Analysis of a New Bivariate Distribution in Reliability Theory

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Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

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ABSTRACT

Freund [1961] introduced a bivariate extension of the exponential distribution that provides a model in which the exponential residual lifetime of one component depends on the working status of another component. We define and study an extension of the Freund distribution in this dissertation.

In the first chapter we define some basic concepts that are needed for later developments. We give the definition of the multivariate conditional hazard rate functions of a nonnegative absolutely continuous random vector and study a characterization of these functions in Section 1.1. Then we study some notions of aging: an increasing failure rate (IFR) distribution, a decreasing failure rate (DFR) distribution, an increasing failure rate average (IFRA) distribution, and a decreasing failure rate average (DFRA) distribution in Section 1.2. In Section 1.3 we study two concepts of multivariate dependence: association and positive quadrant dependence.

In Chapter 2 we construct a shock model and the new bivariate distribution is the joint distribution of the resulting lifetimes. We explicitly compute the density function, survival function, moment generating function, marginal density functions and marginal survival functions. Also in this chapter, we study the correlation coefficient and other senses of positive dependence of the two random variables of the new bivariate distribution. Then we extend the new distribution to multivariate case.

In Chapter 3 we study some aging properties. We obtain two results about the new distribution in n dimensions. The first result says that the marginal distributions of the new multivariate distribution have decreasing failure rate if the conditional hazard rates are decreasing and bounded above by 1. The second one concerns an $(n-1)$-out-of-n system such that the joint distribution of the lifetimes of each component is the new distribution in n dimensions. It gives conditions on the parameters under which the system has an IFRA distribution.

In Chapter 4 we develop some estimation procedure for the parameters of the new bivariate distribution. We apply the method of moments and the maximum likelihood principle to estimate the parameters. We prove that the method of moments estimator is a consistent asymptotically normal estimator. Then we use Mathematica to run simulation and compare the method of moments estimator with the maximum likelihood estimator. We also compute the 95% confidence interval for the parameters from the method of moments estimator.

In the last chapter we study a stochastic ordering problem. We have two nonnegative n dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$. We assume that $\mathbf{X}$ and $\mathbf{Y}$ have the same conditional hazard rates up to a certain level. We give a condition under which the two vectors $\mathbf{X}$ and $\mathbf{Y}$ are stochastically ordered.

Chapter 1

PRELIMINARIES

For any absolutely continuous nonnegative random variable T with distribution function F, survival function $\bar{F}$, and hazard function $\Lambda \equiv -\log \bar{F}$, define the hazard rate function at time t by

$$\lambda(t) = \frac{f(t)}{P\{T \geq t\}} = \frac{f(t)}{\bar{F}(t)} = \frac{d}{dt}\Lambda(t), \quad 0 \leq t, \tag{1.1}$$

where $f(t) = (d/dt)F(t)$ is the density function of T. If T is the lifetime of a device then $\lambda(t)$ can be thought of as the instantaneous failure rate of the device at time t. It is well known that any absolutely continuous life distribution can be characterized by the hazard rate function, that is, F determines λ and vice versa. Indeed, if F is an absolutely continuous distribution function with support on $[0, \infty)$ and hazard rate function λ, then

$$F(t) = 1 - e^{-\int_0^t \lambda(x)dx}, \quad \forall t \geq 0.$$

A similar characterization (see Cox [1972], Cox and Lewis [1972], and Shaked and Shanthikumar [1986]) holds in the multivariate case. That is, any multivariate absolutely continuous distribution with support in $[0, \infty)^n$ determines a system of conditional hazard rate functions (to be defined in Section 1.1) from which the distribution function can be reconstructed. We present this result in Section 1.1.

In Section 1.2 we study some notions of aging of a nonnegative random variable T. We give the definitions of an increasing failure rate (IFR) distribution and a decreasing failure rate (DFR) distribution. If T is absolutely continuous, then aging can be conveniently studied in terms of the hazard rate function, $\lambda(t)$ in (1.1). In this case, we use the hazard rate function to describe the notions of IFR and DFR. This is presented in Theorem 1.2.3.

Although IFR is a very useful concept in reliability theory, it is not closed under the formation of a coherent system with independent components. This leads to the study of an increasing failure rate average (IFRA) distribution. We also study the contrast, a decreasing failure rate average (DFRA) distribution. We present some closure properties of IFRA, DFR, and DFRA in Section 1.2.

In a great many reliability situations, the random variables of interest are not independent, but rather are "associated". As an example, consider structures in which components share the load, so that failure of one component results in increased load on each of the remaining components.

In Section 1.3 we formulate a definition of association appropriate to reliability situations. We also discuss an alternative notion of multivariate dependence and study the relationship between this notion and association. At the end of Section 1.3, we give some applications of association.

Throughout this dissertation, "increasing" and "decreasing" mean, respectively, "nondecreasing" and "nonincreasing".

1.1 Multivariate Conditional Hazard Rate Functions

In the following we define the multivariate conditional hazard rate functions of a nonnegative absolutely continuous random vector. The development follows the works of Shaked and Shanthikumar [1986, 1987].

For $\mathbf{I} = \{i_1, i_2, \cdots, i_k\} \subset \{1, 2, \cdots, n\}$, let $\mathbf{t_I}$ denote $(t_{i_1}, t_{i_2}, \cdots, t_{i_k})$. The complement of $\mathbf{I}$ will be denoted by $\bar{\mathbf{I}} = \{1, 2, \cdots, n\} - \mathbf{I}$ and if $\bar{\mathbf{I}} = \{j_1, j_2, \cdots, j_{n-k}\}$ then $\mathbf{t}_{\bar{\mathbf{I}}} = (t_{j_1}, t_{j_2}, \cdots, t_{j_{n-k}})$. Let $\mathbf{e} = (1, 1, \cdots, 1)$. The dimension of $\mathbf{e}$ will vary from one formula to another, but it will always be possible to determine it from the expression in which $\mathbf{e}$ appears.

Let $\mathbf{T} = (T_1, T_2, \cdots, T_n)$ be a nonnegative random vector with absolutely continuous distribution function. It is beneficial to think about $T_1, T_2, \cdots, T_n$ as the lifetimes of n components $1, 2, \cdots, n$ that make up some system. Suppose that an observer observes the system continuously in time and records the failure times and the identities of the components that fail as time passes. Thus, a typical "history" that the observer has observed by time $t \geq 0$ is of the form

$$h_t = \{\mathbf{T}_{\bar{\mathbf{I}}} = \mathbf{t}_{\bar{\mathbf{I}}}, \mathbf{T}_{\mathbf{I}} > t\mathbf{e}\}, \quad 0\mathbf{e} \leq \mathbf{t}_{\bar{\mathbf{I}}} \leq t\mathbf{e}, \quad \mathbf{I} \subseteq \{1, 2, \cdots, n\}. \tag{1.2}$$

In (1.2), $\mathbf{I}$ is the set of components that are still alive at time t and $\bar{\mathbf{I}}$ is the set of components that have already failed by time t.

Given the history h_t as in (1.2), let $i \in \mathbf{I}$ be a component that is still alive at time t. Its multivariate conditional hazard rate at time t is defined as follows:

$$\lambda_{i|\bar{\mathbf{I}}}(t|\mathbf{t}_{\bar{\mathbf{I}}}) = \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P\{t < T_i < t + \Delta t | \mathbf{T}_{\bar{\mathbf{I}}} = \mathbf{t}_{\bar{\mathbf{I}}}, \mathbf{T}_{\mathbf{I}} > t\mathbf{e}\}, \tag{1.3}$$

where, of course, $0\mathbf{e} \leq \mathbf{t}_{\bar{\mathbf{I}}} \leq t\mathbf{e}$, and $\mathbf{I} \subseteq \{1, 2, \cdots, n\}$. The absolute continuity of $\mathbf{T}$ ensures that this limit exists. For $\bar{\mathbf{I}} = \varnothing$, $\lambda_{i|\bar{\mathbf{I}}}$ is called the initial hazard rate. For $|\bar{\mathbf{I}}| = 1$, $\lambda_{i|\bar{\mathbf{I}}}$ is called the conditional hazard rate. For $|\bar{\mathbf{I}}| = 2$, $\lambda_{i|\bar{\mathbf{I}}}$ is called the second order conditional hazard rate. In general, $\lambda_{i|\bar{\mathbf{I}}}$ is called the k-th order conditional hazard rate when $|\bar{\mathbf{I}}| = k$. The following notation is in force throughout this dissertation. If $\bar{\mathbf{I}} = \varnothing$ then the conditional hazard rate $\lambda_{i|\bar{\mathbf{I}}}$ is denoted by λ_i. If $\bar{\mathbf{I}} = \{j_1, \cdots, j_k\}$ and $i \notin \bar{\mathbf{I}}$, then the $\lambda_{i|\bar{\mathbf{I}}}$ is denoted by $\lambda_{i|j_1, \cdots, j_k}$.

Let us study first the special case $n = 2$. We denote by F and $\bar{F}$ the joint distribution and the survival function of $\mathbf{T}$. Then (1.3) reduces to

$$\lambda_i(t) = -\frac{(\partial/\partial t_i)\bar{F}(t_1, t_2)|_{(t_1,t_2)=(t,t)}}{\bar{F}(t,t)}, \quad t \in \{s : \bar{F}(s,s) > 0\}, \quad i = 1, 2. \tag{1.4}$$

$$\lambda_{1|2}(t|t_2) = \frac{f(t, t_2)}{-(\partial/\partial t_2)\bar{F}(t, t_2)}, \quad t \in \{s \geq t_2 : -\frac{\partial}{\partial t_2}\bar{F}(s, t_2) > 0\},$$
$$t_2 \in \{s : \bar{F}(s,s) > 0\}, \tag{1.5}$$

and

$$\lambda_{2|1}(t|t_1) = \frac{f(t_1, t)}{-(\partial/\partial t_1)\bar{F}(t_1, t)}, \quad t \in \{s \geq t_1 : -\frac{\partial}{\partial t_1}\bar{F}(t_1, s) > 0\},$$
$$t_1 \in \{s : \bar{F}(s,s) > 0\}, \tag{1.6}$$

where f denotes the joint density function of T_1 and T_2. In (1.4), $\lambda_i(t)$ is called the initial hazard rate of T_i at time t. In (1.5), $\lambda_{1|2}(t|t_2)$ is called the conditional hazard rate of T_1 at time t under the conditions that $T_1 > t \geq t_2$ and $T_2 = t_2$. Similarly, $\lambda_{2|1}(t|t_1)$, in (1.6), is called the conditional hazard rate of T_2 at time t under the conditions that $T_2 > t \geq t_1$ and $T_1 = t_1$. The following result is from Lemma 1.1 (page 439) of Shaked and Shanthikumar [1986].

Theorem 1.1.1. *Let $\bar{F}_i(t_j|t_k) = P\{T_i > t_j|T_{3-i} = t_k\}$ be the conditional survival function for $i, j, k \in \{1, 2\}$. Then*

$$f(t_1, t_2) = \begin{cases} \bar{F}(t_1, t_1)\lambda_1(t_1)\frac{\bar{F}_2(t_2|t_1)}{\bar{F}_2(t_1|t_1)}\lambda_{2|1}(t_2|t_1), & \text{if} \quad 0 \leq t_1 \leq t_2, \\ \bar{F}(t_2, t_2)\lambda_2(t_2)\frac{\bar{F}_1(t_1|t_2)}{\bar{F}_1(t_2|t_2)}\lambda_{1|2}(t_1|t_2), & \text{if} \quad 0 \leq t_2 \leq t_1. \end{cases}$$

Applying (1.4), (1.5), and (1.6), we have the following corollary.

Corollary 1.1.2.

$$f(t_1, t_2) = \begin{cases} e^{-\int_0^{t_1}(\lambda_1(t)+\lambda_2(t))dt}\lambda_1(t_1)e^{-\int_{t_1}^{t_2}\lambda_{2|1}(t|t_1)dt}\lambda_{2|1}(t_2|t_1), & \text{if} \quad 0 \leq t_1 \leq t_2, \\ e^{-\int_0^{t_2}(\lambda_1(t)+\lambda_2(t))dt}\lambda_2(t_2)e^{-\int_{t_2}^{t_1}\lambda_{1|2}(t|t_2)dt}\lambda_{1|2}(t_1|t_2), & \text{if} \quad 0 \leq t_2 \leq t_1. \end{cases}$$

This corollary is applied in next chapter. From this corollary, the joint density function of $\mathbf{T}$ can be computed if the initial hazard rate and the conditional hazard rate functions of $\mathbf{T}$ are known. Theorem 1.1.1 can be extended to the general case as follows. For more detail, see Lemma 2.1 (page 441) of Shaked and Shanthikumar [1986].

Theorem 1.1.3. *Let f be the joint density function of $\mathbf{T} = (T_1, T_2, \cdots, T_n)$. Then for $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$,*

$$\begin{aligned} & f(t_1, t_2, \cdots, t_n) \\ & = \lambda_1(t_1)\bar{F}(t_1, t_1, \cdots, t_1) \\ & \times \prod_{i=2}^{n}\left[\lambda_{i|1,2,\cdots,i-1}(t_i|t_1, \cdots, t_{i-1})\frac{\bar{F}_{i,i+1,\cdots,n}(t_i, \cdots, t_i|T_1 = t_1, \cdots, T_{i-1} = t_{i-1})}{\bar{F}_{i,i+1,\cdots,n}(t_{i-1}, \cdots, t_{i-1}|T_1 = t_1, \cdots, T_{i-1} = t_{i-1})}\right] \end{aligned}$$

where

$$\bar{F}_{i,i+1,\cdots,n}(t_i,\cdots,t_i|T_1 = t_1,\cdots,T_{i-1} = t_{i-1})$$
$$= P\{T_i > t_i,\cdots,T_n > t_i|T_1 = t_1,\cdots,T_{i-1} = t_{i-1}\}$$

is the conditional survival function. Similar expression is valid when $0 \le t_{\pi(1)} \le \cdots \le t_{\pi(n)}$ *for any permutation* π *of* $(1,2,\cdots,n)$.

Schechner [1984] studied a load-sharing model and he derived a similar formula from which the joint density function can be reconstructed from the conditional hazard rate functions as follows (see Theorem 2.1, page 139, of Schechner [1984]).

Corollary 1.1.4. *For* $0 \le t_1 \le \cdots \le t_n$, *we have*

$$f(t_1,t_2,\cdots,t_n)$$
$$= \lambda_1(t_1)e^{-\int_0^{t_1}\sum_{i=1}^{n}\lambda_i(t)dt}\prod_{i=2}^{n}\left[\lambda_{i|1,\cdots,i-1}(t_i|t_1,\cdots,t_{i-1})e^{-\int_{t_{i-1}}^{t_i}\sum_{j=i}^{n}\lambda_{j|1,\cdots,i-1}(t|t_1,\cdots,t_{i-1})dt}\right].$$

Similar expression is valid when $0 \le t_{\pi(1)} \le \cdots \le t_{\pi(n)}$ *for any permutation* π *of* $(1,2,\cdots,n)$.

In Chapter 5 we study the stochastic order between two nonnegative absolutely continuous random vectors through their conditional hazard rates. The definition of the stochastic order is as follows.

Definition 1.1.5. Let $\mathbf{X}$ and $\mathbf{Y}$ be two n dimensional random vectors. Then $\mathbf{X}$ is said to be smaller in the stochastic order than $\mathbf{Y}$, denote by $\mathbf{X} \le_{st} \mathbf{Y}$, if

$$E[\phi(\mathbf{X})] \le E[\phi(\mathbf{Y})] \tag{1.7}$$

for all increasing (in the componentwise ordering in $\mathbb{R}^n$) function ϕ for which the expectations in (1.7) exist.

1.2 Notions of Aging

We first give the definitions and some properties of an increasing failure rate (IFR) distribution and a decreasing failure rate (DFR) distribution. Then we study the notions of increasing failure rate average (IFRA) and decreasing failure rate average (DFRA). All the definitions and properties presented here are studied in Barlow and Proschan [1975].

Throughout this section. let T be a nonnegative random variable with distribution function F.

Definition 1.2.1. T is said to possess an increasing failure rate (IFR) distribution if

$$\frac{\bar{F}(t+x)}{\bar{F}(t)} \quad \text{is} \quad \text{decreasing} \quad \text{in} \quad t \geq 0.$$

for all $x \geq 0$. where $\bar{F}$ is the survival function of T. If

$$\frac{\bar{F}(t+x)}{\bar{F}(t)} \quad \text{is} \quad \text{increasing} \quad \text{in} \quad t \geq 0.$$

for all $x \geq 0$. then T is said to possess a decreasing failure rate (DFR) distribution.

Example 1.2.2. Let $F(t) = 1 - e^{-\alpha t}$. for $t \geq 0$. be an exponential distribution with rate $\alpha > 0$. Then

$$\frac{\bar{F}(t+x)}{\bar{F}(t)} = e^{-\alpha x}$$

which is independent of t. Hence F is an IFR distribution and also a DFR distribution. That is. an exponential distribution is both IFR and DFR distributions.

If T is absolutely continuous with hazard rate function $\lambda(t)$ (see (1.1)) then we have the following result.

Theorem 1.2.3. *The absolutely continuous random variable T possesses an IFR distribution if and only if $\lambda(t)$ is increasing. Similarly. T possesses a DFR distribution if and only if $\lambda(t)$ is decreasing.*

Example 1.2.4. Let $F(t) = 1 - e^{-(\beta t)^{\alpha}}$, for $t \geq 0$, be a Weibull distribution with parameters $\alpha > 0$ and $\beta > 0$. Then

$$\lambda(t) = \alpha\beta(\beta t)^{\alpha-1}, \quad \forall t > 0.$$

Thus F is IFR for $\alpha \geq 1$ and DFR for $0 < \alpha \leq 1$.

In Section 3.2 we study some aging properties of a k-out-of-n system of similar components. A k-out-of-n system is a system with n components that functions if and only if at least k of the n components function. For $k = 1$, it is called a parallel system. From above, a parallel system functions if and only if at least one component functions. For $k = n$, it is called a series system and it functions if and only if all components function. These are special cases of a coherent system. For details about coherent system, see Section 1.2 of Barlow and Proschan [1975].

The following example (see Example 2.1, page 83, of Barlow and Proschan [1975]) shows that IFR is not closed under the formation of a coherent system with independent components.

Example 1.2.5. Let F be the distribution of the lifetime of a parallel system of two independent components having respective life distributions $F_1(t) = 1 - e^{-\alpha_1 t}$ and $F_2(t) = 1 - e^{-\alpha_2 t}$. Then

$$\bar{F}(t) = 1 - (1 - e^{-\alpha_1 t})(1 - e^{-\alpha_2 t}),$$

so that

$$\lambda(t) = \frac{\alpha_1 e^{-\alpha_1 t} + \alpha_2 e^{-\alpha_2 t} - (\alpha_1 + \alpha_2)e^{-(\alpha_1+\alpha_2)t}}{e^{-\alpha_1 t} + e^{-\alpha_2 t} - e^{-(\alpha_1+\alpha_2)t}}.$$

It is easy to verify that $\lambda(t)$ is increasing on $[0, t_0)$ and is decreasing on (t_0, ∞), where t_0 is a constant depending on α_1 and α_2. Hence F is not IFR.

In the above example, F is the life distribution of a parallel system with two independent components. The two independent components are both having an IFR distribution. But F is not IFR. This leads to the study of a class of life distributions which is broader than IFR and which is closed under the formation of a coherent system with independent components.

Definition 1.2.6. A nonnegative random variable T is said to possess an increasing failure rate average (IFRA) if $-\frac{1}{t}\log\bar{F}(t)$ is increasing in $t \geq 0$. where $\bar{F}$ is the survival function of T. Similarly. T is said to possess a decreasing failure rate average (DFRA) if $-\frac{1}{t}\log\bar{F}(t)$ is decreasing in $t \geq 0$.

Remark. It is obvious that an IFRA (DFRA) distribution F is characterized by $\bar{F}^{1/t}(t) \downarrow (\uparrow)$ on $[0, \infty)$. Hence F is IFRA (DFRA) if and only if $\bar{F}(\alpha t) \geq (\leq) \bar{F}^{\alpha}(t)$ for all $0 < \alpha < 1$ and $t \geq 0$.

Theorem 1.2.7. *IFR⇒IFRA.*

Proof. From Lemma 4.2. page 27. of Barlow and Proschan [1996]. we have $\bar{F}^{1/t}(t)$ is decreasing in t if F is IFR. Hence F is IFRA. ∎

The class of all IFRA distributions possesses the following desirable closure property (see Theorem 2.6. page 85. of Barlow and Proschan [1975]).

Theorem 1.2.8. *Suppose each of the independent components of a coherent system has an IFRA life distribution. Then the system itself has an IFRA life distribution.*

There is another class of distributions, new better than used (NBU). which is even broader then IFRA. It is defined as follows:

Definition 1.2.9. A distribution F is NBU if

$$\bar{F}(x+y) \leq \bar{F}(x)\bar{F}(y), \quad \forall x, y \geq 0. \tag{1.8}$$

This is equivalent to stating that $\bar{F}(x+y)/\bar{F}(x)$, the conditional survival probability of a unit of age x, is less then $\bar{F}(y)$, the corresponding survival probability of a new unit. Equality in (1.8) holds if and only if F is an exponential distribution.

Theorem 1.2.10. *IFRA⇒NBU.*

Proof. If one of x and y is 0, then we have $\bar{F}(x+y) = \bar{F}(x)\bar{F}(y)$. Now suppose that $0 < y < x$ and $y = \alpha x$ with $0 < \alpha < 1$. Since F is IFRA, we have

$$\frac{-\log \bar{F}(x+y)}{x+y} \geq \frac{-\log \bar{F}(x)}{x}.$$

that is,

$$\log \bar{F}(x+y) \leq \frac{x+y}{x} \log \bar{F}(x).$$

This is equivalent to

$$\begin{aligned} \bar{F}(x+y) &\leq \bar{F}^{(1+\alpha)}(x) \\ &= \bar{F}(x)\bar{F}^{\alpha}(x) \\ &\leq \bar{F}(x)\bar{F}(\alpha x) \quad (IFRA) \\ &= \bar{F}(x)\bar{F}(y). \end{aligned}$$

Hence F is NBU. ■

Note that if F is not NBU, then F is not IFRA. In Section 3.1.1 we apply this to conclude that the marginal distributions of the new bivariate distribution, defined in Chapter 2, is not IFRA.

For the set, say D_1, of all DFR distributions and the set, say D_2, of all DFRA distributions, we have the following closure property.

Theorem 1.2.11. *D_1 and D_2 are closed under mixture. That is, if F is a mixture of some DFR (or DFRA) distributions then F is also a DFR (or DFRA) distribution.*

We apply this result in Chapter 3 to prove one of the main theorems there.

1.3 Association of Random Variables

We define the notion of association between random variables in this section. We also discuss an alternative notion of multivariate dependence and study the relationship between this notion and association. Then, we give some applications of association. The development of this section follows the works of Barlow and Proschan [1975], and Esary, Proschan, and Walkup [1967].

Let $\mathbf{T} = (T_1, T_2, \cdots, T_n)$.

Definition 1.3.1. We say that random variables $T_1, T_2, \cdots, T_n$ are associated if

$$Cov[f(\mathbf{T}), g(\mathbf{T})] \geq 0$$

for all increasing functions f and g for which $Ef(\mathbf{T})$, $Eg(\mathbf{T})$, and $Ef(\mathbf{T})g(\mathbf{T})$ exist.

Association of random variables satisfies the following desirable multivariate properties.

(P_1) Any subset of associated random variables are associated.

(P_2) The set consisting of a single random variable is associated.

(P_3) Increasing functions of associated random variables are associated.

(P_4) If two sets of associated random variables are independent of one another, then their union is a set of associated random variables.

Properties P_2 and P_4 immediately imply the following theorem.

Theorem 1.3.2. *Independent random variables are associated.*

The notion of association among random variables is just one among many notions of multivariate dependence. In the following we describe the notion of positive quadrant dependence and study the relationship between this notion and association. The notion of positive quadrant dependence is introducted by Lehmann [1966].

Definition 1.3.3. Given random variables S and T, we say that S and T are positively quadrant dependent (PQD) if

$$P\{S \le s, T \le t\} \ge P\{S \le s\}P\{T \le t\}, \quad \forall s, t. \tag{1.9}$$

We write PQD(S, T).

Remark. It can be easily checked that the condition (1.9) is equivalent to

$$\bar{F}(s, t) \ge \bar{F}_S(s)\bar{F}_T(t), \quad \forall s, t. \tag{1.10}$$

where $\bar{F}$ is the joint survival function of S and T, $\bar{F}_S$ is the survival function of S, and $\bar{F}_T$ is the survival function of T. Thus, we can check (1.10) instead of (1.9) in order to check whether two random variables are PQD or not.

The following theorem states the relationship between positively quadrant dependent and association. For more detail, see Theorem 4.2, page 143, of Barlow and Proschan [1975].

Theorem 1.3.4. $A(S, T) \Rightarrow PQD(S, T)$.

The notation $A(S, T)$ signifies that S and T are associated random variables. One application of this theorem is that if S and T are not PQD, then S and T are not associated.

Finally, some interesting applications may be obtained as a consequence of the following theorem. See Esary, Proschan, and Walkup [1967].

Theorem 1.3.5. *Let $T_1, \cdots, T_n$ be associated, $S_i \equiv f_i(T_1, \cdots, T_n)$ be increasing, $i = 1, 2, \cdots, k$. Then*

$$P\{S_1 \le s_1, \cdots, S_k \le s_k\} \ge \prod_{i=1}^{k} P\{S_i \le s_i\}$$

and

$$P\{S_1 > s_1, \cdots, S_k > s_k\} \geq \prod_{i=1}^{k} P\{S_i > s_i\}$$

for all $s_1, \cdots, s_k$.

For a series system (with lifetime $\min T_i$) and a parallel system (with lifetime $\max T_i$), we have the following application.

Corollary 1.3.6 (Bounds on System Reliability). *If* $T_1, \cdots, T_n$ *are associated, then*

$$P\{\min_{1 \leq i \leq n} T_i > t\} \geq \prod_{i=1}^{n} P\{T_i > t\}$$

and

$$P\{\max_{1 \leq i \leq n} T_i > t\} \leq \coprod_{i=1}^{n} P\{T_i > t\},$$

where $\coprod_{i=1}^{n} x_i \equiv 1 - \prod_{i=1}^{n}(1 - x_i)$.

The following is an application to the order statistics of a random sample.

Corollary 1.3.7. *Let* $S_1 \leq \cdots \leq S_n$ *be the order statistics in a random sample* $T_1, \cdots, T_n$. *Then*

$$P\{S_{i_1} \leq s_{i_1}, \cdots, S_{i_k} \leq s_{i_k}\} \geq \prod_{j=1}^{k} P\{S_{i_j} \leq s_{i_j}\}$$

and

$$P\{S_{i_1} > s_{i_1}, \cdots, S_{i_k} > s_{i_j}\} \geq \prod_{j=1}^{k} P\{S_{i_j} > s_{i_j}\}$$

for all choice of $1 \leq i_1 < \cdots < i_k \leq n$ *and* $s_{i_1} < \cdots < s_{i_k}$.

Proof. Since $S_1, \cdots, S_n$ are increasing functions of $T_1, \cdots, T_n$, the result follows from property P_3 and Theorem 1.3.5. ■

Chapter 2

The Model and Basic Properties

Exponential distributions play a central role in life testing, reliability, and other fields of application. Though the assumption of independence can often be used to obtain joint distributions, sometimes such an assumption is questionable or clearly false. Thus, an understanding of multivariate exponential distribution is desirable.

A number of such distributions have been studied in the literature. Marshall and Olkin [1967] proposed a bivariate exponential distribution (BVE)

$$P\{X > s, Y > t\} = e^{-\lambda_1 s - \lambda_2 t - \lambda_{12} \max(s,t)}, \quad \forall s, t > 0. \tag{2.1}$$

This distribution has the following desirable properties:

(a) The marginal distributions are exponentials.

(b)

$$P\{X > s_1 + t, Y > s_2 + t\} = P\{X > s_1, Y > s_2\} P\{X > t, Y > t\}, \tag{2.2}$$

for all $s_1, s_2, t \geq 0$.

The univariate exponential distribution is characterized by

$$\bar{F}(s + t) = \bar{F}(s)\bar{F}(t), \quad \forall s, t \geq 0. \tag{2.3}$$

(2.2) can be thought of as an extension of (2.3). Although BVE has properties (a) and (b), it is not absolutely continuous.

Freund [1961] introduced a bivariate extension of the exponential distribution which is absolutely continuous and the conditional hazard rate functions are constant. It applies, in particular, to two component systems, which can function even if one of the components has failed. In this chapter we study an extension of the Freund distribution.

We first describe a new bivariate shock model and we explicitly derive the joint density function of the associated lifetimes. Having the joint density function, we compute the survival and the moment generating functions. Then some moments of this new density are presented; they are needed for the statistical analysis in Chapter 4. We also compute the marginal density functions and survival functions. In Section 2.2 we study the correlation coefficient between the two lifetimes of this new bivariate density. We explicitly describe the parameters for which the correlation coefficient is positive or negative and we use a graph to summarize the result. We also study the notions of association and of positive quadrant dependent(PQD) between the two lifetimes of this new bivariate density. In Section 2.3 we extend this new model to the multivariate case.

2.1 The Model

In this section we use a shock model to construct a new bivariate distribution function.

Consider a two component model such that each component is subjected to shocks occurring randomly in time as events in two point processes N_1 and N_2 respectively. The shocks are fatal, that is, for $i = 1, 2$, at the time of the first occurrence of an event in the process N_i the component i fails. We suppose that initially N_1 and N_2 behave as independent Poisson processes with rate 1. Let X_1 be the first occurrence of an event in either N_1 or N_2. It follows that X_1 is exponentially distributed with rate 2. Suppose that component i ($i = 1$ or $i = 2$) is killed at time X_1. We assume that if the first shock happens early, $X_1 = x_1 \leq a$, for some fixed $a \geq 0$, then the time X_2 of the second shock (from source $3 - i$) has the property that $X_2 - x_1$ conditioned on $X_1 = x_1 \leq a$ and $X_2 > x_1$ also follows an exponential distribution with failure rate 1. If the first shock happens late, $X_1 = x_1 > a$, then $X_2 - x_1$ conditioned on

$X_1 = x_1 > a$ and $X_2 > x_1$ follows an exponential distribution with rate b which may be different from 1. From the construction, it is seen that the lifetimes of component one and two are not independent. They are associated in the sense that the time of failure of one component will affect the hazard rate of the residual life of the other component. Also, the two components have the same stochastic behavior. That is to say, the joint density and survival functions are symmetric with respect to the line $x_1 = x_2$. Scarsini and Shaked [1999] studied a special case ($a = 1$ and $b = 2$) of this new distribution. See Counterexample 3.3, page 51, of Scarsini and Shaked [1999].

2.1.1 The joint density function

Now let us derive the joint density function of the resulting lifetimes. Let T_1 and T_2 denote the lifetime of component one and two, respectively. Thus the initial hazard rates (see (1.4) in Section 1.1) of both T_1 and T_2 are 1. And under the condition that $T_1 > T_2 = t_2$ the conditional hazard rate of $T_1 - t_2$ is 1 if $t_2 \leq a$ or b if $t_2 > a$ (see (1.5) in Section 1.1). Similarly, the conditional hazard rate of $T_2 - t_1$ (see (1.6) in Section 1.1) under the condition that $T_2 > T_1 = t_1$ is 1 if $t_1 \leq a$ or b if $t_1 > a$. Let λ_1 and $\lambda_{1|2}$ denote the initial hazard rate and the conditional hazard rate of T_1, respectively, and λ_2 and $\lambda_{2|1}$ denote the initial and the conditional hazard rate of T_2, respectively. Then

$$\lambda_1(t) = 1, \qquad \lambda_2(t) = 1, \qquad \forall t; \tag{2.4}$$

$$\lambda_{1|2}(t|t_2) = \begin{cases} 1, & \text{if } t_2 \leq a, \\ b, & \text{if } t_2 > a, \end{cases} \tag{2.5}$$

for all $t > t_2 > 0$; and

$$\lambda_{2|1}(t|t_1) = \begin{cases} 1, & \text{if } t_1 \leq a, \\ b, & \text{if } t_1 > a, \end{cases} \tag{2.6}$$

for all $t > t_1 > 0$.

Let $f(t_1, t_2)$ and $\bar{F}(t_1, t_2)$ denote the joint density and survival functions of T_1 and T_2. We have by Corollary 1.1.2 of Section 1.1 that

$$f(t_1, t_2) = e^{-\int_0^{t_1}(\lambda_1(t)+\lambda_2(t))dt} \cdot \lambda_1(t_1) \cdot e^{-\int_{t_1}^{t_2} \lambda_{2|1}(t|t_1)dt} \cdot \lambda_{2|1}(t_2|t_1) \tag{2.7}$$

for all $t_2 \geq t_1 > 0$, and

$$f(t_1, t_2) = e^{-\int_0^{t_2}(\lambda_1(t)+\lambda_2(t))dt} \cdot \lambda_2(t_2) \cdot e^{-\int_{t_2}^{t_1} \lambda_{1|2}(t|t_2)dt} \cdot \lambda_{1|2}(t_1|t_2) \tag{2.8}$$

for all $t_1 \geq t_2 > 0$.

From (2.4), (2.5), (2.6), (2.7), and (2.8), the joint density function $f(t_1, t_2)$ has the form

$$f(t_1, t_2) = \begin{cases} e^{-t_1-t_2}, & \text{if } t_1 \leq a \quad \text{or} \quad t_2 \leq a; \\ be^{-(2-b)t_2-bt_1}, & \text{if } t_1 \geq t_2 > a; \\ be^{-(2-b)t_1-bt_2}, & \text{if } t_2 > t_1 > a. \end{cases} \tag{2.9}$$

The following figure (Figure 2.1) gives a geometric view of the joint density.

2.1.2 The joint survival function

Through the joint density function $f(t_1, t_2)$ we can compute the survival function $\bar{F}(t_1, t_2)$. For $t_2 \geq t_1 > a$,

$$\begin{aligned} \bar{F}(t_1, t_2) &= \int_{t_1}^{t_2}\int_{t_2}^{\infty} be^{-(2-b)x-by}dydx + \int_{t_2}^{\infty}\int_{x}^{\infty} be^{-(2-b)x-by}dydx \\ &\quad + \int_{t_2}^{\infty}\int_{t_2}^{x} be^{-(2-b)y-bx}dydx \\ &= e^{-bt_2}\left(\frac{e^{-(2-b)t_1} - e^{-(2-b)t_2}}{2-b}\right) + \frac{e^{-2t_2}}{2} + \frac{e^{-2t_2}}{2-b} - \frac{be^{-2t_2}}{2(2-b)} \\ &= \frac{e^{-bt_2-(2-b)t_1}}{2-b} + \frac{e^{-2t_2}}{2} - \frac{be^{-2t_2}}{2(2-b)} \\ &= \frac{e^{-2t_2}(e^{(2-b)(t_2-t_1)} + 1 - b)}{2-b}. \end{aligned}$$

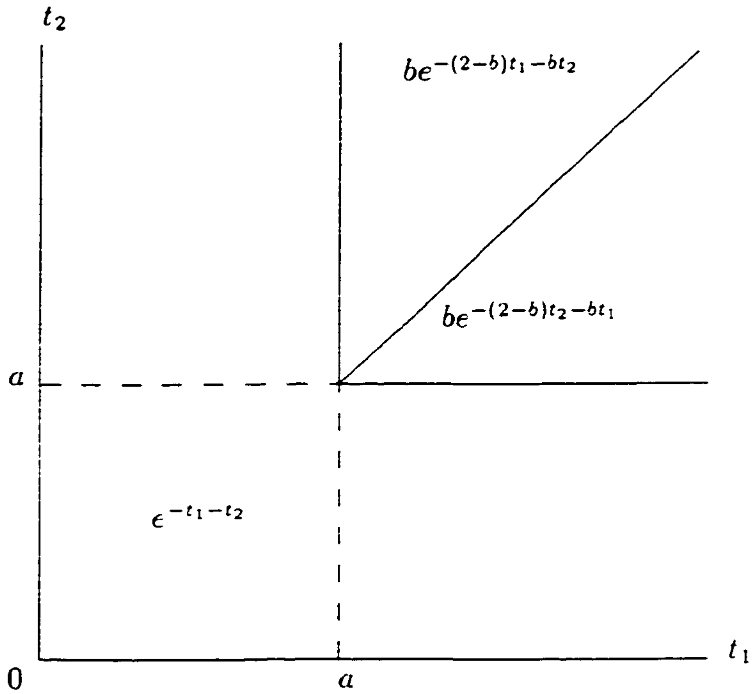

FIGURE 2.1. *The joint density function*

For $t_2 > a \geq t_1$,

$$\begin{aligned}
\bar{F}(t_1, t_2) &= \int_{t_1}^{a}\int_{t_2}^{\infty} e^{-x-y}\,dy\,dx + \int_{a}^{t_2}\int_{t_2}^{\infty} be^{-(2-b)x-by}\,dy\,dx \\
&\quad + \int_{t_2}^{\infty}\int_{x}^{\infty} be^{-(2-b)x-by}\,dy\,dx + \int_{t_2}^{\infty}\int_{t_2}^{x} be^{-(2-b)y-bx}\,dy\,dx \\
&= e^{-t_2}(e^{-t_1} - e^{-a}) + \frac{e^{-bt_2-(2-b)a}}{2-b} - \frac{e^{-2t_2}}{2-b} + \frac{e^{-2t_2}}{2} + \frac{e^{-2t_2}}{2-b} - \frac{be^{-2t_2}}{2(2-b)} \\
&= e^{-t_2}(e^{-t_1} - e^{-a}) + \frac{e^{-2t_2}(e^{(2-b)(t_2-a)} + 1 - b)}{2-b}.
\end{aligned}$$

For $a \geq t_2 > t_1$,

$$\begin{aligned}
\bar{F}(t_1, t_2) &= \int_{t_1}^{a}\int_{t_2}^{\infty} e^{-x-y}\,dy\,dx + \int_{a}^{\infty}\int_{t_2}^{a} e^{-x-y}\,dy\,dx \\
&\quad + \int_{a}^{\infty}\int_{x}^{\infty} be^{-(2-b)x-by}\,dy\,dx + \int_{a}^{\infty}\int_{a}^{x} be^{-(2-b)y-bx}\,dy\,dx
\end{aligned}$$

$$
\begin{aligned}
&= e^{-t_2}(e^{-t_1} - e^{-a}) + e^{-a}(e^{-t_2} - e^{-a}) + \frac{e^{-2a}}{2} + \frac{e^{-2a}}{2-b} - \frac{be^{-2a}}{2(2-b)} \\
&= e^{-t_1-t_2} - e^{-2a} + \frac{e^{-2a}}{2} + \frac{e^{-2a}}{2-b} - \frac{be^{-2a}}{2(2-b)} \\
&= e^{-t_1-t_2}.
\end{aligned}
$$

Therefore the survival function $\bar{F}(t_1, t_2)$ has the form

$$
\bar{F}(t_1, t_2) = \begin{cases}
\frac{e^{-2t_2}(e^{(2-b)(t_2-t_1)}+1-b)}{2-b}, & t_2 > t_1 > a, \\
e^{-t_2}(e^{-t_1} - e^{-a}) + \frac{e^{-2t_2}(e^{(2-b)(t_2-a)}+1-b)}{2-b}, & t_2 > a \geq t_1, \\
e^{-t_1-t_2}, & a \geq t_1, t_2, \\
\frac{e^{-2t_1}(e^{(2-b)(t_1-t_2)}+1-b)}{2-b}, & t_1 \geq t_2 > a, \\
e^{-t_1}(e^{-t_2} - e^{-a}) + \frac{e^{-2t_1}(e^{(2-b)(t_1-a)}+1-b)}{2-b}, & t_1 > a \geq t_2.
\end{cases} \tag{2.10}
$$

Again, we have a figure (Figure 2.2) to represent the survival function.

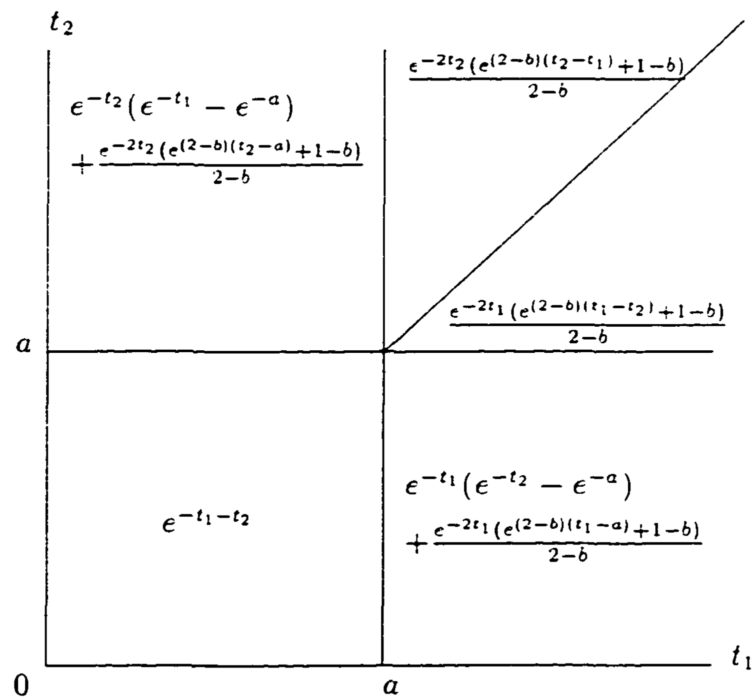

FIGURE 2.2. *The joint survival function*

2.1.3 The moment generating function and some moments

Next we compute the moment generating function $m(s,t)$.

$$\begin{aligned}
m(s,t) &= \int\int e^{sx+ty} f(x,y)\,dy\,dx \\
&= \int_0^a \int_0^\infty e^{sx+ty} e^{-x-y}\,dy\,dx + \int_a^\infty \int_0^a e^{sx+ty} e^{-x-y}\,dy\,dx \\
&+ \int_a^\infty \int_a^x e^{sx+ty} b e^{-(2-b)y-bx}\,dy\,dx + \int_a^\infty \int_x^\infty e^{sx+ty} b e^{-(2-b)x-by}\,dy\,dx \\
&= \frac{1-e^{-(1-s)a}}{(1-s)(1-t)} + \frac{e^{-(1-s)a}(1-e^{-(1-t)a})}{(1-s)(1-t)} + \frac{b e^{-(2-s-t)a}}{(b-t)(2-s-t)} \\
&+ \frac{b e^{-(2-s-t)a}}{(2-b-t)(b-s)} - \frac{b e^{-(2-s-t)a}}{(2-b-t)(2-s-t)} \\
&= \frac{1-e^{-(2-s-t)a}}{(1-s)(1-t)} + \frac{b e^{-(2-s-t)a}}{(b-t)(2-s-t)} + \frac{b e^{-(2-s-t)a}}{(2-b-t)(b-s)} \\
&- \frac{b e^{-(2-s-t)a}}{(2-b-t)(2-s-t)}.
\end{aligned}$$

By differentiating the moment generating function in the above, we obtain

$$\begin{aligned}
ET_1 &= 1 - e^{-2a}(a+1) + \frac{e^{-2a}}{2}(a+\frac{1}{2}) + \frac{e^{-2a}}{2-b}(a+\frac{1}{b}) - \frac{b e^{-2a}}{2(2-b)}(a+\frac{1}{2}) \\
&= 1 + e^{-2a}(\frac{1-b}{2b}); \qquad (2.11)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}ET_1^2 &= 1 - e^{-2a}(\frac{a^2}{2}+a+1) + \frac{e^{-2a}}{2}(\frac{a^2}{2}+\frac{a}{2}+\frac{1}{4}) + \frac{e^{-2a}}{2-b}(\frac{a^2}{2}+\frac{a}{b}+\frac{1}{b^2}) \\
&- \frac{b e^{-2a}}{2(2-b)}(\frac{a^2}{2}+\frac{a}{2}+\frac{1}{4}) \\
&= 1 + \frac{e^{-2a}(1-b)}{2}(\frac{a}{b}+\frac{3b+2}{2b^2}).
\end{aligned}$$

and therefore

$$\begin{aligned}
ET_1^2 &= 2 + e^{-2a}(1-b)(\frac{a}{b}+\frac{3b+2}{2b^2}) \\
&= 2 + e^{-2a}(\frac{1-b}{2b})(2a+3+\frac{2}{b}); \qquad (2.12)
\end{aligned}$$

$$ET_1T_2 = 1 - \epsilon^{-2a}(1 + a + a + 1 + a^2) + \frac{\epsilon^{-2a}}{2}(\frac{1}{2b} + \frac{a}{b} + \frac{1}{2} + \frac{a}{2} + \frac{a}{2} + a^2)$$
$$+ \frac{\epsilon^{-2a}}{2-b}(\frac{1}{b(2-b)} + \frac{a}{2-b} + \frac{a}{b} + a^2)$$
$$- \frac{b\epsilon^{-2a}}{2(2-b)}(\frac{1}{2(2-b)} + \frac{a}{2-b} + \frac{1}{2} + \frac{a}{2} + \frac{a}{2} + a^2)$$
$$= 1 + \epsilon^{-2a}(\frac{1-b}{2b})(2a+1); \qquad (2.13)$$

$$\frac{1}{4}ET_1^2T_2^2 = 1 - \epsilon^{-2a}(1 + \frac{a^2}{2} + a + a + \frac{a^3}{2} + a^2 + \frac{a^2}{2} + \frac{a^4}{4} + \frac{a^3}{2})$$
$$+ \frac{\epsilon^{-2a}}{2}\left(\frac{1}{4b^2} + \frac{a^2}{2b^2} + \frac{a}{2b^2} + \frac{3}{8b} + \frac{a}{4b} + \frac{a}{2b} + \frac{a^2}{2b} + \frac{a^2}{4b} + \frac{a^3}{2b} + \frac{6}{16}\right.$$
$$\left. + \frac{3a}{8} + \frac{3a}{8} + \frac{a^2}{8} + \frac{a^2}{8} + \frac{a^2}{2} + \frac{a^3}{4} + \frac{a^3}{4} + \frac{a^4}{4}\right)$$
$$+ \frac{\epsilon^{-2a}}{2-b}\left(\frac{1}{b^2(2-b)^2} + \frac{a^2}{2(2-b)^2} + \frac{a}{b(2-b)^2} + \frac{a}{b^2(2-b)} + \frac{a^3}{2(2-b)}\right.$$
$$\left. + \frac{a^2}{b(2-b)} + \frac{a^2}{2b^2} + \frac{a^4}{4} + \frac{a^3}{2b}\right)$$
$$- \frac{b\epsilon^{-2a}}{2(2-b)}\left(\frac{1}{4(2-b)^2} + \frac{a^2}{2(2-b)^2} + \frac{a}{2(2-b)^2} + \frac{3}{8(2-b)} + \frac{a}{4(2-b)}\right.$$
$$+ \frac{a}{2(2-b)} + \frac{a^2}{2(2-b)} + \frac{a^2}{4(2-b)} + \frac{a^3}{2(2-b)} + \frac{6}{16}$$
$$\left. + \frac{3a}{8} + \frac{3a}{8} + \frac{a^2}{8} + \frac{a^2}{8} + \frac{a^2}{2} + \frac{a^3}{4} + \frac{a^3}{4} + \frac{a^4}{4}\right)$$
$$= 1 + \epsilon^{-2a}(\frac{1-b}{2b})(a^3 + \frac{a^2(2+5b)}{2b} + \frac{a(2+5b)}{2b} + \frac{2+5b}{4b}).$$

and therefore

$$ET_1^2T_2^2 = 4 + \epsilon^{-2a}(\frac{1-b}{b})(2a^3 + \frac{a^2(2+5b)}{b} + \frac{a(2+5b)}{b} + \frac{2+5b}{2b}); \qquad (2.14)$$

$$
\begin{aligned}
\frac{1}{2}ET_1^2T_2 &= 1 - e^{-2a}(1 + a + a + a^2 + \frac{a^2}{2} + \frac{a^3}{2} \\
&+ \frac{e^{-2a}}{2}(\frac{1}{4b} + \frac{a}{4} + \frac{3}{8} + \frac{a}{2b} + \frac{a^2}{2} + \frac{a}{2} + \frac{a^2}{2b} + \frac{a^2}{4} + \frac{a^3}{2}) \\
&+ \frac{e^{-2a}}{2-b}(\frac{1}{b^2(2-b)} + \frac{a}{b^2} + \frac{a}{b(2-b)} + \frac{a^2}{b} + \frac{a^2}{2(2-b)} + \frac{a^3}{2}) \\
&- \frac{be^{-2a}}{2(2-b)}(\frac{1}{4(2-b)} + \frac{a}{4} + \frac{3}{8} + \frac{a}{2} + \frac{a}{2(2-b)} + \frac{a^2}{2} + \frac{a^2}{2(2-b)} + \frac{a^2}{4} + \frac{a^3}{2}) \\
&= 1 + e^{-2a}(\frac{3a^2(1-b)}{4b} + \frac{a(1-b)(2+5b)}{4b^2} + \frac{(1-b)(2+5b)}{8b^2}).
\end{aligned}
$$

and therefore

$$
ET_1^2T_2 = 2 + e^{-2a}(\frac{1-b}{2b})(3a^2 + \frac{a(2+5b)}{b} + \frac{2+5b}{2b}). \tag{2.15}
$$

2.1.4 The marginal density functions and survival functions

Next, we compute the marginal density functions and survival functions. Let f_1 and $\bar{F}_1$ denote the marginal density and survival functions of component one, respectively. From (2.9), we have

$$
f_1(t) = \int_0^\infty e^{-t-t_2}dt_2 = e^{-t}
$$

for $t \leq a$ and

$$
\begin{aligned}
f_1(t) &= \int_0^a e^{-t-t_2}dt_2 + \int_a^t be^{-(2-b)t_2-bt}dt_2 + \int_t^\infty be^{-(2-b)t-bt_2}dt_2 \\
&= e^{-t}(1-e^{-a}) + \frac{e^{-2t}}{2-b}(e^{(2-b)(t-a)} - 1) + e^{-2t},
\end{aligned}
$$

for $t > a$. Therefore

$$
f_1(t) = \begin{cases} e^{-t}, & \text{if } t \leq a; \\ e^{-t}(1-e^{-a}) + \frac{e^{-2t}}{2-b}(e^{(2-b)(t-a)} - 1) + e^{-2t}, & \text{if } t > a. \end{cases} \tag{2.16}
$$

Let $t_2 = 0$ in (2.10). We obtain

$$
\bar{F}_1(t) = \begin{cases} e^{-t}, & \text{if } t \leq a, \\ e^{-t}(1-e^{-a}) + \frac{e^{-2t}}{2-b}(e^{(2-b)(t-a)} + 1 - b), & \text{if } t > a. \end{cases} \tag{2.17}
$$

Let f_2 and $\bar{F}_2$ denote the marginal density and survival functions of component two, respectively. Then $f_2 \equiv f_1$ and $\bar{F}_2 \equiv \bar{F}_1$.

2.2 Correlation Coefficient

In this section we study the correlation between T_1 and T_2. By applying the results from Section 2.1, we express the covariance of T_1 and T_2 as a function of a and b. We then derive the conditions on a and b under which T_1 and T_2 are positive correlated (respectively, negative correlated). We also make a graph to illustrate these conditions. This section ends with the study of the notions of association and positively quadrant dependence between T_1 and T_2.

Definition 2.2.1. Let X and Y be two random variables defined on the same probability space $(\Omega, \mathfrak{F}, P)$. The covariance of X and Y is defined as follows

$$Cov(X, Y) = EXY - EXEY,$$

where E denotes the expectation. If $Cov(X, Y) > 0$ then X and Y are said to be **positive correlated.** If $Cov(X, Y) < 0$ then X and Y are said to be **negative correlated.**

Theorem 2.2.2.

$$Cov(T_1, T_2) > 0 \quad if \quad \begin{cases} \frac{1}{2} < a, \ \frac{1}{2(2a-1)e^{2a}+1} < b < 1, \\ a_0 < a < \frac{1}{2}, \ 1 < b < \frac{1}{2(2a-1)e^{2a}+1}, \\ 0 < a \leq a_0, \ 1 < b, \end{cases} \tag{2.18}$$

where a_0 satisfies the equation $2(2a-1)e^{2a} + 1 = 0$. Also

$$Cov(T_1, T_2) < 0 \quad if \quad \begin{cases} 0 < a \leq a_0, \ b < 1, \\ a_0 < a, \ b < \min\{1, \frac{1}{2(2a-1)e^{2a}+1}\}, \\ a_0 < a, \ b > \max\{1, \frac{1}{2(2a-1)e^{2a}+1}\}. \end{cases} \tag{2.19}$$

Proof. From Section 2.1, T_1 and T_2 are identically distributed. Then

$$Cov(T_1, T_2) = ET_1T_2 - E^2T_1.$$

Using (2.13) and (2.11). we have

$$\begin{aligned} Cov(T_1.T_2) &= ET_1T_2 - E^2T_1 \\ &= \left[1+e^{-2a}\left(\frac{1-b}{2b}\right)(2a+1)\right] - \left[1+e^{-2a}\left(\frac{1-b}{2b}\right)\right]^2 \\ &= e^{-2a}\left[\frac{(1-b)(2a-1)}{2b} - \frac{(1-b)^2e^{-2a}}{(2b)^2}\right]. \end{aligned} \tag{2.20}$$

Then $Cov(T_1.T_2) > 0$ if and only if

$$\frac{(1-b)(2a-1)}{2b} - \frac{(1-b)^2e^{-2a}}{(2b)^2} > 0.$$

Case1 : $\frac{1}{2} < a.\ 0 < b < 1$. Then

$$\begin{aligned} Cov(T_1.T_2) > 0 &\iff \frac{(1-b)(2a-1)}{2b} - \frac{(1-b)^2e^{-2a}}{(2b)^2} > 0 \\ &\iff 2a-1 > \frac{(1-b)e^{-2a}}{2b} \\ &\iff b > \frac{1}{2(2a-1)e^{2a}+1}. \end{aligned}$$

Therefore. $Cov(T_1.T_2) > 0$ if $\frac{1}{2} < a$ and $\frac{1}{2(2a-1)e^{2a}+1} < b < 1$.

Case2 : $a < \frac{1}{2}.\ 1 < b$. Then

$$\begin{aligned} Cov(T_1.T_2) > 0 &\iff \frac{(1-b)(2a-1)}{2b} - \frac{(1-b)^2e^{-2a}}{(2b)^2} > 0 \\ &\iff 1-2a > \frac{(b-1)e^{-2a}}{2b} \\ &\iff \frac{1}{b} > 2(2a-1)e^{2a}+1. \end{aligned}$$

Let a_0 be such that $2(2a_0-1)e^{2a_0}+1 = 0$. Then $Cov(T_1.T_2) > 0$ if $0 < a \le a_0.\ 1 < b$ or $a_0 < a < \frac{1}{2}.\ 1 < b < \frac{1}{2(2a-1)e^{2a}+1}$.

Combining these two cases, we have

$$Cov(T_1.T_2) > 0 \quad \text{if} \quad \begin{cases} \frac{1}{2} < a.\ \frac{1}{2(2a-1)e^{2a}+1} < b < 1. \\ a_0 < a < \frac{1}{2}.\ 1 < b < \frac{1}{2(2a-1)e^{2a}+1}. \\ 0 < a \le a_0.\ 1 < b. \end{cases}$$

This proves (2.18).

From (2.20), we have $Cov(T_1, T_2) < 0$ if and only if

$$\frac{(1-b)(2a-1)}{2b} - \frac{(1-b)^2 e^{-2a}}{(2b)^2} < 0.$$

Case1 : $a < \frac{1}{2}$, $b < 1$. Then $Cov(T_1, T_2) < 0$.

Case2 : $\frac{1}{2} < a$, $1 < b$. Then $Cov(T_1, T_2) < 0$.

Case3 : $\frac{1}{2} < a$, $0 < b < 1$. Then

$$\begin{aligned} Cov(T_1, T_2) < 0 \quad &\Longleftrightarrow \quad \frac{(1-b)(2a-1)}{2b} - \frac{(1-b)^2 e^{-2a}}{(2b)^2} < 0 \\ &\Longleftrightarrow \quad b < \frac{1}{2(2a-1)e^{2a}+1}. \end{aligned}$$

Case4 : $a < \frac{1}{2}$, $1 < b$. Then

$$\begin{aligned} Cov(T_1, T_2) < 0 \quad &\Longleftrightarrow \quad \frac{(1-b)(2a-1)}{2b} - \frac{(1-b)^2 e^{-2a}}{(2b)^2} < 0 \\ &\Longleftrightarrow \quad b > \frac{1}{2(2a-1)e^{2a}+1}. \end{aligned}$$

Combining these,

$$Cov(T_1, T_2) < 0 \quad \text{if} \quad \begin{cases} 0 < a \le a_0, \ b < 1, \\ a_0 < a, \ b < \min\{1, \frac{1}{2(2a-1)e^{2a}+1}\}, \\ a_0 < a, \ b > \max\{1, \frac{1}{2(2a-1)e^{2a}+1}\}. \end{cases}$$

This completes the proof. ■

The following graph (Figure 2.3) illustrates these conditions.

From above, we have that $Cov(T_1, T_2) > 0$ if the point (a, b) locates in an appropriate region. The property $Cov(T_1, T_2) > 0$ is very close to the concept of association (see Section 1.3). This leads to the following question: are T_1 and T_2 associated? From Theorem 1.3.4, if T_1 and T_2 are not positively quadrant dependent, then they are not associated. We apply (1.10) to answer this question.

Claim: T_1 and T_2 are not *PQD* for any a and b.

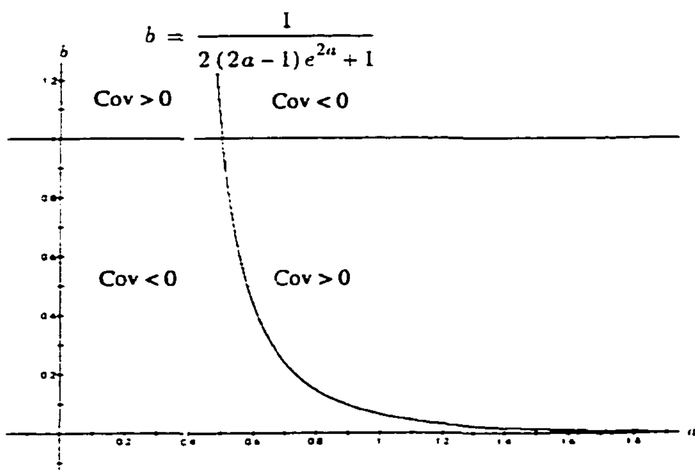

FIGURE 2.3. *Regions for positive and negative correlated*

Proof. From (2.10) and (2.17), we have

$$\begin{aligned}
&\bar{F}(t_1,t_2) \geq 0 \\
&\iff e^{-t_2}(e^{-t_1} - e^{-a}) + \frac{e^{-2t_2}}{2-b}(e^{(2-b)(t_2-a)} + 1 - b) \\
&\quad - e^{-t_1}\left(e^{-t_2}(1-e^{-a}) + \frac{e^{-2t_2}}{2-b}(e^{(2-b)(t_2-a)} + 1 - b)\right) \geq 0 \\
&\iff e^{-2t_2}(1-e^{-t_1})\left(\frac{e^{(2-b)(t_2-a)} + 1 - b}{2-b} - e^{(t_2-a)}\right) \geq 0 \\
&\iff \frac{e^{(2-b)(t_2-a)} + 1 - b}{2-b} - e^{(t_2-a)} \geq 0
\end{aligned} \tag{2.21}$$

for all $t_2 > a \geq t_1$. Also

$$\begin{aligned}
&\bar{F}(t_1,t_2) \geq 0 \\
&\iff e^{-2t_2} - \left(e^{-t_2}(1-e^{-a}) + \frac{e^{-2t_2}}{2-b}(e^{(2-b)(t_2-a)} + 1 - b)\right)^2 \geq 0 \\
&\iff e^{-2t_2}\left(e^{-t_2}(2-e^{-a}) + \frac{e^{-2t_2}}{2-b}(e^{(2-b)(t_2-a)} + 1 - b)\right) \\
&\quad \times \left(e^{(t_2-a)} - \frac{e^{(2-b)(t_2-a)} + 1 - b}{2-b}\right) \geq 0
\end{aligned}$$

$$\Longleftrightarrow e^{(t_2-a)} - \frac{e^{(2-b)(t_2-a)} + 1 - b}{2-b} \geq 0 \tag{2.22}$$

for all $t_2 = t_1 > a$. By comparing (2.21) and (2.22), we have that T_1 and T_2 are not PQD. ■

Hence T_1 and T_2 are not associated.

2.3 Generalization

To fix ideas, consider first an extension of the model of Section 2.1 to a three component model.

Let N_1, N_2, and N_3 be three point processes that govern the occurrence of shocks fatal to component one, two, and three, respectively. Assume that at first they behave as independent Poisson processes with rate 1. Let X_1 denote the first occurrence of an event among N_1, N_2, and N_3. Then X_1 is exponentially distributed with rate 3. Without loss of generality, assume that component one is killed at time X_1. Suppose that after time X_1, N_2 and N_3 still behave as independent Poisson processes but the rate may be different from the initial one. Explicitly, if $X_1 = x_1 \leq a_1$, for some fixed $a_1 \geq 0$, then N_2 and N_3 both have failure rate 1. If $X_1 = x_1 > a_1$ then they have failure rate b_1 which may be different from 1. Let X_2 denote the time of second shock (from N_2 or N_3). Assume that then $X_2 - x_1$ conditioned on $X_1 = x_1 \leq a_1$ and $X_2 > x_1$ follows an exponential distribution with rate 2 and $X_2 - x_1$ conditioned on $X_1 = x_1 > a_1$ and $X_2 > x_1$ is exponentially distributed with rate $2b_1$. Without loss of generality, let us assume that component two is killed at time X_2. We further assume that after time X_2, N_3 still behaves as a Poisson process. The rate of N_3 could be 1, b_1, or b_2; b_2 may be different from 1 and b_1. Explicitly, we assume that if the rate after the first shock is 1 then the rate after second shock is still 1 if $X_2 = x_2 \leq a_2$, for some fixed $a_2 \geq a_1$, or change to b_1 if $X_2 = x_2 > a_2$. While if the rate after the first shock is b_1 then the above rate stays at b_1 or become b_2 depending on whether

$X_2 = x_2 \leq a_2$ or $X_2 = x_2 > a_2$. Let X_3 denote the time of third shock (from $\mathcal{N}_3$). From above, X_3 is exponentially distributed with rate 1, b_1, or b_2 depending on the rate of $\mathcal{N}_3$ being 1, b_1, or b_2. By time X_3 all three components have failed.

From the construction above, component one, two, and three behave stochastically the same. They all have initial hazard rate being 1, conditional hazard rate being 1 or b_1, and second order conditional hazard rate being 1, b_1, or b_2. Let λ_i, $\lambda_{i|j}$, and $\lambda_{i|j,k}$ denote the initial hazard rate, the conditional hazard rate, and the second order conditional hazard rate, respectively. Refer to (1.3) in Section 1.1 for the corresponding definitions. Let I_A denote the indicator function of set A, that is, $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$. Then

$$\lambda_i \equiv 1,$$

for all $i = 1, 2, 3$,

$$\lambda_{i|j}(t|t_1) = \begin{cases} 1, & t_1 \leq a_1; \\ b_1, & t_1 > a_1, \end{cases}$$

for all $t > t_1 \geq 0$ and $1 \leq i \neq j \leq 3$, and

$$\lambda_{i|j,k}(t|t_1, t_2) = \begin{cases} 1, & \sum_{m=1}^{2} I_{(a_m,\infty)}(t_m) = 0; \\ b_1, & \sum_{m=1}^{2} I_{(a_m,\infty)}(t_m) = 1; \\ b_2, & \sum_{m=1}^{2} I_{(a_m,\infty)}(t_m) = 2, \end{cases}$$

for all $t > t_2 > t_1 \geq 0$ and $\{i, j, k\} = \{1, 2, 3\}$.

Let T_1, T_2, and T_3 denote the life length of component one, two, and three, respectively. The joint density for T_1, T_2, and T_3 can be computed through the initial hazard rates, conditional hazard rates, and second order conditional hazard rates (see Theorem 1.1.3 of Section 1.1). For $t_1 \leq t_2 \leq t_3$, we have

$$f(t_1, t_2, t_3) = \begin{cases} e^{-t_1-t_2-t_3}, & \text{if } t_1 \leq a_1,\ t_2 \leq a_2, \\ b_1 e^{-t_1-(2-b_1)t_2-b_1 t_3}, & \text{if } t_1 \leq a_1,\ t_2 > a_2, \\ b_1^2 e^{-(3-2b_1)t_1-b_1 t_2-b_1 t_3}, & \text{if } a_1 < t_1 \leq a_2,\ t_2 \leq a_2, \\ b_1 b_2 e^{-(3-2b_1)t_1-(2b_1-b_2)t_2-b_2 t_3}, & \text{if } a_1 < t_1 \leq a_2,\ t_2 < a_2, \\ & \quad \text{or } t_1 > a_2. \end{cases} \tag{2.23}$$

Because the three marginals are identically distributed, the joint density can be defined symmetrically for all the other orders of t_1, t_2, and t_3. Next we can compute the marginal density from the joint density. Let f_1 denote the marginal density. By simple calculation, we have

$$f_1(t) = \epsilon^{-t}$$

for $t \leq a_1$,

$$f_1(t) = \epsilon^{-t}(1 - \epsilon^{-2a_1}) + \frac{2b\epsilon^{-bt-(3-b)a_1} + 3(1-b)\epsilon^{-3t}}{3-b}$$

for $a_1 < t < a_2$, and

$$\begin{aligned} f_1(t) &= \left(\epsilon^{-t}(1 + \epsilon^{-a_1}) - 2\epsilon^{-t-a_2} + \frac{2b\epsilon^{-bt-(2-b)a_2}}{2-b}\right)(1 - \epsilon^{-a_1}) \\ &+ \frac{4(1-b)\epsilon^{-2t}(1-\epsilon^{-a_1})}{2-b} + \frac{2b\epsilon^{-bt}(\epsilon^{-(3-b)a_1} - \epsilon^{-(3-b)a_2})}{3-b} \\ &+ 2[\frac{bc(\epsilon^{-ct-(2b-c)a_2} - \epsilon^{-2bt})}{2b-c} + b\epsilon^{-2bt} - b\epsilon^{-bt-ba_2}] \cdot \frac{\epsilon^{-(3-2b)a_1} - \epsilon^{-(3-2b)a_2}}{3-2b} \\ &+ \epsilon^{-3t} + \frac{2bc(\epsilon^{-ct-(3-c)a_2} - \epsilon^{-3t})}{(2b-c)(3-c)} + \frac{4b(b-c)(\epsilon^{-2bt-(3-2b)a_2} - \epsilon^{-3t})}{(2b-c)(3-2b)} \end{aligned}$$

for $a_2 < t$.

In a similar fashion, we can describe the general case. Consider an n component system such that the initial hazard rate of each component is 1. We assume that after the time of first failure, the conditional hazard rates of the other alive components are 1 or b_1. If the first failure happens early, before a_1, then they remain one. If the first failure happens late, after a_1, then they become b_1. The conditional hazard rates stay unchanged between failures. At the time of next failure, if it happens early, before a_2, then the second order conditional hazard rates of the other alive components stay the same. If it happens late, after a_2, then the rates change to next b_i. So, every time there is a failure which happens early, before a_i, then the conditional hazard rates remain the same. If it happens late, after a_i, then the rates change. We call $(a_1, a_2, \cdots, a_{n-1})$ the time parameter. Let t_i denote the time of the ith failure among the n components. Then the conditional hazard rate after time t_i will change if and only if $t_i > a_i$. Let λ_i, $i = 1, \cdots, n$, $\lambda_{i|j}$, $1 \le i \ne j \le n$, and $\lambda_{i|j_1,\cdots,j_k}$, $1 \le i \ne j_1 \ne \cdots \ne j_k \le n$, denote the initial hazard rate, the conditional hazard rate, and the k-th order conditional hazard rate, respectively. Formally, we can define the hazard rates as

$$\lambda_i \equiv 1, \quad \forall\, i = 1, \cdots, n,$$

$$\lambda_{i|j}(t|t_j) = \begin{cases} 1 & \text{if } t_j \le a_1, \\ b_1 & \text{if } t_j > a_1, \end{cases}$$

for all $t > t_j$, $1 \le i \ne j \le n$, and

$$\lambda_{i|j_1,\cdots,j_k}(t|t_1,\cdots,t_k) = \begin{cases} 1 & \text{if } \sum_{m=1}^{k} I_{(a_m,\infty)}(t_m) = 0, \\ b_1 & \text{if } \sum_{m=1}^{k} I_{(a_m,\infty)}(t_m) = 1, \\ \vdots & \\ b_i & \text{if } \sum_{m=1}^{k} I_{(a_m,\infty)}(t_m) = i, \\ \vdots & \\ b_k & \text{if } \sum_{m=1}^{k} I_{(a_m,\infty)}(t_m) = k, \end{cases}$$

for all $2 \le k \le n-1$, $t_1 < \cdots < t_k < t$ and $1 \le i \ne j_1 \ne \cdots \ne j_k \le n$.

Given all the hazard rates as the above, we can derive the joint density now. Let $T_1, T_2, \cdots, T_n$ denote the life length of component one, two, to n, respectively, and $t_i, i = 1, 2, \cdots, n$, denote the time at which the i-th component fails. From the construction, this new model has a density function which is invariant under any permutation of all the components. Under this observation, we can compute $f(t_1, t_2, \cdots, t_n)$, the joint density function, with $t_1 \le t_2 \le \cdots \le t_n$. For all the other orders of $t_1, t_2, \cdots, t_n$, it can be defined symmetrically. The joint density can be computed through the following formula (see Corollary 1.1.4 of Section 1.1):

$$\begin{aligned} & f(t_1, t_2, \cdots, t_n) \\ & = e^{-n\int_0^{t_1} \lambda_1(t)dt} \lambda_1(t_1) \\ & \quad \times \Pi_{m=2}^{n} e^{-(n-m+1)\int_{t_{m-1}}^{t_m} \lambda_{m|1,\cdots,m-1}(t|t_1,\cdots,t_{m-1})dt} \lambda_{m|1,\cdots,m-1}(t_m|t_1,\cdots,t_{m-1}) \end{aligned}$$

$$= e^{-nt_1} \prod_{m=2}^{n} e^{-(n-m+1)\lambda_{m|1,\cdots,m-1}(t_m|t_1,\cdots,t_{m-1})(t_m-t_{m-1})} \lambda_{m|1,\cdots,m-1}(t_m|t_1,\cdots,t_{m-1}). \tag{2.24}$$

Since the joint density function is invariant under any permutation of $t_1, \cdots, t_n$, the marginal density function $f_1(t_1)$ can be computed by

$$\begin{aligned} f_1(t_1) &= \int_{t_2=0}^{\infty} \int_{t_3=0}^{\infty} \cdots \int_{t_n=0}^{\infty} f(t_1, t_2, \cdots, t_n) dt_n \cdots dt_3 dt_2 \\ &= (n-1)! \int_{t_2=0}^{\infty} \int_{t_3=t_2}^{\infty} \cdots \int_{t_n=t_{n-1}}^{\infty} f(t_1, t_2, \cdots, t_n) dt_n \cdots dt_3 dt_2. \end{aligned} \tag{2.25}$$

Chapter 3
AGING PROPERTIES

In reliability theory, we often encounter the question: does the distribution or system possess any aging property such as IFR, IFRA? Or any anti-aging property such as DFR, DFRA? In this chapter we identify situations in which the new multivariate distribution described in previous chapter possesses some aging properties. In the first section we show that the marginal density is DFR if the conditional hazard rates are decreasing and bounded above by 1. To prove this we show that the marginal density is a mixture of DFR distributions. Then the result follows from the closure property of DFR distributions. In Section 3.2 we focus on an $(n-1)$-out-of-n system such that the joint density function of all component lifetime is the new one described in previous chapter. We find a necessary and sufficient condition under which the life time of the system is IFRA.

3.1 DFR of marginal density

Recall that an absolutely continuous distribution function F of a nonnegative random variable X is called a *decreasing failure rate* (DFR) distribution if the hazard rate function of X is a decreasing function. The main result of this section is the following theorem:

Theorem 3.1.1. *Let f_1 denote the marginal density function of the new n dimensional distribution constructed in Section 2.3. Then f_1 is DFR if* $0 < b_{n-1} \leq b_{n-2} \leq \cdots \leq b_1 \leq 1$.

The proof of this theorem is based on Theorem 1.2.11 of Section 1.2. It says that the class of DFR distributions is closed under mixture. Therefore it suffices to show that f_1 is a mixture of some DFR distributions. We study the bivariate system, the three component system, and the general case separately. In each case, we show that the marginal distribution or density function is a mixture of some DFR distributions.

Proof of Theorem 3.1.1 in the bivariate case.

In this case we have $0 < b \leq 1$. Define

$$g(\alpha) = e^{-\alpha}, \quad \forall \alpha \geq 0.$$

Also define

$$\bar{H}_\alpha(t) = e^{-t}, \quad \forall t \geq 0,$$

if $0 < \alpha \leq a$, and

$$\bar{H}_\alpha(t) = \begin{cases} e^{-t}, & t \leq \alpha; \\ e^{-\alpha - b(t-\alpha)}, & t > \alpha, \end{cases}$$

when $\alpha > a$. Clearly, g is the density function of an exponential distribution with rate 1 and the hazard rate λ_α of H_α has the form that

$$\lambda_\alpha \equiv 1$$

if $0 < \alpha \leq a$, and

$$\lambda_\alpha(t) = \begin{cases} 1, & t \leq \alpha; \\ b, & t > \alpha, \end{cases}$$

when $\alpha > a$. Therefore H_α is DFR for all $\alpha \geq 0$ since $b \leq 1$.

It suffices to prove the following equation

$$\bar{F}_1(t) = \int_0^\infty \bar{H}_\alpha(t) g(\alpha) d\alpha$$

for all $t \geq 0$. For $t \leq a$ we have

$$\begin{aligned} \int_0^\infty \bar{H}_\alpha(t) g(\alpha) d\alpha &= \int_0^a \bar{H}_\alpha(t) g(\alpha) d\alpha + \int_a^\infty \bar{H}_\alpha(t) g(\alpha) d\alpha \\ &= \int_0^a e^{-t} e^{-\alpha} d\alpha + \int_a^\infty e^{-t} e^{-\alpha} d\alpha \\ &= e^{-t} \\ &= \bar{F}_1(t). \end{aligned}$$

For $t > a$ we have

$$\begin{aligned}\int_0^\infty \bar{H}_\alpha(t)g(\alpha)d\alpha &= \int_0^a \bar{H}_\alpha(t)g(\alpha)d\alpha + \int_a^t \bar{H}_\alpha(t)g(\alpha)d\alpha + \int_t^\infty \bar{H}_\alpha(t)g(\alpha)d\alpha \\ &= \int_0^a e^{-t}e^{-\alpha}d\alpha + \int_a^t e^{-\alpha-b(t-\alpha)}e^{-\alpha}d\alpha + \int_t^\infty e^{-t}e^{-\alpha}d\alpha \\ &= e^{-t}(1-e^{-a}) + \frac{e^{-bt-(2-b)a} - e^{-2t}}{2-b} + e^{-2t} \\ &= \bar{F}_1(t).\end{aligned}$$

This completes the proof. ■

Proof of Theorem 3.1.1 in the 3 dimensional case.

In this case we have $0 < b_2 \leq b_1 \leq 1$. Let $g(\alpha, \beta)$ denote the joint density function of the new bivariate distribution as in Section 2.1. That is,

$$g(\alpha,\beta) = \begin{cases} e^{-\alpha-\beta}, & \alpha \leq a \text{ or } \beta \leq a; \\ be^{-(2-b)\alpha-b\beta}, & \beta \geq \alpha > a; \\ be^{-(2-b)\beta-b\alpha}, & \alpha > \beta > a. \end{cases}$$

Define

$$h_{\alpha,\beta}(t) = e^{-t}, \quad \forall t \geq 0,$$

for $\alpha \leq \beta \leq a_1$ or $\alpha \leq a_1 < \beta \leq a_2$, and

$$h_{\alpha,\beta}(t) = \begin{cases} e^{-t}, & \text{if } t \leq \beta; \\ b_1 e^{-\beta-b_1(t-\beta)}, & \text{if } t > \beta, \end{cases}$$

for $\alpha \leq a_1 \leq a_2 < \beta$, and

$$h_{\alpha,\beta}(t) = \begin{cases} e^{-t}, & \text{if } t \leq \alpha; \\ b_1 e^{-\alpha-b_1(t-\alpha)}, & \text{if } t > \alpha, \end{cases}$$

for $a_1 < \alpha \leq \beta \leq a_2$, and

$$h_{\alpha,\beta}(t) = \begin{cases} e^{-t}, & \text{if } t \leq \alpha; \\ b_1 e^{-\alpha-b_1(t-\alpha)}, & \text{if } \alpha < t \leq \beta; \\ b_2 e^{-\alpha-b_1(\beta-\alpha)-b_2(t-\beta)}, & \text{if } \beta < t. \end{cases}$$

for $a_1 < \alpha \le a_2 < \beta$ or $a_2 < \alpha \le \beta$. For $\beta < \alpha$, we can define $h_{\alpha,\beta}$ symmetrically. Let $\lambda_{\alpha,\beta}$ be the hazard rate of $h_{\alpha,\beta}$. By simple calculation, we have

$$\lambda_{\alpha,\beta} \equiv 1$$

for $\alpha \le \beta \le a_1$ or $\alpha \le a_1 < \beta \le a_2$, and

$$\lambda_{\alpha,\beta}(t) = \begin{cases} 1, & t \le \beta; \\ b_1, & t > \beta, \end{cases}$$

for $\alpha \le a_1 \le a_2 < \beta$, and

$$\lambda_{\alpha,\beta}(t) = \begin{cases} 1, & t \le \alpha; \\ b_1, & t > \alpha, \end{cases}$$

for $a_1 < \alpha \le \beta \le a_2$, and

$$\lambda_{\alpha,\beta(t)} = \begin{cases} 1, & t \le \alpha; \\ b_1, & \alpha < t \le \beta; \\ b_2, & \beta < t, \end{cases}$$

for $a_1 < \alpha \le a_2 < \beta$ or $a_2 < \alpha \le \beta$. Therefore $h_{\alpha,\beta}$ is DFR for all $\alpha \ge 0$ and $\beta \ge 0$ since $0 < b_2 \le b_1 \le 1$. Let f_1 denote the marginal density function of the three component system.

Again, it suffices to show that

$$f_1(t) = \int_0^\infty \int_0^\infty h_{\alpha,\beta}(t) g(\alpha,\beta) d\beta d\alpha$$

for all $t \ge 0$. The idea of this proof is simply to perform the integration. Since $h_{\alpha,\beta}$ is defined symmetrically for $\beta < \alpha$, we only compute the integration over the region that $\beta \ge \alpha$ and then multiply it by 2. For $t \le a_1$ we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty h_{\alpha,\beta}(t) g(\alpha,\beta) d\beta d\alpha \\ &\quad = 2\left(\int_0^\infty \int_\alpha^\infty h_{\alpha,\beta}(t) g(\alpha,\beta) d\beta d\alpha \right) \end{aligned}$$

$$= 2\left(\int_0^{a_1}\int_\alpha^\infty h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha + \int_{a_1}^\infty\int_\alpha^\infty h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha\right)$$
$$= e^{-t}$$
$$= f_1(t).$$

For $a_1 < t \leq a_2$ we have

$$\int_0^\infty\int_0^\infty h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha$$
$$= 2\left(\int_0^\infty\int_\alpha^\infty h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha\right)$$
$$= 2\left(\int_0^{a_1}\int_\alpha^\infty h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha + \int_{a_1}^t\int_\alpha^\infty h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha\right.$$
$$\left. + \int_t^\infty\int_\alpha^\infty h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha\right)$$
$$= e^{-t}(1 - e^{-2a_1}) + \frac{2b_1}{3-b_1}e^{-b_1t-(3-b_1)a_1} + \frac{3(1-b_1)}{3-b_1}e^{-3t}$$
$$= f_1(t).$$

For $a_2 < t$ we have

$$\int_0^\infty\int_0^\infty h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha = 2\left(\int_0^\infty\int_\alpha^\infty h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha\right)$$
$$= 2\left(\int\int_{R_1} + \int\int_{R_2} + \int\int_{R_3} h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha\right).$$

where R_1, R_2, and R_3 are subsets of $\{(\alpha,\beta)|\alpha \leq \beta\}$ such that

$$h_{\alpha,\beta}(t) = e^{-t} \quad \forall(\alpha,\beta) \in R_1.$$

$$h_{\alpha,\beta}(t) = b_1e^{-\alpha-b_1(t-\alpha)} \quad \text{or} \quad b_1e^{-\beta-b_1(t-\beta)} \quad \forall(\alpha,\beta) \in R_2.$$

and

$$h_{\alpha,\beta}(t) = b_2e^{-\alpha-b_1(\beta-\alpha)-b_2(t-\beta)} \quad \forall(\alpha,\beta) \in R_3.$$

Then

$$
\begin{aligned}
\int\int_{R_1} & h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha \\
&= \int_0^{a_1}\int_\alpha^{a_2} h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha + \int_0^{a_1}\int_t^{\infty} h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha \\
&\quad + \int_t^{\infty}\int_\alpha^{\infty} h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha \\
&= \epsilon^{-t}\Big(\frac{1-\epsilon^{-2a_1}}{2}\Big) - \epsilon^{-t-a_2}(1-\epsilon^{-a_1}) + \epsilon^{-2t}(1-\epsilon^{-a_1}) + \frac{\epsilon^{-3t}}{2}.
\end{aligned}
$$

$$
\begin{aligned}
\int\int_{R_2} & h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha \\
&= \int_0^{a_1}\int_{a_2}^{t} h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha + \int_{a_1}^{a_2}\int_\alpha^{a_2} h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha \\
&\quad + \int_{a_1}^{t}\int_t^{\infty} h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha \\
&= \frac{b_1(\epsilon^{-b_1t-(2-b_1)a_2} - \epsilon^{-2t})(1-\epsilon^{-a_1})}{2-b_1} + \frac{b_1(\epsilon^{-b_1t-(3-b_1)a_1} - \epsilon^{-b_1t-(3-b_1)a_2})}{3-b_1} \\
&\quad - \frac{b_1(\epsilon^{-b_1t-b_1a_2-(3-2b_1)a_1} - \epsilon^{-b_1t-(3-b_1)a_2})}{3-2b_1} + \frac{b_1(\epsilon^{-2b_1t-(3-2b_1)a_1} - \epsilon^{-3t})}{3-2b_1}.
\end{aligned}
$$

and

$$
\begin{aligned}
\int\int_{R_3} & h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha \\
&= \int_{a_1}^{a_2}\int_{a_2}^{t} h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha + \int_{a_2}^{t}\int_\alpha^{t} h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha \\
&= b_1b_2\epsilon^{-b_2t}\Big(\frac{\epsilon^{-(2b_1-b_2)a_2} - \epsilon^{-(2b_1-b_2)t}}{2b_1-b_2}\Big)\Big(\frac{\epsilon^{-(3-2b_1)a_1} - \epsilon^{-(3-2b_1)a_2}}{3-2b_1}\Big) \\
&\quad + \frac{b_1b_2\epsilon^{-b_2t}}{2b_1-b_2}\Big(\frac{\epsilon^{-(3-b_2)a_2} - \epsilon^{-(3-b_2)t}}{3-b_2}\Big) - \frac{b_1b_2\epsilon^{-2b_1t}}{2b_1-b_2}\Big(\frac{\epsilon^{-(3-2b_1)a_2} - \epsilon^{-(3-2b_1)t}}{3-2b_1}\Big).
\end{aligned}
$$

By simple comparison, we have

$$
f_1(t) = 2\Big(\int\int_{R_1} + \int\int_{R_2} + \int\int_{R_3} h_{\alpha,\beta}(t)g(\alpha,\beta)d\beta d\alpha\Big).
$$

From above, we have that f_1 is a mixture of some DFR distributions if $0 < b_2 \le b_1 \le 1$. This completes the proof. ■

Proof of Theorem 3.1.1 in the general case.

In this case, we have n nonnegative random variables, $T_1, \cdots, T_n$, and time parameter $(a_1, \cdots, a_{n-1})$ where $0 \leq a_1 \leq a_2 \leq \cdots \leq a_{n-1}$. If the i-th failure among $T_1, \cdots, T_n$ happens at time t_i then the hazard rates of all alive components could stay unchanged if $t_i \leq a_i$ or change to some b_j if $t_i > a_i$. We assume that $0 < b_{n-1} \leq b_{n-2} \leq \cdots \leq b_1 \leq 1$.

Let $g(s_1, \cdots, s_{n-1})$ be the joint density of $(n-1)$ component model with time parameter $(a_1, \cdots, a_{n-2})$. Fix $(s_1, \cdots, s_{n-1})$. Without loss of generality, we assume that $s_1 \leq \cdots \leq s_{n-1}$. Let $k = \sum_{i=1}^{n-1} I_{(a_i,\infty)}(s_i)$. If $k = 0$ then define $h_{s_1,\cdots,s_{n-1}}(t) = e^{-t}$ for all $t \geq 0$. When $k \geq 1$ let j_m, $m = 1, \cdots, k$, be such that $I_{(a_{j_m},\infty)}(s_{j_m}) = 1$ for all $m = 1, \cdots, k$. Define

$$h_{s_1,\cdots,s_{n-1}}(t) = \begin{cases} e^{-t}, & \forall t \in (0, s_{j_1}]; \\ b_1 e^{-s_{j_1} - b_1(t - s_{j_1})}, & \forall t \in (s_{j_1}, s_{j_2}]; \\ \vdots & \\ b_i e^{-s_{j_1} - \sum_{m=1}^{i-1} b_m (s_{j_{m+1}} - s_{j_m}) - b_i (t - s_{j_i})}, & \forall t \in (s_{j_i}, s_{j_{i+1}}]; \\ \vdots & \\ b_k e^{-s_{j_1} - \sum_{m=1}^{k-1} b_m (s_{j_{m+1}} - s_{j_m}) - b_k (t - s_{j_k})}, & \forall t \in (s_{j_k}, \infty). \end{cases}$$

For all the other orders of $s_1, \cdots, s_{n-1}$, $h_{s_1,\cdots,s_{n-1}}$ can be defined symmetrically. By simple computation, we have

$$\int_0^\infty h_{s_1,\cdots,s_{n-1}}(t)dt = 1.$$

Thus $h_{s_1,\cdots,s_{n-1}}$ is a density function and has the hazard rate $\lambda_{s_1,\cdots,s_{n-1}}$ as

$$\lambda_{s_1,\cdots,s_{n-1}}(t) = \begin{cases} 1, & \forall t \in (0, s_{j_1}]; \\ b_1, & \forall t \in (s_{j_1}, s_{j_2}]; \\ \vdots & \\ b_i, & \forall t \in (s_{j_i}, s_{j_{i+1}}]; \\ \vdots & \\ b_k, & \forall t \in (s_{j_k}, \infty). \end{cases} \tag{3.1}$$

So $h_{s_1,\cdots,s_{n-1}}$ is a DFR since $0 < b_{n-1} \le \cdots \le b_1 \le 1$. Let f_1 denote the marginal density of the general n component system.

Again, we show that the following equation holds

$$f_1(t_1) = \int_{s_1=0}^{\infty} \cdots \int_{s_{n-1}=0}^{\infty} h_{s_1,\cdots,s_{n-1}}(t_1) g(s_1,\cdots,s_{n-1}) ds_{n-1} \cdots ds_1$$

for all $t_1 \ge 0$. The idea of the proof is to perform the integration. Although the joint density function and $h_{s_1,\cdots,s_{n-1}}$ are quite complicated, we can use the fact the the hazard rates are constant between failures. This makes the computation possible and easy. Fix t_1. From the permutation symmetry of g and h, it follows

$$\begin{aligned}
&\int_{s_1=0}^{\infty} \cdots \int_{s_{n-1}=0}^{\infty} h_{s_1,\cdots,s_{n-1}}(t_1) g(s_1,\cdots,s_{n-1}) ds_{n-1} \cdots ds_1 \\
&\quad = (n-1)! \int_{s_1=0}^{\infty} \int_{s_2=s_1}^{\infty} \cdots \int_{s_{n-1}=s_{n-2}}^{\infty} h_{s_1,\cdots,s_{n-1}}(t_1) g(s_1,\cdots,s_{n-1}) ds_{n-1} \cdots ds_1.
\end{aligned}$$

From Section 2.3,

$$\begin{aligned}
f_1(t_1) &= \int_{t_2=0}^{\infty} \int_{t_3=0}^{\infty} \cdots \int_{t_n=0}^{\infty} f(t_1,t_2,\cdots,t_n) dt_n \cdots dt_3 dt_2 \\
&= (n-1)! \int_{t_2=0}^{\infty} \int_{t_3=t_2}^{\infty} \cdots \int_{t_n=t_{n-1}}^{\infty} f(t_1,t_2,\cdots,t_n) dt_n \cdots dt_3 dt_2
\end{aligned}$$

where $f(t_1,\cdots,t_n)$ is the joint density of the system. Therefore it suffices to prove that

$$\begin{aligned}
&\int_{t_2=0}^{\infty} \int_{t_3=t_2}^{\infty} \cdots \int_{t_n=t_{n-1}}^{\infty} f(t_1,\cdots,t_n) dt_n \cdots dt_2 \\
&\quad = \int_{s_1=0}^{\infty} \int_{s_2=s_1}^{\infty} \cdots \int_{s_{n-1}=s_{n-2}}^{\infty} h_{s_1,\cdots,s_{n-1}}(t_1) g(s_1,\cdots,s_{n-1}) ds_{n-1} \cdots ds_1. \qquad (3.2)
\end{aligned}$$

We can rewrite the left hand side of of (3.2) as a summation of 2 integrations. One is over the region that $t_1 \le t_2$ and the other is over the region that $0 \le t_2 \le t_1$. Similarly, the right hand side of (3.2) is a summation of 2 integrations that one is over the region $t_1 \le s_1$ and the other is over the region $0 \le s_1 \le t_1$. We show that

$$\begin{aligned}
&\int_{t_2=t_1}^{\infty} \int_{t_3=t_2}^{\infty} \cdots \int_{t_n=t_{n-1}}^{\infty} f(t_1,\cdots,t_n) dt_n \cdots dt_2 \\
&\quad = \int_{s_1=t_1}^{\infty} \int_{s_2=s_1}^{\infty} \cdots \int_{s_{n-1}=s_{n-2}}^{\infty} h_{s_1,\cdots,s_{n-1}}(t_1) g(s_1,\cdots,s_{n-1}) ds_{n-1} \cdots ds_1 \qquad (3.3)
\end{aligned}$$

and that

$$\int_{t_2=0}^{t_1}\int_{t_3=t_2}^{\infty}\cdots\int_{t_n=t_{n-1}}^{\infty} f(t_1,\cdots,t_n)dt_n\cdots dt_2$$
$$= \int_{s_1=0}^{t_1}\int_{s_2=s_1}^{\infty}\cdots\int_{s_{n-1}=s_{n-2}}^{\infty} h_{s_1,\cdots,s_{n-1}}(t_1)g(s_1,\cdots,s_{n-1})ds_{n-1}\cdots ds_1. \qquad (3.4)$$

Then (3.2) follows.

We prove (3.3) first. From Section 2.3 we have the general formula for the joint density function $f(t_1,\cdots,t_n)$ and we know that the conditional hazard rate $\lambda_{m|1,\cdots,m-1}(t_m|t_1,\cdots,t_{m-1})$ does not depend on t_m (once $t_1,\cdots,t_{m-1}$ are determined). In fact, it is a constant b_i for some i. Then

$$\int_{t_2=t_1}^{\infty}\int_{t_3=t_2}^{\infty}\cdots\int_{t_n=t_{n-1}}^{\infty} f(t_1,\cdots,t_n)dt_n\cdots dt_2$$
$$= \int\cdots\int_{t_n=t_{n-1}}^{\infty} \epsilon^{-nt_1}\cdots\epsilon^{-\lambda_{n|1,\cdots,n}(t_n|t_1,\cdots,t_{n-1})t_n}\lambda_{n|1,\cdots,n}(t_n|t_1,\cdots,t_{n-1})dt_n\cdots dt_2$$
$$= \int\cdots\int_{t_{n-1}=t_{n-2}}^{\infty}$$
$$\epsilon^{-nt_1}\cdots\epsilon^{-2\lambda_{n-1|1,\cdots,n-2}(t_{n-1}|t_1,\cdots,t_{n-1})t_{n-1}}\lambda_{n-1|1,\cdots,n-2}(t_{n-1}|t_1,\cdots,t_{n-1})dt_{n-1}\cdots dt_2$$
$$\vdots$$
$$= \frac{1}{(n-2)!}\int_{t_2=t_1}^{\infty} \epsilon^{-nt_1}\epsilon^{-(n-1)\lambda_{2|1}(t_2|t_1)(t_2-t_1)}\lambda_{2|1}(t_2|t_1)dt_2$$
$$= \frac{\epsilon^{-nt_1}}{(n-1)!}.$$

Similarly, we have

$$\int_{s_1=t_1}^{\infty}\int_{s_2=s_1}^{\infty}\cdots\int_{s_{n-1}=s_{n-2}}^{\infty} h_{s_1,\cdots,s_{n-1}}(t_1)g(s_1,\cdots,s_{n-1})ds_{n-1}\cdots ds_1 = \frac{\epsilon^{-nt_1}}{(n-1)!}.$$

This shows (3.3).

Now we show that (3.4) holds. The left hand side of (3.4) can be written as

$$\int_{t_2=0}^{t_1}\int_{t_3=t_2}^{\infty}\cdots\int_{t_n=t_{n-1}}^{\infty} f(t_1,\cdots,t_n)dt_n\cdots dt_2$$
$$=\sum_{i=2}^{n-1}\int\cdots\int_{A_i} f(t_1,\cdots,t_n)dt_n\cdots dt_2$$
$$+\int_{t_2=0}^{t_1}\int_{t_3=t_2}^{t_1}\cdots\int_{t_n=t_{n-1}}^{t_1} f(t_1,\cdots,t_n)dt_n\cdots dt_2$$

with $A_i = \{t_2 \le t_3 \le \cdots \le t_i \le t_1 \le t_{i+1} \le \cdots \le t_n\}$. The right hand side of (3.4) can be written as

$$\int_{s_1=0}^{t_1}\int_{s_2=s_1}^{\infty}\cdots\int_{s_{n-1}=s_{n-2}}^{\infty} h_{s_1,\cdots,s_{n-1}}(t_1)g(s_1,\cdots,s_{n-1})ds_{n-1}\cdots ds_1$$
$$=\sum_{i=1}^{n-2}\int\cdots\int_{B_i} h_{s_1,\cdots,s_{n-1}}(t_1)g(s_1,\cdots,s_{n-1})ds_{n-1}\cdots ds_1$$
$$+\int_{s_1=0}^{t_1}\int_{s_2=s_1}^{t_1}\cdots\int_{s_{n-1}=s_{n-2}}^{t_1} h_{s_1,\cdots,s_{n-1}}(t_1)g(s_1,\cdots,s_{n-1})ds_{n-1}\cdots ds_1$$

with $B_i = \{s_1 \le s_2 \le \cdots \le s_i \le t_1 \le s_{i+1} \le \cdots \le s_{n-1}\}$. Thus it suffices to show that

$$\int\cdots\int_{A_i} f(t_1,\cdots,t_n)dt_n\cdots dt_2$$
$$=\int\cdots\int_{B_{i-1}} h_{s_1,\cdots,s_{n-1}}(t_1)g(s_1,\cdots,s_{n-1})ds_{n-1}\cdots ds_1 \tag{3.5}$$

for all $i = 2,3,\cdots,n-1$ and

$$\int_{t_2=0}^{t_1}\int_{t_3=t_2}^{t_1}\cdots\int_{t_n=t_{n-1}}^{t_1} f(t_1,\cdots,t_n)dt_n\cdots dt_2$$
$$=\int_{s_1=0}^{t_1}\int_{s_2=s_1}^{t_1}\cdots\int_{s_{n-1}=s_{n-2}}^{t_1} h_{s_1,\cdots,s_{n-1}}(t_1)g(s_1,\cdots,s_{n-1})ds_{n-1}\cdots ds_1. \tag{3.6}$$

We prove (3.5) first. By (2.24) and the fact that the conditional hazard rate $\lambda_{m|1,\cdots,m-1}$

is independent of t_m, we have

$$\begin{aligned}
&\int \cdots \int_{A_i} f(t_1, \cdots, t_n) dt_n \cdots dt_2 \\
&= \int_{t_2=0}^{t_1} \int_{t_3=t_2}^{t_1} \cdots \int_{t_i=t_{i-1}}^{t_1} e^{-(n-(n-1)\lambda_{3|2}(t_3|t_2))t_2 - \cdots - (n-i+1)\lambda_{1|2,\cdots,i}(t_1|t_2,\cdots,t_i)t_1} \times \\
&\quad \lambda_{3|2}(t_3|t_2) \cdots \lambda_{1|2,\cdots,i}(t_1|t_2, \cdots, t_i) dt_i dt_{i-1} \cdots dt_2 \\
&\quad = \int \cdots \int_{B_{i-1}} h_{s_1,\cdots,s_{n-1}}(t_1) g(s_1, \cdots, s_{n-1}) ds_{n-1} \cdots ds_1.
\end{aligned}$$

This shows (3.5). To prove (3.6), note that on the set $\{t_2 \leq t_3 \leq \cdots \leq t_n \leq t_1\}$, we have

$$f(t_1, \cdots, t_n) = h_{t_2,\cdots,t_n}(t_1) g(t_2, \cdots, t_n)$$

Hence

$$\begin{aligned}
&\int_{t_2=0}^{t_1} \int_{t_3=t_2}^{t_1} \cdots \int_{t_n=t_{n-1}}^{t_1} f(t_1, \cdots, t_n) dt_n \cdots dt_2 \\
&\quad = \int_{s_1=0}^{t_1} \int_{s_2=s_1}^{t_1} \cdots \int_{s_{n-1}=s_{n-2}}^{t_1} h_{s_1,\cdots,s_{n-1}}(t_1) g(s_1, \cdots, s_{n-1}) ds_{n-1} \cdots ds_1.
\end{aligned}$$

This proves (3.6). Then the proof of (3.4) is complete. Combing (3.3) and (3.4), we complete the proof. ■

Remarks

1. For the case $n = 2$ we may imagine a Poisson process (rate 1) which inflicts shocks on component two. If the first shock of that process occurs at time α ($< a$) then the failure rate of component one is always 1. If the first shock of that process occurs at time α ($> a$) then the failure rate of component one from time point α onward (provided component one is alive at time α) changes from 1 to b. The distribution $g(\alpha) = e^{-\alpha}$ is not the marginal distribution of component two. Nor is H_α the conditional distribution of component one given that component two died at time α.

2. Could the marginal distribution be IFR or IFRA when $1 < b$? No. It suffices to study the case $n = 2$. From (2.17) of Section 2.1.4. we have

$$\begin{aligned}
&\bar{F}_1(x)\bar{F}_1(y) - \bar{F}_1(x+y) \\
&= e^{-x-y}\{-e^{-a}(1-e^{-a}) - \frac{e^{-x-y}}{2-b}(e^{(2-b)(x+y-a)} + 1 - b) \\
&\quad + \frac{(1-e^{-a})}{2-b}[e^{-x}(e^{(2-b)(x-a)} + 1 - b) + e^{-y}(e^{(2-b)(y-a)} + 1 - b)] \\
&\quad + \frac{e^{-x-y}}{(2-b)^2}(e^{(2-b)(x-a)} + 1 - b)(e^{(2-b)(y-a)} + 1 - b)\}
\end{aligned}$$

for $x, y > a$. If x and y are sufficiently large then

$$\begin{aligned}
&\bar{F}_1(x)\bar{F}_1(y) - \bar{F}_1(x+y) \\
&= e^{-x-y}\{-e^{-a}(1-e^{-a}) - \frac{e^{(1-b)(x+y)-a(2-b)} + e^{-x-y}(1-b)}{2-b} \\
&\quad + \frac{(1-e^{-a})}{2-b}\left[e^{(1-b)x-a(2-b)} + e^{-x}(1-b) + e^{(1-b)y-a(2-b)} + e^{-y}(1-b)\right] \\
&\quad + \frac{1}{(2-b)^2}\left[e^{(1-b)x-a(2-b)} + e^{-x}(1-b)\right]\left[e^{(1-b)y-a(2-b)} + e^{-y}(1-b)\right]\} \\
&\quad < 0
\end{aligned}$$

since $1 < b$. This implies that the marginal distribution is not NBU. Hence it could not be IFR or IFRA. (See Theorems 1.2.7 and 1.2.10 in Section 1.2.)

3. When we derive the new distribution in Section 2.3. we proceed "forward in time". That is. if we know that there are i failures at time t and we know $k = \sum_{l=1}^{i} I_{(a_l,\infty)}(t_l)$ where t_l denotes the time of l-th failure. then we can determine the hazard rate of all live components until next failure happens. From the new representation of the marginal distribution. a mixture of some distributions. we have a different view point. It is going "backward in time". That is. if we know the times $t_2, \cdots, t_n$ of failure of component two to component n then we can determine the hazard rate of component one completely. For example. suppose $t_2 < t_3 < \cdots < t_n$. Furthermore. we assume that only components

$j_1 < j_2 < \cdots < j_k$ are such that $t_{j_l} > a_l$. Then the hazard rate r of component one is

$$r(t) = I_{(0,t_{j_1}]}(t) + \sum_{l=1}^{k-1} b_l I_{(t_{j_l},t_{j_{l+1}}]}(t) + b_k I_{(t_{j_k},\infty)}(t)$$

which is (3.1) with $s_1, \cdots, s_{n-1}$ substituted by $t_2, \cdots, t_n$.

3.2 IFRA of an (n-1)-out-of-n system

We study an $(n-1)$-out-of-n system in this section. Assume that the joint density function of all the component lifetimes is the new one described in the previous chapter. We study the necessary and sufficient conditions under which the lifetime T of this system possesses an IFRA distribution. Let $\bar{F}_T$ denote the survival function of T. To prove the main result, we simply compute $-\frac{1}{t}\ln \bar{F}_T$ and show that it is increasing. Once again, we start with the bivariate case, the three component case, and then finish with the general case. Throughout this section, we assume that $a_1 > 0$ and $b_1 \geq 1$.

3.2.1 1-out-of-2 System

In the bivariate case, we have a parallel system and its lifetime T is the maximum of T_1 and T_2 where T_1 and T_2 represent the lifetimes of component one and two, respectively. We assume that the joint density of T_1 and T_2 is given in (2.9). By conditioning on the time t_1 of the first failure among T_1 and T_2, we can compute the density function f_T of T. We have

$$f_T(t) = \int_0^t 2e^{-2t_1}e^{-(t-t_1)}dt_1$$

for $t \leq a$, and

$$f_T(t) = \int_0^a 2e^{-2t_1}e^{-(t-t_1)}dt_1 + \int_a^t 2e^{-2t_1}be^{-b(t-t_1)}dt_1$$

for $t > a$. Therefore

$$f_T(t) = \begin{cases} 2\epsilon^{-t}(1-\epsilon^{-t}), & t \le a; \\ 2\epsilon^{-t}(1-\epsilon^{-a}) + 2b\epsilon^{-bt}\int_a^t \epsilon^{-(2-b)t_1}dt_1, & t > a. \end{cases} \tag{3.7}$$

We will consider the two cases when $b \neq 2$ and $b = 2$ separately.

Case1 : $b \neq 2$.

From (3.7), we have

$$f_T(t) = \begin{cases} 2\epsilon^{-t}(1-\epsilon^{-t}), & t \le a; \\ 2\epsilon^{-t}(1-\epsilon^{-a}) + \frac{2b\epsilon^{-2t}}{2-b}(\epsilon^{(2-b)(t-a)} - 1), & t > a. \end{cases}$$

then

$$\bar{F}_T(t) = \begin{cases} 2\epsilon^{-t} - \epsilon^{-2t}, & \text{if} \quad t \le a; \\ 2(1-\epsilon^{-a})\epsilon^{-t} + \frac{\epsilon^{-2t}}{2-b}(2\epsilon^{(2-b)(t-a)} - b), & \text{if} \quad t > a. \end{cases}$$

For $t \le a$,

$$-\frac{1}{t}\ln \bar{F}_T(t) = -\frac{1}{t}\ln(2\epsilon^{-t} - \epsilon^{-2t}) = \frac{1}{t}\int_0^t \frac{2\epsilon^{-x} - 2\epsilon^{-2x}}{2\epsilon^{-x} - \epsilon^{-2x}}dx = \frac{1}{t}\int_0^t (2 - \frac{2}{2-\epsilon^{-x}})dx.$$

Since $0 \le (2 - \frac{2}{2-\epsilon^{-x}})$ is increasing in x, $\int_0^t (2 - \frac{2}{2-\epsilon^{-x}})dx$ is a star-shaped function. Hence $-\frac{1}{t}\ln \bar{F}_T(t)$ is increasing in t on $(0, a]$. For $t > a$,

$$\begin{aligned} -\frac{1}{t}\ln \bar{F}_T(t) &= -\frac{1}{t}\ln\left(2(1-\epsilon^{-a})\epsilon^{-t} + \frac{\epsilon^{-2t}}{2-b}(2\epsilon^{(2-b)(t-a)} - b)\right) \\ &= 1 - \frac{1}{t}\ln\left(2(1-\epsilon^{-a}) + \frac{\epsilon^{-t}}{2-b}(2\epsilon^{(2-b)(t-a)} - b)\right). \end{aligned}$$

Therefore

$$-\frac{1}{t}\ln \bar{F}_T(t) \uparrow \iff \frac{1}{t}\ln\left(2(1-\epsilon^{-a}) + \frac{\epsilon^{-t}}{2-b}(2\epsilon^{(2-b)(t-a)} - b)\right) \downarrow .$$

Claim : $\frac{1}{t}\ln\left(2(1-\epsilon^{-a}) + \frac{\epsilon^{-t}}{2-b}(2\epsilon^{(2-b)(t-a)} - b)\right)$ is decreasing in t on (a, ∞) if and only if $a \ge \ln 2$.

Proof. First let us assume that $a \ge \ln 2$. The idea of the proof is based on a simple fact from calculus: A differentiable function is decreasing if and only if its first derivative is less than or equal to 0. Clearly,

$$\frac{d}{dt}\left[\frac{1}{t}\ln\left(2(1-e^{-a})+\frac{e^{-t}}{2-b}(2e^{(2-b)(t-a)}-b)\right)\right]$$
$$=\frac{1}{t^2}\left[\frac{\frac{e^{-t}}{2-b}(2(1-b)e^{(2-b)(t-a)}+b)t}{2(1-e^{-a})+\frac{e^{-t}}{2-b}(2e^{(2-b)(t-a)}-b)}\right.$$
$$\left.-\ln\left(2(1-e^{-a})+\frac{e^{-t}}{2-b}(2e^{(2-b)(t-a)}-b)\right)\right].$$

This is less than or equal to 0 if and only if

$$L:=\frac{\frac{e^{-t}}{2-b}(2(1-b)e^{(2-b)(t-a)}+b)t}{2(1-e^{-a})+\frac{e^{-t}}{2-b}(2e^{(2-b)(t-a)}-b)}$$
$$\leq \ln\left(2(1-e^{-a})+\frac{e^{-t}}{2-b}(2e^{(2-b)(t-a)}-b)\right):=R. \tag{3.8}$$

It suffices to find an upper bound for L and a lower bound for R and show that the upper bound for L is less than or equal to the lower bound for R. Note that $\frac{1}{2-b}(2e^{(2-b)(t-a)}-b)>0$ for all $t\geq a$ since $1<b\neq 2$. Then

$$2(1-e^{-a})+\frac{e^{-t}}{2-b}(2e^{(2-b)(t-a)}-b)\geq 2(1-e^{-a})\geq 1$$

on $[a,\infty)$ since $a\geq \ln 2$. This implies that

$$R=\ln\left(2(1-e^{-a})+\frac{e^{-t}}{2-b}(2e^{(2-b)(t-a)}-b)\right)\geq 0$$

on $[a,\infty)$. Now let us examine $\frac{2(1-b)e^{(2-b)(t-a)}+b}{2-b}$(part of the numerator of L). When $t=a$, it has value 1. If $b>2$, then $e^{(2-b)(t-a)}\to 0$ as $t\to\infty$. Then $\frac{2(1-b)e^{(2-b)(t-a)}+b}{2-b}<0$ for t sufficiently large. If $b<2$, then $e^{(2-b)(t-a)}\to\infty$ as $t\to\infty$. Thus, again, $\frac{2(1-b)e^{(2-b)(t-a)}+b}{2-b}<0$ for t sufficiently large since $1<b$. That is,

$$\frac{2(1-b)e^{(2-b)(t-a)}+b}{2-b}\begin{cases}>0, & t\in(a,t_0(b));\\ =0, & t=t_0(b);\\ <0, & t\in(t_0(b),\infty).\end{cases} \tag{3.9}$$

for some $t_0(b)\in(a,\infty)$. Then

$$L=\frac{\frac{e^{-t}}{2-b}(2(1-b)e^{(2-b)(t-a)}+b)t}{2(1-e^{-a})+\frac{e^{-t}}{2-b}(2e^{(2-b)(t-a)}-b)}\leq 0$$

for all $t \geq t_0(b)$. Therefore

$$L \leq 0 \leq R \quad \text{on} \quad [t_0(b), \infty). \tag{3.10}$$

Now we show that $L \leq R$ also holds on the interval $(a, t_0(b))$. Note,

1. If $f(t) = te^{-t}$ on $[0, \infty)$ then $f'(t) = (1-t)e^{-t}$ which is positive on $[0, 1)$ and negative on $(1, \infty)$. Therefore f is increasing on $[0, 1)$ and decreasing on $(1, \infty)$. This implies that f achieves its maximum at $t = 1$, that is,

$$\frac{t}{e^t} \leq \frac{1}{e}, \quad \forall t \geq 0. \tag{3.11}$$

2. If $f(t) = \frac{2(1-b)e^{(2-b)(t-a)}+b}{2-b}$ on $[a, \infty)$ then $f'(t) = 2(1-b)e^{(2-b)(t-a)} < 0$ since $1 < b$. Then $f(t)$ is decreasing on (a, ∞) and in particular on $(a, t_0(b))$. So

$$0 \leq \frac{2(1-b)e^{(2-b)(t-a)}+b}{2-b} \leq 1, \quad \forall t \in (a, t_0(b)), \tag{3.12}$$

since $f(a) = 1$ and $f(t_0(b)) = 0$ (from (3.9)).

3. Since

$$\frac{d}{dt}\left(\frac{e^{-t}}{2-b}(2e^{(2-b)(t-a)} - b)\right) = e^{-t}\left(\frac{2(1-b)e^{(2-b)(t-a)}+b}{2-b}\right) \geq 0$$

on $(a, t_0(b))$ from (3.12), $\frac{e^{-t}}{2-b}(2e^{(2-b)(t-a)} - b)$ in increasing on $(a, t_0(b))$. Then

$$\begin{aligned} 2(1-e^{-a}) + \frac{e^{-t}}{2-b}(2e^{(2-b)(t-a)} - b) &\geq 2(1-e^{-a}) + e^{-a} \quad (\text{with } t = a) \\ &= 2 - e^{-a} \\ &\geq \frac{3}{2} \quad (\text{since } a \geq \ln 2) \end{aligned}$$

on $(a, t_0(b))$. That is,

$$2(1-e^{-a}) + \frac{e^{-t}}{2-b}(2e^{(2-b)(t-a)} - b) \geq \frac{3}{2}, \quad \forall t \in (a, t_0(b)). \tag{3.13}$$

All these imply

$$
\begin{aligned}
L &= \frac{\frac{e^{-t}}{2-b}(2(1-b)e^{(2-b)(t-a)}+b)t}{2(1-e^{-a})+\frac{e^{-t}}{2-b}(2e^{(2-b)(t-a)}-b)} \\
&\leq \frac{\frac{2(1-b)e^{(2-b)(t-a)}+b}{2-b}\frac{1}{e}}{2(1-e^{-a})+\frac{e^{-t}}{2-b}(2e^{(2-b)(t-a)}-b)} \quad \text{(from (3.11))} \\
&\leq \frac{\frac{1}{e}}{2(1-e^{-a})+\frac{e^{-t}}{2-b}(2e^{(2-b)(t-a)}-b)} \quad \text{(from (3.12))} \\
&\leq \frac{2}{3e} \quad \text{(from (3.13))}
\end{aligned}
$$

on $(a, t_0(b))$ and

$$
\begin{aligned}
R &= \ln\left(2(1-e^{-a})+\frac{e^{-t}}{2-b}(2e^{(2-b)(t-a)}-b)\right) \\
&\geq \ln(\frac{3}{2}) \quad \text{(from (3.13))} \\
&> \frac{2}{3e} \quad \text{(by numerical computation)}
\end{aligned}
$$

on $(a, t_0(b))$. Therefore we get an inequality

$$
L \leq \frac{2}{3e} \leq R \quad \text{on} \quad (a, t_0(b)). \tag{3.14}
$$

Together with (3.10), we have (3.8). This completes the proof that

$$
\frac{1}{t}\ln\left(2(1-e^{-a})+\frac{e^{-t}}{2-b}(2e^{(2-b)(t-a)}-b)\right) \tag{3.15}
$$

is decreasing in t on (a, ∞) if $a \geq \ln 2$.

Next, let us assume that $a < \ln 2$. We show that (3.15) is not decreasing in t on (a, ∞). If $a < \ln 2$ then

$$
0 < 2(1-e^{-a}) < 1. \tag{3.16}
$$

Since we assume $b > 1$, we have

$$
\frac{e^{-t}}{2-b}(2e^{(2-b)(t-a)}-b) = \frac{1}{2-b}(2e^{(1-b)t-(2-b)a}-be^{-t}) \to 0 \quad \text{as} \quad t \to \infty. \tag{3.17}
$$

Therefore (3.16) and (3.17) imply

$$\frac{1}{t}\ln\left(2(1-\epsilon^{-a})+\frac{\epsilon^{-t}}{2-b}(2\epsilon^{(2-b)(t-a)}-b)\right)<0$$

for t sufficiently large. Also

$$\frac{1}{t}\ln\left(2(1-\epsilon^{-a})+\frac{\epsilon^{-t}}{2-b}(2\epsilon^{(2-b)(t-a)}-b)\right)\to 0 \quad \text{as} \quad t\to\infty.$$

This means that $\frac{1}{t}\ln\left(2(1-\epsilon^{-a})+\frac{\epsilon^{-t}}{2-b}(2\epsilon^{(2-b)(t-a)}-b)\right)$ can not be decreasing in t on (a,∞). This completes the proof. ■

From the above, we have the following lemma.

Lemma 3.2.1. *Let* $1<b\neq 2$. T *is IFRA if and only if* $a\geq \ln 2$.

Case2 : $b=2$.

In this situation, we have

$$f_T(t)=\begin{cases}2\epsilon^{-t}(1-\epsilon^{-t}), & t\leq a;\\ 2\epsilon^{-t}(1-\epsilon^{-a})+4\epsilon^{-2t}(t-a), & t>a.\end{cases} \tag{3.18}$$

and

$$\bar{F}_T(t)=\begin{cases}2\epsilon^{-t}-\epsilon^{-2t}, & \text{if} \quad t\leq a;\\ 2(1-\epsilon^{-a})\epsilon^{-t}+(2t-2a+1)\epsilon^{-2t}, & \text{if} \quad t>a.\end{cases} \tag{3.19}$$

For $t\leq a$ the survival function is exactly the same as in case 1. This implies that

$$-\frac{1}{t}\ln\bar{F}_T(t) \ \text{ is increasing in } \ t \ \text{ on } \ (0,a].$$

If $t>a$, then

$$\begin{aligned}-\frac{1}{t}\ln\bar{F}_T(t)&=-\frac{1}{t}\ln\left(2(1-\epsilon^{-a})\epsilon^{-t}+(2t-2a+1)\epsilon^{-2t}\right)\\ &=1-\frac{1}{t}\ln\left(2(1-\epsilon^{-a})+(2t-2a+1)\epsilon^{-t}\right)\end{aligned}$$

which is increasing in t on (a, ∞) if and only if

$$\frac{1}{t} \ln \left(2(1 - \epsilon^{-a}) + (2t - 2a + 1)\epsilon^{-t} \right)$$

is decreasing in t on (a, ∞).

Claim: $\frac{1}{t} \ln \left(2(1 - \epsilon^{-a}) + (2t - 2a + 1)\epsilon^{-t} \right)$ is decreasing in t on (a, ∞) if and only if $a \geq \ln 2$.

Proof. Same idea as in previous subsection, we show that the first derivative is less than or equal to 0 if $a \geq \ln 2$. Clearly,

$$\begin{aligned}
&\frac{d}{dt} \left[\frac{1}{t} \ln \left(2(1 - \epsilon^{-a}) + (2t - 2a + 1)\epsilon^{-t} \right) \right] \\
&\qquad = \frac{1}{t^2} \left[\frac{(1 + 2a - 2t)\epsilon^{-t} t}{2(1 - \epsilon^{-a}) + (2t - 2a + 1)\epsilon^{-t}} - \ln \left(2(1 - \epsilon^{-a}) + (2t - 2a + 1)\epsilon^{-t} \right) \right] \\
&\qquad = \frac{1}{t^2} [L - R]
\end{aligned}$$

where

$$L = \frac{(1 + 2a - 2t)\epsilon^{-t} t}{2(1 - \epsilon^{-a}) + (2t - 2a + 1)\epsilon^{-t}} \quad \text{and} \quad R = \ln \left(2(1 - \epsilon^{-a}) + (2t - 2a + 1)\epsilon^{-t} \right).$$

Still, we try to find an upper bound for L and a lower bound for R and show that the upper bound for L is less than or equal to the lower bound for R. Since

$$(1 + 2a - 2t)\epsilon^{-t} t \leq 0, \quad (2t - 2a + 1)\epsilon^{-t} > 0, \quad \text{and} \quad 2(1 - \epsilon^{-a}) \geq 1$$

for all $t \geq a + \frac{1}{2}$, we have

$$L = \frac{(1 + 2a - 2t)\epsilon^{-t} t}{2(1 - \epsilon^{-a}) + (2t - 2a + 1)\epsilon^{-t}} \leq 0 \quad \text{on} \quad [a + \frac{1}{2}, \infty).$$

Also,

$$R = \ln \left(2(1 - \epsilon^{-a}) + (2t - 2a + 1)\epsilon^{-t} \right) > \ln 2(1 - \epsilon^{-a}) \geq 0$$

on $[a + \frac{1}{2}, \infty)$ since $a \geq \ln 2$. Then

$$L \leq 0 \leq R \quad \text{on} \quad [a + \frac{1}{2}, \infty). \tag{3.20}$$

Now we focus on the interval $(a, a+\frac{1}{2})$ and prove that $L \leq R$ still holds over there. Note

1. $e^{-t}t \leq 1/e$ for all $t \geq 0$ from (3.11).

2. Obviously,

$$0 \leq 1+2a-2t \leq 1, \quad \forall t \in (a, a+\frac{1}{2}). \tag{3.21}$$

3. Since $\frac{d}{dt}((2t-2a+1)e^{-t}) = (1+2a-2t)e^{-t}$ which is positive on $(a, a+\frac{1}{2})$, $(2t-2a+1)e^{-t}$ is increasing on $(a, a+\frac{1}{2})$. This implies that the lower bound is located at $t=a$. Hence

$$(2t-2a+1)e^{-t} \geq e^{-a}, \quad \forall t \in [a, a+\frac{1}{2}). \tag{3.22}$$

Therefore

$$\begin{aligned} L &= \frac{(1+2a-2t)e^{-t}t}{2(1-e^{-a})+(2t-2a+1)e^{-t}} \\ &\leq \frac{1+2a-2t}{2(1-e^{-a})+(2t-2a+1)e^{-t}}\frac{1}{e} \quad \text{(from (3.11))} \\ &\leq \frac{1}{2(1-e^{-a})+(2t-2a+1)e^{-t}}\frac{1}{e} \quad \text{(from (3.21))} \\ &\leq \frac{1}{2-e^{-a}}\frac{1}{e} \quad \text{(from (3.22))} \\ &\leq \frac{2}{3e} \quad \text{(from } a \geq \ln 2\text{)}. \end{aligned}$$

Again, by (3.22) and $a \geq \ln 2$

$$R = \ln\left(2(1-e^{-a})+(2t-2a+1)e^{-t}\right) \geq \ln(2-e^{-a}) \geq \ln(\frac{3}{2}) > \frac{2}{3e}.$$

Therefore

$$L \leq \frac{2}{3e} \leq R \quad \text{on} \quad (a, a+\frac{1}{2}). \tag{3.23}$$

From (3.23) and (3.20), we have shown that the first derivative is less than or equal to 0 on (a, ∞) if $a \geq \ln 2$. This completes the first part of the proof.

Now let us assume that $a < \ln 2$. Under this assumption, we have $2(1 - e^{-a}) < 1$ and $(2t - 2a + 1)e^{-t} \to 0$ as $t \to \infty$. Then

$$\frac{1}{t} \ln \left(2(1 - e^{-a}) + (2t - 2a + 1)e^{-t}\right) \longrightarrow \ln 2(1 - e^{-a}) < 0.$$

This implies

$$\frac{1}{t} \ln \left(2(1 - e^{-a}) + (2t - 2a + 1)e^{-t}\right) < 0$$

for t sufficiently large and

$$\frac{1}{t} \ln \left(2(1 - e^{-a}) + (2t - 2a + 1)e^{-t}\right) \longrightarrow 0$$

as $t \to \infty$. Therefore

$$\frac{1}{t} \ln \left(2(1 - e^{-a}) + (2t - 2a + 1)e^{-t}\right)$$

can not be decreasing in t on (a, ∞). This completes the proof. ■

From the above, we have the following lemma.

Lemma 3.2.2. *Let $b = 2$. T is IFRA if and only if $a \geq \ln 2$.*

Combining Lemmas 3.2.1 and 3.2.2, we have the following theorem:

Theorem 3.2.3. *For $a > 0$ and $b > 1$, T is IFRA if and only if $a \geq \ln 2$.*

Remark 1: Could T be IFR? From (3.18) and (3.19), we have

$$\frac{f_T(t)}{\bar{F}_T(t)} = \frac{2e^{-t}(1 - e^{-a}) + 4e^{-2t}(t - a)}{2e^{-t}(1 - e^{-a}) + e^{-2t}(2t - 2a + 1)}$$

for $t > a$. Then

$$\frac{f_T(t)}{\bar{F}_T(t)} = 1 + \frac{(2t - 2a - 1)e^{-2t}}{2e^{-t}(1 - e^{-a}) + e^{-2t}(2t - 2a + 1)}$$

for $t > a$. By simple calculus. $\frac{(2t-2a-1)e^{-2t}}{2e^{-t}(1-e^{-a})+e^{-2t}(2t-2a+1)}$ is not increasing in t. Hence T could not be IFR.

Remark 2: Could T be DFR or DFRA? Again. from (3.18) and (3.19). we have

$$\frac{f_T(t)}{\bar{F}_T(t)} = \frac{2e^{-t}(1-e^{-t})}{2e^{-t}-e^{-2t}} = 1 - \frac{1}{2e^t - 1}$$

for $t \le a$. This is an increasing function of t. Hence T could not be DFR or DFRA.

3.2.2 2-out-of-3 System

Let T_1. T_2. T_3. and T denote the life time of component one. two. three. and the whole system. respectively. Then T is the second order statistics of T_1. T_2. and T_3. By conditioning on t_1. time to the first failure among T_1. T_2. and T_3. the density function f_T of T can be computed by using the hazard rates of T_1. T_2. and T_3. We have

$$f_T(t) = \int_0^t 3e^{-3t_1} 2e^{-2(t-t_1)} dt_1 = 6e^{-2t}(1-e^{-t}) \tag{3.24}$$

for $t \le a_1$. and

$$\begin{aligned} f_T(t) &= \int_0^{a_1} 3e^{-3t_1} 2e^{-2(t-t_1)} dt_1 + \int_{a_1}^t 3e^{-3t_1} 2b_1 e^{-2b_1(t-t_1)} dt_1 \\ &= 6e^{-2t}(1-e^{-a_1}) + 6b_1 e^{-2b_1 t} \int_{a_1}^t e^{-(3-2b_1)t_1} dt_1 \end{aligned}$$

for $t > a_1$. Using the density function. we can compute the survival function $\bar{F}_T$ of T. Then.

$$\bar{F}_T(t) = \begin{cases} 3e^{-2t} - 2e^{-3t}. & t \le a_1; \\ 3(1-e^{-a_1})e^{-2t} + \frac{e^{-3t}}{3-2b_1}(3e^{(3-2b_1)(t-a_1)} - 2b_1). & t > a_1. \end{cases} \tag{3.25}$$

for $b_1 \ne \frac{3}{2}$. and

$$\bar{F}_T(t) = \begin{cases} 3e^{-2t} - 2e^{-3t}. & t \le a_1; \\ 3(1-e^{-a_1})e^{-2t} + e^{-3t}(3t - 3a_1 + 1). & t > a_1. \end{cases} \tag{3.26}$$

for $b_1 = \frac{3}{2}$. The survival function of T has two parameters, a_1 and b_1. This is same as the 1-out-of-2 system in the previous section. Therefore we expect that the IFRA property still holds in our present system.

From (3.24), (3.25), and (3.26), we have

$$\frac{f_T(t)}{\bar{F}_T(t)} = \frac{6\epsilon^{-2t}(1-\epsilon^{-t})}{3\epsilon^{-2t} - 2\epsilon^{-3t}} = 2 - \frac{2}{3\epsilon^{t} - 2},$$

for $0 \le t \le a_1$, which is increasing in t. This implies that T has *increasing failure rate* property on $[0, a_1]$ which is stronger than IFRA. Therefore in order to prove that T is IFRA, we need to show that

$$-\frac{1}{t}\ln \bar{F}_T(t) \uparrow \quad \text{in} \quad t$$

for $t > a_1$. We study this separately for $b_1 \neq \frac{3}{2}$ and $b_1 = \frac{3}{2}$.

Case1 : $3 - 2b_1 \neq 0$.

Lemma 3.2.4. $-\frac{1}{t}\ln \bar{F}_T(t)$ *is increasing in t on (a_1, ∞) if and only if $a_1 \ge \ln \frac{3}{2}$.*

Proof. From (3.25), we have

$$\begin{aligned} -\frac{1}{t}\ln \bar{F}_T(t) &= -\frac{1}{t}\ln\left(3(1-\epsilon^{-a_1})\epsilon^{-2t} + \frac{\epsilon^{-3t}}{3-2b_1}(3\epsilon^{(3-2b_1)(t-a_1)} - 2b_1)\right) \\ &= 2 - \frac{1}{t}\ln\left(3(1-\epsilon^{-a_1}) + \frac{\epsilon^{-t}}{3-2b_1}(3\epsilon^{(3-2b_1)(t-a_1)} - 2b_1)\right). \end{aligned}$$

for $t > a_1$, which is increasing in t if and only if

$$\frac{1}{t}\ln\left(3(1-e^{-a_1}) + \frac{\epsilon^{-t}}{3-2b_1}(3e^{(3-2b_1)(t-a_1)} - 2b_1)\right) \tag{3.27}$$

is decreasing in t. First, let us assume that $a_1 \ge \ln\frac{3}{2}$. Let

$$m(t) = \frac{e^{-t}}{3-2b_1}(3e^{(3-2b_1)(t-a_1)} - 2b_1)$$

be defined on (a_1, ∞). Then $m(t) > 0$ and

$$m(t) = \frac{1}{3-2b_1}(3e^{2(1-b_1)t-(3-2b_1)a_1} - 2b_1\epsilon^{-t}) \to 0 \quad \text{as} \quad t \to \infty \tag{3.28}$$

since $b_1 > 1$. Therefore

$$\frac{1}{t}\ln\left(3(1-e^{-a_1})+\frac{e^{-t}}{3-2b_1}(3e^{(3-2b_1)(t-a_1)}-2b_1)\right)\geq\frac{1}{t}\ln 3(1-e^{-a_1})\geq 0$$

since $a_1 \geq \ln\frac{3}{2}$. Thus

$$\ln\left(3(1-e^{-a_1})+\frac{e^{-t}}{3-2b_1}(3e^{(3-2b_1)(t-a_1)}-2b_1)\right)$$

is a positive function. Therefore, if $m(t)$ is decreasing in t then

$$\ln\left(3(1-e^{-a_1})+\frac{e^{-t}}{3-2b_1}(3e^{(3-2b_1)(t-a_1)}-2b_1)\right)$$

is decreasing in t and then (3.27) is decreasing in t and this will complete the proof. To see whether $m(t)$ is decreasing, we check its first derivative. Clearly

$$\frac{d}{dt}m(t)=2e^{-t}\left(\frac{3(1-b_1)e^{(3-2b_1)(t-a_1)}+b_1}{3-2b_1}\right).$$

Note

1. $e^{-t}>0$ for all t.

2. $e^{(3-2b_1)(t-a_1)}\to\infty$ if $3-2b_1>0$ and $e^{(3-2b_1)(t-a_1)}\to 0$ if $3-2b_1<0$. In both cases, we have $\frac{3(1-b_1)e^{(3-2b_1)(t-a_1)}+b_1}{3-2b_1}<0$ for t sufficiently large since $1<b_1$.

3. $\frac{3(1-b_1)e^{(3-2b_1)(t-a_1)}+b_1}{3-2b_1}$ has value 1 at $t=a_1$.

From above, we have $\frac{dm}{dt}>0$ on $(a_1,t_0(b_1))$, $\frac{dm}{dt}=0$ at $(t_0(b_1))$, and $\frac{dm}{dt}<0$ on $(t_0(b_1),\infty)$ for some $t_0(b_1)\in(a_1,\infty)$. Therefore $m(t)$ is increasing on $(a_1,t_0(b_1))$ and decreasing on $[t_0(b_1),\infty)$. This shows that (3.27) is decreasing on $[t_0(b_1),\infty)$. Now we have to show that (3.27) is decreasing in t on $(a_1,t_0(b_1))$. Still, we check its first derivative. If it is less than or equal to 0 then the result follows. By calculus, the first derivative has the form

$$\frac{1}{t^2}\left[\frac{\frac{2e^{-t}}{3-2b_1}(3(1-b_1)e^{(3-2b_1)(t-a_1)}+b_1)t}{3(1-e^{-a_1})+\frac{e^{-t}}{3-2b_1}(3e^{(3-2b_1)(t-a_1)}-2b_1)}-\ln\left(3(1-e^{-a_1})+\frac{e^{-t}}{3-2b_1}(3e^{(3-2b_1)(t-a_1)}-2b_1)\right)\right].\quad(3.29)$$

Let us denote by $\frac{1}{t^2}(L - R)$, where

$$L = \frac{\frac{2e^{-t}}{3-2b_1}(3(1-b_1)e^{(3-2b_1)(t-a_1)} + b_1)t}{3(1-e^{-a_1}) + \frac{e^{-t}}{3-2b_1}(3e^{(3-2b_1)(t-a_1)} - 2b_1)}$$

and

$$R = \ln\left(3(1-e^{-a_1}) + \frac{e^{-t}}{3-2b_1}(3e^{(3-2b_1)(t-a_1)} - 2b_1)\right).$$

It suffices to find an upper bound for L which is also a lower bound for R. Recall that

1. $\frac{t}{e^t} \le \frac{1}{e}$ for all $t \ge 0$ from (3.11).

2. Since

$$\frac{d}{dt}\left(\frac{3(1-b_1)e^{(3-2b_1)(t-a_1)} + b_1}{3-2b_1}\right) = 3(1-b_1)e^{(3-2b_1)(t-a_1)} < 0$$

   on $(a_1, t_0(b_1))$, $\frac{3(1-b_1)e^{(3-2b_1)(t-a_1)}+b_1}{3-2b_1}$ is a decreasing function. Hence

$$0 \le \frac{3(1-b_1)e^{(3-2b_1)(t-a_1)} + b_1}{3-2b_1} \le 1. \tag{3.30}$$

3. Since $m(t)$ is increasing on $(a_1, t_0(b_1))$, we have $m(t) \ge m(a_1) = e^{-a_1}$ on $(a_1, t_0(b_1))$. Then

$$3(1-e^{-a_1}) + \frac{e^{-t}}{3-2b_1}(3e^{(3-2b_1)(t-a_1)} - 2b_1) \ge 3 - 2e^{-a_1}. \tag{3.31}$$

So

$$\begin{aligned}
L &= \frac{\frac{2e^{-t}}{3-2b_1}(3(1-b_1)e^{(3-2b_1)(t-a_1)}}{3(1-e^{-a_1}) + \frac{e^{-t}}{3-2b_1}(3e^{(3-2b_1)(t-a_1)} - 2b_1)} \\
&\le \frac{\frac{2(3(1-b_1)e^{(3-2b_1)(t-a_1)}+b_1)}{3-2b_1}\frac{1}{e}}{3(1-e^{-a_1}) + \frac{e^{-t}}{3-2b_1}(3e^{(3-2b_1)(t-a_1)} - 2b_1)} \quad \text{(from (3.11))} \\
&\le \frac{2\frac{1}{e}}{3(1-e^{-a_1}) + \frac{e^{-t}}{3-2b_1}(3e^{(3-2b_1)(t-a_1)} - 2b_1)} \quad \text{(from (3.30))}
\end{aligned}$$

$$\leq \frac{2^{\frac{1}{\epsilon}}}{3 - 2\epsilon^{-a_1}} \quad \text{(from (3.31))}$$
$$\leq \frac{6}{5\epsilon} \quad (\text{from } a_1 \geq \ln \frac{3}{2})$$

and

$$\begin{aligned} R &= \ln(3(1 - \epsilon^{-a_1}) + \frac{\epsilon^{-t}}{3 - 2b_1}(3\epsilon^{(3-2b_1)(t-a_1)} - 2b_1)) \\ &\geq \ln(3 - 2\epsilon^{-a_1}) \quad \text{(from (3.31))} \\ &\geq \ln \frac{5}{3} \quad (\text{from } a_1 \geq \ln \frac{3}{2}) \\ &\geq \frac{6}{5\epsilon} \text{(by numerical computation)} \\ &\geq L. \end{aligned}$$

This shows that the first derivative. (3.29). is less than or equal to 0 on $(a_1, t_0(b_1))$. This completes the proof that

$$\frac{1}{t} \ln \left(3(1 - \epsilon^{-a_1}) + \frac{\epsilon^{-t}}{3 - 2b_1}(3\epsilon^{(3-2b_1)(t-a_1)} - 2b_1) \right) \tag{3.32}$$

is decreasing in t on (a_1, ∞) if $a_1 \geq \ln \frac{3}{2}$.

Next. we show this is not the case if $a_1 < \ln \frac{3}{2}$. We have $3(1 - \epsilon^{-a_1}) < 1$ under the assumption. Since $m(t) \to 0$ from (3.28),

$$\ln \left(3(1 - \epsilon^{-a_1}) + \frac{\epsilon^{-t}}{3 - 2b_1}(3\epsilon^{(3-2b_1)(t-a_1)} - 2b_1) \right) \longrightarrow \ln 3(1 - \epsilon^{-a_1}) < 0.$$

Then

$$\frac{1}{t} \ln \left(3(1 - \epsilon^{-a_1}) + \frac{\epsilon^{-t}}{3 - 2b_1}(3\epsilon^{(3-2b_1)(t-a_1)} - 2b_1) \right) < 0$$

for t sufficiently large. But it also approaches to 0 as t approaching ∞. This shows that

$$\frac{1}{t} \ln \left(3(1 - e^{-a_1}) + \frac{e^{-t}}{3 - 2b_1}(3e^{(3-2b_1)(t-a_1)} - 2b_1 \right)$$

can not be decreasing in t if $a_1 < \ln \frac{3}{2}$. ∎

Case2 : $3 - 2b_1 = 0$.

In this case, we have

$$-\frac{1}{t}\ln \bar{F}_T(t) = 2 - \frac{1}{t}\ln\left(3(1-\epsilon^{-a_1}) + \epsilon^{-t}(3t - 3a_1 + 1)\right).$$

for $t > a_1$ from (3.26). It suffices to show that

$$\frac{1}{t}\ln\left(3(1-\epsilon^{-a_1}) + \epsilon^{-t}(3t - 3a_1 + 1)\right)$$

is decreasing in t on (a_1, ∞).

Lemma 3.2.5. $\frac{1}{t}\ln\left(3(1-\epsilon^{-a_1}) + \epsilon^{-t}(3t - 3a_1 + 1)\right)$ *is decreasing in* t *on* (a_1, ∞) *if and only if* $a_1 \geq \ln\frac{3}{2}$.

Proof. The structure of the proof is very similar to the previous one. First, let us assume that $a_1 \geq \ln\frac{3}{2}$. Then $3(1-\epsilon^{-a_1}) \geq 1$. Let $m(t) = \epsilon^{-t}(3t - 3a_1 + 1)$ be defined on (a_1, ∞). Then $m(t) > 0$ and $m(t) \to 0$ as $t \to \infty$. Therefore

$$\frac{1}{t}\ln\left(3(1-\epsilon^{-a_1}) + \epsilon^{-t}(3t - 3a_1 + 1)\right) \geq \frac{1}{t}\ln 3(1-\epsilon^{-a_1}) \geq 0$$

since $a_1 \geq \ln\frac{3}{2}$. Thus $\ln\left(3(1-\epsilon^{-a_1}) + \epsilon^{-t}(3t - 3a_1 + 1)\right)$ is a positive function. Therefore, if $m(t)$ is a decreasing function then $\ln\left(3(1-\epsilon^{-a_1}) + \epsilon^{-t}(3t - 3a_1 + 1)\right)$ is a decreasing function and the result follows. We compute the first derivative of $m(t)$ to determine the monotonicity. Since

$$\frac{dm}{dt} = \epsilon^{-t}(2 + 3a_1 - 3t).$$

$\frac{dm}{dt} > 0$ on $(a_1, a_1 + \frac{2}{3})$, $\frac{dm}{dt} = 0$ at $(a_1, a_1 + \frac{2}{3})$, and $\frac{dm}{dt} < 0$ on $(a_1 + \frac{2}{3}, \infty)$. Then $m(t)$ is increasing on $(a_1, a_1 + \frac{2}{3})$ and decreasing on $(a_1 + \frac{2}{3}, \infty)$. This shows that

$$\frac{1}{t}\ln\left(3(1-\epsilon^{-a_1}) + \epsilon^{-t}(3t - 3a_1 + 1)\right)$$

is decreasing in t on $[a_1 + \frac{2}{3}, \infty)$. Now we show that this property also holds on $(a_1, a_1 + \frac{2}{3})$. Once again, we prove that the first derivative is less than or equal to 0. Indeed,

$$\frac{d}{dt}\left[\frac{1}{t}\ln\left(3(1-e^{-a_1}) + e^{-t}(3t - 3a_1 + 1)\right)\right]$$
$$= \frac{1}{t^2}\left[\frac{e^{-t}(2+3a_1-3t)t}{3(1-e^{-a_1}) + e^{-t}(3t-3a_1+1)} - \ln\left(3(1-e^{-a_1}) + e^{-t}(3t-3a_1+1)\right)\right]$$
$$= \frac{1}{t^2}(L - R),$$

where

$$L = \frac{e^{-t}(2+3a_1-3t)t}{3(1-e^{-a_1}) + e^{-t}(3t-3a_1+1)}$$

and

$$R = \ln\left(3(1-e^{-a_1}) + e^{-t}(3t-3a_1+1)\right).$$

Note

1. $\frac{t}{e^t} \le \frac{1}{e}$ for all t from (3.11).

2. Obviously,

$$0 \le 2 + 3a_1 - 3t \le 2, \quad \forall t \in (a_1, a_1 + \frac{2}{3}). \tag{3.33}$$

3. Since $m(t)$ is increasing on $(a_1, a_1 + \frac{2}{3})$, we have $m(t) \ge e^{-a_1}$ on $(a_1, a_1 + \frac{2}{3})$. Then

$$3(1-e^{-a_1}) + e^{-t}(3t - 3a_1 + 1) \ge 3 - 2e^{-a_1}, \quad \forall t \in (a_1, a_1 + \frac{2}{3}). \tag{3.34}$$

So

$$L = \frac{e^{-t}(2+3a_1-3t)t}{3(1-e^{-a_1}) + e^{-t}(3t-3a_1+1)}$$
$$\le \frac{\frac{1}{e}(2+3a_1-3t)}{3(1-e^{-a_1}) + e^{-t}(3t-3a_1+1)} \quad \text{(from (3.11))}$$

$$\leq \frac{2\frac{1}{e}}{3(1-e^{-a_1})+e^{-t}(3t-3a_1+1)} \quad \text{(from (3.33))}$$
$$\leq \frac{2\frac{1}{e}}{3-2e^{-a_1}} \quad \text{(from (3.34))}$$
$$\leq \frac{6}{5e} \quad (\text{from } a_1 \geq \ln\frac{3}{2})$$

and

$$R = \ln\Big(3(1-e^{-a_1})+e^{-t}(3t-3a_1+1)\Big)$$
$$\geq \ln 3 - 2e^{-a_1} \quad \text{(from (3.34))}$$
$$\geq \ln\frac{5}{3} \quad (\text{from } a_1 \geq \ln\frac{3}{2})$$
$$\geq \frac{6}{5e} \quad \text{(by numerical computation)}$$
$$\geq L$$

for all $t \in (a_1, a_1+\frac{2}{3})$. Hence the first derivative is less than or equal to 0 on that interval. This completes the proof that $\frac{1}{t}\ln\Big(3(1-e^{-a_1})+e^{-t}(3t-3a_1+1)\Big)$ is decreasing in t on (a_1, ∞) if $a_1 \geq \ln\frac{3}{2}$.

Next, if $a_1 < \ln\frac{3}{2}$ then $3(1-e^{-a_1}) < 1$. Then

$$\ln\Big(3(1-e^{-a_1})+e^{-t}(3t-3a_1+1)\Big) \longrightarrow \ln 3(1-e^{-a_1}) < 0.$$

So

$$\frac{1}{t}\ln\Big(3(1-e^{-a_1})+e^{-t}(3t-3a_1+1)\Big) < 0$$

for t sufficiently large and

$$\frac{1}{t}\ln\Big(3(1-e^{-a_1})+e^{-t}(3t-3a_1+1)\Big) \longrightarrow 0$$

as $t \to \infty$. Hence

$$\frac{1}{t}\ln\Big(3(1-e^{-a_1})+e^{-t}(3t-3a_1+1)\Big)$$

can not be decreasing if $a_1 < \ln\frac{3}{2}$. This completes the proof. ■

Combining with Lemma 3.2.4, we have the following theorem:

Theorem 3.2.6. *For $a_1 > 0$ and $b_1 > 1$, T is IFRA if and only if $a_1 \geq \ln \frac{3}{2}$.*

3.2.3 (n-1)-out-of-n System

Let T_i denote the lifetime of component i, $i = 1, \cdots, n$, and T denote the lifetime of the $(n-1)$-out-of-n system. We assume here that $T_1, T_2, \cdots, T_n$ have the joint density described in (2.24). Then T is the second order statistic of $T_1, T_2, \cdots, T_n$. By conditioning on the time t_1 of the first failure among $T_1, T_2, \cdots, T_n$, the density function f_T of T can be computed by using the initial and conditional hazard rates of $T_1, T_2, \cdots, T_n$. We have

$$f_T(t) = \int_0^t n\epsilon^{-nt_1}(n-1)\epsilon^{-(n-1)(t-t_1)}dt_1 = n(n-1)\epsilon^{-(n-1)t}(1-\epsilon^{-t})$$

for $t \leq a_1$, and

$$\begin{aligned} f_T(t) &\\ &= \int_0^{a_1} n\epsilon^{-nt_1}(n-1)\epsilon^{-(n-1)(t-t_1)}dt_1 + \int_{a_1}^t n\epsilon^{-nt_1}(n-1)b_1\epsilon^{-(n-1)b_1(t-t_1)}dt_1 \\ &= n(n-1)\epsilon^{-(n-1)t}(1-\epsilon^{-a_1}) + n(n-1)b_1\epsilon^{-(n-1)b_1 t}\int_{a_1}^t \epsilon^{-(n-(n-1)b_1)t_1}dt_1 \end{aligned}$$

for $t > a_1$. Therefore, for $1 < b_1 \neq \frac{n}{n-1}$ we have

$$f_T(t) = \begin{cases} n(n-1)\epsilon^{-(n-1)t}(1-\epsilon^{-t}), & t \leq a_1; \\ n(n-1)\left[\epsilon^{-(n-1)t}(1-\epsilon^{-a_1}) + \frac{b_1\epsilon^{-nt}(\epsilon^{(n-(n-1)b_1)(t-a_1)}-1)}{n-(n-1)b_1}\right], & t > a_1, \end{cases} \tag{3.35}$$

and for $b_1 = \frac{n}{n-1}$ we have

$$f_T(t) = \begin{cases} n(n-1)\epsilon^{-(n-1)t}(1-\epsilon^{-t}), & t \leq a_1; \\ n(n-1)\left[\epsilon^{-(n-1)t}(1-\epsilon^{-a_1}) + b_1\epsilon^{-nt}(t-a_1)\right], & t > a_1. \end{cases} \tag{3.36}$$

We study these two cases separately.

Case1 : $b_1 \neq \frac{n}{n-1}$.

The survival function $\bar{F}_T$ of T can be computed through the density function (3.35). Indeed.

$$\bar{F}_T(t) = \begin{cases} n\epsilon^{-(n-1)t} - (n-1)\epsilon^{-nt}. & t \leq a_1; \\ n(1-\epsilon^{-a_1})\epsilon^{-(n-1)t} + \frac{\epsilon^{-nt}}{n-(n-1)b_1}\left(n\epsilon^{(n-(n-1)b_1)(t-a_1)} - (n-1)b_1\right). & t > a_1. \end{cases}$$

Lemma 3.2.7. *T is IFRA if and only if $a_1 \geq \ln \frac{n}{n-1}$.*

Proof. We need to show that $-\frac{1}{t}\ln \bar{F}_T(t)$ is increasing in t if and only if $a_1 \geq \ln \frac{n}{n-1}$. From the survival function of T, we have

$$\begin{aligned} -\frac{1}{t}\ln \bar{F}_T(t) &= (n-1) - \frac{1}{t}\ln(n-(n-1)\epsilon^{-t}) \\ &= (n-1) - \frac{1}{t}\int_0^t \frac{(n-1)\epsilon^{-x}}{n-(n-1)\epsilon^{-x}}dx \\ &= \frac{1}{t}\int_0^t (n-1) - \frac{(n-1)\epsilon^{-x}}{n-(n-1)\epsilon^{-x}}dx. \end{aligned}$$

for $t \leq a_1$. Since $0 < (n-1) - \frac{(n-1)\epsilon^{-x}}{n-(n-1)\epsilon^{-x}}$ is increasing in x, $-\frac{1}{t}\ln \bar{F}_T(t)$ is a star-shaped function. This implies that $-\frac{1}{t}\ln \bar{F}_T(t)$ is increasing in t for $t \leq a_1$. For $t > a_1$,

$$\begin{aligned} &-\frac{1}{t}\ln \bar{F}_T(t) \\ &= (n-1) - \frac{1}{t}\ln\left[n(1-\epsilon^{-a_1}) + \frac{\epsilon^{-t}}{n-(n-1)b_1}\left(n\epsilon^{(n-(n-1)b_1)(t-a_1)} - (n-1)b_1\right)\right]. \end{aligned}$$

Then $-\frac{1}{t}\ln \bar{F}_T(t)$ is increasing in t if and only if

$$\frac{1}{t}\ln\left[n(1-\epsilon^{-a_1}) + \frac{\epsilon^{-t}}{n-(n-1)b_1}\left(n\epsilon^{(n-(n-1)b_1)(t-a_1)} - (n-1)b_1\right)\right]$$

is decreasing in t for $t > a_1$. Let

$$m(t) = \frac{\epsilon^{-t}}{n-(n-1)b_1}(n\epsilon^{(n-(n-1)b_1)(t-a_1)} - (n-1)b_1)$$

be defined on (a_1, ∞). Then $m > 0$ for all $t > a_1$ no matter $b_1 > \frac{n}{n-1}$ or $b_1 < \frac{n}{n-1}$. Also

$$m(t) = \frac{1}{n-(n-1)b_1}\left(n e^{n(1-b_1)(t-a_1)} - (n-1)b_1\epsilon^{-t}\right).$$

Thus $m(t) \to 0$ as $t \to \infty$ because $1 < b_1$. If $a_1 < \ln \frac{n}{n-1}$ then $n(1 - \epsilon^{-a_1}) < 1$. This implies that

$$\frac{1}{t} \ln \left[n(1 - \epsilon^{-a_1}) + \frac{\epsilon^{-t}}{n-(n-1)b_1} \left(n\epsilon^{(n-(n-1)b_1)(t-a_1)} - (n-1)b_1 \right) \right] < 0$$

for t sufficiently large because $m(t) \to 0$. But

$$\frac{1}{t} \ln \left[n(1 - \epsilon^{-a_1}) + \frac{\epsilon^{-t}}{n-(n-1)b_1} \left(n\epsilon^{(n-(n-1)b_1)(t-a_1)} - (n-1)b_1 \right) \right] \to 0$$

as $t \to \infty$(again. $m(t) \to 0$). Therefore

$$\frac{1}{t} \ln \left[n(1 - \epsilon^{-a_1}) + \frac{\epsilon^{-t}}{n-(n-1)b_1} \left(n\epsilon^{(n-(n-1)b_1)(t-a_1)} - (n-1)b_1 \right) \right]$$

is not decreasing in t. This means that T is not IFRA if $a_1 < \ln \frac{n}{n-1}$.

Now let us assume that $a_1 \geq \ln \frac{n}{n-1}$. We show that

$$\frac{1}{t} \ln \left[n(1 - \epsilon^{-a_1}) + \frac{\epsilon^{-t}}{n-(n-1)b_1} \left(n\epsilon^{(n-(n-1)b_1)(t-a_1)} - (n-1)b_1 \right) \right]$$

is decreasing in t. Since $a_1 \geq \ln \frac{n}{n-1}$. we have $n(1 - \epsilon^{-a_1}) \geq 1$. Also $m(t) > 0$ for all $t > a_1$. Then

$$\ln \left[n(1 - \epsilon^{-a_1}) + \frac{\epsilon^{-t}}{n-(n-1)b_1} \left(n\epsilon^{(n-(n-1)b_1)(t-a_1)} - (n-1)b_1 \right) \right] \geq \ln n(1 - \epsilon^{-a_1}) \geq 0$$

for all $t > a_1$. If we can show that $m(t)$ is decreasing in t then

$$\frac{1}{t} \ln \left[n(1 - \epsilon^{-a_1}) + \frac{\epsilon^{-t}}{n-(n-1)b_1} \left(n\epsilon^{(n-(n-1)b_1)(t-a_1)} - (n-1)b_1 \right) \right]$$

is a positive and decreasing function and then the result will follow. Clearly.

$$\frac{d}{dt} m(t) = \frac{(n-1)\epsilon^{-t}}{n-(n-1)b_1} \left(n(1-b)\epsilon^{(n-(n-1)b_1)(t-a_1)} + b_1 \right).$$

Note

1. $\epsilon^{-t} > 0$ for all t.

2. $\frac{n(1-b_1)e^{(n-(n-1)b_1)(t-a_1)}+b_1}{n-(n-1)b_1}$ has value 1 at $t = a_1$.

3. $e^{(n-(n-1)b_1)(t-a_1)} \to \infty$ as $t \to \infty$ if $b_1 < \frac{n}{n-1}$. Under this situation.

$$\frac{n(1-b_1)e^{(n-(n-1)b_1)(t-a_1)}+b_1}{n-(n-1)b_1} < 0$$

for t sufficiently large since $1 < b_1$. Or $e^{(n-(n-1)b_1)(t-a_1)} \to 0$ as $t \to \infty$ if $b_1 > \frac{n}{n-1}$. In this case, still.

$$\frac{n(1-b_1)e^{(n-(n-1)b_1)(t-a_1)}+b_1}{n-(n-1)b_1} < 0$$

for t sufficiently large since $b_1 > \frac{n}{n-1}$. Hence

$$\frac{n(1-b_1)e^{(n-(n-1)b_1)(t-a_1)}+b_1}{n-(n-1)b_1} < 0$$

for t sufficiently large no matter $b_1 < \frac{n}{n-1}$ or $b_1 > \frac{n}{n-1}$.

From above, $\frac{d}{dt}m > 0$ on $(a_1, t_0(b_1))$, $\frac{d}{dt}m = 0$ at $t_0(b_1)$, and $\frac{d}{dt}m < 0$ on $(t_0(b_1), \infty)$ for some $t_0(b_1) \in (a_1, \infty)$. Therefore $m(t)$ is increasing in t on the interval $(a_1, t_0(b_1))$ and decreasing in t on the interval $[t_0(b_1), \infty)$. This means that

$$\frac{1}{t}\ln\left[n(1-e^{-a_1}) + \frac{e^{-t}}{n-(n-1)b_1}\left(ne^{(n-(n-1)b_1)(t-a_1)} - (n-1)b_1\right)\right]$$

is decreasing in t on the interval $[t_0(b_1), \infty)$. Now we show that this also holds for $a_1 < t < t_0(b_1)$. We accomplish this task by showing that the first derivative is negative. Clearly.

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{t}\ln\left[n(1-e^{-a_1}) + \frac{e^{-t}}{n-(n-1)b_1}\left(ne^{(n-(n-1)b_1)(t-a_1)} - (n-1)b_1\right)\right]\right) \\
&= \frac{1}{t^2}\left[\frac{\frac{(n-1)e^{-t}}{n-(n-1)b_1}(n(1-b_1)e^{(n-(n-1)b_1)(t-a_1)} + b_1)t}{n(1-e^{-a_1}) + \frac{e^{-t}}{n-(n-1)b_1}(ne^{(n-(n-1)b_1)(t-a_1)} - (n-1)b_1)}\right. \\
&\quad \left. - \ln(n(1-e^{-a_1}) + \frac{e^{-t}}{n-(n-1)b_1}\left(ne^{(n-(n-1)b_1)(t-a_1)} - (n-1)b_1\right)\right]. \qquad (3.37)
\end{aligned}$$

Let us denote the left hand side of (3.37) by $\frac{1}{t^2}D(t)$. We show that $D(t) < 0$ on $(a_1, t_0(b_1))$ and this will complete the proof.

Claim: $D(t) < 0$ on $(a_1, t_0(b_1))$.

We show that the first derivative of D is negative and this implies D is decreasing. Also, we show that $D(a_1) < 0$. Combining these, we have $D < 0$. Indeed,

$$\frac{d}{dt}D(t) = -\frac{\frac{(n-1)e^{-t}}{n-(n-1)b_1}(n(1-b_1)e^{(n-(n-1)b_1)(t-a_1)} + b_1)}{n(1-e^{-a_1}) + \frac{e^{-t}}{n-(n-1)b_1}(ne^{(n-(n-1)b_1)(t-a_1)} - (n-1)b_1)}$$
$$+ \frac{(n-1)e^{-t}n(1-b_1)e^{(n-(n-1)b_1)(t-a_1)}t}{n(1-e^{-a_1}) + \frac{e^{-t}}{n-(n-1)b_1}(ne^{(n-(n-1)b_1)(t-a_1)} - (n-1)b_1}$$
$$- t\left(\frac{\frac{(n-1)e^{-t}}{n-(n-1)b_1}(n(1-b_1)e^{(n-(n-1)b_1)(t-a_1)} + b_1)}{n(1-e^{-a_1}) + \frac{e^{-t}}{n-(n-1)b_1}(ne^{(n-(n-1)b_1)(t-a_1)} - (n-1)b_1)}\right)^2.$$

Note

1. The denominator of the first two fraction is positive since

$$n(1-e^{-a_1}) + \frac{e^{-t}}{n-(n-1)b_1}\left(ne^{(n-(n-1)b_1)(t-a_1)} - (n-1)b_1\right)$$
$$= n(1-e^{-a_1}) + m(t) > 0.$$

2. The numerator of the first term is exactly $\frac{d}{dt}m$ and $\frac{d}{dt}m > 0$ on $(a_1, t_0(b_1))$.

3. The numerator of the second term is negative since $1 < b_1$.

From above, we have $\frac{d}{dt}D < 0$. Now we show that $D(a_1) < 0$ for all $a_1 \geq \ln \frac{n}{n-1}$. It suffices to show that as a function of a_1, $D(a_1)$ is decreasing and $D(\ln \frac{n}{n-1}) < 0$ for all $n \geq 2$. Clearly,

$$D(a_1) = \frac{(n-1)e^{-a_1}a_1}{n-(n-1)e^{-a_1}} - \ln(n-(n-1)e^{-a_1})$$
$$= \frac{(n-1)a_1}{ne^{a_1} - (n-1)} - \ln(n-(n-1)e^{-a_1}).$$

and

$$D'(a_1) = -\frac{n(n-1)a_1e^{a_1}}{(ne^{a_1} - (n-1))^2} < 0.$$

Therefore $D(a_1)$ is a decreasing function in a_1. Now we show that $D(\ln \frac{n}{n-1}) < 0$ for all $n \geq 2$. Since

$$D(\ln \frac{n}{n-1}) = \frac{(n-1)^2 \ln \frac{n}{n-1}}{2n-1} - \ln \frac{2n-1}{n}.$$

we have $D(\ln \frac{n}{n-1}) < 0$ for $n = 2, 3$. We still have to prove this is also true for all $n \geq 4$. Let $E(x)$ be defined as

$$E(x) = \frac{(x-1)^2 \ln \frac{x}{x-1}}{2x-1} - \ln \frac{2x-1}{x}$$

for all $x \geq 3$. So $E(x) = D(\ln \frac{x}{x-1})$. We show that $E(x) < 0$ for all $x \geq 3$. This will complete the proof. Clearly $E(3) < 0$. By calculus,

$$\frac{d}{dx}E = \frac{2x(x-1)\ln \frac{x}{x-1}}{(2x-1)^2} - \frac{1}{2x-1} = \frac{2x(x-1)\ln \frac{x}{x-1} - (2x-1)}{(2x-1)^2}.$$

Then $\frac{d}{dx}E < 0$ if and only if $2x(x-1)\ln \frac{x}{x-1} - (2x-1) < 0$. Since $\ln \frac{x}{x-1} = \ln(1 + \frac{1}{x-1})$ is an analytic function on $(0, \infty)$, we can apply the Taylor Theorem and it shows that

$$\ln(1 + \frac{1}{x-1}) < \frac{1}{x-1} - \frac{1}{2}(\frac{1}{x-1})^2 + \frac{1}{3}(\frac{1}{x-1})^3$$

for all $x \geq 3$. This implies that

$$2x(x-1)\ln \frac{x}{x-1} - (2x-1) < \frac{3-x}{3(x-1)^2} \leq 0$$

for all $x \geq 3$. Thus $\frac{d}{dx}E < 0$. Then E is a decreasing function of x. Hence $E(x) < 0$ for all $x \leq 3$. This proves that $D(\ln \frac{n}{n-1}) < 0$ for all $n \geq 3$. This completes the proof. ■

Case2 : $b_1 = \frac{n}{n-1}$.

In this case, we have

$$\bar{F}_T(t) = \begin{cases} ne^{-(n-1)t} - (n-1)e^{nt}, & t \leq a_1; \\ n(1 - e^{-a_1})e^{-(n-1)t} + e^{-nt}(nt - na_1 + 1), & t > a_1. \end{cases}$$

Under this situation, we still have the IFRA property for T.

Lemma 3.2.8. *T is IFRA if and only if $a_1 \geq \ln \frac{n}{n-1}$.*

Proof. We show that $-\frac{1}{t} \ln \bar{F}_T(t)$ is increasing in t if and only if $a_1 \geq \ln \frac{n}{n-1}$. From the previous subsection, we know that $-\frac{1}{t} \ln \bar{F}_T(t)$ is increasing in t on the interval $(0, a_1]$.Also

$$-\frac{1}{t} \ln \bar{F}_T(t) = (n-1) - \frac{1}{t} \ln \left[n(1 - e^{-a_1}) + e^{-t}(nt - na_1 + 1) \right].$$

Hence it suffices to show that

$$\frac{1}{t} \ln \left[n(1 - e^{-a_1}) + e^{-t}(nt - na_1 + 1) \right]$$

is decreasing in t on (a_1, ∞) if and only if $a_1 \geq \ln \frac{n}{n-1}$. The structure of the proof is similar to the one in previous case. Let $m(t) = e^{-t}(nt - an_1 + 1)$ be defined on (a_1, ∞). Then $m > 0$ and $m(t) \to 0$ as $t \to \infty$. If $a_1 < \ln \frac{n}{n-1}$ then $n(1 - e^{-a_1}) < 1$. This implies that

$$\frac{1}{t} \ln \left[n(1 - e^{-a_1}) + e^{-t}(nt - na_1 + 1) \right] < 0$$

for t sufficiently large since $m(t) \to 0$ as $t \to \infty$. But

$$\frac{1}{t} \ln \left[n(1 - e^{-a_1}) + e^{-t}(nt - na_1 + 1) \right] \to 0$$

as $t \to \infty$ (again, $m(t) \to 0$). This shows that

$$\frac{1}{t} \ln \left[n(1 - e^{-a_1}) + e^{-t}(nt - na_1 + 1) \right]$$

is not decreasing in t. Therefore T is not IFRA if $a_1 < \ln \frac{n}{n-1}$.

Now let us assume that $a_1 \geq \ln \frac{n}{n-1}$. Then $n(1 - e^{-a_1}) \geq 1$. Since $m(t) > 0$ for all $t > a_1$, we have

$$\ln \left[n(1 - e^{-a_1}) + e^{-t}(nt - na_1 + 1) \right] > \ln n(1 - e^{-a_1}) \geq 0.$$

If we can show that $m(t)$ is decreasing in t then

$$\frac{1}{t} \ln \left[n(1 - e^{-a_1}) + e^{-t}(nt - na_1 + 1) \right]$$

is a positive and decreasing function of t and then the result follows. Indeed,

$$\frac{d}{dt}m(t) = \epsilon^{-t}(n-1+na_1-nt)\begin{cases} > 0, & \forall t \in (a_1, a_1+\frac{n-1}{n}); \\ = 0, & t = a_1+\frac{n-1}{n}; \\ < 0, & \forall t \in (a_1+\frac{n-1}{n}, \infty). \end{cases}$$

So $\epsilon^{-t}(nt-na_1+1)$ increases on $(a_1, a_1+\frac{n-1}{n})$ and decreases on $[a_1+\frac{n-1}{n}, \infty)$. Hence we need to show that

$$\frac{1}{t}\ln\left[n(1-e^{-a_1})+\epsilon^{-t}(nt-na_1+1)\right]$$

is decreasing on $(a_1, a_1+\frac{n-1}{n})$. To achieve this aim, we show that the first derivative is negative. By calculus,

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{t}\ln\left(n(1-e^{-a_1})+\epsilon^{-t}(nt-na_1+1)\right)\right] \\ &= \frac{1}{t^2}\left[\frac{\epsilon^{-t}(n-1+na_1-nt)t}{n(1-\epsilon^{-a_1})+\epsilon^{-t}(nt-na_1+1)} - \ln\left(n(1-\epsilon^{-a_1})+\epsilon^{-t}(nt-na_1+1)\right)\right]. \end{aligned}$$

This is negative if and only if the term inside the bracket is negative. Let us denote it by $D(t)$, that is,

$$D(t) = \frac{\epsilon^{-t}(n-1+na_1-nt)t}{n(1-\epsilon^{-a_1})+\epsilon^{-t}(nt-na_1+1)} - \ln\left(n(1-\epsilon^{-a_1})+\epsilon^{-t}(nt-na_1+1)\right).$$

To show $D(t)<0$ it suffices to show that $D'(t)<0$ and $D(a_1)<0$. Once again, by applying calculus

$$\begin{aligned} &D'(t) \\ &= -\frac{\epsilon^{-t}(n-1+na_1-nt)t+\epsilon^{-t}nt}{n(1-\epsilon^{-a_1})+\epsilon^{-t}(nt-na_1+1)} - \frac{(\epsilon^{-t}(n-1+na_1-nt))^2 t}{(n(1-\epsilon^{-a_1})+\epsilon^{-t}(nt-na_1+1))^2} \\ &< 0. \end{aligned}$$

Also, $D(a_1) = \frac{(n-1)a_1}{ne^{a_1}-(n-1)} - \ln(n-(n-1)\epsilon^{-a_1})$. This is exactly the same as the one in previous case. Therefore $D(a_1)<0$. This completes the proof.

Combining Lemmas 3.2.7 and 3.2.8, we have the following theorem:

Theorem 3.2.9. *For $a_1 > 0$ and $b_1 > 1$, T is IFRA if and only if $a_1 \geq \ln\frac{n}{n-1}$.*

Chapter 4
Statistical Analysis

In this chapter, we study the method of moments estimator (MME) and the maximum likelihood estimator (MLE) for (a, b) from the new bivariate distribution with density

$$f(t_1, t_2) = \begin{cases} e^{-t_1-t_2}, & t_1 \le a \text{ or } t_2 \le a, \\ be^{-(2-b)t_1-bt_2}, & a < t_1 \le t_2, \\ be^{-(2-b)t_2-bt_1}, & a < t_2 < t_1. \end{cases}$$

First, we apply the method of moments to derive the estimator, $(\hat{a}, \hat{b})$, and then we prove that it is a consistent asymptotically normal (CAN) estimator. In the second section, we derive the maximum likelihood estimator. We start with a sample of 2 observations. For this case, we use a graph to illustrate how to determine the maximum likelihood estimator. Then we compute the general rule. In Section 4.3, we use the software Mathematica to do some numerical analysis. We compare the MME with the MLE. We also compute the 95% confidence interval from the MME.

4.1 Method of Moments and CAN Estimator

In this section we study the following questions: how to estimate a and b from the new bivariate distribution of Chapter 2 and what property do the estimators possess? First we apply the method of moments to get estimators $\hat{a}$ and $\hat{b}$ for a and b, respectively. Then we prove that $(\hat{a}, \hat{b})$ is a CAN estimator.

From Section 2.1, we have

$$ET_1 = 1 + e^{-2a}\left(\frac{1-b}{2b}\right) \quad \text{and} \quad ET_1T_2 = 1 + e^{-2a}\left(\frac{1-b}{2b}\right)(2a+1). \tag{4.1}$$

Solving a and b from these 2 equations, we get

$$a = \frac{ET_1T_2 - 1}{2(ET_1 - 1)} - \frac{1}{2} \quad \text{and} \quad b = \frac{1}{2e^{2a}(ET_1 - 1) + 1}.$$

Thus, we estimate a and b by

$$\hat{a}_n = \frac{\frac{1}{n}\sum_{i=1}^{n} T_{i,1}T_{i,2} - 1}{2\left(\frac{1}{n}\sum_{i=1}^{n} T_{i,1} - 1\right)} - \frac{1}{2} \tag{4.2}$$

and

$$\hat{b}_n = \frac{1}{2e^{2\hat{a}_n}\left(\frac{1}{n}\sum_{i=1}^{n} T_{i,1} - 1\right) + 1} \tag{4.3}$$

where $(T_{1,1}, T_{1,2}), (T_{2,1}, T_{2,1}), \cdots, (T_{n,1}, T_{n,2})$ are a sample of size n from the new bivariate distribution.

In the following we prove that $(\hat{a}_n, \hat{b}_n)$ is a CAN estimator.

Definition 4.1.1. A 2-dimensional random vector $\mathbf{X}$ is said to have a 2-dimensional normal distribution if and only if for any vector $\mathbf{u}$ the random variable $\mathbf{u}^T\mathbf{X}$ has a normal distribution.

Definition 4.1.2. An estimator $(\hat{a}_n, \hat{b}_n)$ for (a, b) is called a consistent asymptotically normal (CAN) estimator if and only if $\sqrt{n}(\hat{a}_n - a, \hat{b}_n - b)$ converges in distribution to a 2-dimensional normal random vector with mean 0 and variance $(\sigma_1^2(a, b), \sigma_2^2(a, b))$ for some positive functions σ_1 and σ_2.

To prove that $(\hat{a}_n, \hat{b}_n)$ is a CAN estimator, we need the following theorems. These three theorems are from Dudewicz and Mishra (1988) Corollary 6.3.13(iv),p.323, Corollary 6.3.14(iii),p.323, and Theorem 6.3.15(i)(iv)(v),p.324, respectively.

Theorem 4.1.3. *If* $X_n \xrightarrow{p} X$ *and* g *is a continuous function, then* $g(X_n) \xrightarrow{p} g(X)$.

Theorem 4.1.4. *If* $X_n \xrightarrow{p} x$ *and* $Y_n \xrightarrow{P} y$ *for some constants* x *and* $y \neq 0$, *then* $X_n/Y_n \xrightarrow{p} x/y$.

Theorem 4.1.5. *(a) If* $X_n \xrightarrow{d} X$ *and* $X_n - Y_n \xrightarrow{p} 0$, *then* $Y_n \xrightarrow{d} X$.

(b) If $X_n \xrightarrow{d} X$ *and* $Y_n \xrightarrow{p} 0$, *then* $X_nY_n \xrightarrow{p} 0$.

(c) If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c$ for some constant $c \neq 0$, then $X_n/Y_n \xrightarrow{d} X/c$.

Remark: No independence relationship between X_n and Y_n is needed in all these three theorems.

Now we can apply these three theorems, the Law of Large Numbers, and the Central Limit Theorem to prove the main result of this section.

Theorem 4.1.6. *$(\hat{a}_n, \hat{b}_n)$ is a CAN estimator.*

Proof. Let (u_1, u_2) be any vector in $\mathbb{R}^2$. From Definitions 4.1.1 and 4.1.2, it suffices to show that $\sqrt{n}(u_1(\hat{a}_n - a) + u_2(\hat{b}_n - b))$ converges in distribution to a normal distribution. From (4.2) and (4.3), we have

$$\begin{aligned}\hat{a}_n - a &= \frac{\frac{\sum_{i=1}^n T_{i,1}T_{i,2}}{n} - 1}{2(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1)} - \frac{1}{2} - a \\ &= \frac{\frac{\sum_{i=1}^n T_{i,1}T_{i,2}}{n} - 1 - (2a+1)(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1)}{2(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1)}\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\hat{b}_n - b &= \frac{1}{2e^{2\hat{a}_n}(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1) + 1} - b \\ &= \frac{(1-b) - 2be^{2\hat{a}_n}(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1)}{2e^{2\hat{a}_n}(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1) + 1}.\end{aligned} \tag{4.5}$$

Then

$$\begin{aligned}&\sqrt{n}\Big(u_1(\hat{a}_n - a) + u_2(\hat{b}_n - b)\Big) \\ &= \sqrt{n}\Bigg[\frac{u_1\Big(\frac{\sum_{i=1}^n T_{i,1}T_{i,2}}{n} - 1 - (2a+1)(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1)\Big)}{2(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1)} \\ &\qquad + \frac{u_2\Big((1-b) - 2be^{2\hat{a}_n}(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1)\Big)}{2e^{2\hat{a}_n}(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1) + 1}\Bigg] \\ &= \frac{N_n}{D_n}.\end{aligned}$$

where

$$\mathcal{X}_n = \sqrt{n}\Bigg[u_1\left(\frac{\sum_{i=1}^n T_{i,1}T_{i,2}}{n} - 1 - (2a+1)(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1)\right) \times \left(2e^{2\hat{a}_n}(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1) + 1\right) + u_2\left((1-b) - 2be^{2\hat{a}_n}(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1)\right)\cdot 2\left(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1\right)\Bigg] \quad (4.6)$$

and

$$D_n = 2\left(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1\right)\left(2e^{2\hat{a}_n}(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1) + 1\right). \quad (4.7)$$

From Theorem 4.1.5(c), if we can show that: 1. $\mathcal{X}_n$ converges in distribution to some normal random variable and: 2. D_n converges in probability to some nonzero constant then the result follows. We prove these two statements separately in two lemmas.

Lemma 4.1.7. $\mathcal{X}_n \xrightarrow{d} \mathcal{X}$ *for some normal variable* $\mathcal{X}$.

Proof. The idea of this proof is as follow. We rewrite $\mathcal{X}_n$ as a summation of six sums of random variables. Then we show that four of them converge in probability to 0 and the summation of the other two converges in distribution to a normal distribution. These together with Theorem 4.1.5(a) imply that $\mathcal{X}_n$ converges to a normal variable in distribution.

Indeed, from (4.6), we have

$$\begin{aligned}&\mathcal{X}_n\\ &= \sqrt{n}\Bigg[u_1\left(\frac{\sum_{i=1}^n T_{i,1}T_{i,2}}{n} - 1 - (2a+1)(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1)\right)\\ &\times\left(2e^{2\hat{a}_n}(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1) + 1\right)\\ &+ u_2\left((1-b) - 2be^{2\hat{a}_n}(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1)\right)\cdot 2\left(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1\right)\Bigg]\end{aligned}$$

$$= \frac{u_1 \sum_{i=1}^{n} (T_{i,1}T_{i,2} - 1 - (2a+1)(T_{i,1} - 1))}{\sqrt{n}}$$
$$\times \left[\left(2\epsilon^{2\hat{a}_n}(\frac{\sum_{i=1}^{n} T_{i,1}}{n} - 1) + 1 - \frac{1}{b}\right) + \frac{1}{b}\right]$$
$$+ \sqrt{n}u_2 \left((1-b) - 2b\epsilon^{2a}(\frac{\sum_{i=1}^{n} T_{i,1}}{n} - 1) + 2b(\epsilon^{2a} - \epsilon^{2\hat{a}_n})(\frac{\sum_{i=}^{n} T_{i,1}}{n} - 1)\right)$$
$$\times \left[\left(2(\frac{\sum_{i}^{n} T_{i,1}}{n} - 1) - \epsilon^{-2a}(\frac{1-b}{b})\right) + \epsilon^{-2a}\left(\frac{1-b}{b}\right)\right]$$
$$= \frac{u_1 \sum_{i=1}^{n} (T_{i,1}T_{i,2} - 1 - (2a+1)(T_{i,1} - 1))}{\sqrt{n}} \left[2\epsilon^{2\hat{a}_n}(\frac{\sum_{i=1}^{n} T_{i,1}}{n} - 1) + 1 - \frac{1}{b}\right] \tag{4.8}$$
$$+ \frac{u_1 \sum_{i=1}^{n} (T_{i,1}T_{i,2} - 1 - (2a+1)(T_{i,1} - 1))}{\sqrt{n}} \cdot \frac{1}{b} \tag{4.9}$$
$$+ \frac{u_2 \sum_{i=1}^{n} (1 - b - 2b\epsilon^{2a}(T_{i,1} - 1))}{\sqrt{n}} \left[2(\frac{\sum_{i=1}^{n} T_{i,1}}{n} - 1) - \epsilon^{-2a}(\frac{1-b}{b})\right] \tag{4.10}$$
$$+ \frac{u_2 \sum_{i=1}^{n} (1 - b - 2b\epsilon^{2a}(T_{i,1} - 1))}{\sqrt{n}} \cdot \epsilon^{-2a}\left(\frac{1-b}{b}\right) \tag{4.11}$$
$$+ 2bu_2(\epsilon^{2a} - \epsilon^{2\hat{a}_n})\left(\frac{\sum_{i=}^{n}(T_{i,1} - 1)}{\sqrt{n}}\right)\left[2(\frac{\sum_{i=1}^{n} T_{i,1}}{n} - 1) - \epsilon^{-2a}(\frac{1-b}{b})\right] \tag{4.12}$$
$$+ 2bu_2(\epsilon^{2a} - \epsilon^{2\hat{a}_n})\left(\frac{\sum_{i=}^{n}(T_{i,1} - 1)}{\sqrt{n}}\right) \cdot \epsilon^{-2a}\left(\frac{1-b}{b}\right). \tag{4.13}$$

Therefore X_n is a summation of six sums of random variables. We show that (4.8), (4.10), (4.12), and (4.13) all converge in probability to 0. This implies the summation of these four series still converges in probability to 0. We also show that the summation of (4.9) and (4.11) converges in distribution to a normal variable. This completes the proof.

From the Law of Large Numbers, we have

$$\frac{\sum_{i=1}^{n} T_{i,1}}{n} \xrightarrow{p} 1 + \epsilon^{-2a}\left(\frac{1-b}{2b}\right) \tag{4.14}$$

and

$$\frac{\sum_{i=1}^{n} T_{i,1}T_{i,2}}{n} \xrightarrow{p} 1 + \epsilon^{-2a}\left(\frac{1-b}{2b}\right)(2a+1). \tag{4.15}$$

Applying Theorems 4.1.3 and 4.1.4, (4.14) and (4.15) imply

$$\hat{a}_n = \frac{\frac{\sum_{i=1}^{n} T_1^i T_2^i}{n} - 1}{2(\frac{\sum_{i=1}^{n} T_1^i}{n} - 1)} - \frac{1}{2} \xrightarrow{p} a$$

and this combining with Theorem 4.1.3 shows

$$\epsilon^{2\hat{a}_n} \xrightarrow{p} \epsilon^{2a}. \tag{4.16}$$

From Section 1.1, we have

$$ET_1^2 = 2 + \frac{\epsilon^{-2a}(1-b)}{2b}(2a + 3 + \frac{2}{b}). \tag{4.17}$$

$$ET_1^2 T_2^2 = 4 + \frac{\epsilon^{-2a}(1-b)}{2b}\left(4a^3 + \frac{2a^2(2+5b)}{b} + \frac{2a(2+5b)}{b} + \frac{2+5b}{b}\right). \tag{4.18}$$

and

$$ET_1^2 T_2 = 2 + \frac{\epsilon^{-2a}(1-b)}{2b}\left(3a^2 + \frac{a(2+5b)}{b} + \frac{2+5b}{2b}\right). \tag{4.19}$$

Combining with (4.1), these imply that T_1, T_1T_2 and linear combination of T_1 and T_1T_2 all have finite mean and variance then we can apply the Central Limit Theorem (CLT).

By (4.14), (4.15), (4.16), and CLT, we show that (4.8), (4.10), (4.12), and (4.13) all converge in probability to 0 in the following.

1. From (4.1), $T_1T_2 - 1 - (2a+1)(T_1 - 1)$ has mean 0 and finite variance. Then

$$\frac{u_1 \sum_{i=1}^{n} (T_{i,1}T_{i,2} - 1 - (2a+1)(T_{i,1} - 1))}{\sqrt{n}} \xrightarrow{d} \text{some normal variable}$$

   by CLT. Also

$$2\epsilon^{2\hat{a}_n}\left(\frac{\sum_{i=1}^{n} T_{i,1}}{n} - 1\right) + 1 - \frac{1}{b} \xrightarrow{p} 2\epsilon^{2a} \cdot \epsilon^{-2a}(\frac{1-b}{2b}) + 1 - \frac{1}{b} = 0.$$

   Then the product of these two converges to 0 in probability. That is, (4.8) converges to 0 in probability from Theorem 4.1.5(b).

2. From (4.1), $1-b-2b\epsilon^{2a}(T_1-1)$ has mean 0 and finite variance. Then

$$\frac{u_2\sum_{i=1}^n(1-b-2b\epsilon^{2a}(T_{i,1}-1))}{\sqrt{n}} \xrightarrow{d} \text{some normal variable}$$

by CLT. Also

$$2\left(\frac{\sum_{i=1}^n T_{i,1}}{n}-1\right)-\epsilon^{-2a}\left(\frac{1-b}{b}\right) \xrightarrow{p} 2\cdot\epsilon^{-2a}(\frac{1-b}{2b})-\epsilon^{-2a}(\frac{1-b}{b})=0.$$

Then the product of these two converges to 0 in probability. That is, (4.10) converges to 0 in probability from Theorem 4.1.5(b).

3. From (4.1), T_1-1 has mean $\epsilon^{-2a}(\frac{1-b}{2b})$ and finite variance. Then

$$\frac{\sum_{i=}^n(T_{i,1}-1)}{\sqrt{n}} \xrightarrow{d} \text{some normal variable}$$

by CLT. Also

$$2bu_2(\epsilon^{2a}-\epsilon^{2\hat{a}_n})\left[2\left(\frac{\sum_{i=1}^n T_{i,1}}{n}-1\right)-\epsilon^{-2a}\left(\frac{1-b}{b}\right)\right] \xrightarrow{p} 0. \quad (\text{from}(4.16))$$

Then the product of these two converges to 0 in probability. That is, (4.12) converges to 0 in probability from Theorem 4.1.5(b).

4. From (4.1), T_1-1 has mean $\epsilon^{-2a}(\frac{1-b}{2b})$ and finite variance. Then

$$\frac{\sum_{i=}^n(T_{i,1}-1)}{\sqrt{n}} \xrightarrow{d} \text{some normal variable}$$

by CLT. Also

$$2bu_2(\epsilon^{2a}-\epsilon^{2\hat{a}_n})\epsilon^{-2a}\left(\frac{1-b}{b}\right) \xrightarrow{p} 0. \quad (\text{from}(4.16))$$

Then the product of these two converges to 0 in probability. That is, (4.13) converges to 0 in probability from Theorem 4.1.5(b).

Now we show that the summation of (4.9) and (4.11) converges in distribution to a normal variable. The summation is

$$\frac{u_1\sum_{i=1}^n(T_{i,1}T_{i,2}-1-(2a+1)(T_{i,1}-1))}{\sqrt{n}}\cdot\frac{1}{b}$$
$$+\frac{u_2\sum_{i=1}^n(1-b-2b\epsilon^{2a}(T_{i,1}-1))}{\sqrt{n}}\cdot\epsilon^{-2a}\left(\frac{1-b}{b}\right)$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\frac{u_1}{b} (T_{i,1}T_{i,2} - 1 - (2a+1)(T_{i,1} - 1)) \right.$$
$$\left. + u_2 \epsilon^{-2a} \left(\frac{1-b}{b} \right) (1 - b - 2b\epsilon^{2a}(T_{i,1} - 1)) \right]. \qquad (4.20)$$

Let X be defined as

$$X = \frac{u_1}{b}\left(T_1 T_2 - 1 - (2a+1)(T_1 - 1) \right) + u_2 \epsilon^{-2a} \left(\frac{1-b}{b} \right) \left(1 - b - 2b\epsilon^{2a}(T_1 - 1) \right).$$

We use (4.1). (4.17). (4.18). and (4.19) to show that X has mean 0 and finite variance. Therefore (4.20) is convergent in distribution to some normal variable by CLT.

Indeed.

$$\begin{aligned} X &= \frac{u_1}{b}\left(T_1 T_2 - 1 - (2a+1)(T_1 - 1) \right) + u_2 \epsilon^{-2a} \left(\frac{1-b}{b} \right) \left(1 - b - 2b\epsilon^{2a}(T_1 - 1) \right) \\ &= \frac{u_1}{b} T_1 T_2 - \frac{u_1}{b} - \frac{u_1}{b}(2a+1)(T_1 - 1) + \frac{u_2 \epsilon^{-2a}(1-b)^2}{b} - 2u_2(1-b)(T_1 - 1) \\ &= \frac{u_1}{b} T_1 T_2 - \frac{u_1(2a+1) + 2u_2 b(1-b)}{b} T_1 + \frac{2u_1 a + 2u_2 b(1-b) + u_2 \epsilon^{-2a}(1-b)^2}{b}. \end{aligned}$$

Then

$$\begin{aligned} EX &= \frac{u_1}{b} ET_1T_2 - \left(\frac{u_1(2a+1)}{b} + 2u_2(1-b) \right) ET_1 \\ &\quad + \left[\frac{2au_1}{b} + u_2\left(2(1-b) + \frac{\epsilon^{-2a}(1-b)^2}{b} \right) \right] \\ &= \frac{u_1}{b}\left[1 + \epsilon^{-2a}\left(\frac{1-b}{2b} \right)(2a+1) \right] - \left(\frac{u_1(2a+1)}{b} + 2u_2(1-b) \right) \\ &\quad \times \left[1 + \epsilon^{-2a}\left(\frac{1-b}{2b} \right) \right] + \left[\frac{2au_1}{b} + u_2\left(2(1-b) + \frac{\epsilon^{-2a}(1-b)^2}{b} \right) \right] \\ &= \left(\frac{u_1}{b} - \frac{u_1(2a+1)}{b} - 2u_2(1-b) + \frac{2au_1}{b} + 2u_2(1-b) \right) \\ &\quad + \epsilon^{-2a}\frac{(1-b)}{2b}\left(\frac{u_1}{b}(2a+1) - \frac{u_1(2a+1)}{b} - 2u_2(1-b) + 2u_2(1-b) \right) \\ &= 0. \end{aligned}$$

Thus the variance of X is equal to EX^2 and

$$
\begin{aligned}
&EX^2 \\
&= \left(\frac{u_1}{b}\right)^2 ET_1^2T_2^2 + \left(\frac{u_1(2a+1)}{b} + 2u_2(1-b)\right)^2 ET_1^2 \\
&\quad + \left[\frac{2au_1}{b} + u_2\left(2(1-b) + \frac{\epsilon^{-2a}(1-b)^2}{b}\right)\right]^2 \\
&\quad - \frac{2u_1}{b}\left(\frac{u_1(2a+1)}{b} + 2u_2(1-b)\right) ET_1^2T_2 \\
&\quad + \frac{2u_1}{b}\left[\frac{2au_1}{b} + u_2\left(2(1-b) + \frac{\epsilon^{-2a}(1-b)^2}{b}\right)\right] ET_1T_2 \\
&\quad - 2\left(\frac{u_1(2a+1)}{b} + 2u_2(1-b)\right)\left[\frac{2au_1}{b} + u_2\left(2(1-b) + \frac{\epsilon^{-2a}(1-b)^2}{b}\right)\right] ET_1 \\
&= u_1^2\left[\frac{1}{b^2}ET_1^2T_2^2 + \frac{(2a+1)^2}{b^2}ET_1^2 + \frac{4a^2}{b^2} + \frac{4a}{b^2}ET_1T_2 - \frac{4a(2a+1)}{b^2}ET_1\right. \\
&\qquad \left. - \frac{2(2a+1)}{b^2}ET_1^2T_2\right] \\
&\quad + u_1u_2\left[\frac{4(2a+1)(1-b)}{b}ET_1^2 + \frac{4a}{b}\left(2(1-b) + \frac{\epsilon^{-2a}(1-b)^2}{b}\right)\right. \\
&\qquad + \frac{2}{b}\left(2(1-b) + \frac{\epsilon^{-2a}(1-b)^2}{b}\right)ET_1T_2 - \frac{4(1-b)}{b}ET_1^2T_2 \\
&\qquad \left. - \frac{2(2a+1)}{b}\left(2(1-b) + \frac{\epsilon^{-2a}(1-b)^2}{b}\right)ET_1 - \frac{8a(1-b)}{b}ET_1\right] \\
&\quad + u_2^2\left[4(1-b)^2ET_1^2 + \left(2(1-b) + \frac{\epsilon^{-2a}(1-b)^2}{b}\right)^2\right. \\
&\qquad \left. - 4(1-b)\left(2(1-b) + \frac{\epsilon^{-2a}(1-b)^2}{b}\right)ET_1\right].
\end{aligned}
$$

We compute the coefficients of u_1^2, u_1u_2, and u_2^2 separately.

First, the coefficient of u_1^2 is

$$\frac{1}{b^2}ET_1^2T_2^2 + \frac{(2a+1)^2}{b^2}ET_1^2 + \frac{4a^2}{b^2} + \frac{4a}{b^2}ET_1T_2 - \frac{4a(2a+1)}{b^2}ET_1 - \frac{2(2a+1)}{b^2}ET_1^2T_2$$

$$= \frac{1}{b^2}\left[4 + \epsilon^{-2a}\left(\frac{1-b}{2b}\right)\left(4a^3 + \frac{2a^2(2+5b)}{b} + \frac{2a(2+5b)}{b} + \frac{2+5b}{b}\right)\right]$$

$$+ \frac{(2a+1)^2}{b^2}\left[2 + \epsilon^{-2a}\left(\frac{1-b}{2b}\right)\left(2a+3+\frac{2}{b}\right)\right] + \frac{4a^2}{b^2}$$

$$+ \frac{4a}{b^2}\left[1 + \epsilon^{-2a}\left(\frac{1-b}{2b}\right)(2a+1)\right] - \frac{4a(2a+1)}{b^2}\left[1 + \epsilon^{-2a}\left(\frac{1-b}{2b}\right)\right]$$

$$- \frac{2(2a+1)}{b^2}\left[2 + \epsilon^{-2a}\left(\frac{1-b}{2b}\right)\left(3a^2 + \frac{a(2+5b)}{b} + \frac{2+5b}{2b}\right)\right]$$

$$= \frac{1}{b^2}\left[4 + 2(4a^2+4a+1) + 4a^2 + 4a - (8a^2+4a) - (8a+4)\right]$$

$$+ \epsilon^{-2a}\left(\frac{1-b}{2b^3}\right)\left[4a^3 + \frac{2a^2(2+5b)}{b} + \frac{2a(2+5b)}{b} + \frac{2+5b}{b}\right.$$

$$\left. + (2a+1)^2(2a+3+\frac{2}{b}) - 6a^2(2a+1) - 2(2a+1)\left(\frac{a(2+5b)}{b} + \frac{2+5b}{2b}\right)\right]$$

$$= \frac{1}{b^2}(4a^2+2) + \epsilon^{-2a}\left(\frac{1-b}{2b^3}\right)\left[-8a^3 - \frac{2a^2(2+5b)}{b} - \frac{2a(2+5b)}{b}\right.$$

$$\left. + (2a+1)^2(2a+3+\frac{2}{b}) - 6a^2\right]$$

$$= \frac{4a^2+2}{b^2} + \epsilon^{-2a}\left(\frac{1-b}{2b^3}\right)\left[-8a^3 - \frac{2a^2(2+5b)}{b} - \frac{2a(2+5b)}{b}\right.$$

$$\left. + \frac{2}{b}(2a+1)^2 + 8a^3 + 14a^2 + 14a + 3\right]$$

$$= \frac{4a^2+2}{b^2} + \epsilon^{-2a}\left(\frac{1-b}{2b^3}\right)\left[-\frac{4a^2}{b} - 10a^2 - \frac{4a}{b} - 10a\right.$$

$$\left. + \frac{2(4a^2+4a+1)}{b} + 14a^2 + 14a + 3\right]$$

$$= \frac{4a^2+2}{b^2} + \epsilon^{-2a}\left(\frac{1-b}{2b^3}\right)\left[4a^2 + 4a + 3 + \frac{4a^2+4a+2}{b}\right].$$

Second, the coefficient of $u_1 u_2$ is

$$\frac{4(2a+1)(1-b)}{b}ET_1^2 + \frac{4a}{b}\left(2(1-b) + \frac{\epsilon^{-2a}(1-b)^2}{b}\right) - \frac{8a(1-b)}{b}ET_1$$
$$+ \frac{2}{b}\left(2(1-b) + \frac{\epsilon^{-2a}(1-b)^2}{b}\right)ET_1T_2 - \frac{2(2a+1)}{b}\left(2(1-b) + \frac{\epsilon^{-2a}(1-b)^2}{b}\right)ET_1$$
$$- \frac{4(1-b)}{b}ET_1^2T_2$$
$$= \frac{4(2a+1)(1-b)}{b}\left[2 + \epsilon^{-2a}\left(\frac{1-b}{2b}\right)\left(2a+3+\frac{2}{b})\right)\right]$$
$$+ \frac{4a}{b}\left(2(1-b) + \frac{\epsilon^{-2a}(1-b)^2}{b}\right) - \frac{8a(1-b)}{b}\left[1 + \epsilon^{-2a}\left(\frac{1-b}{2b}\right)\right]$$
$$+ \frac{2}{b}\left(2(1-b) + \frac{\epsilon^{-2a}(1-b)^2}{b}\right)\left[1 + \epsilon^{-2a}\left(\frac{1-b}{2b}\right)(2a+1)\right]$$
$$- \frac{2(2a+1)}{b}\left(2(1-b) + \frac{\epsilon^{-2a}(1-b)^2}{b}\right)\left[1 + \epsilon^{-2a}\left(\frac{1-b}{2b}\right)\right]$$
$$- \frac{4(1-b)}{b}\left[2 + \epsilon^{-2a}\left(\frac{1-b}{2b}\right)\left(3a^2 + \frac{a(2+5b)}{b} + \frac{2+5b}{2b}\right)\right]$$
$$= \frac{4(2a+1)(1-b)}{b} - \frac{4(1-b)}{b} + \frac{\epsilon^{-2a}(1-b)}{2b}\left[\frac{4(2a+1)(1-b)}{b}(2a+3+\frac{2}{b})\right.$$
$$\left. - \frac{8a(1-b)}{b} - \frac{4(1-b)}{b}\left(3a^2 + \frac{a(2+5b)}{b} + \frac{2+5b}{2b}\right)\right]$$
$$= \frac{8a(1-b)}{b} + \frac{\epsilon^{-2a}(1-b)}{2b}\left[\frac{4(2a+1)(1-b)(2ab+3b+2)}{b^2} - \frac{8a(1-b)}{b}\right.$$
$$\left. - \frac{4(1-b)(6a^2b+2a(2+5b)+2+5b)}{2b^2}\right]$$
$$= \frac{8a(1-b)}{b} + \frac{\epsilon^{-2a}(1-b)}{4b^3}[8(2a+1)(1-b)(2ab+3b+2) - 16ab(1-b)$$
$$- 4(1-b)(6a^2b+2a(2+5b)+2+5b)]$$
$$= \frac{8a(1-b)}{b} + \frac{\epsilon^{-2a}(1-b)}{4b^3}\left[4(1-b)(2+b+2a^2b+4a+2ab)\right]$$
$$= \frac{8a(1-b)}{b} + \frac{\epsilon^{-2a}(1-b)^2}{b^3}\left[2+4a-6a^2+8a^2b+b+2ab\right].$$

Third, the coefficient of u_2^2 is

$$
\begin{aligned}
&4(1-b)^2 ET_1^2 + \left(2(1-b) + \frac{\epsilon^{-2a}(1-b)^2}{b}\right)^2 \\
&\quad - 4(1-b)\left(2(1-b) + \frac{\epsilon^{-2a}(1-b)^2}{b}\right) ET_1 \\
&\quad = 4(1-b)^2\left[2 + \epsilon^{-2a}\left(\frac{1-b}{2b}\right)\left(2a+3+\frac{2}{b}\right)\right] + \left(2(1-b) + \frac{\epsilon^{-2a}(1-b)^2}{b}\right)^2 \\
&\quad\quad - 4(1-b)\left(2(1-b) + \frac{\epsilon^{-2a}(1-b)^2}{b}\right)\left[1 + \epsilon^{-2a}\left(\frac{1-b}{2b}\right)\right] \\
&\quad\quad = 4(1-b)^2 + \frac{\epsilon^{-2a}(1-b)}{2b}4(1-b)^2(2a+1+\frac{2}{b}) - \frac{\epsilon^{-4a}(1-b)^4}{b^2} \\
&\quad\quad = 4(1-b)^2 + \frac{2\epsilon^{-2a}(1-b)^3}{b}(2a+1+\frac{2}{b}) - \frac{\epsilon^{-4a}(1-b)^4}{b^2}.
\end{aligned}
$$

From above, we have

$$
\begin{aligned}
EX^2 = u_1^2&\left[\frac{4a^2+2}{b^2} + \epsilon^{-2a}\left(\frac{1-b}{2b^3}\right)\left(4a^2+4a+3+\frac{4a^2+4a+2}{b}\right)\right] \\
&+ u_1u_2\left[\frac{8a(1-b)}{b} + \frac{\epsilon^{-2a}(1-b)^2}{b^3}\left(2+4a-6a^2+8a^2b+b+2ab\right)\right] \\
&+ u_2^2\left[4(1-b)^2 + \frac{2\epsilon^{-2a}(1-b)^3}{b}(2a+1+\frac{2}{b}) - \frac{\epsilon^{-4a}(1-b)^4}{b^2}\right]
\end{aligned}
\tag{4.21}
$$

which is finite. Therefore (4.20) is convergent to a normal variable with mean 0 and variance EX^2.

Hence X_n converges in distribution to a normal variable with mean 0 and variance EX^2, that is,

$$
X_n \xrightarrow{d} \mathcal{N}(0, EX^2). \tag{4.22}
$$

This completes the proof. ■

Lemma 4.1.8.

$$
D_n = 2\left(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1\right)\left(2\epsilon^{2\hat{a}_n}\left(\frac{\sum_{i=1}^n T_{i,1}}{n} - 1\right) + 1\right) \xrightarrow{p} \epsilon^{-2a}\left(\frac{1-b}{b^2}\right).
$$

Proof. By (4.14), (4.15), and (4.16),

$$2\left(\frac{\sum_{i=1}^{n} T_{i,1}}{n} - 1\right) \xrightarrow{p} 2 \cdot e^{-2a} \cdot \frac{1-b}{2b}$$

and

$$2e^{2\hat{a}_n}\left(\frac{\sum_{i=1}^{n} T_{i,1}}{n} - 1\right) + 1 \xrightarrow{p} 2 \cdot e^{2a} \cdot e^{-2a} \cdot \frac{1-b}{2b} + 1.$$

But

$$2 \cdot e^{-2a} \cdot \frac{1-b}{2b} \cdot \left(2 \cdot e^{2a} \cdot e^{-2a} \cdot \frac{1-b}{2b} + 1\right) = e^{-2a}\left(\frac{1-b}{b^2}\right).$$

Hence

$$2\left(\frac{\sum_{i=1}^{n} T_{i,1}}{n} - 1\right)\left(2e^{2\hat{a}_n}\left(\frac{\sum_{i=1}^{n} T_{i,1}}{n} - 1\right) + 1\right) \xrightarrow{p} e^{-2a}\left(\frac{1-b}{b^2}\right).$$

that is,

$$D_n \xrightarrow{p} \frac{1-b}{e^{2a}b^2}. \tag{4.23}$$

This completes the proof. ■

Back to the proof of Theorem 4.1.6.

From (4.22), (4.23), and Theorem 4.1.5(c), $\frac{X_n}{D_n}$ converges in distribution to a normal variable, that is,

$$\sqrt{n}(u_1(\hat{a}_n - a) + u_2(\hat{b}_n - b)) = \frac{X_n}{D_n} \xrightarrow{d} N(0, \sigma_{\mathbf{u}}) \tag{4.24}$$

with

$$\sigma_{\mathbf{u}} = EX^2 \cdot \frac{e^{4a}b^4}{(1-b)^2}. \tag{4.25}$$

This completes the proof of Theorem 4.1.6. ■

Remark From the above, (i) if $u_1 = 1$ and $u_2 = 0$, then $\sqrt{n}(\hat{a}_n - a)$ has mean 0, variance σ_a^2, with

$$\sigma_a^2 = \left[\frac{4a^2+2}{b^2} + \epsilon^{-2a}\left(\frac{1-b}{2b^3}\right)\left(4a^2+4a+3+\frac{4a^2+4a+2}{b}\right)\right] \cdot \frac{\epsilon^{4a}b^4}{(1-b)^2}. \quad (4.26)$$

(applying $u_1 = 1$ and $u_2 = 0$ to (4.21) and combing with (4.25)) and

$$\sqrt{n}(\hat{a}_n - a) \xrightarrow{d} \mathcal{N}(0, \sigma_a^2). \quad (4.27)$$

(ii) If $u_1 = 0$ and $u_2 = 1$, then $\sqrt{n}(\hat{b}_n - b)$ has mean 0 and variance σ_b^2, with

$$\sigma_b^2 = \left[4(1-b)^2 + \frac{2\epsilon^{-2a}(1-b)^3}{b}(2a+1+\frac{2}{b}) - \frac{\epsilon^{-4a}(1-b)^4}{b^2}\right] \cdot \frac{\epsilon^{4a}b^4}{(1-b)^2}. \quad (4.28)$$

(applying $u_1 = 0$ and $u_2 = 1$ to (4.21) and combing with (4.25)) and

$$\sqrt{n}(\hat{b}_n - b) \xrightarrow{d} \mathcal{N}(0, \sigma_b^2). \quad (4.29)$$

(iii) Let $u_1 = u_2 = 1$. Together with (i) and (ii), the variance of $\sqrt{n}((\hat{a}_n - a) + (\hat{b}_n - b))$ is equal to the summation of σ_a^2, σ_b^2, and twice the covariance σ_{ab} of $\sqrt{n}(\hat{a}_n - a)$ and $\sqrt{n}(\hat{b}_n - b)$. Applying (4.21) with $u_1 = u_2 = 1$, (4.25), (4.26), and (4.28), we have

$$\sigma_{ab} = \frac{1}{2}\left[\frac{8a(1-b)}{b} + \frac{\epsilon^{-2a}(1-b)^2}{b^3}\left(2+4a-6a^2+8a^2b+b+2ab\right)\right] \cdot \frac{\epsilon^{4a}b^4}{(1-b)^2}. \quad (4.30)$$

From (i), (ii), and (iii), $\sqrt{n}((\hat{a}_n - a), (\hat{b}_n - b))^T$ converges in distribution to a two-dimensional normal random vector with mean **0** and variance matrix Σ where

$$\Sigma = \begin{vmatrix} \sigma_a^2 & \sigma_{ab} \\ \sigma_{ab} & \sigma_b^2 \end{vmatrix}$$

with σ_a^2, σ_b^2, and σ_{ab} from (4.26), (4.28), and (4.30), respectively.

4.2 Maximum Likelihood Estimators

In contrast to Section 4.1, we give the maximum likelihood estimators (MLE) of a and b in this section. Maximum likelihood method is one of the oldest known methods of estimation. It is based on choosing as an estimator that value of which maximizes the probability joint density of the sample.

To reduce complexity, we study a sample of size 2 first. This special case has the advantage of easy computation and still possesses enough information to show how the general case, a sample of size n, will look like.

Let (T_{11}, T_{12}) and (T_{21}, T_{22}) be 2 observations from the new bivariate distribution

$$f(t_1, t_2) = \begin{cases} \epsilon^{-t_1-t_2}, & t_1 \le a \text{ or } t_2 \le a; \\ b\epsilon^{-(2-b)t_1-bt_2}, & a < t_1 \le t_2; \\ b\epsilon^{-(2-b)t_2-bt_1}, & a < t_2 < t_1. \end{cases}$$

The likelihood function L is defined as

$$L = f(T_{1,1}, T_{1,2}) \cdot f(T_{2,1}, T_{2,2}).$$

Since $(T_{1,1}, T_{1,2})$ and $(T_{2,1}, T_{2,2})$ are sample values, L is a function of a and b. The objective is then to find values of a and b that maximize L. Without loss of generality, let us assume that $T_{1,2} \le T_{1,1}$, $T_{2,1} \le T_{2,2}$, and $T_{2,1} < T_{1,2}$. Since the new bivariate distribution is obviously continuous, we may assume that $T_{1,2} \neq T_{2,1}$. Then

$$L(a,b) = \begin{cases} \epsilon^{-T_{1,1}-T_{1,2}-T_{2,1}-T_{2,2}}, & T_{2,1} < T_{1,2} \le a; \\ \epsilon^{-T_{2,1}-T_{2,2}} \cdot b\epsilon^{-(2-b)T_{1,2}-bT_{1,1}}, & T_{2,1} \le a < T_{1,2}; \\ b^2\epsilon^{-(2-b)(T_{1,2}+T_{2,1})-b(T_{1,1}+T_{2,2})}, & a < T_{2,1} < T_{1,2}. \end{cases}$$

By calculus, $b\epsilon^{-(2-b)T_{1,2}-bT_{1,1}}$ is maximized by $b = \frac{1}{T_{1,1}-T_{1,2}}$ and the maximum value is $\frac{\epsilon^{-1-2T_{1,2}}}{T_{1,1}-T_{1,2}}$. That is,

$$b\epsilon^{-(2-b)T_{1,2}-bT_{1,1}} \le \frac{\epsilon^{-1-2T_{1,2}}}{T_{1,1}-T_{1,2}}.$$

Also, $b^2\epsilon^{-(2-b)(T_{1,2}+T_{2,1})-b(T_{1,1}+T_{2,2})}$ is maximized by $b = \frac{2}{(T_{1,1}-T_{1,2})+(T_{2,1}-T_{2,2})}$ and the maximum value is $\frac{4\epsilon^{-2-2(T_{1,2}+T_{2,1})}}{[(T_{1,1}-T_{1,2})+(T_{2,1}-T_{2,2})]^2}$. That is,

$$b^2\epsilon^{-(2-b)(T_{1,2}+T_{2,1})-b(T_{1,1}+T_{2,2})} \le \frac{4\epsilon^{-2-2(T_{1,2}+T_{2,1})}}{[(T_{1,1}-T_{1,2})+(T_{2,1}-T_{2,2})]^2}.$$

Thus.

$$L(a.b) \leq \begin{cases} e^{-T_{1.1}-T_{1.2}-T_{2.1}-T_{2.2}}, & T_{2.1} < T_{1.2} \leq a; \\ e^{-T_{2.1}-T_{2.2}} \cdot \frac{e^{-1-2T_{1.2}}}{T_{1.1}-T_{1.2}}, & T_{2.1} \leq a < T_{1.2}; \\ \frac{4e^{-2-2(T_{1.2}+T_{2.1})}}{[(T_{1.1}-T_{1.2})+(T_{2.1}-T_{2.2})]^2}, & a < T_{2.1} < T_{1.2}. \end{cases} \tag{4.31}$$

Next. we find the value of a that maximizes L. From (4.31). it is enough to find the maximum among the right hand side of (4.31).

Claim:

$$e^{-T_{1.1}-T_{1.2}-T_{2.1}-T_{2.2}} \leq e^{-T_{1.2}-T_{2.2}} \cdot \frac{e^{-1-2T_{1.2}}}{T_{1.1}-T_{1.2}}.$$

Indeed.

$$\begin{aligned} & e^{-T_{11}-T_{12}-T_{21}-T_{22}} \leq e^{-T_{12}-T_{22}} \cdot \frac{e^{-1-2T_{12}}}{T_{11}-T_{12}} \\ \iff & 1 \leq \frac{e^{-T_{11}-T_{12}}}{e(T_{11}-T_{12})} \\ \iff & 0 \leq e^{x} - ex. \quad \forall 0 \leq x \quad (\text{since} T_{1.2} \leq T_{1.1}). \end{aligned}$$

Let $g(x) = e^x - ex$. Then $g'(x) = e^x - e$. Therefore $g' < 0$ on (0.1). $g' = 0$ at $x = 1$. and $g' > 0$ on $(1. \infty)$. Hence $g(x) \geq g(1) = 0$. that is, $g(x) \geq 0$. This shows that

$$e^{-T_{1.1}-T_{1.2}-T_{2.1}-T_{2.2}} \leq e^{-T_{1.2}-T_{2.2}} \cdot \frac{e^{-1-2T_{1.2}}}{T_{1.1}-T_{1.2}}.$$

From the above. the MLE $\hat{a}$ should be in one of the 2 intervals $(0. T_{2.1})$ and $[T_{2.1}. T_{1.2})$ depending upon whether

$$e^{-T_{2.1}-T_{2.2}} \cdot \frac{e^{-1-2T_{1.2}}}{T_{1.1}-T_{1.2}} \leq \frac{4e^{-2-2(T_{1.2}+T_{2.1})}}{[(T_{1.1}-T_{1.2})+(T_{2.1}-T_{2.2})]^2}$$

or not.

By simple computation.

$$\begin{aligned} & e^{-T_{2.1}-T_{2.2}} \cdot \frac{e^{-1-2T_{1.2}}}{T_{1.1}-T_{1.2}} \leq \frac{4e^{-2-2(T_{1.2}+T_{2.1})}}{[(T_{1.1}-T_{1.2})+(T_{2.1}-T_{2.2})]^2} \\ \iff & e(T_{1.1}-T_{1.2}+T_{2.2}-T_{2.1})^2 \leq 4(T_{1.1}-T_{1.2})e^{T_{2.2}-T_{2.1}}. \end{aligned} \tag{4.32}$$

Hence the MLE is

$$(\hat{a}.\hat{b}) = (t'. \frac{2}{T_{1.1} - T_{1.2} + T_{2.2} - T_{2.1}})$$

where $t' \in (0.T_{2.1})$. if $(T_{1.1}.T_{1.2})$ and $(T_{2.1}.T_{.22})$ satisfy (4.32). Note that $\hat{a}$ is not unique. This follows from (4.31) since for any $a \in (0.T_{2.1})$ L is bounded above by $\frac{4e^{-2-2(T_{1.2}+T_{2.1})}}{[(T_{1.1}-T_{1.2})+(T_{2.1}-T_{2.2})]^2}$. If $(T_{1.1}.T_{1.2})$ and $(T_{2.1}.T_{.2.2})$ do not satisfy (4.32) then

$$(\hat{a}.\hat{b}) = (t'. \frac{1}{T_{1.1} - T_{1.2}})$$

where $t' \in [T_{2.1}.T_{1.2})$. Again. $\hat{a}$ is not unique.

We summarize these in the following theorem.

Theorem 4.2.1. *Let $(T_{1.1}.T_{1.2})$ and $(T_{2.1}.T_{2.2})$ be 2 observations.*

1. *If* $\min(T_{1.1}.T_{1.2}) > \min(T_{2.1}.T_{2.2})$ *and* (4.32) *is satisfied. then*

$$(\hat{a}.\hat{b}) = (t'. \frac{2}{T_{1.1} - T_{1.2} + T_{2.2} - T_{2.1}}) \tag{4.33}$$

with t' being any value in $(0.T_{2.1})$.

2. *If* $\min(T_{1.1}.T_{1.2}) > \min(T_{2.1}.T_{2.2})$ *and* (4.32) *is not satisfied. then*

$$(\hat{a}.\hat{b}) = (t'. \frac{1}{T_{1.1} - T_{1.2}}) \tag{4.34}$$

with t' being any value in $[T_{2.1}.T_{1.2})$.

Remark Let $x = |T_{1.1} - T_{1.2}|$ and $y = |T_{2.1} - T_{2.2}|$. Then (4.32) could be rewritten as

$$e(x+y)^2 \leq 4xe^y.$$

The graph of this region, say I, is as follow (Figure 4.1).

Therefore. if $(|T_{1.1} - T_{1.2}|.|T_{2.1} - T_{2.2}|) \in I$. then (4.33) gives the MLE for $(a.b)$. If $(|T_{1.1} - T_{1.2}|,|T_{2.1} - T_{.22}|) \notin I$. then (4.34) gives MLE for $(a.b)$.

FIGURE 4.1. *MLE of 2 observations*

Following the same idea, we study the general case now. Let $(T_{1,1}, T_{1,2}), (T_{2,1}, T_{2,2}), \cdots, (T_{n,1}, T_{n,2})$ be a sample of size n from the new bivariate distribution. By rearranging the n observations, we can assume that

$$\min(T_{n,1}, T_{n,2}) < \cdots \min(T_{2,1}, T_{2,2}) < \min(T_{1,1}, T_{1,2}).$$

Once again, we may assume that any two consecutive numbers in the above are not equal since they come from an absolutely continuous distribution. Let $\min(T_{i,1}, T_{i,2}) = t_i$, $i = 1, \cdots, n$. Then $t_n < \cdots t_2 < t_1$. Define $t_{n+1} = 0$. Thus

$$L(a,b) = \begin{cases} e^{-\sum_{i=1}^{n}\sum_{j=1}^{2} T_{i,j}}, & t_1 < a; \\ \vdots & \\ b^l e^{-\sum_{i=l+1}^{n}\sum_{j=1}^{2} T_{i,j} - (2-b)\sum_{i=1}^{l}\min(T_{i,1},T_{i,2}) - b\sum_{i=1}^{l}\max(T_{i,1},T_{i,2})}, & t_{l+1} \le a < t_l; \\ \vdots & \\ b^n e^{-(2-b)\sum_{i=1}^{n}\min(T_{i,1},T_{i,2}) - b\sum_{i=1}^{n}\max(T_{i,1},T_{i,2})}, & a < t_n. \end{cases}$$

By calculus,

$$b^l e^{-(2-b)\sum_{i=1}^{l}\min(T_{i,1},T_{i,2}) - b\sum_{i=1}^{l}\max(T_{i,1},T_{i,2})}$$

is maximized by $b = \frac{l}{\sum_{i=1}^{l} |T_{i,1} - T_{i,2}|}$ and the maximum value is

$$\left(\frac{l}{\sum_{i=1}^{l} |T_{i,1} - T_{i,2}|}\right)^{l} e^{-l-2\sum_{i=1}^{l} \min(T_{i,1}, T_{i,2})}.$$

Therefore

$$L(a,b) \leq \begin{cases} e^{-\sum_{i=1}^{n} \sum_{j=1}^{2} T_{i,j}}, & t_1 \leq a; \\ \vdots & \\ \left(\frac{l}{\sum_{i=1}^{l} |T_{i,1} - T_{i,2}|}\right)^{l} e^{-l-2\sum_{i=1}^{l} \min(T_{i,1}, T_{i,2}) - \sum_{i=l+1}^{n} \sum_{j=1}^{2} T_{i,j}}, & t_{l+1} \leq a < t_l; \\ \vdots & \\ \left(\frac{n}{\sum_{i=1}^{n} |T_{i,1} - T_{i,2}|}\right)^{n} e^{-n-2\sum_{i=1}^{n} \min(T_{i,1}, T_{i,2})}, & a < t_n. \end{cases} \tag{4.35}$$

Similar to the case of 2 observations, we can easily prove that

$$e^{-\sum_{i=1}^{n} \sum_{j=1}^{2} T_{i,j}} \leq \left(\frac{1}{|T_{1,1} - T_{1,2}|}\right)^{1} e^{-l-2\min(T_{1,1}, T_{1,2}) - \sum_{i=2}^{n} \sum_{j=1}^{2} T_{i,j}}.$$

This leads to the following theorem.

Theorem 4.2.2. *The MLE for (a, b) is*

$$(\hat{a}, \hat{b}) = \left(t', \frac{l}{\sum_{i=1}^{l} |T_{i,1} - T_{i,2}|}\right),$$

with t' being any value in $[t_{l+1}, t_l)$, for some $l \in \{1, 2, \cdots, n\}$. The maximum value of $L(a, b)$ is

$$\left(\frac{l}{\sum_{i=1}^{l} |T_{i,1} - T_{i,2}|}\right)^{l} e^{-l-2\sum_{i=1}^{l} \min(T_{i,1}, T_{i,2}) - \sum_{i=l+1}^{n} \sum_{j=1}^{2} T_{i,j}}. \tag{4.36}$$

This is an "operational" theorem. That is, we compute all the expressions except the first one on the right hand side of (4.35). Then we can determine the maximum value among them. Then we can determine the MLE. Once again, $\hat{a}$ is not unique.

Remark. Let $s_i = |T_{i,1} - T_{i,2}|$, $i = 1, 2, \cdots, n$. Since $t_i = \min(T_{i,1}, T_{i,2})$, $i = 1, 2, \cdots, n$, we rewrite (4.36) as

$$\left(\frac{l}{\sum_{i=1}^{l} s_i}\right)^{l} e^{-l-2\sum_{i=1}^{l} t_i - \sum_{i=l+1}^{n} (2t_i + s_i)} = \left(\frac{l}{\sum_{i=1}^{l} s_i}\right)^{l} e^{-l-\sum_{i=1}^{l} s_i - 2\sum_{i=1}^{n} t_i - \sum_{i=1}^{n} s_i}. \tag{4.37}$$

In practice, we only have to compute

$$\left(\frac{l}{\sum_{i=1}^{l} s_i}\right)^l e^{-l-\sum_{i=1}^{l} s_i}, \quad \forall l = 1, 2, \cdots, n,$$

to determine the MLE.

4.3 Data analysis

In this section, we use software Mathematica to generate a random sample from the new bivariate distribution with $a = 0.1$ and $b = 1.8$. Then we compute the MME and MLE and we make a table to compare these two estimators. Then we compute the 95% confidence interval from MME.

From the construction of the new bivariate distribution of (T_1, T_2), we know that $\min(T_1, T_2)$ possesses an exponential distribution with $\lambda = 2$ ($Exp(2)$) and $E[T_1|T_1 > T_2 = t]$ (or $E[T_2|T_2 > T_1 = t]$) possesses an exponential distribution with $\lambda = 1$ ($Exp(1)$) or $\lambda = b$ ($Exp(b)$) depending upon $t \leq a$ or $a < t$. Therefore we need to generate an $Exp(2)$ first then use it to generate another $Exp(1)$ or $Exp(b)$. For this purpose, we need the following theorem.

Theorem 4.3.1. *Let U be a random variable with the uniform distribution on $(0,1)$. Let F be and $Exp(\lambda)$. Then $-\log(1-U)/\lambda \stackrel{d}{\equiv} F$.*

Proof. Since $U(\Omega) = (0, 1)$, $(-\log(1-U)/\lambda)(\Omega) = (0, \infty)$. For any $x \in (0, \infty)$, we have

$$P(-\log(1-U)/\lambda < x) = P(U < 1 - e^{-\lambda x}) = 1 - e^{-\lambda x}.$$

This completes the proof. ∎

Using Mathematica, we can generate a random variable that is uniformly distributed on $(0,1)$. Then the above theorem enables us to generate an $Exp(2)$. Once $\min(T_1, T_2)$ is generated, we compare it with a. If $\min(T_1, T_2) \leq a$, then we generate an $Exp(1)$. Otherwise, we generate an $Exp(b)$. Therefore, we have $\min(T_1, T_2)$ and $|T_1 - T_2|$ been generated. Next, we generate a random variable X such that

$$P(X = 1) = P(X = 2) = \frac{1}{2}$$

and use it to determine $T_1 > T_2$ (when $X = 1$) or $T_1 < T_2$ (when $X = 2$). The random variable X could be generated by Mathematica. Hence we have (T_1, T_2). By repeating this process, we have $(T_{1,1}, T_{1,2}), \cdots, (T_{n,1}, T_{n,2})$. Now applying (4.2) and (4.3), we have the MME.

For the MLE, we have to sort $(T_{1,1}, T_{1,2}), \cdots, (T_{n,1}, T_{n,2})$ such that

$$\min(T_{n,1}, T_{n,2}) < \cdots < \min(T_{2,1}, T_{2,2}) < \min(T_{1,1}, T_{1,2}).$$

Then we apply Theorem 4.2.2. That is, we compute

$$v_l = \left(\frac{l}{\sum_{i=1}^{l} |T_{i,1} - T_{i,2}|} \right)^l e^{-l - \sum_{i=1}^{l} |T_{i,1} - T_{i,2}|}, \quad \forall l = 1, 2, \cdots, n.$$

Then find l such that $v_l = \max(v_1, \cdots, v_n)$ and the MLE is $(t, \frac{l}{\sum_{i=1}^{l} |T_{i,1} - T_{i,2}|})$ for some $t \in [t_{l+1}, t_l)$. The following is the actual code that we run on Mathematica.

```
a=0.1;
b=1.8;
f[{x_,y_}/;x<=a] :=-Log[1-y]
f[{x_,y_}/;x>a] :=-Log[1-y]/b
  g[{x_,y_,z_}/;z<=1.5] :={x,x+y}
g[{x_,y_,z_}/;z>1.5] :={x+y,x}
Do[
 s=1000*i;
 s1=330+i;
 s2=430+i;
 s3=530+i;
```

```
 SeedRandom[s1];
 data1=Table[Random[],{j,1,s,1}];
 SeedRandom[s2];
 data2=Table[Random[],{j,1,s,1}];
 SeedRandom[s3];
 id=Table[Random[Integer],{j,1,s,1}]+1;
 firstf=-Log[1-data1]/2;
 firstf=-Sort[-firstf];
 data3=Table[{firstf[[j]],data2[[j]]},{j,1,s,1}];
 secondf=Map[f,data3];
 data4=Table[{firstf[[j]],secondf[[j]],id[[j]]},{j,1,s,1}];
 life=Map[g,data4];
 mean=Apply[Plus,life]/s;
 t1t2=Table[life[[j,1]]*life[[j,2]],{j,1,s,1}];
 meant1t2=Apply[Plus,t1t2]/s;
 A=(meant1t2-1)/(2*(mean[[1]]-1))-0.5;
 B=1/(Exp[2*A]*2*(mean[[1]]-1)+1);
 temp=FoldList[Plus,0,secondf];
 L=Table[Exp[-j+temp[[j+1]]]*((j/temp[[j+1]])^j),{j,1,s,1}];
 pos=-Sort[-L];
 p=Table[Position[L,pos[[j]]],{j,1,s,1}];
 p=Level[p,{3}];
 mleA=firstf[[p[[1]]]];
 mleB=p[[1]]/temp[[p[[1]]+1]];
 Print["Size",s,": ",A," , ",B," , ",mleA," , ",mleB],
 {i,1,4,1}]
Do[
 s=1000*i;
```

```
s1=110+i;
s2=210+i;
s3=310+i;
SeedRandom[s1];
data1=Table[Random[],{j,1,s,1}];
SeedRandom[s2];
data2=Table[Random[],{j,1,s,1}];
SeedRandom[s3];
id=Table[Random[Integer],{j,1,s,1}]+1;
firstf=-Log[1-data1]/2;
firstf=-Sort[-firstf];
data3=Table[{firstf[[j]],data2[[j]]},{j,1,s,1}];
secondf=Map[f,data3];
data4=Table[{firstf[[j]],secondf[[j]],id[[j]]},{j,1,s,1}];
life=Map[g,data4];
mean=Apply[Plus,life]/s;
t1t2=Table[life[[j,1]]*life[[j,2]],{j,1,s,1}];
meant1t2=Apply[Plus,t1t2]/s;
A=(meant1t2-1)/(2*(mean[[1]]-1))-0.5;
B=1/(Exp[2*A]*2*(mean[[1]]-1)+1);
Print["Size",s,": ",A," , ",B],
{i,5,10,1}]
```

The results are given in the following table (Table 4.1)

	MME		MLE	
sample size	$\hat{a}_n$	$\hat{b}_n$	$\hat{a}_n$	$\hat{b}_n$
1000	-0.0601104	1.4781	0.0979948	1.89022
2000	0.160453	2.1548	0.100429	1.84433
3000	0.0717592	1.70921	0.0995225	1.78585
4000	0.0927433	1.78522	0.0977703	1.83881
5000	0.0656966	1.693		
6000	0.120798	1.97606		
7000	0.132387	1.86748		
8000	0.115897	1.80265		
9000	0.139199	1.8394		
10000	0.0737492	1.81137		

TABLE 4.1. MME and MLE

Comparison:

1. The MLE is more accurate than the MME. The following table (Table 4.2) shows the error from both estimators.

	MME		MLE	
sample size	$\|\hat{a}_n - a\|$	$\|\hat{b}_n - b\|$	$\|\hat{a}_n - a\|$	$\|\hat{b}_n - b\|$
1000	0.1601104	0.3219	0.0020052	0.09022
2000	0.060453	0.3548	0.000429	0.04433
3000	0.0282408	0.09079	0.0004775	0.01415
4000	0.0072567	0.01478	0.0022297	0.03881

TABLE 4.2. Error from both MME and MLE

2. The advantage of MME is easy computation. By comparing (4.2) and (4.3) to Theorem 4.2.2, it is clear that MME is easier to compute then MLE. This is why we do not compute MLE for sample size 5000 to 10000. It is too much time consuming.

In the following, we compute the 95% confidence intervals from MME. From (4.27) and (4.29). we have

$$\sqrt{n}(\hat{a}_n - a) \xrightarrow{d} \mathcal{N}(0, \sigma_a^2)$$

and

$$\sqrt{n}(\hat{b}_n - b) \xrightarrow{d} \mathcal{N}(0, \sigma_b^2)$$

where σ_a^2 and σ_b^2 are from (4.26) and (4.28). respectively. Then

$$P(\frac{\sqrt{n}|\hat{a}_n - a|}{\sigma_a} \leq 1.96) \approx 0.95$$

and

$$P(\frac{\sqrt{n}|\hat{b}_n - b|}{\sigma_b} \leq 1.96) \approx 0.95.$$

With $a = 0.1$ and $b = 1.8$. the 95% confidence intervals are

$$|\hat{a}_n - a| \leq \frac{8.3403}{\sqrt{n}} \quad \text{and} \quad |\hat{b}_n - b| \leq \frac{8.58918}{\sqrt{n}}.$$

We have the following table.

sample size	$(\hat{a}_n - \frac{8.3403}{\sqrt{n}}, \hat{a}_n + \frac{8.3403}{\sqrt{n}})$	$(\hat{b}_n - \frac{8.58918}{\sqrt{n}}, \hat{b}_n + \frac{8.58918}{\sqrt{n}})$
1000	(-0.3238504,0.2036296)	(1.20649,1.74971)
2000	(-0.026037,0.346943)	(1.96274,2.34686)
3000	(-0.0805108,0.2240292)	(1.5524,1.86602)
4000	(-0.0391267,0.2246133)	(1.64941,1.92103)
5000	(-0.0522534,0.1836466)	(1.57153,1.81447)
6000	(0.013108,0.228488)	(1.86517,2.08695)
7000	(0.032697,0.232077)	(1.76482,1.97014)
8000	(0.022647,0.209147)	(1.70662,1.89868)
9000	(0.051289,0.227109)	(1.74886,1.92994)
10000	(-0.0096508,0.1571492)	(1.725478,1.897262)

TABLE 4.3. 95% confidence intervals

Chapter 5

Stochastic Comparison Between 2 Random Vectors of Lifetimes

We study the following question in this chapter. Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional nonnegative random vectors such that $\mathbf{X} = (X_1, \cdots, X_n)$ and $\mathbf{Y} = (Y_1, \cdots, Y_n)$. We can assume that $X_1, \cdots, X_n$ are lifetimes of n components that make up some system and $Y_1, \cdots, Y_n$ are also lifetimes of another n components that make up another system. Assume that $\mathbf{X}$ and $\mathbf{Y}$ behave the same stochastically until there are $k < n$ components failed in each of them. The question is how can we compare the whole of $\mathbf{X}$ and $\mathbf{Y}$ stochastically? That is, under what condition can we say that $\mathbf{X} \leq_{st} \mathbf{Y}$? This question is originated from Scarsini and Shaked (1999). In that paper, they considered the special case when $n = 2$ and $k = 1$. We extend the result to the general situation here.

In the first section, we have $n = 3$ and assume that $\mathbf{X}$ and $\mathbf{Y}$ have the same initial hazard rates. We give a condition under which $\mathbf{X} \leq_{st} \mathbf{Y}$. In Section 5.2, we still have $n = 3$ and assume that $\mathbf{X}$ and $\mathbf{Y}$ have the same initial hazard rates and first order conditional hazard rates. We then give a condition under which $\mathbf{X} \leq_{st} \mathbf{Y}$. In the last section, we consider the general case that $\mathbf{X} = (X_1, \cdots, X_n)$ and $\mathbf{Y} = (Y_1, \cdots, Y_n)$. We assume that $\mathbf{X}$ and $\mathbf{Y}$ have the same conditional hazard rates up to certain level, say k. We prove that if the residual lifetimes after k failures of these two random vectors are stochastically ordered then $\mathbf{X}$ and $\mathbf{Y}$ are comparable in the same stochastic order.

5.1 Initial hazard rates

Let $\mathbf{X} = (X_1, X_2, X_3)$ and $\mathbf{Y} = (Y_1, Y_2, Y_3)$ be two nonnegative random vectors. We assume that $\mathbf{X}$ and $\mathbf{Y}$ both have absolutely continuous distribution functions F and G, respectively. We denote by $\bar{F}$ and $\bar{G}$ the survival functions and by $\lambda_i^{\mathbf{X}}$, $\lambda_i^{\mathbf{Y}}$, $i = 1, 2, 3$, the initial hazard rates of $\mathbf{X}$ and $\mathbf{Y}$, respectively. For the definition of the initial hazard rate, see (1.3) in Section 1.1. The following assumption is in force throughout this section.

Assumption 1. The random vectors $\mathbf{X}$ and $\mathbf{Y}$ satisfy

$$\lambda_i^{\mathbf{X}}(t) = \lambda_i^{\mathbf{Y}}(t),$$

for all t and $i = 1, 2, 3$.

If we think of both $\mathbf{X}$ and $\mathbf{Y}$ as the lifetime vectors of 3 devices, then Assumption 1 says that $\mathbf{X}$ and $\mathbf{Y}$ behave the same stochastically before the first failure. That is, there is no stochastic difference between $\mathbf{X}$ and $\mathbf{Y}$ before there is a failure. Under this assumption we have the following theorem.

Theorem 5.1.1. *Suppose that Assumption 1 holds. If*

$$[(X_j, X_k) | X_i = t, X_j > t, X_k > t] \\ \leq_{st} [(Y_j, Y_k) | Y_i = t, Y_j > t, Y_k > t] \tag{5.1}$$

for all t, $\{i, j, k\} = \{1, 2, 3\}$ then

$$(X_1, X_2, X_3) \leq_{st} (Y_1, Y_2, Y_3).$$

Let $X_{(1)} \equiv \min(X_1, X_2, X_3)$ and $Y_{(1)} \equiv \min(Y_1, Y_2, Y_3)$. These are the first failure times associated with X_1, X_2, X_3, and Y_1, Y_2, Y_3, respectively. Theorem 5.1.1 says that under Assumption 1 if we know the first failure time and its identity, say $X_{(1)} = X_1 = t$ and $Y_{(1)} = Y_1 = t$, then $\mathbf{X}$ and $\mathbf{Y}$ are stochastically ordered if the residual lives of

X_2, X_3 and Y_2, Y_3 are ordered. We need the following 2 lemmas to prove Theorem 5.1.1.

Lemma 5.1.2. *Under Assumption 1.*

$$P(X_{(1)} = X_i) = P(Y_{(1)} = Y_i)$$

for all $i = 1, 2, 3$.

Proof. The hazard rates for $X_{(1)}$ and $Y_{(1)}$ are $\lambda_1^{\mathbf{X}} + \lambda_2^{\mathbf{X}} + \lambda_3^{\mathbf{X}}$ and $\lambda_1^{\mathbf{Y}} + \lambda_2^{\mathbf{Y}} + \lambda_3^{\mathbf{Y}}$, respectively, and they are equal. Therefore $X_{(1)} =_{st} Y_{(1)}$. Also, we have

$$\begin{aligned} P(X_{(1)} = X_i | X_{(1)} = t) &= \frac{\lambda_i^{\mathbf{X}}(t)}{\lambda_1^{\mathbf{X}}(t) + \lambda_2^{\mathbf{X}}(t) + \lambda_3^{\mathbf{X}}(t)} \\ &= \frac{\lambda_i^{\mathbf{Y}}(t)}{\lambda_1^{\mathbf{Y}}(t) + \lambda_2^{\mathbf{Y}}(t) + \lambda_3^{\mathbf{Y}}(t)} = P(Y_{(1)} = Y_i | Y_{(1)} = t). \end{aligned}$$

These imply

$$\begin{aligned} P(X_{(1)} = X_i) &= \int P(X_{(1)} = X_i | X_{(1)} = t) d\mathbf{F}_{X_{(1)}}(t) \\ &= \int_{\tau} \lambda_i^{\mathbf{X}}(u) \exp(-\int_0^u \sum_{j=1}^{3} \lambda_i^{\mathbf{X}}(t) dt) du \\ &= \int_{\tau} \lambda_i^{\mathbf{Y}}(u) \exp(-\int_0^u \sum_{j=1}^{3} \lambda_i^{\mathbf{Y}}(t) dt) du \\ &= \int P(Y_{(1)} = Y_i | Y_{(1)} = t) d\mathbf{F}_{Y_{(1)}}(t) \\ &= P(Y_{(1)} = Y_i). \end{aligned}$$

This completes the proof. ■

Lemma 5.1.3. *Under Assumption 1,*

$$[X_i | X_{(1)} = X_i] =_{st} [Y_i | Y_{(1)} = Y_i]$$

for all $i = 1, 2, 3$.

Proof. For any $t > 0$ and $i = 1, 2, 3$, we have

$$P(X_i > t | X_{(1)} = X_i) = \frac{P(X_{(1)} = X_i > t)}{P(X_{(1)} = X_i)}.$$

From Lemma 5.1.2 we have

$$P(X_{(1)} = X_i) = P(Y_{(1)} = Y_i).$$

So we only have to prove that

$$P(X_{(1)} = X_i > t) = P(Y_{(1)} = Y_i > t).$$

Similar to the proof of Lemma 5.1.2, we have

$$\begin{aligned} P(X_{(1)} = X_i > t) &= \int P(X_{(1)} = X_i > t | X_{(1)} = y) d\mathbf{F}_{X_{(1)}}(y) \\ &= \int_{\tau \bigcap (t,\infty)} P(X_{(1)} = X_i | X_{(1)} = y) d\mathbf{F}_{X_{(1)}}(y) \\ &= \int_{\tau \bigcap (t,\infty)} P(Y_{(1)} = Y_i | Y_{(1)} = y) d\mathbf{F}_{Y_{(1)}}(y) \\ &= \int P(Y_{(1)} = Y_i | Y_{(1)} = y) d\mathbf{F}_{Y_{(1)}}(y) \\ &= P(Y_{(1)} = Y_i > t). \end{aligned}$$

This completes the proof. ∎

Proof of Theorem 5.1.1

From Lemma 5.1.2 it follows that for any function ϕ we have

$$E[\phi(\mathbf{X})] = \sum_{i=1}^{3} \alpha_i E[\phi(\mathbf{X}) | X_{(1)} = X_i].$$

and

$$E[\phi(\mathbf{Y})] = \sum_{i=1}^{3} \alpha_i E[\phi(\mathbf{Y}) | Y_{(1)} = Y_i].$$

where $\alpha_i = P(X_{(1)} = X_i) = P(Y_{(1)} = Y_i)$, $i = 1, 2, 3$.

Now write

$$E[\phi(\mathbf{X}) | X_{(1)} = X_1] = \int_{\tau} E[\phi(t, X_2, X_3) | X_{(1)} = X_1 = t] \, d\mathbf{F}_{[X_1 | X_{(1)} = X_1]}(t).$$

and

$$E[\phi(\mathbf{Y})|Y_{(1)} = Y_1] = \int_\tau E[\phi(t, Y_2, Y_3)|Y_{(1)} = Y_1 = t]\, d\mathbf{F}_{[Y_1|Y_{(1)}=Y_1]}(t).$$

where $\mathbf{F}_{[X_1|X_{(1)}=X_1]}$ and $\mathbf{F}_{[Y_1|Y_{(1)}=Y_1]}$ are, respectively, the distribution functions of $[X_1|X_{(1)} = X_1]$ and $[Y_1|Y_{(1)} = Y_1]$. The expressions $E[\phi(\mathbf{X})|X_{(1)} = X_i]$, $i = 2, 3$, and $E[\phi(\mathbf{Y})|Y_{(1)} = Y_i]$, $i = 2, 3$ can be written in a similar manner.

From Lemma 5.1.3 it follows that $\mathbf{F}_{[X_1|X_{(1)}=X_1]} = \mathbf{F}_{[Y_1|Y_{(1)}=Y_1]}$. Let ϕ be an increasing function on $\mathbb{R}^3$. Then, for each t, we have that $\phi(t, x + t, y + t)$ is increasing on $\mathbb{R}^2$. Therefore from (5.1) we get

$$\begin{aligned} E[\phi(t, X_2, X_3)|X_{(1)} = X_1 = t] \\ &= E[\phi(t, (X_2 - t) + t, (X_3 - t) + t)|X_{(1)} = X_1 = t] \\ &\le E[\phi(t, (Y_2 - t) + t, (Y_3 - t) + t)|Y_{(1)} = Y_1 = t] \\ &= E[\phi(t, Y_2, Y_3)|Y_{(1)} = Y_1 = t],\ t \in \tau. \end{aligned}$$

Hence we have

$$E[\phi(\mathbf{X})|X_{(1)} = X_1] \le E[\phi(\mathbf{Y})|Y_{(1)} = Y_1].$$

Similarly,

$$E[\phi(\mathbf{X})|X_{(1)} = X_2] \le E[\phi(\mathbf{Y})|Y_{(1)} = Y_2].$$

and

$$E[\phi(\mathbf{X})|X_{(1)} = X_3] \le E[\phi(\mathbf{Y})|Y_{(1)} = Y_3].$$

Then

$$(X_1, X_2, X_3) \le_{st} (Y_1, Y_2, Y_3).$$

This completes the proof. ∎

5.2 Conditional hazard rates

Let $\mathbf{X}$, $\mathbf{Y}$, $\lambda_i^{\mathbf{X}}$, and $\lambda_i^{\mathbf{Y}}$ be the same as in Section 5.1. In previous section, we assumed that $\mathbf{X}$ and $\mathbf{Y}$ have the same behavior before the first failure. In this section we further assume the following

Assumption 2. In addition to Assumption 1, assume also that $\mathbf{X}$ and $\mathbf{Y}$ satisfy

$$\lambda_{i|j}^{\mathbf{X}}(t|t_j) = \lambda_{i|j}^{\mathbf{Y}}(t|t_j),$$

for all $t \geq t_j \geq 0$ and for all $i, j \in \{1, 2, 3\}$ with $i \neq j$, where $\lambda_{i|j}^{\mathbf{X}}$ and $\lambda_{i|j}^{\mathbf{Y}}$ are the conditional hazard rates of $\mathbf{X}$ and $\mathbf{Y}$, respectively. See (1.3) in Section 1.1 for the definition.

Under Assumptions 1 and 2, we know that the two random vectors have the same behavior up to the first two failures. Then we should have a stochastic order if the remaining ones are ordered. In the proof of the following theorem, we need the following notation. Let $X_{(2)}$ = the second failure within X_1, X_2, and X_3, that is, $X_{(2)} = X_j$ if $X_i < X_j < X_k$. $Y_{(2)}$ is defined in a similar way.

Theorem 5.2.1. *Under Assumptions 1 and 2, if*

$$\begin{aligned}[X_k|X_i = t_i, X_j = t_j, X_k > \max(t_i, t_j)]& \\ \leq_{st}[Y_k|Y_i = t_i, Y_j = t_j, Y_k > \max(t_i, t_j)].& \qquad (5.2)\end{aligned}$$

then

$$(X_1, X_2, X_3) \leq_{st} (Y_1, Y_2, Y_3).$$

That is, there are two failures at time t_i and t_j and the identities are devices i and j. If the residual lives of device k from both $\mathbf{X}$ and $\mathbf{Y}$ are stochastically ordered then $\mathbf{X}$ and $\mathbf{Y}$ will be comparable in the same order.

Proof. We have to show that

$$E[\phi(\mathbf{X})] \le E[\phi(\mathbf{Y})]$$

for any increasing function ϕ on $\mathbb{R}^3$. Similar to the proof of Theorem 5.1.1, we have

$$E[\phi(\mathbf{X})]$$
$$= \sum_{1\le i_1 \ne i_2 \le 3} P(X_{(1)} = X_{i_1}, X_{(2)} = X_{i_2}) E[\phi(\mathbf{X}) | X_{(1)} = X_{i_1}, X_{(2)} = X_{i_2}]$$

and

$$E[\phi(\mathbf{Y})]$$
$$= \sum_{1\le i_1 \ne i_2 \le 3} P(Y_{(1)} = Y_{i_1}, Y_{(2)} = Y_{i_2}) E[\phi(\mathbf{Y}) | Y_{(1)} = Y_{i_1}, Y_{(2)} = Y_{i_2}].$$

It suffices to show that

$$P(X_{(1)} = X_1, X_{(2)} = X_2) = P(Y_{(1)} = Y_1, Y_{(2)} = Y_2)$$

and

$$E[\phi(\mathbf{X}) | X_{(1)} = X_1, X_{(2)} = X_2] \le E[\phi(\mathbf{Y}) | Y_{(1)} = Y_1, Y_{(2)} = Y_2].$$

For the equality, we have

$$P(X_{(1)} = X_1, X_{(2)} = X_2) = P(X_{(1)} = X_1) P(X_{(2)} = X_2 | X_{(1)} = X_1)$$

and

$$P(Y_{(1)} = Y_1, Y_{(2)} = Y_2) = P(Y_{(1)} = Y_1) P(Y_{(2)} = Y_2 | Y_{(1)} = Y_1).$$

From Section 5.1, we have that $P(X_{(1)} = X_1) = P(Y_{(1)} = Y_1)$. Therefore, we have to prove that $P(X_{(2)} = X_2 | X_{(1)} = X_1) = P(Y_{(2)} = Y_2 | Y_{(1)} = Y_1)$.

Since $\mathbf{X}$ and $\mathbf{Y}$ have the same conditional hazard rates, we have

$$P(X_{(2)} = X_2 | X_{(1)} = X_1 = t, X_{(2)} = t_2) = \frac{\lambda^{\mathbf{X}}_{2|1}(t_2|t)}{\lambda^{\mathbf{X}}_{2|1}(t_2|t) + \lambda^{\mathbf{X}}_{3|1}(t_2|t)}$$
$$= \frac{\lambda^{\mathbf{Y}}_{2|1}(t_2|t)}{\lambda^{\mathbf{Y}}_{2|1}(t_2|t) + \lambda^{\mathbf{Y}}_{3|1}(t_2|t)} = P(Y_{(2)} = Y_2 | Y_{(1)} = Y_1 = t, Y_{(2)} = t_2).$$

Also $[X_{(2)}|X_{(1)} = X_1 = t]$ and $[Y_{(2)}|Y_{(1)} = Y_1 = t]$ have the same hazard rates, since $\lambda^{\mathbf{X}}_{2|1}(t_2|t) + \lambda^{\mathbf{X}}_{3|1}(t_2|t) = \lambda^{\mathbf{Y}}_{2|1}(t_2|t) + \lambda^{\mathbf{Y}}_{3|1}(t_2|t)$, this implies

$$[X_{(2)}|X_{(1)} = X_1 = t] =_{st} [Y_{(2)}|Y_{(1)} = Y_1 = t].$$

Therefore,

$$\begin{aligned}\int_{\tau_t} & P(X_{(2)} = X_2|X_{(1)} = X_1 = t, X_{(2)} = t_2)d\mathbf{F}_{[X_{(2)}|X_{(1)}=X_1=t]}(t_2) \\ &= \int_{\tau_t} P(Y_{(2)} = Y_2|Y_{(1)} = Y_1 = t, Y_{(2)} = t_2)d\mathbf{F}_{[Y_{(2)}|Y_{(1)}=Y_1=t]}(t_2).\end{aligned}$$

This means that

$$P(X_{(2)} = X_2|X_{(1)} = X_1 = t) = P(Y_{(2)} = Y_2|Y_{(1)} = Y_1 = t).$$

Hence by Lemma 5.1.3 we have

$$\begin{aligned}P(X_{(2)} &= X_2|X_{(1)} = X_1) \\ &= \int_{\tau} P(X_{(2)} = X_2|X_{(1)} = X_1 = t)d\mathbf{F}_{[X_1|X_{(1)}=X_1]}(t) \\ &= \int_{\tau} P(Y_{(2)} = Y_2|Y_{(1)} = Y_1 = t)d\mathbf{F}_{[Y_1|Y_{(1)}=Y_1]}(t) \\ &= P(Y_{(2)} = Y_2|Y_{(1)} = Y_1).\end{aligned}$$

Next, we will show that

$$E[\phi(\mathbf{X})|X_{(1)} = X_1, X_{(2)} = X_2] \leq E[\phi(\mathbf{Y})|Y_{(1)} = Y_1, Y_{(2)} = Y_2].$$

Indeed,

$$\begin{aligned}E[\phi(\mathbf{X})&|X_{(1)} = X_1, X_{(2)} = X_2] \\ &= \int E[\phi(\mathbf{X})|X_{(1)} = X_1 = t_1, X_{(2)} = X_2 = t_2]d\mathbf{F}_{[(X_1,X_2)|X_{(1)}=X_1,X_{(2)}=X_2]}(t_1, t_2).\end{aligned}$$

and

$$\begin{aligned}E[\phi(\mathbf{Y})&|Y_{(1)} = Y_1, Y_{(2)} = Y_2] \\ &= \int E[\phi(\mathbf{Y})|Y_{(1)} = Y_1 = t_1, Y_{(2)} = Y_2 = t_2]d\mathbf{F}_{[(Y_1,Y_2)|Y_{(1)}=Y_1,Y_{(2)}=Y_2]}(t_1, t_2).\end{aligned}$$

For any fixed $t_1 \le t_2$, $\phi(t_1, t_2, t)$ is an increasing function of t. By (5.2), we have

$$E[\phi(\mathbf{X})|X_{(1)} = X_1 = t_1, X_{(2)} = X_2 = t_2] \le E[\phi(\mathbf{Y})|Y_{(1)} = Y_1 = t_1, Y_{(2)} = Y_2 = t_2].$$

The result will follow if we can prove that

$$[(X_1, X_2)|X_{(1)} = X_1, X_{(2)} = X_2] =_{st} [(Y_1, Y_2)|Y_{(1)} = Y_1, Y_{(2)} = Y_2].$$

From above, we have $P(X_{(1)} = X_1, X_{(2)} = X_2) = P(Y_{(1)} = Y_1, Y_{(2)} = Y_2)$. Therefore, we only have to show that

$$P(X_{(1)} = X_1 = t_1, X_{(2)} = X_2 = t_2) = P(Y_{(1)} = Y_1 = t_1, Y_{(2)} = Y_2 = t_2)$$

for any fixed $t_1 < t_2$. Now we write

$$\begin{aligned} &P(X_{(1)} = X_1 = t_1, X_{(2)} = X_2 = t_2) \\ &\quad = P(X_{(2)} = X_2|X_{(1)} = X_1 = t_1, X_{(2)} = t_2)P(X_{(1)} = X_1 = t_1, X_{(2)} = t_2) \\ &\quad = P(X_{(2)} = X_2|X_{(1)} = X_1 = t_1, X_{(2)} = t_2)P(X_{(2)} = t_2|X_{(1)} = X_1 = t_1) \\ &\qquad \times P(X_{(1)} = X_1 = t_1). \end{aligned}$$

Similarly

$$\begin{aligned} &P(Y_{(1)} = Y_1 = t_1, Y_{(2)} = Y_2 = t_2) \\ &\quad = P(Y_{(2)} = Y_2|Y_{(1)} = Y_1 = t_1, Y_{(2)} = t_2)P(Y_{(1)} = Y_1 = t_1, Y_{(2)} = t_2) \\ &\quad = P(Y_{(2)} = Y_2|Y_{(1)} = Y_1 = t_1, Y_{(2)} = t_2)P(Y_{(2)} = t_2|Y_{(1)} = Y_1 = t_1) \\ &\qquad \times P(Y_{(1)} = Y_1 = t_1). \end{aligned}$$

From Assumption 2, we have

$$\begin{aligned} &P(X_{(2)} = X_2|X_{(1)} = X_1 = t_1, X_{(2)} = t_2) \\ &\quad = \frac{\lambda^{\mathbf{X}}_{2|1}(t_2|t_1)}{\lambda^{\mathbf{X}}_{2|1}(t_2|t_1) + \lambda^{\mathbf{X}}_{3|1}(t_2|t_1)} \\ &\quad = \frac{\lambda^{\mathbf{Y}}_{2|1}(t_2|t_1)}{\lambda^{\mathbf{Y}}_{2|1}(t_2|t_1) + \lambda^{\mathbf{Y}}_{3|1}(t_2|t_1)} \\ &\quad = P(Y_{(2)} = Y_2|Y_{(1)} = Y_1 = t_1, Y_{(2)} = t_2). \end{aligned}$$

Next, from above we have

$$[(X_{(2)}|X_{(1)} = X_1 = t_1] =_{st} [Y_{(2)}|Y_{(1)} = Y_1 = t_1].$$

Hence

$$P(X_{(2)} = t_2|X_{(1)} = X_1 = t_1) = P(Y_{(2)} = t_2|Y_{(1)} = Y_1 = t_1).$$

Finally, from Lemmas 5.1.2 and 5.1.3 we have

$$P(X_{(1)} = X_1 = t_1) = P(Y_{(1)} = Y_1 = t_1).$$

This completes the proof. ■

5.3 The general case

In this section, we generalize the ideas of Theorems 5.1.1 and 5.2.1 to n dimensional random vectors. In the followings, we have $k \in \{0, 1, \cdots, n-1\}$.

Assumption 3. Let $\mathbf{X}$ and $\mathbf{Y}$ be such that

$$\lambda^{\mathbf{X}}_{i|\bar{\mathbf{I}}}(t|\mathbf{t}_{\bar{\mathbf{I}}}) = \lambda^{\mathbf{Y}}_{i|\bar{\mathbf{I}}}(t|\mathbf{t}_{\bar{\mathbf{I}}})$$

for all $0\mathbf{e} \leq \mathbf{t}_{\bar{\mathbf{I}}} \leq t\mathbf{e}$, and $\mathbf{I} \subset \{1, \cdots, n\}$ with $|\bar{\mathbf{I}}| \leq k$.

From Assumption 3, we know $\mathbf{X}$ and $\mathbf{Y}$ have the same behavior before there are k devices failed. Hence, if the residual lives of the other devices are stochastically ordered, it is reasonable to expect that $\mathbf{X} \leq_{st} \mathbf{Y}$.

Theorem 5.3.1. *Under Assumption 3, if*

$$[\mathbf{X}_{\mathbf{I}}|\mathbf{X}_{\bar{\mathbf{I}}} = \mathbf{t}_{\bar{\mathbf{I}}}, \mathbf{X}_{\mathbf{I}} > t\mathbf{e}] \leq_{st} [\mathbf{Y}_{\mathbf{I}}|\mathbf{Y}_{\bar{\mathbf{I}}} = \mathbf{t}_{\bar{\mathbf{I}}}, \mathbf{Y}_{\mathbf{I}} > t\mathbf{e}]$$

for all $0\mathbf{e} \leq \mathbf{t}_{\bar{\mathbf{I}}} \leq t\mathbf{e}$, *and* $\mathbf{I} \subset \{1, \ldots, n\}$ *with* $|\bar{\mathbf{I}}| = k$, *then* $\mathbf{X} \leq_{st} \mathbf{Y}$.

Proof. By applying mathematical induction to n, we will show that this theorem holds for all n-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$ that satisfy the given conditions.

From Sections 5.1 and 5.2, we have that the theorem holds for $n = 3$.

Now suppose that it holds for $n = m$ and let $n = m + 1$. We will show that

$$E[\phi(\mathbf{X})] \leq E[\phi(\mathbf{Y})]$$

for any increasing function ϕ in $\mathbb{R}^{m+1}$.

Same as before, we can write

$$E[\phi(\mathbf{X})] = \sum_{i=1}^{m+1} \alpha_i E[\phi(\mathbf{X})|X_{(1)} = X_i].$$

and

$$E[\phi(\mathbf{Y})] = \sum_{i=1}^{m+1} \alpha_i E[\phi(\mathbf{Y})|Y_{(1)} = Y_i].$$

where $\alpha_i = P(X_{(1)} = X_i) = P(Y_{(1)} = Y_i)$, from Section 5.1, for all $i = 1, \cdots, m+1$.

Therefore, if we can prove that $E[\phi(\mathbf{X})|X_{(1)} = X_i] \leq E[\phi(\mathbf{Y})|Y_{(1)} = Y_i]$ for all i then the result follows.

Clearly, for any i we have

$$E[\phi(\mathbf{X})|X_{(1)} = X_i] = \int E[\phi(\mathbf{X}; t_i)|X_{(1)} = X_i = t_i]\, d\mathbf{F}_{[X_i|X_{(1)}=X_i]}(t_i).$$

and

$$E[\phi(\mathbf{Y})|Y_{(1)} = Y_i] = \int E[\phi(\mathbf{Y}; t_i)|Y_{(1)} = Y_i = t_i]\, d\mathbf{F}_{[Y_i|Y_{(1)}=Y_i]}(t_i).$$

From Section 5.1, $[X_i|X_{(1)} = X_i] =_{st} [Y_i|Y_{(1)} = Y_i]$. Also $(\mathbf{X}; t_i)$ and $(\mathbf{Y}; t_i)$ are random vectors with both i-th coordinate being fixed and equal to t_i. This means they are m-dimensional random vectors. With i-th coordinate being fixed and equal to t_i, ϕ is an increasing function in $\mathbb{R}^m$. By induction assumption, we have

$$E[\phi(\mathbf{X}; t_i)|X_{(1)} = X_i = t_i] \leq E[\phi(\mathbf{Y}; t_i)|Y_{(1)} = Y_i = t_i].$$

This implies

$$E[\phi(\mathbf{X})|X_{(1)} = X_i] \leq E[\phi(\mathbf{Y})|Y_{(1)} = Y_i].$$

This completes the proof. ■

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