SYNTHESIS OF NONLINEAR ELECTRICAL NETWORKS

by

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I hereby recommend that this dissertation prepared under my direction by ALFRED JOSEPH HOEHN, JR.

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be accepted as fulfilling the dissertation requirement of the degree of DOCTOR OF PHILOSOPHY

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CHAPTER 1

INTRODUCTION AND LITERATURE SURVEY

1.1 Introduction

This dissertation is concerned with the synthesis of nonlinear networks for operation in the steady state with outputs harmonically related to the input. This category includes generators of power at frequencies which are harmonics or subharmonics of the input frequency, and those which generate pulses, triangular waves, and the like from a sinusoidal or other periodic input. Included also are filters designed to pass such periodic waves with a minimum of distortion. Only inputs with frequency components which are not harmonically related, and which would therefore generate modulation products or combination tones, are eliminated from consideration. Unlike many previous approaches to nonlinear problems, no restrictions are placed on the departure from linearity which may be exhibited by a network element, nor upon the number of nonlinear elements, nor upon the number of loops or degrees of freedom in the network.

1.2 The Underlying Problems

It is well known that the general superposition theorem of linear networks is inapplicable when nonlinearities are introduced. Any mathematical description of the nonlinear elements must therefore preserve their amplitude-sensitive characteristics. To be useful the description must also be normalized or generalized to the extent that the elements can be combined into networks without specifying, per se, a particular amplitude. This has been the fundamental problem.

Most of the effort in the solution of this problem has been directed toward the use of nonlinear energy-storage elements, with the inductor as the main object of investigation. Nonlinear capacitive devices are, of course, subject to similar treatment. Resistance which is nonlinear with frequency has been introduced rather simply in one of the applications. Resistance exhibiting nonlinear voltage-current relations has not been treated in this study, but this problem should yield to a similar type of analysis and representation.

The second problem involved the combination of the elements into a network which would relate the specified input to the desired output. For this purpose the ladder network was chosen as the basic configuration. While not the most general, the ladder network is probably the most useful, and it affords a wealth of practical applications to which the method can be applied.

1.3 Method of Solution

The method of solution may be described briefly as follows:

1) The magnetization curve of the material is approximated by straight-line segments as shown in Figure 1.1. This approximation is an excellent one for the so-called "rectangular-hysteresis-loop" iron materials such as Deltamax and Orthonol. Even for the more gently varying transformer irons, reasonable agreement with experimental values can be achieved. In the derivations the magnetization curve is expressed in terms of flux linkage versus current.

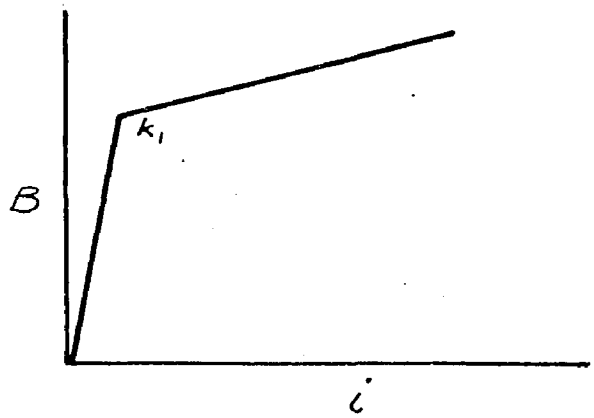

Fig. 1.1 Boyajian's Representation of Saturation Curve

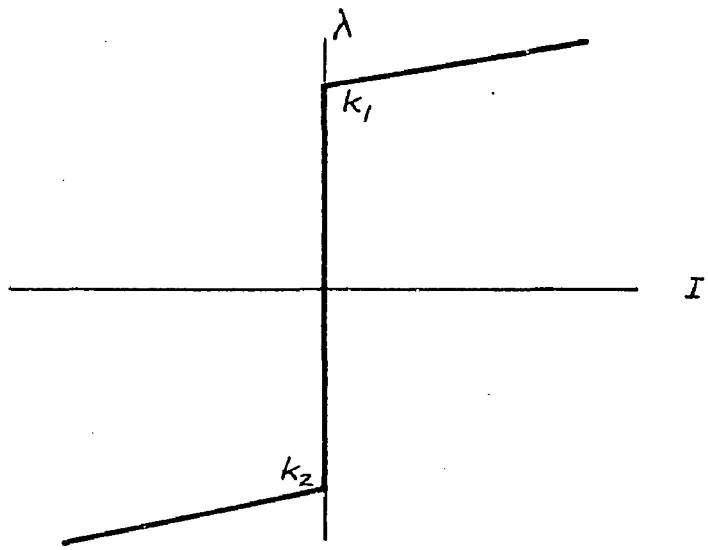

Fig. 1.2 Approximation of Travis and Weygandt

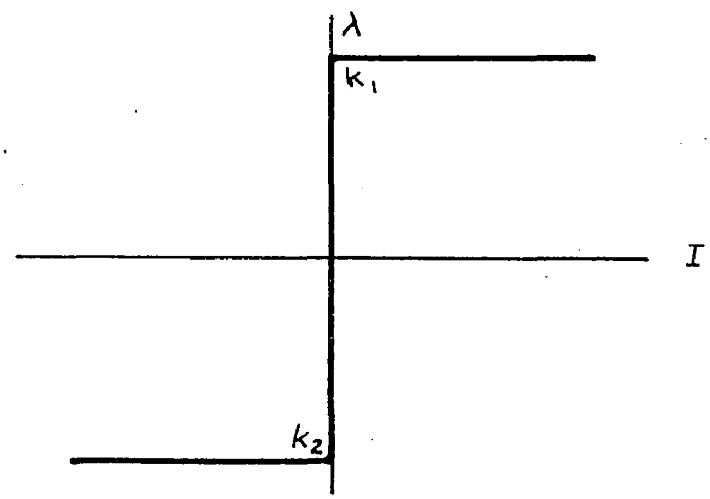

Fig. 1.3 Switch Approximation to Magnetization Curve

2) By means of Fourier series the relations between voltage across the inductor and current through it are expressed for as many harmonics of the fundamental as are necessary to the problem at hand. They are expressed in terms of the breakpoints k_1 and k_2 of the magnetization curve (Figure 1.2). The trigonometric form of the series is used, which results in two families of equations--one set relating the sine components of voltage to the cosine components of current, and a second relating the cosine components of voltage to the sine components of current. Phase relations are thus implicit in the two sets of equations.

3) The two sets of equations are expressed as two matrix equations, in each of which a column matrix of currents is equal to the product of a triangular matrix and a column matrix of voltages. The elements of the triangular matrices are the Fourier coefficients mentioned above, which have the dimensions of admittance. The effects of nonlinearity in the admittance matrix appear primarily in the off-diagonal terms. In contrast, a similar matrix for a linear inductor would have non-zero elements only along the main diagonal. It can be said, then, that the two admittance matrices describe the electrical properties of the inductor under conditions of operation corresponding to the particular values of k_1 and k_2 selected.

4) The nonlinear network elements are combined with other elements, either linear or nonlinear, into networks. Through straight-forward application of Kirchhoff's Laws, the matrices describing the various elements are manipulated into matrix equations

relating input and output. Such equations for ladder networks are presented in Tables 3.1 and 3.2.

5) If the problem were one of analysis, values for the various elements would be substituted in the matrices at this point, and indicated computations performed. In the synthesis problem, a choice of values must be made so that the output matrix will match the product of the network-admittance matrix and the input-voltage matrix. No formal, routine process exists for this. Much of the success of the synthesis depends on the choice of k_1 and k_2, for which criteria are usually available (cf. Sec. 2.2.2). Because of the matrix notation, choices are often obvious from inspection. To expedite these choices and minimize the tedium and calculation, tables covering a wide range of values of k_1 and k_2 have been precomputed by the Numerical Analysis Laboratory of the University of Arizona and are available from them. An abridged version of these tables is presented in Appendix B of this dissertation.

1.4 Literature Survey

The synthesis of nonlinear systems provides a modern touch to a problem as old as science itself. Engineers and scientists have always been very much aware of the essential nonlinearity of nature, and from the very earliest days have made use of the nonlinear properties of various materials in their designs. Because of the inherent difficulties of nonlinear analysis, however, the approaches were generally of an experimental and/or graphical nature.

Although Poincare[1], Liapounoff[2], van der Pol[3], Duffing[4] and others had laid the foundations for the study of nonlinear mechanics in the late 1800's and early 1900's, the analytic approach to the subject was not pursued actively until there became available in 1937 the publications by Andronow and Chaikin[5], representing a group of mathematicians in Moscow, and by Kryloff and Bogoliuboff[6] representing a similar group in Kiev.

Since the beginning of World War II the literature on nonlinear phenomena has burgeoned. Ku's book[7], for example, contains a bibliography of 735 listings, of which all but 153 have appeared since 1941. A great assist to this effort at the applied level was given by the appearance in 1947 of a series of reports edited by Minorsky[8], which essentially were translations of the works of the Russians mentioned above. Another factor contributing to the interest in nonlinear analysis has been the success of the linear theorists, notably Wiener[9] in establishing an upper theoretical limit to the performance of purely linear systems.

The various approaches to nonlinear analysis (and synthesis) have been rather neatly classified by Kaplan[10] in a round table discussion following a symposium on Nonlinear Circuit Analysis at the Polytechnic Institute of Brooklyn in 1956. According to Kaplan, all nonlinear methods may be put into three groups:

(a) "the essentially linear methods applicable to slightly nonlinear problems, such as sinusoidal describing functions, perturbation and iteration processes, and the like;"

(b) "the phase-plane method which is simple and informative even for highly nonlinear problems as long as their order is low enough to permit its application;"

(c) "the nonlinear problems of higher dimensions for which the only effective tools appear to be the series expansions, piecewise linearization and possible operational methods."

Most methods which have been proposed for engineering use have been confined to groups (a) and (b), and further have been restricted to single degree of freedom (one-loop) systems. None of these are applicable to the problem at hand.

A method of dealing with nonlinear multiloop circuits has been advanced by Klotter[11] and another by Ku[12]. Klotter's method makes use of the Ritz-Galerkin approximation which is based on the calculus of variations, but in Klotter's own words "is unable to exhibit any unknown or unexpected features of the solution." The so-called "acceleration plane" method of Ku is essentially a graphical approach. It is quite laborious and must be individually tailored to each problem presented to it.

Piecewise-linear methods seem to offer the greatest possibilities for synthesis of multiloop systems. Vallese[13] and Stern[14] have developed methods dealing with the synthesis of function generators made up of nonlinear resistive elements (diodes). Application of their methods appears to be primarily for use in computing devices.

Energy-storage devices also have received the piecewise-linear treatment over the years. Many analyses based on linearized hysteresis loops and magnetization curves have appeared in the literature. This approach was probably first used by Boyajian[15] in

1931 in a general analytical study of several nonlinear engineering problems. He represented the magnetization curve in the form shown in Figure 1.1. In Boyajian's approach the problem is set up in terms of as many linear equations as there are segments to the magnetization curve. The linear equations are solved in straightforward fashion, and the solutions matched at the junctions, or transition points where two linear segments join. The matching of solutions is a most tedious process and not suited to the synthesis of multiloop systems.

Travis[16,17] and Weygandt used the slightly different representation shown in Figure 1.2 in their classic study of subharmonic generation. More recent investigations of subharmonic and ferroresonant processes by Brenner[18] and Dennard[19] also have been based on piecewise approximations with matched solutions at the transitions. All studies were made on one-loop circuits.

Another very popular technique depends on the idealization of the hysteresis loop in the form shown in Figure 1.3. In this form the magnetic core is regarded as a switch which turns current on or off at some point in the cycle as determined by the design of the reactor. Melville[20] used this analysis in his source paper on the generation of pulses by magnetic means. Other investigators[21,22] utilized the same approach in later papers refining and expanding Melville's work. Salihi[23] made use of the switch representation in still another study of the ferroresonant circuit, in which he likened the magnetic core to a gas tube. Finally, Johnson and Rauch[24] made use of the "on-off" representation in the preliminary phases of an explanation of the operation of harmonic generators. Objections to the switch approach are that it is unnecessarily crude and that it suppresses information during the design stage. No information is available, for example, on the relative size or phase of harmonics generated in the core. Also, even for rectangular-loop irons, the departure from the actual situation can be quite significant, thus limiting the on-off approach to qualitative, rather than quantitative, consideration. It is believed, however, that the switch approach can be used to considerable advantage as an adjunct to the method described in this dissertation. It should be of particular value as a rough guide in choosing values of the breakpoints, k_1 and k_2.

Methods based on a trigonometric-series representation of linearized characteristics appear intermittently in the literature. The tendency, however, has always been to brush them off as being unwieldy, cumbersome, and excessively laborious. Use of the matrix notation in this dissertation removes this objection, which is not completely valid in the first place. Two methods which have appeared in recent years are worthy of note.

In 1956 Neal[25] introduced his "phasor method", in which he obtains a set of circuit equations for each significant harmonic frequency. The method requires that significant harmonics be selected prior to its application, with the selection to be based on "previous experience, experiment, and preliminary analysis." While the method offers a relatively simple approach to analysis, the dependence on the choice of the significant harmonics again makes it undesirable for synthesis. Like Klotter's approach mentioned earlier, the method does not draw attention to unknown or unexpected features in the network. One of the features of Neal's method is a precomputed set of equations which may be used directly, thus eliminating much of the alleged tedium from the trigonometic approach. Also, the method is not limited to the usual small departure from linearity.

In the same year Gohar[26] presented an analysis of the single-loop circuit containing a nonlinear reactor. His representation of the voltage-current relations in the reactor is similar to that used in the first steps of the method of this dissertation. Gohar confined himself to the analysis of single-loop circuits and made no attempt to generalize or normalize his results, so that combinations of elements might be made. Multiloop circuits and the synthesis problem were not considered. Considerable experimental evidence was offered in justification of the Fourier-series representation of the voltage-current relations in the reactor. Because of the similarities in the basic analysis, this justification would apply also to the voltage-current relations derived in the first part of this dissertation.

CHAPTER 2

DEVELOPMENT OF BASIC CONCEPTS

2.1 Current Generated by a Voltage Impressed on a Nonlinear Element

In Figures 2.1 through 2.4 curves are presented showing the variation of flux linkage (λ) with current (I). These curves represent a piecewise-linear approximation to the true λ-I characteristics. As pointed out in the introduction, this approximation is excellent for the rectangular-hysteresis-loop materials, and gives fair agreement with experimental results when transformer iron is the core material.

In all figures the characteristics marked (a) include hysteresis losses while those marked (b) are for the idealized lossless case. Within the degree of precision afforded by the linear approximations already made, the (a) characteristics might also be regarded as dynamic hysteresis loops (with appropriate changes in width to represent the increased losses, of course). No attempt is made in this presentation to account for non-uniform saturation of core material as it affects core losses, or causes slight variations in the location of breakpoints on the curve. As it turns out, the core losses have a very small effect on the generation of harmonic currents, which is the main object of this investigation. In the applications of Chapter 3, core loss is neglected completely.

Figure 2.1 shows curves having equal areas above and below the abscissa, which correspond to symmetrical excitation of the core.

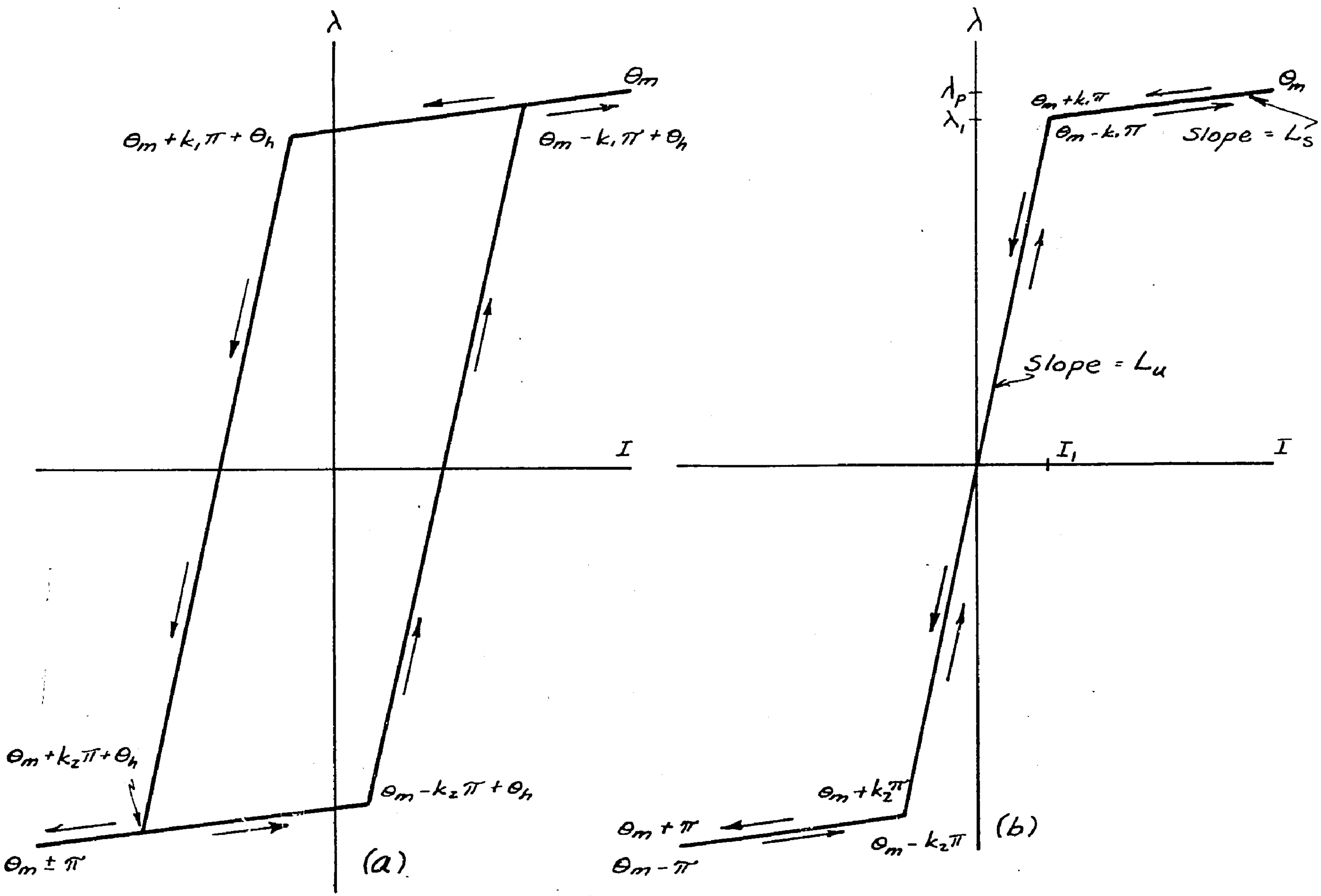

Fig. 2.1 Hysteresis Loops for Symmetrical Excitation

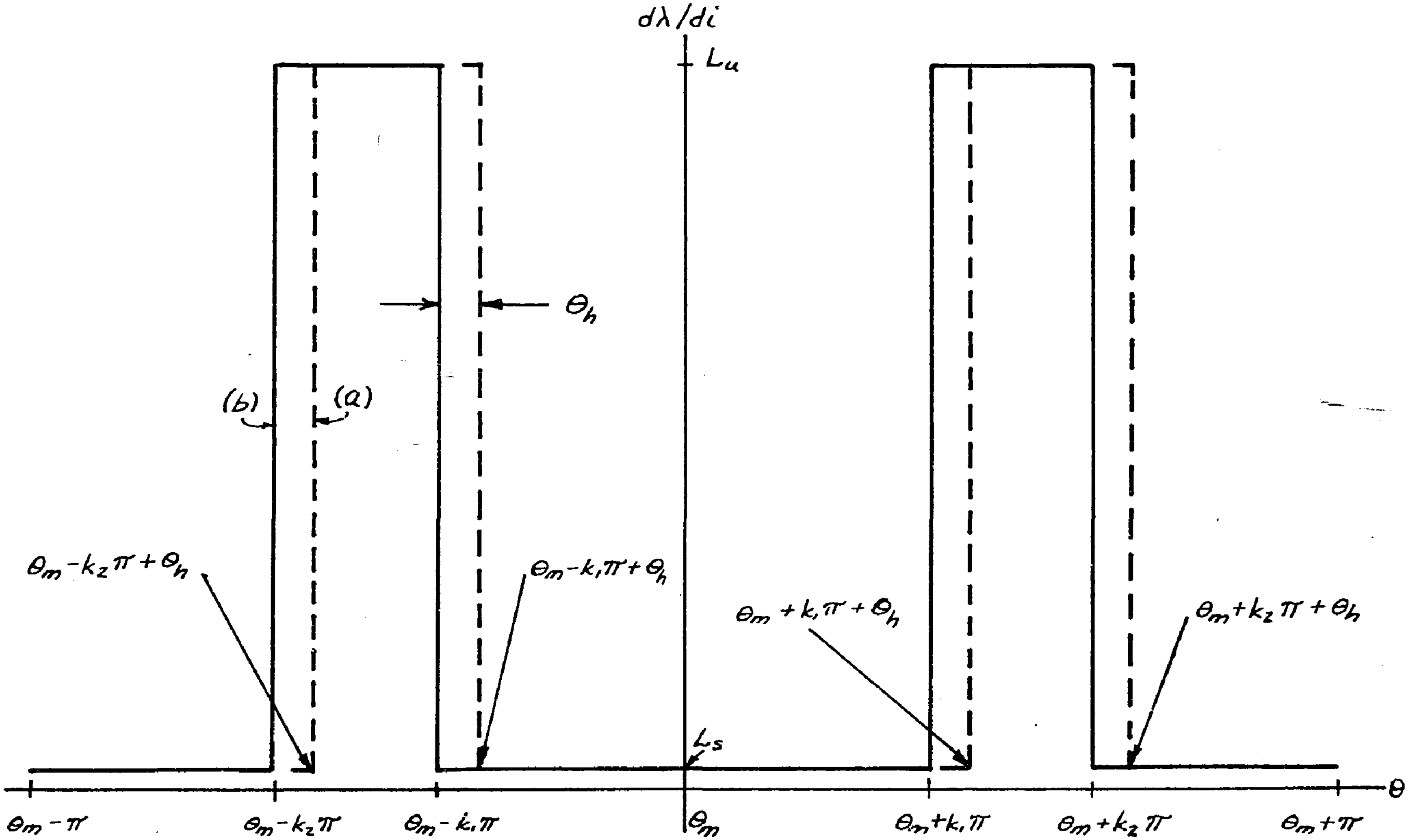

Fig. 2.2 Slope dλ/di as a Function of θ

(a) – – – **With Hysteresis** (b) ——— **No Hysteresis**

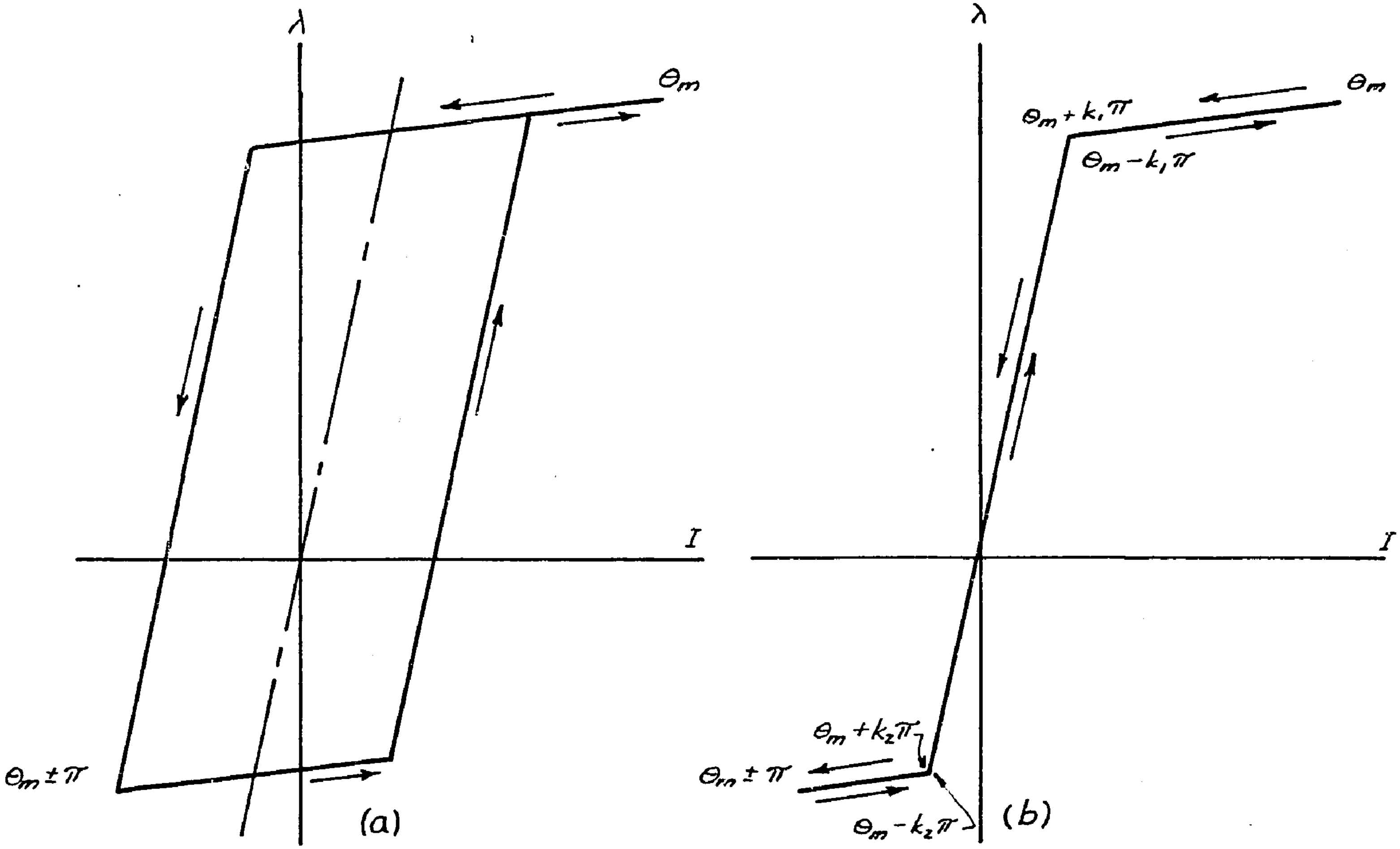

Fig. 2.3 Unsymmetrical Excitation

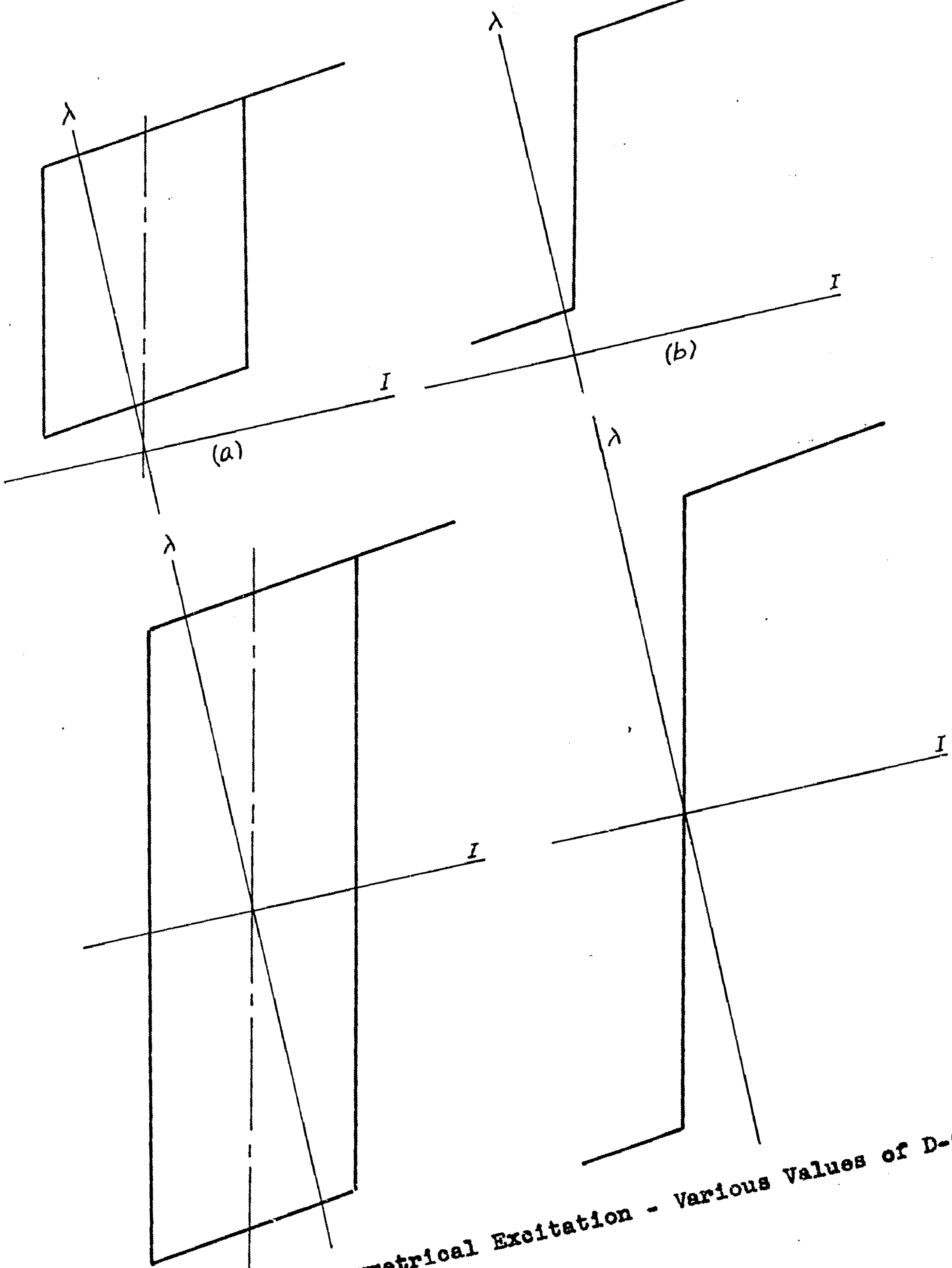

Fig. 2.4 Unsymmetrical Excitation - Various Values of D-C Bias

Figures 2.3 and 2.4 show curves corresponding to excitation of a core biased by a fixed number of d-c ampere turns. This method is normally used for the generation of even harmonics of the fundamental excitation frequency.

2.1.1 Derivation of the Fourier Coefficients

Consider a voltage e from a generator of zero impedance impressed on an inductor with a λ versus I characteristic such as that of Figure 2.1. The voltage e is expressed by

$$e = d\lambda/dt = \sum_{n=1}^{N} (A_n \cos n\omega t + B_n \sin n\omega t) \tag{2-1}$$

The resulting current may be expressed in a Fourier series as

$$i = \sum_{m=1}^{\infty} (a_m \cos m\omega t + b_m \sin m\omega t) \tag{2-2}$$

To evaluate the a_m and b_m in the expression for current, an expression is first obtained for di/dt, whose constants are easily evaluated. These constants are then compared with differentiated values of (2-2).

Since i is a function of λ,

$$di/dt = (di/d\lambda)(d\lambda/dt) \tag{2-3}$$

The di/dt may be expressed in a Fourier series as

$$di/dt = \sum_{m=1}^{\infty} (b_m' \cos m\omega t + a_m' \sin m\omega t) \tag{2-4}$$

The a_m' and b_m' of (2-4) are evaluated in the usual fashion based on the facts that

(a) $d\lambda/dt$ is given by (2-1), and

(b) $di/d\lambda$ takes on different values over the range of the λ versus I characteristic.

In Figure 2.1, λ may also be regarded as a function of θ for a complete cycle of the fundamental, where

$$\theta = \omega t \tag{2-5}$$

Figure 2.2 shows $d\lambda/di$ as a function of θ. Both λ and $d\lambda/di$ are seen to vary from $(\theta_m - \pi)$, corresponding to negative peak current, to θ_m, corresponding to positive peak current, and back to $(\theta_m + \pi)$ which completes the cycle at the negative peak-current point.

From Figure 2.2(b) and 2.3(b) it may be seen that

$$di/d\lambda = (d\lambda/di)^{-1} = 1/L_s \begin{cases} (\theta_m - \pi) < \theta < (\theta_m - k_2\pi) \\ (\theta_m - k_1\pi) < \theta < (\theta_m + k_1\pi) \\ (\theta_m + k_2\pi) < \theta < (\theta_m + \pi) \end{cases} \tag{2-6a}$$

$$di/d\lambda = 1/L_u \qquad \begin{cases} (\theta_m - k_2\pi) < \theta < (\theta_m - k_1\pi) \\ (\theta_m + k_1\pi) < \theta < (\theta_m + k_2\pi) \end{cases} \tag{2-6b}$$

where θ_m is the angle corresponding to maximum current. Reference to Figure 2.1, Figure 2.3, or Figure 2.4 also reveals that L_u is the coil inductance during operation on the unsaturated portion of the magnetization curve, while L_s is the inductance of the coil when operated on the saturated portion of the curve.

Evaluation of constants a_m' and b_m' proceeds, with

$$f(\theta) = (1/L_s)\sum_{n=1}^{N} (A_n \cos n\theta + B_n \sin n\theta) \tag{2-7}$$

over the range of values given in (2-6a), and

$$f(\theta) = (1/L_u)\sum_{n=1}^{N} (A_n \cos n\theta + B_n \sin n\theta) \tag{2-8}$$

over the range of values given in (2-6b)

In evaluating the a_m' and b_m', it is noted that the angle θ_m merely reflects an arbitrary choice of zero time or zero radians on the abscissa. No generality will be lost if it is required that maximum current occur at

$$\theta_m = 0 \tag{2-9}$$

Comparison of (2-2) and (2-4) indicates further that

$$b_m' = m\omega b_m \tag{2-10}$$

and that

$$a_m' = -m\omega a_m \tag{2-11}$$

Values of the a_m and b_m are thus obtained from consideration of the foregoing equations for the case with hysteresis neglected. Under these conditions the $(d\lambda/di)^{-1}$ characteristics are symmetrical about the axis of ordinates ($\theta = 0$) so that the Fourier half-range expression may be used. Further, since $d\lambda/di$ is an even function, the cosine terms in e will produce only cosine terms in di/dt and the sine terms in e will produce only sine terms in di/dt. Thus,

$$b_m = \sum_{n=1}^{N} (A_n/m\omega_1) \left[(2/L_s\pi) \int_0^{k_1\pi} \cos m\theta \cos n\theta \, d\theta + (2/L_u\pi) \int_{k_1\pi}^{k_2\pi} (\text{ditto}) \right.$$

$$\left. + (2/L_s\pi) \int_{k_2\pi}^{\pi} (\text{ditto}) \right] \qquad (2\text{-}12)$$

from which

$$b_m = \sum_{n=1}^{N} \frac{A_n}{m\omega_1} \left\{ \frac{2}{L_s\pi} \left[\frac{\sin (m - n)\theta}{2(m - n)} + \frac{\sin (m + n)\theta}{2(m + n)} \right]_0^{k_1\pi} + \left[\frac{2}{L_u\pi} \text{ ditto} \right]_{k_1\pi}^{k_2\pi} \right.$$

$$\left. + (2/L_s\pi) \Big[\text{ditto} \Big]_{k_2\pi}^{\pi} \right\} \quad (m^2 \neq n^2) \qquad (2\text{-}13)$$

and

$$b_m = (A_n/m\omega_1) \left\{ \frac{2}{L_s\pi} \left[\frac{\theta}{2} + \frac{\sin 2m\theta}{4m} \right]_0^{k_1\pi} + \frac{2}{L_u\pi} \Big[\text{ditto} \Big]_{k_1\pi}^{k_2\pi} \right.$$

$$\left. + \frac{2}{L_s\pi} \Big[\text{ditto} \Big]_{k_2\pi}^{\pi} \right\} \quad (m = n) \qquad (2\text{-}14)$$

Values for the a_m are obtained in similar fashion from the sine equations corresponding to (2-12). Integration and substitution of limits, followed by application of trigonometric identities produces the following equations for the a_m and b_m:

$$a_m = \sum_{n=1}^{N} \frac{+B_n}{m\omega_1} \left[\frac{1}{L_s} - \frac{1}{L_u} \right] \left[\frac{\sin (m - n)(\pi/2)\xi \cos (m - n)(\pi/2)\psi}{(m - n)(\pi/2)} \right.$$

$$\left. - \frac{\sin (m + n)(\pi/2)\xi \cos (m + n)(\pi/2)\psi}{(m + n)(\pi/2)} \right] \quad (m > n) \qquad (2\text{-}15)$$

$$a_m = \frac{-B_m}{m\omega_1}\left[\frac{1}{L_s} - \left(\frac{1}{L_s} - \frac{1}{L_u}\right)\xi + \left(\frac{1}{L_s} - \frac{1}{L_u}\right)\left(\frac{\sin m\pi\xi \cos m\pi\psi}{m\pi}\right)\right]_{(m = n)} \tag{2-16}$$

$$b_m = \sum_{n=1}^{N} \frac{-A_n}{m\omega_1}\left(\frac{1}{L_s} - \frac{1}{L_u}\right)\left[\frac{\sin (m-n)(\pi/2)\xi \cos (m-n)(\pi/2)\psi}{(m-n)(\pi/2)}\right.$$

$$(m > n)$$

$$\left. + \frac{\sin (m+n)(\pi/2)\xi \cos (m+n)(\pi/2)\psi}{(m+n)(\pi/2)}\right] \tag{2-17}$$

$$b_m = \frac{+A_m}{m\omega_1}\left[\frac{1}{L_s} - \left(\frac{1}{L_s} - \frac{1}{L_u}\right)\xi - \left(\frac{1}{L_s} - \frac{1}{L_u}\right)\left(\frac{\sin m\pi\xi \cos m\pi\psi}{m\pi}\right)\right]_{(m = n)} \tag{2-18}$$

In the above equations

$$\xi = k_2 - k_1 \qquad (0 < \xi < 1) \tag{2-19}$$

$$\psi = k_2 + k_1 \qquad (1 \leqq \psi < 2) \tag{2-20}$$

and ω_1 is the angular frequency of the fundamental.

The quantity ξ is defined so that $\pi\xi$ corresponds to the number of radians per half-cycle during which the reactance is equal to $\omega_1 L_u$. If L_u were infinite (an idealization often used in approximate analyses), $\pi\xi$ would correspond to that portion of the half-cycle during which zero current flows. The quantity ψ has unit value for the case of symmetrical excitation (cf. Sec. 2.2.1) and other values for unsymmetrical excitation, depending on the biasing ampere turns.

2.1.2 Normalized Form of Equations

Equations (2-15) through (2-18) are not in the most satisfactory form for computation and experimental verification. The unsaturated inductance, L_u, may be easily measured and the ratio of unsaturated inductance to saturated inductance (L_u/L_s) is obtainable, but direct measurement of L_s, either from a graph or electrically, is difficult because of its small value. Accordingly the ratio

$$\delta = L_u/L_s \tag{2-21}$$

is defined.

In the applications considered later in this dissertation, the following per-unit notation of Russell[27,28] is adopted:

a. Base impedance for the system is the reactance of the inductor when operating in the unsaturated region at fundamental frequency; i.e.,

$$\omega_1 L_u = 1 \tag{2-22}$$

b. Base current is that which flows in the nonlinear reactor at the point of saturation.

c. Base volts are then equal to the product of $\omega_1 L_u$ and the current in the nonlinear reactor at saturation. Unit peak base volts would be just sufficient to cause the peak flux linkage to operate between breakpoints k_1 and k_2.

The equations (2-15) through (2-18) are arranged in Table 2.1 in the form best suited for computation. Also, the quantity $(1/\omega_1 L_u)$ is factored out so that use of the per-unit system is optional. In the equations of Table 2.1 and in further work in this dissertation, it is assumed that

$$\delta >> 1 \tag{2-23}$$

TABLE 2.1

Relationship of Fourier-Series Coefficients for the Reactor

$$a_m = \sum_{n=1}^{N} \frac{B_n}{\omega_1 L_u} \frac{\delta}{m} \xi \left\{ \frac{\sin (m-n)(\pi/2)\xi \cos (m-n)(\pi/2)\psi}{(m-n)(\pi/2)\xi} \right. \qquad (m > n)$$

$$\left. - \frac{\sin (m+n)(\pi/2)\xi \cos (m+n)(\pi/2)\psi}{(m+n)(\pi/2)\xi} \right\} \qquad (2\text{-}24)$$

$$a_m = \frac{-B_m}{\omega_1 L_u} \left\{ \frac{\delta}{m} \left[1 - \xi \left(1 - \frac{\sin m\pi\xi \cos m\pi\psi}{m\pi\xi}\right) \right] \right\} \qquad (m = n) \qquad (2\text{-}25)$$

$$b_m = \sum_{n=1}^{N} - \frac{A_n}{\omega_1 L_u} \frac{\delta}{m} \xi \left\{ \frac{\sin (m-n)(\pi/2)\xi \cos (m-n)(\pi/2)\psi}{(m-n)(\pi/2)\xi} \right. \qquad (m > n)$$

$$\left. + \frac{\sin (m+n)(\pi/2)\xi \cos (m+n)(\pi/2)\psi}{(m+n)(\pi/2)\xi} \right\} \qquad (2\text{-}26)$$

$$b_m = \frac{+A_m}{\omega_1 L_u} \left\{ \frac{\delta}{m} \left[1 - \xi \left(1 + \frac{\sin m\pi\xi \cos m\pi\psi}{m\pi\xi}\right) \right] \right\} \qquad (m = n) \qquad (2\text{-}27)$$

$$\xi = k_2 - k_1 \qquad (0 < \xi < 1) \qquad (2\text{-}19)$$

$$\psi = k_2 + k_1 \qquad (1 \leq \psi < 2) \qquad (2\text{-}20)$$

$$\delta = L_u / L_s \qquad (2\text{-}21)$$

$$\delta \gg 1 \qquad (2\text{-}23)$$

ω_1 is the radian frequency of the fundamental

On the per-unit basis

$$\omega_1 L_u = 1 \qquad (2\text{-}22)$$

2.1.3 The Fourier Expressions in Matrix Notation

Computations for the a_m, for example, from (2-24) and (2-25) result in equations of the form

$$a_1 = S_{B11}B_1 \tag{2-28a}$$

$$a_2 = S_{B21}B_1 + S_{B22}B_2 \tag{2-28b}$$

....

$$a_m = S_{Bm1}B_1 + S_{Bmm}B_m \tag{2-28c}$$

In the above S_{B11} is obtained by setting $(m = 1)$ in (2-25); S_{B31} is obtained by setting $(m = 3)$ and $(n = 1)$ in (2-24) and so on. Values of the b_m would be computed in the same way from (2-26) and (2-27).

By use of the matrix notation the equations relating the a's, b's, A's, B's, and S's may be put into a form which facilitates combination of the various nonlinear elements with other elements, both linear and nonlinear. Network equations in matrix form are developed in Chapter 3. These equations allow the relations developed in this chapter to be applied to network analysis and synthesis. Notation to be used in the matrix developments is as follows:

The input voltages are column matrices, written

$$A_n] = \begin{Vmatrix} A_1 \\ A_2 \\ \cdots \\ A_n \end{Vmatrix} \tag{2-29}$$

$$B_n] = \begin{Vmatrix} B_1 \\ B_2 \\ \cdots \\ B_n \end{Vmatrix} \qquad (2\text{-}30)$$

The currents, a_n and b_n, are also column matrices

$$a_m] = \begin{Vmatrix} a_1 \\ a_2 \\ \cdots \\ a_m \end{Vmatrix} \qquad (2\text{-}31)$$

$$b_n] = \begin{Vmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{Vmatrix} \qquad (2\text{-}32)$$

The coefficients of the A_n and B_n in (2-24) through (2-27) may be written compactly in triangular matrices as follows:

$$[S_B] = \begin{Vmatrix} S_{B11} & 0 & 0 & 0 \\ S_{B21} & S_{B22} & 0 & 0 \\ \cdots & \cdots & \cdots & 0 \\ S_{Bm1} & S_{Bm2} & \cdots & S_{Bmn} \end{Vmatrix} \delta \qquad (2\text{-}33)$$

$$[S_A] = \begin{Vmatrix} S_{A11} & 0 & 0 & 0 \\ S_{A21} & S_{A22} & 0 & 0 \\ \cdots & \cdots & \cdots & 0 \\ S_{Am1} & S_{Am2} & \cdots & S_{mn} \end{Vmatrix} \delta \qquad (2\text{-}34)$$

Occasionally, where no confusion will result, the scalar multiplier, δ, will be absorbed into the S's to minimize space and writing. Tables of values of $[S_A]$ and $[S_B]$ and their inverses up to the 25th harmonic have been calculated and tabulated for increments of 0.1 in ξ and Ψ by the Numerical Analysis Laboratory of the University of Arizona. Much of the tedium of any design is removed by the availability of the pre-computed values of $[S_A]$ and $[S_B]$.

By standard matrix multiplication techniques it is readily demonstrated that (2-2) is satisfied by values obtained from the matrix equations

$$a_m] = [S_B] \; B_n] \qquad (2\text{-}35)$$

$$b_m] = [S_A] \; A_n] \qquad (2\text{-}36)$$

It may also be seen in the notation that the subscripts A and B associated with the [S] matrices refer to the voltages upon which they operate.

Although there are theoretically an infinite number of the a_m and b_m, the usual practice of terminating the series where judgment and experience justify it may be followed. Care must be taken, however, always to keep the number of rows equal to the number of columns in the [S] matrices. It will be necessary to use the inverse matrix, written $[S]^{-1}$, quite often, and only non-singular matrices have inverses. A matrix is non-singular only if the determinant of its elements is non-zero. In matrices such as (2-33) and (2-34) in which all terms above the main diagonal are zero, the value of the determinant is simply the product of the elements in the main diagonal. From (2-25) and (2-27) it can be seen that the main-diagonal elements will never be zero. As mentioned above, the inverse matrices for a large number of operating conditions also have been computed and tabulated by the Numerical Analysis Laboratory. A short table is presented in Appendix B of this dissertation.

2.1.4 Physical Interpretation of Matrices

The $[S_A]$ and $[S_B]$ of (2-33) and (2-34) are dimensionless. If the $(\delta/\omega_1 L_u)$ term is absorbed into the matrices, they have dimensions of admittance. The inverse of the admittance matrix will, of course, have dimensions of impedance. All terms below the main diagonal for both the admittance and impedance matrices are the result of nonlinearity. Similar matrices describing a linear reactance or susceptance would have non-zero terms only along the main diagonal. Although the nonlinearities in the [S] matrices affect their main-diagonal terms somewhat, the major effect appears in the S_{mn} terms.

2.2 Design Interpretation of Equation Parameters

2.2.1 Interpretation of ξ and ψ in Terms of Reactor Current and Voltage

The breakpoints k_1 and k_2 on the λ-I relations, together with the quantities ξ and ψ derived from them are seen from (2-24) through (2-27) to be critical quantities in the derivation of the voltage and current relations. According to the basic derivation of Section 2.1.1, $\pi\xi$ is the number of radians which the fundamental component of the exciting flux wave spends on the near-vertical portion (L_u) of the λ-I curve. Values of ξ and ψ will be chosen from many different considerations, and it is essential that the designer be able to translate these parameters into information suitable to the design and testing of the inductor. The obvious choice is one based on excitation of the core by a single-frequency sine wave of voltage.

Consider Figure 2.5(a) where a cosinusoidal flux wave is impressed upon a symmetrical λ-I characteristic and Figure 2.5(b) where the same wave is impressed on an unsymmetrical characteristic. Reference to the figures shows that for both the symmetrical and unsymmetrical cases

$$\lambda_1 = \lambda_p \cos k_1\pi \qquad (2\text{-}37)$$

$$\lambda_2 = \lambda_p \cos k_2\pi \qquad (2\text{-}38)$$

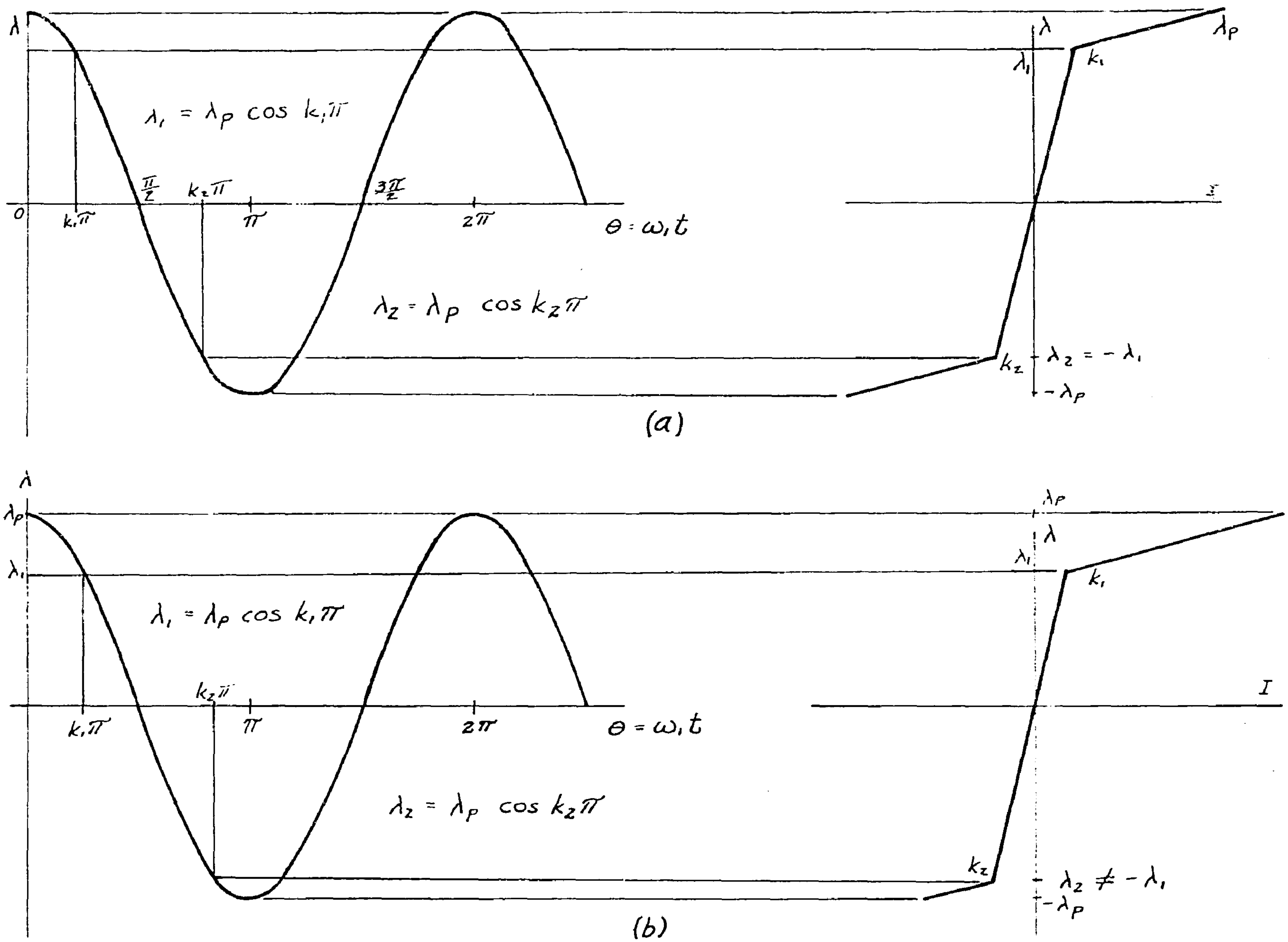

Fig. 2.5 Single-Frequency Flux versus Angle

and it is readily shown from (2-19) and (2-20) that

$$k_1 = (1/2)(\Psi - \xi) \tag{2-39}$$

$$k_2 = (1/2)(\Psi + \xi) \tag{2-40}$$

For the case of symmetrical excitation

$$\lambda_2 = -\lambda_1 \tag{2-41}$$

It immediately follows that for symmetrical excitation

$$k_1 + k_2 = \Psi = 1 \qquad \text{(Symmetrical Excitation)} \tag{2-42}$$

since only angles which are supplements of one another have cosines which are equal but opposite in sign. Under the prescribed conditions of sinusoidal excitation voltage, (2-37) and (2-38) apply with the voltages E_1 and E_p substituted for λ_1 and λ_p.

Currents under these conditions are given by

$$I_1 = \lambda_1/L_u = E_1/\omega_1 L_u \tag{2-43}$$

and

$$\begin{aligned} I_p &= I_1 + (\lambda_p - \lambda_1)/L_s \\ &= (E_1/\omega_1 L_u) + (E_p - E_1)/\omega_1 L_s \end{aligned} \tag{2-44}$$

In terms of δ defined in (2-21)

$$I_p = (1/\omega_1 L_u)\,[E_1 + \delta(E_p - E_1)] \tag{2-45}$$

and, if δ is large, which is the usual case

$$I_p \approx (\delta/\omega_1 L_u)(E_p - E_1) \qquad (2\text{-}46)$$

Since I_2 and I_1 may have different magnitudes when excitation is unsymmetrical

$$I_2 = \lambda_2/L_u = E_2/\omega_1 L_u \qquad (2\text{-}47)$$

It remains to determine the amount of d-c bias current, I_o, required for satisfactory operation under conditions of unsymmetrical excitation. Unless nonlinear resistive elements (rectifiers) are present in the circuit, one of the requirements for stable operation with a-c core excitation is a zero average value for reactor current. Asymmetrical excitation of the core will, because of the nonlinear characteristic of the iron, produce a constant or "d-c" term in the Fourier expression for reactor current. An amount of direct current equal and opposite to that required by the Fourier expansion must therefore be supplied to the core. This is the d-c bias or saturating direct current, which most often is supplied through a separate winding, suitably filtered to minimize any effect it might have on the a-c output. The magnitude of this current is obtained from the equations developed below, following choice of ξ and ψ, for the operating conditions to produce the desired a-c wave.

In this derivation it is necessary to deal with current and flux linkage, rather than with the time derivative of current as in the original analysis. Current is expressed by (2-2) as before and flux linkage by

$$\lambda = \sum_{n=1}^{N} [(-B_n/n\omega_1) \cos n\omega_1 t + (A_n/n\omega_1) \sin n\omega_1 t] \qquad (2\text{-}48)$$

The function is even as before. From Figure 2.5(b) it can be seen that current takes on the values listed below over the range from zero to π radians, where θ is written for ωt.

$$
\begin{aligned}
i &= \lambda/L_u & k_1\pi &\leqq \theta \leqq k_2\pi \\
i &= (\lambda_1/L_u) + (\lambda/L_s) & 0 &\leqq \theta \leqq k_1\pi \\
i &= (\lambda_2/L_u) + (\lambda/L_s) & k_2\pi &\leqq \theta \leqq \pi
\end{aligned}
\tag{2-49}
$$

Because the function is even, the sine terms from (2-48) do not contribute and the expression for a_o is

$$
\begin{aligned}
a_o = \sum_{n=1}^{N} - \Big\{ &(2/L_u\pi)\int_{k_1\pi}^{k_2\pi}(B_n/n\omega_1)\cos n\theta\, d\theta + (2\lambda_1/L_u\pi)\int_0^{k_1\pi} d\theta \\
&+ (2/L_s\pi)\int_0^{k_1\pi}(B_n/n\omega_1)\cos n\theta\, d\theta + (2\lambda_2/L_u\pi)\int_{k_2\pi}^{\pi} d\theta \\
&+ (2/L_s\pi)\int_{k_2\pi}^{\pi}(B_n/n\omega_1)\cos n\theta\, d\theta \Big\}
\end{aligned}
\tag{2-50}
$$

After performing the indicated integrations, substituting limits and rearranging, the expression becomes

$$
\begin{aligned}
-a_o = \sum_{n=1}^{N} &+(2B_n/n\omega_1)\Big[(1/L_u) - (1/L_s)\Big]\left[\frac{\sin n(\pi/2)\xi \cos n(\pi/2)\psi}{n(\pi/2)}\right] \\
&+ 2\Big[\lambda_1 k_1 - \lambda_2\,(1 - k_2)\Big]
\end{aligned}
\tag{2-51}
$$

Finally an expression for I_o is obtained in terms of δ (large), ξ, and the voltages E_1 and E_2 corresponding to λ_1 and λ_2.

$$I_o = -(a_o/2) = (1/\omega_1 L_u)\left\{E_1 k_1 + E_2(1 - k_2)\right.$$
$$\left. + \delta \sum_{n=1}^{N} B_n\ (\xi/n)\left[\frac{\sin n(\pi/2)\xi}{n(\pi/2)\xi}\right] \cos n(\pi/2)\psi\right\} \qquad (2\text{-}52)$$

The quantity included in the summation sign has been precomputed over the same ranges of ξ and ψ as the [S] matrix values by the Numerical Analysis Laboratory.

2.2.2 Choice of ξ and ψ

Choice of ξ and ψ in general will be governed by different considerations for circuits with different duties. In certain cases such as the harmonic generators discussed in Section 3.2, the choice can be made through maximizing techniques; in polyphase harmonic generators the choice is largely dictated by cross-firing considerations (cf. Sec. 3.2.2); in the design of pulse generators, the pulse duty cycle is a strong determining factor.

In every case, however, there is a certain area of optimization. Availability of the precomputed matrix values for $[S_A]$ and $[S_B]$ allows the designer to improve his design over one which might be made from linear considerations plus intuition.

In this section conditions will be derived whereby a given, single-frequency sinusoidal excitation voltage will produce maximum or minimum values of current at any given harmonic frequency. Conditions for symmetrical and unsymmetrical excitation will be considered separately.

2.2.2.1 Symmetrical Excitation

Assuming pure sinusoidal excitation

$$A_n] = b_n] = 0 \tag{2-54}$$

$$\psi = 1 \tag{2-55}$$

and the equation for a_m from Table 2.1 is

$$a_m = (B_1/\omega_1 L_u)(\delta \xi /m)\left\{ \frac{\sin (m-1)(\pi/2)\xi \cos (m-1)(\pi/2)\psi}{(m-1)(\pi/2)\xi} - \frac{\sin (m+1)(\pi/2)\xi \cos (m+1)(\pi/2)\psi}{(m+1)(\pi/2)\xi} \right\} \tag{2-56}$$

It is worthy of note that a_m vanishes for all even values of m, which is in agreement with experimental observation. Since the core is excited with a single frequency, a_m will be examined for maxima and minima for various values of m. It should also be noted that $\cos (m+1)(\pi/2)$ and $\cos (m-1)(\pi/2)$ are always of opposite sign, since for all odd m they differ by π radians. With this fact in mind, a_m is then differentiated with respect to ξ and the result set equal to zero. This gives

$$\cos (m-1)(\pi/2)\xi + \cos (m+1)(\pi/2)\xi = 0 \tag{2-57}$$

The use of trigonometric identities leads finally to the extremum condition

$$\cos (m\pi\xi/2) = 0 \tag{2-58}$$

which reduces to

$$\xi = 1/m, \text{ or } 3/m, \text{ or } 5/m, \ldots \qquad (0<\xi<1) \tag{2-59}$$

The determination as to whether the extremum in a particular case is a maximum or minimum is most easily determined by direct substitution in (2-24).

Occasionally, when considering a higher harmonic, a maximum will occur for more than one value of ξ . Choice of the value closest to unity will minimize the total current flow, since largest total current (i.e., the fundamental plus all its harmonics) corresponds to the smallest value of ξ .

Still another condition which exists when ξ is set equal to 1/m must be mentioned. It can readily be shown from equations (2-25) and (2-27) that, as a result of this choice of ξ ,

$$S_{Amm} = -S_{Bmm}$$

This equality is of particular importance in the applications to harmonic generators which are discussed in Section 3.2.1 and Section 3.2.3. As it turns out, such choice of ξ is desirable in the first case, but must be avoided in the second.

2.2.2.2 Unsymmetrical Excitation

Unsymmetrical excitation is used to obtain even harmonics of the fundamental exciting frequency. This type of excitation is most generally achieved by the application of biasing d-c ampere turns to the core from an additional winding. The bias winding must, of course, be adequately filtered and designed to prevent undesirable interaction with the a-c windings. In certain applications, however, a direct current may actually be allowed to flow in the main winding.

For the case of unsymmetrical excitation, both ξ and ψ .

must be examined to determine a combination which will produce maximum even-harmonic output. Differentiating (2-24) with respect to ψ and equating the result to zero produces

$$- \sin (m - 1)(\pi/2)\xi \ \sin (m - 1)(\pi/2)\psi + \sin (m + 1)\xi \ \sin (m + 1)(\pi/2)\psi = 0 \qquad (2\text{-}60)$$

Equation (2-60) is satisfied when the following relations apply simultaneously:

$$\psi = 1 + \xi \qquad (2\text{-}61)$$

$$\xi = 1/m \qquad (2\text{-}62)$$

The direct bias current required to establish these conditions in the core can be computed from (2-52).

Before applying (2-52), however, it is necessary to choose a peak value for E and to compute values of E_1 and E_2 from (2-37) and (2-38), where E of course is written instead of λ in those equations. The relations existing among k_1, k_2, ξ, and ψ in equations (2-39) and (2-40) must also be used before it is possible to determine I_o from (2-52).

In cases where a separate bias winding is used, this value of I_o, which applies to the main winding, must be translated into current in the bias winding from consideration of the relative number of turns on the two windings.

Thus, in summary it is seen that the condition on ξ for the generation of a particular harmonic is identical whether the desired harmonic is even or odd; namely, that ξ have a value equal to the reciprocal of the harmonic number. The value of ψ is unity for symmetrical excitation which produces odd harmonics; the unit value of ψ is increased by ξ for optimum production of an even harmonic.

2.3 Effect of Hysteresis

Reference to Figure 2.2(a) shows the effect of hysteresis to be simply an advancement in phase of the current with respect to the exciting voltage. In other words, because of hysteresis, current lags the voltage by a lesser amount than in the lossless case. The amount is θ_h as measured on the fundamental scale. The generation of harmonic currents is affected only to a very minor degree by hysteresis.

Consider the power input to an inductor with core loss taken into account, but with fundamental current taken as phase reference. In this case current is given by (2-2) as before, and the a_m and b_m are determined from (2-24) through (2-27) as before. The expression for voltage now reflects the phase shift; that is,

$$e = \sum_{n=1}^{N} \left[A_n \cos n(\omega_1 t - \theta_h) + B_n \sin n(\omega_1 t - \theta_h) \right] \tag{2-63}$$

$$= \sum_{n=1}^{N} \Big[(A_n \cos n\theta_h - B_n \sin n\theta_h) \cos n\omega_1 t$$

$$+ (A_n \sin n\theta_h + B_n \cos n\theta_h) \sin n\omega_1 t \Big]$$

The average power to the inductor is then

$$P_{av} = (1/2) \sum_{n=1}^{N} \Big[(A_n \cos n\theta_h - B_n \sin n\theta_h)a_m$$

$$+ (A_n \sin n\theta_h + B_n \cos n\theta_h)b_m\Big] \tag{2-64}$$

In what follows, no generality is lost if the input is considered to be a series of sinusoids, in which case,

$$A_n = b_m = 0 \tag{2-65}$$

and the average power is given by

$$P_{av} = (1/2) \sum_{n=1}^{N} (-B_n \sin n\theta_h)a_m \tag{2-66}$$

As expected, (2-66) yields a positive result since a_m is commonly negative.

It is interesting to see if an equivalent circuit can be established which will reflect the effect of hysteresis (and eddy current losses, if included) as given in (2-66). An approach can be made by writing out the first few terms of (2-66) and grouping the sine terms with the current. Thus

$$P_{av} = (1/2)[B_1(-a_1 \sin \theta_h) + B_2(-a_2 \sin 2\theta_h) + B_3(-a_3 \sin 3\theta_h) + \ldots] \tag{2-67}$$

The terms in parentheses can be regarded as a current in phase with the voltage $B_n \sin n\omega_1 t$, and therefore passing through a resistor in parallel with the inductor. The resistor is a function of both amplitude and frequency and has the general term

$$R_m = B_m/(-a_m \sin m\theta_h) \tag{2-68}$$

Thus the three terms written out in (2-67) would have resistances

$$R_1 = B_1/(-a_1 \sin \theta_h) = B_1/(-S_{B11}B_1 \sin \theta_h) = 1/(-S_{B11}\sin \theta_h) \tag{2-69}$$

$$R_2 = \frac{B_2}{-a_2 \sin 2\theta_h} = \frac{B_2}{-(S_{B21}B_1 + S_{B22}B_2) \sin 2\theta_h} \tag{2-70}$$

$$R_3 = \frac{B_3}{-(S_{B31}B_1 + S_{B32}B_2 + S_{B33}B_3) \sin 3\theta_h} \tag{2-71}$$

Equations (2-69), (2-70), and (2-71) demonstrate rather graphically the dependence of R on frequency, amplitude of harmonic voltage, and point of operation on the λ--I curve. A first approximation to a non-linear circuit element representing hysteresis loss might be a fixed resistor computed from (2-69). A very conservative, and nonlinear, representation would be a diagonal resistance matrix $[R_{mm}]$ whose terms would be given by

$$R_{mm} = \frac{1}{-S_{Bmm} \sin m\theta_h} \tag{2-72}$$

Measured values of hysteresis loss would undoubtedly be less than values computed with the aid of (2-72). A third possibility always exists; namely, to measure losses in a sample at the frequencies of interest, and use these values, appropriately normalized, in the diagonal matrix $[R_{mm}]$.

Recent efforts to minimize hysteresis and eddy-current losses through metallurgy and manufacturing processes have been very successful, particularly for the rectangular-loop core materials. Losses of less than 0.05 watt per cubic centimeter of material (3 watts per pound) at 400 cycles per second are consistently attained. At higher frequencies, where filter uses might arise, the powdered irons have acceptably low losses, even to frequencies in the megacycle range.

In situations where it is desired to account for iron losses in the design, the suggested procedure is first to synthesize the network on a lossless basis, then add the matrices representing the losses. At this point the situation can be treated as a problem of analysis, rather than synthesis.

Throughout the remainder of this dissertation, hysteresis and eddy-current losses will be neglected.

CHAPTER 3

NETWORK EQUATIONS AND APPLICATIONS

3.1 Network Matrix Equations

The $[S_A]$ and $[S_B]$ matrices made up of terms calculated by the methods of Chapter 2 relate the voltage across a nonlinear reactor to the current through it. The next step is to establish the input-output relations in a network composed of these nonlinear elements plus associated linear ones.

A few points pertinent to the matrix description of linear network elements to be associated with the nonlinear ones are discussed below:

1) A linear network element is one which obeys the law of superposition of effects, even though its properties may vary with frequency. If it is not amplitude-sensitive, it is linear.

2) Matrices describing the behavior of linear network elements have non-zero terms only along the main diagonal.

3) If the linear element is also insensitive to frequency, all of these main-diagonal terms are identical.

4) Terms in the matrix representing variation with frequency of linear, reactive elements are still governed by Foster's reactance theorem. An individual term cannot be chosen at will to satisfy a particular network requirement with no thought of its

relations with other terms in the matrix.

5) In the notation of this dissertation, voltage and current are expressed in terms of their sine and cosine components, rather than the usual magnitude and phase angle. The sign of terms in the admittance (or, more correctly, susceptance) matrices $[Y_n]$ will be governed by the type of voltage component upon which they operate. Since sign conventions in Electrical Engineering are based on sinusoidal voltages, terms of an admittance matrix operating on a sinusoidal voltage (written $[Y_n^B]$) will have the conventional signs; i.e., $+Y_{mn}^B$ indicates capacitive and $-Y_{mn}^B$ indicates inductive susceptance. An admittance matrix for the same network element(s) operating on a cosinusoidal voltage would be written $[Y_n^A]$ and would have the opposite sign; i.e.,

$$[Y_n^B] = -[Y_n^A] = [Y_n] \tag{3-1}$$

Throughout this entire dissertation, capital A_n and B_n refer respectively to cosinusoidal and sinusoidal voltage components while the lower case a_m and b_m refer to corresponding current components. These letters, used as subscripts or superscripts to a matrix symbol refer to the type of voltage upon which the matrix operates in the basic equations.

3.1.1 Elementary Combinations of Network Elements

Since admittances add directly in parallel systems, a matrix of lossless elements in parallel with the nonlinear reactor may be combined directly by addition. Suppose, for example, that the capacitor in Figure 3.1 is described in the frequency range of interest by the matrix

$$[Y] = \begin{Vmatrix} Y_{11} & 0 & 0 \\ 0 & Y_{22} & 0 \\ 0 & 0 & Y_{33} \end{Vmatrix} \tag{3-2}$$

Then the admittance operator connected between points 1 and 2 of Figure 3.1 is given by

$$[S] + [Y] = \begin{Vmatrix} (S_{11} + Y_{11}) & 0 & 0 \\ S_{21} & (S_{22} + Y_{22}) & 0 \\ S_{31} & & (S_{33} + Y_{33}) \end{Vmatrix} \tag{3-3}$$

Such a connection affords an excellent way to change the value of the main-diagonal elements without affecting the essential nonlinearity of the inductor. If distributed capacitance in the inductor were important, its effect could be taken into account in this fashion.

Similar possibilities exist for the inverted matrix. Let

$$[S]^{-1} = \begin{Vmatrix} W_{11} & 0 & 0 \\ W_{21} & W_{22} & 0 \\ W_{31} & W_{32} & WW_{33} \end{Vmatrix} \tag{3-4}$$

and the reactance of the linear reactor L_o in Figure 3.2 be described by the matrix

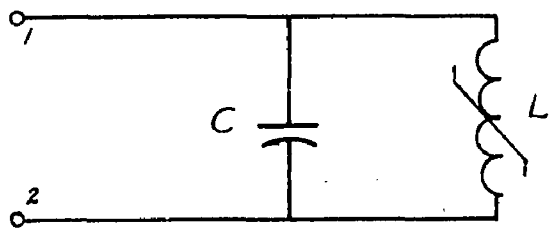

Fig. 3.1 Capacitor in Parallel with Nonlinear Inductor

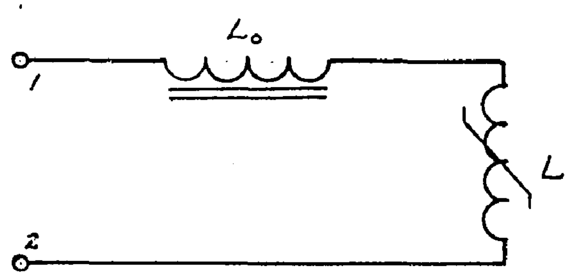

Fig. 3.2 Linear Inductor in Series with Nonlinear Inductor

$$[X] = \begin{Vmatrix} X_{11} & 0 & 0 \\ 0 & X_{22} & 0 \\ 0 & 0 & X_{33} \end{Vmatrix} \qquad (3\text{-}5)$$

Then the impedance operator between points 1 and 2 of Figure 3.2 is

$$[S]^{-1} + [X] = \begin{Vmatrix} (X_{11} + W_{11}) & 0 & 0 \\ W_{21} & (X_{22} + W_{22}) & 0 \\ W_{31} & & (X_{33} + W_{33}) \end{Vmatrix} \qquad (3\text{-}6)$$

3.1.2 <u>General Equations for the Ladder Network</u>

Of several network configurations available to the designer the simplest and most useful probably is the unbalanced ladder network. Equations for a multi-mesh network of this type incorporating nonlinear elements are developed in this section.

Consider Figure 3.3 which shows a nonlinear inductor in series with its inherent resistance r and with a parallel combination of the load resistor R_o and a lossless network of admittance Y_o. It is excited by an input voltage

$$e_1 = \sum_{n=1}^{N} (A_{1n} \cos n\omega_1 t + B_{1n} \sin n\omega_1 t) \qquad (3\text{-}7)$$

and produces across R_o an output voltage

$$e_o = \sum_{p=1}^{P} (A_{op} \cos p\omega_1 t + B_{op} \sin p\omega_1 t) \qquad (3\text{-}8)$$

An input current is established, given by

$$i_1 = \sum_{m=1}^{\infty} (a_{1m} \cos m\omega_1 t + b_{1m} \sin m\omega_1 t) \qquad (3\text{-}9)$$

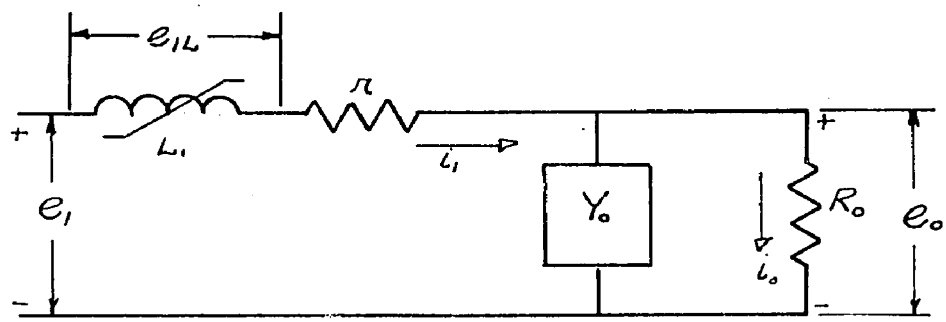

Fig. 3.3 Single-Stage Network Containing Nonlinear Element

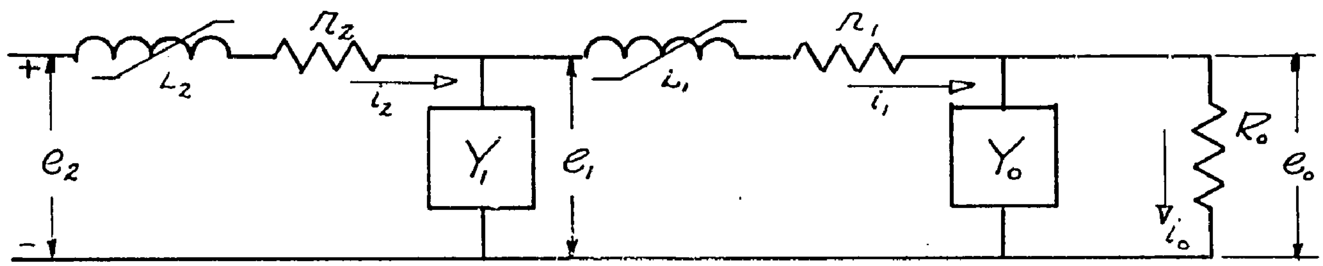

Fig. 3.4 Two-Stage Network with Two Nonlinear Elements

while the output current through R_o is

$$i_o = \sum_{p=1}^{P} (a_{op} \cos p\omega_1 t + b_{op} \sin p\omega_1 t) \qquad (3\text{-}10)$$

In (3-7) through (3-10) different summation indices have been used to point up the fact that the number of terms in the expression for input voltage, output current, and output voltage may be chosen by the designer, but the number of frequency components in the input current is theoretically infinite. When considering any particular harmonic, however

$$m = n = p = \text{the number of the harmonic}$$

so that no distinction need be made between indices. To simplify the notation throughout the rest of this dissertation, the general harmonic number for any voltage or current is denoted by the index, n, and written as the second subscript index. The first subscript index refers to the location of the current or voltage in the network, as indicated in Figure 3.3 and those following.

The inductor is described by the matrix admittance operators $[S_A]$ and $[S_B]$. The series resistor, r, is considered constant at all frequencies so that it may be represented by a scalar matrix, or scalar multiplier of another matrix. The admittance matrix $[Y_o]$ is made up of lossless reactive elements. The $[Y_o]$ matrix could represent either linear or nonlinear lossless elements. The equations to be derived in this section are valid for both. Finally, the load resistance may in some cases be a function of frequency, hence for generality will be described by a diagonal matrix $[R_{on}]$.

Straightforward application of Kirchhoff's laws to the circuit gives the following equations, in matrix form, for the currents and voltages:

$$a_{1n}] = [U]\ a_{on}] + [Y_{on}^{B}]\ B_{on}] \tag{3-11}$$

$$b_{1n}] = [U]\ B_{on}] + [Y_{on}^{A}]\ A_{on}] \tag{3-12}$$

$$A_{1L}] = A_{1n}] - r_1 a_{1n}] - [R_{on}]\ a_{on}] \tag{3-13}$$

$$B_{1L}] = B_{1n}] - r_1 b_{1n}] - [R_{on}]\ b_{on}] \tag{3-14}$$

where

$$A_{on}] = [R_{on}]\ a_{on}] \tag{3-15}$$

$$B_{on}] = [R_{on}]\ b_{on}] \tag{3-16}$$

In the above, $[U]$ is the unity matrix or idemfactor, the $[R_{on}]$ and $[Y_{on}]$ are the diagonal matrices described in Section 3.1, and $A_n]$, $B_n]$, $a_n]$, and $b_n]$ are column matrices of the voltages and currents. In Section 2.1.3 the following relations were derived:

$$[S_A]\ A_{1L}] = a_{1n}] \tag{2-35}$$

$$[S_B]\ B_{1L}] = b_{1n}] \tag{2-36}$$

therefore

$$[S_A]\ (A_{1n}] - r_1 a_{1n}] - [R_{on}]a_{on}]) = [U]a_{on}] + [Y_{on}^{B}]B_{on}] \tag{3-17}$$

$$[S_B]\ (B_{1n}] - r_1 b_{1n}] - [R_{on}]b_{on}]) = [U]b_{on}] + [Y_{on}^{A}]A_{on}] \tag{3-18}$$

It was shown in Section 2.1 that the matrices $[S_A]$ and $[S_B]$ are triangular, and in general will have non-vanishing terms along their main diagonals. They are therefore nonsingular and may be inverted, such that

$$[S_A]^{-1}\,[S_A] = [S_A][S_A]^{-1} = [U] \tag{3-19}$$

and similarily for $[S_B]$.

By combining and rearranging equations (3-11) through (3-18) and making use of the inverse relations, equations (3-17) and (3-18) may be put in the form

$$B_{1n}] = [S_B]^{-1}\; a_{on}] + r[Y^A_{on}]\; A_{on}] + ([U] + [S_B]^{-1}[Y^B_{on}])B_{on}] + rb_{on}] \tag{3-20}$$

$$A_{1n}] = [S_A]^{-1}\; b_{on}] + r[Y^B_{on}]B_{on}] + ([U] + [S_A]^{-1}[Y^A_{on}])A_{on}] + \; ra_{on}] \tag{3-21}$$

Equations (3-11), (3-12), (3-20), and (3-21) may be written concisely in a matrix of matrices as follows:

$$\begin{Vmatrix} a_{1n}] \\ B_{1n}] \\ b_{1n}] \\ A_{1n}] \end{Vmatrix} = \begin{Vmatrix} [U] & [Y^B_{on}] & 0 & 0 \\ [S_B]^{-1} & [U] + [S_B]^{-1}[Y^B_{on}] & r & r[Y^A_{on}] \\ 0 & 0 & [U] & [Y^A_{on}] \\ r & r[Y^B_{on}] & [S_A]^{-1} & [U] + [S_A]^{-1}[Y^A_{on}] \end{Vmatrix} \begin{Vmatrix} a_{on}] \\ B_{on}] \\ b_{on}] \\ A_{on}] \end{Vmatrix} \tag{3-22}$$

It will be seen that (3-20) and (3-21) are similar except for changes in the subscript and superscript letters. Also in (3-22) the input currents and voltages are written as the product of a matrix of passive elements and the corresponding output current and voltage. In a multi-mesh ladder network the output from stage one is the input to stage 2, and so on. By use of this property and (3-22) plus straightforward matrix multiplication, the input to a ladder network of any number of stages may be expressed in terms of the product of a matrix of passive elements and the output. Internal currents and voltages need not be considered. The process is similar to the cascading of linear four-terminal networks.

Results of such a multiplication are summarized in Table 3.1 for single-stage and two-stage networks. Both of these results include the effect of coil resistance, r, but in such position in the equations that it may be easily ignored if considered negligible. For most applications it can be neglected. Equations for the three-stage network are given in Table 3.2 without consideration of coil resistance.

The equations of Table 3.1 and Table 3.2 are the basic network matrix equations, and will be used repeatedly in the sections to follow. The three-stage equations also are arranged in such fashion that the progressive effect of adding stages is made evident. This is helpful in many designs, since an additional stage may be introduced without undoing the effect of the stages already considered.

TABLE 3.1

Matrix Equations Describing the Circuit of Figure 3.3

Single-Stage Equations

$$a_{1n}] = a_{on}] + [Y_{on}^{B}]B_{on}] \tag{3-11}$$

$$b_{1n}] = b_{on}] + [Y_{on}^{A}]A_{on}] \tag{3-12}$$

$$B_{1n}] = [S_{B1}]^{-1} a_{on} + ([U] + [S_{B1}]^{-1} [Y_{on}^{B}])B_{on}]$$

$$+ r_1(b_{on}] + [Y_{on}^{A}]A_{on}]) \tag{3-20}$$

$$A_{1n}] = [S_{A1}]^{-1} b_{on}] + ([U] + [S_{A1}]^{-1}[Y_{on}^{A}])A_{on}]$$

$$+ r_1(a_{on}] + [Y_{on}^{B}]B_{on}]) \tag{3-21}$$

TABLE 3.1 (Cont.)

Matrix Equations Describing the Circuit of Figure 3.4

Two-Stage Equations

$$
\begin{aligned}
a_{2n}] &= ([U] + [Y^{B}_{1n}][S_{B1}]^{-1})a_{on}] \\
&+ \left\{ [Y^{B}_{1n}] + ([U] + [Y^{B}_{1n}][S_{B1}]^{-1})[Y^{B}_{on}] \right\} B_{on}] \\
&+ r_1([Y^{B}_{1n}]b_{on}] + [Y^{B}_{1n}][Y^{A}_{on}]A_{on}])
\end{aligned}
\qquad (3\text{-}22)
$$

$$
\begin{aligned}
b_{2n}] &= ([U] + [Y^{A}_{1n}][S_{A1}]^{-1})b_{on}] \\
&+ \left\{ [Y^{A}_{1n}] + ([U] + [Y^{A}_{1n}][S_{A1}]^{-1})[Y^{A}_{on}] \right\} A_{on} \\
&+ r_1([Y^{A}_{1n}]a_{on}] + [Y^{A}_{1n}][Y^{B}_{on}]B_{on}])
\end{aligned}
\qquad (3\text{-}23)
$$

TABLE 3.1 (Cont.)

Matrix Equations Describing the Circuit of Figure 3.4

Two-Stage Equations

$$
\begin{aligned}
B_{2n} = & \left\{[S_{B2}]^{-1}([U] + [Y^B_{1n}][S_{B1}]^{-1}) + [S_{B1}]^{-1}\right\} a_{on} \\
& + \Big[\left\{[S_{B2}]^{-1}([U] + [Y^B_{1n}][S_{B1}]^{-1}) + [S_{B1}]^{-1}\right\}[Y^B_{on}] \\
& + \left\{[U] + [S_{B2}]^{-1}[Y^B_{1n}]\right\}\Big] B_{on}] \\
& + r_1 r_2([Y^A_{1n}]a_{on}] + [Y^A_{1n}][Y^B_{on}]B_{on}]) \\
& + \left\{r_1([U] + [S_{B2}]^{-1}[Y^B_{1n}]) + r_2([U] + [Y^A_{1n}][S_{A1}]^{-1})\right\} b_{on} \\
& + \left\{r_1([U] + [S_{B2}]^{-1}[Y^B_{1n}]) Y^A_{on}\right. \\
& \left. + r_2([U] + [Y^A_{1n}][S_{A1}]^{-1})[Y^A_{on}] + r_2[Y^A_{1n}]\right\} A_{on}]
\end{aligned}
\tag{3-24}
$$

$$
\begin{aligned}
A_{2n} = & \left\{[S_{A2}]^{-1}([U] + [Y^A_{1n}][S_{A1}]^{-1}) + [S_{A1}]^{-1}\right\} b_{on}] \\
& + \Big[\left\{[S_{A2}]^{-1}([U] + [Y^A_{1n}][S_{A1}]^{-1}) + [S_{A1}]^{-1}\right\}[Y^A_{on}] \\
& + \left\{[U] + [S_{A2}]^{-1}[Y^A_{1n}]\right\}\Big] A_{on}] \\
& + r_1 r_2([Y^B_{1n}]b_{on}] + [Y^B_{1n}][Y^A_{on}]A_{on}]) \\
& + \left\{r_1([U] + [S_{A2}]^{-1}[Y^A_{1n}]) + r_2([U] + [Y^B_{1n}][S_{B1}]^{-1})\right\} a_{on}] \\
& + \left\{r_1([U] + [S_{A2}]^{-1}[Y^A_{1n}]) [Y^B_{on}]\right. \\
& \left. + r_2([U] + [Y^B_{1n}][S_{B1}]^{-1})[Y^B_{on}] + r_2[Y^B_{1n}]\right\} B_{on}
\end{aligned}
\tag{3-25}
$$

TABLE 3.2

Matrix Equations Describing the Circuit of Figure 3.5

Three-Stage Equations

(3-26)

$$B_{3n}] = [S_{B1}]^{-1} a_{on}] \qquad \text{1st stage}$$

$$+ [S_{B2}]^{-1}([U] + [Y^B_{1n}]^{-1}) a_{on}] \qquad \text{added by 2nd}$$

$$+ [S_{B3}]^{-1}\Big[[U] + [Y^B_{1n}][S_{B1}]^{-1} \qquad \text{added by 3rd}$$

$$+ [Y^B_{2n}]\Big\{[S_{B1}]^{-1} + [S_{B2}]^{-1}([U] + [Y^B_{1n}][S_{B1}]^{-1})\Big\}\Big] \; a_{on}]$$

$$+$$

$$([U] + [S_{B1}]^{-1}[Y^B_{on}]) B_{on}] \qquad \text{1st stage}$$

$$+ [S_{B2}]^{-1}\{[Y^B_{on}] + [Y^B_{1n}]([U] + [S_{B1}]^{-1}[Y^B_{on}])\} \; B_{on}] \qquad \text{added by 2nd}$$

$$+ [S_{B3}]^{-1}\Big([Y^B_{on}] + [Y^B_{1n}]([U] + [S_{B1}]^{-1}[Y^B_{on}]) + [Y^B_{2n}]\Big[\{[U] \qquad \text{added by 3rd}$$

$$+ [S_{B1}]^{-1}[Y^B_{on}]\} + [S_{B2}]^{-1}\{[Y^B_{on}] + [Y^B_{1n}]([U] + [S_{B1}]^{-1}[Y^B_{on}])\}\Big]\Big) \; B_{on}]$$

TABLE 3.2 (Cont.)

Matrix Equations Describing the Circuit of Figure 3.5

Three-Stage Equations

$$(3\text{-}27)$$

$$a_{3n}] = [U]a_{on}] \qquad \text{1st stage}$$

$$+ [Y^B_{1n}]\,[S_{B1}]^{-1}a_{on}] \qquad \text{added by 2nd}$$

$$+ [Y^B_{2n}]\,\left\{[S_{B1}]^{-1} + [S_{B2}]^{-1}([U] + [Y^B_{1n}][S_{B1}]^{-1})\right\}a_{on} \qquad \text{added by 3rd}$$

$$+$$

$$[Y^B_{on}][B_{on}] \qquad \text{1st stage}$$

$$+ [Y^B_{1n}]([U] + [S_{B1}]^{-1}[Y^B_{on}])B_{on} \qquad \text{added by 2nd}$$

$$+ [Y^B_{2n}]\,\Big[\left\{[U] + [S_{B1}]^{-1}[Y^B_{on}]\right\} \qquad \text{added by 3rd}$$

$$+ [S_{B2}]^{-1}\left\{[Y^B_{on}] + [Y^B_{1n}]([U] + [S_{B1}]^{-1}[Y^B_{on}])\right\}\Big]\,B_{on}]$$

Equations for A_{3n} and b_{3n} are obtained by writing A for B and b for a in (3-26) and (3-27)

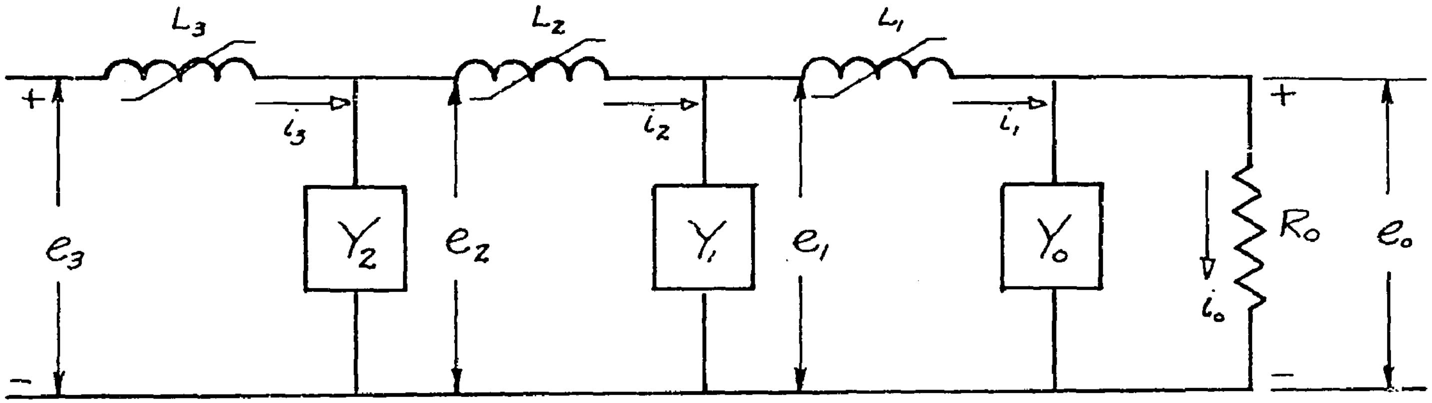

Fig. 3.5 Three-Stage Network with Three Nonlinear Elements

3.1.3 Nonlinear Element in Series with Resistor

The power of the matrix approach is demonstrated through its use in the solution of the series circuit of Figure 3.6, which contains a nonlinear reactor and linear resistor. Because of the nonlinearities in the circuit, its solution is impossible by standard methods, and approximation techniques involving integral equations and the calculus of variations[29] have been used to effect a solution. As is demonstrated below, this problem yields very readily to the matrix approach.

The basic one-stage equations of Table 3.1 apply. Assume r absorbed into the output resistor R_o, and, of course, no by-passing network. Also assume single-frequency excitation. The equations for this situation are

$$a_{1n}] = a_{on}] \tag{3-28}$$

$$b_{1n}] = b_{on}] \tag{3-29}$$

$$[S_B]B_{1n}] = a_{on}] + [S_B]R_o b_{on}] \tag{3-30}$$

$$[S_A]A_{1n}] = b_{on}] + [S_A]R_o a_{on}] \tag{3-31}$$

Combining (3-30) and (3-31)

$$[S_B]B_{1n}] - [S_B]R_o[S_A]A_{1n}] = ([U] - R_o^2[S_B][S_A])\ A_{on}] \tag{3-32}$$

When written out in matrix form to the fifth harmonic

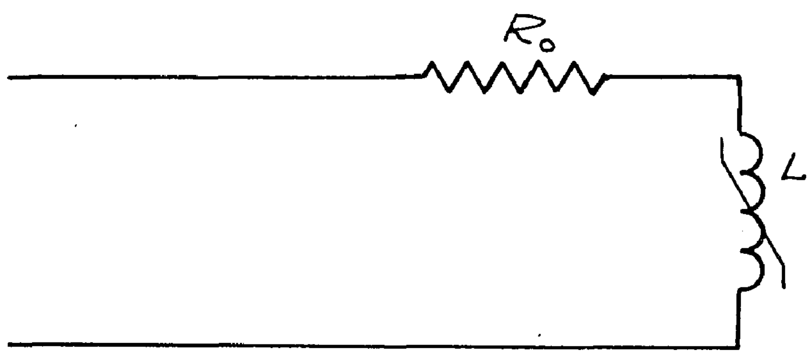

Fig. 3.6 Nonlinear Inductor in Series with Linear Resistor

$$\begin{Vmatrix} S_{B11}B_{11} \\ S_{B31}B_{11} \\ S_{B51}B_{11} \end{Vmatrix} - \begin{Vmatrix} S_{B11} & 0 & 0 \\ S_{B31} & S_{B33} & 0 \\ S_{B51} & S_{B53} & S_{B55} \end{Vmatrix} R_o \begin{Vmatrix} S_{A11} & 0 & 0 \\ S_{A31} & S_{A33} & 0 \\ S_{A51} & S_{A53} & S_{A55} \end{Vmatrix} \begin{Vmatrix} A_{11} \\ 0 \\ 0 \end{Vmatrix} \tag{3-33}$$

$$= \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} - R_o^2 \begin{Vmatrix} S_{B11} & 0 & 0 \\ S_{B31} & S_{B33} & 0 \\ S_{B51} & S_{B53} & S_{B55} \end{Vmatrix} \begin{Vmatrix} S_{A11} & 0 & 0 \\ S_{A31} & S_{A33} & 0 \\ S_{A51} & S_{A53} & S_{A55} \end{Vmatrix} \begin{Vmatrix} a_{01} \\ a_{03} \\ a_{05} \end{Vmatrix}$$

Equation (3-33 expanded produces three linear algebraic equations

$$S_{B11}B_{11} - R_o S_{B11} S_{A11} A_{11} = (1 - R_o^2 S_{B11} S_{A11}) a_{01} \tag{3-34}$$

$$\begin{aligned} &S_{B31}B_{11} - R_o(S_{B31}S_{A11} + S_{B33}\, S_{A31})A_{11} \\ &= -R_o^2(S_{B31}S_{A11} + S_{B33}S_{A31})a_{01} + (1 - R_o^2 S_{B33}S_{A33})a_{03} \end{aligned} \tag{3-35}$$

$$\begin{aligned} &S_{B51}B_{11} - R_0(S_{B51}S_{A11} + S_{B53}S_{A31} + S_{B55}S_{A51})A_{11} \\ &= h_{51}a_{01} - (S_{B53}S_{A33} + S_{B55}S_{A53})R_o^2 a_{03} \\ &\quad + (1 - R_o^2 S_{B55}S_{A55})a_{05} \end{aligned} \tag{3-36}$$

where

$$h_{51} = -R_o^2(S_{B51}S_{A11} + S_{B53}S_{A31} + S_{B55}S_{A51})$$

By progressive solution a_{01}, a_{03}, ..., a_{0n} may be obtained in terms of A_{11} and B_{11}. The b_{on} may be obtained from a similar expansion of (3-31). The relative values of A_{11} and B_{11} can be determined by equating input and output power, where

$$P_{in} = (1/2)(A_{11}a_{01} + B_{11}b_{01}) \tag{3-37}$$
$$= P_{out} = (R_o/2)(a_{01}^2 + a_{03}^2 + \ldots + b_{01}^2 + b_{03}^2 + \ldots)$$

As further a_{0n} terms are considered, the error resulting from termination of the series of (3-37) becomes less and less. Since a_{01}, a_{03}, ..., a_{0n} are each expressed in terms of A_{11} and B_{11}, an excellent idea of the approximation involved can be obtained before substitution into the power equation. Finally, the fact that peak input voltage is given by

$$E_p = A_{11} + j\,B_{11} = (A_{11}^2 + B_{11}^2)^{1/2}$$

relates the entire computation to a single value of source voltage.

The computation may be approached from either of two viewpoints. In one case a value of ξ might be assumed and the source voltage obtained from the equation above. In the second case the source voltage might be assumed, in which case the appropriate value of ξ, which determines the S values, could be obtained from (2-37) and (2-38). A sample computation is worked out in Appendix A.

3.2 Application to Harmonic-Generator Design

One very practical use to which the matrix description of the nonlinear inductor may be put is the design of harmonic generators. It has been known for many years that power could be extracted from nonlinear devices at a frequency which is an integral multiple of the frequency of the impressed voltage. Many circuits and designs have evolved and been optimized, based primarily on past experience and experiment. As a result, a large body of knowledge concerning these devices is currently available in the literature. The availability of these data, thoroughly documented as they are by experiment, affords an opportunity to test the equations derived in the preceding sections in comparison with known facts. It is demonstrated below that the results of the matrix equations agree quite closely with the experimental data.

It is also shown that the additional information contained in the matrix equations suggests other design possibilities, enables analytic evaluations of the effect of non-ideal circuit elements, and opens the possibility of design optimization by analytic, rather than experimental, means.

3.2.1 Single-Stage, Single-Phase Harmonic Generators

Let it be assumed that the current desired is an integral harmonic of the input fundamental voltage, and that the load is resistive. No generality is lost in requiring that the desired output current be cosinusoidal, since this merely fixes the zero point on the time or radian axis. If, for example, the desired output is a third harmonic, the only component of output current would be a_{03}. All other components generated in the inductor must be bypassed by the lossless networks.

Note here that b_{03} is the only b_{on} value which can arbitrarily be set equal to zero. This results from the fact that the desired output is only a single-frequency current. Requiring the other b_{on}

values to be zero without additional investigation might limit the design, although this procedure is permitted and often followed in preliminary steps, which are then followed by more thorough investigation. This situation is in contrast with one wherein a multifrequency output current is the desired quantity (e.g., the pulse generator of Section 3.4). In the latter case the b_{on} value corresponding to every desired a_{on} could be set equal to zero without limiting the design any more than specification of the multifrequency output wave had already done.

It is further understood that the input voltage contains only fundamental components, and is of the form

$$e_g = A_{11} \cos \omega_1 t + B_{11} \sin \omega_1 t \qquad (3\text{-}38)$$

The design procedure is first to examine the situation under the assumption of idealized lossless circuit elements. Because of the large amount of information contained in the matrix which describes the inductor, necessary associated circuits are made evident by inspection. Also, critical design areas are easily recognized and the design arranged to accommodate them. Following the initial choice of network elements, computations in the circuit may be made on the basis of actual, non-ideal inductors and capacitors, and the complete performance of the circuit examined without going into the laboratory. In general, if the quality factor (Q) of the coils involved is 25 or better, this second step is probably unnecessary.

The basic one-stage matrix equations for the circuit of Figure 3.3 with series resistance neglected are

$$[S_B]B_{1n}] = a_{on}] + ([Y^B_{on}] + [S_B])B_{on}] \tag{3-39}$$

$$[S_A]A_{1n}] = b_{on}] + ([Y^A_{on}] + [S_A])\, A_{on}] \tag{3-40}$$

Under the assumptions discussed above

$$B_{on}] = b_{on}] = 0 \tag{3-41}$$

and the matrix equations (3-39) and (3-40) to the seventh harmonic appear as shown in (3-42) and (3-43) on the following page. In (3-42) and (3-43) the multiplier $(\delta/\omega_1 L_s)$ has been included in the S terms to conserve space. The results of the matrix products at the left of the equals sign in (3-39) and (3-40) were written down immediately for the same reason.

In terms of linear, algebraic equations (3-42) becomes

$$\begin{aligned} S_{B11}B_{11} &= (Y^B_{01} + S_{B11}) \times 0 \\ S_{B31}B_{11} &= a_{03} + S_{B31} \times 0 + (Y^B_{03} + S_{B33}) \times 0 \\ S_{B51}B_{11} &= S_{B51} \times 0 + S_{B53} \times 0 + (Y^B_{05} + S_{B55}) \times 0 \\ S_{B71}B_{11} &= S_{B71} \times 0 + S_{B73} \times 0 + S_{B75} \times 0 + (Y^B_{07} + S_{B77}) \times 0 \end{aligned} \tag{3-44}$$

while (3-43) takes the form

$$\begin{Vmatrix} S_{B11}B_{11} \\ S_{B31}B_{11} \\ S_{B51}B_{11} \\ S_{B71}B_{11} \end{Vmatrix} = \begin{Vmatrix} 0 \\ a_{03} \\ 0 \\ 0 \end{Vmatrix} + \begin{Vmatrix} (Y^B_{01} + S_{B11}) & 0 & 0 & 0 \\ S_{B31} & (Y^B_{03} + S_{B33}) & 0 & 0 \\ S_{B51} & S_{B53} & (Y^B_{05} + S_{B55}) & 0 \\ S_{B71} & S_{B73} & S_{B75} & (Y^B_{07} + S_{B77}) \end{Vmatrix} \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Vmatrix} \tag{3-42}$$

and

$$\begin{Vmatrix} S_{A11}A_{11} \\ A_{A31}A_{11} \\ S_{A51}A_{11} \\ S_{A71}A_{11} \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Vmatrix} + \begin{Vmatrix} (Y^A_{01} + S_{A11}) & 0 & 0 & 0 \\ S_{A31} & (Y^A_{03} + S_{A33}) & 0 & 0 \\ S_{A51} & S_{A53} & (Y^A_{05} + S_{A55}) & 0 \\ S_{A71} & S_{A73} & S_{A75} & (Y^A_{07} + S_{A77}) \end{Vmatrix} \begin{Vmatrix} 0 \\ R_o a_{03} \\ 0 \\ 0 \end{Vmatrix} \tag{3-43}$$

$$S_{A11}A_{11} = (Y_{01}^{A} + S_{A11}) \times 0$$

$$S_{A31}A_{11} = S_{A31} \times 0 + (Y_{03}^{A} + S_{A33})R_o a_{03}$$

$$S_{A51}A_{11} = S_{A51} \times 0 + S_{A53}R_o a_{03} + (Y_{05}^{A} + S_{A55}) \times 0 \qquad (3\text{-}45)$$

$$S_{A71}A_{11} = S_{A71} \times 0 + S_{A73}R_o a_{03} + S_{A75} \times 0 + (Y_{07}^{A} + S_{A77}) \times 0$$

Since the S's are admittance operators, it may be seen from (3-44) and (3-45) that currents of magnitudes $S_{A11}A_{11}$, $S_{B11}B_{11}$, $S_{A31}A_{11}$, $S_{B31}B_{11}$, etc. will be generated by the impressed voltage. In order that the equations be satisfied, admittance at the fundamental frequency and at the fifth and higher harmonics must be very large, i.e.,

$$Y_{01} = Y_{05} = Y_{07} = Y_{09} = \ldots \rightarrow \infty \qquad (3\text{-}46)$$

The remaining equations are

$$S_{B31}B_{11} = a_{03} \qquad (3\text{-}47)$$

$$S_{A31}A_{11} = (Y_{03}^{A} + S_{A33})R_o a_{03} \qquad (3\text{-}48)$$

These two equations suggest that

1) the maximum achievable value of a_{03} is one approaching the product of the input voltage with the admittance operator S_{B31}, and that

2) the voltage component A_{11} could be minimized by setting

$$Y_{03}^{A} = - S_{A33} \qquad (3\text{-}49)$$

In order to satisfy the requirements of the law of conservation of energy, there must be input power at least equal to the generated output power. The situation leading to (3-47), (3-48), and (3-49) is so idealized that no provision is made for input power. Thus the equations are used merely to suggest the possibilities in the circuit. In actual operation, adjustments of voltage and phase would occur so that the law of conservation of energy would be observed. If it is desired to obtain the complete picture by calculation, this may be done by inserting non-idealized, realizable values in the complete equations and solving. This is done in the equations below, where it will be noticed that no significant changes occur in the design values suggested by the idealized procedure.

The output voltages will be given by

$$A_{on}] = [R_{on}]a_{on}] \tag{3-50}$$

$$B_{on}] = [R_{on}]b_{on}] \tag{3-51}$$

In (3-50) and (3-51) $[R_{on}]$ is taken to be the total effective output resistance. In the case of a resistor in parallel with an anti-resonant tuned circuit (linear elements), the value of R_{on} at each frequency would be the parallel combination of the actual output resistor and the coil losses referred to the equivalent parallel position. Since such referred resistances may be a function of frequency, $[R_{on}]$ becomes a diagonal matrix. Thus the effect of relatively low-Q coils may be taken into account.

With the above substitutions the equations (3-39) and (3-40) may be written

$$a_{on}] = [S_B]B_{1n}] - ([S_B] + [Y^B_{on}])\ [R_{on}]b_{on}] \tag{3-52}$$

$$b_{on}] = [S_A]A_{1n}] - ([S_A] + [Y^A_{on}])\ [R_{on}]a_{on}] \tag{3-53}$$

Since the $[Y^B]$ take on conventional signs for reactive admittance elements (e.g., positive for capacitive susceptance) and

$$[Y^B_{on}] = - [Y^A_{on}] = + [Y_{on}] \tag{3-1}$$

equations (3-52) and (3-53) may be combined to give the following single matrix equation linking the $a_{on}]$ and $B_{on}]$:

$$\begin{aligned} &[S_B]B_{1n}] - ([S_B] + [Y_{On}])\ [R_{On}][S_A]A_{1n}] \\ &= \ [U] - ([S_B] + [Y_{On}])\ [R_{On}]([S_A] - [Y_{On}])[R_{On}]\ a_{On}] \end{aligned} \tag{3-54}$$

While this equation may look somewhat ponderous, it yields readily to straightforward matrix multiplication and addition. When numbers rather than literal coefficients are used, the solution is readily obtained. For purposes of comparison with the idealized case the matrices and resulting algebraic equations are written out below to the third harmonic only.

$$\left\|\begin{matrix} S_{B11}B_{11} \\ S_{B31}B_{11} \end{matrix}\right\| = \left\|\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right\| - \left\|\begin{matrix} (S_{B11} + Y_{01})R_{01} & 0 \\ S_{B31}R_{01} & (S_{B33} + Y_{03})R_{03} \end{matrix}\right\| \tag{3-55}$$

$$x \quad \begin{Vmatrix} (S_{A11} - Y_{01})R_{01} & 0 \\ S_{A31}R_{01} & (S_{A33} - Y_{03})R_{03} \end{Vmatrix} \begin{Vmatrix} a_{01} \\ a_{03} \end{Vmatrix}$$

where all the Y's take the signs of the $[Y^B]$. Matrix equation (3-55) produces the algebraic equations

$$\begin{aligned} S_{B11}B_{11} - R_{01}(S_{B11} + Y_{01})S_{A11}A_{11} \\ = [1 - R_{01}^2(S_{B11} + Y_{01})(S_{A11} - Y_{01})]a_{01} \end{aligned} \qquad (3\text{-}56)$$

$$\begin{aligned} S_{B31}B_{11} - [R_{01}S_{B31}S_{A11} + R_{03}(S_{B33} + Y_{03})S_{A31}]A_{11} \\ = [S_{B31}R_{01}^2(S_{A11} - Y_{01}) + S_{A31}R_{01}R_{03}(S_{B33} + Y_{01})]a_{01} \\ + [1 - R_{03}^2(S_{B33} + Y_{03})(S_{A33} - Y_{03})]a_{03} \end{aligned} \qquad (3\text{-}57)$$

Considerably more information can be obtained from (3-56) and (3-57) under the condition that Y_{01} is large and that $(S_{B33} + Y_{03})$ vanishes. Due to the latter condition, (3-57) becomes

$$S_{B31}B_{11} - R_{01}S_{B31}S_{A11}A_{11} = -R_{01}^2S_{B31}(S_{A11} - Y_{01})a_{01} + a_{03} \qquad (3\text{-}58)$$

and since Y_{01} is large

$$R_{01}^2(S_{B11} + Y_{01})(S_{A11} - Y_{01}) \gg 1$$

so that (3-56) may then be solved for a_{01}.

$$a_{01} = \frac{-S_{B11}B_{11}}{R_{01}^2(S_{B11} + Y_{01})\ (S_{A11} - Y_{01})} + \frac{S_{A11}A_{11}}{R_{01}(S_{A11} - Y_{01})} \qquad (3\text{-}59)$$

If (3-58) and (3-59) are now solved for a_{03}, the amazingly simple result is

$$a_{03} = S_{B31}B_{11}\left[1 - \frac{S_{B11}}{S_{B11} + Y_{01}}\right] \approx S_{B31}B_{11} \qquad (3\text{-}60)$$

So long as Y_{01} is large and $(S_{B33} + Y_{01})$ vanishes, the cosine component of third-harmonic current is essentially the product $S_{B31}B_{11}$. In general a_{03} will be the dominant component of third-harmonic current. Under light-load conditions or when a non-optimum choice of ξ may have been dictated by other constraints on the system, b_{03} may become large enough to merit consideration. By writing out (3-53) the following are obtained:

$$b_{01} = S_{A11}A_{11} - R_{01}(S_{A11} - Y_{01})a_{01} \qquad (3\text{-}61)$$

$$b_{03} = S_{A31}A_{11} - S_{A31}R_{01}a_{01} - (S_{A33} - Y_{03})R_{03}a_{03} \qquad (3\text{-}62)$$

Substituting the value of a_{01} from (3-59) in (3-61), the expression for b_{01} becomes

$$b_{01} = \frac{S_{B11}B_{11}}{R_{01}(S_{B11} + Y_{01})} \qquad (3\text{-}63)$$

It has been pointed out that when a harmonic current is generated under optimum conditions with ξ set at 1/m, the magnitudes of S_{A33} and S_{B33} are equal. Thus the last term of (3-62) drops out completely.. With this term omitted and the value of a_{01} from (3-59) again substituted, (3-62) gives for b_{03}

$$b_{03} = S_{A31}A_{11}\left[1 - \frac{S_{A11}}{S_{A11} - Y_{01}}\right] \approx S_{A31}A_{11} \qquad (3\text{-}64)$$

Values of the input current components a_{11} and b_{11} are occasionally of interest. These are obtained directly from (3-11) and (3-12):

$$a_{11} = a_{01} + Y_{01}^{B}R_{01}b_{01}$$

$$b_{11} = b_{01} + Y_{01}^{A}R_{01}a_{01}$$

The value of a_{01} from (3-59) and of b_{01} from (3-63) are substituted in the above equations, and the results are rather long and complicated. Application of the condition that Y_{01} is large causes them to reduce to the very simple form

$$a_{11} = S_{B11}B_{11} \tag{3-65}$$

$$b_{11} = S_{A11}A_{11} \tag{3-66}$$

The size of these currents merely reflects the fact that near-infinite admittance at the fundamental frequency has been established by the lossless network, and therefore essentially the entire source voltage (minus any IR drop in the inherent series resistance) is impressed across the nonlinear inductor.

Power-balance relations establish another limiting condition on the design of the inductor and associated circuits. In what follows only fundamental and third-harmonic power are considered. Reduced harmonic generation efficiency at the higher frequencies plus action of the filter effectively reduce higher-harmonic power to negligible proportions.

In order to consider inherent resistance, distinction will be made between volts across the system (A_{11}, B_{11}) and volts across the nonlinear inductor (A_{1L}, B_{1L}) where

$$A_{1L} = A_{11} - ra_{11} \tag{3-67}$$

$$B_{1L} = B_{11} - rb_{11} \tag{3-68}$$

From the fundamental relations, assuming perfect by-passing of the load resistor at fundamental frequency

$$a_{11} = S_{B11}(B_{11} - rb_{11}) = S_{B11}B_{1L} \tag{3-69}$$

$$b_{11} = S_{A11}(A_{11} - ra_{11}) = S_{A11}A_{1L} \tag{3-70}$$

Simultaneous solution of (3-69) and (3-70) produces the following expressions for a_{11} and b_{11}

$$a_{11} = \frac{S_{B11}B_{11} - rS_{A11}S_{B11}A_{11}}{1 - r^2 S_{A11}S_{B11}} \approx S_{B11}(B_{11} - rS_{A11}A_{11}) \tag{3-71}$$

$$b_{11} = \frac{S_{A11}A_{11} - rS_{A11}S_{B11}B_{11}}{1 - r^2 S_{A11}S_{B11}} \approx S_{A11}(A_{11} - rS_{B11}B_{11}) \tag{3-72}$$

In the remainder of the derivation, the term

$$\epsilon = r^2 S_{A11}S_{B11}$$

will be neglected.

By solving (3-69) through (3-72) for A_{11} and B_{11} in terms of A_{1L} and B_{1L} the following are obtained.

$$A_{11} = A_{1L} + rS_{B11}B_{1L} + \epsilon A_{11} \approx A_{1L} + rS_{B11}B_{1L} \tag{3-73}$$

$$B_{11} = B_{1L} + rS_{A11}A_{1L} + \epsilon B_{11} \approx B_{1L} + rS_{A11}A_{1L} \tag{3-74}$$

Had the small quantity ϵ been considered the expression for A_{11} would be

$$A_{11} = [(1 + \epsilon) / (1 - \epsilon)][A_{1L} + rS_{B11}B_{1L}] \tag{3-75}$$

with a similar expression for B_{11}.

Using (3-73) and (3-74) the expression for input power would be

$$\begin{aligned} P_{in} &= (1/2)[(B_{1L} + rS_{A11}A_{1L})(S_{A11}A_{1L}) + (A_{1L} + rS_{B11}B_{1L})(S_{B11}B_{1L})] \\ &= (1/2)\,[A_{1L}B_{1L}(S_{A11} + S_{B11}) + rS^2_{A11}A^2_{1L} + rS^2_{B11}B^2_{1L}] \\ &= (1/2)[A_{1L}B_{1L}(S_{A11} + S_{B11}) + P_W] \end{aligned} \tag{3-76}$$

where P_W is the power lost in the winding of the nonlinear inductor. Third-harmonic power to the load resistor is thus

$$P_3 = (1/2)(S_{A11} + S_{B11})A_{1L}B_{1L} \tag{3-77}$$

Third-harmonic power is also given by

$$\begin{aligned} P_3 &= (1/2)R_{03}\,(a^2_{03} + b^2_{03}) \\ &= (1/2)R_{03}\,(S^2_{B31}B^2_{1L} + S^2_{A31}A^2_{1L}) \end{aligned} \tag{3-78}$$

By equating (3-77) and (3-78) and solving for the ratio A_{1I}/B_{1L} there results

$$(A_{1I}/B_{1L}) = \frac{(S_{A11} + S_{B11}) - [(S_{A11} + S_{B11})^2 - 4R_{03}^2 S_{A31}^2 S_{B31}^2]^{1/2}}{2R_{03}S_{A31}^2} \tag{3-79}$$

where the minus sign in front of the radical is taken since smaller values of R_{03} will require smaller values of A_{11}. It will also be noticed from (3-76) that A_{11} and B_{11} must have the same sign, since S_{A11} is greater than S_{B11} and is positive (cf. Table 3.3). The requirement that the discriminant in (3-79) be real establishes the following relation among the parameters of the nonlinear inductor and the various resistance values:

$$|(S_{A11} + S_{B11})| \geqq |2R_{03}S_{A31}S_{B31}| \tag{3-80}$$

If this relation is not satisfied, it is impossible to operate the nonlinear inductor at the point on its λ-I characteristic which is optimum for the generation of the particular harmonic under consideration--in this case the third. This means that the inductor will be forced into an operating range which is less efficient for the generation of third-harmonic currents, and that application of (3-60) and (3-64) would give values of a_{03} and b_{03} in excess of the true amount unless new values of S_{A31} and S_{B31} were used in the equations. Equation (3-80) is of interest to the designer since it enables him to choose component sizes such that his device will operate most efficiently over the entire range of interest. As it

TABLE 3.3

Values of the $[S_A]$ and $[S_B]$ Matrices for $\xi = 1/3$, $\psi = 1$

$$[S_B] = \begin{array}{c|ccc} m \backslash n & 1 & 3 & 5 \\ \hline 1 & -0.391 & 0 & 0 \\ 3 & -0.138 & -0.222 & 0 \\ 5 & -0.028 & -0.041 & -0.144 \end{array}$$

$$[S_A] = \begin{array}{c|ccc} m \backslash n & 1 & 3 & 5 \\ \hline 1 & +0.943 & 0 & 0 \\ 3 & +0.046 & +0.222 & 0 \\ 5 & +0.028 & +0.069 & +0.122 \end{array}$$

TABLE 3.4

Values of the $[S_A]$ and $[S_B]$ Matrices for = 2/3, = 1

m \ n	1	3	5
1	−0.058	0	0
3	−0.046	−0.111	0
5	+0.028	−0.069	−0.078

	m \ n	1	3	5
	1	+0.609	0	0
$[S_A]$ =	3	+0.138	+0.111	0
	5	−0.028	+0.041	+0.055

turns out, (3-80) imposes no very severe limitation.

A further remark is in order concerning the distribution of a fixed source voltage between components A_{11} and B_{11}. According to the derivation of Chapter 1, B_{11} is the dominant term for the generation of harmonic current; the A_{11} term is necessary merely to satisfy the power-balance relation. If a fixed voltage E_p is applied to the harmonic generator circuit and the load resistor is varied, the relative size of A_{11} and B_{11} will vary with the load resistor so that the power balance requirement will be satisfied, as well as that which requires the complex sum $A + j B$ to remain constant at E_p. Appendix A contains an illustrative example which demonstrates the mechanics of a calculation of these quantities for a fixed E_p and varying R_{03}. Figure 3.7 on the following page gives a comparison of values calculated from these equations with those obtained by Russell[27] on an analog computer. Agreement is seen to be excellent.

It is instructive at this point to consider a few of the numerical values of the $[S_A]$ and $[S_B]$ matrices as given in Tables 3.3 and 3.4 for the chosen values of ξ and ψ.
Examination of the table shows

$$S_{B11} = 2.8\, S_{B31} = 13.9\, S_{B51}$$

$$S_{A11} = 20.5\, S_{A31} = 33.7\, S_{A51}$$

or, in other words, a sinusoidal input voltage will generate a fundamental current 2.83 times as large as the third-harmonic

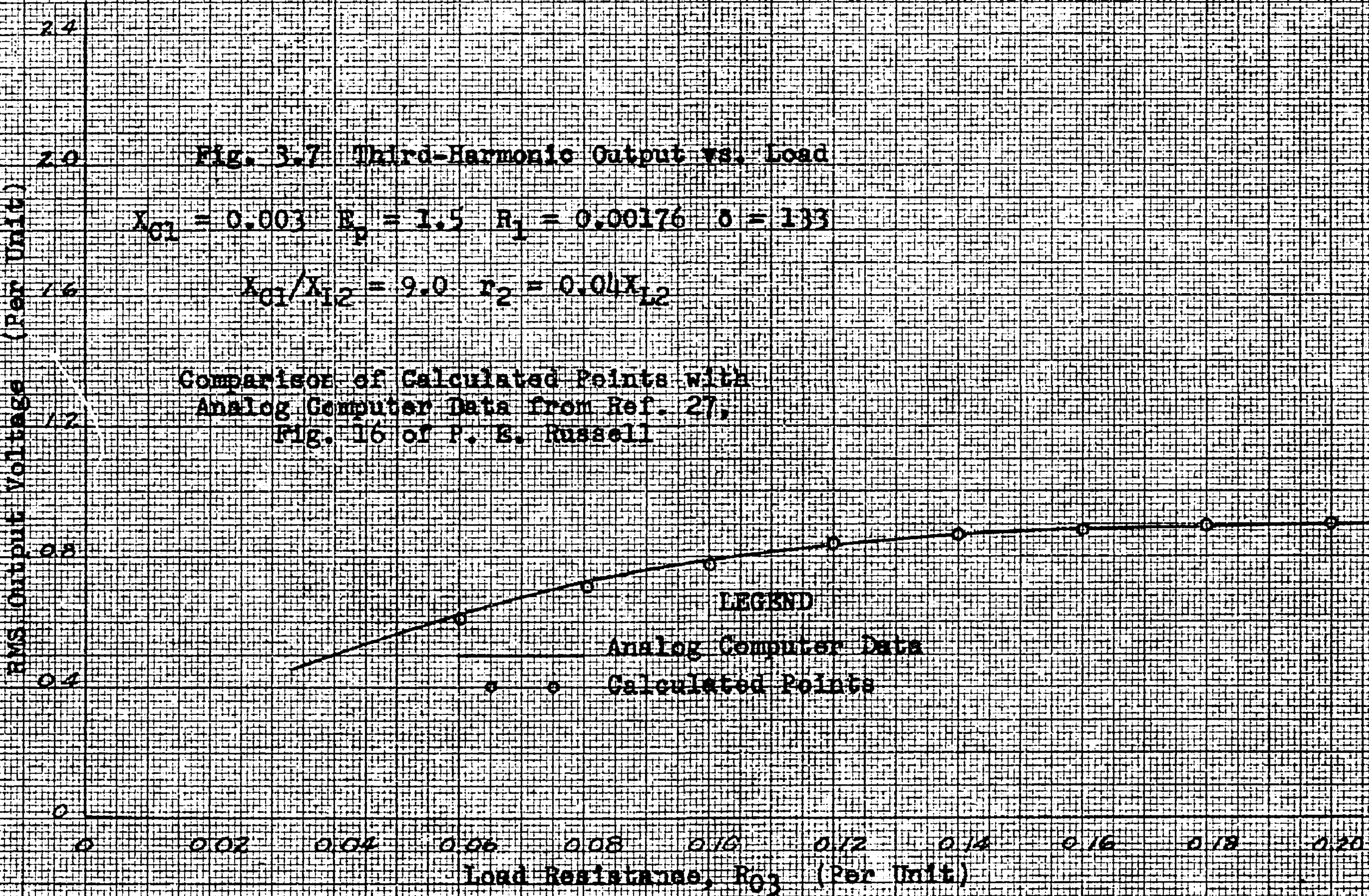

Fig. 3.7 Third-Harmonic Output vs. Load

$X_{C1} = 0.003$ $E_p = 1.5$ $R_1 = 0.00176$ $\delta = 133$

$X_{C1}/X_{L2} = 9.0$ $r_2 = 0.04X_{L2}$

Comparison of Calculated Points with Analog Computer Data from Ref. 27, Fig. 16 of P. E. Russell current and 13.9 times as large as the fifth harmonic. A cosinusoidal voltage generates a fundamental larger than the third harmonic by a factor of 20.5, and larger than the fifth by a factor of 33.7. This suggests that the fundamental component of current, rather than the higher harmonics, will be the most troublesome to remove by filtering in the design of harmonic generators.

3.2.2 Single-Stage, Polyphase Harmonic Generators

One method much used commercially to minimize the filter problem is to connect polyphase units in such fashion that the fundamental component is canceled, while the desired harmonic components re-inforce each other. The use of the matrix equations in such a connection again affords much more information to the designer than the conventional, somewhat intuitive approach.

Since it has been shown that the fundamental component is much larger than any other, and is by far the largest unwanted one which is generated, a connection such as that of Figure 3.8 offers the possibility of building a harmonic generator without output filter. It is known from experience, and demonstrated analytically below, that the fundamental component will be canceled. The possibility of using an unfiltered load resistor will therefore be investigated.

Before plunging into the design equations for the three-phase harmonic generator, it is necessary to consider the choice of ξ , since a limiting situation exists for multiphase generators which is not present for the single-phase case. Reference to Figure 3.9 shows

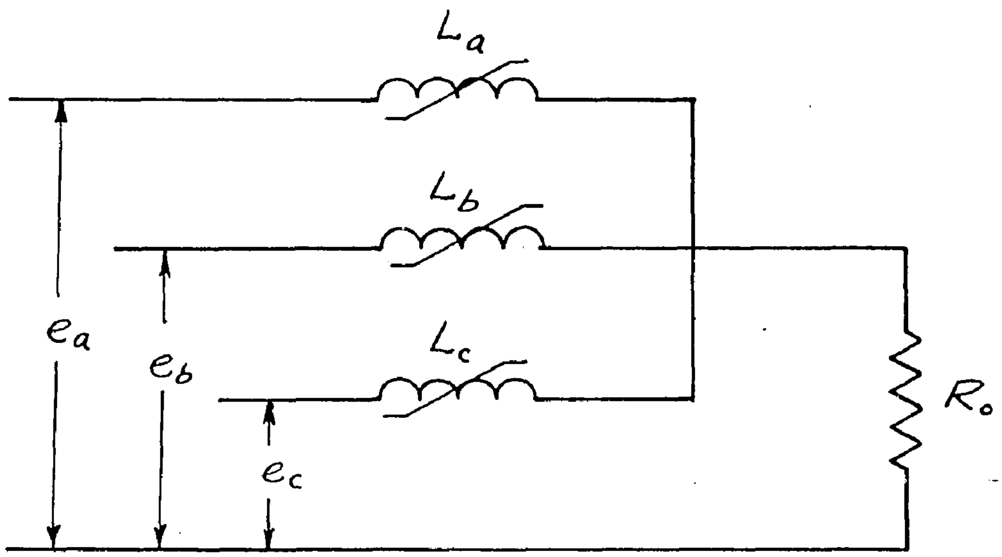

Fig. 3.8 Three-Phase Nonlinear Circuit

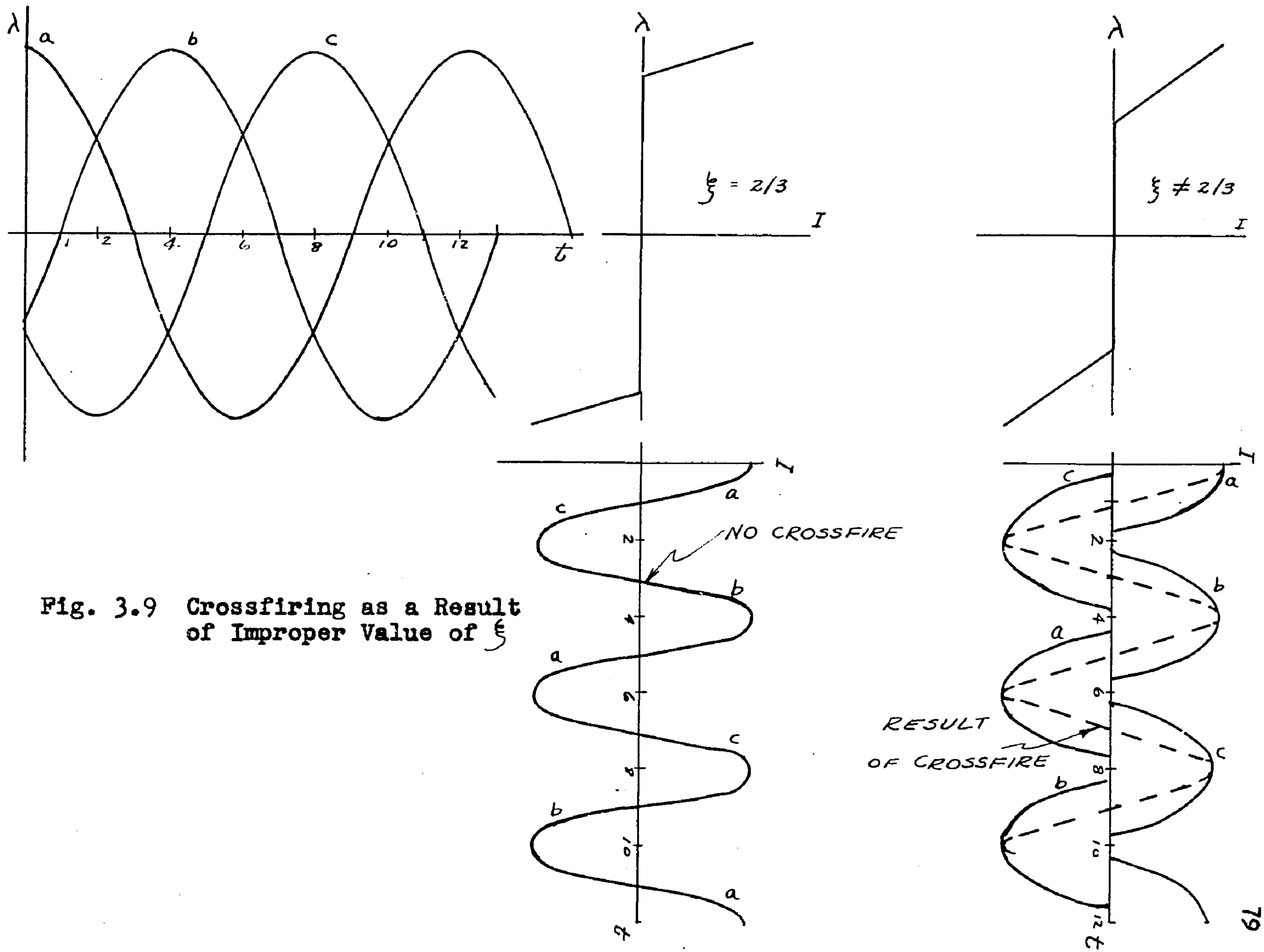

Fig. 3.9 Crossfiring as a Result of Improper Value of ξ

that high impedance must be presented to phases b and c of the three-phase system during the time that phase a is working into low impedance. If this requirement is not met, a condition known as "cross-firing" will exist, which results in considerable distortion and loss of third harmonic power. It is further evident from Figure 3.9 that the period of high impedance (which corresponds to $\xi\pi$ in our notation) in the three-phase case must have a minimum value of $2\pi/3$ radians to prevent cross-firing. It is shown elsewhere [24] that in the production of an nth harmonic from an n-phase supply, the minimum value of this angle, sometimes called the "firing angle, θ_f" is given by

$$\theta_f = \xi\pi = (n - 1)\pi/n$$

Due to this forced choice of firing angle, the reactor no longer works in its optimum range for generation of third harmonics. Suppression of the fundamental component of current more than makes up for the difference, however.

Consider first the case wherein the load resistor, R_o, is short-circuited, so that

$$R_o = 0$$

The magnitude of current in each inductor excited from a single frequency source is given by

$$A_{1n}] = [S_B]B_{1n}] \tag{3-81}$$

$$b_{1n}] = [S_A]A_{1n}] \tag{3-82}$$

In order to account for the difference in current phase due to the 120° separation in phase of exciting voltage, the following equations are written:

$$\begin{aligned} a_{1na}] &= [\phi_a][S_B]B_{1n}] \\ a_{1nb}] &= [\phi_b][S_B]B_{1n}] \\ a_{1nc}] &= [\phi_c][S_B]B_{1n}] \end{aligned} \tag{3-83}$$

where, taking phase a as reference

$$\phi_a = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix} \tag{3-84a}$$

$$\phi_b = \begin{Vmatrix} \exp\ (j2\pi/3) & 0 & 0 & 0 \\ 0 & \exp\ (j2\pi) & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \exp\ (j2n\pi/3) \end{Vmatrix} \tag{3-84b}$$

$$\phi_c = \begin{Vmatrix} \exp\ (j4\pi/3) & 0 & 0 & 0 \\ 0 & \exp\ (j4\pi) & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \exp\ (j4n\pi/3) \end{Vmatrix} \tag{3-84c}$$

By Kirchhoff's second law

$$a_{1na}] + a_{1nb}] + a_{1nc}] = a_{on}] \tag{3-85}$$

Application of (3-83), (3-84), and (3-85) to a three-phase, single-frequency excitation voltage demonstrates quickly the well-known fact that all resulting currents cancel, except for the third harmonic and its integer multiples.

As R_o is allowed to take on finite values, however, the situation changes. A voltage will appear across R_o which will modify the excitation voltages so that the equations (3-83) now become

$$\begin{aligned} a_{1na}] &= [\phi_a][S_B]B_{1n}] - [S_B]R_o b_{on}] \\ a_{1nb}] &= [\phi_b][S_B]B_{1n}] - [S_B]R_o b_{on}] \\ a_{1nc}] &= [\phi_c][S_B]B_{1n}] - [S_B]R_o b_{on}] \end{aligned} \tag{3-86}$$

The corresponding b_{1n} currents, since the A_{1n} are no longer zero, are written

$$\begin{aligned} b_{1na}] &= [\phi_a][S_A]A_{1n}] - R_O[S_A]a_{On}] \\ b_{1nb}] &= [\phi_b][S_A]A_{1n}] - R_O[S_A]a_{On}] \\ b_{1nc}] &= [\phi_c][S_A]A_{1n}] - R_O[S_A]a_{On}] \end{aligned} \tag{3-87}$$

Application of (3-85) to (3-86) and (3-87) gives

$$a_{on}] = ([\phi_a] + [\phi_b] + [\phi_c])\ [S_B]B_{1n}] - 3\ [S_B]R_o b_{on}] \tag{3-88}$$

and

$$b_{on}] = ([\phi_a] + [\phi_b] + [\phi_c])[S_A]A_{1n}] - 3[S_A]R_0 a_{on}] \qquad (3\text{-}89)$$

from which

$$([U] - 9R_o^2[S_B])a_{on}] = ([\phi_a] + [\phi_b] + [\phi_c])([S_B](B_{1n}] - 3R_0[S_A]A_{1n}]) \qquad (3\text{-}90)$$

If only the fundamental, third, and fifth harmonic terms in (3-90) are considered, the matrices become

$$\begin{Vmatrix} (1 - 9R_o^2 S_{B11}S_{A11}) & 0 & 0 \\ -9R^2(S_{B31}S_{A11} + S_{B33}S_{A31}) & (1 - 9R^2 S_{B33}S_{A33}) & 0 \\ H & -9R^2(S_{B53}S_{A33} + S_{B55}S_{A53}) & (1 - 9R^2 S_{B55}S_{A55}) \end{Vmatrix}$$

$$\times \begin{Vmatrix} a_{01} \\ a_{03} \\ a_{05} \end{Vmatrix} = \begin{Vmatrix} 0 & 0 & 0 \\ 3S_{B31} & 3S_{B33} & 0 \\ 0 & 0 & 0 \end{Vmatrix} \begin{Vmatrix} B_{11} - 3R_o S_{A11}A_{11} \\ - 3R_o S_{A31}A_{11} \\ - 3R_o S_{A51}A_{11} \end{Vmatrix} \qquad (3\text{-}91)$$

where

$$H = -9R_o^2(S_{B51}S_{A11} + S_{B53}S_{A31} + S_{B55}S_{A51})$$

From (3-91) and (3-89) the following are obtained:

$$(1 - 9R_o^2 S_{B11}S_{A11})a_{01} = 0 \qquad (3\text{-}92)$$

$$(1 - 9R_o^2 S_{B33}S_{A33})a_{03} = 3[S_{B31}B_{11} - 3R_o(S_{B31}S_{A11} + S_{B33}S_{A31})A_{11}] \qquad (3\text{-}93)$$

$$- 9R_o^2(S_{B53}S_{A33} + S_{B55}S_{A53})a_{03} + (1 - 9R_o^2S_{B55}S_{A55})a_{05} = 0 \qquad (3\text{-}94)$$

From these are obtained the equations for the currents

$$a_{03} = \frac{3[S_{B31}B_{11} - 3R_o(S_{B31}S_{A11} + S_{B33}S_{A31})A_{11}]}{(1 - 9R_o^2S_{B33}S_{A33})} \qquad (3\text{-}95)$$

$$a_{05} = \frac{+9R_o^2(S_{B53}S_{A33} + S_{B55}S_{A53})a_{03}}{1 - 9R_o^2S_{B55}S_{A55}} \qquad (3\text{-}96)$$

$$b_{03} = -3R_oS_{A33}a_{03} + 3S_{A31}A_{11} \qquad (3\text{-}97)$$

$$b_{05} = -3R_o(S_{A53}a_{03} + S_{A55}a_{05}) \qquad (3\text{-}98)$$

Thus it is seen that inclusion of the load resistor causes some fifth harmonic current to flow in the neutral circuit. Power balance relations again may be used to determine the A_{11}/B_{11} ratio.

3.2.3 Multi-Stage, Single-Phase Harmonic Generators

So far all considerations have concerned single-stage devices excited from both single- and three-phase sources. Examples have been confined to third-harmonic generators, but the method is valid for any generator of harmonics of the exciting frequency, be they even or odd, prime or non-prime. The above techniques may be used also for the design of higher-harmonic generators where the harmonic is the product of two or more integer numbers. For example, a ninth-harmonic generator might be made up of two cascaded stages, the first a generator of third harmonics, and the second a generator of third harmonics of the output from the first. This is the obvious approach.

Other possibilities of two-stage generators are also suggested by the matrix description of the nonlinear element. Consider, for example, the basic two-stage equations from Table 3.1:

$$\begin{aligned}[S_{B2}]B_{2n}] = \Big\{[U] + ([S_{B2}] + [Y^{B}_{1n}])[S_{B1}]^{-1}\Big\}a_{on}] \\ + \Big\{[S_{B2}] + [Y^{B}_{on}] \\ + [Y^{B}_{1n}] + ([S_{B2}] + [Y^{B}_{1n}])[S_{B1}]^{-1}[Y^{B}_{on}]\Big\}B_{on}]\end{aligned} \tag{3-99}$$

$$\begin{aligned}[S_{A2}]A_{2n}] = \Big\{[U] = ([S_{A2}] + [Y^{A}_{1n}])[S_{A1}]^{-1}\Big\}b_{on}] \\ + \Big\{[S_{A2}] + [Y^{A}_{on}] \\ + [Y_{1n}] + ([S_{A2}] + [Y^{A}_{1n}])[S_{A1}]^{-1}[Y^{A}_{on}]\Big\}A_{on}]\end{aligned} \tag{3-100}$$

Preliminary design procedure is similar to that for the single-stage case; viz., assume

$$B_{on}] = b_{on}] = 0$$

and, in addition, assume both nonlinear inductors to be described electrically by the same matrix. The equations then become

$$[S_B]B_{2n}] = (2[U] + [Y^{B}_{1n}][S_B]^{-1})a_{on}] \tag{3-101}$$

$$[S_A]A_{2n}] = \Big\{(2[U] + [Y_{1n}][S_A]^{-1})[Y^{A}_{on}] + ([Y^{A}_{1n}] + [S_A])\Big\}[R_{on}]a_{on}] \tag{3-102}$$

The second step in the design is the substitution of possibilities suggested by (3-101) and (3-102) into more complete equations. Assuming single-frequency excitation as in the single-stage case and a resistive output the following modifications may be made to (3-99) and (3-100).

$$A_{on}] = [R_{on}]a_{on}]$$

$$B_{on}] = [R_{on}]b_{on}]$$

It should also be recalled that

$$[Y^B_{kn}] = -[Y^A_{kn}] = [Y_{kn}] \qquad (3\text{-}1)$$

With these modifications (3-99) and (3-100) may be combined into the following single equation which completely describes the two-stage harmonic generator:

$$\begin{aligned}[S_B]B_{2n}] &- ([S_B] + [Y^B_{1n}] + [U_B][Y^B_{0n}])[R_{0n}][U_A]^{-1}[S_A]A_{2n}] \\ &= [U_B]a_{0n}] - ([S_B] + [Y^B_{1n}] + [U_B][Y^B_{0n}])[R_{0n}] \\ &\times \{[Y_{0n}] + [U_A]^{-1}([S_A] + [Y^A_{1n}])\}[R_{0n}]a_{0n}] \qquad (3\text{-}103)\end{aligned}$$

where the dimensionless quantities $[U_A]$ and $[U_B]$ are defined as

$$[U_A] = (2[U] + [Y^A_{1n}][S_A]^{-1})$$

$$[U_B] = (2[U] + [Y^V_{1n}][S_B]^{-1})$$

As an example consider the design of a third-harmonic generator in which the trap coils all have high Q's so that the matrix $[R_{on}]$ becomes practically the scalar multiplier R_o. Elements of the matrices $[S_A]$ and $[S_B]$ will be denoted by S_{A11}, S_{A31}, etc. and the elements of the corresponding inverted matrices by W_{A11}, W_{A31}, etc. In actual design these numbers are available from pre-computed tables. In particular, the only inverse matrix which must be computed is $[U_A]^{-1}$.

Equations (3-101) and (3-102) produce the matrix relations

$$\begin{Vmatrix} S_{B11}B_{21} \\ S_{B31}B_{21} \end{Vmatrix} = \begin{Vmatrix} 2 + Y_{11}^{B}W_{B11} & 0 \\ W_{B31}\,Y_{13}^{B} & 2 + Y_{13}^{B}W_{B33} \end{Vmatrix} \begin{Vmatrix} a_{01} \\ a_{03} \end{Vmatrix} \tag{3-104}$$

and

$$\begin{Vmatrix} S_{A11}A_{21} \\ S_{A31}A_{21} \end{Vmatrix} = \tag{3-105}$$

$$\begin{Vmatrix} (2 + Y_{11}^{A}W_{A11})Y_{01}^{A} + (Y_{11}^{A} + S_{A11}) & 0 \\ Y_{13}^{A}W_{A31}Y_{01}^{A} & (2 + Y_{13}^{A}W_{A33})Y_{03}^{A} + (Y_{13}^{A} + S_{A33}) \end{Vmatrix} \begin{Vmatrix} R_o a_{01} \\ R_o a_{03} \end{Vmatrix}$$

These in turn produce the four algebraic equations

$$S_{B11}B_{21} = (2 + Y_{11}^{B}W_{B11})a_{01} \tag{3-106}$$

$$S_{B31}B_{21} + Y_{13}^{B}W_{B31}a_{01} + (2 + Y_{13}^{B}W_{B33})a_{03} \tag{3-107}$$

$$S_{A11}A_{21} = [(2 + Y_{11}^{A}W_{A11})Y_{01}^{A} + (Y_{11}^{A} + S_{A11})]R_o a_{01} \tag{3-108}$$

$$S_{A31}A_{21} = (Y^A_{13}W_{A31}Y^A_{01})R_o a_{01} + [(2 + Y^A_{13}W_{A33})Y_{03} + (Y^A_{33} + S_{A33})]R_o a_{03} \qquad (3\text{-}109)$$

Solving for a_{01} and a_{03} from (3-106) and (3-107)

$$a_{01} = \frac{S_{B11}B_{21}}{2 + Y^B_{11}W_{B11}} \qquad (3\text{-}110)$$

$$a_{03} = \frac{S_{B31}B_{21} - Y_{13}W_{B31}a_{01}}{2 + Y^B_{13}W_{B33}} \qquad (3\text{-}111)$$

Equation (3-110) indicates the necessity for Y_{11} to be large if a_{01} is to be minimized, while (3-111) suggests the possibility of adjusting Y^B_{13} so that a_{03} could become very large. Methods of calculating the dependence of a_{03} on the inherent resistance of the nonlinear inductor are presented in Section 3.2.4.

From (3-109) it may also be seen that if a_{01} is negligible, and if

$$(2 + Y^A_{13}W_{A33})Y^A_{03} + (Y^A_{13} + S_{S33}) = 0 \qquad (3\text{-}112)$$

then A_{21} would be minimized. Some A_{21} must be present in order to satisfy the power-balance requirements.

The two-stage design technique is thus similar to that for a single-stage harmonic generator, and the results are similar. The difference is in the series resonant type of condition which may be established in the two-stage case.

Once design values have been selected, numerical expansion of the complete equation (3-103) is quite straightforward and not difficult. Expansion of (3-103) in literal terms will not be presented here since the results are quite cumbersome and confusing. Such an expansion has been made, however, and the results are as listed in the paragraph below.

If the design condition (3-112) is met and the denominator of (3-111) is caused to be zero,

(a) the effect of the A_{21} term in producing a_{0n} currents may be neglected, and

(b) a rather complete description of the voltage-current relations may be obtained by considering the main-diagonal terms only on the right-hand side of the expanded equation (3-103).

The reason for statement (b) may be understood from consideration of the simpler matrix equation (3-84) and the algebraic equations (3-106) and (3-107) resulting from it. Equation (3-106) is exact as it stands, and is a function of the upper left term on the main diagonal of the admittance matrix. Equation (3-107) is a function of both a_{01} and a_{03}. Now if a_{01} is caused to be small, which is an essential goal of the design anyway, the $Y_{13}^{B}\ W_{B31}a_{01}$ term will also be small, and (3-107) becomes a function of only a_{03} and the lower right term of the main diagonal.

Results of such a computation are given in (3-113) and (3-114).

$$S_{B11}B_{21} + \begin{matrix}\text{(neglected mul-)}\\ \text{(tiplier of } A_{21})\end{matrix} = \Big\{(2 + Y_{11}W_{B11})$$

$$+ [(2 + Y_{11}W_{B11})Y_{01} + (Y_{11} + S_{B11})]\left[\frac{S_{A11} - Y_{11}}{Y_{11}W_{A11} - 2}\right] + Y_{01}\,R_{01}^2\Big\}a_{01} \tag{3-113}$$

$$S_{B31}B_{21} + \begin{matrix}\text{(neglected mul-)}\\ \text{(tiplier of } A_{21})\end{matrix} = \begin{matrix}\text{(neglected mul-)}\\ \text{(tiplier of } a_{01})\end{matrix} + \Big\{(2 + Y_{13}W_{B33})$$

$$+ [(2 + Y_{13}W_{B33})Y_{03} + (Y_{13} + S_{B33})]\left[\frac{S_{A33} - Y_{13}}{Y_{13}W_{A33} - 2}\right] + Y_{03}\,R_{03}^2\Big\}a_{03} \tag{3-114}$$

It is evident that all products of main-diagonal terms with a_{0n} terms are similar, with merely the subscript numbers changing to agree with a_{0n}. Thus, subject to the restriction that all currents of lower harmonic frequency in R_{0n} be negligibly small, the design equation for any order harmonic may be set down immediately from the pattern set by (3-113) and (3-114). It is also clear that if the denominator of (3-111) were set equal to zero and if (3-112) were satisfied, the entire portion of (3-112) which is within braces (curly brackets) would vanish, leaving a_{03} to be limited only by S_{B31}, B_{21}, and the inherent series resistance of the nonlinear inductor. The effect of the latter quantity is considered in the next section.

A note of warning should be injected here concerning choice of S_{B31}. It has been noted previously (Sec. 2.2.2.1) that when ξ is set equal to 1/n for generation of maximum <u>n</u>th harmonic current

$$S_{Bnn} = -S_{Ann}$$

If this condition exists, it is impossible simultaneously to satisfy (3-112) and the requirement that the denominator in (3-111) vanish. Another value of ξ should be chosen so that the design equations can be satisfied without critical adjustment. Values of ξ such that S_{Bnn} is 1.5 to 2.0 times the magnitude of S_{Ann} work quite satisfactorily. Such choices can easily be attained with slightly detuned antiresonant circuits, which simultaneously satisfy the requirement for large admittance at undesired frequencies.

3.2.4 Consideration of Series Resistance in Inductor

This section will be primarily concerned with the effect of series resistance in the two-stage generator, since the single-stage case has been treated in equations (3-67) through (3-80).

From Table 3.1 of Section 3.1.2 the basic design equations are written with series resistance considered, but under the assumption that

$$b_{0n}] = B_{0n}] = A_{2n}] = 0$$

For the two-stage case these are

$$\begin{aligned}
[S_{B2}]B_{2n}] = ([U] + [Y^B_{1n}][S_{B1}]^{-1} + [S_{B2}][S_{B1}]^{-1})a_{0n}] \\
+ r_1 r_2 [S_{B2}][Y^A_{1n}]a_{0n}] \\
+ \Big\{ r_1[S_{B2}]([U] - [S_{B2}]^{-1}[Y^B_{1n}])[Y^A_{0n}] \\
+ r_2[S_{B2}]([U] + [Y^A_{1n}][S_{A1}]^{-1})[Y^A_{0n}] + r_2[S_{B2}][Y^A_{1n}]\Big\}[R_{0n}]a_{0n}]
\end{aligned} \qquad (3\text{-}115)$$

If both nonlinear inductors have similar electrical descriptions and if the $r_1 r_2 [Y^A_{1n}] a_{0n}]$ term is neglected, (3-115) simplifies to

$$
\begin{aligned}
[S_B] B_{2n}] &= (2[U] + [Y^B_{1n}][S_B]^{-1}) a_{0n}] \\
&\quad + [S_{B2}]\{([U] + [S_B]^{-1}[Y^B_{1n}])[Y^A_{0n}] \\
&\quad + ([U] + [Y_{1n}][S_A]^{-1})[Y^A_{0n}] + [Y^A_{1n}]\} r[R_{0n}] a_{0n}]
\end{aligned}
\qquad (3\text{-}116)
$$

Since r_1 and r_2 are both very small, neglect of their product will have little effect on the results.

Comparison of equation (3-47) with (3-111) indicates that the two-stage harmonic generator is inferior to the single-stage one unless the resonant condition is obtained for the two-stage case. Assuming that resonance is obtained, then (3-117) will give the amount of current to be expected at the desired frequency. For an *n*th-harmonic, two-stage resonant generator

$$
a_{0n} = \frac{S_{Bn1} B_{21}}{S_{Bnn} r R_{0n}(1 + W_{Bnn} Y^B_{1n}) Y^A_{0n} + (1 + Y^A_{1n} W_{Ann}) Y^A_{0n} + Y^A_{1n}} \qquad (3\text{-}117)
$$

From (3-117) the desirability of causing Y^A_{0n} to be small is obvious, and inspection of (3-114) indicates that this will cause no conflict with the requirements of that equation.

As a matter of design procedure, it is desirable to cause

$$
Y_{11} \longrightarrow \infty \qquad (3\text{-}118)
$$

since this essentially removes the fundamental component of current from the final inductor. Thus the I^2R losses in the final inductor are minimized. Also, since the fundamental frequency terms have been eliminated from the final inductor, it may be a much smaller unit than the input inductor.

Even when the cascading process of harmonic generation is used (e.g.; making a ninth-harmonic generator by cascading a third-harmonic generator with one which generates third harmonics of the output from the first), calculations indicate that in the "touching-up" process, components are actually caused to satisfy (3-112). In the cascaded design, however, there is no guarantee that the fundamental component has been eliminated from the final inductor.

If the design is such that the fundamental is eliminated from the second stage, the voltage drop and power loss due to the fundamental component will be given by (3-73), (3-74), (3-76) just as for the single-stage case.

It is possible to obtain performance from a two-stage unit which is improved over its single-stage equivalent. The price, in addition to more components, is that the resonant condition may limit the variety and type of load which might be connected to it.

3.2.5 Harmonic Generators—Design Summary

1) Choose type of generator. This will be predicated on available power sources, economic factors, size, space, etc. Equations (3-47), (3-95), and (3-111) will be helpful in estimating the relative size of voltages required for a given output current.

2) Choose the optimum combination of ξ and ψ. Where it is possible to maximize the harmonic output, application of the methods of Section 2.2.2 will achieve the optimum condition. If a polyphase supply is being designed, cross-firing considerations will dictate the choice of ξ , as they did in the three-phase example above. For the two-stage case, the warning in the paragraphs following (3-114) should be observed.

3) Choose the values of associated linear circuit elements with the help of the equation listed:

Single-stage, single-phase	Equations (3-47) and (3-48)
Single-stage, polyphase	Probably none needed
Two-stage, single-phase	Equations (3-111) and (3-112)

4) Substitute values chosen in Step 3 in the more complete equations describing the chosen type of generator. These are

Single-stage	Equation (3-54)
Two-stage	Equation (3-103)

5) Check voltage loss and power loss in the inductor according to (3-73), (3-74) and (3-76). This will give an idea of the minimum size wire which can be tolerated in the inductor.

3.3 Application to the Generation of Subharmonics

Considerable study has been made of the series RLC circuit containing a nonlinear element, usually the inductor, which may be used to generate subharmonics. According to the theoretical work of Trefftz [30] and experimental observations over the years, as reported in general fashion by Rudenberg [31], "an oscillatory circuit of nonlinear characteristic is able to respond in its fundamental natural oscillation to voltages having the frequency of any and all of its natural higher harmonics."

Subharmonic generation has been studied with an eye toward suppression of subharmonic currents as well as toward their generation. In particular the addition of capacitors to a power system for power-factor improvement may produce a ferroresonant condition leading to irregular fluctuations of voltage and current in the system. It is possible that these fluctuations may lead to damage to subscribers' equipment or to the system itself. Thus there is considerable interest in measures which may be taken to suppress subharmonic currents on the power system.

A classic example of the intentional generation of subharmonic is the "sub-cycle ringer" in the telephone plant. In this application, energy at 20 cycles per second is generated from the 60-cycle power source by means of a ferroresonant circuit.

As a result of previous work on the generation of subharmonic frequencies it is fairly well known that

1) If resistance in the circuit is too high, subharmonics will not be generated.

2) If voltage is fixed, there are certain ranges, or bands, of capacitance values which permit subharmonics to be generated.

3) If capacitance is fixed, there is a range of voltage in which subharmonics can be generated.

4) The capacitance or voltage range which is permissive of subharmonic generation decreases as resistance increases.

5) The natural frequency of the circuit should be close to the desired subharmonic frequency.

In the study of subharmonics perhaps more interest has been shown in the conditions which permit their generation than in actual values of the currents themselves. The matrix description of the inductor, as used in this dissertation, lends itself very well to specification of such conditions in a general, normalized fashion which may be universally applied. Previous studies have been tied to a given inductor, and only general qualitative statements could be made concerning situations other then the one specifically studied. It is also possible to get quantitative estimates of the amount of subharmonic, as well as harmonic, current which would flow in a given or chosen situation.

An inherent weakness of the piecewise linear approach as applied to subharmonic generation lies in the fact that the best condition for the generation of subharmonics turns out to occur when the core is only lightly or moderately saturated. It is under these conditions that the approximations basic to the derivation in Chapter 2 are most liable to error. A second drawback to accurate comparison with experiment is the fact that most data have been taken on inductors with "gradually varying" λ-I characteristics. Thus the precision with which comparisons may be made with data from transformer iron is dependent to a large extent on judicious choice of slope and operating point, since these values change continuously and sizeably about the knee of the curve. In spite of these difficulties, however, surprisingly good agreement has been obtained with published data, both experimental and calculated, of Mostafa and El-Karaksi[32]. Their paper is one of the very few to give both test data and calculations to check test data. A comparison of results calculated from the methods of this dissertation and experimental results of the above authors will be given following derivation of the equations.

3.3.1 Equations for the Subharmonic Generator

The configuration considered is the RLC circuit of Figure 3.10. The procedure is similar to that used before; i.e., to write the Kirchhoff equations around the circuit, then apply the current-voltage relations in matrix form to the voltage, E_L, across the inductor.

$$E_L = E_{in} - E_c - E_R \tag{3-119}$$

The cosine and sine components of current in the circuit then becomes

$$[S_B]\{B_{1n}] - Rb_{1n}] - [X_c^A]a_{1n}]\} = a_{1n}] \tag{3-120}$$

$$[S_A]\{A_{1n}] - Ra_{1n}] - [X_c^B]b_{1n}]\} = b_{1n}] \tag{3-121}$$

Rearranging, and solving for a_{1n} and b_{1n},

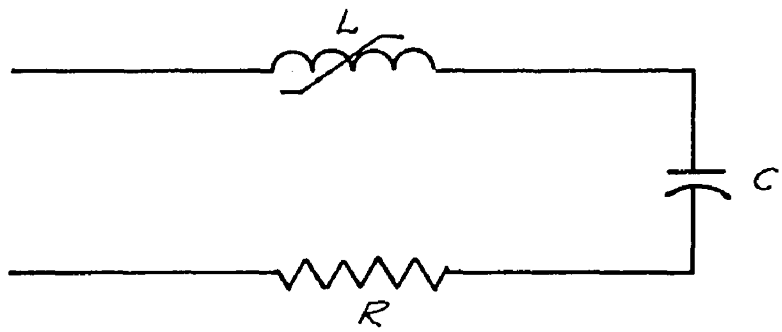

Fig. 3.10 Nonlinear RLC Circuit

$$b_{1n}] = (1/R)\left\{B_{1n}] - ([S_B]^{-1} + [X_c^A\ ])a_{1n}]\right\} \tag{3-122}$$

$$a_{1n}] = (1/R)\left\{A_{1n}] - ([S_A]^{-1} + [X_c^B\ ])b_{1n}]\right\} \tag{3-123}$$

$$= (1/R)A_{1n}] - (1/R^2)([S_A]^{-1} + [X_c^B])\left\{B_{1n}] - ([S_B]^{-1} + [X_c^A\ ])a_{1n}]\right\}$$

After rearranging (3-123)

$$([S_A]^{-1} + [X_c^B])B_{1n}] - RA_{1n}] = \left\{([S_A]^{-1} + [X_c^B])(S_B]^{-1} + [X_c^A]) - R^2[U]\right\} a_{1n}] \tag{3-124}$$

Equations (3-122) and (3-124) are the basic equations of the circuit. The equations are written out below in matrix form to the fifth harmonic of the lowest subharmonic current. To be specific it is assumed that the desired subharmonic is one-third the frequency of the source voltage, so the source-voltage components A_{13} and B_{13} are placedd in the middle row of the matrix equation. Had it been desired to investigate a subharmonic of order one-fifth, the source voltage components would have been placed in the bottom row.

$$\begin{Vmatrix} W_{A11}+X_{11} & 0 & 0 \\ W_{A31} & W_{A33}+X_{13} & 0 \\ W_{A51} & W_{A53} & W_{A55}+X_{15} \end{Vmatrix} \begin{Vmatrix} 0 \\ B_{13} \\ 0 \end{Vmatrix} - \begin{Vmatrix} 0 \\ RA_{13} \\ 0 \end{Vmatrix}$$

$$= \left[\begin{Vmatrix} W_{A11}+X_{11} & 0 & 0 \\ W_{A31} & W_{A33}+X_{13} & 0 \\ W_{A51} & W_{A53} & W_{A55}+X_{15} \end{Vmatrix} \begin{Vmatrix} W_{B11}-X_{11} & 0 & 0 \\ W_{B31} & W_{B33}-X_{13} & 0 \\ W_{B51} & W_{B53} & W_{B55}-X_{15} \end{Vmatrix} \right.$$

$$- \left\| \begin{matrix} R^2 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & R^2 \end{matrix} \right\| \Bigg] \left\| \begin{matrix} a_{11} \\ a_{13} \\ a_{15} \end{matrix} \right\| \qquad (3\text{-}125)$$

where

$$-[X_c^A] = [X_c^B] = [X_c]$$

The matrix equation expands to give the following algebraic equations:

$$0 = [(W_{A11} + X_{11})(W_{B11} - X_{11}) - R^2]a_{11} \qquad (3\text{-}126)$$

$$(W_{A33} + X_{13})B_{13} - (1/R)A_{13} = [W_{A31}(W_{B11} - X_{11}) + W_{B31}(W_{A33} + X_{13})]a_{11}$$
$$+ [(W_{A33} + X_{13})(W_{B33} - X_{13}) - R^2]a_{13} \qquad (3\text{-}127)$$

$$W_{A53}B_{13} = [W_{A51}(W_{B11} - X_{11}) + W_{A53}W_{B31} + (W_{A55} + X_{15})W_{B51}]a_{11}$$
$$+ [W_{A53}(W_{B33} - X_{13}) + W_{B53}(W_{A55} + X_{15})a_{13}$$
$$+ [(W_{A55} + X_{15})(W_{B55} - X_{15}) - R^2]a_{15} \qquad (3\text{-}128)$$

It will be noticed that (3-126), which describes the lowest-frequency current, is not explicitly a function of the source voltage, and its subharmonic order number is not explicitly related to the source voltage either. Order of the subharmonic would be reflected in the relative size of the X components. Thus the conclusions drawn below may be applied to any order of subharmonic by adjusting the size of X_{11}.

Again consider (3-126). If a_{11} is not to be zero, then the bracketed term must be zero. This gives the conditions under which the subharmonic will exist, and the following may be written:

$$R^2 = W_{A11}W_{B11} + X_{11}(W_{B11} - W_{A11}) - X_{11}^2 \qquad (3\text{-}129)$$

Now the inverse W_{A11} is simply $1/S_{A11}$ and W_{B11} is $1/S_{B11}$ so that (3-129) becomes

$$R^2 = (1/S_{A11}S_{B11}) + X_{11}(S_{A11} - S_{B11})/S_{A11}S_{B11} - X_{11}^2 \qquad (3\text{-}130)$$

From Table 2.1

$$S_{A11} = \delta[1 - \xi + (1/\pi)\sin \pi\xi] = [(\sin \pi\xi/\pi) + (1 - \xi)] \qquad (3\text{-}131)$$

$$S_{B11} = \delta[(\sin \pi\xi/\pi) - (1 - \xi)] \qquad (3\text{-}132)$$

so that

$$S_{A11}S_{B11} = \delta^2[(\sin^2 \pi\xi/\pi^2) - (1 - \xi)^2] \qquad (3\text{-}133)$$

$$S_{A11} - S_{B11} = 2\delta(1 - \xi) \qquad (3\text{-}134)$$

Equation (3-130) may now be written in terms of δ and ξ, the fundamental quantities of the inductor

$$(X_{11}^2 + R^2) = \frac{1 + 2\delta(1 - \xi)X_{11}}{\delta^2[(\sin^2 \pi\xi/\pi^2) - (1 - \xi)^2]} \qquad (3\text{-}135)$$

Let

$$R^2 = p^2X_{11}^2 \tag{3-136}$$

Then (3-135) may be solved for X_{11}, giving

$$\delta X_{11} = \frac{1 - \xi + \sqrt{(1/\pi^2)(\sin^2 \pi\xi) + p^2\{(1/\pi^2)(\sin^2 \pi\xi) - (1 - \xi)^2\}}}{(1 + p^2)\{(1/\pi^2)(\sin^2 \pi\xi) - (1 - \xi)^2\}} \tag{3-137}$$

The requirement that (3-137) be real places the following restriction on the term under the radical:

$$(1/\pi^2)(\sin^2\pi\xi) \geqq p^2\{(1/\pi^2)\sin^2\pi\xi - (1 - \xi)^2\} \tag{3-138}$$

This restriction, together with (3-136) gives the range of R which permits subharmonics of the chosen order to be generated by the circuit. In Figure 3.11, ξ is presented as a function of p_{max}. In order for subharmonics to be possible in the circuit the operating range must be above and to the left of the curve. Figure 3.12 is a plot of ξ as a function of δX_{11}. The lower curve represents values of X_{11} for the maximum permissible value of R, while the upper curve is based on zero resistance. Operation which will generate subharmonics must therefore lie in the shaded portion between the two curves. Values of X_{11} for values of R between zero and maximum may be computed from (3-137) for the particular value of R which was chosen, or with less accuracy, read directly from the curves of Figure 3.12.

Probably the most useful computation is the above one to determine conditions which permit or suppress the generation of

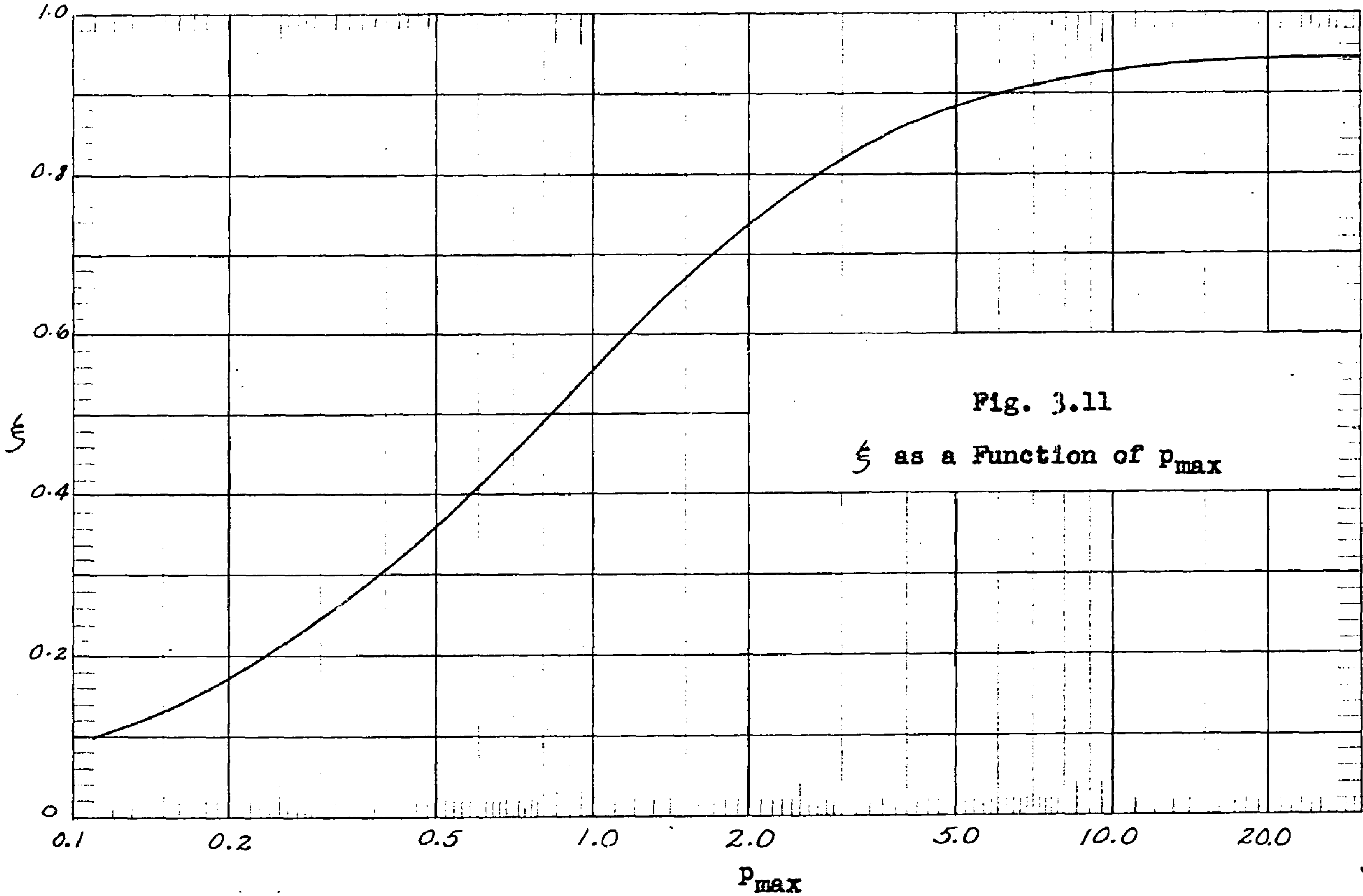

Fig. 3.11

ξ as a Function of p_{max}

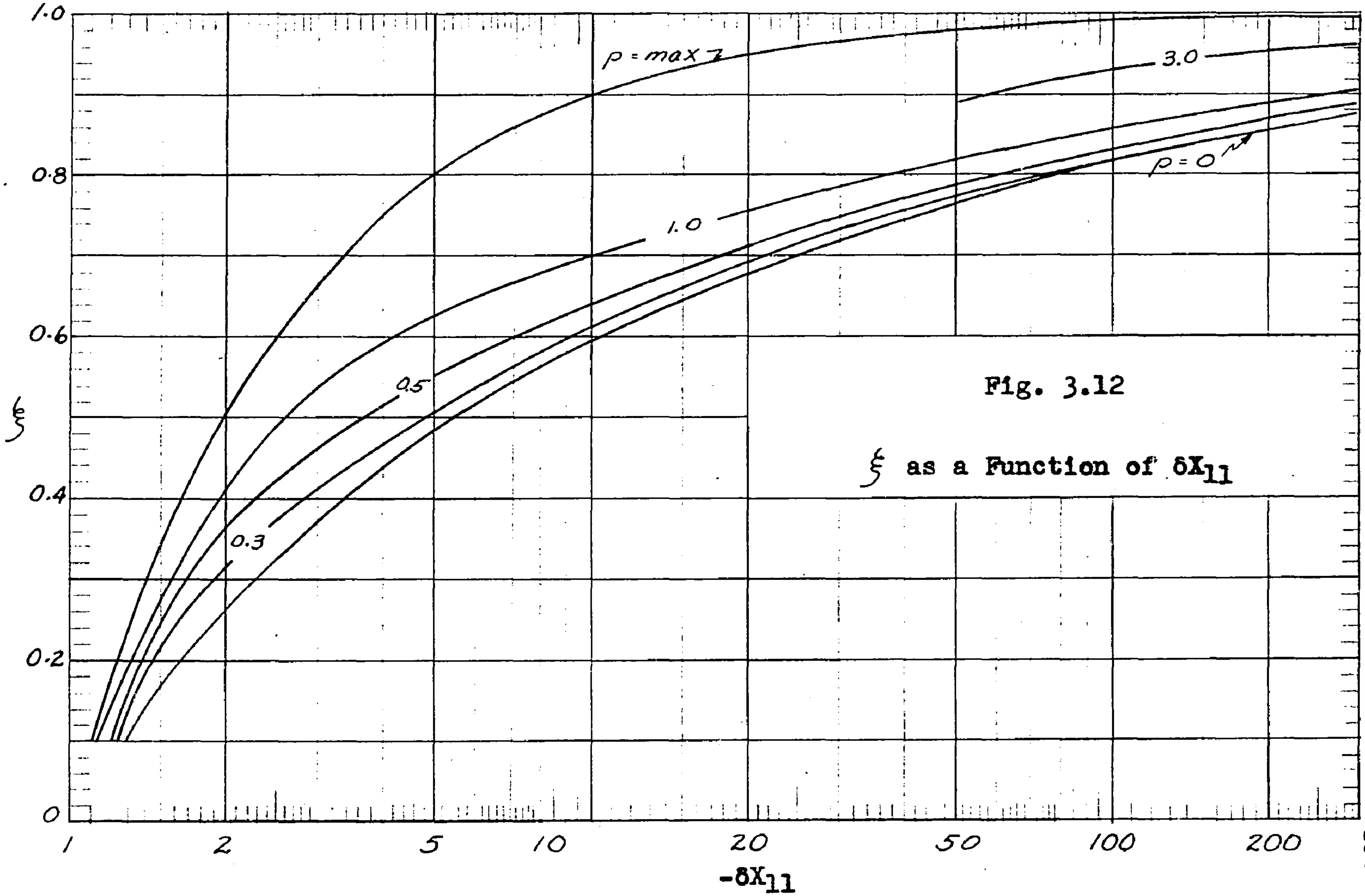

Fig. 3.12

ξ as a Function of δX_{11}

subharmonics. Determination of voltages and currents in the circuit is similar to the calculation of these values for the harmonic generator and for the resistor in series with the reactor. There is one departure from those procedures. Since a_{11} and b_{11} are not determined from (3-126), some manner of determining the division of current between a_{11} and b_{11} must be devised. The needed relation is the fact that the sum of all the a_{1n} currents must equal the maximum current. This was the premise upon which the values of the matrix elements were derived. It is usually adequate to consider only the subharmonic current plus that at the frequency of the source voltage. As in other examples, power-balance equations may be used to determine the division between the source voltages A_{1n} and B_{1n}. An example of this type of calculation is presented in Appendix A.

Results of that calculation may be compared with experimental results from Figure 10 of the paper of Mostafa and El-Karaksi[32] plus their calculations as reported on p. 110 of the same article. For one point on the curve these are as follows:

	Experimental Results	Calculated Results
Source Voltage (Volts, rms).............	43.8	43.6
Third Subharmonic Current Amperes, rms).	0.38	0.16
Total Circuit Current (Amperes, rms)....	0.52	0.98
Maximum Flux Linkage (Weber-Turns)......	0.53	0.576

Conditions for both the experiment and calculations were

Resistance (Ohms).. 30

Capacitance (Microfarads)................................ 100

Frequency (Cycles per Second)............................ 50

Agreement is seen to be quite good except for the division of current between the subharmonic and source-frequency components. The discrepancy is believed to be reasonable with the approximations made to bring the problem within the scope of the methods of this dissertation. The flux linkage versus current curve of the authors cited is reproduced in Figure 3.13, and the piecewise linear approximation superimposed on it. It may be seen that the departure from the experimental relations around the knee of the curve is sizeable. Had the core been driven further into saturation, the approximation would have caused less error.

Even with approximations as gross as that of Figure 3.13 the method may be expected to give order-of-magnitude results or better. Determination of conditions favorable to the generation of a particular subharmonic is a simple computation, and certainly reliable within an order of magnitude for transformer iron. A similar computation on rectangular-loop iron might be expected to give results correct within a few percent.

Determination of all currents and voltages by computation is a rather long and tiresome process, although straightforward. Fortunately the usual interest in subharmonic currents is in the measures which must be taken to suppress them. Computations to this end are simple and reasonably accurate, even for transformer iron.

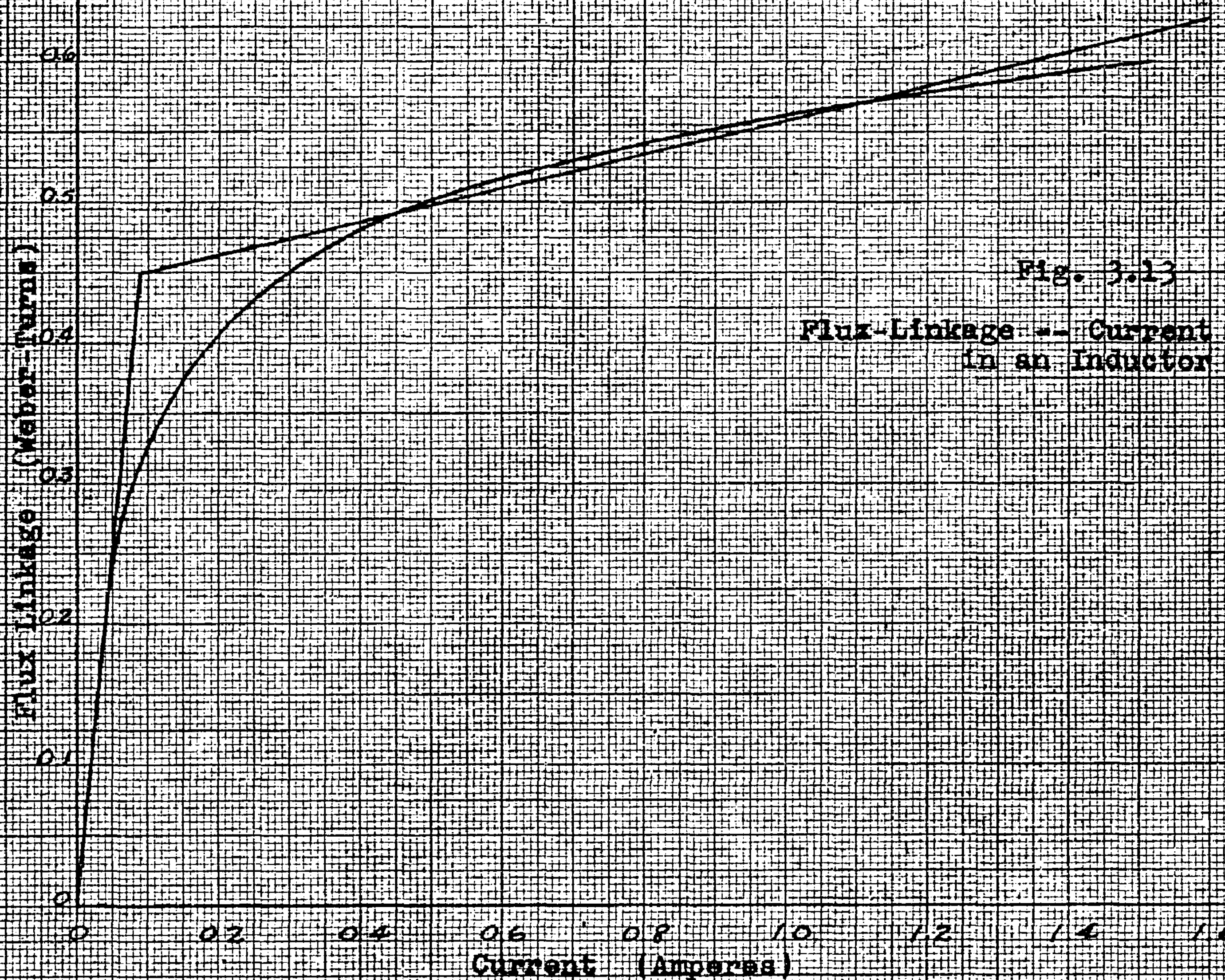

Fig. 3.13

Flux-Linkage -- Current Relations
in an Inductor

3.4 Application to Pulse Generation

As a final illustration of the possible applications of the matrix equations, consider their use in the design of magnetic pulse generators. A sizeable amount of design information is already available from the work of Melville[20] and others[21,22], and very satisfactory designs can be produced from their equations. The best use of the matrix approach in this application is probably in a sharpening or polishing of the design after basic parameters have been chosen by techniques described in the papers above.

All of the design equations previously developed have been based on the switch, or on-off approach, whereby the inductor characteristic resembles that of Figure 1.2. The basic circuit of the device is given in Figure 3.14. In that figure, the inductor, L_g, is linear and all the rest are nonlinear. All capacitances are usually of equal value, although this is neither necessary nor completely desirable. An essential to all designs is that the unsaturated inductances

$$L_g/L_{un} = L_{un}/L_{u(n-1)} = \dots = L_{u2}/L_{u1} \geqq 20 \tag{3-139}$$

and that

$$L_{u(n-1)}/L_{sn} \geqq 20 \tag{3-140}$$

This means for each individual inductor that

$$\delta = L_u/L_s \geqq 400 \tag{3-141}$$

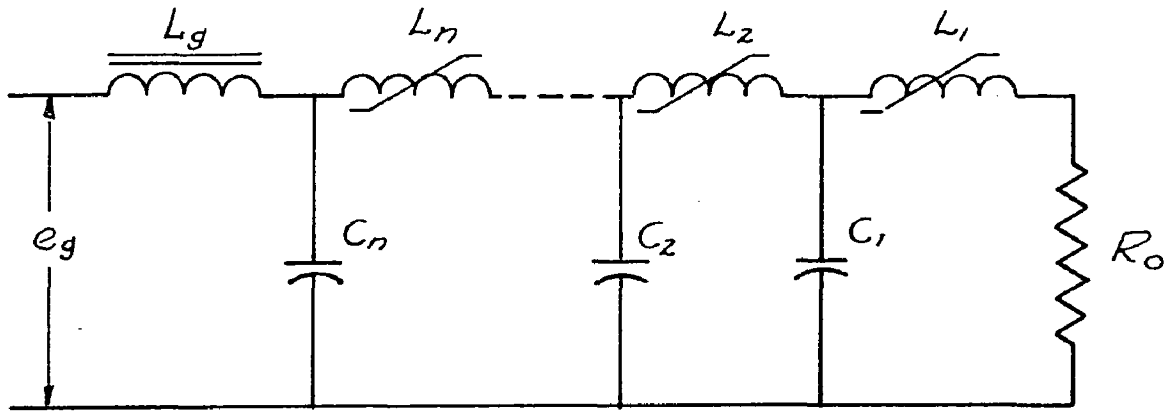

Fig. 3.14 Pulse Generator Circuit

This places a very severe requirement on the type of iron to be used. Only a very few meet the requirements.

Reference to Table 2.1 indicates that the [S] matrices describing L_1 are twenty times as large as those of L_2. More significant is consideration of the inverse matrices, where the situation is just reversed. The inductor closest to the load has but one-twentieth the impedance of its neighbor toward the generator, and so on. Essentially the pulse is formed by the L_g-C_n-L_n combination closest to the generator. The remaining reactive elements serve as pulse sharpeners.

In the problem under consideration, output current may be considered an even function so that

$$b_{on}] = B_{on}] = 0 \tag{3-142}$$

From Table 3.2, which gives the matrix relations for the three-stage circuit, the input-output relations might be written

$$B_{gn}] = [W_B]a_{on}] \tag{3-143}$$

$$A_{gn}] = [W_A]A_{on}] = R\ [W_A]a_{on}]$$

where the matrix [W] represents the complicated assemblage of [Y] and inverse [S] matrices making up the multipliers of $a_{on}]$ in (3.26). The problem may be stated quite simply, although it is not so easily solved. A solution has been obtained when values of the [W] matrices are chosen so that (3-143) is satisfied for specified values of

B_{gn} and a_{on}.

Fortunately, in addition to Melville's equations, there are additional factors determined immediately by the physical requirements. For example, the ξ value of the final inductor L_1 is determined by the pulse duty cycle. Since $\xi\pi$ represents the number of radians during which the inductor is essentially nonconducting in a half cycle, the conducting fraction, $(1 - \xi)\pi/\pi$, may be equated to the quotient of the pulse width divided by half the period of the fundamental-frequency component; i.e.,

$$(1 - \xi) = 2f_1\tau \qquad (3\text{-}144)$$

where τ is the pulse length in seconds and f_1 is the fundamental frequency in cycles per second. This relation holds for both the uni-cycle and the bi-cycle cases. The uni-cycle case occurs when the core is unsymmetrically excited to give one unidirectional pulse per fundamental period; the bi-cycle case occurs when the core is symmetrically excited and produces two oppositely polarized pulses in the fundamental period.

Input voltage is specified, usually as a single-frequency quantity. For pulse generation the output current might be specified from the Fourier expansion of a periodic rectangular pulse of appropriate amplitude and width. It is of interest in this connection, however, to note that the output pulse actually produced at present by these devices more nearly approaches the form

$$e_R = (Vt/\sqrt{L_{s1}C_1}) \exp(-t/\sqrt{L_{s1}C_1}) \qquad (3\text{-}145)$$

So far as pulse shape is concerned, considerable room exists for improvement.

A large number of the quantities required by Table 3.2 are seen to be fixed, either by specification or physical requirements. Considerable latitude remains for optimizing the design through selection of ideal ξ values and variation of capacitance size. With a table of precomputed values of the $[S]^{-1}$ matrices, this optimization is possible by means of slide-rule computation. It is also tedious. Equations (3-143) also lend themselves very readily to machine computation techniques. It is worthy of note that one machine routine would handle a tremendous variety of applications, merely by changing the input and output quantities.

CHAPTER 4

SUMMARY, CONCLUSIONS, AND SUGGESTIONS FOR FURTHER WORK

A method has been presented by which the nonlinear magnetization curve can be described mathematically in a normalized form which contains information on both the linear and nonlinear properties of the device. The representation is such that linear and nonlinear network elements may be combined directly in single- or multi-loop circuits. Since information on the effect of the nonlinearity is available in the tables of matrix values prior to the design of the circuit, it is not necessary to select in advance an amplitude or range over which the nonlinear element is to work, then experiment to see if the guess was correct. To be sure, amplitude specification is inherent in the matrix description of the reactor, but in terms such that the circuit may be analyzed or synthesized on the basis of desired performance. The reactor can then be built to these needs. The method also is not limited to small perturbations about an operating point. On the contrary, best results are obtained when the departure from linearity is large.

The mathematics required is quite simple, being really nothing more than elementary algebra. Expression of the quantities in matrix notation also is compatible with the use of high-speed computing machines to optimize designs. Since these devices have become almost ubiquitous, the tedium and manual labor formerly associated with nonlinear design can be practically eliminated. Even without machine computation, many problems utilizing the method can be handled by slide-rule computation. All of the examples presented in this dissertation were computed on the slide rule.

There is one obvious weakness in the technique which additional research could probably correct. The approximation is poor around the knee of the curve of materials with gradually varying characteristics, such as transformer iron. It is believed that the approximation could be improved by the introduction of additional segments into the piecewise linear approximation. Undue complexity in the resulting matrices might be avoided by expressing the additional breakpoints in terms of the original k_1 and k_2.

Additional study might also be profitably directed toward effecting a closer relation between the nonlinear methods of this dissertation and the methods of modern linear network synthesis.

The basic ideas may be extended to other electrical network elements and, by analogy, to their counterparts in other areas such as mechanics, acoustics, etc. The methods of this dissertation would apply directly to nonlinear capacitance merely by changing the quantities involved from inductance, flux-linkage, and current to capacitance, charge, and voltage, taking care, of course, to observe any changes in sign which might result.

Although an additional Fourier analysis would be necessary, it is believed that the same type of mathematical descriptive technique could be applied to nonlinear resistive devices such as diodes, thyrite elements, and even electric arcs. Voltage-current relations could be described in terms of tabulated values based on the breakpoint of the curve. Such piecewise linear approximations have been often used throughout the years, but simultaneous presentation of all the information in matrix form would be new. The results could be presented in normalized, tabular form which would be applicable to any nonlinear resistive device fitting the general outline of the approximation.

Several applications to demonstrate use of the method have been worked out in Chapter 3 with numerical computations in Appendix A. It is believed that these techniques may be applied to many more specific problems.

One possible application would be to the design of filters having fast rise times without overshoot. The fact that the matrix equations are based on periodic excitation poses no problem. It has been shown[33,34] that, for practical circuits, transient analysis based on response to a periodic square wave or pulse is equivalent, and often superior to the classical Laplace Transform or Fourier Integral approaches. The reason is that departures from idealized conditions may be more easily accommodated in the Fourier Series than in the other techniques.

In such a problem, input voltage and output current or voltage would be specified, through their Fourier-Series representations, as being equal to within a multiplicative constant. That is

$$B_{1n}] = KR_o[W]a_{on}]$$

with the specification that

$$B_{1n}] = kR_o a_{on}]$$

The problem then would be to assemble a network such that the matrix W describing it would approach the unity matrix to some specified tolerance.

A similar type of problem is presented in the design of circuits which include pulse transformers. In such circuits it is desired that the output wave form approach quite closely that of the input. It is sometimes required that the output be an improvement over the input. Both problems should be solvable by the techniques presented herein.

It is believed that matrix synthesis and analysis techniques might also be applied to magnetic-amplifier design. Such an investigation could well provide material for several theses or dissertations. In fact the nonlinear matrix approach might be profitably applied to many of the applications of saturable reactors[35], most of which have evolved through experimental and intuitive developments.

APPENDIX A

ILLUSTRATIVE EXAMPLES

A-1 Nonlinear Element in Series with Resistor

In this example the value of currents to the fifth harmonic will be determined, in terms of B_{11}, for the case of the nonlinear element in series with a resistor (Figure 3.6). For this example, ξ will be assumed to be 2/3 and δ taken at 133. Excitation of the core is symmetrical so that ψ is unity. Values of the $[S_A]$ and $[S_B]$ matrices are given below computed to the fifth harmonic for these conditions:

$$[S_B] = \begin{Vmatrix} -7.7 & 0 & 0 \\ -6.12 & -14.8 & 0 \\ +3.72 & -9.18 & -10.9 \end{Vmatrix} ; \quad [S_A] = \begin{Vmatrix} +81 & 0 & 0 \\ +18.4 & +14.8 & 0 \\ -3.72 & +5.45 & +7.3 \end{Vmatrix}$$

The output resistor, R_0 will be taken as 0.1 per-unit ohm.

The value of a_{01} is obtained by direct substitution of values into (3-34); viz.,

$$-7.7\, B_{11} - (0.1)(-7.7)(+81)\, A_{11} = [1 - (0.01)(-7.7)(+81)]a_{01} \qquad \text{(A-1)}$$

from which

$$a_{01} = 8.64\, A_{11} - 1.06\, B_{11} \qquad \text{(A-2)}$$

Direct substitution of this value of a_{01}, together with the other values called for by (3-35) produces

$$a_{03} = 3.26\ A_{11} + 0.636\ B_{11} \tag{A-3}$$

A similar operation with (3.36) gives

$$a_{05} = -4.88\ A_{11} + 0.362\ B_{11} \tag{A-4}$$

Expansion of (3-31) gives the following equations for the b_{0n}:

$$b_{01} = S_{A11}(A_{11} - R_0 a_{01}) \tag{A-5}$$

$$b_{03} = S_{A31}(A_{11} - R_0 a_{01}) - S_{A33} R_0 a_{03} \tag{A-6}$$

$$b_{05} = S_{A51}(A_{11} - R_0 a_{01}) - S_{A53} R_0 a_{03} - S_{A55} R_0 a_{05} \tag{A-7}$$

Straightforward substitution of values into these equations gives for the b_{0n}:

$$b_{01} = +11.00\ A_{11} + 8.60\ B_{11} \tag{A-8}$$

$$b_{01} = -\ 2.33\ A_{11} + 1.01\ B_{11} \tag{A-9}$$

$$b_{01} = +\ 1.28\ A_{11} - 1.11\ B_{11} \tag{A-10}$$

Power output is given by (3-37) and is

$$P_{out} = (1/2)(23.6A_{11}^2 + 16.4A_{11}B_{11} + 7.8B_{11}^2) \tag{A-11}$$

Input power also is given by (3-37) and is computed as

$$P_{1n} = (1/2)(8.6A_{11}^2 - 10A_{11}B_{11} + 8.6B_{11}^2) \tag{A-12}$$

After equating the two and combining, the following relation between A_{11} and B_{11} results:

$$15\,A_{11}^2 + 26.4A_{11}B_{11} - 0.8B_{11}^2 = 0 \tag{A-13}$$

from which the ratio A_{11}/B_{11} is determined to be

$$A_{11}/B_{11} = -1.79 \tag{A-14}$$

In terms of B_{11} the currents then become

$$a_{01} = (-15.45 - 1.06)\ B_{11} = -16.51\ B_{11} \tag{A-15}$$

$$a_{03} = (-\ 5.84 + 0.64)\ B_{11} = -5.20\ B_{11} \tag{A-16}$$

$$a_{05} = (+\ 8.74 + 0.36)\ B_{11} = +9.10\ B_{11} \tag{A-17}$$

$$b_{01} = (-19.70 + 8.60)\ B_{11} = -11.10\ B_{11} \tag{A-18}$$

$$b_{03} = (+\ 4.17 + 1.01)\ B_{11} = +5.18\ B_{11} \tag{A-19}$$

$$b_{05} = (-\ 2.28 + 1.10)\ B_{11} = -3.38\ B_{11} \tag{A-20}$$

A-2 Single-Stage Harmonic Generator

Let it be required to design a single-stage generator of third harmonics. Let it be further assumed that core material is available which has a maximum achievable δ value of 133. Since an odd harmonic is being generated, the value of ψ is unity from (2-42). Maximum third-harmonic current would be generated, according to (2-59) with ξ set equal to one-third. In order for ξ to be one third when ψ is unity, the value of k_1 from (2-39) is

$$k_1 = (1/2)(\psi - \xi) = (1/2)[1 - (1/3)] = 1/3 \qquad \text{(A-21)}$$

Substitution of this value of ξ into (2-37) with E's replacing λ's gives

$$E_1/E_p = \cos\ (\pi/3) = 0.5 \qquad \text{(A-22)}$$

so that

$$E_p = 2E_1$$

where E_1 is the value at the breakpoint, and is taken as unity in the per unit system. Russell[27] chose a value of 1.5 per unit for E_p in his study. This was the voltage value which produced minimum distortion in his circuits without too much loss of output energy. For comparison purposes, his value of 1.5 per unit for E_p will be used in this example.

Thus, again utilizing (2-37), $k_1\pi$ is computed at 48° and k_1 is then 0.267. From (2-39), ξ is found to be 0.467. Values of the matrices for this value of ξ, with the multiplier δ included are given below:

$$[S_B] = \begin{Vmatrix} -28.8 & 0 \\ -15.6 & -28 \end{Vmatrix} \quad ; \quad [S_A] = \begin{Vmatrix} +113 & 0 \\ +12.6 & +19.2 \end{Vmatrix}$$

From (3-57) it is seen that the output trap should be tuned so that

$$Y_{03} = +28 \quad \text{(A-23)}$$

which requires that a capacitive susceptance of +28 exist at the third-harmonic frequency. Equation (3-46) requires large admittance at the fundamental frequency and at harmonics higher than the third. Let it also be required, as a matter of engineering judgment, that this trap circuit present an admittance at the fundamental frequency such that

$$Y_{trap} \approx 20S_{A11} \quad \text{(A-24)}$$

since S_{A11} is the largest of the S values.

If $(Y_C)_3$ and $(Y_L)_3$ represent admittances at the third-harmonic frequency, and $(Y_C)_1$ and $(Y_L)_1$ the corresponding admittances at the fundamental, the following equations may be written:

$$(Y_C)_3 + (Y_L)_3 = +29.5 \quad \text{(A-25)}$$

$$|(Y_C)_1 + (Y_L)_1| = |(Y_C)_3/3 + 3(Y_L)_3| \approx |\ 20 \times 113.0| \quad \text{(A-26)}$$

Since the antiresonant circuit is capacitive above resonance and inductive below resonance, the sign of (A-6) will be negative. The parallel-resonant LC circuit must then be tuned to resonance slightly below the third-harmonic frequency.

Simultaneous solution of (A-5 and (A-6) gives values of

+823 for $(Y_C)_3$ and −851 for $(Y_L)_3$. Choice of the following slightly different values will be in the conservative direction and also will allow comparison with Russell's results[27]:

$$(Y_C)_3 = +1000 \tag{A-27}$$

$$(Y_L)_3 = -972 \tag{A-28}$$

The chosen Y values as combined in the trap network give the matrices

$$[Y_0^B] = \begin{Vmatrix} -2583 & 0 \\ & +28.0 \end{Vmatrix} \; ; \qquad [Y_0^A] = \begin{Vmatrix} +2583 & \\ 0 & -28.0 \end{Vmatrix}$$

These permit immediate computation of the matrices

$$([S_B] + [Y_0^B]) = \begin{Vmatrix} -2612 & 0 \\ -15.6 & 0 \end{Vmatrix}$$

and

$$([S_A] + [Y_0^A]) = \begin{Vmatrix} +2670 & 0 \\ +12.6 & 19 \end{Vmatrix}$$

The terms to which reference is made in (3-58) through (3-80) are then

$$\begin{aligned} S_{B11} + Y_{01} &= -2612 \\ S_{B31} &= -15.6 \\ S_{A11} - Y_{01} &= +2670 \\ S_{A31} &= +12.6 \end{aligned} \tag{A-29}$$

and so on.

Consideration will next be given to computation of the third-harmonic voltage appearing across the 0.22 per-unit-ohm resistor, and comparison made with the analog-computer values mentioned above. The trap coil used in Russell's study was assumed to have a series resistance given by

$$r_2 = 0.04\, X_{L2} \,\Big|\, \text{fundamental} \tag{A-30}$$

where the circuit and symbols are those of Figure A-1. The inherent series resistance of this coil may be converted to an equivalent parallel resistor, which would then be in parallel with the load resistor. Equations relating the series and parallel equivalent representations of this circuit are

$$R_p = (r^2 + X_s^2)/r \tag{A-31}$$

$$X_p = (r^2 + X_s^2)/X_s \tag{A-32}$$

Because of the small size of r

$$X_p \approx X_s \tag{A-33}$$

to a very good approximation, and

$$R_p \approx X_s^2/r \tag{A-34}$$

Interest in this computation is in the value of R_p at third-harmonic frequency. At the fundamental

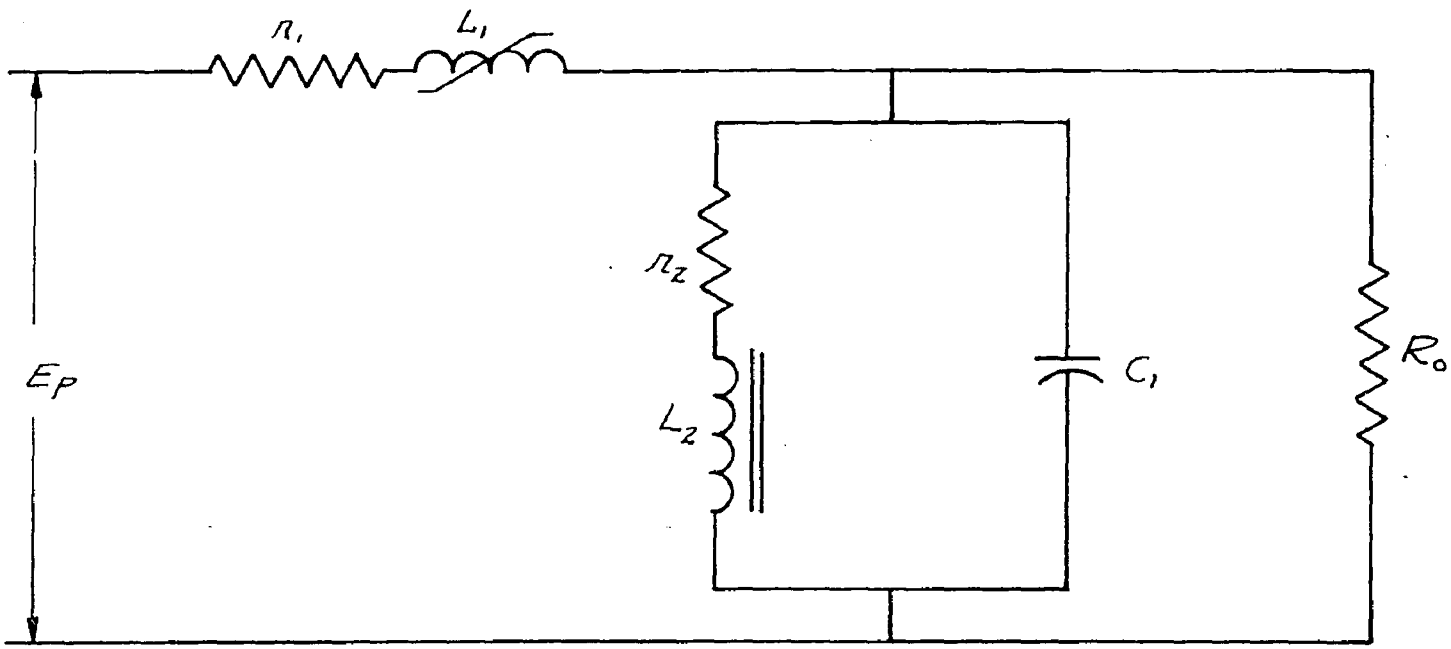

Fig. A-1 Schematic Diagram of Russell's Basic Circuit

$$X_{L2}|_{fund} = 1/3(970.5) = 0.000344 \qquad \text{(A-35)}$$

At the third harmonic

$$R_{p3} = (3X^2_{L2fund}) / 0.04X_{L2\ fund} = 9 \times 25\ X_{L2\ fund} \qquad \text{(A-36)}$$

so that R_{p3} becomes

$$R_{p3} = (225)(0.000344) = 0.0744 \qquad \text{(A-37)}$$

This value of resistance is then in parallel with any value of load resistor under consideration. It is immediately obvious that the losses in the trap coil are sufficient to keep the parallel-equivalent value of R_{O3} quite low for any value of load.

For an output resistor of 0.22 per-unit ohm, the equivalent parallel combination of trap and load resistance is 0.0556. Substitution of this value in (3-79) along with values of the inductor parameters from (A-29) gives for the required A_{1L}/B_{1L} ratio

$$A_{1L}/B_{1L} = 0.164 \qquad \text{(A-38)}$$

The effect of inherent resistance of the nonlinear inductor must next be considered. Russell's inductor had a value of 0.00176 per-unit ohm and this value is used in the computation below. A_{11} is computed by direct substitution in (3-73).

$$A_{11} = (0.164)B_{1L} + (0.00176)(-28.8)B_{1L} = 0.113B_{1L} \qquad \text{(A-39)}$$

while a similar computation for B_{11} gives

$$B_{11} = 1.0\ B_{1L} + (0.00176)(113)(0.164)\ B_{1L} = 1.02\ B_{1L} \tag{A-40}$$

The peak value of input voltage is then

$$E_p = A_{11} + j\ B_{11} = 0.113\ B_{1L} + j\ 1.02\ B_{1L} = 1.03\ B_{1L} \tag{A-41}$$

Russell's value of peak input voltage was

$$E_p = 1.5$$

For this computation, then, the value of B_{1L} is

$$B_{1L} = 1.5/1.03 = 1.457 \tag{A-42}$$

By direct substitution of values from (A-42) the voltage across the inductor in complex form is

$$A_{1L} + j\ B_{1L} = 0.165 + j\ 1.485 \tag{A-43}$$

Third-harmonic current is then computed from (3-60) and (3-64), where it must be recalled that series resistance was not considered in these equations so that A_{11} must be interpreted as A_{1L} and similarly for B_{11}.

$$a_{03} = S_{B31}B_{11} = (-15.6)(1.485) = -25.8 \text{ per-unit amperes} \tag{A-44}$$

$$b_{03} = S_{A31}A_{11} = (12.6)(0.165) = +\ 2.24 \text{ per-unit amperes} \tag{A-45}$$

The rms value of current is therefore

$$I_{03} = 16.4 \text{ per-unit amperes} \tag{A-46}$$

and the output voltage across the parallel combination of load resistor and trap coil is

$$E_{03} = (0.0556)(16.4) = 0.912 \text{ per-unit volt} \tag{A-47}$$

which compares favorably with Russell's value of 0.900.

A-3 Calculation of Subharmonic Currents

The given quantities are those of Section 3.3; viz.,

$$R = 30 \text{ ohms}$$

$$C = 100 \text{ microfarads}$$

$$f = 50 \text{ cycles per second}$$

plus the curve of Figure 3.13. The piecewise-linear approximation to the experimental curve will be used.

From the curve:

$$L_u = 0.25/0.05 = 5.0 \text{ henrys}$$

$$L_s = (0.625 - 0.45)/(1.55 - 0.09 = 0.12 \text{ henry}$$

$$\delta = L_u/L_s = 41.7$$

The basis for per-unit values, since the lowest frequency is one third the source frequency is

$$\omega_1 L_u = (6.28)(50)(5)/3 = 524 \text{ ohms}$$

From the curve

λ_1 = 0.45 weber turn corresponds to 1.0 p-u weber turn

I_1 = 0.09 ampere corresponds to 1.0 p-u ampere

E_1 = $\omega\lambda$ = 47.1 volts corresponds to 1.0 p-u volt

R = 30/524 = 0.055 p-u ohm

X_{11} = 95.5 ohms at 16-2/3 cycles/second

= 95.5/524 = 0.182 p-u ohm

X_{13} = 0.061 p-u ohm

$$[X_c^B] = \begin{Vmatrix} -0.182 & 0 \\ 0 & -0.061 \end{Vmatrix} \quad ; \quad [X_c^A] = \begin{Vmatrix} +0.182 & 0 \\ 0 & +0.061 \end{Vmatrix}$$

$$X_1 = (41.7)(0.183) = 7.60$$

$$p = 0.055/0.182 = 0.3$$

From Figure 3.11, the above value of p requires a value of ξ greater than 0.24. From Figure 3.12, the above values of p and X_1 require a value of ξ of 0.57. This value is greater than 0.24 so that the RC combination, together with an inductor capable of producing a ξ-value of 0.57 under the given excitation conditions will generate subharmonics.

For greatest accuracy the S-matrix values were computed directly from the equations of Table 2.1 for the indicated value of ξ. These values, including the absorbed δ value of 41.7, are:

$$[S_A] = \begin{Vmatrix} +30.9 & 0 \\ +5.26 & +4.8 \end{Vmatrix} \quad ; \quad [S_B] = \begin{Vmatrix} -4.96 & 0 \\ -3.38 & -7.14 \end{Vmatrix}$$

The inverted matrices are

$$[S_A]^{-1} = \begin{Vmatrix} +0.032 & 0 \\ -0.036 & +0.208 \end{Vmatrix} \quad ; \quad [S_B]^{-1} = \begin{Vmatrix} -0.202 & 0 \\ +0.096 & -0.14 \end{Vmatrix}$$

The sum of the inverted matrices and the capacitive reactance is

$$[W_{AX}] = [S_A]^{-1} + [X^B] \qquad\qquad [W_{BX}] = [S_B]^{-1} + [X^A]$$

$$= \begin{Vmatrix} -0.15 & 0 \\ -0.036 & +0.148 \end{Vmatrix} \quad ; \quad = \begin{Vmatrix} -0.02 & 0 \\ +0.096 & -0.08 \end{Vmatrix}$$

The product $[W_{AX}][W_{BX}]$ is

$$[W_{AX}][W_{BX}] = \begin{Vmatrix} +0.003 & 0 \\ +0.015 & -0.012 \end{Vmatrix}$$

so that, with R^2 taken at 0.003 per-unit ohm

$$([S]^{-1} + [X^B])([S_B]^{-1} + [X^A]) - R^2[U] = \begin{Vmatrix} 0 & 0 \\ +0.015 & -0.015 \end{Vmatrix}$$

Thus (3-124) becomes

$$\begin{Vmatrix} -0.150 & 0 \\ -0.036 & +0.148 \end{Vmatrix} \begin{Vmatrix} 0 \\ B_{13} \end{Vmatrix} - \begin{Vmatrix} 0 \\ 0.055A_{13} \end{Vmatrix} = \begin{Vmatrix} 0 & 0 \\ +0.015 & -0.015 \end{Vmatrix} \begin{Vmatrix} a_{11} \\ a_{13} \end{Vmatrix}$$

while (3-122) in matrix form is

$$0.055 \begin{Vmatrix} b_{11} \\ b_{13} \end{Vmatrix} = \begin{Vmatrix} 0 \\ B_{13} \end{Vmatrix} - \begin{Vmatrix} -0.02 & 0 \\ +0.096 & -0.08 \end{Vmatrix} \begin{Vmatrix} a_{11} \\ a_{13} \end{Vmatrix}$$

These matrix equations produce the algebraic equations

$$0.148B_{13} - 0.055A_{13} = 0.015a_{11} + 0.015a_{13}$$

$$0.055b_{11} = 0.02a_{11}$$

$$0.055b_{13} = B_{13} - 0.096a_{11} + 0.08a_{13}$$

The peak flux linkage λ_p computed from (2-39) and (2-37) for a ξ value of 0.57 with ψ set at unity is

$$\lambda_p = \lambda_1/\sin 51.3^\circ = 0.45/0.78 = 0.576$$

From Figure 3.13, corresponding to $\lambda = 0.576$

$$I_p = 1.15 \text{ amperes}$$

$$= 1.15/0.09 = 12.8 \text{ per-unit amperes}$$

Therefore

$$a_{11} + a_{13} = 12.8$$

By combining the above equations with the algebraic equations obtained from the matrix expression of (3-122) and (3-124) the following are obtained:

$$a_{11} = 5B_{13} - 1.86A_{13} + 7.83$$

$$b_{11} = 1.82B_{13} - 6.76A_{13} + 2.85$$

$$a_{13} = 1.86A_{13} - 5B_{13} + 4.97$$

$$b_{13} = 5.93 + 2.25B_{13} - 6.35$$

Next for the power-balance considerations

$$P_{in} = (1/2)(A_{13}a_{13} + B_{13}b_{13})$$

$$P_{out} = (R/2)(a_{11}^2 + a_{13}^2 + b_{11}^2 + b_{13}^2)$$

Equating input and output power and solving the resulting equations for A_{13} produces

$$A_{13} = (1/1.02)[1.64B_{13} + 1.63 \pm (0.75B_{13}^2 - 8.75B_{13} - 12.4)^{1/2}]$$

The minimum magnitude of B_{13} will occur when the quantity to the one-half power is zero. It must be zero or larger since A_{13} must be real. Setting the aforementioned quantity equal to zero and solving gives the minimum value for B_{13}.

This value is either

$$B_{13} = 12.9 \text{ or } -1.245$$

The first quantity is ridiculously large, and B_{13} is permitted to be negative, hence the second value is taken for B_{13}.

From the previous equation

$$A_{13} = -0.4$$

Substitution of these values into the equations for the a's and b's gives

$$a_{11} = 2.34 \text{ p-u amperes}$$

$$a_{13} = 10.46 \text{ p-u amperes}$$

$$b_{11} = 0.85 \text{ p-u amperes}$$

$$b_{13} = -11.32 \text{ p-u amperes}$$

This gives the corresponding rms per-unit values

$$I_{11} = 1.76 \text{ p-u amperes}$$

$$I_{13} = 10.9 \text{ p-u amperes}$$

Multiplication by 0.09 gives the actual current

$$I_{11} = 0.159 \text{ ampere}$$

$$I_{13} = 0.98 \text{ ampere}$$

Rms source voltage in per-unit volts is computed at

$$E = 0.927 \text{ p-u volt}$$

Actual voltage is obtained by multiplying this value by the normalizing factor 47.1, which gives

$$E_{13} = 43.6 \text{ volts}$$

APPENDIX B

TABLES

On the following pages tables are presented which give values of the S_A and S_B matrices for $\psi = 1.0$, which is the case of symmetrical excitation of the core. Values of the matrices are given through the tenth harmonic, with ξ advancing in increments of 0.1.

Tables of values encompassing a greater range of values of ξ and ψ may be obtained from the Numerical Analysis Laboratory, University of Arizona.

ψ = 1.0 S_A Values ξ = 0.1

m \ n	1	2	3	4	5	6	7	8	9	10
1	+.99836									
2	.00000	+.40323								
3	+.00160	.00000	+.32861							
4	.00000	+.04605	.00000	+.20608						
5	-.00154	.00000	+.00454	.00000	+.19273					
6	.00000	-.02821	.00000	+.02700	.00000	+.14159				
7	+.00145	.00000	-.00427	.00000	+.00684	.00000	+.13383			
8	.00000	+.01869	.00000	-.01800	.00000	+.01689	.00000	+.10958		
9	-.00134	.00000	+.00393	.00000	-.00631	.00000	+.00833	.00000	+.10121	
10	.00000	-.01261	.00000	+.01226	.00000	-.01169	.00000	+.01093	.00000	+.09000

S_B Values

m										
1	-.80164									
2	.00000	-.49677								
3	-.06397	.00000	-.27139							
4	.00000	-.00313	.00000	-.24392						
5	+.03588	.00000	-.03481	.00000	-.16727					
6	.00000	+.00298	.00000	-.00578	.00000	-.15841				
7	-.02307	.00000	+.02246	.00000	-.02126	.00000	-.12332			
8	.00000	-.00277	.00000	+.00539	.00000	-.00770	.00000	-.11542		
9	+.01548	.00000	-.01514	.00000	+.01448	.00000	-.01353	.00000	-.09879	
10	.00000	+.00252	.00000	-.00491	.00000	+.00702	.00000	-.00874	.00000	-.09000

$\psi = 1.0$ — **S_A Values** — $\xi = 0.2$

m \ n	1	2	3	4	5	6	7	8	9	10
1	+.98710									
2	.00000	+.32432								
3	+.01191	.00000	+.30030							
4	.00000	+.07200	.00000	+.18831						
5	-.01009	.00000	+.02806	.00000	+.16000					
6	.00000	-.03302	.00000	+.03118	.00000	+.13853				
7	+.00773	.00000	-.02162	.00000	+.03118	.00000	+.10811			
8	.00000	+.01261	.00000	-.01502	.00000	+.01798	.00000	+.10473		
9	-.00520	.00000	+.01468	.00000	-.02162	.00000	+.02499	.00000	+.08658	
10	.00000	-.00156	.00000	+.00577	.00000	-.01135	.00000	+.01663	.00000	+.08000

S_B Values

m \ n	1	2	3	4	5	6	7	8	9	10
1	-.61290									
2	.00000	-.47568								
3	-.11282	.00000	-.23303							
4	.00000	-.02155	.00000	-.21169						
5	+.05046	.00000	-.04677	.00000	-.16000					
6	.00000	+.01743	.00000	-.03118	.00000	-.12814				
7	-.02110	.00000	+.02162	.00000	-.02227	.00000	-.12046			
8	.00000	-.01261	.00000	+.02282	.00000	-.02879	.00000	-.09527		
9	+.00520	.00000	-.00775	.00000	+.01201	.00000	-.01658	.00000	-.09120	
10	.00000	+.00780	.00000	-.01442	.00000	+.01892	.00000	-.02079	.00000	-.08000

$\Psi = 1.0$ S_A Values $\xi = 0.3$

m \ n	1	2	3	4	5	6	7	8	9	10
1	+.95752									
2	.00000	+.27432								
3	+.03538	.00000	+.24426							
4	.00000	+.07258	.00000	+.18669						
5	-.02372	.00000	+.06086	.00000	+.12727					
6	.00000	-.01743	.00000	+.03231	.00000	+.12186				
7	+.01137	.00000	-.03072	.00000	+.04124	.00000	+.10201			
8	.00000	-.00386	.00000	-.01502	.00000	+.03395	.00000	+.08277		
9	-.00188	.00000	+.00711	.00000	-.01526	.00000	+.02441	.00000	+.08096	
10	.00000	+.00780	.00000	+.00468	.00000	-.01892	.00000	+.02861	.00000	+.07000

S_B Values

m										
1	-.44248									
2	.00000	-.42568								
3	-.13629	.00000	-.22240							
4	.00000	-.05618	.00000	-.16331						
5	+.03683	.00000	-.04215	.00000	-.15273					
6	.00000	+.03302	.00000	-.05353	.00000	-.11147				
7	+.00200	.00000	+.01253	.00000	-.03233	.00000	-.09799			
8	.00000	-.01206	.00000	+.02282	.00000	-.03043	.00000	-.09223		
9	-.01227	.00000	-.00018	.00000	+.01838	.00000	-.03282	.00000	-.07460	
10	.00000	-.00156	.00000	-.00187	.00000	+.01135	.00000	-.02289	.00000	-.07000

$\psi = 1.0$ S_A Values $\xi = 0.4$

m \ n	1	2	3	4	5	6	7	8	9	10
1	+.90273									
2	.00000	+.25323								
3	+.06973	.00000	+.17921							
4	.00000	+.06009	.00000	+.16892						
5	-.03118	.00000	+.07568	.00000	+.12000					
6	.00000	-.00298	.00000	+.05046	.00000	+.09159				
7	+.00190	.00000	-.01336	.00000	+.03604	.00000	+.08953			
8	.00000	-.00780	.00000	-.01800	.00000	+.04118	.00000	+.07792		
9	+.00841	.00000	-.01254	.00000	-.00742	.00000	+.03624	.00000	+.06293	
10	.00000	+.00252	.00000	-.00356	.00000	-.00702	.00000	+.02691	.00000	+.06000

S_B Values

m \ n	1	2	3	4	5	6	7	8	9	10
1	-.29727									
2	.00000	-.34677								
3	-.13209	.00000	-.22079							
4	.00000	-.09127	.00000	-.13108						
5	+.00624	.00000	-.04541	.00000	-.12000					
6	.00000	+.02821	.00000	-.05046	.00000	-.10841				
7	+.01972	.00000	+.01336	.00000	-.05046	.00000	-.08190			
8	.00000	+.00780	.00000	+.00539	.00000	-.03450	.00000	-.07208		
9	-.00841	.00000	+.00132	.00000	+.01336	.00000	-.03104	.00000	-.07040	
10	.00000	-.01261	.00000	+.00891	.00000	+.01169	.00000	-.03364	.00000	-.06000

$\psi = 1.0$ **S_A Values** $\xi = 0.5$

m \ n	1	2	3	4	5	6	7	8	9	10
1	+.81831									
2	.00000	+.25000								
3	+.10610	.00000	+.13130							
4	.00000	+.05305	.00000	+.12500						
5	-.02122	.00000	+.06366	.00000	+.11273					
6	.00000	.00000	.00000	+.06366	.00000	+.08333				
7	-.01516	.00000	+.00909	.00000	+.04547	.00000	+.06493			
8	.00000	-.00531	.00000	.00000	.00000	+.03410	.00000	+.06250		
9	+.00707	.00000	-.01179	.00000	-.00505	.00000	+.03537	.00000	+.05949	
10	.00000	.00000	.00000	-.01516	.00000	.00000	.00000	+.03537	.00000	+.05000

S_B Values

m \ n	1	2	3	4	5	6	7	8	9	10
1	-.18169									
2	.00000	-.25000								
3	-.10610	.00000	-.20203							
4	.00000	-.10610	.00000	-.12500						
5	-.02122	.00000	-.06366	.00000	-.08727					
6	.00000	.00000	.00000	-.04244	.00000	-.08333				
7	+.01516	.00000	+.00909	.00000	-.04547	.00000	-.07792			
8	.00000	+.02122	.00000	.00000	.00000	-.04547	.00000	-.06250		
9	+.00707	.00000	+.01179	.00000	-.00505	.00000	-.03537	.00000	-.05163	
10	.00000	.00000	.00000	+.00606	.00000	.00000	.00000	-.02829	.00000	-.05000

ψ = 1.0 $\qquad$ S_A Values $\qquad$ ξ = 0.6

m \ n	1	2	3	4	5	6	7	8	9	10
1	+.70273									
2	.00000	+.24677								
3	+.13209	.00000	+.11254							
4	.00000	+.06009	.00000	+.08108						
5	+.00624	.00000	+.04541	.00000	+.08000					
6	.00000	+.00298	.00000	+.05046	.00000	+.07508				
7	-.01972	.00000	+.01336	.00000	+.05046	.00000	+.06096			
8	.00000	-.00780	.00000	+.01800	.00000	+.04118	.00000	+.04708		
9	-.00841	.00000	-.00132	.00000	+.01336	.00000	+.03104	.00000	+.04071	
10	.00000	-.00252	.00000	-.00356	.00000	+.00702	.00000	+.02691	.00000	+.04000

S_B Values

m										
1	-.09727									
2	.00000	-.15323								
3	-.06973	.00000	-.15412							
4	.00000	-.09127	.00000	-.11892						
5	-.03118	.00000	-.07568	.00000	-.08000					
6	.00000	-.02821	.00000	-.05046	.00000	-.05826				
7	-.00190	.00000	-.01336	.00000	-.03604	.00000	-.05332			
8	.00000	+.00780	.00000	-.00539	.00000	-.03450	.00000	-.05292		
9	+.00841	.00000	+.01254	.00000	-.00742	.00000	-.03624	.00000	-.04818	
10	.00000	+.01261	.00000	+.00891	.00000	-.01169	.00000	-.03364	.00000	-.04000

$\psi = 1.0$ S_A Values $\xi = 0.7$

m \ n	1	2	3	4	5	6	7	8	9	10
1	+.55752									
2	.00000	+.22568								
3	+.13629	.00000	+.11093							
4	.00000	+.07258	.00000	+.06331						
5	+.03683	.00000	+.04215	.00000	+.04727					
6	.00000	+.01743	.00000	+.03231	.00000	+.04480				
7	-.00200	.00000	+.01253	.00000	+.03233	.00000	+.04486			
8	.00000	-.00386	.00000	+.01502	.00000	+.03395	.00000	+.04223		
9	-.01227	.00000	+.00018	.00000	+.01838	.00000	+.03282	.00000	+.03651	
10	.00000	-.00780	.00000	+.00468	.00000	+.01892	.00000	+.02861	.00000	+.03000

S_B Values

m										
1	-.04248									
2	.00000	-.07432								
3	-.03538	.00000	-.08907							
4	.00000	-.05618	.00000	-.08669						
5	-.02372	.00000	-.06086	.00000	-.07273					
6	.00000	-.03302	.00000	-.05353	.00000	-.05520				
7	-.01137	.00000	-.03072	.00000	-.04124	.00000	-.04085			
8	.00000	-.01206	.00000	-.02282	.00000	-.03043	.00000	-.03277		
9	-.00188	.00000	-.00711	.00000	-.01526	.00000	-.02441	.00000	-.03015	
10	.00000	+.00156	.00000	-.00187	.00000	-.01135	.00000	-.02289	.00000	-.03000

$\psi = 1.0$ $\quad$ S_A Values $\quad$ $\xi = 0.8$

m \ n	1	2	3	4	5	6	7	8	9	10
1	+.38710									
2	.00000	+.17568								
3	+.11282	.00000	+.10030							
4	.00000	+.07200	.00000	+.06169						
5	+.05046	.00000	+.04677	.00000	+.04000					
6	.00000	+.03302	.00000	+.03118	.00000	+.02814				
7	+.02110	.00000	+.02162	.00000	+.02227	.00000	+.02239			
8	.00000	+.01261	.00000	+.01502	.00000	+.01798	.00000	+.02027		
9	+.00520	.00000	+.00775	.00000	+.01201	.00000	+.01658	.00000	+.01991	
10	.00000	+.00156	.00000	+.00577	.00000	+.01135	.00000	+.01663	.00000	+.02000

S_B Values

m										
1	-.01290									
2	.00000	-.02432								
3	-.01191	.00000	-.03303							
4	.00000	-.02155	.00000	-.03831						
5	-.01009	.00000	-.02806	.00000	-.04000					
6	.00000	-.01743	.00000	-.03118	.00000	-.03835				
7	-.00773	.00000	-.02162	.00000	-.03118	.00000	-.03475			
8	.00000	-.01261	.00000	-.02282	.00000	-.02879	.00000	-.02973		
9	-.00720	.00000	-.01468	.00000	-.02162	.00000	-.02499	.00000	-.02453	
10	.00000	-.00780	.00000	-.01442	.00000	-.01892	.00000	-.02079	.00000	-.02000

$\psi = 1.0$ **S_A Values** $\xi = 0.9$

m \ n	1	2	3	4	5	6	7	8	9	10
1	+.19836									
2	.00000	+.09677								
3	+.06397	.00000	+.06195							
4	.00000	+.04605	.00000	+.04392						
5	+.03588	.00000	+.03481	.00000	+.03273					
6	.00000	+.02821	.00000	+.02700	.00000	+.02508				
7	+.02307	.00000	+.02246	.00000	+.02126	.00000	+.01954			
8	.00000	+.01869	.00000	+.01800	.00000	+.01689	.00000	+.01542		
9	+.01548	.00000	+.01514	.00000	+.01448	.00000	+.01353	.00000	+.01233	
10	.00000	+.01261	.00000	+.01226	.00000	+.01169	.00000	+.01093	.00000	+.01000

S_B Values

m	1	2	3	4	5	6	7	8	9	10
1	-.00164									
2	.00000	-.00323								
3	-.00160	.00000	-.00472							
4	.00000	-.00313	.00000	-.00608						
5	-.00154	.00000	-.00454	.00000	-.00727					
6	.00000	-.00298	.00000	-.00778	.00000	-.00826				
7	-.00145	.00000	-.00427	.00000	-.00684	.00000	-.00903			
8	.00000	-.00277	.00000	-.00539	.00000	-.00770	.00000	-.00958		
9	-.00134	.00000	-.00393	.00000	-.00631	.00000	-.00833	.00000	-.00990	
10	.00000	-.00252	.00000	-.00491	.00000	-.00702	.00000	-.00874	.00000	-.01000

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