=1,reps
             n=1
   40        if (flagm1.or.flagm2) then
c
c *** generate a multivariate vector.
c
               x1=trpnrm(ix)
               x2=trpnrm(ix)
               x(1)=a*x1+b*x2+h
               x(2)=b*x1+a*x2
               nn=n
c
c *** call subroutine musum1 to evaluate data using multivariate
c *** cumulative sum technique number 1 if run length has not yet
c *** been found.
c
               if (flagm1) call musum1(nn,x,dsq2,rho)
c
c *** call subroutine musum2 to evaluate data using multivariate
c *** cumulative sum technique number 2 if run length has not yet
c *** been found.
c
               if (flagm2) call musum2(nn,x,dsq2,rho)
               n=n+1
               go to 40
c
c*** reset the counters and flags for the beginning of the next
c *** replication.
c
             else
               count1=0
               count2=0
```

```
              flagm1=.true.
              flagm2=.true.
              if(icnum.eq.1) flagm2=.false.
              if(icnum.eq.2) flagm1=.false.
            end if
   70     continue
c

c
c *** calculate the average run lengths
c
          arlm1=summ1/reps
          arlm2=summ2/reps
c
c *** write the results
c
          write(1,90)arlm1
   90     format(1x,'average run length cusum method 1 = ',e16.8)
     /      write(1,*)
          write(2,*)arlm1,arlm2
          write(1,110)arlm2
  110     format(1x,'average run length cusum method 2 = ',e16.8)
          write(1,165)ix
  165     format(' next random number seed=',i15)
  900   continue
        close(1)
        close(2)
      end
****************************************************************
      subroutine musum1(n,x,dsq2,rho)
c
        real arlm1,summ1,x(2)
        logical flagm1
        double precision dchisq,cs,limit
        integer count,count1,reps
        common arlm1,arlm2,count,count1,count2,flagm1,flagm2,
     /          limit,reps,summ1,summ2
c
c *** initialize array at beginning of each run.
c
        if (count .eq. 1) then
          arlm1=0.0
          count1=0
          summ1=0.0
        end if
c
```

```
c ***
c
        if (n.eq.1) then
          nn=1
          diff1=0.0
          diff2=0.0
        else
        end if
        diff1=diff1+x(1)
        diff2=diff2+x(2)
        cs=(diff1**2+diff2**2-2.0*rho*diff1*diff2)/(nn*(1.-
     β    rho**2))
c
        if (1.0d0-dchisq(cs,2).le.limit) then
            count1=n
            flagm1=.false.
            summ1=summ1+count1
        end if
c
c *** zeroing out rule uses h1: mu=delta
c *** idf=2
c
        if (cs-2.-dsq2.le.0.0)then
          nn=1
          diff1=0.0
          diff2=0.0
        else
          nn=nn+1
        end if
        return
      end
*****************************************************************
      subroutine musum2(n,x,dsq2,rho)
c
        real arlm2,summ2,x(2)
        logical flagm2
        double precision dchisq,cs,limit
        integer count,count2,reps
        common arlm1,arlm2,count,count1,count2,flagm1,flagm2,
     β        limit,reps,summ1,summ2
c
c *** initialize array at beginning of each run.
c
        if (count .eq. 1) then
          arlm2=0.0
          count2=0
```

```
          summ2=0.0
        end if
c ***
        if (n .eq. 1) then
          cusum=0.0
          nn=1
        else
        end if
c
        chisq=( x(1)**2+x(2)**2-2.0*rho*x(1)*x(2) )/(1.-rho**2)
        cusum=cusum+chisq
c
        cs=cusum
        idf=2*nn
        if (1.0d0-dchisq(cs,idf).le.limit) then
          count2=n
          flagm2=.false.
          summ2=summ2+count2
        end if
c
c ** zeroing out rule uses h1: mu=delta
c
        if (cusum-(idf+nn*dsq2).le.0.0) then
          cusum=0.0
          nn=1
        else
          nn=nn+1
        end if
        return
      end
*******************************************************************
      function trpnrm(ix)
c
c *** generates unit normal deviate by composition method of
c ***ahrens and dieter.  the area under normal curve is divided
c *** into 5 different areas
c *** input:
c ***   ix = random number seed
c *** auxiliary routines:
c ***   unif
c
        u=unif(ix)
        u0=unif(ix)
        if(u.ge. 0.919544 )goto 160
c
c *** area a, the trapezoid in the middle
```

```
c
        trpnrm=2.40376*(u0+u*.825339)-2.11403
        return
 160    if(u.lt. 0.965487)goto 210
c
c *** area b
c
 180    trptmp=sqrt(4.46911-2*alog(unif(ix)))
        if(trptmp*unif(ix).gt.2.11403)goto 180
        goto 340
 210    if(u.lt. 0.949991)goto 260
c
c *** area c
c
 230    trptmp=1.8404+unif(ix)*.273629
        if(.398942*exp(-trptmp*trptmp/2)-.443299+trptmp*.209694
     β    .lt.unif(ix)*4.27026e-02)goto 230
        goto 340
 260    if(u.lt. 0.925852)goto 310
c
c *** area d
c
 280    trptmp=.28973+unif(ix)*1.55067
        if(.398942*exp(-trptmp*trptmp/2)-.443299+trptmp*.209694
     β     .lt.unif(ix)*1.59745e-02)goto 280
        goto 340
 310    trptmp=unif(ix)*.28973
c
c *** area e
c
        if(.398942*exp(trptmp*trptmp/2)-.382545.lt.unif(ix)
     β    *1.63977e-02) goto 310
 340    if(u0.gt.0.5)goto 370
        trptmp=-trptmp
 370    trpnrm=trptmp
        return
      end
*****************************************************************
      function unif(ix)
c
c *** portable random number generator using the recursion:
c*** ix=16807*ix mod (2**31-1)  using only 32 bits, including
c *** sign.
c *** input:
c ***   ix = integer greater than 0 and less than 2147483647
c *** outputs:
```

```
c ***   ix= new pseudorandom value,
c ***   unif= uniform fraction between 0 and 1.
c
        k1=ix/127773
        ix=16807*(ix-k1*127773)-k1*2836
        if(ix.lt.0)ix=ix+2147483647
        unif=ix*4.656612875e-10
        return
      end
******************************************************************
      double precision function dchisq(x,n)
c
c *** computes the cum distr. function p[y<=x] for a chi sq r.v.
c *** y w/n df auxiliary function:
c ***   dnml
c
        double precision a,y,s,e,c,z,x1,x,dnml,pi,one,zero,half,
     β                       largex
        logical bigx
c
c*** largex is the largest value of x such that dexp(-x/2) is
c *** accurate
c
        data largex,pi/177.44567822335,3.141592653589793/
        data one,zero,half/1.d0,0.d0,.5d0/
        a=half*x
        i=mod(n+1,2)
        bigx=.false.
        if(x.gt.largex)bigx=.true.
        y=dexp(-a)
        if(n.eq.1.or.bigx)y=zero
        s=y
        if(i.eq.0)s=2*dnml(-dsqrt(x))
        if(n.eq.1)goto 30
        x1=half*(n-1)
        z=(one+i)/2
        if(.not.bigx)goto 40
        e=zero
        if(i.eq.0)e=dlog(dsqrt(pi))
        c=dlog(a)
 20     if(z.gt.x1)goto 30
        e=e+dlog(z)
        s=s+dexp(c*z-a-e)
        z=z+one
        goto 20
 30     dchisq=one-s
```

```
      goto 9999
 40   e=one
      if(i.eq.0)e=one/dsqrt(pi*a)
      c=zero
 50   continue
      if(z.gt.x1)goto 60
      if(e.lt.2.d18)goto 55
      dchisq=dnml( dsqrt(2*x)-dsqrt(s*n-1) )
      goto 9999
 55   continue
      e=e*a/z
      c=c+e
      z=z+one
      goto 50
 60   dchisq=one-c*y-s
 9999 continue
      return
     end
*******************************************************************
     double precision function dnml(x)
       double precision x,y,s,rn,zero,one,erf,sqrt2,pi
       data sqrt2,one/1.414213562373095,1.d0/
       data pi,zero/3.141592653589793,0.d0/
       y = x/sqrt2
       if (x.lt.zero) y=-y
       s = zero
       do 1 n = 1,37
          rn = dfloat(n)
 1        s = s + dexp(-rn*rn/25.)/n * dsin(2.*n*y/5.)
       s = s + y/5
       erf = 2*s/pi
       dnml = (one + erf)/2
       if (x.lt.zero)  dnml = (one - erf)/2
       if (x.lt.-8.3d0) dnml = zero
       if (x.gt.8.3d0) dnml = one
       return
     end
```

```
      program  ci
        character*64 nfile
        sqrt61=sqrt(61.)
2       write(*,*)'Enter name of arl data file'
        read(*,*)nfile
        open(unit=1,file=nfile,status='OLD')
        sum1=0.0
        sum2=0.0
        sq1=0.0
        sq2=0.0
        do 10 i=1,61
          read(1,*)arl1,arl2
          sum1=sum1+arl1
          sum2=sum2+arl2
          sq1=sq1+arl1**2
          sq2=sq2+arl2**2
10      continue
        ave1=sum1/61.
        ave2=sum2/61.
        std1=sqrt(( sq1-61.*(ave1**2) )/60.)
        std2=sqrt(( sq2-61.*(ave2**2) )/60.)
        temp1=2.000*std1/sqrt61
        temp2=2.000*std2/sqrt61
        cl1=ave1-temp1
        cl2=ave2-temp2
        ch1=ave1+temp1
        ch2=ave2+temp2
        write(1,*)
        write(1,*)cl1,' <arl1< ',ch1,'  arl1=',ave1,' sd=',std1
        write(*,*)cl1,' <arl1< ',ch1,'  arl1=',ave1,' sd=',std1
        write(1,*)
        write(1,*)cl2,' <arl2< ',ch2,'  arl2=',ave2,' sd=',std2
        write(*,*)cl2,' <arl2< ',ch2,'  arl2=',ave2,' sd=',std2
        write(1,*)
        close(1)
        goto 2
      end
```

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