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VECTOR BUNDLES ON AN ELLIPTIC CURVE OVER A DISCRETE VALUATION RING

by
Seog Young Kim

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2001
As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Seoy Young Kim entitled \textit{Vector Bundles on an Elliptic Curve over a Discrete Valuation Ring} and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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**ABSTRACT**

We classify rank 2 vector bundles on a smooth curve $\mathcal{X}$ of genus 1 over a discrete valuation ring $R$. Atiyah\[5\] classified rank 2 vector bundles on elliptic curves over algebraically closed fields. The fact that a genus 1 curve over a discrete valuation ring has a codimension 2 subscheme prevents us from applying Atiyah's work directly. We find that genus 1 curve over an arbitrary field can have three types of rank 2 vector bundles.

We classify rank 2 vector bundles on a curve of genus 1 over a discrete valuation ring using the classification on a curve of genus 1 over a field and quadruples $(\mathcal{L}, \mathcal{M}, \mathcal{Z}, \eta)$ where $\mathcal{L}$ and $\mathcal{M}$ are line bundles on $\mathcal{X}$ and $\mathcal{Z}$ is a local complete intersection subscheme of codimension 2 and $\eta$ is an orbit in $\text{Ext}^1(\mathcal{M} \otimes I_\mathcal{Z}, \mathcal{L})$ under the $R^*$ action.
Chapter 1

INTRODUCTION

Our primary purpose is to classify vector bundles, $E$ over a smooth curve, $\mathcal{X}$, of genus 1, defined over the discrete valuation ring, $R$. Grothendieck[6] showed that any vector bundle on a genus zero curve over a field splits as the direct sum of line bundles. Atiyah[5] classified nontrivial vector bundles over elliptic curves, defined over algebraically closed fields.

Recall that a smooth curve of genus 1, $\mathcal{X}$, over a discrete valuation ring is a flat, projective scheme of dimension 2 whose fibers are curves of genus 1. In particular, it is an arithmetic surface. For the remainder, we assume that fibers are smooth. When we have a subbundle $\mathcal{L}$ of a rank 2 vector bundle $E$ on $\mathcal{X}$, we can make a short exact sequence $0 \rightarrow \mathcal{L} \rightarrow E \rightarrow E/\mathcal{L} \rightarrow 0$. Although $E/\mathcal{L}$ may not be torsion free, after saturation we obtain the exact sequence $0 \rightarrow \mathcal{L}' \rightarrow E \rightarrow \mathcal{Q}' \rightarrow 0$ where $\mathcal{Q}'$ is torsion free. Unfortunately, $\mathcal{Q}'$ may not be a line bundle as there may be a locally complete intersection of codimension 2 subscheme, $\mathcal{Z}$, in $\mathcal{X}$. Such a $\mathcal{Z}$ makes the classification of vector bundles on $\mathcal{X}$ different from the one of vector bundles on a smooth curve defined over a field $k$.

In Chapter 2, we recall some general facts about vector bundles. Although
a smooth curve $X$ of genus 1 defined over a field, $k$, may not have a rational point, it has a rational divisor say of degree $a$. We are able to assume that $a$ is minimal. For example if $k = \overline{k}$, then $a = 1$. We have a line bundle of degree $a$. Using this line bundle we can reduce the classification to the problem of classification of vector bundles of degree between 1 and $2a$. We generalize Atiyah’s work[5] to the classification of vector bundles on a smooth curve of genus 1 over a non-algebraically closed field. Here rank 2 vector bundles on a smooth surface may be constructed as an extension of two line bundles.

In Chapter 3, we consider arithmetic surfaces. These surfaces possess two types of divisors, the so called, horizontal and fiber divisors. The fiber divisors are in the special fiber. Consequently, an arithmetic surface over a discrete valuation ring has one special fiber, $X_1$ which we are assuming to be smooth. As $X_1$ is smooth, it is principal, thus we need only to consider fiber divisors of the form $nX_1$, for some $n \in \mathbb{Z}$. Given a divisor on a smooth curve over a field, we can extend it over an arithmetic surface whose generic fiber is the smooth curve. This allows us to make use of the classification of vector bundles on a smooth curve of genus 1 over a field from Chapter 2.
Accordingly, we assign to each bundle $\mathcal{E}$ a canonical extension

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{E} \rightarrow \mathcal{M} \otimes I_Z \rightarrow 0$$

where $\mathcal{L}$ and $\mathcal{M}$ are line bundles and $Z$ is a locally complete intersection subscheme of codimension 2 and $I_Z$ is the ideal sheaf of $Z$.

We see that a rank 2 vector bundle on an arithmetic surface produces $\mathcal{L}$, $\mathcal{M}$, and $Z$. On the other hand, non-isomorphic bundles may yield the same $\mathcal{L}$, $\mathcal{M}$, and $Z$, as there may be many distinct, non-isomorphic extensions of $\mathcal{M} \otimes I_Z$ by $\mathcal{L}$. So we are led to study the group of such extensions, $\text{Ext}^1_Y(\mathcal{M} \otimes I_Z, \mathcal{L})$.

As we have that $\text{Ext}^1_Y(\mathcal{M} \otimes I_Z, \mathcal{L}) \simeq \text{Ext}^1_Y(I_Z, \mathcal{M}^{-1} \otimes \mathcal{L})$, we associate to a rank 2 vector bundle $\mathcal{E}$ an element $[\mathcal{E}]$ of $\text{Ext}^1_Y(I_Z, \mathcal{F})$, where $\mathcal{F} = \mathcal{M}^{-1} \otimes \mathcal{L}$ and $\mathcal{O}_Z$ is the sheaf of rings on $Z$, and after some calculation we obtain the exact sequence:

$$0 \rightarrow H^1(\mathcal{X}, \mathcal{F}) \rightarrow \text{Ext}^1_Y(I_Z, \mathcal{F}) \rightarrow H^0(Z, \mathcal{O}_Z) \rightarrow 0.$$

Also we need to study $R^*$-action on $\text{Ext}^1_Y(I_Z, \mathcal{F})$. Then every extension belongs to an orbit $\eta$ under $R^*$-action. Given an indecomposable rank 2
vector bundle $E$, we can make a quadruple $(\mathcal{L}, \mathcal{M}, \mathcal{Z}, \eta)$. Here we find the relationship between isomorphism classes of indecomposable rank 2 vector bundles $E$ and isomorphism classes of quadruples $(\mathcal{L}, \mathcal{M}, \mathcal{Z}, \eta)$.

We have our main result:

**THEOREM** Let $R$ be a discrete valuation ring and $\mathcal{X}$ an arithmetic surface over $\text{Spec}(R)$ whose fibers are smooth curves of genus 1.

1. An isomorphism class of indecomposable rank 2 vector bundle $E$ over $\mathcal{X}$ can produce an isomorphism class of $(\mathcal{L}, \mathcal{M}, \mathcal{Z}, \eta)$ where $\mathcal{L}$ and $\mathcal{M}$ are line bundles and $\mathcal{Z}$ is a locally complete intersection on $\mathcal{X}$ of codimension 2 and $\eta$ is an orbit of isomorphism classes in $\text{Ext}^1_{\mathcal{X}}(\mathcal{M} \otimes \mathcal{I}_Z, \mathcal{L})$ which has a unit image in $H^0(Z, \mathcal{O}_Z)$.

2. These data in 1 also determine $E$ uniquely as the middle term of the exact sequence

$$0 \rightarrow \mathcal{L} \rightarrow E \rightarrow \mathcal{M} \otimes \mathcal{I}_Z \rightarrow 0$$

corresponding to an extension in the orbit $\eta$.

In Chapter 4, we discuss the structure of $\text{Ext}^1_{\mathcal{X}}(\mathcal{I}_Z, \mathcal{F})$ in case that $\mathcal{Z}$ is one point. Let $U$ be the subset of elements in $\text{Ext}^1_{\mathcal{X}}(\mathcal{I}_Z, \mathcal{F})$ whose images in
\( H^0(\mathcal{Z}, \mathcal{O}_\mathcal{Z}) \) are units. Then an element in \( U \) has a rank 2 vector bundle as the middle term. We need to find the relationship between \( \text{Ext}^1_{\mathcal{X}}(I_{\mathcal{Z}}, \mathcal{F}) \) and the cohomological terms of an exact sequence. We know that \( \text{Ext}^1_{\mathcal{X}}(I_{\mathcal{Z}}, \mathcal{F}) \) and the cohomological terms of exact sequence are \( R \)-modules. In Lemma 18, we show that, as an \( R \)-module, \( H^1(\mathcal{X}, \mathcal{F}) \simeq R^d \) if \(-d = \text{deg} \mathcal{F} < 0\), or \( R \) if \( \mathcal{F} \) is trivial, or 0 if \( \text{deg} \mathcal{F} = 0 \) and \( \mathcal{F} \) is not trivial, or 0 if \( \text{deg} \mathcal{F} > 0 \).

Assuming that \( \mathcal{Z} \) is a point, we have \( H^0(\mathcal{Z}, \mathcal{O}_\mathcal{Z}) \simeq k \) where \( k \) is a residue field of \( R \). Then we will give more explicit computations for \( U \) which depend on \( \mathcal{F} \).

In this paper, we assume that \( R \) is a discrete valuation ring unless otherwise specified.
Chapter 2
VECTOR BUNDLE

2.1 Vector Bundles over a Scheme

Let $X$ be a scheme. There is a one-to-one correspondence between isomorphism classes of locally free sheaves of rank $n$ on $X$, and isomorphism classes of vector bundles of rank $n$ over $X$. Because of this, we may use the words locally free sheaf and vector bundle interchangeable if no confusion seems likely to result.

2.2 Vector Bundles on a Smooth Curve of Genus 1 defined over a Field

In this section we recall the classification of vector bundles over a smooth curve of genus 1 defined over a field as in Atiyah[5]. For this, Let $C$ be a smooth curve of genus 1 over a field $k$, and $E$ a rank 2 vector bundle over $C$. First we recall the

**Theorem 1** (Serre)[11, II.5.17] Let $X$ be a projective scheme over a noetherian ring $A$. Let $O(1)$ be a very ample invertible sheaf on $X$, and let $\mathcal{F}$ be a coherent sheaf of $O_X$-modules. Then there is an integer $n_0$ such that for all $n \geq n_0$, the sheaf $\mathcal{F}(n)$ can be generated by a finite number of global sections.
Serre’s Theorem asserts that for every vector bundle $E$ and a sufficiently large $n$, depending on $E$, $E(n)$ has a global section and there is an inclusion $\mathcal{O}(-n) \hookrightarrow E$.

The divisor group of a curve $C$, denoted $\text{Div}(C)$ is the free abelian group generated by the points of $C$. Thus a divisor $D \in \text{Div}(C)$ is a formal sum

$$D = \sum_{p \in C} n_p(p)$$

with $n_p \in \mathbb{Z}$, and $n_p = 0$ for all but finitely many $p \in C$. The degree of $D$ is defined by

$$\deg D = \sum_{p \in C} n_p(p) \deg(p).$$

We also say that $D$ is defined over $k$ if $D^\sigma = D$ for all $\sigma \in G_{\overline{k}/k}$.

Now we need a Riemann-Roch Theorem over a field $k$.

**Proposition 2** Let $k$ be a any field. Let $X/k$ be a nonsingular complete curve of genus $g$. Then there exists a canonical divisor class $K$ of $X$ such that for all divisors $D$ of $X$, the $k$-vector space $H^1(D)^\vee$ is isomorphic to the space $H^0(K - D)$. In particular, where $h^i(D) = \dim_k H^i(X, D)$

$$h^1(D) = h^0(K - D) \quad \text{and} \quad h^0(D) = \deg(D) + 1 - g + h^0(K - D).$$
We write $\deg(E) = \deg(\Lambda^2 E)$ for degree of $E$.

For any point $p$ of a nonsingular curve $C/k$, $\mathcal{O}_p$ is a regular local ring and discrete valuation ring since $C$ is nonsingular. Denoting by $k_p$ the residue class field, $\mathcal{O}_p/m_p$ of $p$, for the maximal ideal $m_p$ we define

$$\deg(p) = [k_p : k].$$

If $\overline{k} = k$, then every point $p$ in $C$ has a degree 1.

Suppose that we have a rational divisor $p$ in $C$ with degree $a > 0$ and we assume that $p$ has a minimal degree $a$. Then we have a line bundle $L$ of degree $a$. We know that

$$\deg(E \otimes M) = \deg(\Lambda^2 (E \otimes M)) = \deg(\Lambda^2 E) + 2 \deg(M) \text{ for } M \in \text{Pic}C.$$

If we tensor $E$ with a line bundle $L^n$ for some $n \in \mathbb{Z}$ then $\deg(E \otimes L) = \deg(\Lambda^2 E) + 2n \deg(L)$ may be one of $1, 2, \cdots, 2a$. Thus we may assume $\deg(E) = 1, 2, \cdots, 2a$.

Now we need[4]

**Lemma 3** Let $E$ be a rank 2 vector bundle on a smooth curve $C$ of genus 1 over a field $k$. 
1. There exists an exact sequence

\[ 0 \to L \to E \to M \to 0 \]

with \( L, M \) in Pic \( C \).

2. If \( h^0(E) \geq 1 \), we can take \( L = \mathcal{O}(D) \), with \( D \geq 0 \).

3. If \( h^0(E) \geq 2 \) and \( \text{deg}(E) > 0 \), we may further assume \( D > 0 \).

**Proof.** By Serre’s Theorem, we may replace \( E \) by \( E \otimes N \) for some \( N \in \text{Pic}(C) \), and we can assume that we have \( h^0(E) \geq 1 \). Then \( E \) admits a non-zero section \( s \), i.e., \( \mathcal{O}_C \xrightarrow{s} E \). From which we have that \( E^\vee \xrightarrow{s^\vee} \mathcal{O}_C^\vee = \mathcal{O}_C \) is a non-zero morphism. The image of \( s^\vee \) is an ideal of \( \mathcal{O}_C \), that is a sheaf \( \mathcal{O}_C(-D) \), for some effective divisor \( D \) of \( C \). Thus we have a surjective morphism \( E^\vee \to \mathcal{O}_C(-D) \) and one can show that its kernel is an invertible sheaf. Dualizing we have

\[ 0 \to \mathcal{O}_C(D) \to E \to M \to 0. \]

If \( h^0(E) \geq 2 \), we can choose \( s \) and \( t \) two linearly independent sections. The section \( s \wedge t \) of \( \wedge^2 E \) must vanish at some point \( p \) of \( C \) since \( \text{deg}(E) > 0 \).
This guarantees that there exist \( \mu \) and \( \lambda \) (not both zero) such that \( \mu s(p) + \lambda t(p) = 0 \). Then section \( \mu s + \lambda t \) vanishes at \( p \). Hence \( D > 0 \). ■

**Corollary 4** (Riemann-Roch for rank 2 vector bundle) For a rank 2 vector bundle \( E \), we have

\[
h^0(E) - h^1(E) = \deg(E) + 2 - 2g.
\]

**Proof.** By Lemma 2.1 and Riemann-Roch for invertible sheaf, we have that

\[
\chi(E) = \chi(L) + \chi(M)
\]

\[
= \deg(L) + \deg(M) + 2(1 - g) = \deg(E) + 2 - 2g. \quad \Box
\]

We may assume that \( d = \deg(E) \) is one of \( 1, \ldots, 2a \), so that \( h^0(E) \geq 1 \) by Riemann-Roch. Thus there are invertible sheaves \( L_k \) and \( M_{d-k} \) of degree \( k \geq 0 \) and \( d - k \) respectively such that

\[
0 \to L_k \to E \to M_{d-k} \to 0.
\]

If \( d \geq 2 \), then by Lemma 3.3 we can assume \( k \geq 1 \).

If \( k = 0 \), then \( L_k = \mathcal{O}_C \).

The class of the exact sequence lives in
$$\text{Ext}_C^1(M_{d-k}, L_k) \cong H^1(C, L_k \otimes M_{d-k}^\vee) \cong H^0(C, M_{d-k} \otimes L_k^\vee \otimes K_C)^\vee,$$

and $\deg(M_{d-k} \otimes L_k^\vee \otimes K_C) = d - k - k + 2g - 2 = d - 2k$.

We have

(1) $d - 2k = 0$ or
(2) $d - 2k > 0$ or
(3) $d - 2k < 0$.

In (1), we have a sequence

$$0 \to L_k \to E \to M_k \to 0.$$ 

By tensoring by $L_k^{-1}$, we have

$$0 \to \mathcal{O}_C \to E \otimes L_k^{-1} \to M_k \otimes L_k^{-1} \to 0$$

with $\deg M_k \otimes L_k^{-1} = 0$.

If $h^0(C, M_k \otimes L_k^{-1}) \neq 0$, then $M_k \otimes L_k^{-1} = \mathcal{O}_C$, and we have

$$0 \to \mathcal{O}_C \to E \otimes L_k^{-1} \to \mathcal{O}_C \to 0.$$ 

If $h^0(C, M_k \otimes L_k^{-1}) = 0$, then $h^0(C, M_k \otimes L_k^{-1} \otimes K) = 0$ and $\text{Ext}_C^1(M_k, L_k) = 0$, 

which implies $E = L_k \oplus M_k$.

In (2), if we have $d = 1$ and $k = 0$, then we have an indecomposable sequence

$$0 \to \mathcal{O}_C \to E \to M_1 \to 0.$$

If $d = 2$, then by Lemma 3.3 $k > 0$, which is impossible as we would have that $2 = d > 2k \geq 2$. For $d > 2$, then again Lemma 3.3 $k > 0$ and

$$0 \to L_k \to E \to M_{d-k} \to 0.$$

By tensoring with $L_k^{-1}$, we obtain the exact sequence $0 \to \mathcal{O}_C \to E \otimes L_k^{-1} \to M_{d-k} \otimes L_k^{-1} \to 0$.

In (3), since $d-2k < 0$, $\text{Ext}^1_C(M_{d-k}, L_k) = 0$, and we have a decomposable sequence, i.e., $E \simeq L_k \oplus M_{d-k}$.

Therefore we have

**Theorem 5** Let $E$ be a rank 2 vector bundle of degree $d$, $1 \leq d \leq 2a$, on an elliptic curve $C$ over a field $k$. Then $E$ satisfies

1. $E \simeq L_k \oplus M_{d-k}$, or

2. $E \simeq E' \otimes L_k$ with $0 \to \mathcal{O}_C \to E' \to \mathcal{O}_C \to 0$, or
3. \( E \simeq E' \otimes L_k \) with \( 0 \to \mathcal{O}_C \to E' \to M_{d-2k} \to 0 \) with \( d - 2k > 0 \).

The exact sequence in Theorem 5.3 is in one-to-one correspondence with an element in \( \text{Ext}^1_C(M_{d-2k}, \mathcal{O}_C) \). Since \( \text{Ext}^1_C(M_{d-2k}, \mathcal{O}_C) \simeq H^0(C, M_{d-2k})^* \), isomorphism classes of \( E' \) in such an extension are in one-to-one correspondence with elements in the space \( (H^0(C, M_{d-2k}) - \{0\})/\mathbb{R}^* \). So the set \( D \) of isomorphism classes of \( E' \) in such an extension carries the structure of the set of rational points of a projective space \([11, II.7.7]\) and the dimension of \( D \) is \( d - 2k - 1 \). Therefore the set of isomorphism classes of indecomposable rank 2 vector bundles of degree \( d, 1 \leq d \leq 2a \) is in bijection with the points of a projective bundle over the Picard variety \( \text{Pic}_d(C) \) of line bundles of degree \( d \) with fiber dimension \( d - 1 \).

Notice if \( k = \overline{k} \), then all of our rational points have degree 1 and thus the degree of \( E \) reduces to 0 or 1 and the projective bundles have fiber dimension 0. This reduces our description to Atiyah’s classification.

2.3. Vector Bundle over a Surface

In this section we describe a general method to construct a rank 2 vector bundle over a smooth surface \( X \). Although we can take a direct sum of two line bundles, \( V = L_1 \oplus L_2 \), this is not interesting. Another way is to consider an extension of line bundles which is defined below, in other words rank 2.
vector bundles $V$ such that there is an exact sequence

$$0 \to L_1 \to V \to L_2 \to 0$$

for some line bundles $L_i$.

**Definition 6** Let $\mathcal{F}$ and $\mathcal{F}'$ be coherent sheaves on $X$. An extension of $\mathcal{F}$ by $\mathcal{F}'$ is given by a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{E} \to \mathcal{F} \to 0.$$  

The trivial or split extension is $\mathcal{F} \oplus \mathcal{F}'$. We will denote this extension by $[\mathcal{E}]$.

When $X$ is a smooth curve, then all rank 2 bundles can be obtained in this way and we showed it in case of elliptic curves. Note that every rank 2 vector bundle has a subline bundle by Serre’s Theorem.

We will make use of the local description of a subline bundle. For a smooth surface $X$ let $R$ be the local ring of $X$ at $x$; it is a UFD since $X$ is regular. Let $L \to V$ be a subline bundle. After choosing local trivializations $\varphi$ corresponds to an inclusion $R \to R \oplus R$. Thus $\varphi$ is locally determined by $\varphi(1) = (f, g) \in R \oplus R$. Now either $f$ and $g$ are relatively prime elements of
$R$ or they are not relatively prime. If they are relatively prime consider the map $\psi : R \oplus R \to R$ given by $\psi(a, b) = ag - bf$. Clearly, $\text{Im } \psi$ is the ideal generated by $f$ and $g$. The following is an easy special case of the exactness of the Koszul complex:

**Claim 7** If $f$ and $g$ are relatively prime and $I$ is an ideal generated by $f$ and $g$, the sequence

$$0 \to R \to R \oplus R \to I \to 0$$

is exact.

**Proof.** If $(a, b) \in \text{ker } \psi$, then $ag = bf$. Since $f$ and $g$ are relatively prime, $f|a$ and $g|b$. Thus $a = fh$ and $b = gh'$. On the other hand, $fg h = f gh'$, so that $h = h'$. Then, $(a, b) = \varphi(h) \in \text{Im } \varphi$. \[ \blacksquare \]

However if $f$ and $g$ are not relatively prime, we let $t = \gcd(f, g)$, where $t$ is not a unit in $R$, and write $f = tf'$, $g = tg'$. Then $(R \oplus R)/\text{Im } \varphi$ has torsion, since $(f', g') \notin \text{Im } \varphi$ but $t(f', g') \in \text{Im } \varphi$. Let the map $\varphi' : R \to R \oplus R$ be defined by $\varphi'(1) = (f', g')$ and $\varphi$ is the composition of this map with multiplication by $t$. Alternatively, $\varphi$ extends to a map from $(1/t)R$ to $R \oplus R$. Globally, $t$ defines an effective divisor $D$ on $X$ and we can summarize these calculations in a coordinate free way as follows:
Proposition 8 1. Let $\phi : L \to V$ be a subline bundle. Then there exists a unique effective divisor $D$ on $X$, possibly $0$, such that the map $\phi$ factors through the inclusion $L \to L \otimes \mathcal{O}_X(D)$ and such that $V/(L \otimes \mathcal{O}_X(D))$ is torsion free.

2. In the above situation, if $V/L$ is torsion free, i.e., if $D = 0$, then there exists a local complete intersection codimension two subscheme $Z$ of $X$ and an exact sequence

$$0 \to L \to V \to M \otimes I_Z \to 0.$$ 

This Proposition is still true if $X$ is a smooth projective variety and Lemma 3.1 is a special case of this Proposition. Thus, since every rank 2 vector bundle has a subline bundle, every rank 2 bundle over a smooth curve can be written as an extension of line bundles as before.
Chapter 3

ARITHMETIC SURFACE WITH ELLIPTIC FIBERS

3.1 Definition of an Arithmetic Surface

Let \( A \) be a Dedekind domain with fraction field \( K \), and \( C/K \) a non-singular projective curve of genus \( g \). Let \( \mathcal{X} \) be a scheme, flat and proper over \( \text{Spec}(A) \). If \( \mathcal{X} \) is integral and fiber are curves, then we say that \( \mathcal{X} \) is an arithmetic surface over \( \text{Spec}(A) \). The generic fiber is a non-singular connected projective curve \( C \) over the fraction field \( K \) of \( A \) and the special fibers are unions of curves over the residue field of \( A \). Each fiber is connected.

Thus an arithmetic surface \( \mathcal{X} \) is a one-dimensional family of one-dimensional varieties, so it is a scheme of dimension two. By [1,2] and [8,9], there exists a regular arithmetic surface \( \mathcal{X} \) over \( \text{Spec}(A) \) whose generic fiber is isomorphic to \( C/K \).

Let \( R \) be a discrete valuation ring. We assume in this chapter that \( \mathcal{X} \) is a regular arithmetic surface over \( \text{Spec}(R) \) whose fibers are smooth curves of genus 1. In this case we have two fibers, the generic fiber \( X_0 \) which is a smooth curve of genus 1 over the fraction field of \( R \), and the special fiber \( X_1 \) which is a smooth curve of genus 1 over the residue field of \( R \).
Let $Y = \text{Spec}(R) = \{a, m\}$, where $a$ and $m$ are the generic and closed points of $\text{Spec}(R)$. We have a diagram

$$
\begin{array}{ccl}
X_0 & \xrightarrow{i} & X \\
\downarrow & & \downarrow \pi \\
\text{Spec}(K) & \xrightarrow{j} & \text{Spec}(R) \xleftarrow{i} \text{Spec}(k)
\end{array}
$$

where $K$ is a fraction field of $R$ and $k$ is a residue field. Then $j$ is a closed immersion, so we may identify $X_1$ with the closed subscheme $\pi^{-1}(m)$ in $X$. Moreover $i$ is a bijection of $X_0$ with the subset $\pi^{-1}(a)$ of $X$, and for all $x \in X_0$, $i : \mathcal{O}_{i(x), X} \to \mathcal{O}_{x, X_0}$ is an isomorphism.

We know[11,II.6] that on the scheme $X$ for any Cartier divisor $D$, $\mathcal{L}(D)$ is an invertible sheaf on $X$, and the map $D \to \mathcal{L}(D)$ gives a 1-1 correspondence between Cartier divisors on $X$ and invertible sheaves. As $X$ is regular, so Cartier and Weil divisors need not be distinguished. For $X \to \text{Spec}(R)$, we may classify our divisors into two types.

**Horizontal divisors:** These are finite linear combinations of prime horizontal divisors, where a prime horizontal divisor is merely the Zariski closure in $X$ of a rational point of $X_0$ in some finite extension of $k$.

**Fiber divisors:** These are finite linear combinations of irreducible com-
ponents of the special fibers. As $X_1$ is the only special fiber and it is smooth, then all fiber divisors will be of the form $nX_1$ for some $n \in \mathbb{Z}$. Since $R$ is a discrete valuation ring and the maximal ideal is generated by $p \in R$, $X_1$ is a principal divisor.

Thus divisors on $\mathcal{X}$ are the sum of horizontal divisors and fiber divisors and so we may write them as

$$\mathcal{D} = \sum D_h + \sum D_f,$$

where $D_h$ are prime horizontal divisors and $D_f$ are fiber divisors. Since a prime horizontal divisor comes from the divisor of $X_0$, it can be written $\overline{i_*(D)}$, where $D$ is a divisor on $X_0$. As we have one special fiber and each $D_f$ is a multiple of it, we can rewrite

$$\mathcal{D} = \sum \overline{i_*(D)} + nX_1,$$

where $D$ is a divisor on $X_0$. If two divisor $\mathcal{D}_1$ and $\mathcal{D}_2$ on $\mathcal{X}$ are linearly equivalent on the generic fiber, then the associated line bundles are isomorphic since $X_1$ is a principal divisor and $\mathcal{O}_\mathcal{X}(nX_1) \simeq \mathcal{O}_\mathcal{X}$. Thus for a given line bundle $L$ on $X_0$ there is a line bundle $\mathcal{L}$ on $\mathcal{X}$ unique up to isomorphism.
3.2 Vector Bundles over an Arithmetic Surface

In this section we wish to extend the result of Theorem 5 to a case of an arithmetic surface \( \mathcal{X} \), where we will assign to each rank 2 vector bundle \( \mathcal{E} \) a canonical extension by using the canonical extension on the smooth curve \( X_0 \) of genus 1.

Let \( \mathcal{E} \) be a rank 2 vector bundle on \( \mathcal{X} \) with fiber degree \( d \). Then \( \mathcal{E} \) is decomposable or indecomposable. If \( \mathcal{E} \) is decomposable, then \( \mathcal{E} \) is isomorphic to \( \mathcal{L} \oplus \mathcal{M} \), where \( \mathcal{L} \) and \( \mathcal{M} \) are line bundles on \( \mathcal{X} \), each of which corresponds to a divisor on \( \mathcal{X} \). A fiber divisor is a principal divisor and a horizontal divisor is induced from a divisor on \( X_0 \). Thus two divisors on \( X_0 \) determine all isomorphism classes of decomposable rank 2 vector bundles on \( X_0 \) and all isomorphism classes of decomposable rank 2 vector bundles on \( \mathcal{X} \).

Since \( X_0 \) is a smooth curve of genus 1 over the fraction field \( K \) of \( R \), by Theorem 5, \( \mathcal{E}|_{X_0} = E \) is isomorphic to one of

1. \( L_k \oplus M_{d-k} \), or

2. \( E' \otimes L_k \) with \( 0 \to \mathcal{O}_{X_0} \to E' \to \mathcal{O}_{X_0} \to 0 \), or

3. \( E' \otimes L_k \) with \( 0 \to \mathcal{O}_{X_0} \to E' \to M_{d-2k} \to 0 \) with \( d - 2k > 0 \), \( M_{d-2k} \)
is a line bundle of degree $d - 2k$.

Now suppose that on $X_0$ we have an injective map $i : L \hookrightarrow E$, where $L$ is a subline bundle over $X_0$. Then $E/L = M$ is a line bundle on $X_0$ and we have an exact sequence

$$0 \rightarrow L \rightarrow E \rightarrow M \rightarrow 0.$$

Let $D$ be a divisor on $X_0$ corresponding to $\mathcal{O}_{X_0}(D) = L$. We can extend $D$ over $\mathcal{X}$ to be $\overline{i_*(D)}$. Thus there is a unique line bundle $\mathcal{L} = \mathcal{O}_\mathcal{X}(\overline{i_*(D)})$ over $\mathcal{X}$ up to isomorphism with $\mathcal{L}|_{X_0} = L$. Since $\mathcal{O}_{X_0} \xrightarrow{i} E \otimes L^{-1}$, we have any nonzero section $s$ in $\Gamma(X_0, E \otimes L^{-1})$. We may extend it over $\mathcal{X}$. But it may have a pole over $X_1$ of degree $n$ and we obtain a nonzero section $\overline{s}$ in $\Gamma(\mathcal{X}, E(nX_1) \otimes L^{-1})$. Hence we have an injective map $\overline{i} : \mathcal{L}(-nX_1) \hookrightarrow E$. But as $nX_1$ is a principal divisor we have $\mathcal{L}(-nX_1) \simeq \mathcal{L}$. Whereby we have an injective map $g : \mathcal{L} \hookrightarrow E$ and the exact sequence

$$0 \rightarrow \mathcal{L} \xrightarrow{g} E \rightarrow Q \rightarrow 0$$

where $Q$ is a $\text{Coker}(g)$.

By Proposition 8, there exists a unique effective divisor $D$ on $\mathcal{X}$ such that
the map \( g \) factors through the inclusion \( \mathcal{L} \rightarrow \mathcal{L} \otimes \mathcal{O}_X(\mathcal{D}) \) and such that \( \mathcal{E}/(\mathcal{L} \otimes \mathcal{O}_X(\mathcal{D})) \) is torsion free. From this we obtain the exact sequence

\[
0 \rightarrow \mathcal{L} \otimes \mathcal{O}_X(\mathcal{D}) \rightarrow \mathcal{E} \rightarrow \mathcal{Q}' \rightarrow 0,
\]

where \( \mathcal{Q}' \) is torsion free.

On the generic fiber we have \( 0 \rightarrow \mathcal{L} \rightarrow \mathcal{E} \rightarrow \mathcal{M} \rightarrow 0 \) and \( \mathcal{M} \) is torsion free. It follows that \( \mathcal{D} \) is a fiber divisor \( nX_1 \) for some \( n \in \mathbb{Z} \) and it is a principal divisor. We have \( \mathcal{L} \otimes \mathcal{O}_X(\mathcal{D}) \simeq \mathcal{L} \). Since \( \mathcal{Q}' \) is torsion free, there exists a line bundle \( \mathcal{M} \) and a local complete intersection \( \mathcal{Z} \) such that \( \mathcal{Q}' \simeq \mathcal{M} \otimes I_\mathcal{Z} \).

We have proved

**Lemma 9** Let \( \mathcal{E} \) be a rank 2 vector bundle on \( X \) with \( \mathcal{E}|_{X_0} = \mathcal{E} \). Suppose that we have an injective map \( i : \mathcal{L} \hookrightarrow \mathcal{E} \), where \( \mathcal{L} \) is a line bundle over \( X_0 \). Then there exist unique line bundles up to isomorphism \( \mathcal{L} \rightarrow \mathcal{E} \), \( \mathcal{M} \) and a local complete intersection \( \mathcal{Z} \) on \( X \) such that

\[
0 \rightarrow \mathcal{L} \rightarrow \mathcal{E} \rightarrow \mathcal{E}/\mathcal{L} \rightarrow 0,
\]
where $\mathcal{L}|_{X_0} = L$ and $\mathcal{E}/\mathcal{L}$ is torsion free and $\mathcal{E}/\mathcal{L} \simeq \mathcal{M} \otimes I_Z$.

By Lemma 9, given a line bundle $L$ of $E = \mathcal{E}|_{X_0}$, we can extend it to $\mathcal{L}$ canonically on $X$. Hence we can assign to each $\mathcal{E}$ a canonical extension according to classification of $E$ on $X_0$. We have three different kinds of maximal subline bundles of $E = \mathcal{E}|_{X_0}$ in Theorem 5. Now we check cases, one by one.

3.2.1 Case 1

We assume that $\mathcal{E}|_{X_0} = E \simeq L_k \oplus M_{d-k}$ with $k \geq d - k$. We have an injective map $g : L_k \hookrightarrow E$. By Lemma 9, we have

$$0 \rightarrow \mathcal{L}_k \rightarrow \mathcal{E} \rightarrow \mathcal{M} \otimes I_Z \rightarrow 0$$

with $\mathcal{L}_k|_{X_0} = L_k$.

Clearly $\mathcal{M}|_{X_0} = M_{d-k}$.

We also have an injective map $g' : M_{d-k} \hookrightarrow E$. By Lemma 9, we have

$$0 \rightarrow \mathcal{M}_{d-k} \rightarrow \mathcal{E} \rightarrow \mathcal{L} \otimes I_Z \rightarrow 0$$

with $\mathcal{M}_{d-k}|_{X_0} = M_{d-k}$. Clearly $\mathcal{L}|_{X_0} = L_k$.

If $\mathcal{L} = \mathcal{L}_k$, then with composition we have $\mathcal{L} \otimes I_Z \rightarrow \mathcal{L}_k \rightarrow \mathcal{E} \rightarrow \mathcal{L} \otimes I_Z$. 
which is an identity. Hence $E$ is decomposable, i.e., $E \simeq (L \otimes I_\mathcal{Z'}) \oplus \mathcal{M}_{d-k}$.

Since $E$ is locally free, $Z' = 0$ and $E \simeq \mathcal{L} \oplus \mathcal{M}_{d-k}$.

If $\mathcal{M} = \mathcal{M}_{d-k}$, then with composition we have $\mathcal{M} \otimes I_\mathcal{Z} \to \mathcal{M}_{d-k} \to E \to \mathcal{M} \otimes I_\mathcal{Z}$ which is an identity. Hence $E$ is decomposable, i.e., $E \simeq (\mathcal{M} \otimes I_\mathcal{Z}) \oplus \mathcal{L}_k$. Since $E$ is locally free, $Z' = 0$ and $E \simeq \mathcal{L}_k \oplus \mathcal{M}$.

Clearly $\mathcal{M} \simeq \mathcal{M}_{d-k}$ and $\mathcal{L} \simeq \mathcal{L}_k$.

In summary, when $E|_{X_0} = E \simeq L_k \oplus M_{d-k}$ with $k \geq d-k$, one of following is true

1. $E$ is isomorphic to $L_k \oplus M_{d-k}$

2. $E$ is the middle term in the non trivial exact sequence

   $$0 \to \mathcal{M}_{d-k} \to E \to L_k \otimes I_\mathcal{Z'} \to 0$$

3. $E$ is the middle term in the non trivial exact sequence

   $$0 \to L_k \to E \to \mathcal{M}_{d-k} \otimes I_\mathcal{Z} \to 0.$$
3.2.2 Case 2

We assume that $E = \mathcal{E}|_{X_0}$ such that $E \simeq E' \otimes L_k$ with

$$0 \to \mathcal{O}_{X_0} \to E' \to \mathcal{O}_{X_0} \to 0.$$  

Since $L_k$ is a line bundle on $X_0$, we have a line bundle $\mathcal{L}_k$ with $\mathcal{L}_k|_{X_0} = L_k$.

Let $\mathcal{E}' = \mathcal{E} \otimes L_k^{-1}$. Then $\mathcal{E}'|_{X_0} = \mathcal{E} \otimes L_k^{-1}|_{X_0} = E \otimes L_k^{-1} = E'$. Since $E'$ has degree 0 and $\mathcal{O}_{X_0}$ is a maximal subline bundle of $E'$, by Lemma 9 we have the exact sequence

$$0 \to \mathcal{L} \to \mathcal{E}' \to \mathcal{M} \otimes I_z \to 0$$

with $\mathcal{L}|_{X_0} = \mathcal{O}_{X_0}$. Then $\mathcal{L} = \mathcal{O}_X \otimes \mathcal{O}_X(nX_1) \simeq \mathcal{O}_X$ and $\mathcal{M} \simeq \mathcal{O}_X$.

3.2.3 Case 3

We assume that $E = \mathcal{E}|_{X_0}$ is such that $E \simeq E' \otimes L_k$ with

$$0 \to \mathcal{O}_{X_0} \to E' \to M_{d-2k} \to 0$$

and $d - 2k > 0$.

Since $L_k$ is a line bundle on $X_0$, we have a line bundle $\mathcal{L}_k$ with $\mathcal{L}_k|_{X_0} = L_k$. 
Let $\mathcal{E}' = \mathcal{E} \otimes L_k^{-1}$. Then $\mathcal{E}'|_{X_0} = \mathcal{E} \otimes L_k^{-1}|_{X_0} = E \otimes L_k^{-1} = \mathcal{E}'$. Since $\mathcal{E}'$ has degree $d - k$ and $\mathcal{O}_{X_0}$ is a maximal subline bundle of $\mathcal{E}'$, by Lemma 9 we have

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{E}' \rightarrow \mathcal{M} \otimes I_Z \rightarrow 0$$

with $\mathcal{L}|_{X_0} = \mathcal{O}_{X_0}$.

Clearly $\mathcal{L} = \mathcal{O}_X \otimes \mathcal{O}_X(nX_1) \simeq \mathcal{O}_X$ and $\mathcal{M}|_{X_0} = M_{d-2k}$.

In summary, we have proved

**Theorem 10** Let $\mathcal{X}$ be an arithmetic surface over $\text{Spec}(R)$ whose fibers are smooth curves of genus 1. Let $\mathcal{E}$ be a rank 2 vector bundle on $\mathcal{X}$. Then $\mathcal{E}$ is one of following:

1. $\mathcal{E}|_{X_0} = E \simeq L_k \oplus M_{d-k}$ and $\mathcal{E}$ is one of following

   (a) $\mathcal{E} \simeq \mathcal{L} \oplus \mathcal{M}$.

   (b) $\mathcal{E}$ is in the non-trivial exact sequence, i.e.,

$$0 \rightarrow M_{d-k} \rightarrow \mathcal{E} \rightarrow \mathcal{L}_k \otimes I_Z' \rightarrow 0.$$
(c) $\mathcal{E}$ is in the non-trivial exact sequence, i.e.,

$$0 \rightarrow \mathcal{L}_k \rightarrow \mathcal{E} \rightarrow \mathcal{M}_{d-k} \otimes I_Z \rightarrow 0.$$ 

2. $\mathcal{E}|_{x_0} = E \simeq E' \otimes L_k$, and $0 \rightarrow \mathcal{O}_{x_0} \rightarrow E' \rightarrow \mathcal{O}_{x_0} \rightarrow 0$, and $\mathcal{E}' \simeq \mathcal{E} \otimes \mathcal{L}_k^{-1}$ in the non-trivial exact sequence,

$$0 \rightarrow \mathcal{O}_{x} \rightarrow \mathcal{E}' \rightarrow \mathcal{O}_{x} \otimes I_Z \rightarrow 0.$$ 

3. $\mathcal{E}|_{x_0} = E \simeq E' \otimes L_k$, and $0 \rightarrow \mathcal{O}_{x_0} \rightarrow E' \rightarrow \mathcal{M}_{d-2k} \rightarrow 0$ with $d - 2k > 0$, and $\mathcal{E}' \simeq \mathcal{E} \otimes \mathcal{L}_k^{-1}$ in the non-trivial exact sequence,

$$0 \rightarrow \mathcal{O}_{x} \rightarrow \mathcal{E}' \rightarrow \mathcal{M} \otimes I_Z \rightarrow 0.$$ 

3.3 Existence of Rank 2 Vector Bundles

As stated in Theorem 10, any rank 2 vector bundle $\mathcal{E}$ on $\mathcal{X}$ determines line bundles $\mathcal{L}$ and $\mathcal{M}$ and a local complete intersection $\mathcal{Z}$ uniquely, and $\mathcal{E}$ was described as the middle term in the extension. It is natural to ask if, given line bundles $\mathcal{L}$, $\mathcal{M}$ and a local complete intersection $\mathcal{Z}$ of codimension
2, there is a rank 2 vector bundle $\mathcal{E}$ as the middle term in the extension $[\mathcal{E}]$

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{M} \otimes I_Z \longrightarrow 0.$$ 

We recall [7, V.4.] that there is a bijective correspondence between equivalence classes of extensions and $Ext^1_{\mathcal{X}}(\mathcal{M} \otimes I_Z, \mathcal{L})$ with zero corresponding to the trivial extension. We will use same notation $[\mathcal{E}]$ for the extension and the element in $Ext^1_{\mathcal{X}}(\mathcal{M} \otimes I_Z, \mathcal{L})$ corresponding to the extension. Thus our question is to determine an element $[\mathcal{E}] \in Ext^1_{\mathcal{X}}(\mathcal{M} \otimes I_Z, \mathcal{L})$ which has a rank 2 vector bundle $\mathcal{E}$ as the middle term. Because $Ext^1_{\mathcal{X}}(\mathcal{M} \otimes I_Z, \mathcal{L}) = Ext^1_{\mathcal{X}}(I_Z, \mathcal{M}^{-1} \otimes \mathcal{L})$, we investigate the extension of $I_Z$ by $\mathcal{F} = \mathcal{M}^{-1} \otimes \mathcal{L}$.

For this we need a Lemma from [10].

**Lemma 11** Let $A$ be a noetherian local ring, $I \subset A$ an ideal with a free resolution of length 1

$$0 \rightarrow A^p \rightarrow A^q \rightarrow I \rightarrow 0.$$ 

Let $e \in Ext^1_{A}(I, A)$ be represented by the extension

$$0 \rightarrow A \rightarrow M \rightarrow I \rightarrow 0.$$
Then $M$ is a free $A$-module if and only if $e$ generates the $A$-module $\text{Ext}_A^1(I, A)$.

We need the spectral sequence

$$E_2^{p,q} = H^p(X, \text{Ext}_{O_X}^q(I_Z, \mathcal{F})) \to E^{p+q} = \text{Ext}^p_X(I_Z, \mathcal{F})$$

i.e.,

$$0 \to H^1(X, \text{Hom}_{O_X}(I_Z, \mathcal{F})) \to \text{Ext}^1_X(I_Z, \mathcal{F}) \to H^0(X, \text{Ext}_{O_X}^1(I_Z, \mathcal{F})) \to H^2(X, \text{Hom}(I_Z, \mathcal{F})) \to . \quad (1)$$

$$0 \to H^0(X, \text{Ext}^1_{O_X}(I_Z, \mathcal{F})) \to H^1(X, \text{Ext}_{O_X}^1(I_Z, \mathcal{F})) \to . \quad (2)$$

By Lemma 11 it follows that the extension corresponding to an element $e \in \text{Ext}^1_X(I_Z, \mathcal{F})$ will give a rank 2 vector bundle as the middle term if and only if the image of $e$, through the mapping $\text{Ext}^1_X(I_Z, \mathcal{F}) \to H^0(X, \text{Ext}^1_{O_X}(I_Z, \mathcal{F}))$ generates the sheaf $\text{Ext}^1_{O_X}(I_Z, \mathcal{F})$.

Therefore we need to show that $H^2(X, \text{Hom}(I_Z, \mathcal{F})) = 0$ for the map to be surjective. We further find a unit in $H^0(X, \text{Ext}^1_{O_X}(I_Z, \mathcal{F}))$ to find the element in $\text{Ext}^1_X(I_Z, \mathcal{F})$ whose middle term is the rank 2 vector bundle.
First in order to simplify (2) and (3), we make use of the exact sequence

\[ 0 \to I_Z \to O_X \to O_Z \to 0 \]

which gives us

\[ 0 \to \text{Hom}_{O_X}(O_Z, F) \to \text{Hom}_{O_X}(O_X, F) \to \]

\[ \to \text{Hom}_{O_X}(I_Z, F) \to \text{Ext}^1_{O_X}(O_Z, F) \to . \]

Because \( Z \) is the local complete intersection of codimension 2 [7,V.3]

\[ \text{Ext}^i_{O_X}(O_Z, F) = 0 \text{ for } i = 0, 1. \]

We have \( \text{Hom}(I_Z, F) \simeq \text{Hom}(O_X, F) \simeq F \). We rewrite (2) and (3) to obtain

\[ 0 \to H^1(X, F) \to \text{Ext}^1_X(I_Z, F) \to H^0(X, \text{Ext}^1_{O_X}(I_Z, F)) \to H^2(X, F). \quad (3) \]

Claim 12 \( H^2(X, F) = 0. \)

**Proof.** From Leray spectral sequence, \( H^2(X, F) = H^0(Y, R^2\pi_*(F)) \) since \( Y \) is an affine space. \( H^2(X_0, F|_{X_0}) = H^2(X_1, F|_{X_1}) = 0 \) since \( X_i \) is a curve.

By Grauert’s Theorem[11,III.12.9] \( R^2\pi_*(F) \simeq H^2(X_0, F|_{X_0}). \)

Hence \( H^2(X, F) = 0. \)
Now we are able to calculate $Ext_{O_x}^1(I_Z, \mathcal{F})$.

From $0 \to I_Z \to O_X \to O_Z \to 0$, we have

$\to Ext_{O_x}^1(O_Z, \mathcal{F}) \to Ext_{O_x}^2(O_X, \mathcal{F}) \to Ext_{O_x}^1(I_Z, \mathcal{F}) \to$

$\to Ext_{O_x}^2(O_Z, \mathcal{F}) \to Ext_{O_x}^2(O_X, \mathcal{F}) \to$.

Since $O_X$ is locally free, $Ext_{O_x}^i(O_X, \mathcal{F}) = 0$ for $i = 1, 2$ and we have $Ext_{O_x}^2(I_Z, \mathcal{F}) \simeq Ext_{O_x}^2(O_Z, \mathcal{F})$. Since $Z$ is a local complete intersection of codimension 2, there is an isomorphism [3, I.4.5]

$$Ext_{O_x}^2(O_Z, \mathcal{F}) \simeq Hom_{O_Z}(det(I_Z/I_Z^2), \mathcal{F}|_Z).$$

Since $Z$ is of dimension 0, $det(I_Z/I_Z^2) \simeq \mathcal{F}|_Z$ as an $O_Z$-module. Hence $Ext_{O_x}^2(I_Z, \mathcal{F}) \simeq Ext_{O_x}^2(O_Z, \mathcal{F}) \simeq Hom_{O_Z}(\mathcal{F}|_Z, \mathcal{F}|_Z) \simeq O_Z$.

We have $H^0(X, Ext_{O_x}^2(I_Z, \mathcal{F})) = H^0(X, O_Z) = H^0(Z, O_Z)$ and from (4)

$$0 \to H^1(X, \mathcal{F}) \to Ext_{\mathcal{X}}^1(I_Z, \mathcal{F}) \to H^0(Z, O_Z) \to 0.$$

Since a unit in $H^0(Z, O_Z)$ generates $O_{X,x}$-mod $O_{Z,x} \simeq Ext_{O_x}^1(I_Z, \mathcal{F})_x$ for $x \in \mathcal{X}$, by Lemma 11, any element in $Ext_{\mathcal{X}}^1(I_Z, \mathcal{F})$ with image $1 \in H^0(Z, O_Z)$ will give a rank 2 vector bundle as the middle term in the exact sequence.

Thus we have proved the following result:
Theorem 13 Let $X$ be an arithmetic surface over $\text{Spec}(R)$, $\mathcal{L}$, $\mathcal{M}$ line bundles on $X$, and let $\mathcal{Z}$ be a locally complete intersection of codimension 2 in $X$. Then there exists a rank 2 vector bundle $\mathcal{E}$ on $X$ which fits into an exact sequence

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \otimes I_\mathcal{Z} \to 0.$$ 

3.4 Isomorphism of Extensions

In this section we will talk about isomorphisms of extensions. We showed in Theorems 10 and 13 that a rank 2 vector bundle $\mathcal{E}$ gives line bundles $\mathcal{L}$, $\mathcal{M}$, and a locally complete intersection $\mathcal{Z}$ and an extension. Further two line bundles $\mathcal{L}$, $\mathcal{M}$, and a locally complete intersection $\mathcal{Z}$ gives $\mathcal{E}$ and an extension. However, it is possible that two non-isomorphic rank 2 vector bundles $\mathcal{E}$ and $\mathcal{E}'$ give the same $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{Z}$, or two non-isomorphic extensions of $\mathcal{M} \otimes I_\mathcal{Z}$ by $\mathcal{L}$ give the same rank 2 vector bundle $\mathcal{E}$. Thus we need to find the relationship between the isomorphisms of rank 2 vector bundles and isomorphisms of extensions. An isomorphism between two extensions $[\mathcal{E}]$ and $[\mathcal{E}']$ is an isomorphism of bundles $\alpha : \mathcal{E} \to \mathcal{E}'$ with
automorphisms of $\mathcal{L}$ and $\mathcal{M} \otimes I_Z$ such that the following diagram commutes

$$
\begin{array}{ccc}
0 & \to & \mathcal{L} & \to & \mathcal{E} & \to & \mathcal{M} \otimes I_Z & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{L} & \to & \mathcal{E}' & \to & \mathcal{M} \otimes I_Z & \to & 0.
\end{array}
$$

Since $\mathcal{L}$ and $\mathcal{M} \otimes I_Z$ are torsion free of rank 1, their only automorphism are $R^*$ where $R^*$ is a set of units in $R$. And $R^* \times R^*$ acts on the set of all isomorphism classes of extensions. The diagonal subgroup of $R^* \times R^*$ acts trivially and the equivalence classes of extension form the quotient of $\text{Ext}^1_X(M, L)$ by the action of $R^*$.

In the previous section, we saw that given an indecomposable rank 2 vector bundle $\mathcal{E}$, there exist unique isomorphism classes of line bundles $\mathcal{L}$, $\mathcal{M}$ and a local complete intersection $\mathcal{Z}$ of codimension 2 on $\mathcal{X}$ such that

$$
0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \otimes I_Z \to 0. \quad [\mathcal{E}]
$$

Then $[\mathcal{E}]$ is an element in $\text{Ext}^1_X(M \otimes I_Z, \mathcal{L})$. Now we consider the $R$ action on $\text{Ext}^1_X(M \otimes I_Z, \mathcal{L})$. For $r \in R$ and $[\mathcal{E}]$ in $\text{Ext}^1_X(M \otimes I_Z, \mathcal{L})$ $r[\mathcal{E}]$ is an
element in $\text{Ext}^1(M \otimes I_Z, \mathcal{L})$ satisfying following commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{L} & \rightarrow & \mathcal{E} & \rightarrow^I M \otimes I_Z & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow r & & \\
0 & \rightarrow & \mathcal{L} & \rightarrow & r\mathcal{E} & \rightarrow M \otimes I_Z & \rightarrow & 0
\end{array}
\]

where $r\mathcal{E} = f^{-1}(Im r)$. Since $M \otimes I_Z$ is torsion free of rank 1, $r$ is a just multiplication by $r$. If $r$ is a unit element in $R$ then the map $r$ is an isomorphism and $\mathcal{E}$ and $r\mathcal{E}$ are isomorphic. For a given element $[\mathcal{E}]$ in $\text{Ext}^1(M \otimes I_Z, \mathcal{L})$ there exists a unique orbit $\eta$ containing $[\mathcal{E}]$.

If two rank 2 vector bundles $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic, then $\mathcal{E}|_{X_0}$ is isomorphic to $\mathcal{E}'|_{X_0}$. We can choose a maximal subline bundle $L$ of $\mathcal{E}|_{X_0}$ and $L'$ of $\mathcal{E}'|_{X_0}$, where $L$ is isomorphic to $L'$. It follows that $\mathcal{L}$, induced from $L$, is isomorphic to $\mathcal{L}'$, induced from $L'$. Since $Z$ and $Z'$ are locally complete intersection, they were induced from $\mathcal{L} \rightarrow \mathcal{E}$ and $\mathcal{L}' \rightarrow \mathcal{E}'$ and $Z = Z'$ and $M \simeq M'$ which commute a following diagram, i.e., $[\mathcal{E}] \simeq [\mathcal{E}']$.

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{L} & \rightarrow & \mathcal{E} & \rightarrow M \otimes I_Z & \rightarrow & 0 \\
\downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{L}' & \rightarrow & \mathcal{E}' & \rightarrow M' \otimes I_{Z'} & \rightarrow & 0
\end{array}
\]
We have proved

**Lemma 14** If two rank 2 vector bundles $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic, then we can make two extension $[\mathcal{E}]$ and $[\mathcal{E}']$ where $[\mathcal{E}] \simeq [\mathcal{E}']$.

If two rank 2 vector bundles $\mathcal{E}$ and $\mathcal{E}'$ determines same line bundles $\mathcal{L}$, $\mathcal{M}$ and a local complete intersection $\mathcal{Z}$ and an orbit $\eta$, then we have

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{E} \rightarrow \mathcal{M} \otimes I_\mathcal{Z} \rightarrow 0 \quad \xrightarrow{\eta} \quad a \uparrow$$

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{E}' \rightarrow \mathcal{M} \otimes I_\mathcal{Z} \rightarrow 0.$$

$a$ is just multiplication by scalar $a$. If $a$ is a unit, then $\mathcal{E} \simeq \mathcal{E}'$. If $a$ is not a unit, then $[\mathcal{E}'] = a[\mathcal{E}]$ in $\text{Ext}^1_\mathcal{X}(\mathcal{M} \otimes I_\mathcal{Z}, \mathcal{L})$ does not have a vector bundle as the middle term, which will be proved in Chapter 4.4. But $\mathcal{E}'$ is a rank 2 vector bundle and it is a contradiction.

We have proved

**Lemma 15** If two rank 2 vector bundles $\mathcal{E}$ and $\mathcal{E}'$ determine the same $\mathcal{L}$, $\mathcal{M}$, $\mathcal{Z}$ and $\eta$ as above, then $\mathcal{E} \simeq \mathcal{E}'$. 

3.5 Classification of Rank 2 Vector Bundles over an Arithmetic Surface.

We define an isomorphism of quadruples \((\mathcal{L}, \mathcal{M}, \mathcal{Z}, \eta)\) where \(\mathcal{L}\) and \(\mathcal{M}\) are line bundles and \(\mathcal{Z}\) is a locally complete intersection on \(\mathcal{X}\) and \(\eta\) is an orbit of isomorphism classes in \(\text{Ext}^{1}_{\mathcal{X}}(\mathcal{M} \otimes \mathcal{I}_{\mathcal{Z}}, \mathcal{L})\). \((\mathcal{L}, \mathcal{M}, \mathcal{Z}, \eta)\) is isomorphic to \((\mathcal{L}', \mathcal{M}', \mathcal{Z}', \eta')\) if \(\mathcal{L}\) is isomorphic to \(\mathcal{L}'\) and \(\mathcal{M}\) is isomorphic to \(\mathcal{M}'\) and \(\mathcal{Z} = \mathcal{Z}'\) and \(\eta\) corresponds to \(\eta'\) by \(\text{Ext}^{1}_{\mathcal{X}}(\mathcal{M} \otimes \mathcal{I}_{\mathcal{Z}}, \mathcal{L}) \cong \text{Ext}^{1}_{\mathcal{X}}(\mathcal{M}' \otimes \mathcal{I}_{\mathcal{Z}'}, \mathcal{L}')\).

Given an indecomposable rank 2 vector bundle \(\mathcal{E}\), by Theorem 10, there are line bundles \(\mathcal{L}, \mathcal{M}\) and a locally complete intersection \(\mathcal{Z}\) such that \(0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \otimes \mathcal{I}_{\mathcal{Z}} \to 0\). Under \(R^*\)-action there is an orbit \(\eta\) in \(\text{Ext}^{1}_{\mathcal{X}}(\mathcal{M} \otimes \mathcal{I}_{\mathcal{Z}}, \mathcal{L})\) containing \([\mathcal{E}]\). Hence \(\mathcal{E}\) produces a quadruple \((\mathcal{L}, \mathcal{M}, \mathcal{Z}, \eta)\).

If \(\mathcal{E} \cong \mathcal{E}'\), then by Lemma 14, \(\mathcal{E}\) and \(\mathcal{E}'\) can produce isomorphic quadruples, i.e., \((\mathcal{L}, \mathcal{M}, \mathcal{Z}, \eta) \sim (\mathcal{L}', \mathcal{M}', \mathcal{Z}', \eta')\).

Given two line bundles \(\mathcal{L}, \mathcal{M}\) and a locally complete intersection \(\mathcal{Z}\) of codimension 2, we can consider an orbit in \(\text{Ext}^{1}_{\mathcal{X}}(\mathcal{M} \otimes I_{\mathcal{Z}}, \mathcal{L})\) which has an element of the orbit whose image in \(H^{0}(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})\) has a rank 2 vector bundle \(\mathcal{E}\) as the middle term. Then by Theorem 13 and Lemma 15, there is a unique isomorphism class of rank 2 vector bundle \(\mathcal{E}\).

We thus have our main
Theorem 16  Let $R$ be a discrete valuation ring and $X$ an arithmetic surface over $\text{Spec}(R)$ whose fibers are smooth curves of genus 1.

1. An isomorphism class of indecomposable rank 2 vector bundle $\mathcal{E}$ over $X$ can produce an isomorphism class of $(\mathcal{L}, \mathcal{M}, \mathcal{Z}, \eta)$ where $\mathcal{L}$ and $\mathcal{M}$ are line bundles and $\mathcal{Z}$ is a locally complete intersection on $X$ of codimension 2 and $\eta$ is an orbit of isomorphism classes in $\text{Ext}^1_X(\mathcal{M} \otimes I_Z, \mathcal{L})$ which has a unit image in $H^0(Z, \mathcal{O}_Z)$.

2. These data in 1 also determine $\mathcal{E}$ uniquely as the middle term of the exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{M} \otimes I_Z \longrightarrow 0$$

corresponding to an extension in the orbit $\eta$. 
Chapter 4
EXAMPLES

4.1 Cohomological Computation

We now discuss the structure of $\text{Ext}^1_X(\mathcal{M} \otimes I_Z, \mathcal{L})$ as an $R$-module. We will give an explicit description in the case where $Z$ is a point. We assume that $Z$ is a point throughout this Chapter. Since $\text{Ext}^1_X(\mathcal{M} \otimes I_Z, \mathcal{L}) \simeq \text{Ext}^1_X(I_Z, \mathcal{M}^{-1} \otimes \mathcal{L})$, it is enough to compute $\text{Ext}^1_X(I_Z, \mathcal{F})$ for an arbitrary line bundle $\mathcal{F}$ in $X$. We make use of the short exact sequence from Section 3.3

\[ 0 \rightarrow H^1(X, \mathcal{F}) \xrightarrow{\phi} \text{Ext}^1_X(I_Z, \mathcal{F}) \xrightarrow{\psi} H^0(Z, \mathcal{O}_Z) \rightarrow 0. \]

We know that $H^1(X, \mathcal{F}), H^0(Z, \mathcal{O}_Z)$, and $\text{Ext}^1_X(I_Z, \mathcal{F})$ are finitely generated $R$-modules, and so we can compute the canonical generators and orbits of $\text{Ext}^1_X(I_Z, \mathcal{F})$ under $R^*$-action, where $R^*$ is a subset of units in $R$.

**Definition 17** We call an element $[\mathcal{E}]$ in $\text{Ext}^1_X(I_Z, \mathcal{F})$ a unit element if $[\mathcal{E}]$ has a rank 2 vector bundle as the middle term in the extension. Let $U$ be the subset of unit elements in $\text{Ext}^1_X(I_Z, \mathcal{F})$.

By Lemma 11, a unit element in $\text{Ext}^1_X(I_Z, \mathcal{F})$ is an element whose image in $H^0(Z, \mathcal{O}_Z)$ is a unit. Now $Z$ is a locally complete intersection of
codimension 2 in $\mathcal{X}$ and it is a point on the special fiber. So we can have

$$H^0(\mathcal{Z}, \mathcal{O}_\mathcal{Z}) \simeq k \text{ as } R \text{- module},$$

where $k$ is a residue field of $R$ and $p$ is a generator of the maximal ideal $m$ of $R$.

Since $k$ is a field, every element in $\text{Ext}^1(I_\mathcal{Z}, \mathcal{F})$ which is not in $\ker \psi$ is a unit element by Lemma 11. And the $\ker \psi$ is the image of $\phi$. Thus we need to investigate $H^1(\mathcal{X}, \mathcal{F})$ and the map $\phi$.

Now we have the following result about $H^1(\mathcal{X}, \mathcal{F})$.

**Lemma 18** Let $\mathcal{F}$ be a line bundle on $\mathcal{X}$ with degree $-d$ on generic fiber.

Then as $R$-module

$$H^1(\mathcal{X}, \mathcal{F}) \simeq \begin{cases} 
R^d & \text{if } d > 0 \\
R & \text{if } \mathcal{F} \text{ is trivial} \\
0 & \text{if } d = 0 \text{ and } \mathcal{F} \text{ is not trivial} \\
0 & \text{if } d < 0.
\end{cases}$$

**Proof.** $H^1(\mathcal{X}, \mathcal{F})$ is a finitely generated $R$-module. We know that $\mathcal{F}|_{\mathcal{X}_0}$ and $\mathcal{F}|_{\mathcal{X}_1}$ have same degree. Thus Grauert's Theorem[11, III.12.9] implies
that $h^1(X_0, \mathcal{F}|_{X_0}) = \dim_K(H^1(\mathcal{X}, \mathcal{F}) \otimes K)$, where $K$ is a fraction field of $R$. Also $H^1(\mathcal{X}, \mathcal{F})$ is torsion free. Since $R$ is a discrete valuation ring, $H^1(\mathcal{X}, \mathcal{F})$ is a free $R$-module. We have

$$h^1(\mathcal{X}, \mathcal{F}) = h^1(X_0, \mathcal{F}|_{X_0}) = h^0(X_0, \mathcal{F}|_{X_0}) - \deg((\mathcal{F}|_{X_0}))$$

$$= 0 \text{ if } \deg(\mathcal{F}|_{X_0}) = -d > 0$$

$$= 0 \text{ if } \deg(\mathcal{F}|_{X_0}) = 0 \text{ and } \mathcal{F} \text{ is not trivial}$$

$$= 1 \text{ if } \mathcal{F} \text{ is trivial}$$

$$= d \text{ if } \deg(\mathcal{F}|_{X_0}) = -d < 0,$$

which implies the result. ■

Therefore we have an exact sequence of $R$-modules

$$0 \rightarrow R^* \xrightarrow{\phi} Ext^1_X(I_Z, \mathcal{F}) \xrightarrow{\psi} k \rightarrow 0,$$

where $R^*$ is $R^d$ or $R$ or 0. Because of $H^1(\mathcal{X}, \mathcal{F}) \simeq Ext^1_X(\mathcal{O}_X, \mathcal{F})$ [11, III.6.3] one element in $H^1(\mathcal{X}, \mathcal{F})$ corresponds to an exact sequence $[\mathcal{E}]$

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{O}_X \rightarrow 0.$$
Since we have a map $I_Z \xrightarrow{i} \mathcal{O}_X$, the map $\phi$ is just making a new exact sequence by pull-back as following diagram

$$
\begin{array}{cccc}
0 & \to & \mathcal{F} & \to \mathcal{E} & \xrightarrow{i} & \mathcal{O}_X & \to & 0 \\
\ || & \uparrow & \uparrow i \\
0 & \to & \mathcal{F} & \to \mathcal{E}' & \to & I_Z & \to & 0,
\end{array}
$$

where $[\mathcal{E}'] = \phi([\mathcal{E}]) \in Ext^1_{\mathcal{X}}(I_Z, \mathcal{F})$.

We note that $\mathcal{E}' = f^{-1}(i(I_Z))$. Since $\text{Im} \phi = \ker \psi$, $\mathcal{E}'$ is not a rank 2 vector bundle.

Now we check the same 3 cases in Lemma 18.

4.2 Case $H^1(\mathcal{X}, \mathcal{F}) = 0$

We assume that $\deg \mathcal{F} = -d > 0$ or $d = 0$ and $\mathcal{F}$ is not trivial in this section.

By Lemma 18, $H^1(\mathcal{X}, \mathcal{F}) = 0$ and we have an isomorphism

$$
Ext^1_{\mathcal{X}}(I_Z, \mathcal{F}) \cong H^0(Z, \mathcal{O}_Z).
$$

Since $H^0(Z, \mathcal{O}_Z) \cong k$, we have $Ext^1_{\mathcal{X}}(I_Z, \mathcal{F}) \cong k$ as $R$-module. Every nonzero element in $k$ is a unit and every nonzero element or non-trivial extension
in $\text{Ext}_X^1(I_Z, F)$ is a unit element. Canonically we can take one element $[\mathcal{E}_1]$ in $\text{Ext}_X^1(I_Z, F)$ corresponding to $1 \in k$. Then every unit element in $\text{Ext}_X^1(I_Z, F)$ can be generated by $R^*$ action on $[\mathcal{E}_1]$. That is, we have an isomorphism $I_Z \xrightarrow{u} I_Z$ from multiplication by a unit element $u$ in $R$ and we have the following commutative diagram:

$$
\begin{array}{cccccc}
0 & \rightarrow & F & \rightarrow & \mathcal{E}_1 & \rightarrow & I_Z & \rightarrow & 0 \\
& & \| & \uparrow & \uparrow u & & \\
0 & \rightarrow & F & \rightarrow & u\mathcal{E}_1 & \rightarrow & I_Z & \rightarrow & 0.
\end{array}
$$

Now $u\mathcal{E}_1$ is a rank 2 vector bundle and there are $\#(k^*)$ unit elements in $\text{Ext}_X^1(I_Z, F)$. We have proved

**Theorem 19** Let $F$ be a line bundle with $\text{deg } F > 0$ or a non-trivial line bundle with $\text{deg } F = 0$. Then under $R^*$ action, all unit elements in $\text{Ext}_X^1(I_Z, F)$ are generated by $[\mathcal{E}_1]$, which corresponds to $1$ in $k$, and there are $\#(k^*)$ unit elements in $\text{Ext}_X^1(I_Z, F)$.

### 4.3 Case $H^1(X, F) = R$

In this section, we assume that $F$ is trivial.

By Lemma 18, we have $H^1(X, F) \cong R$ and we have an exact sequence of
$R$-module

$$0 \to R \xrightarrow{\phi} \text{Ext}^1_X(I, \mathcal{F}) \xrightarrow{\psi} k \to 0.$$ 

So we can have $\text{Ext}^1_X(I, \mathcal{F}) \cong R$ or $R \oplus k$.

If we have $\text{Ext}^1_X(I, \mathcal{F}) \cong R$, then the map $\phi$ is a multiplication by $p$. Then image of $\phi$ is the maximal ideal $m = (p)$ of $R$ which is $\ker \psi$. Since $R$ is a discrete valuation ring, elements which are not in $m$ are unit elements in $R$. Thus unit elements in $\text{Ext}^1_X(I, \mathcal{F})$ are corresponding to unit elements in $R$. And we can choose $[\mathcal{E}_1]$ in $\text{Ext}^1_X(I, \mathcal{F})$ corresponding to 1 in $R$ and we can generate all elements in $U$ under $R^*$ action.

If we have $\text{Ext}^1_X(I, \mathcal{F}) \cong R \oplus k$, then $\ker \psi = \{(a, 0) \in R \oplus k \mid a \in R\}$ and any element $(a, b)$ with $b \neq 0$ in $R \oplus k$ corresponds to a unit element in $\text{Ext}^1_X(I, \mathcal{F})$. Let $S$ be a subset of elements $(a, b)$ with $b \neq 0$ in $R \oplus k$. Then $R^*$ acts on a set $S$ by $r(a, b) = (ra, rb)$. Each orbit contains an element $(a, 1)$ for some $a \in R$ since $(x, y) \sim y^{-1}(xy, 1)$ for any $(x, y) \in R \oplus k$. Let $R_1^* = \{r \in R^* \mid r \equiv 1 \mod m\}$ where $m$ is a maximal ideal of $R$. We have $r(a, 1) = (ra, 1)$ for $r \in R_1^*$ and $(a, 1) \sim (ra, 1)$. Since $S$ is equivalent to a subset $U$ of unit elements in $\text{Ext}^1_X(I, \mathcal{F})$ which $R^*$ acts on, there is a one-to-one correspondence between orbits in $U$ and elements in $R^*/R_1^*$.

We have
Theorem 20 Let $\mathcal{F}$ be a trivial bundle. With same notation as above,

1. if we have $\text{Ext}_{\mathcal{X}}^{1}(I_{Z}, \mathcal{F}) \simeq R$, then there exists $[\mathcal{E}]$ in $\text{Ext}_{\mathcal{X}}^{1}(I_{Z}, \mathcal{F})$ which generates all unit elements in $\text{Ext}_{\mathcal{X}}^{1}(I_{Z}, \mathcal{F})$ by $R^{*}$-action,

2. if we have $\text{Ext}_{\mathcal{X}}^{1}(I_{Z}, \mathcal{F}) \simeq R \oplus k$, then any elements in $\text{Ext}_{\mathcal{X}}^{1}(I_{Z}, \mathcal{F})$ corresponding to elements in $S$ are unit elements and there is a one-to-one correspondence between orbits in $U$ and elements in $R^{*}/R_{1}$.

4.4 Case $H^{1}(\mathcal{X}, \mathcal{F}) = R^{d}$

In this section, we assume that $-d = \text{deg} \mathcal{F} < 0$.

By Lemma 18, we have $H^{1}(\mathcal{X}, \mathcal{F}) \simeq R^{d}$ and we have an exact sequence of $R$-module

$$0 \to R^{d} \xrightarrow{\phi} \text{Ext}_{\mathcal{X}}^{1}(I_{Z}, \mathcal{F}) \xrightarrow{\psi} k \to 0.$$ 

So we can have $\text{Ext}_{\mathcal{X}}^{1}(I_{Z}, \mathcal{F}) \simeq R^{d}$ or $R^{d} \oplus k$. If we have $\text{Ext}_{\mathcal{X}}^{1}(I_{Z}, \mathcal{F}) \simeq R^{d}$, then since $\text{coker} \phi = k$ we can rewrite $\phi$ as follows: $\phi : R^{d} \to R^{d}$:

$$(x_{1}, \ldots, x_{d}) \to (x_{1}, \ldots, x_{i-1}, px_{i}, x_{i+1}, \ldots, x_{d})$$

for some $i$.

For convenience, say $i = 1$. Then image of $\phi$ is $m \times R^{d-1}$ which is $\text{ker} \psi$. As in Section 4.3, unit elements in $\text{Ext}_{\mathcal{X}}^{1}(I_{Z}, \mathcal{F})$ correspond to elements in $R^{*} \times R^{d-1}$. We can choose $[\mathcal{E}_{x_{2}, \ldots, x_{d}}]$ in $\text{Ext}_{\mathcal{X}}^{1}(I_{Z}, \mathcal{F})$ corresponding to $(1, x_{2}, \ldots, x_{d})$ in $R^{*} \times R^{d-1}$ and we can generate all elements in $U$ under $R^{*}$-action.
If we have $\text{Ext}_A^1(I_Z, \mathcal{F}) \cong R^d \oplus k$, then $\ker \psi = \{(a, 0) \in R^d \oplus k | a \in R^d\}$ and any element $(a, b)$ with $b \neq 0$ in $R^d \oplus k$ corresponds to a unit element in $\text{Ext}_A^1(I_Z, \mathcal{F})$. As in Section 4.3, let $S'$ be a subset of elements $(a, b)$ in $R^d \oplus k$ with $b \neq 0$. Then $R^*$ acts on a set $S'$ by $r(a, b) = (ra, rb)$. Each orbit contains an element $(a, 1)$ for some $a \in R^d$ since $(x, y) \sim y^{-1}(xy, 1)$. We have $r(a, 1) = (ra, 1)$ for $r \in R^*_1$ and $(a, 1) \sim (ra, 1)$. Since $S'$ is equivalent to $U$ on which $R^*$ acts, thus there is a one to one correspondence between orbits in $U$ and elements in $R^*/R^*_1$.

We have

**Theorem 21** Let $\mathcal{F}$ be a line bundle with $\deg \mathcal{F} < 0$. With same notation as above,

1. if we have $\text{Ext}_A^1(I_Z, \mathcal{F}) \cong R^d$, then there exists a set of elements 
   $$\{e_{x_2, \ldots, x_d} \in \text{Ext}_A^1(I_Z, \mathcal{F}) | e_{x_2, \ldots, x_d} \text{ corresponds to } (1, x_2, \ldots, x_d) \in R^d, \ x_i \in R, 2 \leq i \leq d\}$$
   which generates all unit elements in $\text{Ext}_A^1(I_Z, \mathcal{F})$ under $R^*$-action,

2. if we have $\text{Ext}_A^1(I_Z, \mathcal{F}) \cong R^d \oplus k$, then any elements in $\text{Ext}_A^1(I_Z, \mathcal{F})$ corresponding to elements in $S'$ are unit elements and there is a one-to-one correspondence between orbits in $U$ and elements in $R^*/R^*_1$. 
Note if $r$ is not a unit element in $R$, then $r$ is in $m$ and any element multiplied by $r$ is $0 \mod m$ which means that the image in $H^0(\mathcal{Z}, \mathcal{O}_Z)$ is $0$. Thus for any $\mathcal{F}$ and $\mathcal{Z}$ and a unit element $[\mathcal{E}]$ in $\text{Ext}^1_\mathcal{X}(I_Z, \mathcal{F})$, its image in $H^0(\mathcal{Z}, \mathcal{O}_Z)$ is not a zero and it is not $0 \mod m$ as $R$-modules. If $r$ is not a unit element in $R$, then $r[\mathcal{E}]$ is $0 \mod m$ as $R$-modules which means $r[\mathcal{E}]$ is not a unit element and the middle term $r\mathcal{E}$ is not a vector bundle. If $r$ is a unit element in $R$, then $r[\mathcal{E}]$ is not $0 \mod m$ as $R$-modules which means $r[\mathcal{E}]$ is a unit element and the middle term $r\mathcal{E}$ is a vector bundle. It completes the proof of Lemma 15.
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