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A variable sampling frequency cumulative sum control chart scheme

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A VARIABLE SAMPLING FREQUENCY CUMULATIVE SUM CONTROL CHART SCHEME

by

Stefan Leopold Myslicki

A Thesis Submitted to the Faculty of the

DEPARTMENT OF SYSTEMS AND INDUSTRIAL ENGINEERING

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For the Degree of

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In the Graduate College

THE UNIVERSITY OF ARIZONA

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ABSTRACT

This study uses Monte Carlo simulation to examine the performance of a variable frequency sampling cumulative sum control chart scheme for controlling the mean of a normal process. The study compares the performance of the method with that of a standard fixed interval sampling cumulative sum control chart scheme.

The results indicate that the variable frequency sampling cumulative sum control chart scheme is superior to the standard cumulative sum control chart scheme in detecting a small to moderate shift in the process mean.

CHAPTER 1

INTRODUCTION

Control charts are used to monitor the quality level of repetitive production processes through statistical means. Control charts try to differentiate variation in the quality of the process between variation due to assignable causes and that due to random behavior. Since variation caused by random behavior conforms to statistical laws, if data from a process is found to have a statistical pattern which might reasonably be produced by such random behavior, then it is assumed that the process is operating without assignable causes present. The process is then said to be in control. Otherwise, it is concluded that one or more assignable causes are present and that the process is operating out-of-control.

Cumulative sum (CUSUM) control charts were first introduced by Page in 1954 [19]. The concept of this control chart method is based on the sequential sampling theories of Wald. CUSUM control charts have been shown to be extremely effective when there is a small to moderate shift from the target value of the process for the quality characteristic of interest [4,8,25].

In this study, Monte Carlo Simulation methods are used to study the performance of a variable frequency sampling (VFS) CUSUM control chart scheme in which the time interval between samples depends on where the last sample plots on the control chart. The concept of average time to detection (ATTD) will be introduced to compare the VFS scheme to the standard fixed sampling interval CUSUM scheme. The VFS CUSUM scheme will be shown to respond much more quickly to shifts in the process mean than the traditional fixed sampling frequency CUSUM scheme.

The following chapter contains a review of standard control chart procedures and enhancements based on these schemes. A discussion of run length and time to detection as measures of performance and the various analytical procedures used to compute the distribution and average of run lengths of a standard CUSUM is contained in Chapter 3. The VFS CUSUM procedure is detailed in Chapter 4 and the techniques used in the Monte Carlo simulation explored in Chapter 5. The results of the simulation are presented and summarized in Chapter 6. Conclusions and areas for further study are explored in Chapter 7.

CHAPTER 2

REVIEW OF SOME CONTROL CHART SCHEMES

The purpose of this chapter is to review some standard process control chart schemes and describe some modifications and enhancements of these schemes. For a sequence of successive samples from a process with an unknown population parameter θ of some measurable quality characteristic, X, from a continuous production process, control charts can be thought of as repeated tests of significance with the null and alternative hypotheses in the form:

$$H_0 : \theta = \theta_0$$

versus

$$H_1 : \theta = \theta_1$$

where θ_0 represents the target value for the quality characteristic and $\theta_1 = \theta_0 + D$, where D is some specified non-zero shift in the process parameter. The purpose of these charts is to control the process parameter θ.

In the strictest sense, control charts are not true statistical hypotheses tests because the null hypothesis H_0 is never accepted and can only be rejected. When the control chart shows no evidence that the process is out-of-control the process is allowed to continue to operate. Yashchin (1985) [29] has described control charts as dynamic tests of hypotheses wherein no acceptance of H_0 can ever occur because, as the process operates, shifts in the process parameter may occur. The only decision is whether or not to conclude H_1 is true. Thus, for the purpose of this review and in the study which follows, let

$$H_0 : \text{denote process is in control}$$

and

$$H_1 : \text{denote process is out-of-control with } \theta = \theta_1.$$

Furthermore, the process mean μ will be considered as the parameter of interest with the assumption that the quality characteristic is normally distributed with known and constant variance σ^2. Also, without loss of generality, it will be assumed that when the process is in control, the process mean μ_0 is equal to zero and that the process standard deviation σ is one.

The Shewhart $\bar{X}$ Chart

The simplest form of a control chart is an $\bar{X}$ chart or Shewhart chart. In the procedure for using a Shewhart chart, a sample of size n is taken at regular (i.e., fixed) time intervals. The sample mean of the quality characteristic of interest ($\bar{X}$) is measured and plotted on the control chart versus time. If the value falls outside predetermined limits, known as control limits, then the process is deemed to be out-of-control.

Two-sided Shewhart charts are used when both positive and negative changes in the process parameter are to be detected. Two-sided Shewhart charts have a centerline and both upper and lower control limits. One-sided Shewhart charts have only one control limit depending on whether changes in the positive direction (with a single upper control limit) or the negative direction (with a single lower control limit) are of interest.

Shewhart charts are constructed with the centerline set to the target value μ_0 and the upper and lower control limits, UCL and LCL respectively, are taken as

$$UCL = \mu_0 + k\sigma / \sqrt{n}$$

$$LCL = \mu_0 - k\sigma / \sqrt{n}$$

The width parameter k is often taken to be 3 [7].

$\bar{X}$ Control Chart Schemes with Warning Limits

Warning limits have been proposed by Page (1955) for use in conjunction with $\bar{X}$ control chart schemes [20]. Page recommended that when the $\bar{X}$ value is between 2 $\sigma_{\bar{X}}$ and 3 $\sigma_{\bar{X}}$, the process be declared out-of-control if t consecutive sample means fell above the 2 $\sigma_{\bar{X}}$ warning limit.

The Western Electric Statistical Quality Control Handbook (1956) [27] recommends the following conditions for declaring the process out-of-control using a Shewhart scheme;

1. If 1 value > 3 $\sigma_{\bar{X}}$
2. If 2 of 3 consecutive values are > 2 $\sigma_{\bar{X}}$
3. If 4 of 5 consecutive values are > 1 $\sigma_{\bar{X}}$
4. If 7 of 8 consecutive values plot above the centerline.

In both Page's and the Western Electric scheme runs tests are applied to make an out-of-control decision. Page refers to the $2\sigma_{\bar{X}}$ threshold in his version as a warning limit and this has become the standard definition of a warning limit in control chart schemes.

The Cumulative Sum (CUSUM) Chart

The cumulative sum (CUSUM) control chart procedures were introduced by Page in 1954 [20]. The cumulative sum scheme is based on the sequential sampling theories of Wald. In the CUSUM procedure, information from previous sample results is incorporated into the decision statistic. The CUSUM chart has some distinct advantages over the Shewhart procedure. These include:

1. For an equal number of false alarms, CUSUM tests can be made to react more quickly to a small shift in the population mean of the quality characteristic.

2. When and if an out-of-control situation is signaled, the CUSUM chart can easily estimate when the change first occurred.

One-sided and two-sided CUSUM schemes can be devised to detect either upward or downward shifts in the process mean or to detect a shift in either direction from the specified target value. This study is confined to the one-sided CUSUM procedure to detect shifts in the positive (upward) direction from the target value. Since the two-sided procedure can be thought of as two one-sided procedures operating simultaneously, the concepts discussed can be applied to the two-sided scheme, although the various statistical measures involved will not be the same as in the one-sided case.

The procedure for using a CUSUM control chart to control the mean of a process consists of taking samples of size n at regular fixed intervals of f time units. The sample average ($\bar{X}$) is calculated and the value of the CUSUM statistic

$$S_r = \max[0, \bar{X}_r - k + S_{r-1}],$$

is plotted against time, where r is the sample number, k is a specified constant reference value, and $\bar{X}_r$ is the mean of the r^{th} sample. The initial starting value of the CUSUM statistic, S_0, is usually taken to be zero. Lucas and Crosier (1982) discuss non zero values for S_0 (see next section). If S_r exceeds a specified control limit h, the process is deemed to be out-of-control. The "zeroing out" feature is to ensure the quick response of the chart to any shift in the process average and to periodically eliminate older observations from the data base.

This study uses the tabular version of the CUSUM scheme. Another form, known as the V-mask, was proposed by Page (1954). The equivalence of the V-mask and tabular versions is shown in Kemp (1960) [11] and Van Dobben de Bruyn (1968) [25].

The values of h and k are chosen to optimize the detection of a specific shift D, in the process mean from μ_0 to $\mu_1 = \mu_0 + D$. These parameters should be chosen so that such a shift is detected with sufficient rapidity without an excess of false out-of-control signals given when $\mu = \mu_0$. For a control scheme designed to detect such a shift from μ_0 to $\mu_1 = \mu_0 + D$, a value $k = D/2$ has been shown to be optimal by Kemp (1961) and others [11,5].

To select the value of the control limit h, given the optimal choice for k, the concept of run length is needed. Run length is a random variable which represents the total number of samples needed to first signal an out-of-control condition given the value of the process parameter. The expectation or mean of this random variable is called the average run length or ARL. The value of h is chosen to give the longest ARL when the process is in control ($\mu = \mu_0$), (i.e., maximize the average time between false alarms), while yielding an adequately small ARL when the process is out-of-control ($\mu = \mu_1$) (i.e., minimize the average time to detection of a shift in the process parameter).

Improvement and Enhancements of CUSUM Control Chart Schemes

Numerous schemes have been developed based on the classical CUSUM scheme of Page. Perhaps the simplest modification was made by Lucas and Crosier (1982) [14] in which they proposed a fast initial response (FIR) feature. This feature consists of not starting the CUSUM statistic, S_0, at zero at start-up, or after an out-of-control signal, but rather to an initial "head start" value. For one-sided CUSUM schemes, they recommend a head-start value of half the decision interval or h/2. Using ARL values calculated by the Markov chain approach of Brook and Evans, they show that with only a small (<5%) decrease in ARL when $\mu = \mu_0$ they can obtain a 30-40% decrease in the ARL's when $\mu = \mu_1$ (depending on the difference D = ($\mu_1 - \mu_0$)).

Munford (1980) [16] proposed a control chart scheme based on cumulative scores in which $\bar{X}_r$ values less than a specified negative constant (i.e., $\bar{X}_r \leqslant -c$) are assigned a score of -1, values of $\bar{X}_r$ near the target value (i.e., $-c < \bar{X}_r < +c$) are assigned a score of zero and large positive values $\bar{X}_r$ (i.e., $\bar{X}_r \geqslant c$) assigned a score of +1. Munford's cumulative score procedure works in a manner similar to a CUSUM scheme with a running sum of the scores and a "zeroing out" procedure.

Lucas (1982) [15] presents a combined Shewhart-CUSUM quality control scheme which include Shewhart control limits in the standard CUSUM procedure. This modification simply adds the qualification that if any $\bar{X}_j \geqslant 3\sigma$ then the process is declared to be out-of-control, regardless of the status of the CUSUM statistic value. Ncube and Woodall (1984) [17] use Munford's cumulative score procedure combined with the addition of Shewhart control limits in a similar fashion to the Lucas scheme.

In this thesis another modification to the CUSUM scheme will be introduced involving variable frequency sampling dependent on the value of the CUSUM statistic. This scheme is described in Chapter 4.

CHAPTER 3

RUN LENGTHS FOR CUSUM CHARTS

In this chapter the justification for using run length as the measure of performance for CUSUM control charts will be explored. Also described are the various methods used to approximate the probability distribution of CUSUM run lengths. Finally, the reasons for using average run length (ARL) as the basis for comparing the effectiveness of modified CUSUM schemes will be summarized and various techniques listed for computing CUSUM ARL approximations.

An out-of-control signal can be thought of as an alarm detecting a shift in the process mean. A false alarm would be given if the control chart signals an out-of-control condition when, in fact, no change in the process mean has occurred. The number of samples taken after a shift in the mean μ to μ_1 would be a measure of the responsiveness of the scheme. Therefore, a good CUSUM control scheme would signal a minimum of false alarms when $\mu = \mu_0$ and quickly signal an out-of-control situation after a shift to μ_1 has occurred.

These two goals require conflicting levels of h, the out-of-control limit. A large value for h would keep the number of false alarms low (i.e., long run length when $\mu = \mu_0$) but would increase the run length needed to signal a true out-of-control condition. Correspondingly, a low value for h would increase the responsiveness of the scheme to a shift in the process mean but would also trigger a high rate of false alarms. Therefore, a compromise in the value of h is required to keep the run length as long as possible when the process is in control with an adequately small run length when the process mean shifts to $\mu = \mu_1$.

Recall that run length has been previously defined to be a random variable (now denoted as RL), so its probability distribution will need to be considered. Since successive points on a CUSUM chart are not independent, the exact probability distribution of the run length cannot easily be determined analytically and has to be obtained numerically. However, the run length probability distribution for a given combination of μ and h has been approximated by a number of different approaches. Ewan and Kemp (1960) [8] provide two expressions that can be evaluated recursively to determine the run length distribution. Brook and Evans (1972) use a different approach in which the CUSUM scheme is regarded as forming a Markov chain [6]. A transition probability matrix is derived for this chain and its properties used to determine the moments and percentage points along with the expected value (ARL) of the run length. Waldmann (1986) uses an extrapolation method based on a series of iterates (Pr (RL>0), Pr (RL>1), etc.) to derive upper and lower bounds for the probability distribution of run length [26].

Woodall (1983) [28] uses a method closely related to the method of Aroian (1976) [2] by considering CUSUM schemes as series of sequential probability ratio tests. Aroian's method usually requires numerical integration to find the required probabilities at each stage of a sequential probability ratio test. Woodall uses a system of algebraic linear expressions as approximations to these integral equations in his approach.

The references cited above do not disagree with the conclusion of Brook and Evans (1972) that the probability distribution of run length is closely analogous to the univariate geometric form:

$$P(RL = N) = q^{N-1}p \quad (N=1,2,3,\ldots)$$

Figure 1 (after Brook and Evans, 1972) shows, for comparative purposes, a run length distribution along with the corresponding probabilities for a geometric distribution with the same mean of 3.0.

Since the mean of the run length distribution is the ARL, the percentage points and the bounds of the run length distribution can be approximated by the geometric distribution equivalents if the ARL is known since p=1/ARL.

Because of the difficulty in comparing percentage points, bounds, etc., of the run length distribution and because the expected or average run length value can be computed with relative ease, ARL is the most widely accepted measure of performance for control charts and the standard for comparison of various CUSUM schemes. Several techniques are now considered for finding the ARL of CUSUM schemes.

Methods for Determining ARL of CUSUM Schemes

Page (1954) [19] has shown that the ARL for a CUSUM chart can be expressed as the ratio:

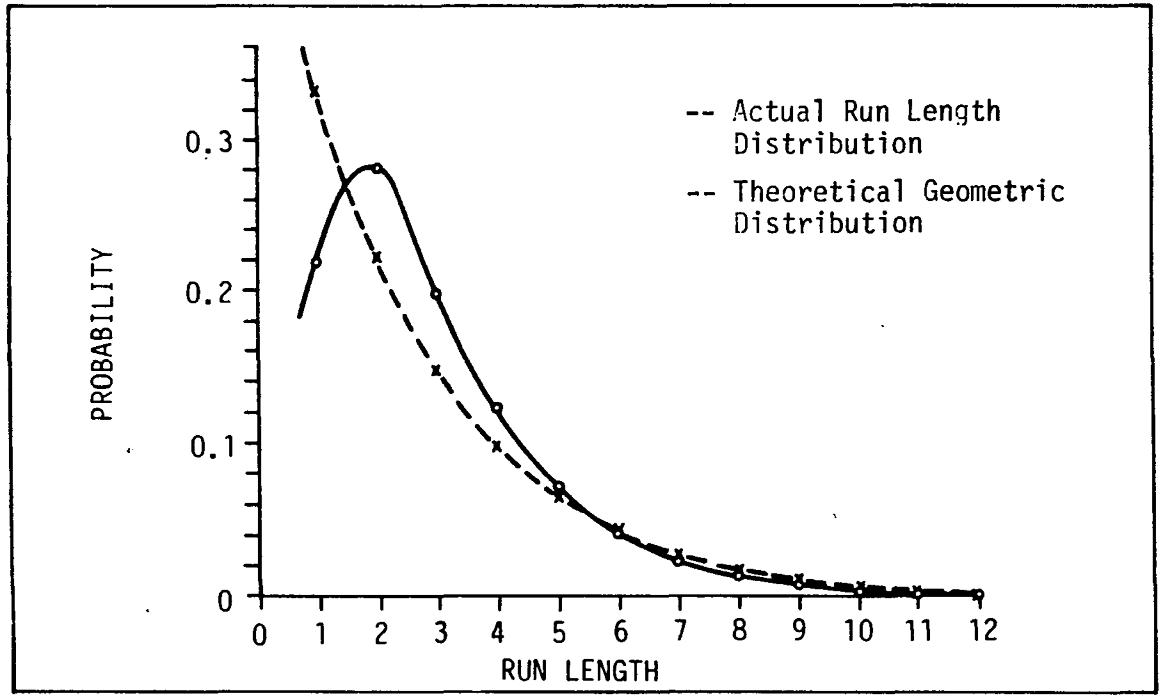

Figure 1. A Comparison of Actual and Theoretical Run Length Distributions with Identical Means (after Brook and Evans, 1972).

$$ARL = N(0)/(1-P(0))$$

where P(0) and N(0) are special cases for Z=0 of P(Z) and N(Z) where

$$P(z) = \int_{-\infty}^{-z} f(x)dx + \int_0^h P(x)f(x-z)dx, \quad 0 \leqslant z \leqslant h$$

$$N(z) = 1 + \int_0^h N(x)f(x-z)dx, \qquad 0 \leqslant z \leqslant h$$

and f(x) is the probability density function of X, the quality characteristic of interest. For the case when X is normal with mean μ and variance unity,

$$P(z) = \Phi(-z-\mu) + \int_0^h P(x)(2\pi)^{-1/2}\exp\{-(x-z-\mu)^2/2\}dx$$

and

$$N(z) = 1 + \int_0^h N(x)(2\pi)^{-1/2}\exp\{-(x-z-\mu)^2/2\}dx$$

where

$$\Phi(z) = (2\pi)^{-1/2} \int_{-\infty}^{z} \exp(-t^2/2)dt \text{ [24].}$$

Ewan and Kemp (1960) [8] took the expectation of their approximation of the run length distribution mentioned in the previous section and constructed a nomogram for determining the value of the control limit h when μ_0, μ_1, are specified along with the ARL desired under H_0 and the ARL desired under H_1.

Goel and Wu (1971) [9] solved the above integral equations of Page for determining ARL of a CUSUM scheme by converting the integral expressions to a series of simultaneous algebraic linear equations. These approximations to the integral expressions were then solved numerically by Gaussian quadrature using 15 Gaussian points. They also constructed a nomogram similar to that of Ewan and Kemp.

Vance (1986) [24] used the method of Goel and Wu (but with 24 Gaussian points) and wrote a FORTRAN program that calculates the ARL for a specified combination of μ and h. This program was used to validate the results of simulation procedures used in this study.

Reynolds (1975) [22] and Van Dobbyn de Bruyn (1968) [25] have both concluded computer simulation methods are an easy and sufficient means for approximating ARL's of CUSUM schemes. This was the methodology used in this study. A complete description of the simulation procedure is found in Chapter 5.

Simulation methods were used in this study to calculate ARL values for the standard and VFS CUSUM schemes presented. Simulation was used because the VFS CUSUM scheme did not lend itself to any of the other methods mentioned. It was thought best to use the same methods in determining ARL's for the standard CUSUM as the VFS scheme so that fair and accurate comparisons between the two schemes could be made.

CHAPTER 4

VARIABLE SAMPLING FREQUENCY CONTROL CHARTS

In this chapter the variable sampling frequency (VSF) control charts will be introduced. VSF control charts feature a variable time interval between samples. When the control chart statistic exceeds a specified value or warning limit but has not yet yielded an out-of-control signal, the process will be sampled sooner than if the control chart statistic indicates a very high probability of the process being in control. Thus, depending on the value of the control chart statistic, the time between samples will vary.

The concept behind VSF control charts is that when the process is in control, observations should cluster near the target value while if a shift in the process mean to $\mu = \mu_1$ should occur, there will be a corresponding upward shift in the observations from the process. If the process appears to be out-of-control, the process should be sampled again quickly, so that if a shift in the process mean has occurred, a full f time units have not elapsed before again sampling the process. (Recall that the standard or fixed sampling frequency charts require that f time units transpire between samples.) If the process appears highly likely to be in control, the time at which the process is again sampled can be increased.

The control limit, h, for a VSF control chart is defined in the same manner as for a standard fixed sampling interval control. That is, if the value of the control chart statistic exceeds h, the process is declared out-of-control and an assignable cause is sought. In addition to the control limit, a warning limit value g ($g \leqslant h$) is defined. Also, the warning zone is defined as the values between g and h. If the control chart statistic is in the warning zone; that is, if the statistic exceeds g but is still less than h, the sampling interval (time to the next sample) is shortened compared to a standard fixed sampling interval control chart. The time between samples when the statistic is in the warning zone will be designated as ε. Finally, the sampling interval when the control chart statistic is below the warning limit will be denoted as f'. This interval may be different than that of a comparable fixed sampling interval control chart scheme.

Although the ARL will be the same for VFS and traditional charts, the average time until detection of a shift in the process parameter may be different. A method of comparison will be considered based on time of detection.

The concept of average time to detection or ATTD will now be introduced and defined. Without loss of generality, assume that the time between samples in a standard fixed sampling interval control chart is one time unit. Since the run length has been defined as the number of samples required to first signal out-of-control, the run length is equivalent to the time required to give an out-of-control signal or time to detection. Consequently, average run length (ARL) is equivalent to average time to detection or ATTD. In a VSF control chart scheme, however, the time interval between samples depends on the value of the control chart statistic. Thus, the total time elapsed until an out-of-control signal is given is an appropriate measure of performance for these types of control chart schemes. The average time to detection (ATTD) is then defined as the average time required for a control chart scheme to signal an out-of-control condition.

It might be advantageous to clarify the concept of time in a control chart scheme. In a standard fixed interval CUSUM scheme, it is assumed that samples are taken at a constant interval from a production process, (i.e., one sample every 100 items). If it is assumed the process is actually producing items at a constant rate, then the time between samples corresponds to a constant sampling rate. Therefore, the time involved in control chart schemes refers to production time or time during which the process is actually producing items at a constant rate.

The VSF $\bar{X}$ control chart scheme uses the sample mean ($\bar{X}$) as the control chart statisitc. The control limit is set as before (UCL = $\mu_0 + k\sigma_{\bar{X}}$) and the warning limit g set to $\mu_0 + c\sigma_{\bar{X}}$ (where $0<c\leq k$).

The VSF CUSUM scheme is the subject of investigation in this study. It uses the CUSUM statistic, $S_r = \max(0, \bar{X}_r - k + S_{r-1})$ as the control chart statistic. Here again r is the sample number. The constant k is the same as a standard CUSUM and is determined in exactly the same fashion.

The values for ε and f', the sampling intervals when the value of the control chart statistic is above or below the warning limit g respectively, are determined prior to control chart implementation. The value for ε is usually the minimum sampling interval possible based and the particular process and inspection procedure. Determining the value for f', the sampling interval when the control chart statistic lies below the warning limit g, is related to how VSF and fixed time control charts can be compared. This issue is discussed in the following section.

Adjusting Time Between Samples to Compare Various Control Chart Schemes at Equal False Alarm Rates

To compare VFS and fixed sampling interval schemes fairly, the false alarm rate must be the same. Recall that a false alarm is an out-of-control signal that is given while $\mu = \mu_0$. Therefore, the false alarm rate for a control chart scheme can be defined as 1/ATTD when $\mu = \mu_0$.

The sampling interval for a VSF scheme is shortened to ε when the control chart statistic is in the warning zone between g and h. Therefore, the sampling interval when the statistic is below the warning limit g must be adjusted to a value of f' to compensate for the shortened interval ε such that the ATTD for a VSF scheme is exactly equal to the ATTD (ARL) of a standard control chart scheme when both operate with $\mu = \mu_0$. This adjustment is done assuming that both the standard and VSF control charts have identical out-of-control limits h. The value of f', for any combination of g and h, is based on the average number of samples falling below the warning limit g before the control chart scheme gives an out-of-control signal while $\mu = \mu_0$. The exact methodology and procedure used to determine the value of f' for the various VSF CUSUM schemes presented in this study is discussed in the next chapter along with a description of the simulation procedures and methods used to compare these VSF CUSUM schemes with standard fixed interval CUSUM schemes.

CHAPTER 5

DISCUSSION OF THE SIMULATION STUDY

Simulation can be defined as a computer-based numerical technique for the experimental study of a stochastic or deterministic process over time. Monte Carlo simulation denotes any numerical procedure utilizing random or pseudo-random numbers. Reynolds (1975) and Van Dobben de Bruyn (1968) both agree Monte Carlo simulation is an appropriate, sufficient and easy technique to estimate the ARL of classical fixed sampling interval CUSUM schemes.

This study involves Monte Carlo stochastic simulation of both the fixed sampling interval and VFS CUSUM schemes described in the previous chapter. Standard normal variables were generated to estimate the values of ARL and ATTD for various values of μ, h, k and g. The sample size n, was kept constant at one throughout all simulation procedures. Consequently, X will be used instead of $\bar{X}$ in the following descriptions. The simulation studies performed in this study were run on the University of Arizona's DEC VAX 8600 minicomputer.

Generation of Standard Normal Random Variables

The UNIF function of Bratley, Fox, and Schrage (1983) (a linear congruential generator) was used to obtain pseudo-random standard unit uniform values. The TRPNRM function (also Bratley, Fox and Schrage), [5] which calls UNIF, was used to generate standard normal deviates using the composition method of Ahrens and Dieter (1972) [1]. TRPNRM divides the standard normal curve into a set of three trapezoidal functions and two residual functions to exactly decompose the standard normal curve. This method for generating normal variables was used to avoid the cycling and correlation that occurs when a linear congruential generator is used with the Box-Muller method (Sanchez (1986), personal communication).

Replications

The number of replications to be performed for the simulation of each CUSUM scheme was limited by the cost and amount of available computer time. The theoretical probability distribution of CUSUM run lengths derived by methods such as the Markov chain approach of Brook and Evans (1972) [6] has its mean approximately equal to its standard deviation. Therefore, pilot runs were conducted to find the minimum number of replications necessary so that the average run length of a fixed time sampling CUSUM scheme was approximately equal to its sample standard deviation of the run length.

Simulation Procedure

The procedure used in the simulation of a VFS CUSUM scheme and methodology involved in assuring accurate comparisons to a classical fixed sampling interval CUSUM scheme are described in this section. Flowcharts of algorithms employed in the simulation procedure of both the fixed interval and VFS CUSUM scheme are presented.

The initial phase of the study was the simulation of a fixed time sampling CUSUM scheme to determine two values of h, the out-of-control limit, such that the ARL when the process is in control yielded "standard" ARL values of around 200 and 700, respectively. These two values were taken to be the standard values for comparative purposes. An ARL approximately equal to 200 when $\mu = \mu_0$ was used by Crosier (personal communication), and by Pignatiello, Korpela and Runger (1987) [21] for comparing new multivariate CUSUM schemes with others. A $3\sigma_{\bar{X}}$ control limit Shewhart $\bar{X}$ chart has an ARL of approximately 700 so this value is often a literature standard. Values for the out-of-control mean μ_1 were taken to be $\mu_1 = 1.0$ in one set of simulations and $\mu_1 = 0.5$ in another. These values were chosen because a CUSUM scheme is usually used when it is desired to detect small changes in the process mean. A Shewhart control chart is more responsive to large shifts ($\mu_1 \geqslant 3\sigma$) in the process mean than a CUSUM scheme and is simpler to operate. Consequently, the Shewhart chart is usually employed in those situations when large shifts are anticipated.

The flow chart of the procedure used in the simulation of the fixed sampling interval CUSUM scheme is shown in Figure 2. First, the value of the CUSUM statistic, S_0, was initialized at zero. Then a pseudo-random standard normal deviate was generated. This value was taken to be a sample of size n=1 from a process. The reference value k was taken as 0.5 since shifts from $\mu = 0$ to $\mu = 1.0$ are of interest.

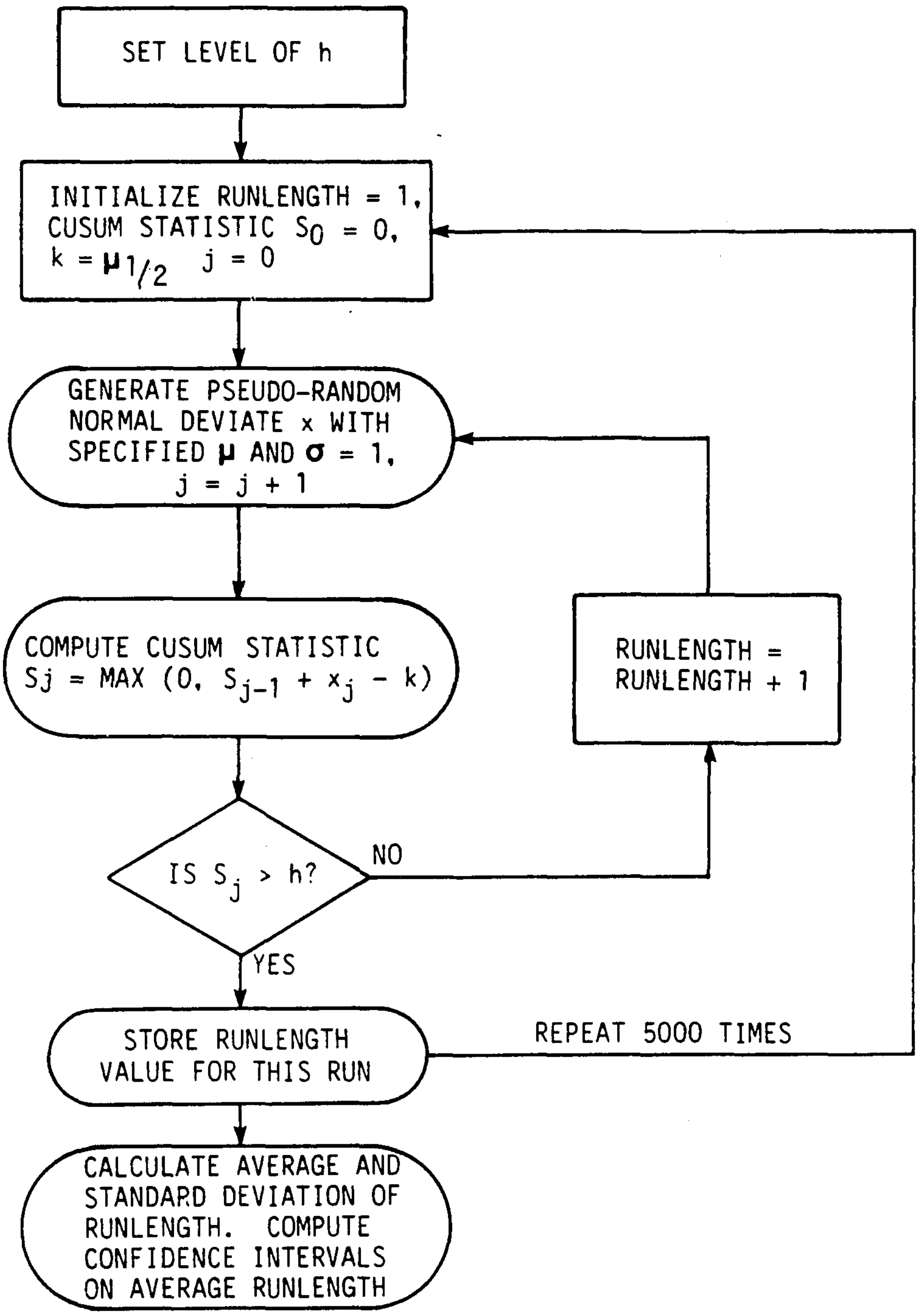

Figure 2. Flowchart Algorithm of Monte Carlo Simulation of a Standard Fixed Sampling Interval CUSUM Scheme.

Pseudo-random standard normal deviates with a mean of zero were generated and the CUSUM statistic was computed until the value of CUSUM statistic exceeded the control limit h. When this occurred, the number of samples taken (i.e., the number of times a standard normal value was generated) was recorded. This value is the run length for this particular simulated run. A total of 5000 replications were made and an average run length and standard deviation was calculated. This average is the estimated ARL. The estimated standard deviation was used in calculation of confidence limits on the ARL. The simulation procedure was run for all levels of h from 0.1 to 5.0 in increments of 0.1. The simulation was repeated for the case where $\mu = 1.0$ to give a basis for comparison of the ATTD's found in the fixed interval case with those of the VFS CUSUM scheme. For fixed interval CUSUM simulation the values of h that yielded close to the "standard" ARL result of 200 and 700 were h=3.5 and h=4.7, respectively.

A simulation of a VFS CUSUM scheme was run using the above values of h for values of g, the warning limit, ranging from 0.1 to h in increments of 0.1.

First consider the case of the VFS CUSUM when $\mu = 0.0$. Figure 3 shows a flowchart of the algorithm used in the simulation of a VFS CUSUM scheme for a given combination of g and h when $\mu = \mu_0$.

A standard normal deviate is generated and the CUSUM statistic computed. However, if the value of the CUSUM statistic is in the warning limit zone between g and h, it is not counted in the running total of samples taken before the procedure gave an out-of-control

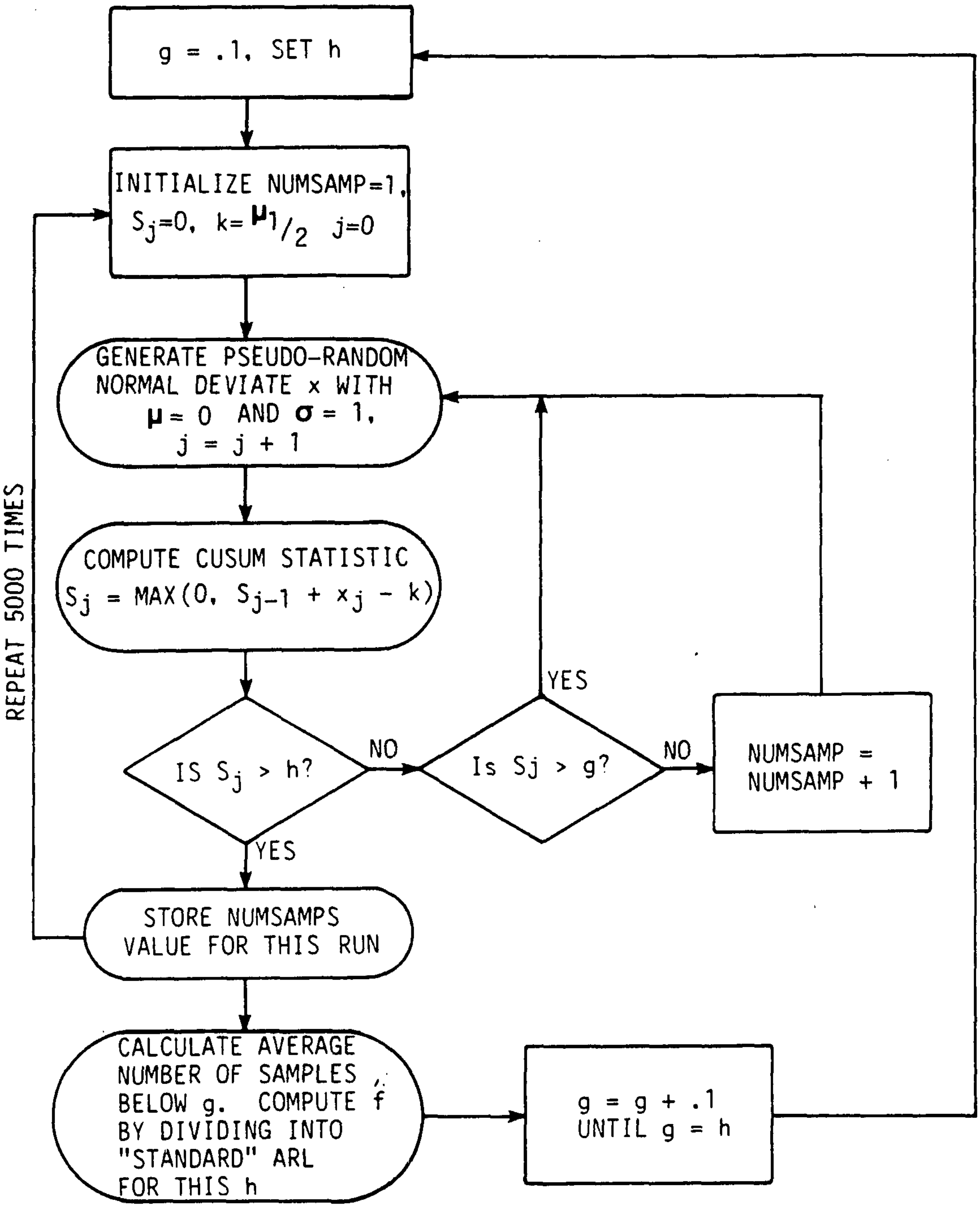

Figure 3. Algorithm of VFS CUSUM Monte Carlo Simulation Procedure when $\mu = \mu_o = 0.0$.

signal and stopped. In other words, samples are only counted in the "run length" of this scheme if the value of the CUSUM statistic at that point is below g. An average of the number of samples per run falling below the warning limit g is then calculated using the results of 5000 replications. The "standard" ARL for the same value of h was divided by this average to estimate f', the sampling interval between samples when the CUSUM statistic is below g.

Since the procedure to estimate f' is at the heart of this study, it might be helpful to give an example. Say it was found, for a particular value of g, that an average of 100 points fell below the warning limit before the scheme gave an out-of-control signal. If h=3.5, the "standard" ARL would be 200. Consequently, the estimated f' for this particular value of g would be 200/100 = 2.00. This means that the sampling interval when the CUSUM statistic's value is below g should be doubled from that employed in the standard fixed sampling interval CUSUM scheme. This assumes that no time elapses between samples when the CUSUM statistic lies in the warning zone between g and h (i.e., $\varepsilon=0$). This procedure for obtaining the estimate of f' (hereafter designated $\hat{f}'$) allows the value for the ATTD for a VFS CUSUM scheme to be set equal to the ARL for a fixed interval scheme when $\mu=\mu_0$. (Again, this assumes that the sampling interval for a classical fixed interval scheme is one time unit.) This is the necessary condition for comparison of the two schemes at identical false alarm rates.

The second phase for $\mu=\mu_1$ of the simulation of the VFS CUSUM schemes involved using the value for $\hat{f}'$ found in the first phase when $\mu = \mu_0$. The algorithm used in the simulation of the VFS CUSUM scheme when $\mu = \mu_1$ is shown in Figure 4. For every combination of g, h and $\hat{f}'$, the VFS simulation procedure was run in a manner similar to the first phase but with $\mu = 1.0$. A pseudo-random normal deviate with μ $\mu = 1.0$ was generated, the CUSUM statistic computed and the number of times the statistic fell below g before an out-of-control signal is counted. The procedure was repeated for 5000 replications. An average for the number of samples per run falling below g was calculated. This number was then multiplied by $\hat{f}'$ to estimate the ATTD when $\mu = \mu_1$.

An identical set of simulations were run for the case $\mu = 0.5$. The h values yielding the "standard" ARL values were found to be h=5.8 and h=7.9 for ARL's of 200 and 700, respectively.

A summary of the results of all simulations is found in the next chapter.

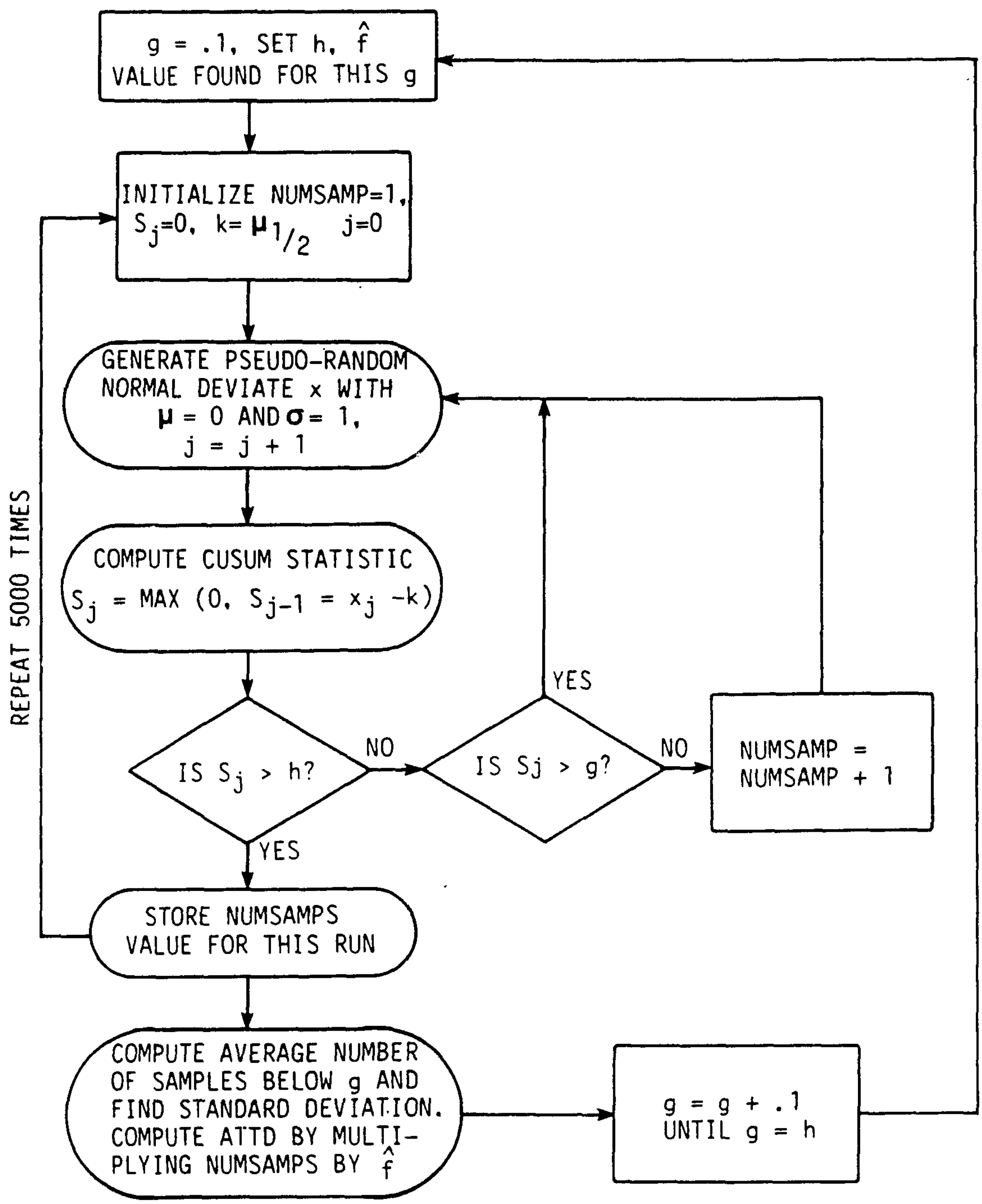

Figure 4. Algorithm of VFS CUSUM Monte Carlo Simulation Procedure when $\mu = \mu_1$.

CHAPTER 6

RESULTS AND DISCUSSION OF THE SIMULATION STUDY

This chapter presents and discusses the results of the Monte Carlo simulation of the standard fixed sampling interval CUSUM scheme and the variable frequency sampling CUSUM schemes. Results of average run length values for a standard fixed sampling interval CUSUM calculated by the FORTRAN program of Vance (1986) are also reported for comparative purposes.

The simulation of the fixed sampling interval CUSUM was done to determine the values of h (given k) that yield, when $\mu = \mu_0$, the "standard" ARL's of near 200 and 700, as described in the previous chapter. The simulation of the fixed sampling interval CUSUM for the case when $\mu = \mu_1$ was also done to determine the ARL's and confidence levels for comparing the response to a shift in the process mean of a standard CUSUM with the VFS CUSUM scheme.

The VFS CUSUM simulation results are presented with the estimated sampling interval $\hat{f}'$ factor needed to adjust the average time to detection of the VFS CUSUM to the ATTD values of the classical CUSUM scheme when both are run with $\mu = \mu_0$. The exact procedure involved and reasons for calculating an $\hat{f}'$ factor are explored in the previous chapter.

Results of Simulation of Fixed Sampling Interval CUSUM Scheme and Comparison of Results with Average Run Length Approximations of Vance

Table I shows the results of the Monte Carlo simulation of the standard fixed sampling interval CUSUM scheme when $\mu = \mu_0$ for values of the control limit, h, from .1 to 5.5. Without loss of generality, μ_0 is assumed to be zero. For this table the out-of-control mean is taken to be $\mu_1 = 1.0$. Therefore, the constant k used in computation of the CUSUM statistic was set at k=0.5. For each value of h, 5000 replications were made to calculate a point estimate of the average run length for that given control limit. The sample variance (using the unbiased estimator) was also calculated for the purpose of determining confidence limits on ARL, assuming this mean is normally distributed as postulated by the Central Limit Theorem. Table I shows the ARL, the lower and upper confidence bounds of a 90% two-sided confidence interval for each specified level of h along with the ARL values derived by the Vance program for identical h values.

Figure 5 shows the relationship between h and ARL obtained by simulation when the process is in control (i.e., $\mu = \mu_0 = 0.0$). The upper confidence bound (U.C.B.), lower confidence bound (L.C.B.), and the ARL values obtained from the approximations generated by the computer program of Vance were not plotted as they all lie too close to the ARL values for the scale used. Figure 5 shows that h increases rapidly with ARL values less than 200 but that the slope of the line increases less rapidly for ARL values greater than 200.

TABLE I. FIXED INTERVAL CUSUM

Average Run Lengths

$\mu = 0.0$

h	90% Two-sided Confidence Interval L.C.B.	ARL	U.C.B.	ARL Vance's Approximation
0.1	3.5118	3.5824	3.6530	3.64
0.2	4.0099	4.0942	4.1786	4.10
0.3	4.5536	4.6490	4.7444	4.62
0.4	5.0120	5.1194	5.2268	5.23
0.5	5.8571	5.9834	6.1097	5.93
0.6	6.5986	6.7434	6.8882	6.72
0.7	7.4207	7.5790	7.7373	7.63
0.8	8.5044	8.6886	8.8728	8.68
0.9	9.5607	9.7700	9.9793	9.86
1.0	10.7425	10.9758	11.2091	11.21
1.1	12.2205	12.4892	12.7579	12.74
1.2	14.0579	14.3724	14.6869	14.47
1.3	16.2259	16.5876	16.9493	16.42
1.4	18.3783	18.7774	19.1765	18.62
1.5	20.1572	20.6044	21.0516	21.09
1.6	23.9128	24.4476	24.9824	23.85
1.7	26.5471	27.1468	27.7465	26.95
1.8	29.2046	29.8654	30.5262	30.41
1.9	33.9095	34.6760	35.4425	34.26
2.0	36.9690	37.8122	38.6554	38.55
2.1	42.4402	43.4092	44.3782	43.31
2.2	48.6459	49.7270	50.8081	48.60
2.3	52.9292	54.1090	55.2888	54.47
2.4	59.5276	60.8744	62.2212	60.98
2.5	65.7597	67.2924	68.8251	68.19
2.6	75.9018	77.6484	79.3950	76.17

TABLE I: continued

h	90% Two-sided Confidence Interval L.C.B.	ARL	U.C.B.	ARL Vance's Approximation
2.7	82.3807	84.2926	86.2045	85.01
2.8	92.8636	94.9772	97.0908	94.79
2.9	103.5828	105.9784	108.3740	105.61
3.0	117.0197	119.7730	122.5264	117.60
3.1	127.7303	130.6830	133.6357	130.85
3.2	142.6155	145.8552	149.0949	145.52
3.3	157.9247	161.6644	165.4041	161.75
3.4	174.8769	178.9160	182.9551	179.71
3.5	197.0709	201.8044	206.5379	199.57
3.6	213.5280	218.5958	223.6636	221.55
3.7	236.0244	241.5832	247.1420	245.86
3.8	274.3343	280.7936	287.2529	272.74
3.9	292.0853	298.8216	305.5579	302.48
4.0	331.6344	339.4964	347.3584	335.37
4.1	362.3826	370.7218	379.0610	371.74
4.2	396.8899	406.2308	415.5717	411.95
4.3	437.4636	448.0888	458.7141	456.42
4.4	492.7648	504.2704	515.7760	505.59
4.5	548.9067	561.5968	574.2869	559.95
4.6	604.3763	618.4306	632.4849	620.05
4.7	670.5361	686.5118	702.4875	686.49
4.8	741.4073	759.0190	776.6307	759.94
4.9	828.3122	848.0190	867.7258	841.13
5.0	883.1874	904.6750	926.1626	930.89
5.1	985.7119	1009.3584	1033.0049	1030.11
5.2	1118.6576	1145.0974	1171.5372	1139.78
5.3	1239.3846	1268.6670	1297.9493	1261.00
5.4	1366.4001	1398.6718	1430.9434	1395.00
5.5	1532.0813	1568.0786	1604.0759	1543.11

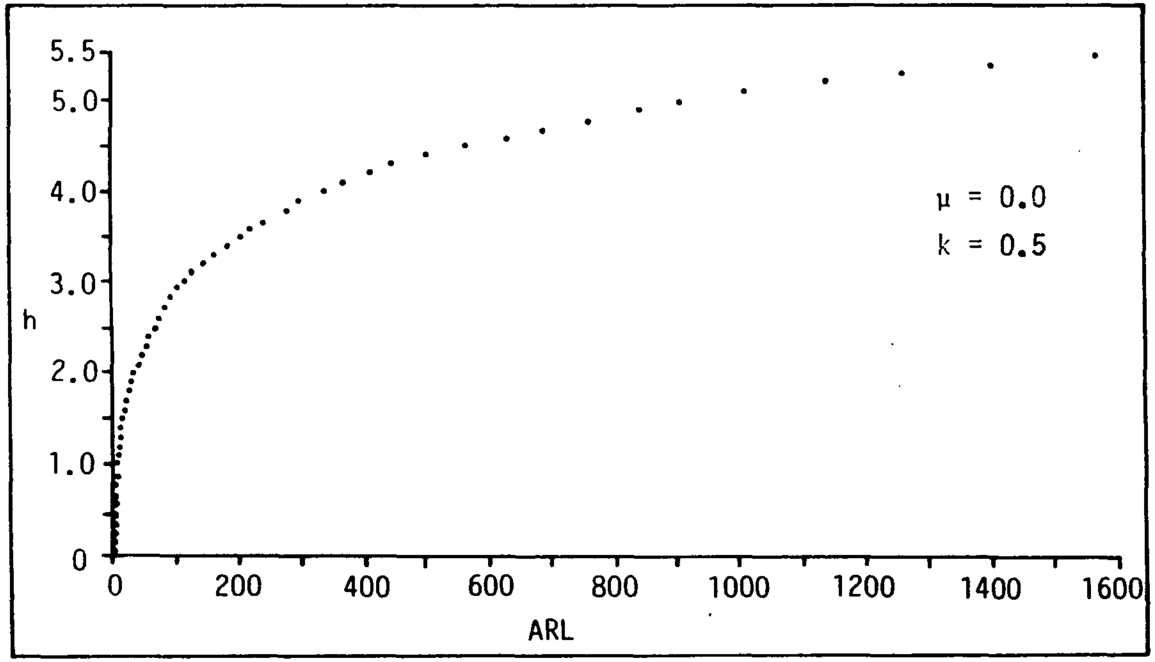

Figure 5. Average Run Length (ARL) vs. h
Results of Monte Carlo Simulation.

The average run lengths derived by Monte Carlo simulation agree with the ARL's given by the Vance program. This shows that the methods employed both in the simulation of the fixed interval CUSUM scheme and the VSF CUSUM scheme yield similar results when compared to a literature standard. It must be kept in mind that the Vance program involves only approximations of the integral equations involved in the calculation of the ARL and that the values generated by the Vance program do not necessarily reflect the "true" value of the ARL for a specified particular value of h. The high degree of correlation between the simulation results and those obtained through the approximation methods used in the Vance program lend validity to the Monte Carlo methods.

The values of the control limit h that yield the specified "standard" ARL values when $\mu = \mu_0$ for a standard CUSUM were found. The value of h equal to 3.5 gives an ARL of approximately 200. An ARL of approximately 700 was yielded by setting h to 4.7. A value for h equal to 4.71 using the Vance program gave an ARL of approximately 701. The value of h equal to 4.7 actually is 686.5. However, it was decided to use this value of h equal to 4.7 because the overall precision of the various CUSUM schemes were limited to .1. The 90% confidence interval calculated on the ARL when h=4.7 does include 700.

The ARL of a standard fixed sampling interval CUSUM with k=0.5 and $\mu = \mu_1 = 1.0$ for the above values of h are presented in Table II. Also presented are the lower and upper 90% two-sided confidence interval limits and the ARL values computed by the Vance FORTRAN program.

TABLE II. FIXED INTERVAL CUSUM

Average Run Lengths

$\mu = 1.0$

h	90% Two-sided Confidence Interval L.C.B.	ARL	U.C.B.	ARL Vance's Approximation
3.5	7.2694	7.3702	7.4710	7.39
4.7	9.7617	9.8862	10.0107	9.78

It will be assumed for the purposes of this study that if the one-sided 95% lower confidence limit on the difference between the ATTD for any VFS CUSUM scheme and the ARL for a fixed sampling interval CUSUM (both with identical levels of the control limit h) is greater than zero, then the VFS scheme is significantly more responsive to shifts in the process mean than the standard CUSUM. The confidence limit on the difference in ARL from ATTD was calculated using the method outlined in Hoel (1984) [10]. The difference between two normally distributed means, X and Y, can be shown to be also normally distributed with a mean equal to the difference in the individual means and a variance equal to the sum of the variances of the means. Or, in equation form:

$$\mu_{\bar{X}-\bar{Y}} = \mu_X - \mu_Y$$
$$\sigma^2_{\bar{X}-\bar{Y}} = \sigma^2_{\bar{X}} + \sigma^2_{\bar{Y}}$$

Given the large sample size (5000), we can replace the population mean and variance by their unbiased estimates. Therefore:

$$\mu_{\bar{X}-\bar{Y}} \cong \bar{X}-\bar{Y}$$
$$\sigma^2_{\bar{X}-\bar{Y}} \cong S^2_{\bar{X}} + S^2_{\bar{Y}}$$

The 95% lower one-sided confidence limit on the difference between ARL and ATTD values therefore is:

$$Pr(\bar{X}-\bar{Y}-1.645 \quad \sigma_{\bar{X}-\bar{Y}} > \mu_{\bar{X}-\bar{Y}}) = .95$$
$$Pr(ARL-ATTD-1.645\ S_{ARL-ATTD} > \mu_{ARL-ATTD}) = .95$$
$$95\%\ L.C.B. = ARL-ATTD-1.645 \sqrt{S^2_{ARL} + S^2_{ATTD}}$$

Results of Simulation of the VFS Cusum Scheme

Results of the simulation of the variable frequency sampling CUSUM scheme with the control limit h held at 3.5 are presented in Table III. The value of the constant k was set to be 0.5 (as in the fixed interval sampling case). The value of g, the warning limit, ranged from 0.1 to 3.5 in increments of 0.1. The ATTD and $\hat{f}'$ values were calculated in the manner described in the previous chapter. The values for the adjusted $\hat{f}'$ were found by simulating the VFS CUSUM scheme for the above set of conditions (h=3.5, k=0.5, g for values varying from .1 to 3.5) for the case where $\mu = \mu_0 = 0.0$.

The ATTD when $\mu = \mu_1 = 1.0$ is shown in Table III along with the lower and upper confidence bounds of the 90% two-sided confidence interval on these average values. A plus sign (+) signifies the 95% lower one-sided confidence limit on the difference between the ARL value and these ATTD values is greater than zero. The ATTD values of the VFS scheme were found to be significantly different than the ARL's of a standard CUSUM for all values of the warning limit g less than or equal to 3.3.

A series of graphs (Figures 6a,b,c) are presented for the results in Table III for the following set of variables; g versus $\hat{f}'$, g versus ATTD when $\mu = 1.0$ and $\hat{f}'$ versus ATTD when $\mu = 1.0$. The plot of g versus $\hat{f}'$ shows that there is little decrease in the $\hat{f}'$ values for warning limits greater than 2.0 but a steady increase in $\hat{f}'$ values as the warning limit tends toward 0.1. The relationship between the

TABLE III. VFS CUSUM

h = 3.5

		ATTD $\mu = 1.0$			
		90% Two-sided Confidence Interval			
g	$\hat{f}'$	L.C.B.	ATTD	U.C.B.	
0.1	1.731476	3.338321	3.394385	3.450450	+
0.2	1.617921	3.399770	3.457822	3.515873	+
0.3	1.533305	3.380084	3.438590	3.497095	+
0.4	1.497450	3.497683	3.560037	3.622391	+
0.5	1.422995	3.552280	3.613554	3.674829	+
0.6	1.346957	3.568444	3.631936	3.695427	+
0.7	1.342042	3.711104	3.776775	3.842447	+
0.8	1.250050	3.718981	3.787151	3.855322	+
0.9	1.232013	3.864020	3.931352	3.998684	+
1.0	1.220626	4.026940	4.098128	4.169317	+
1.1	1.197942	4.153268	4.224422	4.295577	+
1.2	1.161718	4.297814	4.372475	4.447135	+
1.3	1.129575	4.316314	4.391335	4.466355	+
1.4	1.145157	4.683521	4.761103	4.838685	+
1.5	1.125994	4.757105	4.838397	4.919689	+
1.6	1.088633	4.739976	4.817420	4.894864	+
1.7	1.065245	4.942339	5.022418	5.102498	+
1.8	1.067097	5.013113	5.091760	5.170406	+
1.9	1.060615	5.264503	5.348895	5.433238	+
2.0	1.037343	5.255573	5.340980	5.426387	+
2.1	1.057158	5.662585	5.752211	5.841835	+
2.2	1.028797	5.622904	5.711881	5.800858	+
2.3	1.008992	5.692807	5.779507	5.866207	+
2.4	1.037380	6.056868	6.148136	6.239404	+

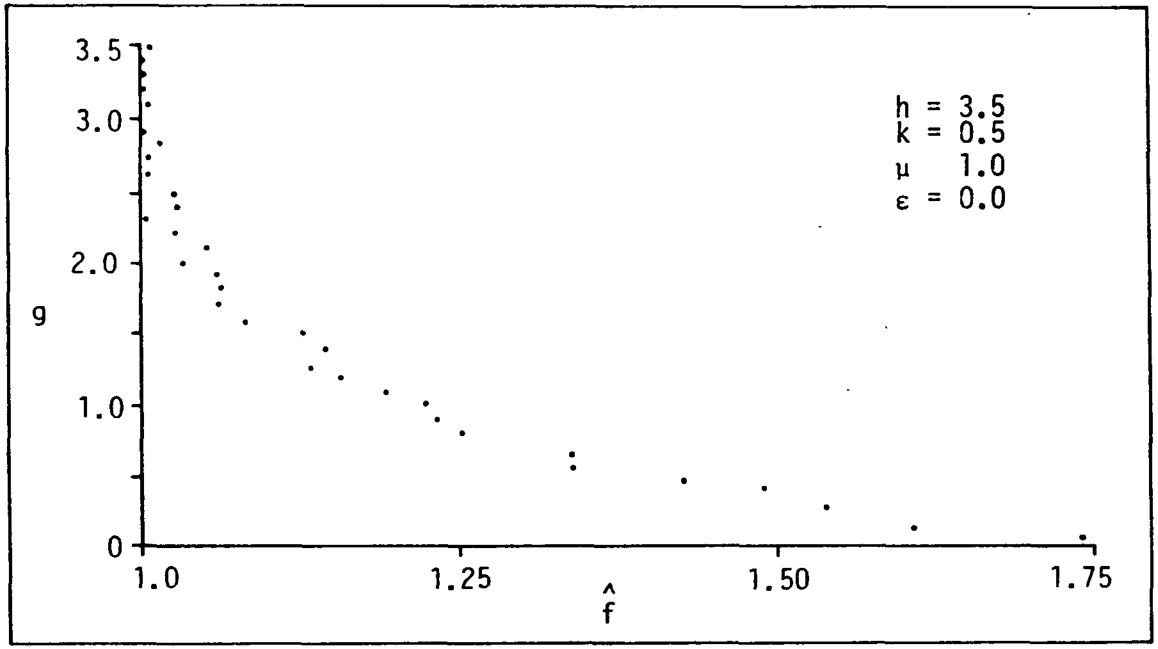

Figure 6a. $\hat{f}'$ vs. g, ARL = 200 when $\mu = \mu_o$
VFS CUSUM Scheme.

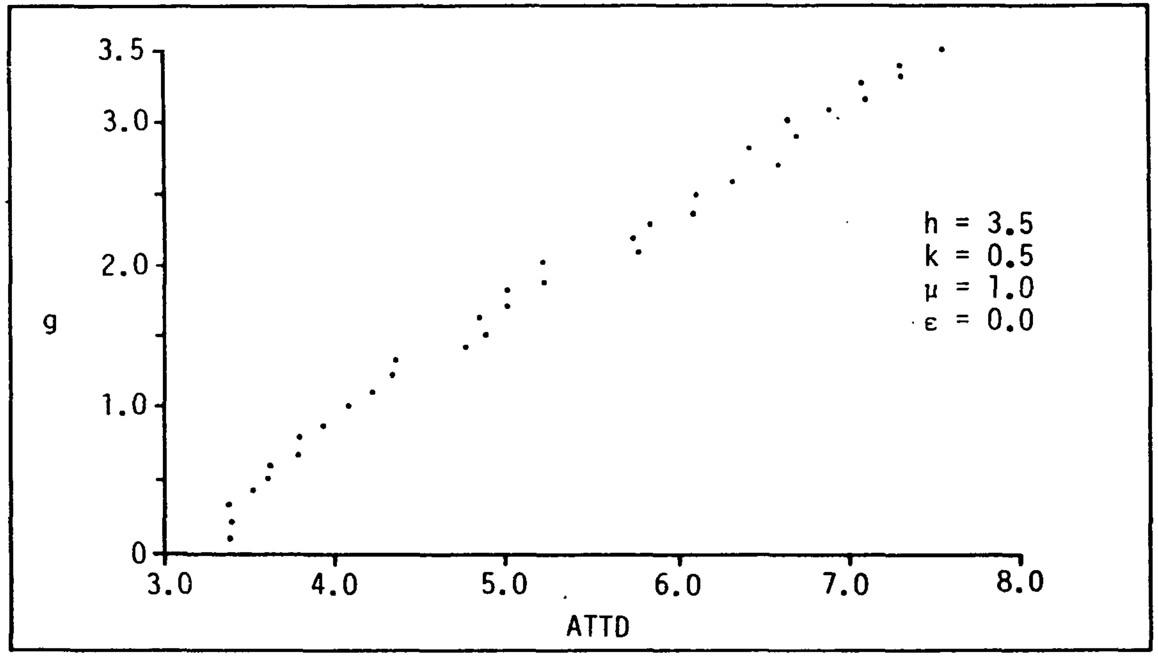

Figure 6b. ATTD vs. g, ARL = 200 when $\mu = \mu_o$
VFS CUSUM Scheme.

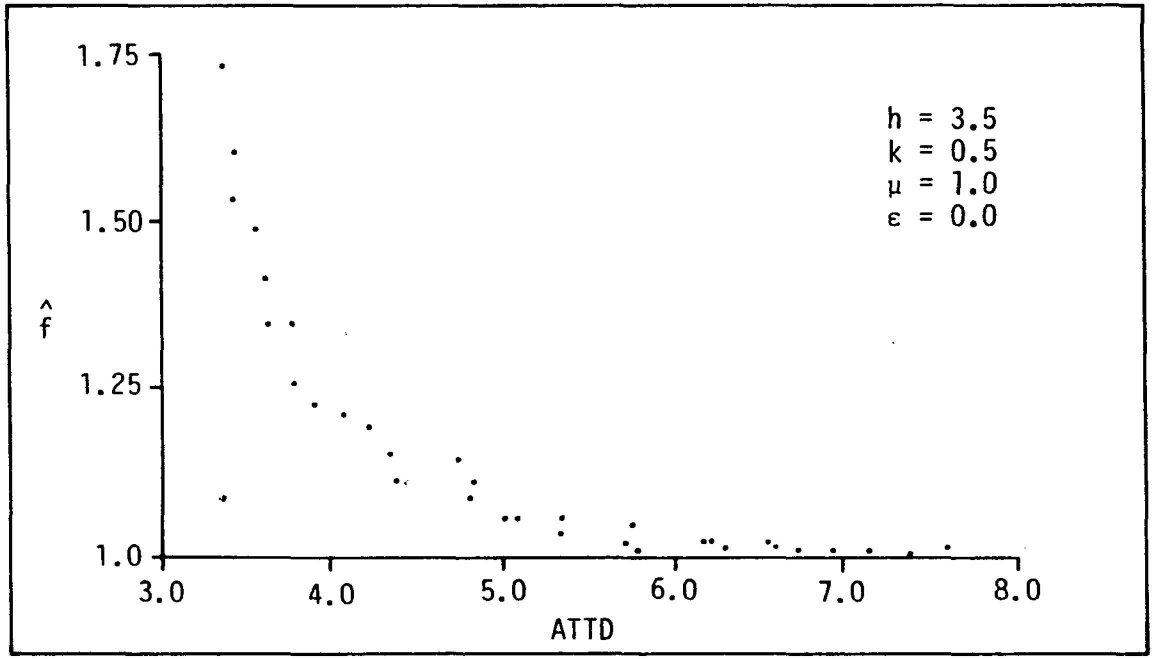

Figure 6c. ATTD vs. $\hat{f}'$, ARL = 200 when $\mu = \mu_o$
VFS CUSUM Scheme.

warning limit g and ATTD appears to be linear with a positive correlation. Except for the points where g equals 0.1 and g equal to 0.2, the ATTD steadily increases with the value for the warning limit g. The two points mentioned do not fit this trend: a situation found in almost all simulation runs of the VFS CUSUM scheme. It was also found that the shortest ATTD values for any particular combination of h and μ were usually for values of the warning limit g between .2 and .3. In the case of $\hat{f}'$ versus ATTD, the value of $\hat{f}'$ decreases rapidly as the ATTD increases when the ATTD is less than 4.5 but almost is level when the ATTD is greater than 5.0.

Table IV is presented for the VFS CUSUM schemes when h=4.7, k=0.5 and g varying from .1 to 4.5 in increments of .2. These increments of g for these VFS CUSUM schemes were used because of the large amount of computer time and money involved in making one simulation for a particular value of g. Also, the case for g=h was not run, as this is the same as a fixed sampling interval classical CUSUM scheme.

In the above case with h=4.7, the ATTD when $\mu = \mu_0$ was calculated using $\hat{f}'$ values to the "standard" ARL of 700 found in the fixed interval CUSUM simulation for h=4.7. Plots of the data in Table IV are presented in Figures 7a,b,c for the same sets of variables as Figures 6a,b,c, respectively. Figures 7a,b,c show the same general trends as Figures 6a,b,c, respectively. Figure 7a shows $\hat{f}'$ increases only slightly when g decreases from 4.5 to around 2.0, then increases rapidly as g nears .1. The plot of g versus ATTD is linear like Figure 6b. The plot of ATTD versus $\hat{f}'$ shows the rapid decrease in $\hat{f}'$ when the

TABLE IV. VFS CUSUM

h = 4.7

		ATTD $\mu = 1.0$			
		90% Two-sided Confidence Interval			
g	$\hat{f}'$	L.C.B.	ATTD	U.C.B.	
0.1	1.818029	3.508499	3.593152	3.677805	+
0.3	1.621947	3.629687	3.719450	3.809212	+
0.5	1.531116	3.801425	3.896385	3.991344	+
0.7	1.363626	3.864043	3.964879	4.065715	+
0.9	1.282946	4.150703	4.254763	4.358822	+
1.1	1.243366	4.365802	4.472635	4.579468	+
1.3	1.230318	4.706930	4.822845	4.938760	+
1.5	1.138003	4.760608	4.875659	4.990709	+
1.7	1.098405	5.172600	5.296071	5.419541	+
1.9	1.099596	5.459432	5.580232	5.701032	+
2.1	1.094305	5.833120	5.960896	6.088673	+
2.3	1.078129	6.377820	6.520094	6.662368	+
2.5	1.071849	6.535072	6.677189	6.819306	+
2.7	1.056665	6.896801	7.050069	7.203337	+
2.9	1.042618	7.153896	7.303330	7.452765	+
3.1	1.055374	7.783675	7.946545	8.109414	+
3.3	1.056062	8.205027	8.374998	8.544969	+
3.5	1.014514	8.101432	8.261796	8.422160	+
3.7	1.015423	8.418065	8.579515	8.740965	+
3.9	1.036562	8.862692	9.032598	9.202503	+
4.1	1.025416	9.143282	9.310365	9.477447	+
4.3	1.018836	9.175751	9.344359	9.512967	+
4.5	1.021860	9.656219	9.834377	10.01254	

STANDARD CUSUM ARL, h=4.7

9.7617 9.8862 10.0107

+95% Lower one-sided confidence interval.

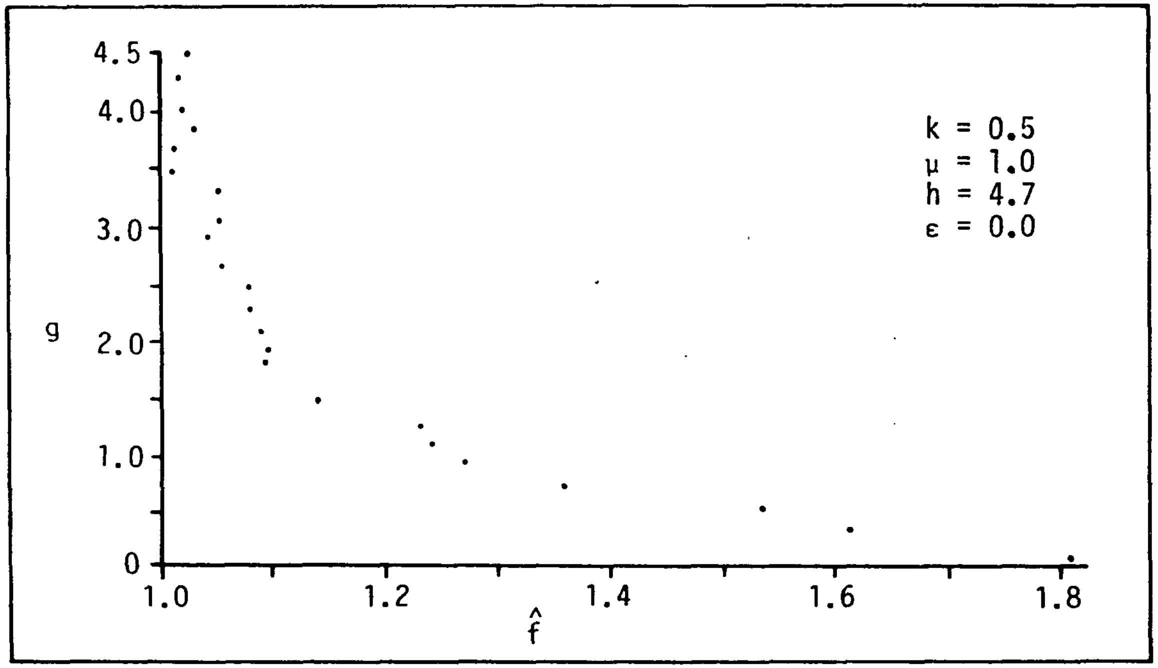

Figure 7a. $\hat{f}'$ vs. g, ARL = 700 when $\mu = \mu_o$
VFS CUSUM Scheme.

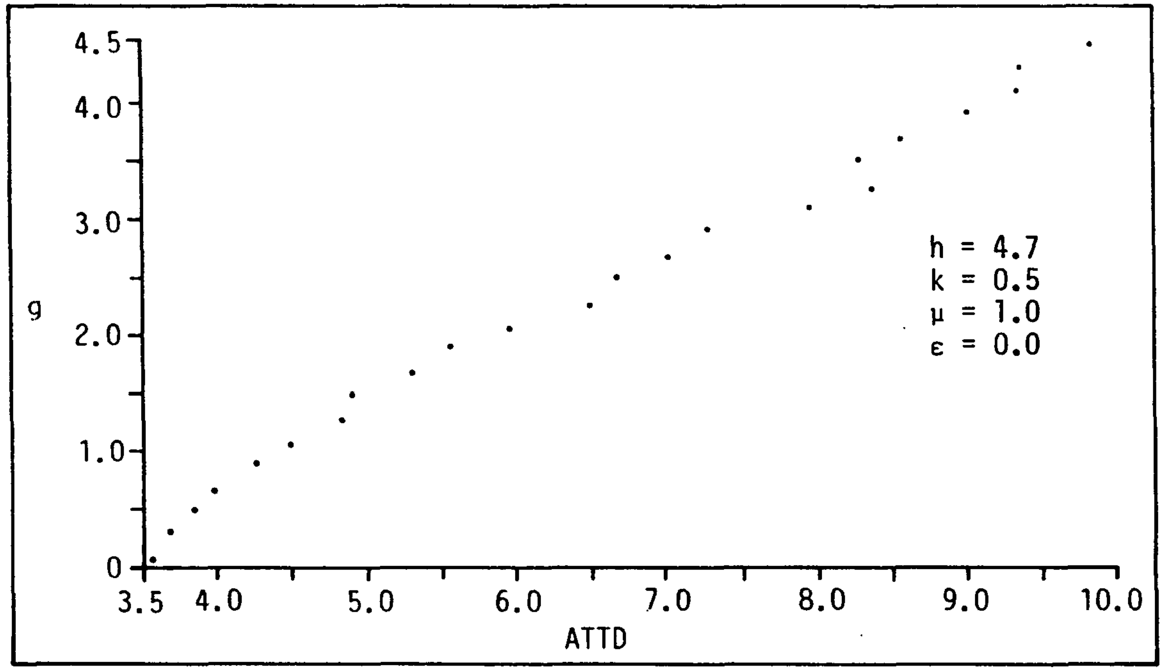

Figure 7b. ATTD vs. g, ARL = 700 when $\mu = \mu_o$
VFS CUSUM Scheme.

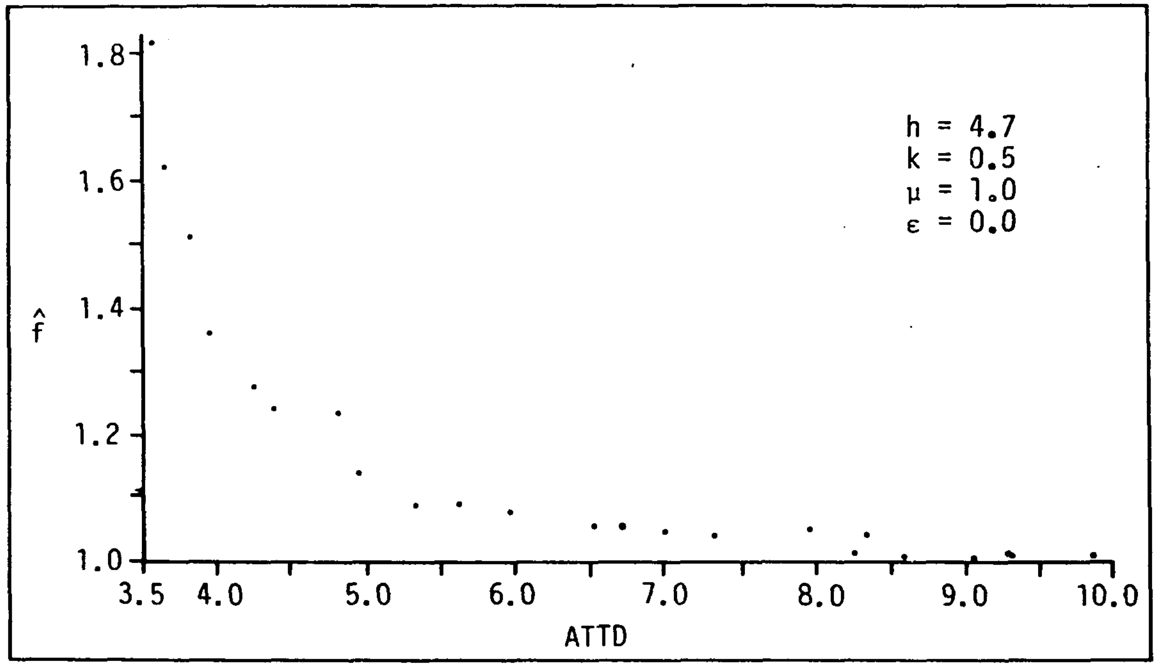

Figure 7c. ATTD vs. $\hat{f}'$, ARL = 700 when $\mu = \mu_o$
VFS CUSUM Scheme.

ATTD is less than 5 but a near level slope when the ATTD is greater than 5.

Because CUSUM schemes are used primarily for the detection of small shifts in the process mean, the VFS CUSUM schemes' sensitivity to a shift in the population mean to $\mu = 0.5$ were tested. In this set of runs then the reference value of k was taken to be $k = (\mu_1 - \mu_0/2) = .25$. Because of the previously found highly significant agreement between the Vance computer program and the results of the simulation of a fixed interval scheme and the amount of computer time involved in simulating the fixed-interval classical CUSUM scheme, the Vance program was used to determine the values of h given ARL numbers close to the "standard" figures of 200 and 700. The value of h found to yield an ARL of approximately 200 for the above set of assumptions was found to be an h of 5.6 and for an ARL of 700 was h equal to 7.9.

The results of the Monte Carlo simulation of the VFS CUSUM schemes for h=5.6, k=0.25 and g values going from .1 to 5.6 by .1 increments are given in Table V. This table has, for every value of g, an estimate of $\hat{f}'$, the ATTD, and the lower and upper confidence limits of the 90% two-sided confidence interval.

Figure 8a is a plot of the $\hat{f}'$ versus g. It shows the same shape as other $\hat{f}'$ versus g plots, namely very slow increasing values of $\hat{f}'$ as the warning zone expands until h is approximately 2.5, then a rapid increase in $\hat{f}'$ as g approaches zero. The ATTD versus g plot shows the same linear relationship as previous ATTD versus g plots. The ATTD

TABLE V. VFS CUSUM

h = 5.6

		ATTD $\mu = 0.5$			
		90% Two-sided Confidence Interval			
g	$\hat{f}'$	L.C.B.	ATTD	U.C.B.	
0.1	2.686699	9.008711	9.191733	9.374758	+
0.2	2.493592	8.971101	9.151482	9.331861	+
0.3	2.311508	9.108610	9.292725	9.476839	+
0.4	2.209066	9.170629	9.457895	9.645162	+
0.5	2.010301	9.174174	9.361567	9.548961	+
0.6	1.931449	9.296785	9.486505	9.676225	+
0.7	1.852569	9.678662	9.879750	10.08084	+
0.8	1.730933	9.578944	9.776817	9.974690	+
0.9	1.746203	10.10995	10.30993	10.50992	+
1.0	4.658977	10.15293	10.35733	10.56172	+
1.1	1.593798	10.45817	10.66761	10.87705	+
1.2	1.470143	9.983736	10.17956	10.37539	+
1.3	1.480858	10.63354	10.84640	11.05926	+
1.4	1.443483	11.02808	11.24588	11.46369	+
1.5	1.390457	11.27607	11.49797	11.71986	+
1.6	1.340572	11.09977	11.31148	11.52319	+
1.7	1.326990	11.53038	11.75209	11.97380	+
1.8	1.308074	11.85932	12.08555	12.31178	+
1.9	1.297182	12.35637	12.59278	12.82919	+
2.0	1.264434	12.25307	12.47996	12.70685	+
2.1	1.217508	12.24993	12.47410	12.69826	+
2.2	1.190454	12.36109	12.59666	12.83224	+
2.3	1.173347	12.57671	12.81224	13.04777	+
2.4	1.166332	12.97693	13.21314	13.44935	+
2.5	1.172527	13.48288	13.73310	13.98332	+
2.6	1.131698	13.08607	13.32574	13.56542	+
2.7	1.148285	13.65448	13.90206	14.14964	+
2.8	1.097890	13.60279	13.84505	14.08732	+
2.9	1.114532	14.24219	14.50050	14.75881	+
3.0	1.081646	14.22107	14.47004	14.71902	+
3.1	1.085938	14.60205	14.85780	15.11355	+
3.2	1.066989	14.63481	14.89004	15.14527	+
3.3	1.082729	15.21750	15.48736	15.75722	+
3.4	1.078001	15.06327	15.32314	15.58300	+

TABLE V: continued

		ATTD $\mu = 0.5$			
		90% Two-sided Confidence Interval			
g	$\hat{f}'$	L.C.B.	ATTD	U.C.B.	
3.5	1.087603	15.75780	16.05629	16.32477	+
3.6	1.053160	15.85115	16.12157	16.39198	+
3.7	1.071094	16.24570	16.51798	16.79026	+
3.8	1.039282	16.23572	16.50442	16.77312	+
3.9	1.058033	16.39653	16.66698	16.93744	+
4.0	1.029792	16.27598	16.54155	16.80711	+
4.1	1.058555	16.75416	17.02813	17.30210	+
4.2	1.036144	16.92757	17.19978	17.47198	+
4.3	1.025059	16.73874	17.00798	17.27721	+
4.4	1.015348	17.05419	17.32570	17.59720	+
4.5	1.019426	17.53583	17.81590	18.09597	+
4.6	1.019204	17.84486	18.12776	18.41066	+
4.7	1.011119	17.96642	18.24807	18.52973	+
4.8	0.9980468	17.80224	18.07922	18.35620	+
4.9	1.011378	18.12852	18.41694	18.70537	+
5.0	1.009421	18.24773	18.52530	18.80286	+
5.1	1.015014	18.46051	18.74284	19.02518	+
5.2	1.019450	19.02686	19.31613	19.60540	
5.3	1.003487	18.58045	18.85913	19.13782	+
5.4	1.003320	18.73004	19.00930	19.28856	
5.5	0.9944539	18.76426	19.04141	19.31855	
5.6	0.9987975	19.00329	19.28778	19.57227	

STANDARD CUSUM ARL, h=5.6

19.2446
(No. L.C.B. or U.C.B.)

+95% Lower one-sided confidence interval > 0.

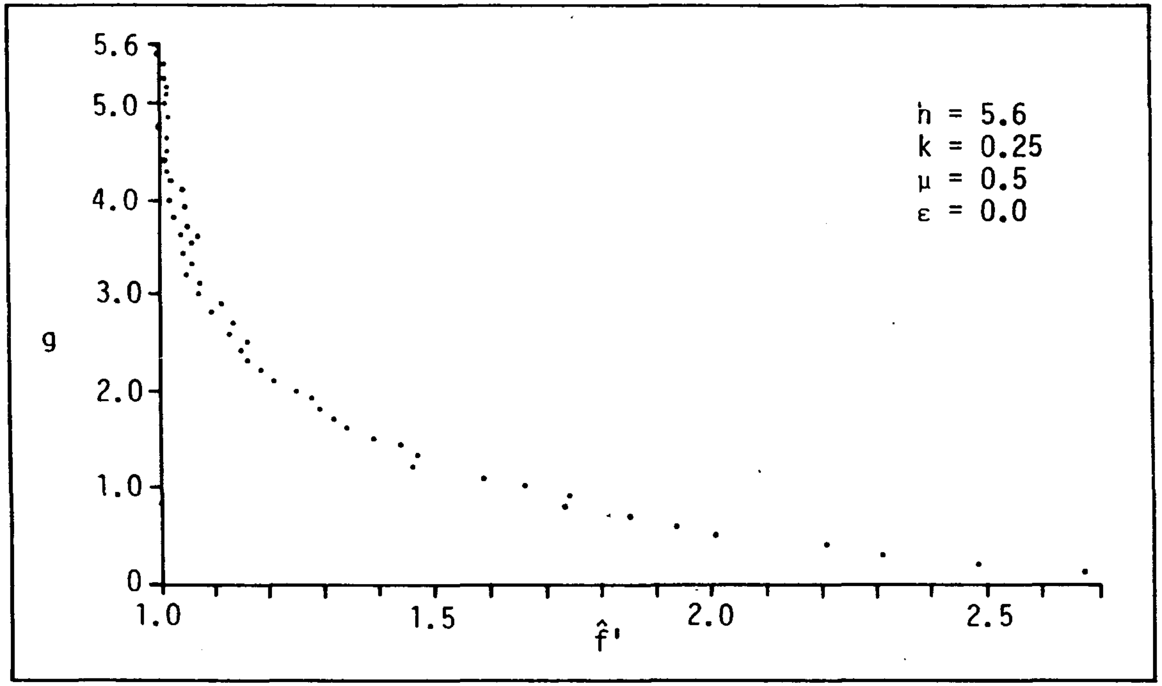

Figure 8a. $\hat{f}'$ vs. g, ARL = 200 when $\mu = \mu_o$
VFS CUSUM Scheme.

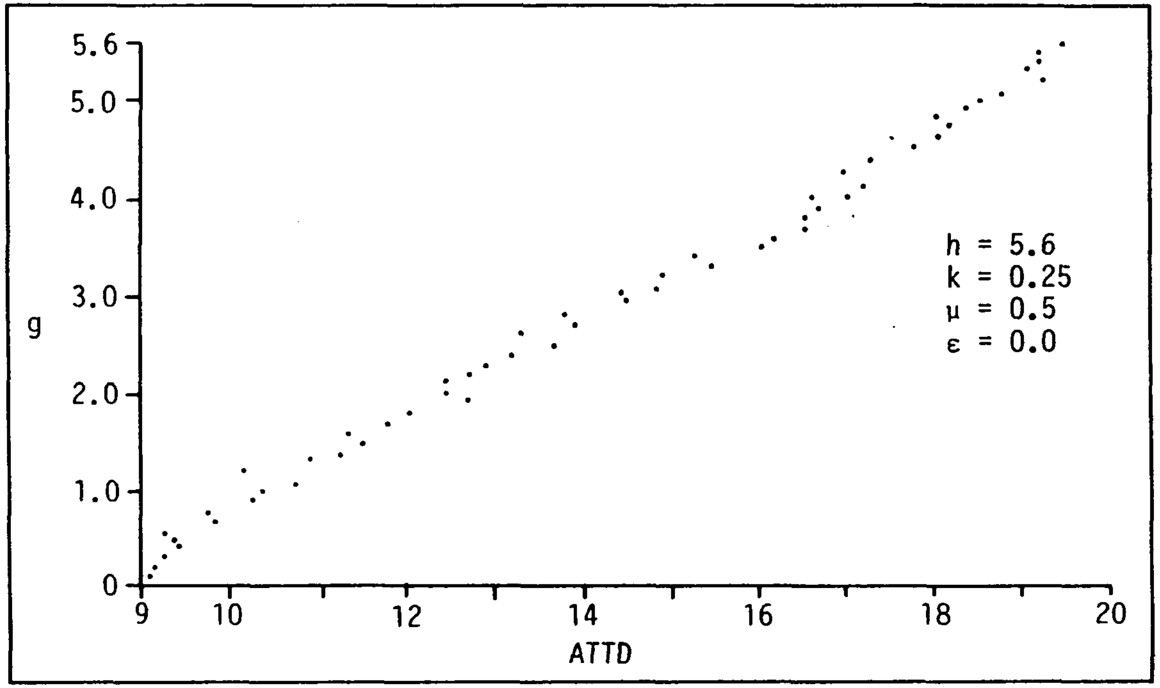

Figure 8b. ATTD vs. g, ARL = 200 when $\mu = \mu_o$
VFS CUSUM Scheme.

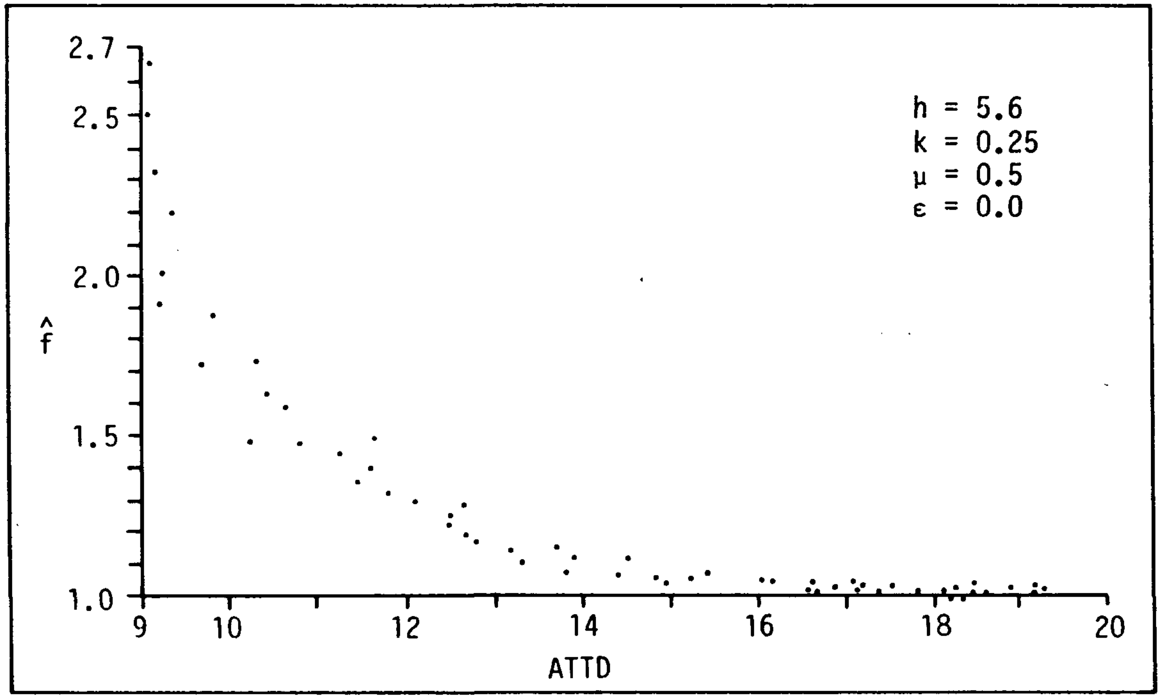

Figure 8c. ATTD vs. $\hat{f}'$, ARL = 200 when $\mu = \mu_0$
VFS CUSUM Scheme.

versus $\hat{f}'$ plot has the same shape as the g versus $\hat{f}'$ because of the linear correlation between g and ATTD that was mentioned earlier.

Table VI displays the data obtained from the Monte Carlo simulation of a VFS CUSUM scheme when h=7.9, k=0.25 and g varies from .1 to 7.9 in increments of .2. Here $\hat{f}'$ values yield the ATTD of 700 when $\mu = \mu_0 = 0.0$.

The plots shown in Figures 9a,b, and c are of the same combination of variables as previous figures. The plot of $\hat{f}'$ versus g show the very small increase in $\hat{f}'$ necessary when h > g > 4.0 and then rapid increase in $\hat{f}'$ is found as g trends toward .1. The ATTD versus g graph in Figure 9b shows the positive linear correlation that all plots of these variables exhibit and the ATTD versus $\hat{f}'$ graph is similar to the ATTD versus g graph, as expected.

Example

It may be helpful to give an example of how to use the sets of tables and figures presented to derive a VFS CUSUM scheme. First, assume that the shift in the process mean that is to be detected is from $\mu = 0.0$ to $\mu = 1.0$. Now select the ATTD (ARL) value that is desired when the process is in control. For the purpose of this example suppose it is desired to have an ATTD value when $\mu = \mu_0$ to be equal to 200. Now determine the control limit h by using Figure 2 or Table I. For an ATTD of 200 when $\mu = \mu_0$, a value of h=3.5 is needed. Now go to Figures 6a,b, and c or Table III. If it is desired that a maximum ATTD

TABLE VI. VFS CUSUM

h = 7.9

		ATTD $\mu = 0.5$			
		90% Two-sided Confidence Interval			
g	$\hat{f}'$	L.C.B.	ATTD	U.C.B.	
0.1	2.939590	10.01687	10.21978	10.42269	+
0.3	2.533108	10.22562	10.43438	10.64314	+
0.5	2.177538	10.06603	10.26709	10.46815	+
0.7	2.018340	10.89110	11.11419	11.33729	+
0.9	1.829185	10.82498	11.04682	11.26865	+
1.1	1.685640	11.47540	11.71014	11.94487	+
1.3	1.559745	11.75253	11.98976	12.22699	+
1.5	1.461483	11.83864	12.07535	12.31207	+
1.7	1.422220	13.08530	13.34185	13.59840	+
1.9	1.324979	12.99076	13.23946	13.48815	+
2.1	1.316927	13.96969	14.23282	14.49595	+
2.3	1.279338	14.50936	14.77892	15.04848	+
2.5	1.255049	14.83551	15.10778	15.38004	+
2.7	1.200870	15.32729	15.60939	15.89148	+
2.9	1.168811	16.10179	16.38626	16.67074	+
3.1	1.149558	16.32920	16.62951	16.92981	+
3.3	1.137693	17.18884	17.50409	17.81934	+
3.5	1.118957	17.67595	17.97448	18.27301	+
3.7	1.094992	18.32408	18.63786	18.95163	+
3.9	1.077446	18.52918	18.84841	19.16765	+
4.1	1.078708	19.54235	19.86636	20.19036	+
4.3	1.041967	19.32420	19.64316	19.96212	+

TABLE VI: continued

		ATTD $\mu = 0.5$			
		90% Two-sided Confidence Interval			
g	$\hat{f}'$	L.C.B.	ATTD	U.C.B.	
4.5	1.064784	20.65081	20.98646	21.32211	+
4.7	1.049943	21.04305	21.38629	21.72953	+
4.9	1.056716	21.80170	22.15743	22.51317	+
5.1	1.020140	21.62721	21.97076	22.31431	+
5.3	1.024031	22.55531	22.90676	23.25821	+
5.5	1.001324	22.45678	22.79935	23.14192	+
5.7	1.021844	23.87671	24.24591	24.61512	+
5.9	1.024599	24.08536	24.45104	24.81671	+
6.1	1.005467	24.60938	24.98042	25.35147	+
6.3	1.016967	24.99389	25.36234	25.73078	+
6.5	1.019901	26.18167	26.56842	26.95516	+
6.7	1.008609	25.63735	26.00637	26.37538	+
6.9	0.9963269	25.97890	26.34607	26.71324	+
7.1	1.019244	27.21745	27.59785	27.97826	
7.3	1.002329	27.07139	27.46280	27.85421	+
7.5	1.025042	28.14596	28.53369	28.92142	
7.7	0.9948975	27.44427	27.82828	28.21229	
7.9	0.9896371	27.77436	28.16032	28.54628	

STANDARD CUSUM ARL, h=7.9

28.0332

+95% Lower one-sided confidence interval > 0.

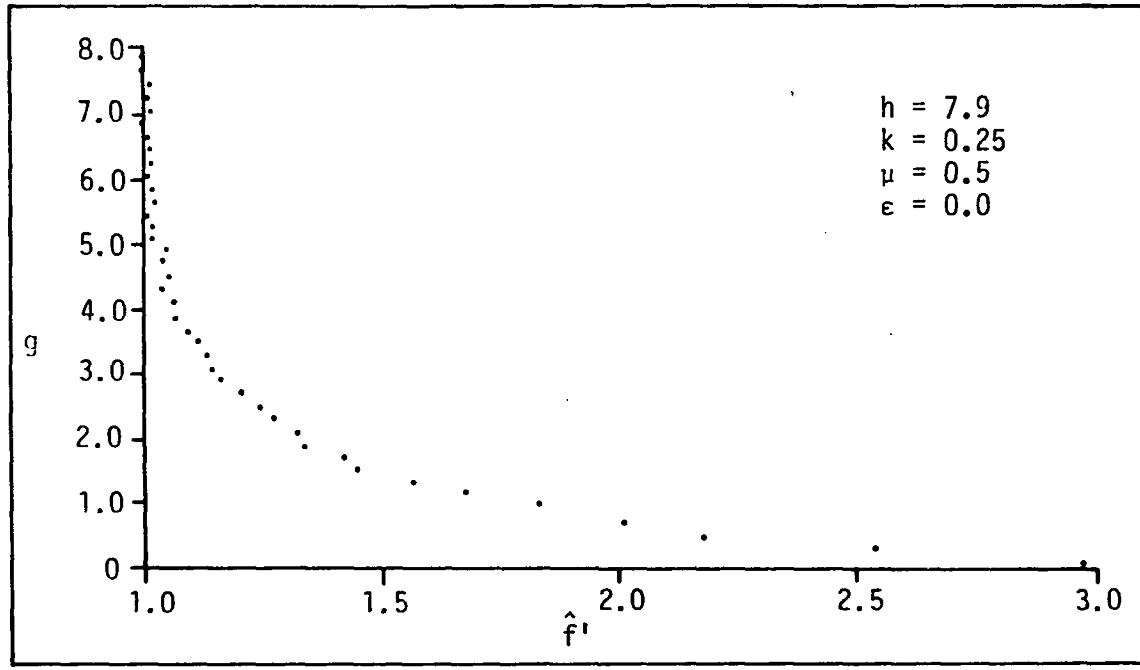

Figure 9a. $\hat{f}'$ vs. g, ARL = 700 when $\mu = \mu_o$
VFS CUSUM Scheme.

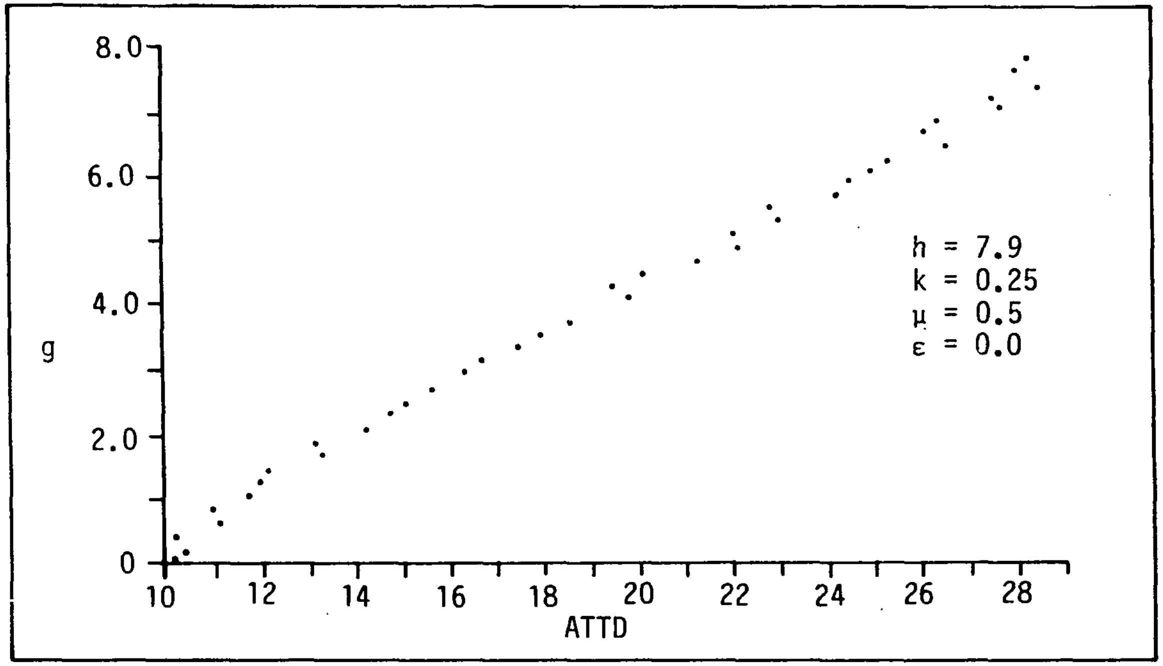

Figure 9b. ATTD vs. g, ARL = 700 when $\mu = \mu_o$
VFS CUSUM Scheme.

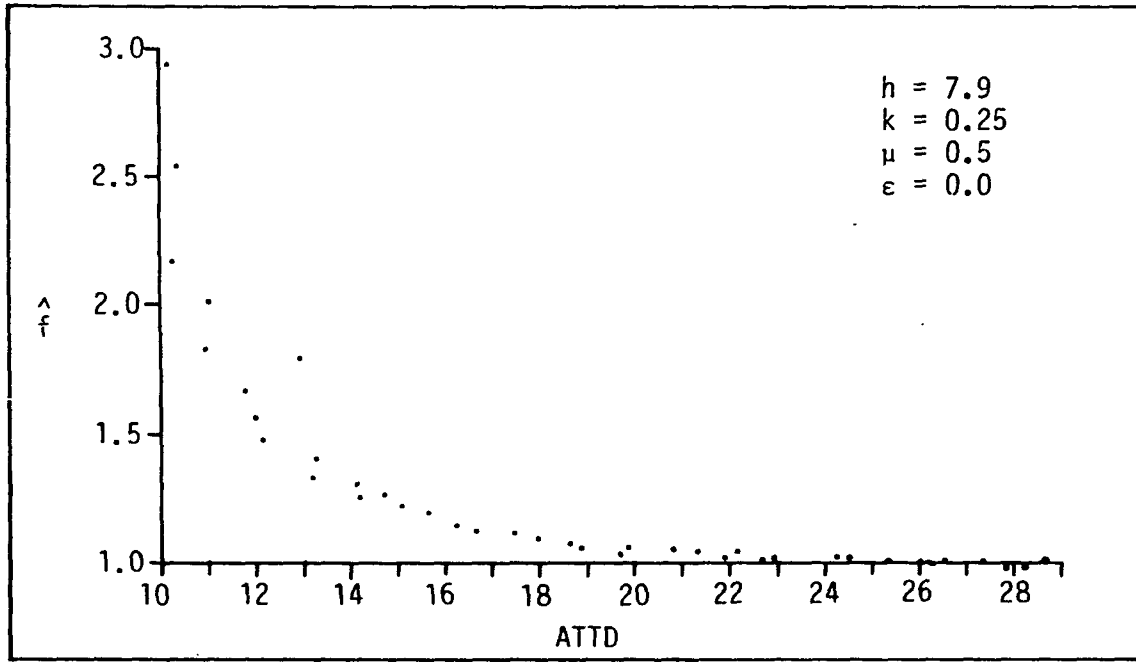

Figure 9c. ATTD vs. $\hat{f}'$, ARL = 700 when $\mu = \mu_o$
VFS CUSUM Scheme.

value when a shift in the process mean occurs be no greater than 6.0, use Figure 6b. From the ATTD value of 6.0, go upwards to determine the setting of the warning limit g that gives an ATTD value that is less than or equal to 6.0. That value of g is found to be 2.3. Then find the value of $\hat{f}'$ that corresponds to this combination of g and ATTD from Figure 6a,c, or Table III. A value of $\hat{f}' = 1.008992$ is found.

Therefore, a VFS CUSUM scheme that has an ATTD of approximately 200 when the process is in control and a maximum ATTD after the process mean shifts to $\mu = \mu_1 = 1.0$ of less than or equal to 6.0 has the following characteristics;

h, the control limit, = 3.5;

g, the warning limit, = 2.3;

k, the constant in the CUSUM statistic, = .5

f, the sampling interval when the value of the CUSUM statistic < g, $\cong$ 1.009.

Sampling Interval

These results for the VFS CUSUM scheme assume that the sampling interval ε is zero when the value of the CUSUM statistic is between g and h (i.e., no time expires between samples). If this is not the case, the value of $\hat{f}'$ and ATTD when $\mu = \mu_1$ can be calculated easily. It is obvious that when the sampling interval ε is one time unit then a value of $\hat{f}'$ equal to one is needed to give a "standard" ATTD when $\mu = \mu_0$. Note that when the sampling interval ε is equal to one, the VFS CUSUM becomes a standard fixed sampling interval CUSUM (i.e., sampling intervals are fixed and equal both above and below the warning limit). Therefore, two cases can be considered: a sampling interval ε of zero or one time unit. The tables give the ATTD for $\mu = \mu_1$ and a sampling interval ε equal to zero. The ATTD for $\mu = \mu_1$ and ε equal to one is the ARL of a standard fixed sampling interval CUSUM with $\mu = \mu_1$. This ARL can be determined from Table II or by using the Vance program. From Table II we find that for h=3.5 and $\mu = 1.0$ the ARL (ATTD) of a standard fixed interval CUSUM is 7.38. The ATTD is 5.78 when $\mu = \mu_1 = 1.0$ of the VFS CUSUM scheme with an identical control limit h = 3.5 and g set to 2.3 and $\varepsilon = 0$. For any other value of ε between zero and 1.0 the corresponding ATTD of a VFS CUSUM is a simple linear interpolation between the ATTD when $\varepsilon = 0$ and ATTD when $\varepsilon = 1.0$. The same procedure is true for the value of $\hat{f}'$ needed to compensate for ε. In this case the new value of $\hat{f}'$ is a simple linear interpolation between $\hat{f}'$ given assuming $\varepsilon = 0$ and $\hat{f}' = 1.0$ which is the value when $\varepsilon = 1.0$.

As an example, if the time needed to sample one item is 0.5 time units, that is $\varepsilon = 0.5$, the ATTD, when $\mu = \mu_1$, would be halfway between 5.78 and 7.38, or 6.58. The $\hat{f}'$ value that would be needed will be halfway between 1.009 and 1.000, or 1.0045.

If for some reason it is thought advisable to have some maximum value on $\hat{f}'$, the graphs and tables can be used to derive a VFS CUSUM scheme that minimizes ATTD. For example with h=3.5 (i.e., ATTD when the process is in control is approximately 200), g equal to 0.9 is the lowest value of g that yields an $\hat{f}'$ value less than or equal to 1.25. When $\mu = \mu_1$ and g = .9, the ATTD is found to be 3.931.

All the results of the Monte Carlo simulation show the VFS CUSUM scheme yields ATTD values that are significantly smaller than the corresponding ATTD values for a standard fixed sampling interval CUSUM scheme when the warning zone was sufficiently large. It was extremely surprising to observe such a large decrease in ATTD with a minimum corresponding increase in the sampling interval $\hat{f}'$.

The results of this simulation are summarized in the next chapter. Also discussed are optimal placement of the warning limit g and areas for further study.

CHAPTER 7

SUMMARY AND AREA FOR FURTHER STUDY

The purpose of this chapter is to summarize the results of this study of the Variable Frequency Sampling CUSUM control chart scheme and discuss possible factors involved in the placement of the warning limit g. Also discussed are areas for further study arising from this examination of VFS CUSUM schemes.

The results of the simulation indicate that average time to detection, the measure of the responsiveness of a particular control chart scheme to a shift in the process mean, is significantly less for a VFS CUSUM scheme than for a standard fixed interval sampling chart when compared at identical control limits h. The VFS CUSUM schemes were calibrated so that they had identical false alarm rates with the fixed sampling interval scheme. Therefore, it can be reasonably concluded that the VFS CUSUM scheme is a superior method for detecting a change in the mean of a process for at least the small differences $D = (\mu_1 - \mu_0)$ tested.

The quicker response of the VFS CUSUM scheme is obviously the major advantage it has over the standard fixed sampling interval CUSUM scheme. However, there are some disadvantages to the practical operation of a VFS CUSUM scheme. First, there is no quick and easy method to calculate the ATTD values for any particular g and h combination. The Monte Carlo simulation methods used in this study would be too expensive and time consuming for more than two or three values of the control limit h. It might be possible to use the linear relationship that exists between the warning limit g and ATTD to interpolate approximate ATTD values between the ATTD when g=h (i.e., the ARL of a standard CUSUM with the identical h value) and any other value of g.

It would be useful to modify the FORTRAN computer program of Vance that determines ARL of standard fixed sampling interval CUSUM schemes to compute the ATTD of VFS CUSUM scheme. This would yield a fast and straightforward means of determining the characteristics of such a scheme also. However, attempts to modify the Vance program to determine ATTD values of a VFS CUSUM scheme has not been successful so far.

We are not so naive to believe a scheme calling for a sampling interval of 1.0038 time units will precisely be implemented. However, if the sampling interval $\hat{f}'$ is standardized to one time unit, the ATTD after a shift in the process mean to $\mu = \mu_1$ would be less than the ATTD computed in an exact VFS CUSUM scheme. Of course the ATTD when $\mu = \mu_0$ would be slightly less than the "standard" values.

The VFS CUSUM scheme will always have a lower cost structure than a standard fixed sampling interval CUSUM with the same control limit h. This is because the number of samples needed to detect a shift is the same in both schemes, so the variable and fixed sampling costs are identical. Because the average time to detection of when the process is operating out-of-control is significantly less with the VFS CUSUM, the losses attributable to the process being out-of-control will always be less. The gain or profit from the process operating in control without interruption is the same for both the standard and VFS CUSUM scheme because the false alarms rates are adjusted to be identical. The only condition under which the VFS CUSUM scheme could show a higher cost factor than a standard fixed interval scheme would be if there were some additional costs associated with variable time sampling.

Placement of the warning limit g is the major decision involved in constructing a VFS CUSUM appropriate to a particular process application. If the warning limit is placed too low, the control chart will signal a warning of a possible out-of-control condition too often and a large number of consecutive samples will unnecessarily be taken. In addition, when the warning limit g is set low, the corresponding sampling interval $\hat{f}'$ may be longer than desired. (A large $\hat{f}'$ may be undesirable because the long time between samples would let the process run in an out-of-control state before another sample is taken.) If the warning g is quite close to h, the control chart statistic may not give any warning when a shift in the process mean occurs. The CUSUM statistic may immediately signal the process is out-of-control because the probabilities of its value being between g and h are small when g and h are close together. This negates the advantages of a VFS CUSUM scheme. It is suggested that, without any prior knowledge of the process behavior, a value for the warning limit g equal to h/2 be considered. This value gave significant reductions in the ATTD when $\mu = \mu_1$ for the value of μ_1 tested without large increases in $\hat{f}$' values.

Many areas for further study and investigation have been opened up by the work on VFS $\bar{X}$ and CUSUM schemes. First and foremost, an easier way of determining ATTD values for any given combination of μ, g and h must be developed if the VFS concept is to gain wide acceptance in practical situations. The Monte Carlo methods used in this study were extremely expensive in terms of computer resources. Secondly, an investigation of the functions of ATTD versus the warning limit g should be explored to determine where the ideal placement of the warning limit should be.

Perhaps an investigation of the probabilities that the value of the CUSUM statistic, when in control, falls within certain boundaries or states might be useful. Taylor (1968) [23] has said that the value of the CUSUM statistic is very nearly independent of time (or the number of previous samples taken) when $\mu = \mu_0$. Using this concept, it is possible to derive a transitional probability matrix (TPM) using a Markov chain approach. Traditionally, the Markov chain concept of a CUSUM chart has been of a random walk (in the discrete case or Brownian motion in the continuous case) between a reflecting and absorbing barrier. The reflecting barrier is zero, below which the value of the

CUSUM statistic is not allowed to fall, and the absorbing barrier h, above which the statistic is not allowed to return. This TPM then represents a phase-type distribution of Neuts (1975) [18], and steady-state probabilities of the value of the CUSUM statistic are not possible.

Higle (1986, personal communication) has suggested overcoming this problem by considering a CUSUM scheme as a Markov chain between two reflective barriers because, if the CUSUM statistic ever goes above h and it is assumed the process mean has not changed, then the value of the statistic is reset to zero, thereby creating a reflective barrier when state of the CUSUM statistic is greater than h. A steady-state TPM can be easily derived for a Markov chain with two reflective barriers. Such a TPM could be used to assist in the placement of the warning limit g, derive ATTD values, and other factors of a VFS CUSUM scheme.

Assaf (1987) [3] has developed a process control scheme utilizing Bayesian techniques and assuming an exponentially distributed time to the shift in the process mean with known expected value of λ. His scheme calls for minimum sampling when the value of his statistic is below a critical value y_0 and maximum sampling when it exceeds y_0. Because of the obvious similarities to the VFS schemes, and his method of determining y_0 by Bayesian methods, his approach should be investigated to find which areas would be helpful in developing optimal VFS schemes.

APPENDIX A

FORTRAN CODE

```
      PROGRAM MAIN
C     THE PURPOSE OF THE THIS PROGRAM IS TO SIMULATE A
C     CUMULATIVE SUM CHART FOR CONTROLLING A NORMAL MEAN
C     DETERMINE THE NUMBER OF REPETIONS OF THE PROCESS
C     NEEDED TO DETERMINE THE AVERAGE RUN LENGTH.
       N=100
C     WRITE THE HEADINGS FOR THE OUTPUT DATAFILE
      WRITE(48, 10)
   10 FORMAT (/10X,'RUNLENGTH')
C     INITIALIZE THE SUM OF RUNLENGTHS OF ALL REPETIONS
C     USED TO DETERMINE THE AVERAGE RUN LENGTH.
      SUMRL=0.0
C     INITIALIZE THE MEAN OF THE PROCESS
      MU=0.0
C     SET THE UPPER AND LOWER OUT-OF-CONTROL LIMITS FOR
C     CUSUM CHART.
      H= 3.0
      G= -H
C     OBTAIN A NORMAL VARIABLE WITH MEAN = MU AND VARIANCE OF
C     ONE AND ADD IT TO A RUNNING SUM OF SIMILAR VARIABLES UNTIL
C     THE RUNNING SUM (CUSUM) EXCEEDS ONE OF THE OUT-OF-CONTROL LIMITS
C     (G OR H).  THE NUMBER OF TIMES A NEW OBSERVATION IS MADE
C     IS DEFINED AS THE RUN LENGTH OF THAT PARTICULAR CUSUM CHART.
C     THIS PROCEDURE IS REPEATED N TIMES.
       DO 20 I=1,N
C      RESET THE RUNLENGTH COUNTER AND THE INITIAL VALUE OF THE
C      CUMULATIVE SUM EVERY REPETITION.
       CUSUM = 0.0
       RUNLENGTH=0.0
C      OBTAIN A RANDOM STANDARD NORMAL VARIABLE FROM SUBROUTINE
C      NORM.
   30  CALL NORM(R)
C      ADD ONE TO THE NUMBER OF RUNS MADE FOR THIS REPETITION
       RUNLENGTH = RUNLENGTH + 1.0
C      FIND THE NEW VALUE OF THE CUSUM INCLUDING THIS NEW OBSERVATION
       WRITE (48,35)R
   35  FORMAT(/10X,F10.5)
       CUSUM= CUSUM + ( R+MU)
C      IF THE CUMULATIVE SUM OF'ALL OBSERVATIONS TO THIS POINT IS NOT
```

```
      return
 160  if(u.lt. 0.965487)goto 210
c area b
 180  trptmp=sqrt(4.46911-2*alog(unif(ix)))
      if(trptmp*unif(ix).gt.2.11403)goto 180
      goto 340
 210  if(u.lt. 0.949991)goto 260
c area c
 230  trptmp=1.8404+unif(ix)*.273629
      if(.398942*exp(-trptmp*trptmp/2)-.443299+trptmp*.209694
     &  .lt.unif(ix)*4.27026e-02)goto 230
      goto 340
 260  if(u.lt. 0.925852)goto 310
c area d
 280  trptmp=.28973+unif(ix)*1.55067
      if(.398942*exp(-trptmp*trptmp/2)-.443299+trptmp*.209694
     &   .lt.unif(ix)*1.59745e-02)goto 280
      goto 340
 310  trptmp=unif(ix)*.28973
c area e
      if(.398942*exp(trptmp*trptmp/2)-.382545.lt.unif(ix)*1.63977e-02)
     &   goto 310
 340  if(u0.gt.0.5)goto 370
      trptmp=-trptmp
 370  trpnrm=trptmp
      return
      end
************************
       function unif(ix)
c portable random number generator using the recursion:
c ix=16807*ix mod (2**31-1)  using only 32 bits, including sign.
c input:
c  ix = integer greater than 0 and less than 2147483647
c outputs:
c  ix= new pseudorandom value,
c  unif= uniform fraction between 0 and 1.
c
      k1=ix/127773
      ix=16807*(ix-k1*127773)-k1*2836
      if(ix.lt.0)ix=ix+2147483647
      unif=ix*4.656612875e-10
      return
      end
*******************************************
```

```
C          WITHIN THE CONTROL LIMITS THEN TERMINATE THIS RUN AND COUNT THE
C          RUNLENGTH.

           IF (CUSUM .LT. G ) THEN
              GO TO 40
           ELSE IF (CUSUM .GT. H ) THEN
              GO TO 40

C          OTHERWISE OBTAIN ANOTHER OBSERVATION AND REPEAT TEST OF THE
C          CUSUM VALUE UNTIL AN OUT-OF-CONTROL SITUATION EXISTS.

           ELSE
                GO TO 30
           END IF

C          KEEP A RUNNING TOTAL OF THE RUN LENGTHS FOR THE PURPOSE OF
C          OBTAINING AN AVERAGE RUNLENGTH.

   40      SUMRL=  SUMRL + RUNLENGTH

C          WRITE THE RUNLENGTH FOR THIS REPETITION TO THE DATA FILE

           WRITE (48,50) RUNLENGTH
   50      FORMAT(/15X,F10.3)

   20      CONTINUE

C          CALCULATE THE AVERAGE RUNLENGTH FOR THIS NUMBER OF REPETITIONS
C          AND WRITE IT TO THE DATA FILE

           ARL= SUMRL/N
           WRITE (48,60)ARL
  60      FORMAT(//,/5X,'THE AVERAGE RUN LENGTH IS',2X,F10.3)

          STOP
          END


C         THIS SUBROUTINE GENERATES TWO RANDOM IID STANDARD NORMAL VARIABLES

          SUBROUTINE NORM(R)

  70      U1=RAN(ISEED)
          U2=RAN(ISEED)

          V1=(2*U1) - 1
          V2=(2*U2) -1

          W1=(V1**2)+(V2**2)
          Z=1.00

          IF (W1.GT.Z)THEN
           GO TO 70
          ELSE
           CONTINUE

          END IF

          Y=((-2*ALOG(W1))/W1)**0.5
```

```
            PROGRAM MAIN
            common iseed
            REAL K,MU,G,H
C           THIS PROGRAM SIMULATES AN ADAPTIVE ONE-SIDED CUSUM
C           ON A PROCESS WITH A NORMALLY DISTRIBUTED MEAN.
C           FIRST, SPECIFY THE NUMBER OF REPETITIONS REQUIRED.
            WRITE (6,11)
   11       FORMAT(2X,'SPECIFY THE NUMBER OF REPETIONS DESIRED')
            READ (5,12) N
   12       FORMAT(I4)
            WRITE(6,13)
   13       FORMAT(2X,'SPECIFY THE REFERENCE VALUE K (A REAL NUMBER)')
            READ (5,14)K
   14       FORMAT(F5.3)
            WRITE(6,15)
   15       FORMAT(2X,'SPECIFY THE MEAN OF THE PROCESS AVERAGE, MU ')
            READ(5,16)MU
   16       FORMAT(F4.3)
            WRITE(6,22)
   22       FORMAT(2X,'PICK A RANDOM SEED WITH AN 8-DIGIT INTEGER')
            READ(5,23)ISEED
   23       FORMAT(I8)
C           KEEP A RUNNING SUM OF THE RUNLENGTHS FOR EACH REPETION
C           FOR THE PURPOSE OF DETERMING THE AVERAGE RUN LENGTH.
C           INITIALIZE ITS VALUE.
            DO 55 H=1.0,4.0,0.1
            SUMRL=0.0
            SSRL=0.0
C           SET UP A LOOP FOR EVERY REPETION OF A RUN
            DO 20 I=1,N
C           RESET THE RUNLENGTH COUNTER AND THE VALUE OF THE CUSUM TO
C           ZERO EVERY REPETION.
            CUSUM=0.0
            RUNLENGTH=0.0
C           OBTAIN A NORMAL (0,1) RANDOM VARIABLE GENERATED BY THE
C           FUNCTION TRPNRM.
   30       r=trpnrm(iseed)
C           ADD THE PROCESS MEAN MU TO THE N(0,1) RANDOM VARIABLE
C           TO OBTAIN A RANDOM NORMAL VARIABLE WITH MEAN MU.
            X = R + MU
```

```
            Z=0.0
C           OBTAIN THE CUSUM STATISTIC

            CUSUM= CUSUM + (X-K)

C           FIRST "ZERO OUT" THE CUSUM IF IT IS LESS THAN OR EQUAL TO ZERO

            IF (CUSUM.LE.Z) THEN
             CUSUM= Z
            ELSE
             CONTINUE
            END IF

C          NOW SEE IF IT IS BELOW THE OUT OF CONTROL LIMIT.

            IF (CUSUM.LT.H) THEN
             RUNLENGTH = RUNLENGTH + 1.0
             GO TO 30
            ELSE
             CONTINUE
            END IF

C          TO GET TO THIS PORTION OF THE CODE MEANS THAT THIS
C          REPETION IS IN THE OUT OF CONTROL ZONE.  ESTABLISH
C          THE NUMBER OF SAMPLING INTERVALS THAT TOOK PLACE
C          BEFORE THE CHART WENT OUT OF CONTROL AND ESTABLISH A
C          SAMPLING INTERVAL.


         SUMRL=SUMRL + RUNLENGTH
         SQRL=(RUNLENGTH**2)
         SSRL=SSRL+SQRL

   20    CONTINUE

         ARL=SUMRL/N
         VARRL= SSRL/N-(ARL**2)
         STDDEV=(VARRL**0.5)
         CL5= ARL-(1.645*(STDDEV/(N**0.5)))
         CL95= ARL+(1.645*(STDDEV/(N**0.5)))
         WRITE(66,*) H,CL5,ARL,CL95
   55    CONTINUE
         STOP
         END

*************************
      function trpnrm(ix)
c generates unit normal deviate by composition method of ahrens and dieter
c the area under normal curve is divided into 5 different areas
c input:
c  ix = random number seed
c auxiliary routines:
c  unif
      u=unif(ix)
      u0=unif(ix)
      if(u.ge. 0.919544 )goto 160
c area a, the trapezoid in the middle
      trpnrm=2.40376*(u0+u*.825339)-2.11403
```

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