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U-MI
A SYSTEMATIC, EXPERIMENTAL METHODOLOGY
FOR DESIGN OPTIMIZATION

by
Paul Andrew Ritchie

A Thesis Submitted to the Faculty of the
DEPARTMENT OF SYSTEMS AND INDUSTRIAL ENGINEERING
In Partial Fulfillment of the Requirements
For the Degree of
MASTER OF SCIENCE
WITH A MAJOR IN RELIABILITY ENGINEERING
In the Graduate College
THE UNIVERSITY OF ARIZONA

1988
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This thesis has been approved on the date shown below:

R.G. ASKIN
Associate Professor of Systems and Industrial Engineering

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ACKNOWLEDGEMENTS

I would first like to thank my adviser, Dr. R.G. Askin for his invaluable help and advice throughout the course of my work on this thesis. Secondly, I would like to express my appreciation to my wife, Candis, for her patience, understanding and support during a long and, for her, uneventful summer.

Finally, I would like to thank Mr. Ed Linard, of Aurora Software Development Unit, Canadian Forces Base Greenwood, Nova Scotia, for providing information on DIAC software.
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ABSTRACT

Much attention has been directed at off-line quality control techniques in recent literature. This study is an refinement of and an enhancement to one technique, the Taguchi Method, for determining the optimum setting of design parameters in a product or process.

In place of the signal-to-noise ratio, the mean square error (MSE) for each quality characteristic of interest is used. Polynomial models describing mean response and variance are fit to the observed data using statistical methods. The settings for the design parameters are determined by minimizing a statistical model. The model uses a multicriterion objective consisting of the MSE for each quality characteristic of interest.

Minimum bias central composite designs are used during the data collection step to determine the settings of the parameters where observations are to be taken. Included is the development of minimum bias designs for various cases. A detailed example is given.
CHAPTER 1

INTRODUCTION

Off line quality control techniques are becoming increasingly important to product design. In particular, knowledge of the relationship between design parameters and the performance of a product can be exploited in order to arrive at a product design that is driven by the requirement to perform well under a wide range of expected conditions of use. The Taguchi method embraces this concept.

With the Taguchi Method, an understanding of the relationship between design parameters and performance is gained through planned experimentation. The design parameters are 'factors' in the experiment. Instead of replicating each design point under identical outside (experimental) conditions, the conditions under which the observations are taken are varied in order to obtain an estimate of the mean performance and variance under expected conditions of use. Taguchi summarizes the performance at each design point in a single statistic, known as the signal to noise ratio. The actual form of the ratio varies from problem to problem. A product design is chosen which optimizes the signal to noise ratio. More detail about the Taguchi method is contained in Chapter 2.
The advantage of the Taguchi approach to parameter design is that quality, as measured by product performance, is the determining factor in the choice of parameter settings.

1.1 Inherent Shortcomings of the Taguchi Method

Taguchi's approach entails accepting the design point which optimizes the signal to noise ratio. This does not make maximum use of the data collected during the experiment. Contemporary techniques allow inferences to be made about the functional relationship between response and design parameters over the entire space (convex hull) from which observations were taken. This allows all points in the design space to become candidates for consideration, not just those considered in the experiment.

The Taguchi method measures performance by averaging observations over the expected conditions of use (noise parameters). This does not necessarily provide unbiased estimates of mean and variance of performance, since the noise parameters are not randomly sampled.

A strict application of the Taguchi Method would lead to a single summary statistic for each design. This may not be desirable, or even possible. In many cases, multiple objectives exist, each with an accompanying performance measure. In order to use a single summary statistic, the individual performance measures must be combined into one number using some type of (subjective) weighting system. Thus the final solution is very sensitive to the weighting system. Moreover,
information about the contribution of individual performance measures is completely lost, and there is no 'feel' for the contribution of individual performance measures.

Consider, for example, the design of an electronics package for an aircraft. Assume that the total volume is limited. Measures of quality for the package could be defined as follows:

a. reliability for a specified mission time (maximize); and
b. weight (minimize).

Objective (1) could be improved by introducing redundancy into the system, but at the expense of (2). In this example, the two objectives are incompatible. In many cases, competing objectives can be replaced with constraints (i.e. limiting weight to $X$ pounds, or placing a lower bound on the reliability), but should this not be possible, then the multiple criterion approach is most appropriate.

The multiple criterion approach is also a better representation of the design problem. An optimal solution from a model will rarely satisfy the (unquantifiable) opinions of the designers. The multiple criterion approach lends itself well to this requirement, as it highlights the trade-offs that must occur between competing objectives under different product designs. This is also a more effective tool during sensitivity analysis.
1.2 An Alternate Approach

This thesis proposes a variation to the Taguchi Method that addresses the shortcomings outlined above. Briefly, for each design point, observations are taken under different conditions representative of those in actual usage. For each performance criteria, an equation is estimated that describes the performance over the observed range of conditions. Estimates of mean performance and variance are obtained for each design point. These statistics are then used to obtain a predictive equation for mean performance and variance over the range of designs. The global predictive equations of performance (one for each criteria) are then used in a response surface model which is optimized to determine the final design.

1.2.1 Formal Problem Statement

It has been determined that a product has \( p=1, \ldots, P \) design parameters that could have a bearing on \( t=1, \ldots, T \) measures of performance (quality characteristics). The set of all possible combinations of the \( P \) design parameters is known as the design space. It is planned to take observations at \( N \) points in the design space. The \( N \) points comprise the design matrix. Let \( \theta \) represent the region of interest for the current experiment. Let the settings of design parameters be denoted by \( X_h = \{ x_{h1}, x_{h2}, \ldots, x_{hp} : h=1, \ldots, N \} \). A total of \( j=1, \ldots, q_h \) noise parameters have been identified that describe the environment under which the product is manufactured and operated at
design point h. Some of these parameters represent variability in the design parameters, while the rest represent the variability in other conditions of use.

The set of all possible combinations of the \( q_h \) noise parameters is known as the noise space, \( \mathcal{N}_h \). Note that the noise space need not be the same for each observed combination of the design parameters, as denoted by the subscript 'h'. At each design point h, \( r_h \) observations are taken under noise conditions \( W = \{ w_{k1}, w_{k2}, \ldots, w_{kq_h} : k=1, \ldots, r_h \} \), which, for experimental purposes, are controlled. The set of \( r_h \) observations is called the noise matrix. The T performance measures result in a vector \( Y_{hk} \), where:

\[
Y_{hk} = \begin{bmatrix}
Y_{(hk)1} \\
Y_{(hk)2} \\
\vdots \\
Y_{(hk)T}
\end{bmatrix}
\]

This scheme is illustrated in Figure 1.1.

Assuming that an unknown functional relationship between \((X,W)\) and \( Y_t(X,W) = E[Y_t \mid (X=x_{hp}, W=w_{kq_h})] \) exists, second order statistical relationships are fit for each of the T quality characteristics at each of the N design points. These equations seek to describe the behavior of each quality characteristic over their respective \( \mathcal{N}_h \). Denote these equations by \( \hat{Y}_{ht}(X, W) \).

Estimates of the mean performance (\( \mu_{ht} \)) and variance (\( \sigma_{ht}^2 \)) are obtained from:
Figure 1.1 The Experiment

<table>
<thead>
<tr>
<th>$w_{11}$</th>
<th>$\ldots$</th>
<th>$w_{k1}$</th>
<th>$\ldots$</th>
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<tr>
<td>$w_{12}$</td>
<td>$\ldots$</td>
<td>$w_{k2}$</td>
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</tr>
<tr>
<td>$w_{1g}$</td>
<td>$w_{kq}$</td>
<td>$w_{xq}$</td>
<td>$\ldots$</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>$x_{11}, x_{12}, \ldots, x_{1P}$</th>
<th>$y_{11}$</th>
<th>$y_{1k}$</th>
<th>$y_{1r}$</th>
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<td>$\ldots$</td>
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<td>$\ldots$</td>
</tr>
<tr>
<td>$x_{h1}, x_{h2}, \ldots, x_{hP}$</td>
<td>$y_{h1}$</td>
<td>$y_{hk}$</td>
<td>$y_{hr}$</td>
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<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$x_{n1}, x_{n2}, \ldots, x_{nP}$</td>
<td>$y_{n1}$</td>
<td>$y_{nk}$</td>
<td>$y_{nr}$</td>
</tr>
</tbody>
</table>

$X^n, X^1P$
\[ \hat{\mu}_{ht} = \int_{\Omega_h} \hat{y}_{ht}(X, W) g(W | X) dW \]  

\[ \hat{\sigma}^2_{ht} = \int_{\Omega_h} [\hat{y}_{ht}(X, W) - \hat{\mu}_{ht}]^2 g(W | X) dW \]

where \( g(X, W) \) represents the probability distribution function for \( X \) on \( \Omega_h \), that is, the probability of being manufactured and used under conditions \( \Omega_h \). We assume that the \( Y_{htk} \), \( t=1, \ldots, T \) are independent.

The calculated values for \( \hat{\mu}_{ht} \) and \( \hat{\sigma}^2_{ht} \) are used to estimate the global functions \( M_t(X) \) and \( H_t(X) \), describing mean response and variance over \( \theta \) respectively. Note that \( \phi \) and \( \Omega_h \) are assumed to be continuous.

One possible objective is the expected squared deviation from the target value. This is the mean square error for the \( t^{th} \) quality characteristic (MSE, MSE in general), consisting of mean square error plus bias, and will be used throughout this study. The objective can be generalized to any relevant performance metric, including those independent of adjustment (Leon, et. al., 1987). Further discussion will be postponed until Chapter 4.

The design problem is invariably influenced by many outside factors. Thus, it is not enough to minimize MSE without constraint. The multiple criterion problem has the form:
Pl: Minimize

(1) MSE_1
(2) MSE_2

Subject To:

(1) \text{continuous design parameters}

(2) \text{discrete design parameters}

(3) cost, complexity, and tolerance constraints

Constraints (1) and (2), are imposed by practical limits on adjustment, and the boundaries of \( \theta \). The inferential equations \( M_t(X) \) and \( H_t(X) \) are not necessarily valid outside the limits observed during the experiment.

If the optimal solution to \( Pl \) has no tight constraints of the form (1) or (2), and the response functions are convex, then no improvement in the objectives is possible by adjusting the design parameters outside the range of \( \theta \) under current consideration (bounds of the current experiment). However, if some constraints of type (1) or (2) are tight, and if they are tight because they represent the limits of inference of \( M_t(X) \) and \( H_t(X) \), then a direction of improved response exists and should be explored to locate the minimum, followed by a confirmation experiment.

Typically, there is a limit to the number of observations that can be taken. Some of these observations may be devoted to screening experiments to determine important effects. The problem is then to
decide where to take observations (W) in order to best estimate $Y_{ht}(X, W)$. Experimental designs for both the design and noise matrices must then be chosen, each having their own special considerations. The choice of experimental design is critical to the accuracy of the fitted equations ($Y_{ht}(X, W)$), and therefore very important to the accuracy of the overall solution procedure.

This thesis then makes two basic contributions to the subject of optimal parameter selection. First, a valid systematic procedure is developed for determining optimal settings. Second, appropriate experimental designs are developed for the design and noise matrices.

The following chapter contains a review of applicable literature and sufficient detail on selected topics to ensure continuity in the text. Chapter 3 contains the development of minimum bias central composite designs for most cases that will be encountered when applying the solution methodology. Chapter 4 contains the detailed development of the method. Chapter 5 is an example of the method applied to a practical problem, the determination of parameter settings in a computer operating system environment. Finally, Chapter 6 presents the summary and concluding material.
CHAPTER 2

LITERATURE REVIEW AND BACKGROUND MATERIAL

Taguchi's methodology was not well understood in the Western world until recently. This was largely due to indifference on the part of Westerners to Japanese quality control methods, and language difficulties. Competition from Japan has damaged industry, causing many to take Japan more seriously. Understanding was not easy: poor or conflicting translations clouded comprehension. One major contribution to better understanding was the October 1985 (Vol 17, No 4) issue of the Journal of Quality Technology, which was almost exclusively devoted to Taguchi's methodology.

Kackar's (1985) paper in this issue provided, perhaps, the best interpretation to date of the Taguchi methodology. Equally significant were the discussions following this paper, one main theme of which was the shortcomings of the Taguchi methods. Although mentioned in Chapter 1 as background to the problem, a more complete coverage of these shortcomings will be provided here. Also, the Taguchi Method will be covered in sufficient detail to provide continuity in the text. Other references may be consulted for more detail (for example, Kackar, 1985, and Taguchi, 1987).
2.1 The Taguchi Method

The Taguchi method is a system for determining the settings of design parameters that maximize a given performance measure.

Step 1. Identify the $X_p$ design parameters, their ranges, and the noise parameters, $W_q$ and their ranges.

Step 2. Plan the parameter design experiment: construct the design and noise matrices.

Step 3. Conduct the experiment. This can be either physical experimentation, or computer simulation. Calculate the performance statistic, the estimate of the corresponding performance measure. Taguchi's performance statistics are all signal to noise ratios.

Step 4. Using the observed values for the performance statistic, determine the 'optimum' settings for the design parameters. Taguchi advocates a stepwise optimization of the parameters, rather than a simultaneous determination. Parameters are first divided into three categories and adjusted accordingly:

a. Variables that affect the variance of the performance and possibly the mean performance. These are set so as to minimize variance.

b. Variables that affect the mean performance, and not the variance. These are set so as to place the mean on target.

c. Variables which have no effect on mean performance or variance. These are placed at their most economical settings.
21

Step 5. Run confirmation experiments to determine if the new settings improve the performance measure. This serves to highlight Taguchi's views about data interpretation. He is not concerned with understanding the physical process, but rather uses the results as a signpost, highlighting directions of improved response.

2.1.1 Criticism of Taguchi's Tactics

Most criticism is not directed at Taguchi's ideas, but rather at the specific methods used. The consensus is that many of his methods are unsophisticated, in that better methods exist, or that the supporting theory is poorly developed, or non-existent. Box (1985) points out that currently available constrained optimization procedures are far superior to Taguchi's two step optimization approach. Box also expresses concern that many of the vital ingredients of contemporary data analysis are omitted or ignored:

a. the sequential nature of experimentation, where information gained at early stages enriches the experiment at later stages,

b. his method of analyzing the data is too complicated,

c. techniques such as data transformations are not used which could lead to a more informative analysis, and;

d. the use of the signal to noise ratio is unconvincing.
Easterling (1985) is particularly concerned with the issue of interactions in the context of using Taguchi's recommended designs. These designs are highly fractionated and frequently do not allow for the estimation of anything but a few main effects. In practice, significant interactions are bound to occur which Taguchi ignores. It is entirely possible that Taguchi's recommended designs, being highly fractionated factorials, are inadequate in identifying design parameters (those which affect the variance and possibly the mean) and adjustment parameters (those which affect the mean performance only).

Easterling objects to the signal to noise ratio as a performance metric on the grounds that no model is specified that would lead to the signal to noise ratio being a valid analytical tool. Although Taguchi has a vast array of signal to noise ratios that are tailored to the problem at hand, this criticism is no less valid.

Lucas (1985), echoing Box's criticism, disagrees with Taguchi's signal to noise ratio as an overall performance statistic on the grounds that computers can be used much more effectively to analyze multiple responses. Lucas also advocates analyzing signal and noise responses separately, rather than using the combined signal-to-noise ratio.

Hunter (1987) issues a condemnation of the signal to noise ratio on sound theoretical grounds, along with the recommendation that they not be used. In particular, he states that most forms of the signal to noise ratio do not separate signal from noise and are useless in determining which factors affect the mean and variance. He suggests analyzing the statistic
\[
\log(\bar{y}/\bar{s}) = \log(\bar{y}) - \log(\bar{s}) \quad (2.1.1.1)
\]
in the form of the right hand side above, which points out that Taguchi's recommended statistic (left expression in 2.1.1.1) is totally inadequate for determining which factors affect the mean performance, and those that affect the variance of the mean performance.

The consensus is that Taguchi's philosophy is an excellent vehicle for incorporating quality considerations into the design of a product, but the analytical methods leave something to be desired.

2.2 Design of Experiments

Consider the problem of designing an experiment in order to determine the relationship between a response variable \( Y \), and \( P \) explanatory variables \( X_1, \ldots, X_P \). Assume a model of the form
\[
Y = X\beta + \epsilon, \quad \epsilon \sim N(0, \sigma^2 I) \quad (2.2.1)
\]
is to be fitted to the results. The coefficients, \( \beta \), must be estimated with some procedure. Continuous parameters (explanatory variables) are approximated for experimental purposes with a number of discrete levels. Discrete parameters, of course, are represented by their available levels.
2.2.1 Obtaining the Estimators of $p$

In this study, the estimators of $p$ are obtained via least squares estimation. The estimators of $p$ have minimum variance among all unbiased linear estimators (Graybill, 1961, p. 115).

**Weighted Least Squares.** This is used when the error term variance is not equal for all observations. Let $W$ be a diagonal matrix with

$$w_{ii} = \frac{1}{\sigma_i^2}.$$  \hspace{1cm} (2.2.1.1)

The normal equations for the weighted least squares method are:

$$X'WXb = X'WY$$  \hspace{1cm} (2.2.1.2)

and the weighted least squares estimators are:

$$b = (X'WX)^{-1}X'WY.$$  \hspace{1cm} (2.2.1.3)

**Unweighted Least Squares.** If the assumption of homoscedasticity (constant error variance) is valid, then $W = I$ and the appropriate estimators are:

$$b = (X'X)^{-1}X'y$$  \hspace{1cm} (2.2.1.4)

and the normal equations are:

$$X'Xb = X'y$$  \hspace{1cm} (2.2.1.5)

2.2.2 Minimum Bias Design

The performance an experimental design may be measured for comparative purposes using an independent metric. In this section, the metric is bias (to be explained in detail shortly). We would like to minimize the mean square error integrated over the entire region of
interest, but as will be seen shortly, it is impossible to construct such a design without prior information.

In response surface estimation, a polynomial of degree $d_1$ is used to estimate a polynomial of (assumed) degree $d_2$, where $d_2 > d_1$. Formally, if the set of explanatory variables is $\{X\}$, then over the region of interest $\theta$, the model is:

$$ Y = f(X) + \epsilon $$  \hspace{1cm} (2.2.2.1)

but the true functional relationship is

$$ Y = \alpha(X) + \epsilon $$  \hspace{1cm} (2.2.2.2)

or

$$ Y = f(X) + \delta(X) + \epsilon $$  \hspace{1cm} (2.2.2.3)

where

$$ \delta(X) = \alpha(X) - f(X) $$  \hspace{1cm} (2.2.2.4)

This demonstrates the two types of errors in the model:

a. Bias, $\delta(X)$, the systematic difference between the true value and the approximated value.

b. Random error, $\epsilon$.

If $\epsilon$ has a mean of 0 and variance $\sigma^2$, then the expected value of the true functional relationship (2.2.2.2) is $E(Y) = \alpha(X)$, which is approximated by $f(X)$. Estimates of the parameters for model (2.2.2.1) are obtained by least squares methods (section 2.2.1). Let $\hat{Y}$ be the estimated model. The MSE (squared bias plus variance) for this model, after standardization for $N$, the number of observations, and $\sigma^2$, is (Box and Draper, 1986):
\[
\frac{N\text{E}(\hat{Y} - \alpha)^2}{\sigma^2} = \frac{N\text{Var}(\hat{Y})}{\sigma^2} + \frac{N(\text{E}(\hat{Y}) - \alpha)^2}{\sigma^2}
\] (2.2.5)

Stated another way:

Mean Square Error (MSE) = Variance Error (V) +

Squared Bias Error (B) (2.2.6)

Suppose a weight function is used to indicate the analyst's interest in the accuracy of the estimates (greater weight implies greater accuracy over \( \theta \)). For example, if it is known the values a noise parameter can assume in practice are normally distributed, then the experimenter will want high accuracy of the estimators at the values that occur most frequently, and can tolerate less accuracy in the less frequently occurring values. Denote the weight function \( w(x) \).

Then, for a given design:

\[
\text{WMSE} = \frac{N}{\sigma^2} \int_{\theta} w(x) \text{E}\{\hat{Y}(x) - \alpha(x)\}^2 dx,
\] (2.2.7)

\[
V = \frac{N}{\sigma^2} \int_{\theta} w(x) \text{E}\{(\hat{Y}(x) - \text{E}Y(x))^2 dx,
\] (2.2.8)

\[
B = \frac{N}{\sigma^2} \int_{\theta} w(x) \{(\text{E}Y(x) - \alpha(x))^2 \} dx.
\] (2.2.9)

Where \( \text{WMSE} \) indicates weighted MSE.

The weight function is chosen so that:

\[
\int_{\theta} w(x) = 1.
\] (2.2.10)

A good design will minimize \( \text{WMSE} \), but in practice this can only be achieved when an accurate estimate of \( \sigma \) is available (Meyers, 1971). Box and Draper (1986) show that designs that minimize \( B \) are very close
to the designs that minimize WMSE unless the effect of variance error, V, is quite large compared to the effect of bias error, B. Thus, if σ is not known, the most conservative course of action is to use a design that minimizes B. This approach is taken throughout this thesis.

The Box-Draper Condition. Box and Draper (1959) found necessary and sufficient conditions for the minimization of B. Another condition, simpler to achieve in practice, was also found which is sufficient for the minimization of B.

Let a polynomial model of degree d₁ be used to approximate a polynomial model of degree d₂. For all N data points, this is formally expressed as:

\[ \hat{Y}(x) = X_1 \beta_1 \]  \hspace{1cm} (2.2.2.11)

and

\[ \kappa(x) = X_1 \beta_1 + X_2 \beta_2 , \]  \hspace{1cm} (2.2.2.12)

where Y is an (N X 1) response vector, X₁ is an (N X p) matrix of explanatory variables, β₁ is a (p X 1) vector of parameters, X₂ is an (N X p₂) of explanatory variables not in the model, and β₂ is a (p₂ X 1) vector of parameters that are not estimated.

Defining

\[ M_{11} = N^{-1} X_1' X_1, \quad M_{12} = N^{-1} X_1' X_2, \]
\[ \mu_{11} = \int w(x) x_1 x_1' dx, \quad \mu_{12} = \int w(x) x_1 x_2' dx. \]  \hspace{1cm} (2.2.2.13)

Then necessary and sufficient conditions to minimize B are:
\[ M^{-1}_{11} M_{12} = \mu^{-1}_{11} \mu_{12}. \]  

(2.2.2.14)

A sufficient condition (hereafter called the Box-Draper condition) is therefore:

\[ M_{11} = \mu_{11} \text{ and } M_{12} = \mu_{12}. \]  

(2.2.2.15)

Simply stated, condition 2.2.2.15 means that for all moments up to and including order \( d_1 + d_2 \), the moments of the weight function must equal the moments of the design. This can be seen more clearly when one considers that the elements of \( M_{11} \) and \( M_{12} \) are of the form:

\[
N^{-1} \sum_{u=1}^{N} \alpha_1 x_1^2 \ldots x_k^2
\]

and the elements of \( \mu_{11} \) and \( \mu_{12} \) are of the form:

\[
\int_{\Omega} w(x) x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_k^{\alpha_k} dx
\]

and 2.2.2.15 is equated on an element by element basis.

Draper and Lawrence (1967) investigate minimum bias designs where the region of interest is a sphere or hyper-sphere. Draper and Lawrence (1965) investigate designs with a cuboidal region of interest and a uniform weight function. The minimum bias designs developed in Chapter 3 are an original extension to this work. The difference is that the multivariate normal function is used as a weight function, and cuboidal and unbounded regions of interest are used.
2.2.3 Other Design Criteria

There is a wide selection of design criteria that can be used for comparative purposes in addition to the minimum bias criteria discussed previously. Design points may be selected so that the experimental design meets some type of 'optimal' criteria. The most frequently used optimality criteria are (Steinberg and Hunter, 1984, Zemrock, 1986):

a. D-optimal: Minimize $|X'X|^{-1}$

b. A-optimal: Minimize $\text{trace}(X'X)^{-1}$

c. E-optimal: Minimize the maximum eigenvalue of $(X'X)^{-1}$

d. G-optimal: Minimize $\{\text{Maximum Var}(\hat{y}(x))\}$ where $x$ is some point $x = (1, x_1, \ldots, x_p)^T$, and

$$\text{Var}(\hat{y}(x)) = x'(X'X)^{-1}x\sigma^2.$$  

The minimization is performed over all possible values for $x$.

A D-optimal design minimizes the maximum of the standard errors of the predicted responses, and results in a joint confidence region for the $p$'s which has the smallest hyper-volume of designs meeting the above listed criteria (Hahn, Meeker and Feder, 1976). Welch (1984) points out that D-optimality is not the best criteria if one is interested in response predictions. Also, care must be to ensure that each parameter (explanatory variable) is given equal weight in calculating $|X'X|^{-1}$, since it is scale dependent. A common technique is to transform each parameter $X_i$ to a scaled parameter $X_i'$, such that $-1 \leq X_i' \leq 1$. 

The D-optimal and G-optimal criteria are subject to criticism on the grounds that they assume a model form is known completely in advance. One shortcoming of this assumption is that the optimal designs do not usually allow for checking model adequacy (Zemroch, 1986, p 40), since the optimal design places one-half of the points at -1 and the other half at 1 (assuming $-1 \leq X \leq 1$).

**Application Specific Criteria.** There are instances when the experimenter is interested in applying some application-specific criteria for determining the eventual design. For example, the design space may not be a regular figure because some combinations of explanatory variables are infeasible. One such situation is mixture experiments, where the proportion of ingredients is always 100%.

For the mixture experiment example, cluster analysis has been successfully used to produce designs which cover the region of interest uniformly (Snee, 1985, Zemroch, 1985). The point here is that the metric must be chosen commensurate with the problem at hand. In this case, the metric is uniform coverage. Consequently, no single technique dominates the solution to choosing experimental designs.

In addition to the aforementioned optimality measures, an experimenter may desire that an experimental design have certain properties. Some of those properties are defined here and will be used throughout this study. All definitions are from Montgomery (1984).
Rotatability. An experimental design is said to be rotatable when the variance of the predicted response, \( \hat{Y} \), is a function only of the distance from the design center, and not direction. Rotatability is important when the subsequent search direction for improved settings is unknown.

Orthogonality. A design is orthogonal when the off-diagonal elements of the \( X'X \) matrix are zero. These designs minimize \( \text{Var}(\hat{b}) \), the variance of the estimators of \( \beta \). This property is desirable when the effects of the factors are to be estimated. Tests concerning orthogonal elements are also independent.

Uniform Precision. A uniform precision design is one in which \( \text{Var}(\hat{Y}) \) at a unit distance from the origin is equal to \( \text{Var}(\hat{Y}) \) at the design center. The uniform precision design provides better protection against bias in the regression coefficients caused by the presence of higher order terms than does an orthogonal design. In that sense it may serve as a surrogate for minimum bias.

Factorial Design. These are designs for studying the effects of \( k \) factors on some observed response. Each of the factors are studied at the same number of levels, usually 2 or 3. (Subsequent discussion will concern a two level factorial only). There are \( 2^k \) possible points at which to take observations. The two levels are denoted \(-a\) (low level) and \(a\) (high level), where 'a' is the distance along an axis from the origin of the co-ordinate system. For \( k=2 \), the points comprising the \( 2^2 \) design are \((-a, -a), (-a, a), (a, -a) \) and \((a, a)\).

Sometimes it is not possible, or not required, to take all \( 2^k \) observations, and only a fraction of the possible observations are
taken. In general, a $2^{-p}$, $(p \leq k)$ fraction ($1/2$, $1/4$, $1/8$, etc) of a $k$ factorial is taken. This is denoted $2^{k-p}$.

Central Composite Design (CCD). The central composite design is formed by combining a $2^k-p$ factorial design with $2k$ axial points at $(\pm d, 0, \ldots, 0)$, $(0, \pm d, 0, \ldots, 0)$, \ldots, $(0, \ldots, \pm d)$, (where $d$ is the distance of an observation along an axis from the origin of the coordinate system), and $n_0$ center points at $(0, \ldots, 0)$. Geometrically, this looks like a cube (factorial portion) plus a star (axial portion). A CCD is a widely used design for response surface models.

2.2.4 The Experimental Design Philosophy

The preceding is suggestive of a broader view to experimental design, more suited to the considerations of the real world. Stigler (1971) felt that a good design should meet three conditions:

a. Allow for checking lack of fit in the current model.

b. If the model is adequate (judged from the lack of fit test), reasonably efficient inferences should be possible.

c. The model design should not depend on unknown parameters.

The development of a design is modified by any unique conditions imposed by a particular situation, as well as the unique objectives of the experiment. Once these influences are known, the development of a design proceeds as follows (Snee, 1985):

a. generation of feasible points for the experiment,

b. generation of the experimental design i.e. a collection of points from (a); and,
c. evaluation of alternate designs from (b) to choose the best one, commensurate with the objectives of the experiment.

The evaluation of alternate designs is done according to the design criteria discussed in this Chapter. The experimenter must make the decision of which criteria to use based on his professional judgment and knowledge of the problem at hand.

The remainder of this thesis concentrates on minimum bias central composite designs with rotatability as a secondary objective. Such designs have appeal for their logical simplicity, efficiency in use of the data, can be built sequentially from first order \(2^k\) designs, and generally provide good predictive models.
CHAPTER 3

MINIMUM BIAS DESIGNS

This chapter contains the theoretical development of minimum bias central composite designs (CCDs) for the following cases:

a. unbounded region of interest with a multivariate normal weight function,
b. cuboidal region of interest with a multivariate normal weight function, and;
c. cuboidal region of interest with a uniform weight function.

These situations have not been covered previously in the literature and, as such, are an original contribution to the field.

The implied situation in this chapter is as follows. An experiment is to be designed. Based on experimental observations, a (polynomial) regression model will be fit to approximate the response variables over the design region. Our interest is in a model which yields minimum (weighted) bias response estimates.

This chapter is a digression from the theme of a unified approach to the parameter design problem, in that a specific issue required to support the methodology is examined in detail. The thread is picked up again in Chapter 4, where minimum bias designs will be employed to facilitate the building of polynomial models which relate...
parameter settings to performance. The region of interest will be the feasible parameter space or a sub-region thereof.

The designs in this chapter are non-sequential. Since multiple quality characteristics are being measured, and there is only one experimental design, it must accommodate models of varying degree. Thus, it is reasonable to assume that a second order model will be required by at least one quality characteristic with prior knowledge. Nevertheless, suitable \((2^k-p + \text{center point})\) first order designs may be used to check for non-linearity. Sequential designs can be obtained by imposing the additional constraint that the moments of the first order design equal the moments of the weight function — that is, applying the Box Draper condition to the first order design.

For notational purposes, let the experimental design variables (factors) be \(X_1, \ldots, X_k\). The CCD will be represented as follows:

a. \(2^{k-p}\) factorial points formed from \(p\) defining relations of \((x_1, \ldots, x_k) = (i\alpha, \ldots, i\alpha)\),

b. \(2k\) axial points at \((\pm\alpha, 0, \ldots, 0), (0, \pm\alpha, 0, \ldots, 0), (0, \ldots, 0, \pm\alpha)\); and

c. \(n_0\) center points.

The values for \(a\), \(\alpha\), and \(n_0\) are determined by the minimum bias conditions. We describe how to select these values in the remainder of this chapter.
3.1 Weight Functions

A weight functions expresses the experimenter's interest in the accuracy of the estimators. Two weight functions will be discussed: the multivariate normal and the uniform weight function.

3.1.1 The Multivariate Normal Weight Function

The \( k \)-factor multivariate (joint) normal distribution is formally denoted

\[
N_k(\mu, \Sigma) = \frac{1}{C} \exp\left\{-0.5x^T\Sigma^{-1}x\right\}
\]

where:

\[
C = (2\pi)^{k/2} |\Sigma|^{1/2}
\]

\( \mu \) is the \( k \times 1 \) vector

\[
\mu = \begin{bmatrix}
\mu_1 \\
\mu_2 \\
\vdots \\
\mu_k
\end{bmatrix}
\]

and \( \Sigma \) is the matrix

\[
\Sigma = \begin{bmatrix}
\sigma_{11} & \cdots & \cdots & \sigma_{1k} \\
\vdots & \sigma_{22} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{k1} & \cdots & \cdots & \sigma_{kk}
\end{bmatrix}
\]

also known as the variance-covariance matrix.
If the experimenter is more interested in estimating the value of the quality characteristics in the region where they are most likely to occur rather than in the extreme areas, and the likelihood of occurrence is $N_k(\mu, \Sigma)$, then the multivariate normal function outlined above is the appropriate weight function for use in equations 1.2.1.1 and 1.2.1.2 when computing the mean and variance of performance at each design point. For the purposes of this study, it is assumed that $\mu = 0$, or the condition can be satisfied through a linear transformation. That is, the design center coincides with the most likely values of the noise parameters. It is further assumed that $\Sigma$, the variance and covariance of the parameters as they occur in nature, is known through prior knowledge of the environment in which the product is produced and used.

3.1.2 The Uniform Weight Function

When a uniform weight function is used, the experimenter is implicitly stating that he has no preference of one point in the region of interest over another. Thus, in order to satisfy the condition

$$\int_{\mathcal{D}} w(x) = 1$$  \hspace{1cm} (3.1.2.1)$$

the weight function must be the inverse of the volume (area, for 2 parameters) of the region of interest.
3.2 Unbounded Region of Interest - Multivariate Normal Weight Function

In this section, it is assumed that all parameters in the design either have infinite ranges or, as a practical matter, have bounds outside of $\mu \pm 3\sigma$. The multivariate normal weight function is used to express our interest in specific points in the design region. A design is determined by equating the moments of the design to the region moments (the Box-Draper condition) and solving for the design parameters.

3.2.1 Moments of the Design

(Property 1). If the design is symmetric about some point $c$, all odd order moments about $c$ are zero.

Proof. For any set of points $\{x\}$, a general expression for the moments of $\{x\}$ about $c$ is (assume $c = (0, ..., 0)^T$ for simplicity):

$$\sum_{u=1}^{N} x_1^\alpha_1 x_2^\alpha_2 ... x_k^\alpha_k$$

For any point in the design a perpendicular distance $d$ from an axis, there exists a corresponding point of distance $-d$ (i.e. in opposite direction) from the same axis. For any $\alpha_i$ odd, the contribution of that point to the sum (3.2.1.1) is zero since:

$$(x_1^\alpha_1 ... (d)^\alpha_1 ... x_k^\alpha_k) + (x_1^\alpha_1 ... (-d)^\alpha_1 ... x_k^\alpha_k) = 0.$$ 

As this is true for all points, all terms of equation 3.2.1.1 are zero, and the corresponding moment is zero.
The consequence of property 1 is that one only need to be concerned with the even order moments. Assuming \( d_1 = \) degree of model = 2 and \( d_2 = \) true degree of functional relationship = 3, the design moments (of even order) for a CCD are:

\[
\begin{align*}
N^{-1} \sum_{u=1}^{N} \sum_{i=1}^{2} x_i^2 &= \frac{2^{k-p} a^2 + 2^2 a^2}{2^{k-p} + 2k + n_0} \\
N^{-1} \sum_{u=1}^{N} \sum_{i=1}^{4} x_i^4 &= \frac{2^{k-p} a^4 + 2^4 a^4}{2^{k-p} + 2k + n_0} \\
N^{-1} \sum_{u=1}^{N} \sum_{i=1}^{2} x_i^2 w(x) &= \frac{2^{k-p} a^4}{2^{k-p} + 2k + n_0}
\end{align*}
\]

These results follow directly from the CCD definition provided earlier.

3.2.2 Moments of the Weight Function

The \( \mu(\alpha_1, ..., \alpha_k) \) moment of the weight function is defined as:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1^{\alpha_1} \cdots x_k^{\alpha_k} w(x) dx_1 \cdots dx_k
\]

It can be shown that the moments for the weight function

\[
w(x) = C \exp(-0.5 x^t \Sigma^{-1} x)
\]

where:

\[
\mu = 0 \text{ and } C = \frac{1}{(2\pi)^{k/2} |\Sigma|^{1/2}}
\]

are:

\[
\mu(0, ..., 0, 2, 0 ..., 0) = \sigma_{11}
\]
\[ \mu(0, \ldots, 0, 4, 0\ldots, 0) = 3\sigma_{ij} \]  \hspace{1cm} (3.2.2.4)

\[ \mu(0, \ldots, 0, 2, 2, 0\ldots, 0) = \sigma_{ij}\sigma_{jj}. \]  \hspace{1cm} (3.2.2.5)

A set of transformations is introduced in Chapter 4 that reduce \( E \) to an identity matrix, thus allowing the above to be applied in all situations.

The next property, when combined with Property 1, allow the odd order moments to be disregarded entirely. (Property 2). All odd moments of the weight function about the mean are zero.

Proof. Assume \( \mu_1 = 0 \) for simplicity. Expanding the 1\textsuperscript{th} term of 3.2.2.1:

\[
K \int_{-a_1}^{a_1} x_1^{\alpha_1} f_1(x_1) dx_1 = K \int_{-a_1}^{a_1} x_1^{\alpha_1} f_1(x_1) dx_1 + K \int_{0}^{a_1} x_1^{\alpha_1} f(x_1) dx_1 - K \int_{-a_1}^{a_1} x_1^{\alpha_1} f(x_1) dx_1
\]

where \( K \) is a constant, and \( f_1(x_1) \) are the terms of \( w(x) \) containing \( x_1 \). Setting \( x_1 = -x_1 \) in the first term of the right side above, the limits of integration become \([0, a_1]\) (since the function is symmetric), and the term can be expressed as:

\[
-K \int_{0}^{a_1} x_1^{\alpha_1} f(x_1) dx_1
\]

since \( f_1(x_1) = f_1(-x_1) \) and \( \alpha_1 \) is odd. Then,

\[
K \int_{-a_1}^{a_1} x_1^{\alpha_1} f(x_1) dx_1 = 0. \]

\[ \square \]
3.2.3 Solution

The Box-Draper condition is applied, equating the design moments to the region moments. The expression for $a$ is obtained by equating 3.2.1.4 to 3.2.2.5. Setting $N = 2^{k-p} + \frac{2k + n_0}{n_0}$ and simplifying:

$$a = (\frac{N}{2^{k-p}})^{0.25}$$

(3.2.3.1)

We then choose the value for $\alpha$ so the design is rotatable. For rotatability (Montgomery, 1984):

$$\alpha = (2^{k-p})^{1/4}$$

(3.2.3.2)

It can be confirmed by back substituting into any of the equations produced by the Box-Draper condition that the solution is rotatable.

Finally, the expression for $N$ is obtained by equating 3.2.1.2 to 3.2.2.3. Substituting 3.2.3.1 and 3.2.3.2 for $a$ and $\alpha$ respectively, and solving for $N$:

$$N = \frac{(2^{k-p} + 2^{(k-p)/2+1})^2}{2^{k-p}}$$

(3.2.3.3)

Table 3.1 contains the minimum bias normally weighted CCDs for $k=2$ through 7. It can be seen that for $k$ moderately large, $N$ is quite large. Thus a fractional factorial would be required to produce a design that made sense practically. Any design with a factorial portion of at least resolution V meets the conditions of equation 2.2.4.15 (Box and Draper, 1986). Thus smaller designs are quite possible. A number of designs with a fractional factorial portion are included as Table 3.2.
Table 3.1 Minimum Bias Central Composite Designs
Unbounded Region of Interest

<table>
<thead>
<tr>
<th>k</th>
<th>n₀</th>
<th>N</th>
<th>a</th>
<th>α</th>
<th>αa</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>8</td>
<td>16</td>
<td>1.414</td>
<td>1.414</td>
<td>2.000</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>23</td>
<td>1.306</td>
<td>1.682</td>
<td>2.197</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td>36</td>
<td>1.225</td>
<td>2.000</td>
<td>2.449</td>
</tr>
<tr>
<td>5</td>
<td>17</td>
<td>59</td>
<td>1.163</td>
<td>2.375</td>
<td>2.776</td>
</tr>
<tr>
<td>6</td>
<td>24</td>
<td>100</td>
<td>1.118</td>
<td>2.828</td>
<td>3.162</td>
</tr>
<tr>
<td>7</td>
<td>35</td>
<td>177</td>
<td>1.085</td>
<td>3.364</td>
<td>3.649</td>
</tr>
</tbody>
</table>

Table 3.2 Some Minimum Bias CCDs With A Fractional Factorial Portion

<table>
<thead>
<tr>
<th>k</th>
<th>Fraction</th>
<th>N</th>
<th>n₀</th>
<th>a</th>
<th>α</th>
<th>αa</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>2⁶⁻¹</td>
<td>59</td>
<td>17</td>
<td>1.163</td>
<td>2.375</td>
<td>2.776</td>
</tr>
<tr>
<td>7</td>
<td>2⁷⁻¹</td>
<td>100</td>
<td>24</td>
<td>1.118</td>
<td>2.828</td>
<td>3.162</td>
</tr>
<tr>
<td>7</td>
<td>2⁷⁻²</td>
<td>59</td>
<td>17</td>
<td>1.163</td>
<td>2.375</td>
<td>2.776</td>
</tr>
</tbody>
</table>

Note the similarity in designs, i.e. a design for k=7, 2⁷⁻² factorial portion uses the same design as k=5, 2⁵ factorial portion.

3.3 Cuboidal Region of Interest - Multivariate Normal Weight Function

For this case, the region of interest is a cube or a hypercube. The dimensions of the region of interest are determined by the feasible limits of the experimental parameters, or by a decision on the part of the experimenter.
Moments of the Design. The moments of the design are those specified in equations 3.2.1.2 through 3.2.1.4 as a central composite design is again being used.

3.3.1 Moments of the Weight Function

The weight function is the truncated multivariate normal distribution. For \( \mu = 0 \):

\[
W(x) = \phi(x) = \frac{1}{C}, \quad \text{Exp}\{-0.5x'S^{-1}x\} \quad (3.3.1.1)
\]

where

\[
C' = (2\pi)^{k/2} |\Sigma|^{1/2} \prod_{i=1}^{k} \{\bar{c}(b_{iu}) - \bar{c}(b_{il})\} \quad (3.3.1.2)
\]

and

\( b_{iu}(i=1, \ldots, k) \) is the upper truncation point for the \( i \)th parameter;
\( b_{il}(i=1, \ldots, k) \) is the lower truncation point for the \( i \)th parameter.

Consider the case where \( b_{il} = -b_{iu} \) for \( i=1, \ldots, k \). Equal areas are truncated from the tails of the distribution. (The case \( b_{il} \neq -b_{iu} \) will be addressed later in this chapter). As the probability density function is still symmetric, all moments of odd order are zero. Even moments are expressed as:

\[
\mu(0, \ldots, 0, 2t, 0, \ldots, 0) = \frac{\sigma^{2t}P(t + \frac{1}{2}; \frac{(b-\mu)^2}{\sigma^2})(t + \frac{1}{2})}{C''} \quad (3.3.1.3)
\]

for \( t = 0, 1, 2, 3 \ldots ; \)

\[
C'' = (2\pi)^{k/2} \{\bar{c}(b_{iu}) - \bar{c}(b_{il})\}, \quad (3.3.1.4)
\]

and \( P(v;x) \) is the incomplete Gamma function:
\[ P(v;x) = \int_0^x t^{y-1} e^{-t} dt \quad (3.3.1.5) \]

See Appendix A for a proof.

The moments \( \mu(0, \ldots, 0, 2, 2, 0, \ldots, 0) \) can be obtained by multiplying the individual second order moments. By way of example, if \( k=2, \mu(2,2) = \mu(0,2) x \mu(2,0) \). This is possible as the variables are independent since \( \Sigma \) is diagonal, and the area defined by the convex hull of \((b_{11}, b_{21}), (b_{1u}, b_{21}), (b_{11}, b_{2u})\) and \((b_{1u}, b_{2u})\) is contained within the region of interest.

Before proceeding with the solution, it should be noted that a minimum bias design may have some points outside of the region of interest. This is, unfortunately, a mathematical outcome that cannot be circumvented. Many times, the region of interest is arbitrarily chosen by the experimenter and may be violated without problem. However, if the region of interest represents the region of feasibility, then the experimenter must reduce the size of the cube of interest such that all design points are feasible. The experimenter is taking a calculated risk in doing so, since the design has minimum bias over the cube only. However, he has one other choice if this is unpalatable and a CCD is to be used. (Nothing that has been done so far precludes the use of some other type of design). If a fractional factorial portion is used, the distance of the axial points from the design center will be reduced. This can be seen by comparing tables 3.1 and 3.2 for the same value of \( k \), as increasingly fractionated factorials are used.
3.3.2 Solution

The Box-Draper condition is applied, equating the moments of the design to the moments of the weight function. This produces a system of equations with no apparent closed form solution. The suggested solution method will be illustrated by example.

This rather intractable system of equations results because the distance from the design center to the factorial points, measured parallel to an axis \( (a_i) \) cannot be the same for all \( i \) as the second order moments are no longer all equal. This applies to the \( d_i \) also. This can be clearly seen by examining the right hand sides of the system of equations for the following example.

**Example.** Assume that an experimenter is interested in a region of interest with the following boundaries on a \( N_3(0, I) \) distribution: \((-\infty, \infty), (-1.5, 1.5), (-2.0, 2.0)\). With \( k=3 \) and a full factorial portion, the following system of equation results from applying the sufficient conditions:

\[
\begin{align*}
8a_1^2 + 2d_1^2 &= \mu(2, 0, 0)N = 1.0N \\
8a_2^2 + 2d_2^2 &= \mu(0, 2, 0)N = 0.9091N \\
8a_3^2 + 2d_3^2 &= \mu(0, 0, 2)N = 0.9998N \\
8a_1^4 + 2d_1^4 &= \mu(4, 0, 0)N = 3.0N \\
8a_2^4 + 2d_2^4 &= \mu(0, 4, 0)N = 1.801N \\
8a_3^4 + 2d_3^4 &= \mu(0, 0, 2)N = 2.653N
\end{align*}
\]
\[ 8a_1^2 a_2^2 = \mu(2, 2, 0)N = 0.9091N \]
\[ 8a_1^2 a_3^2 = \mu(2, 0, 2)N = 0.9998N \]
\[ 8a_2^2 a_3^2 = \mu(0, 2, 2) = 0.9089N \]

Note both sides of the system have been multiplied by \( N \). The left hand sides of 3.3.2.1 are obtained by directly applying equations 3.2.1.2 through 3.2.1.4. As an example of how the right hand side is derived, consider the calculation of \( \mu(0, 2, 0) \). Using 3.3.1.4 to calculate \( C'' \):

\[ C'' = (2\pi)^{\frac{k}{2}}[\Phi(1.5) - \Phi(-1.5)] \]
\[ = (2\pi)^{\frac{k}{2}}[0.9332 - 0.0668] \]
\[ = 2.172 \]

and

\[ P\left(t + \frac{\mu}{\sigma^2}; \frac{(b-\mu)^2}{\sigma^2}\right) = P(1.5; 2.25) = 0.6981 \]

(obtained from program PPGamma in Appendix B).

Finally,

\[ \mu(0, 2, 0) = \frac{\sigma^2tP\left(t + \frac{\mu}{\sigma^2}; \frac{(b-\mu)^2}{\sigma^2}\right)}{C''} \]
\[ = \frac{(1)(0.6981)(2.25)}{2.172} = 0.9091. \]

The \( d_i \) are not chosen for rotatability. Rotatability is desirable but this may not be possible (Draper and Lawrence, 1965), and minimum bias is more important.

This system has \( n = 2k + 1 \) unknowns and \( m = 2k + \left\lfloor \frac{k}{2} \right\rfloor \) conditions to
be satisfied. Since the system is not linear, there can be a finite number of non-unique solutions. There is no guarantee that a solution even exists.

Near 'exact' solutions were obtained in the following way. It is assumed that good solutions to the system of equations produced by the Box-Draper condition are ones that come close to satisfying all equations, if they are not satisfied exactly. The measure of closeness, for comparison purposes, is the norm, or the sum of the squared difference between the left and right side of the m conditions.

Solutions were obtained numerically by using the IMSL library procedure ZXSSQ, which solves the non-linear least squares problem:

\[
\text{Minimize } f_1(x)^2 + \ldots + f_m(x)^2
\]

(\(x \in X\))

(3.3.2.2)

where

\[
x = [x_1, \ldots, x_n]^T.
\]

Solution was obtained with the program Min_Squares (Appendix B), which calls the IMSL library procedure ZXSSQ:

\[
a_1 = 1.27; a_2 = 1.23; a_3 = 1.28
\]

\[
d_1 = 2.16; d_2 = 1.79; d_3 = 2.05
\]

\[
N = 21.33 \quad (n_0 = 7.33)
\]

\[
\text{Norm} = \sum_{i=1}^{n} f_i(x)^2 = 1.83.
\]

(3.3.2.3)

An experiment with the above settings for \(a_1\) and \(d_1\) and \(n_0 = 7\) would be very close to a minimum bias design.
3.3.3 Unequal Areas of Truncation

Consider the case where $b_{11} < -b_{11}$ for at least one 1. The odd order moments are no longer zero, as the function is no longer symmetric, and finding a solution that satisfies the Box-Draper condition becomes very difficult. Rather than dealing with the problem in this manner, the mean and variance can be recalculated using (Johnson and Kotz, 1970):

$$\mu_{it} = \mu_1 + \sigma_{it}^2 \left( \frac{\hat{f}(h) - \hat{f}(k)}{\hat{f}(k) - \hat{f}(h)} \right)$$

$$\sigma_{it}^2 = \sigma^2 \left( 1 + \frac{h^2(h) - k^2(k)}{\hat{f}(k) - \hat{f}(h)} \right) - (\mu_{it} - \mu_1)^2$$

where $h = (b_{11} - \mu_1)/\sigma_1$ and $k = (b_{11} - \mu_1)/\sigma_1$. The analysis can then proceed as before with $\mu_1 = \mu_{it}$ and $\sigma_1^2 = \sigma_{it}^2$. This cannot be used if there is any covariance i.e. if any of the off diagonal entries of $E$ are non-zero. A simplification that could be used in this case could be to limit the region of interest so that the tails are of equal size and proceed with the analysis using the solution technique in section 3.3.2.

Example. Continuing with the example in section 3.3.2, assume that the region of interest on the third parameter was (-2.0, 1.5) in place of (-2.0, 2.0). Then,

$$h = -2.0 \quad \hat{f}(h) = 0.05399 \quad \hat{f}(h) = 0.0226$$

$$k = 1.5 \quad \hat{f}(k) = 0.1295 \quad \hat{f}(k) = 0.9322$$

$$\mu_3 = -0.08303, \text{ and } \sigma_3^2 = 0.7604.$$ 

These values for $\mu_3$ and $\sigma_3^2$ would be used to determine the moments of the weight function using equations 3.3.1.3.
The region of interest is a cube or hyper-cube. The design moments are those in section 3.2.1 when a CCD is employed.

3.4.1 Moments of the Weight Function

It is useful to rescale all variables so the boundaries are (-1, 1) in the new scale as it avoids the problem of having different moments about each axis due to different lengths of the sides of the cube. This is possible by applying the transformation:

\[ x'_i = x_i / ((b_{1u} - b_{11})/2) \]
\[ b_{11} \leq x_i \leq b_{1u}. \]  \hspace{1cm} (3.4.1.1)

Draper and Lawrence (1965) found a useful relation between the design moments:

\[ 3c = 5e = 9f \]  \hspace{1cm} (3.4.1.2)

with

\[ c = N^{-1} \sum_{u=1}^{N} x_{1u}^2 \]
\[ e = N^{-1} \sum_{u=1}^{N} x_{1u}^4 \]
\[ f = N^{-1} \sum_{u=1}^{N} x_{1u}^2 x_{3u}^2. \]  \hspace{1cm} (3.4.1.3)

3.4.2 Solution

This is the general form of the solution, allowing fractional factorials. Setting \( 3c = 5e \) and solving for \( a \) gives:
Similarly, setting $5e = 9f$ and solving for $d$ gives:

$$d = a\left(2^k p + 1/5\right)^{1/4} \quad (3.4.2.2)$$

Relation 3.4.1.2 cannot be used to solve for $N$ as the term $N^{-1}$ appears in each part of the relation. Since $\mu(0, \ldots, 0, 2, 2, 0, \ldots 0) = 1/9$ for the uniform weight function, equating it to the design moment $M_{k-p/4}$ gives:

$$N^{-1} \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = 2^{k-p/4} \cdot \frac{a^4}{N} = \frac{1}{9}$$

$$N = 9(2^k a^4). \quad (3.4.2.3)$$
CHAPTER 4

DEVELOPMENT OF THE METHOD

The solution method proposed in this thesis represents, it is believed, a refining of and an enhancement to Taguchi's methodology. Four concerns with the standard Taguchi methodology are directly addressed. These are: Taguchi's use of $3^k$ factorial designs that are often inadequate in the presence of interactions, limited flexibility of the signal-to-noise ratio, biased estimation of the performance coefficients, and freedom to choose parameter combinations not explicitly included in the experiment.

The format for this chapter is as follows: section 4.1 presents the steps comprising the solution method. The ensuing sections investigate each step in the method, justifying the approach taken and developing any further work as required.

4.1 The Solution Method

Figure 4.1 is a flowchart of the solution method. It is assumed the parameter ranges have been specified. The iterative approach to the problem is quite similar to the approach used when applying response surface methodology. The multiple criterion optimization method is not
Figure 4.1 The Solution Method (1 of 2)
Calculate $\mu_{ht}, \sigma_{ht}^{2}, M_{L}(X), H_{L}(X)$. (Section 4.5)

Solve PI. (Chapter 1)

Set new design center at solution point of PI

Does improving direction exist? (Y)

Follow direction until objectives increase.

Solution Complete.

Figure 4.1 (2 of 2)
addressed in detail. For illustrative purposes, a criteria weighting method will be used in Chapter 4.

It is recommended that this flowchart be kept at hand when reading the remainder of the chapter. This enables one to relate the subject under discussion to the corresponding step in the solution method.

One note on notation before proceeding. As defined in Chapter 1, there are $p=1, \ldots, P$ design parameters and $j=1, \ldots, q^h$ noise parameters for the $h=1, \ldots, N$ points comprising the design matrix. In Chapter 3, '$k'$ was used to denote the factors in an experiment in a general sense. We now return to the $p$ and $j$ notation for the design and noise matrix respectively. In addition, $q^h$ becomes simply $q$ if the generalization is made that at each point in the design matrix, there are an equal number of noise parameters.

4.2 Loss Function and Objective Function

Variation in the output of a process or the performance of a product causes a loss to the consumer. For example, the maintenance program for a machine is based on the assumption that preventative maintenance is required at certain intervals based on a specified reliability. Any variation from the specification causes loss due to unscheduled downtime or unnecessary maintenance. In the foregoing example, the loss is monetary, but it can also mean loss of safety, security, customer satisfaction, or other forms of non-monetary loss.
Define the 'loss function', $L(Y)$, to be the relationship between the deviation of output or performance from specification and the loss incurred. Loss being a nebulous term at best, it is frequently impossible to determine the form of the loss function. One must usually approximate the unknown (and frequently unknowable) loss function. A quadratic approximation is usually adequate within a limited range, and is also computationally convenient. Let $Y$ represent a continuous performance characteristic, with a target value of $\tau$. (Under the multicriterion approach, $Y$ and $\tau$ are replaced by $Y_h$ and $\tau_h$ respectively). A symmetric quadratic loss function (Figure 4.2) can be expressed formally as:

$$L(Y) = K(Y - \tau)^2.$$  \hspace{1cm} (4.2.1)
The value for $K$ can be determined by measuring the loss when $Y$ is at some distance $\delta$ from $\gamma$:

$$L(\delta) = K\delta^2$$

$$K = \frac{L(\delta)}{\delta^2}. \quad (4.2.2)$$

The objective function (the quantity to be optimized), is the expected loss, which is risk or MSE, comprised of variance plus squared bias. Apart from a constant,

$$E[L(Y)] = E[(Y - \gamma)^2] = \text{Var}(Y) + [E(Y - \gamma)]^2. \quad (4.2.3)$$

Recalling from Chapter 1 that $M_t(X)$ and $H_t(X)$ are the predictive equations for mean response and variance of the mean response respectively, the possible forms for the MSE are:

a. Quality characteristics where a specified target value exists ($\gamma_t = \text{target value}$).

$$\text{MSE}_t = H_t(X) + [M_t(X) - \gamma_t]^2. \quad (4.2.4)$$

b. Quality characteristics where the smallest value ($> 0$) is the target value ($\gamma_t = 0^+$) and $Y$ is positively distributed

$$\text{MSE}_t = H_t(X) + [M_t(X)]^2. \quad (4.2.5)$$

c. Quality characteristics where the largest value is the target value ($\gamma_t = _\infty$). Kackar (1985, p. 183) suggests using $Y^{-1}$ as the performance statistic with $\gamma_t = 0$:

$$\text{MSE}_t = H_t(X) + M_t(X)^{-2}. \quad (4.2.6)$$
Note the multiplier $K$ has been dropped in all cases. Being a constant, it can be placed outside the expected value operator, and in this analysis is replaced by criterion weights in a multicriterion environment. The composite objective function for P1 (Chapter 1) is:

$$\text{Minimize } E(L(Y)) = \sum_{t=1}^{T} \Delta_t \text{MSE}_t(x)$$

where $\Delta_t = \text{multicriterion weights}$. Note the analyst may choose to model $\text{MSE}_t$ directly instead of using the additive model forms 4.2.4 through 4.2.6.

### 4.3 Design Matrix

The design matrix is the set of observations taken from the design space. Each observation contains another designed experiment whose region of interest ($\Omega_h$) is the noise space that applies at that point. The purpose of a design matrix is to produce input to a regression model for predicting quality throughout a portion of the design space ($\Theta$). In this study, a polynomial model is assumed appropriate. The fitted model is then examined to find improving response directions as a part of formulation P1 (Chapter 1). Taguchi recommends the use of orthogonal arrays for the design matrix, with three levels for each factor to determine if non-linearities are present. As detailed in Chapter 2, there is concern whether these designs are effective considering the data analysis that Taguchi carries out. Furthermore, there are better designs to use for response surface estimation (Lucas, 1976).
The central composite design (CCD) is the most frequently used design for fitting a second order model. The CCD is composed of a factorial design (full or partial), a center point (usually repeated), and a complete set of $2k$ axial points. This study recommends the use of CCDs for the design array.

One advantage of CCDs is that they can be made sequential. That is, a full or partial factorial can be run, and if it is found to be inadequate, the axial points added to allow estimation of the parameters for a second order model. Another advantage is the that number of center points ($n_Q$) determines certain properties of the CCD. For example, $n_Q$ can be used to determine whether the CCD is orthogonal or has uniform precision.

4.3.1 Minimum Bias Designs for the Design Matrix

As the design matrix will be used to determine where the observations are taken to estimate $M_\xi(X)$ and $H_\xi(X)$, it is prudent to use a minimum bias design, as this minimizes the effect of bias between the predictive equations and the true functional relationships. The choice of a weight function is also straightforward. A uniform weight function should be used as this does not prejudice any one product design against another.

Since condition 3.1.2.1 must be satisfied, a region of interest must be defined explicitly. The experimenter shall then use his judgement to ensure that the region of interest is neither too small,
thereby requiring extra iterations of the solution process to be made, nor too large, in which case third order or higher effects may be present to a large degree.

Given these conditions, the experimenter should use a minimum bias design with a cuboidal region of interest and a uniform weight function. Such designs can be constructed using the methods of section 3.4.

Since there is a large amount of experimentation required for each point in the design matrix, \( n_0 \) (the number of repetitions at the center point) may be smaller than its' optimum value due to constraints on experimental resources. Finding values for \( X_1 \) and \( X_2 \) in 2.2.2.14 such that it is satisfied with \( n_0 \) determined by outside factors is extremely difficult. As an alternative, the design produced by applying the methods in section 3.4 can be used with \( n_0 \) set to the largest possible value.

4.3.2 Other Designs for the Design Matrix

There is nothing stopping the experimenter from using a CCD as he would normally, that is, disregarding any minimum bias considerations. This is counter productive, since the advantage of having minimum bias over the space of integration of equations 1.2.1.1 and 1.2.1.2 is better accuracy in the results. However, should he decide to disregard minimum bias considerations, the following principles should be followed.
<table>
<thead>
<tr>
<th>Factors $(k)$</th>
<th>Fraction $2^{-p}$</th>
<th>$n_0$</th>
<th>$\kappa$</th>
</tr>
</thead>
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</tr>
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<td>10</td>
<td>2.000</td>
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<tr>
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<td>2.828</td>
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</tr>
<tr>
<td>8</td>
<td>1/2</td>
<td>33</td>
<td>3.364</td>
</tr>
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</table>

Table 4.1 Rotatable, Uniform Precision Central Composite Designs

Rotatability is important because the location of optimal parameters is unknown when selecting a design matrix. Thus, rotatability provides a low risk approach to design. In a CCD, the distance of the axial points $(\kappa)$ determines rotatability. When $\kappa = (F)^{.25}$ where $F$ is the number of points in the factorial portion of the design, then the overall design is rotatable.

Number of Center Points. If at all possible, $n_0$ should be chosen for uniform precision as this is congruent with the objective of response surface estimation. As was stated previously, $n_0$ may be different from its' optimum value, due to shortages in experimental resources. Lucas (1976) found that this affects the G-efficiency of the design. That is, designs using $n_0$ less than the optimal value had variances of the predicted responses larger than the G-optimal case. As before, $n_0$ should be as large as possible.
In summary, one should use a rotatable CCD with \( n_0 \) chosen for uniform precision if the region of interest is unbounded. Table 4.1 summarizes some designs meeting these criteria. If constraints on experimental resources limit \( n_0 \) to a smaller figure than optimum for uniform precision, it should be as large as possible.

4.3.3 Reducing the Total Number of Observations

One method of reducing the total number of observations and consequently allow for a larger \( n_0 \) in the design matrix is to use fractional factorials in the noise matrix for each replicate at the design center. The design generators need not be the same, nor the fractions \( p \) be the same.

4.4 Noise Matrix

The noise matrix represents a complete planned experiment for each point in the design matrix, where the region of interest is the noise space. By fitting a second order response surface to the results obtained from the experiment in the noise matrix, and the use of equations 1.2.1.1 and 1.2.1.2, one obtains an estimate of the mean performance and variance of the mean performance. The estimates are better than those obtained under normal experimentation (where the laboratory conditions are identical, producing very conservative estimates), as they are calculated from the performance under conditions likely to be encountered in practice. These estimates can
then be combined into a signal to noise ratio, other performance matrix, or maintained as multiple objectives.

**Why Use Minimum Bias Designs?** Minimum bias designs are highly recommended for the noise matrix, and will be the only type used in this study. Using the same argument as was advanced for the design matrix, it is best to minimize the error due to bias between the true functional relationship and \( \hat{Y}_{ht}(X, W) \), as this is later used to estimate \( \hat{\mu}_{ht} \) and \( \hat{\sigma}_{ht}^2 \).

It is intuitive to use \( g(W \mid X) \) as the weight function as it is our best estimate of the distribution of the noise parameters. In this study, it is assumed that \( g(W \mid X) \) is reasonably approximated in practice by either the multivariate normal or uniform distribution. Hence, the results of Chapter 3 may be used.

### 4.4.1 Normalizing Transformations

The following transformations allow the experimenter to use the designs in Chapter 3, regardless of the components of \( \mu \) and \( \Sigma \). The first transformation centers \( \mu \) at \([0, ..., 0]^T\). Let:

\[
X_1' = x_1 - \mu_1. \tag{4.4.1.1}
\]

The second transformation diagonalizes \( \Sigma \). This is done with principal components (Jackson, 1980). Geometrically, this procedure is nothing more than the rotation of the axis such that the new variables are uncorrelated. The principal components approach relies on the fact
that a non-singular and symmetric matrix, \( \Sigma \), can be reduced to a diagonal matrix \( L \) if a matrix \( U \) exists such that:

\[
U' \Sigma U = L \tag{4.4.1.2}
\]

The diagonal elements of \( L \) \( (l_1) \) are the eigenvalues of \( \Sigma \), while the columns of \( U \) \( (u_1) \) are the called the eigenvectors of \( \Sigma \).

One method of obtaining \( l_1 \) and \( u_1 \), not used often but useful for illustrative purposes, is by obtaining the \( q \) solutions to the equation:

\[
|\Sigma - l_1 I| = 0 \tag{4.4.1.3}
\]

while \( u_1, \ldots, u_q \) are the vector solutions to:

\[
[\Sigma - l_1 I] t_1 = 0
\]

where

\[
u_i = \frac{t_1}{\sqrt{\langle t_1^2 \rangle}} . \tag{4.4.1.4}
\]

For the purposes of this study, a third transformation will be applied to the vector \( u_1 \):

\[
v_1 = u_1 / \sqrt{l_1} \tag{4.4.1.5}
\]

Variables obtained in this way are uncorrelated and have unit variance. This transformation is required to comply with the assumption of section 3.2.3.

Defining \( V = [v_1, \ldots, v_q] \), the transformations of the original \( q \times q \) variables to the principal components with unit variance may be written:

\[
z = V'(X - \mu) . \tag{4.4.1.6}
\]
The designs in Chapter 3 are obtained by assuming the weight function were \( N_q(0, \Sigma) \). The points comprising the experiment can be determined in terms of the original variables by the inverse of equation 4.4.1.6.

Example. Consider the following: an experiment consists of 2 parameters \((q=2)\) with an unbounded region of interest. The weight function is multivariate normal with the following parameters:

\[
\mu = [10.00 \ 10.00]^T
\]
\[
\Sigma = \begin{bmatrix} 0.7986 & 0.6793 \\ 0.6793 & 0.7343 \end{bmatrix}
\]

If the boundaries of the region of interest lie outside of \( \mu \pm 3\sigma \), what is the minimum bias design, in terms of the original variables?

Solving 4.4.1.3 for the eigenvalues yields \( \lambda_1 = 1.4465; \lambda_2 = 0.0864 \) and solving 4.4.1.4 for the eigenvectors produces:

\[
U = \begin{bmatrix} 0.7236 & -0.6902 \\ 0.6902 & 0.7236 \end{bmatrix}
\]

The above figures are from Jackson, 1980. The next step is to calculate \( V \):

\[
V_{11} = 0.7236/\sqrt{1.4465} = 0.6016 \quad V_{21} = 0.6902/\sqrt{1.4465} = 0.5739
\]
\[
V_{12} = -0.6902/\sqrt{0.0864} = 2.348 \quad V_{22} = 0.7236/\sqrt{0.0864} = 2.462
\]

\[
V = \begin{bmatrix} 0.6016 & -2.348 \\ 0.5737 & 2.462 \end{bmatrix}
\]
To determine the co-ordinates (in terms of the original variables) of the points comprising the experimental design, the inverse of 4.4.1.6 is applied:

\[ x = Vz + \mu \] (4.4.1.7)

This transformation is applied to the points in the central composite design from Table 3.1, and the resulting points are the settings of the original variables at which to take observations. Table 4.2 tabulates these values for the example problem. Note that the minimum bias criterion now relates to the z orientation. Hence, it is recommended that this parameterization be used throughout the remainder of the procedure.
4.4.2 Design Choices

Minimum bias designs for many situations that are likely to be encountered in practice were developed in Chapter 3. The application of these design to the noise matrix will now be discussed.

Region of Interest. The region of interest is determined by the environment of the experiment. There are times when bounds on the possible values for the noise parameters exist, thus dictating a cuboidal region of interest. In other cases, certain parameters are unbounded. Practically, variables with bounds outside of $\mu \pm 3\sigma$ can be treated as unbounded as the resulting designs will be very close by either method. If there is a mix of bounded and unbounded parameters, the unbounded parameters can be treated as bounded outside of $\mu \pm 3\sigma$ with little effect on the minimum bias solution.

Weight Function. The multivariate normal weight function was introduced earlier, and it was stated that the parameters were determined by the occurrence of the noise variables in nature. Put another way, the experimenter is interested in a weight function that directly reflects the distribution of the environment which the product or process will encounter in use, and his interest in points (weights) is proportional to the probability that a set of conditions will occur in nature.

On the other hand, the uniform weight function is used when the experimenter has little information about the parameters of the usage environment, or when it is more appropriate than the multivariate normal.
4.4.3 Fitting Response Surface Models

For efficiency, the polynomial model for performance $\hat{Y}_{ht}(X, W)$ should be estimated using a multiple linear regression routine. The SAS (Statistical Analysis System) software package was used to estimate the response surface models for the example in Chapter 5.

4.5 Integration of Equations

1.2.1.1 and 1.2.1.2

The goal is to estimate the mean performance and variance of the mean performance for all points in the design matrix, contingent on the observed performance in the noise matrices. This is done by integrating the response surface models over their respective regions of interest.

The general form of equation 1.2.1.1 when a polynomial of degree 2 is fit and $g(W \mid X)$ is multivariate normal is:

$$\hat{\mu}_{ht} = \frac{1}{C} \int \left\{ \sum_{i=1}^{q_h} \sum_{j=1}^{q_h} \rho_{ij} x_{i} x_{j} + \sum_{i=1}^{q_h} \rho_{i} x_{i} \right\} \exp\left(-0.5 x^T \Sigma^{-1} x\right) dx_{1} \cdots dx_{q_h}$$

(4.5.1)

$$\hat{\sigma}_{ht}^2 = \frac{1}{C} \int \left\{ \left( \sum_{i=1}^{q_h} \sum_{j=1}^{q_h} \rho_{ij} x_{i} x_{j} + \sum_{i=1}^{q_h} \rho_{i} x_{i} - \hat{\mu}_{ht} \right)^2 \times \exp\left(-0.5 x^T \Sigma^{-1} x\right) dx_{1} \cdots dx_{q_h} \right\}$$

(4.5.2)

where $C$ is as defined in equation 3.2.2.2 or equation 3.3.1.4 for the truncated case.
These integrations are performed over the transformed space, as the procedure is simplified immensely when $\Sigma$ is diagonal. After squaring, the general form of the bracketed term is (without constants):

$$
\alpha_1 x_1 \alpha_2 x_2 \alpha_3 x_3 \alpha_4 x_4
$$

(4.5.3)

where $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq 4$; $\alpha_i (i = 1, \ldots, 4) \in \{0, 1, 2, 3, 4\}$.

If any of $i$, $j$, $k$ or $l$ are odd, and $a_{1u} = -a_{1u}$, then the contribution of that term to the integral is zero by Property 1 (Chapter 3). The remaining terms are either of the form of equation 4.5.2 with the exception of the constant ($C$ or $C'$), in which case the integral is easily calculated from equation 3.2.2.3, or from equation 3.3.1.3 if the distribution is truncated to a cube.

**Limits of Integration.** The limits of integration are the boundaries of the region of interest. If $\Sigma$ has any off-diagonal non-zero elements (i.e. there is at least one instance of covariation), then a cuboidal region of interest in the original variables will not transform to a cuboidal region with all edges parallel to an axis in the transformed space, and the integration becomes somewhat more difficult. This can be seen in the example to section 4.4.1.

**4.6 Unbiasedness of Estimator - Equation 1.2.1.1**

This purpose of this section is to prove the unbiasedness of the estimator given in equation 1.2.1.1.

**Fact:** $Y(X, W) = f(X, W) + \xi$ is a random variable.
$$E(\hat{\mu}) = E\left[ \int Y(X,W)q(W|X)dW \right]$$
$$\quad W \in \Omega$$
$$= \int \int Y(X,W)q(W|X)dWf(Y)dy$$  \(\text{(where } f(Y) = \text{p.d.f of } Y \text{ over } \Theta)\)

Since \(X\) and \(W\) are unrelated in their limits,

$$= \int E[\hat{Y}(X,W)]g(W|X)dW$$
$$\quad W \in \Omega$$

Assume \(\text{Bias}^2\) is 0. Then \(E[\hat{Y}(X, W)] = \mu_y = f(X, W) \text{ over } \Omega\). So,

$$= \int f(X, W)q(W|X)dW = \mu_x.$$  \(\blacksquare\)
CHAPTER 5

AN APPLIED EXAMPLE

An example will be used to illustrate the effectiveness of the method in practice. This example was motivated by the author's personal experiences in the software support section for a military computer installation.

5.1 Background

In this example, the 'product' is a computer operating system. The objective of the study is to determine the settings of certain operating system parameters so as to maximize a multicriterion objective consisting of response time to keyboard inputs and overall throughput.

5.1.1 The DIAC/CP-140 System

The CP-140 Aircraft is a long-range anti-submarine warfare (ASW) aircraft, built on a P-3 Orion airframe but with electronic systems built to suit Canadian requirements. On board tactical systems are driven by a central computer. Information gathered during patrol on significant events such as sonobuoy drops, fixes, and contacts via
radar, electronic support measures (ESM), magnetic anomaly detector (MAD), weapons deployments, and other sensors are recorded on a cassette. Information such as time, position, aircraft speed and heading are also recorded with each event. When the aircraft returns to base, this information can be played back as an aid to mission analysis and debriefing.

The tape cassette is also used to load the aircraft computer at mission start with information that is required during flight: weapons loads and sonobuoy loads, ESM signature libraries, weather, known activity in the patrol area, and mission briefings. The Data Interpretation and Analysis Centre (DIAC) is the ground based installation that prepares cassettes for the mission, and loads and replays the recorded mission after return. The DIAC performs many other functions. Chiefly, it is a briefing tool for aircrews, and, in that regard, it maintains data bases consisting of disposition of hostile and friendly forces (maritime and air).

It also provides computational aids to mission planning such as simulated sonobuoy patterns, geographic aids and the like. The DIAC is also the local terminal for the NATO message system relating to ASW and provides message editing, sending, retrieval, storage and archival.

The DIAC computer is a heavily used machine. Through the evolutionary process that all computer installations undergo, the computer is tasked to the very limit of its' capabilities. The computer hardware is not equal to the task. Being of mid-70's architecture, it consists of a Sperry-Univac AN/UYK-7 32 bit digital computer with a throughput of approximately .7 Mips and 96K of magnetic core memory.
Replacement of this equipment is about 5-6 years away and purchasing additional equipment (memory and CPUs, as the architecture allows multi-processing) is out of the question due to long lead times (5 years) and high cost compared to replacement.

In addition to the equipment described in the previous paragraph, a DIAC installation includes peripherals such as printers, tape drives, plotters, operator stations (2) and 6 Direct View Consoles (DVCs) which are multipurpose work stations. The DVC is a multi-function device consisting of a keyboard, a panel of switches for activation of software controlled functions, and 2 screens: one for the display of alpha-numeric information and the other for geographic information (missions are replayed on this screen, and appear as if one were watching the scene from above.) See Figure 5.1

When a DVC is in use, the operator typically performs a primary function, such as retrieving information on subsurface contacts. This creates many sub-tasks, such as displaying symbols for the subsurface contacts on the geographic screen, updating the display if another console modifies the data base, and continuous time display.

Some functions can bring the system to its' knees single-handedly due to the large program segments involved, small available memory, and the need for swapping tasks in and out of memory. Even under a typical workload, the systems runs very slowly, resulting in user frustration, wasted personnel time and the curtailment of non-essential but desirable activities.
Legend

A. Geographic display screen
B. Alphanumeric display screen
C. Multi-function switch sets
D. Standard keyboard

Figure 5.1 DVC Layout
5.2 Problem Description

The software maintenance team has known for a long time that certain changes could be made to the basic operating system in order to alleviate the slowdown problem. Since the DIAC computer runs 'captive' software (much like an arcade video game), software maintenance tasks such as editing, compiling and linking must be done off-line. Thus, software maintenance is a more exhaustive process than is available on contemporary computers. New ideas, such as a virtual memory enhancement to the operating system and restructuring of task priorities were promising but human and computer resources intensive.

As an alternative to the development of the new software mentioned previously, this study views the operating system as a process which requires 'tuning' of its' parameters in order to function in an efficient state. Planned experimentation had never been used by the software maintainers. Operating system parameters are, for the most part, using the default settings assigned when the operating system was originally written. In this example, due to the size of the problem, only a portion of the operating system will be modeled.

5.2.1 Objective Functions

The functions of interest are the response time of the system to user inputs and the turn around time for tasks. The response time of the system to user inputs is the elapsed time between when the switch/keyboard input is received and:
a. the time the keyboard interrupt is serviced, for keyboard interrupts; or
b. the time associated button activated software receives control of the central processor for the first time.

The turn around time is indirectly measured as the average number of tasks that were waiting for an event (sub-task completion or I/O processing) to complete before continuing processing themselves (WAITQ length). The turn around time was originally measured as the average elapsed time for a job to be processed from activation to termination, but it was discovered during the data collection step that, when tasks requiring keyboard inputs were simulated at high priorities, they were quickly processed at the expense of all other tasks, and the system became backlogged. Thus, measuring the WAITQ length more accurately recorded what was actually happening.

5.2.2 Design Parameters

The first design parameter was the priority at which tasks requiring keyboard input would be processed. In the SMAC-7 (System Management and Control for the AN/UYK-7 Computer) operating system, priorities run from 0-31 (integer values only). In the simulation, the range was 0-30, as priority 31 (the highest) was used to ensure the Executive (that part of the operating system related to task control and scheduling) was scheduled before anything else. In this context, a task is scheduled when control of the central processing resource is passed to it. Thus, the allowable range for priorities was 0-30.
Although the priorities must be integer valued, the range of priorities is quite wide and the interval can be viewed as continuous.

The second design parameter is the size of the time slice used by the operating system to enforce time sharing. This allowable range is 100-500 milliseconds (msec). All times in the remainder of this chapter are in milliseconds unless otherwise noted.

5.2.3 Noise Parameters

Since a large noise matrix would make for many replications at each point in the design matrix, the noise matrix will be kept small as all experimental results are obtained by simulation. The level of system activity was chosen as the first noise parameter. This was governed by the mean time between switch activations (the arrival of switch activations was an exponential process in the simulation). Each switch activation would activate a task. The allowable range was set as $\tau = 30,000$ msec (low activity) to $\tau = 3,000$ msec (high activity).

The second noise parameter was the type of task that was being processed. When an I/O bound task was being processed (a task requiring much input and output), the time between class IV interrupts is $N(150,50)$. (Class IV interrupts will be discussed in section 5.2.4). When a task is compute bound, that is, most instructions are arithmetic in nature, the time between class IV interrupts is $N(400,50)$. The possible values for $\mu$ is continuous on the interval [150, 400].
5.2.4 Operating System Logic

This section is included to give the reader an insight into the philosophy of SMAC-7 in order to better understand the simulation model to be presented shortly.

SMAC-7 is a multi-processing, multi-tasking, operating system. It is also a time sharing system, but time sharing is attained in a roundabout way. A contemporary time sharing system endeavors to give users with equal priority an equal share of the central processing resource by enforcing time slicing and an egalitarian scheduling algorithm, such as round-robin scheduling.

The DIAC computer, however runs a program that is composed of hundreds of closely related elements, both horizontally and vertically. Elements that require the services of elements not in the linkage chain, or system services, signal their requirements to SMAC-7 with a class IV interrupt. When SMAC-7 regains control of the central processor after a class IV interrupt, it must load and schedule the elements that provide the requested service (called a sub-task), re-activating the parent task only after the sub-task has terminated. If the interrupt was generated to request a system service, SMAC-7 services the request itself. Otherwise, a new task is created to service the task. The point of the matter is the class IV interrupts occur very often in reality, and thus provide a self imposed method of time slicing. This is true because when a task signals a class IV interrupt, it cannot continue further processing and must relinquish the central processor. Note that SMAC-7 will support time slicing at
the request of a task, in order to prevent compute bound tasks from monopolizing the central processor. The requirement to time slice a task is decided when the system environment is created, by cognizant programmers.

5.2.5 The Model

The functioning of the DIAC computer, and the management of system resources provided by SMAC-7 is modeled as per figure 5.2. This figure will be explained from the top down.

Exec periodic tasks simulate the executive overhead required for periodic duties (clock updating, etc.) as well as the work to run the system separate of that required for task control and scheduling. This can represent queue maintenance, I/O, and asynchronous interrupt handling.

Background tasks are applications programs that function on a regular basis and do not require operator input, for example, updating geographic displays, geographic aids, and data base maintenance.

Tasks initiated by switch depression are the heart of the model. The premise is that all functions required by users (people) are initiated by switch depression or a menu selection, which has the same effect as a switch. The parent task may sequentially call many sub-tasks, which in turn can call sub-tasks, all of a finite duration.

Each time a task is scheduled, a certain amount of Exec overhead is incurred. Once the task is executing, it may leave that state in a number of ways:
Figure 5.2 Model of SMAC-7 Used in Simulation (1 of 2)
Figure 5.2 (2 of 2)
a. the task was time sliced to enforce time sharing;
b. the task has completed processing (terminated); or,
c. the task was interrupted, either synchronously (it generated the interrupt itself to request an outside service), or asynchronously, by pre-emption due to a higher priority task requiring the central processor, or an event requiring immediate Executive action.

If a task can execute again immediately after interruption, it is placed in the BLOCKEDQ. If another event must occur before it can be rescheduled, it is placed on the WAITQ, and is unloaded (dequeued) when the sub-task has terminated.

When a task is terminated, if there is a successor task, it is released from the WAITQ and placed on the BLOCKEDQ, as the successor can continue processing. By way of example, if task A needed the service of task B, A is put to sleep (placed on the WAITQ) until B has terminated. From B's point of view, A is the successor (it is also B's parent) since it is to be woken up when B terminates.

5.3 Solution of Example Problem

The calculations for the solution are long and involved; accordingly, only the salient points will be presented.
5.3.1 Design and Noise Matrices

**Design Matrix.** A central composite design with P=2 was used with one center point and a full factorial arrangement. The settings for a and d were chosen for minimum bias with a uniform weight function. Applying equations 3.4.2.1 and 3.4.2.2, $a = 0.7376$ and $d=0.8296$. The design parameters are (1) time slice length in milliseconds, and (2) priority of tasks requiring user input.

**Noise Matrix.** The same design as the design matrix was used with $n_0 = 2.65 \approx 3$. The noise parameters were (1) mean time between switch arrivals (30,000 to 3,000) and (2) time between interrupts (150 to 400 msec). The weight function, $g(W|X)$ is uniform.

5.3.2 Experimental Results and Calculations

The experimental results were obtained by simulation with SIMAN software, a simulation package for microcomputers. Since our interest is in the performance of the system in steady state, for each point in each noise matrix the simulation was run for 150,000 milliseconds, then summary statistics re-initialized, but not the state of the system, and summary statistics collected for the next 150,000 milliseconds. This was done only once for each point. One might well think that one run is not enough, however, it is a very long period of time when one considers the dimension of the units (milliseconds).
The steady state condition was verified by running the simulation under various starting conditions, and examining the performance statistics at regular intervals. When the statistics appeared to produce roughly the same values over successive intervals, the system was deemed to be in steady state. Since this is a subjective decision, the time was increased by a further 25% as a safety factor.

The second order polynomial models for \( Y_{ht}(X, W) \) for each noise matrix were obtained by least squares estimators of the parameters using procedure STEPWISE, a part of the SAS software package. Estimators were retained in the model at a significance level of \( \alpha = 0.15 \).
Table 5.1 contains the estimates of $\mu$ and $\sigma^2$ for all points in the design matrix, obtained by applying equations 1.2.1.1 and 1.2.1.2 to the second order predictive model obtained from a least squares fit of the data for each design matrix point. The method used to integrate 1.2.1.2 is worthy of note. The bracketed expression has a maximum of 6 terms, providing all the $\hat{b}$ are significant. The squared expression could have up to 36 terms, before simplification. Rather than integrating term by term, which is tedious and error prone, the integration was done using MACSYMA, a package available on the University of Arizona's VAX-11 computers. MACSYMA can perform the integration symbolically and produce an exact answer. This proved to be an accurate and quick way of obtaining results.

The next step was to obtain predictive equations, valid over the entire design space, $\Theta$, for mean response ($M_t(X)$) and variance $H_t(X)$. This was again done using least squares estimators for a second order polynomial model. Estimators were retained in the model at a significance level of $\alpha = 0.15$. It was found that the best predictive models had very low efficiencies as measured by their $R^2$ values for both $M_1(X)$ - response time, and $M_2(X)$ - WAITQ size. One possible reason for this outcome is that $\Theta$, the region of interest for the design matrix for this step, was too broad.

The models for variance alone were consequently used for the objectives functions of Pl. The parameters for the models $H_t(X)$ are contained in table 5.2.
Table 5.2 Parameters for $H_t(X)$

<table>
<thead>
<tr>
<th></th>
<th>$b_0$</th>
<th>$b_{11}$</th>
<th>$b_{22}$</th>
<th>$b_{12}$</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Response Time</td>
<td>11229.30</td>
<td>-</td>
<td>-</td>
<td>22465.71</td>
<td>114115.29</td>
<td>11104.94</td>
<td>0.72</td>
</tr>
<tr>
<td>WAITQ Size</td>
<td>1811.87</td>
<td>-</td>
<td>-822.38</td>
<td>374.10</td>
<td>-</td>
<td>-91.95</td>
<td>0.97</td>
</tr>
</tbody>
</table>

Formulation P1 (Chapter 1) then becomes:

Minimize:

\[
\begin{align*}
\text{MSE}_1 & : 22465.71X_1X_2 + 14115.29X_1 + 11104.94X_2 \\
\text{MSE}_2 & : -822.38X_2 + 374.10X_1X_2 - 91.95X_2
\end{align*}
\]

Subject To:

\[
\begin{align*}
100 & \leq X_1 \leq 500 \\
0 & \leq X_2 \leq 30
\end{align*}
\]

The method for optimizing a multicriterion objective is not at issue here. This fast developing field is treated elsewhere in the literature. Osyczka (1984) is an excellent and compact treatment of the subject. A simple approach is to use a weighted objective. For example, if weights of 0.20 and 0.80 are placed on the objectives respectively, then an improving direction is \([0.39, -0.93]'\) and the optimum is on the boundary of the feasible region at $X_1 = \text{time slice} = 100$ msec and $X_2 = \text{priority} = 19$. 
If the models for $M_t(X)$ and $H_t(X)$ do not have a high $R^2$ value, and the experimenter does not completely trust their validity, then the solution to P1 should be used to indicate a direction of descent, which can be followed until the objectives increase. If no improvement is observed, then the solution point should be used as the center of a confirmation experiment, to determine if further descent is possible, or to decide if the current point is indeed the true minimum.

5.3.3 Comment on Experimental Findings

The model used to simulate the SMAC-7 operating system in this chapter is, of necessity, a simplified version of the true system. All of the parameters for the model were chosen from experience, as the author does not have access to the DIAC system at the present time. However, certain results have been borne out by experience.

The above solution shortens the time slice value to 100 msec, the shortest feasible value. In practice, shortening the time slice usually produces better turnaround, as it allows shorter jobs to be processed quicker, and longer ones (which will take a great deal of elapsed time to complete anyway) will take longer to complete, but not proportionally longer.

Section 5.3.2 presents the first iteration of the method. Time limitations precluded further experimentation. Since the optimal point is at a boundary, the next step would be to recenter the region of interest at the optimal point and proceed with another iteration.
The purpose of this chapter is to summarize the major accomplishments of the study, note any areas of further work and record any miscellaneous points that cannot be conveniently noted elsewhere.

6.1 Major Accomplishments

This study has presented a new methodology for design optimization. Borrowing from the Taguchi method, the objective is to minimize MSE, but unlike the Taguchi method, it uses multiple objectives, composed of the MSE for each quality characteristic of interest, rather than collapsing the data into a single summary statistic.

This makes better use of the data than Taguchi's treatment. The fitting of a response surface gives us a better understanding of what is happening to the process, in terms of how the responses of interest are affected by changes in the design parameters. Thus, one can locate an exact optimum rather than just improved points.

The development of minimum bias central composite designs is crucial to the success of the method. Using both an infinite and
cuboidal region of interest, the methods for finding minimum bias designs with a multivariate normal and uniform weight function are developed. This requires solving the Box-Draper conditions for each case.

We have presented a complete experimental design methodology for finding the optimal settings with respect to the desired performance measure. This measure, either as a linear weighting of multivariate criteria (as used in Chapter 5), or as a true multivariate model, can include such factors such as production costs and manufacturability, as well as the quality measure.

6.2 Further Study

This study has assumed the use of central composite designs throughout. The central composite design has many desirable properties which make it useful when a response surface is to be estimated. It does, however, have certain drawbacks which limit its use in certain situations. As noted in section 3.5.2, with a cuboidal region of interest, the minimum bias CCD may produce some points outside of the region of interest. One alternative the experimenter can follow is to use another type of design suitable for estimating a response surface, for example, a Box-Behnken or Uniform Shell design.

Other situations where the experimenter may be reluctant to use a CCD is when it is expensive to run the experiment at 5 levels for each parameter, or the CCD simply requires more observations than can be afforded.
Thus, one area for further work is to continue the work of minimum bias design with designs other than the CCD.

The parameters for minimum bias designs for a cuboidal region of interest with a multivariate normal weight function were obtained by minimizing the squared difference between the right and left sides of the system of equations in 3.5.2.6. A more exact, but demanding method is to solve the necessary and sufficient conditions, originally presented as equation 2.2.3.14:

\[ M_{11}^{-1} M_{12} = \mu_{11}^{-1} \mu_{12} \]  \hspace{1cm} (6.2.1)

6.3 Miscellaneous Points

**Screening Experiments.** One should not neglect the usefulness of screening experiments or confirmation experiments. Since an understanding of the underlying process is not stressed (effect estimation), one must ensure that non-significant factors are not included in the experiment, through the use of screening experiments. This is also useful in reducing the number of observations required.

**Confirmation experiments** should be conducted to ensure that the process response is indeed improved. This should be done by observing the behavior of the system at each point in the noise matrix anew, with the parameter settings at their optimum levels, and comparing to the predicted results.
The probability density function of the truncated normal distribution, truncated at \(-a = (h-\mu)/\sigma\) and \(a = (k-\mu)/\sigma\) respectively (equidistant from \(\mu\)) is:

\[
q(x) = \frac{1}{C_\sigma} \int_{h}^{k} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\} \; ; \; C = (2\pi)^{\frac{1}{4}}[\bar{z}(a) - \bar{z}(-a)]
\]

The moment about the mean is, for \(t = 0, 1, 2, 3, \ldots\)

\[
\mu(2t) = \frac{1}{C_\sigma} \int_{h}^{k} (x-\mu)^{2t} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\} dx
\]

(let \(z = (x-\mu)/\sigma\);
then \(\sigma dz = dx\))

\[
= \frac{2\sigma^{2t}}{C} \int_{0}^{a} z^{2t} e^{-z^2/2} d\sigma
\]

(let \(w = z^2\);
then \(dz = \frac{1}{2\sqrt{w}} d\sigma\))

\[
= \frac{2\sigma^{2t}}{C} \int_{0}^{a^2} \frac{w^{2t}}{2} e^{-w/2} d\sigma
\]
\[
\begin{align*}
&= \frac{s^2t^2}{c} \int_0^w w^{t-\frac{1}{2}} e^{-w/2} dw \\
&= \frac{s^2t}{c} P(t + \frac{1}{2}; a^2) 2^{t+\frac{1}{2}} \quad \text{(Spanier and Oldham, 1987)}
\end{align*}
\]

where

\[
P(v; x) = \int_0^x t^{v-1} e^{-t} dt
\]
APPENDIX B

COMPUTER PROGRAM LISTINGS

These programs are either referenced in the main body of the thesis, or were used to obtain figures contained therein. Each program contains a narrative giving descriptive details.
PROGRAM Min_Squares

P.A. Ritchie, August 1987, for Masters' thesis.

The IMSL function ZXSSQ is used to find a minimum of the sum of squared deviations of M functions in N variables. This program follows the suggested format in the IMSL language manual very closely. It should be consulted for the usage of variables and tables.


IMPLICIT NONE
INTEGER VM/9/, VN/7/, VIXJAC/9/, VNSig/3/, VMaxFn/500/, VIOpt/1/
REAL TParm(4), TX(7)/1.2, 1.3, 1.2, 2.2, 1.6, 2.1, 20.9/, TF(9), TXJAC(9,7), TXJTJ(28), TWork(81), VEPS/0.0/, VDelta/0.0/, VSSQ
EXTERNAL FCN

CALL ZXSSQ(FCN, VM, VN, VNSig, VEPS, VDelta, VMaxFn, VIOpt, 
TParm, TX, VSSQ, TF, TXJAC, VIXJAC, TXJTJ, TWork, 
VINFER, VIER)
WRITE(6,*)
DO 20 AX=1, VN
WRITE(6,30) AX, TX(AX)
FORMAT(' TX(',12,', deviation = ', F9.5)
CONTINUE
WRITE(6,*)
END
Subroutine FCN evaluates the system of equations to be minimized. They are evaluated in the form

\[
\text{TF}(\text{BX}) = F(X_1, \ldots, X_n) - \text{(Right hand side)}
\]

SUBROUTINE FCN(TX_2, VM, VN, TF)

INTEGER VM, VN
REAL TF(9), TX(7), TX_2(7)
REAL A1, A2, A3, B1, B2, B3, N
EQUIVALENCE (A1, TX(1))
EQUIVALENCE (A2, TX(2))
EQUIVALENCE (A3, TX(3))
EQUIVALENCE (B1, TX(4))
EQUIVALENCE (B2, TX(5))
EQUIVALENCE (B3, TX(6))
EQUIVALENCE (N, TX(7))

DO 10 AX = 1, 7
10    TX(AX) = TX_2(AX)

TF(1) = 8*(A1**2) + 2*(B1**2) - 1.0*N
TF(2) = 8*(A2**2) + 2*(B2**2) - 0.9091*N
TF(3) = 8*(A3**2) + 2*(B3**2) - 0.9998*N
TF(4) = 8*(A1**4) + 2*(B1**4) - 3.0*N
TF(5) = 8*(A2**4) + 2*(B2**4) - 1.8007*N
TF(6) = 8*(A3**4) + 2*(B3**4) - 2.653*N
TF(7) = 8*(A1**2)*(A2**2) - 0.9091*N
TF(8) = 8*(A1**2)*(A3**2) - 0.9998*N
TF(9) = 8*(A2**2)*(A3**2) - 0.9089*N

RETURN
END
PROGRAM PPGamma (INPUT, OUTPUT);

{ P.A. Ritchie, August 1987, for Masters' Thesis. }

Evaluate the incomplete gamma function, defined via the integral:

\[ P(v;x) = \int_0^x \frac{t^{v-1}}{v} e^{-t} dt \]

The property \( P(v;x) = x^v \Gamma(v;x) \Gamma(v) \) is used to evaluate \( P(v;x) \). This algorithm is valid for \( x \geq 0 \) and \( v > 0 \). The reference defines \( P(v;x) \) for \( v < 0 \), but this algorithm will not work correctly under these conditions. Accuracy is about 1 part in 10E-08.


VAR

VGX : REAL; { Upper limit of integration. }
VGV : REAL; { Parameter in the gamma function. }

{ **************************** FGamma **************************** }

FUNCTION FGamma(VX : REAL): REAL;

{ Calculate the value for the gamma function, defined by the integral: }

\[ \Gamma(x) = \int_0^\infty \frac{t^{x-1}}{x} e^{-t} dt \]

CONST

Infinity = 1.0E+37;

VAR

VF : REAL; { Gamma(X). }
VG : REAL;
{ VX : REAL; 'x' in the above integral. }
BEGIN
VF := Infinity;
VG := 1.0;

WHILE (VX <> 0.0) AND { If X is a negative integer, Gamma(X) = Inf. } (VX < 3.0) DO
BEGIN
VG := VG*VX;
VX := VX + 1;
END;

IF VX <> 0.0 THEN
BEGIN
VF := 1-2/(7*SQR(VX))*(1-2/(3*SQR(VX)));
VF := VF/(30*SQR(VX));
VF := (1-VF)/(12*VX) + VX*(LN(VX) - 1);
VF := EXP(VF)/VG*SQR((2*PI)/VX);
END;

FGamma := VF;
END;

{ ***************************** FEI Gamma *************************** }
FUNCTION FEIGamma (VV, VX :REAL) : REAL;
{ Evaluate the entire incomplete gamma function, defined by the integral:
1
* 1 / v-1 -xt
q (v;x) = ------- | t e dt
Gamma(x) /
0
}

VAR
AX : INTEGER; { Counter. }
VFn : REAL; { Function value. }
VG : REAL;
VP : REAL;
{ VV : REAL; 'v' in the integral defined above. }
{ VX : REAL; 'x' in the integral defined above. }

BEGIN
VG := 1.0;
VP := 1.0;
$W := W + 1;$

IF $VV \leq 2$ THEN

BEGIN

$VG := VG*VX;$
$VP := VP*VV + VG;$

END;

UNTIL $VV > 2;$

$AX := \text{TRUNC}(5*(3 + \text{ABS}(VX))/2);$
$VFn := 1/(AX + VV - VX);$ 

REPEAT

$AX := AX - 1;$
$VFN := (VFn*VX + 1)/(AX + VV);$  
UNTIL $AX \leq 0;$

$VP := VP + VFn*VG*VX;$
$VG := 1 - (2/(7*\text{SQR}(VV)))*(1 - (2/(3*\text{SQR}(VV))));$
$VG := VG/(30*\text{SQR}(VV));$
$VG := (VG - 1)/(12*VV) - VV*(\text{LN}(VV) - 1);$ 
$\text{FEI Gamma} := VP*\exp(VG - VX)*\sqrt{W/(2*\pi)};$

END;

{ *********************************************** Main Program *********************************************** }

BEGIN
WRITELN;
WRITE('Enter V and X: ');
READLN(VGV, VGX);
WRITELN;

{ EXP and LN are required since Turbo does not have exponentiation. }

WRITELN(\exp(VGV*\text{LN}(VGX))*\text{FEI Gamma}(VGV, VGX)\times\text{FGamma}(VGV):10:7);
END.
BEGIN;
PROJECT,SMAC7 EXECUTIVE,PAUL RITCHIE,8/21/87;
DISCRETE, 600, 7, 5, 0;
TALLIES:1,SW KB RESP. TIME:2,ELAPSED TIME;
RESOURCES:1, EXEC, 1:2, CPU,1:3, CLASSIVI, 1;
PARAMETERS:1, 750:2, 2500:3, 400, 100:
    4, 5300:5, 2500, 800:6, 5, 25:
    7, 250, 50:8,200, 50:9, 3000, 500:
    10, 60, 600;
COUNTERS:1,EXEC PERIODICS:2,BACKGROUND TASKS:3,SWITCH TASKS:
    4,TOTAL INTERRUPTS:5,SPAWNED TASKS:6,KEYBOARD TASKS:
    7,TASK TERMS.;
DSTAT:1,NR(1),EXEC UTIL.:2,NR(2),CPU UTIL.:3,NR(3),INTERRUPT BUSY:
    4,NQ(1),EXEC QUEUE:5,NQ(2),TASK CPU QUEUE:6,NQ(3),CLASS IV QUEUE:
    7,NQ(4),WAIT QUEUE:8,NQ(5),EXEC CPU QUEUE;
RANKINGS:1-2,HVF(3);
SEEDS:1,5902:2,6850:3,9248;
REPLICATE,2, 0, 150000, NO, YES;
END;
BEGIN;

; P.A. RITCHIE, AUGUST 1987, FOR MASTERS' THESIS

; SIMULATE THE BACKGROUND PROCESSING, TASKS INITIATED BY SWITCH
; DEPRESSIONS, AND SPAWNED TASKS IN THE SMAC-7 OPERATING SYSTEM, WITH
; THE OBJECTIVE BEING TO DETERMINE THE COMBINATION OF PARAMETERS THAT
; MAXIMIZES THE RESPONSE TIME TO SWITCH DEPRESSION AND KEYBOARD INPUT,
; AND THE CPU TIME AVAILABLE TO USER TASKS.

; INIT
CREATE:
ASSIGN:X(1)=4;
ASSIGN:X(2)=465;
ASSIGN:X(3)=15:DISPOSE;

EXECPER CREATE:EX(1,1):MARK(1);
COUNT:1,1;
ASSIGN:A(2)=0;
ASSIGN:A(3)=31;
ASSIGN:A(4)=0;
ASSIGN:A(6)=0;
ASSIGN:A(7)=0:NEXT(BLOCKEDQ);

BCKGRND CREATE:EX(2,1):MARK(1);
COUNT:2,1;
ASSIGN:A(2)=RN(3,2);
ASSIGN:A(3)=0;
ASSIGN:A(4)=0;
ASSIGN:A(6)=0;
ASSIGN:A(7)=0;
DELAY:UN(10,3):NEXT(BLOCKEDQ);

SWDEP CREATE:EX(4,1):MARK(1);
COUNT:3,1;
ASSIGN:A(2)=RN(5,2);
ASSIGN:A(3)=X(3);
ASSIGN:A(4)=0;
ASSIGN:A(6)=1;
ASSIGN:A(7)=0;
DELAY:UN(10,3):NEXT(BLOCKEDQ);

BLOCKEDQ QUEUE,1;
SEIZE:EXEC;
QUEUE,5;
SEIZE:CPU;
DELAY:UN(6,3);
RELEASE:CPU;
RELEASE:EXEC;
BRANCH,1:
IF, A(3) .LT. 31, NEXTBR ; EXEC PERIODIC TASKS ARE
DESTROYED HERE SINCE THE
BRANCH IS NOT TAKEN.

NEXTBR

BRANCH,1:
  IF, A(6) .NE. 1, NOSTATS:
  ELSE, STATS;
STATS
  ASSIGN:A(6)=TNOW-A(1);
  TALLY:1,A(6);
  ASSIGN:A(6)=0:NEXT(NOSTATS);

NOSTATS
BRANCH,1:
  IF, A(7) .GT. 0, USEOLDA7:
  ELSE, GETNEW;
GETNEW
  ASSIGN:A(7)=RN(7,2):NEXT(USEOLDA7):TIME - NEXT CL IV INTERRUPT

USEOLDA7
  ASSIGN:A(5)=A(7):
BRANCH,1:
  IF, A(2) .LE. A(5), LASTRUN:
  IF, X(2) .LE. A(5), SLICED:
  ELSE, XAQT;

LASTRUN
  ASSIGN:A(5)=A(2):NEXT(XAQT);

SLICED
  ASSIGN:A(5)=X(2):NEXT(XAQT);

XAQT
  ASSIGN:A(7)=A(7)-A(5);
  ASSIGN:A(2)=A(2)-A(5);
  QUEUE,2;
  SEIZE:CPU;
  DELAY:A(5);
  RELEASE:CPU;
BRANCH,1:
  IF, A(2) .LE. 0, TERM:
  IF, A(5) .LT. X(2), CLIVI:
  ELSE, BLOCKEDQ;

CLIVI
  COUNT:4,1;
BRANCH,1:
  WITH, 0.9, CL1:
  ELSE, BLOCKEDQ;

CL1
  COUNT:5,1;
  QUEUE,3;
  SEIZE:CLASSIVI;
  ASSIGN:X(1)=X(1)+1;
  ASSIGN:A(4)=X(1);  
  RELEASE:CLASSIVI;
BRANCH,2:
  ALWAYS, WAITQ:
  ALWAYS, CL2;

CL2
  ASSIGN:A(2)=RN(8,3);

DELAY FROM TIME SWITCH PRESSED

CLASS IV INTERRUPT WILL OCCUR

DECREMENT NEXT INTERRUPT TIME

DECREMENT REMAINING JOB TIME

QUEUE FOR CENTRAL PROCESSOR

SIMULATE JOB PROCESSING

TIME SLICED, RETURN TO QUEUE

MODEL ASYNCHRONOUS INTERRUPT

DONT LET ANYONE ELSE DIDDLE WITH X(1).

A(4) IS THE SIGNAL TO WAKE UP THE PARENT TASK IN THE WAITQ.

ALLOW NEW INTERRUPT

SPAWNED SUB-PROCESS

CPU TIME FOR SPAWNED TASK
ASSIGN: A(3) = 16;  
BRANCH, 1:  
    WITH, 0.5, MEDPRI:  
    ELSE, HIPRI:  
        ASSIGN: A(3) = A(3) + 8; NEXT(MEDPRI);  
    END;  
HIPRI ASSIGN: A(3) = A(3) + 8; NEXT(MEDPRI);  
MEDPRI ASSIGN: A(1) = TNOW;  
BRANCH, 1:  
    WITH, 0.6, BLOCKEDQ:  
    ELSE, KBINT;  
    COUNT: 6, 1;  
    DELAY: RN(9, 2);  
    ASSIGN: A(2) = 75;  
    ASSIGN: A(6) = 1;  
    ASSIGN: A(1) = TNOW; NEXT(BLOCKEDQ);  
    END;  
KBINT COUNT: 6, 1;  
DELAY: RN(9, 2);  
ASSIGN: A(2) = 75;  
ASSIGN: A(6) = 1;  
ASSIGN: A(1) = TNOW; NEXT(BLOCKEDQ);  
WAITQ QUEUE, 4;  
WAIT: A(4); NEXT(BLOCKEDQ);  
TERM COUNT: 7, 1;  
ASSIGN: A(6) = TNOW - A(1);  
TALLY: 2, A(6);  
SIGNAL: A(4); DISPOSE;  
END;
This appendix tabulates the data from the simulation and the intermediate results not reported in the main body of the thesis. Table C.1 tabulates the results of 150,000 milliseconds of simulation for each point in each noise matrix.

Table C.2 contains the estimators of second order polynomial models fit to the responses observed in the noise matrices. The results are ordered by design matrix point.

Table C.3 contains the estimates $\hat{\mu}$ and $\hat{\sigma}^2$, obtained by applying equations 1.2.1.1 and 1.2.1.2 to the results contained in table C.2 for each point in the design matrix.
### Noise Matrix Points

<table>
<thead>
<tr>
<th>Time Slice</th>
<th>300</th>
<th>153</th>
<th>153</th>
<th>447</th>
<th>447</th>
<th>300</th>
<th>300</th>
<th>134</th>
<th>465</th>
</tr>
</thead>
<tbody>
<tr>
<td>Priority</td>
<td>15</td>
<td>4</td>
<td>26</td>
<td>4</td>
<td>26</td>
<td>28</td>
<td>3</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>165.26**</td>
<td>145.79</td>
<td>325.42</td>
<td>142.60</td>
<td>135.38</td>
<td>228.40</td>
<td>152.95</td>
<td>149.83</td>
<td>193.52</td>
</tr>
<tr>
<td>(0, 0)</td>
<td>220.41</td>
<td>221.37</td>
<td>117.51</td>
<td>107.46</td>
<td>159.87</td>
<td>91.68</td>
<td>195.58</td>
<td>115.18</td>
<td>148.82</td>
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<tr>
<td></td>
<td>42.00</td>
<td>51.24</td>
<td>36.45</td>
<td>38.63</td>
<td>26.72</td>
<td>61.99</td>
<td>48.47</td>
<td>38.72</td>
<td>40.33</td>
</tr>
<tr>
<td>Design</td>
<td>(0, 0)</td>
<td>130.87</td>
<td>78.75</td>
<td>132.17</td>
<td>131.74</td>
<td>98.09</td>
<td>152.87</td>
<td>108.61</td>
<td>129.10</td>
</tr>
<tr>
<td></td>
<td>56.08</td>
<td>66.76</td>
<td>52.51</td>
<td>54.73</td>
<td>44.32</td>
<td>81.79</td>
<td>59.29</td>
<td>52.81</td>
<td>56.20</td>
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<tr>
<td>Matrix</td>
<td>(-a,-a)</td>
<td>78.36</td>
<td>79.32</td>
<td>53.94</td>
<td>103.85</td>
<td>72.49</td>
<td>95.82</td>
<td>115.97</td>
<td>81.02</td>
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<tr>
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<td>77.00</td>
<td>61.65</td>
<td>78.36</td>
<td>66.45</td>
<td>64.00</td>
<td>77.32</td>
<td>81.63</td>
<td>97.38</td>
<td>113.35</td>
</tr>
<tr>
<td>Points</td>
<td>(-a, a)</td>
<td>109.26</td>
<td>55.97</td>
<td>141.40</td>
<td>171.02</td>
<td>97.32</td>
<td>159.46</td>
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<td>1.73</td>
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<td>0.42</td>
<td>0.77</td>
<td>0.39</td>
<td>1.59</td>
<td>1.48</td>
<td>1.68</td>
<td>1.23</td>
</tr>
<tr>
<td>(a,-a)</td>
<td>193.49</td>
<td>102.88</td>
<td>110.82</td>
<td>181.61</td>
<td>133.33</td>
<td>143.66</td>
<td>172.26</td>
<td>145.70</td>
<td>151.51</td>
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<tr>
<td></td>
<td>155.12</td>
<td>109.40</td>
<td>89.38</td>
<td>104.83</td>
<td>124.63</td>
<td>115.15</td>
<td>117.42</td>
<td>131.97</td>
<td>106.69</td>
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<tr>
<td>(a, a)</td>
<td>419.62</td>
<td>223.50</td>
<td>681.04</td>
<td>400.52</td>
<td>971.53</td>
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<td>234.42</td>
<td>258.24</td>
<td>468.77</td>
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<tr>
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<td>2.88</td>
<td>2.51</td>
<td>2.62</td>
<td>1.69</td>
<td>3.70</td>
<td>2.33</td>
<td>3.34</td>
<td>4.24</td>
<td>3.53</td>
</tr>
</tbody>
</table>

* Average switch response time. ** Average WAITQ size.
### Noise Matrix Points

<table>
<thead>
<tr>
<th>Time Slice</th>
<th>300</th>
<th>153</th>
<th>153</th>
<th>447</th>
<th>447</th>
<th>300</th>
<th>300</th>
<th>134</th>
<th>465</th>
</tr>
</thead>
<tbody>
<tr>
<td>Priority</td>
<td>15</td>
<td>4</td>
<td>26</td>
<td>4</td>
<td>26</td>
<td>28</td>
<td>3</td>
<td>15</td>
<td>15</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Design Points</th>
<th>(0, d)</th>
<th>287.27*</th>
<th>139.15</th>
<th>248.49</th>
<th>233.75</th>
<th>218.33</th>
<th>248.44</th>
<th>217.69</th>
<th>148.16</th>
<th>198.87</th>
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<tr>
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<td>1.11**</td>
<td>1.94</td>
<td>2.10</td>
<td>2.01</td>
<td>0.89</td>
<td>0.90</td>
<td>1.22</td>
<td>0.72</td>
<td>1.80</td>
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<td>(0, -d)</td>
<td>108.55</td>
<td>110.04</td>
<td>117.78</td>
<td>150.14</td>
<td>83.26</td>
<td>97.71</td>
<td>98.85</td>
<td>106.49</td>
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<tr>
<td>Matrix Points</td>
<td>99.95</td>
<td>131.60</td>
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<td>93.41</td>
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<tr>
<td></td>
<td>129.03</td>
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<td>92.63</td>
<td>125.77</td>
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<td>102.20</td>
<td>133.25</td>
<td>30.59</td>
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<tr>
<td></td>
<td>29.36</td>
<td>14.71</td>
<td>20.28</td>
<td>20.05</td>
<td>20.53</td>
<td>24.12</td>
<td>31.16</td>
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<td>13.48</td>
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<tr>
<td>(d, 0)</td>
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<td>53.09</td>
<td>46.79</td>
<td>62.62</td>
<td>52.59</td>
<td>30.89</td>
<td>44.62</td>
<td>48.79</td>
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</tbody>
</table>

* Average switch response time. ** Average WAITQ size.

Table C.1 (2 of 2)
Design Matrix Points

(0, 0) (-a, -a) (-a, a) (a, -a) (a, a) (0, d) (0, -d) (-d, 0) (d, 0)

<table>
<thead>
<tr>
<th></th>
<th>SW</th>
<th>NS</th>
<th>NS</th>
<th>NS</th>
<th>84.56</th>
<th>951.78</th>
<th>NS</th>
<th>NS</th>
<th>NS</th>
<th>100.30</th>
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</thead>
<tbody>
<tr>
<td>$P_{11}$</td>
<td>WQ</td>
<td>NS</td>
<td>NS</td>
<td>NS</td>
<td>77.33</td>
<td>18.67</td>
<td>NS</td>
<td>NS</td>
<td>NS</td>
<td>32.22</td>
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<tr>
<td>$P_{22}$</td>
<td>SW</td>
<td>69.73</td>
<td>373.75</td>
<td>NS</td>
<td>NS</td>
<td>51.19</td>
<td>149.26</td>
<td>NS</td>
<td>NS</td>
<td>NS</td>
</tr>
<tr>
<td>$P_{12}$</td>
<td>WQ</td>
<td>-17.21</td>
<td>-26.34</td>
<td>NS</td>
<td>NS</td>
<td>-14.72</td>
<td>NS</td>
<td>NS</td>
<td>NS</td>
<td>NS</td>
</tr>
<tr>
<td>$P_{1}$</td>
<td>SW</td>
<td>14.40</td>
<td>23.10</td>
<td>14.66</td>
<td>12.87</td>
<td>80.73</td>
<td>79.78</td>
<td>96.45</td>
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<tr>
<td>$P_{2}$</td>
<td>WQ</td>
<td>-65.17</td>
<td>-57.84</td>
<td>-59.67</td>
<td>-61.85</td>
<td>-71.69</td>
<td>-70.78</td>
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<td></td>
<td></td>
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<tr>
<td>$P_{0}$</td>
<td>SW</td>
<td>14.72</td>
<td>131.03</td>
<td>141.72</td>
<td>172.88</td>
<td>164.29</td>
<td>134.18</td>
<td>157.48</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R^2$</td>
<td>SW</td>
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<td>0.96</td>
<td>0.96</td>
<td>0.88</td>
<td>0.77</td>
<td>0.54</td>
<td>0.94</td>
<td>0.96</td>
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<tr>
<td></td>
<td>WQ</td>
<td>0.83</td>
<td>0.93</td>
<td>0.96</td>
<td>0.85</td>
<td>0.90</td>
<td>0.97</td>
<td>0.94</td>
<td>0.94</td>
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</tr>
</tbody>
</table>

SW-Switch response time model. WQ-WAITQ size model. NS-Not significant at $\alpha = 0.15$.

Table C.2 Second Order Models for Mean Response
<table>
<thead>
<tr>
<th>Design Matrix Point</th>
<th>Response Time Model $\hat{\mu}$</th>
<th>$\hat{\sigma}^2$</th>
<th>WAITQ Size Model $\hat{\mu}$</th>
<th>$\hat{\sigma}^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>188.29</td>
<td>8068.50</td>
<td>46.58</td>
<td>1789.75</td>
</tr>
<tr>
<td>(-a, -a)</td>
<td>137.04</td>
<td>2519.03</td>
<td>48.73</td>
<td>1558.40</td>
</tr>
<tr>
<td>(-a, a)</td>
<td>272.78</td>
<td>3502.68</td>
<td>40.75</td>
<td>974.16</td>
</tr>
<tr>
<td>(a, -a)</td>
<td>184.99</td>
<td>6921.14</td>
<td>41.77</td>
<td>1242.90</td>
</tr>
<tr>
<td>(a, a)</td>
<td>227.40</td>
<td>56795.00</td>
<td>42.74</td>
<td>1472.78</td>
</tr>
<tr>
<td>(0, d)</td>
<td>172.88</td>
<td>4459.93</td>
<td>48.99</td>
<td>1196.45</td>
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<tr>
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<tr>
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<td>3762.32</td>
<td>48.71</td>
<td>1693.54</td>
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<tr>
<td>(d, 0)</td>
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<td>12913.48</td>
<td>47.87</td>
<td>1768.86</td>
</tr>
</tbody>
</table>

Table C.3 Results from Applying Equations 1.2.1.1 and 1.2.1.2
REFERENCES


