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THE USE OF VARIATIONAL TECHNIQUES IN THE OPTIMIZATION OF FLIGHT TRAJECTORIES

by

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Thomas L. Vincent

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I hereby recommend that this dissertation prepared under my direction by Thomas L. Vincent entitled "The Use of Variational Techniques in the Optimization of Flight Trajectories" be accepted as fulfilling the dissertation requirement of the degree of Doctor of Philosophy

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ABSTRACT

General equations which describe optimum aircraft flight trajectories are obtained in this paper by the use of the methods of the calculus of variations. These general equations are specialized for minimum time, maximum range, and minimum fuel consumption flight paths and are presented in a form which lend themselves to analog computer solution. A few brachistochronic flight trajectories, obtained from the analog computer, are illustrated.

The approach and approximations used to obtain optimum flight trajectory equations and the analog computer method of solution represent the distinctive features of this investigation. An energy equation, with arc length as the independent variable, is employed as the dynamical constraint equation in place of the equations of motion customarily used. The optimizing Euler - Lagrange equations are developed on a basis which includes the effects of thrust, parasite drag, gravity, mass flow, and wind on an aircraft. Thrust is assumed to be a specified function of altitude and velocity. By neglecting induced drag and limitations on lift, the optimizing equations are significantly simplified.

Analog computer solutions to brachistochronic problems indicate that the shape of an optimum flight path is a strong function of normal forces applied to the aircraft, but only weakly dependent on the applied tangential forces.

ACKNOWLEDGMENT

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TABLE OF CONTENTS

SYMBOLS

a	constant defined on page 11
A	constant defined on page 30
B	constant defined on pages 24 and 75
C	constant defined on page 41
C_L	lift coefficient
D	drag
F	functional form defined on pages 23 and 67
F_{qi}	functional operator defined on page 24
F_t	tangential forces
g	acceleration of gravity 32.2 ft/sec^2
G	general integrand function
G_{qi}	functional operator defined on page 24
h	fixed altitude
I	general integral
k	drag constant
L	lift
m	mass
$\bar{m}$	mass parameter defined on page 23
N	normal force of constraint
P_i	functions defined on page 76
q_i	variables defined on pages 23 and 66
Q	function defined on page 68

r_j	functions defined on page 68
R	function defined on page 76
s	arc length
S	wing area, function defined on page 68
t	time
T	thrust
u	fixed velocity defined on page 37
v	inertial velocity
v_r	relative velocity
V_w	wind velocity
w	finite quantities defined on page 68
x	range coordinate
x_o	fixed range defined on page 37
y	altitude coordinate
z	lateral range coordinate
α	thrust deflection angle (see figure 2.1, page 18)
β	mass flow rate
γ	flight path angle (see figure 2.1, page 18)
ϵ	arbitrary constant (see page 68)
λ	Lagrange multiplier
Λ	function defined on page 68
μ	functions defined on page 68
ξ	variable defined on page 21
π	constant = 3.1415 . . .

ρ	atmospheric density
ϕ	constraint equation (see pages 66 and 78)
Φ	functional form relating inertial and relative velocities
ψ	functional form defined on page 28
ω	functional form defined on page 28

SUBSCRIPTS

1	initial conditions
2	final conditions

A prime denotes differentiation with respect to the independent variable (see pages 10 and 15).

A dot denotes differentiation with respect to time.

CHAPTER I

INTRODUCTION

To varying degrees, man can regulate his environment. When he is able to manipulate certain situations to his profit, he usually strives to maximize this profit. For example, the old practice of rewarding exceptional military or civil achievement with a land grant, which could be completely enclosed by a ploughed furrow in a specified time, left the receiver of the grant with the problem of finding the form of the furrow that would contain the greatest possible area. A more recent situation is the pilot who tries to control his aircraft so that it arrives at a given objective using a minimum amount of fuel.

Formulated mathematically, problems of this nature can usually be classified into two categories. The first category contains direct problems involving attainment of a maximum or minimum of some function in which case the solution is obtained directly using the elementary calculus theory of maxima and minima. The second category contains indirect problems, where a function is to be determined which in turn maximizes or minimizes some functional form. The solutions in this latter case are obtained by using variational methods.

The impulse to solve the indirect type of problem mathematically was initiated by John Bernoulli when he propounded his famous brachistochrone problem in 1696.

Since that time James Bernoulli, Euler, Lagrange, Legendre, Jacobi, and Weierstrass have contributed to the development of methods of solution which today go under the name of the Calculus of Variations.

Prior to the last few decades, the main role played by the variational calculus has been in the reformulation of many of the fundamental mathematical models of nature. This reformulation was based on an apparent economy in nature. A famous example is Hamilton's formulation of the equations of mechanics. Finding the shortest distance between two points on a sphere and determining a curve of given length which will enclose the maximum area are typical of geometric problems which were also solved using the variational calculus. The calculus of variations was seldom applied to problems directly related to engineering interest. However, Newton's problem of finding a solid of revolution with the least resistance to motion and Bernoulli's brachistochrone problem represent two early engineering problems which could be solved using the calculus of variations.

Extensions of Newton's least resistance problem and Bernoulli's brachistochrone problem are of current interest and have been studied recently by Miele (1)[1], Kelley (2), and others. The problem of finding a brachistochronic (minimum time) flight between two points

[1]Numbers in parenthesis refer to the list of References.

for a glider in a dragless atmosphere is identical to the brachistochrone problem as formulated by Bernoulli. This problem is extended to an airplane by including the effects of thrust and drag.

Brachistochronic and other optimum maneuvers for high performance aircraft determined by the use of the calculus of variations are of current interest since the old performance methods based on steady flight are too restrictive. This was recognized with the advent of the jet interceptor. For this aircraft the time and fuel used to accelerate up to airspeed is not negligible and necessitates that the acceleration of the airplane be accounted for in performance analysis.

Previous Investigations

General problems of the calculus of variations may be formulated as the problems of Bolza, Mayer, and Lagrange (see reference 3). The problem of Lagrange (the one usually discussed in elementary texts on the calculus of variations) and Mayer are particular cases of the more general problem of Bolza.

Among the first individuals to analyze aircraft performance problems by the use of the calculus of variations were Hestenes (4) and Garfinkel (5) in 1951. Hestenes set up minimum time problems with a variety of constraints using the general Bolza formulation. He obtained the optimizing set of Euler - Lagrange equations but did not solve them. Garfinkel set up optimum path problems using the Mayer formulation, but omitted the acceleration terms in the equations of motion. The Mayer formulation is now used by most investigators who apply the calculus of variations to optimize aircraft performance.

In the numerous papers which have been published since the works of Hestenes and Garfinkel, complete analytical solutions to only a few simple problems are found. The set of nonlinear optimizing Euler - Lagrange equations plus the equations of constraint, for the more complicated problems, are too difficult to solve by analytical means. Except for some restricted problems, numerical integration must be used to solve the optimizing differential equations. Numerical integration is difficult to apply however, since boundary conditions for the optimizing equations are mixed. The boundary conditions are specified partly at the initial point and partly at the final point. A complicated iterative process must be used to determine unknown initial conditions for some of the variables. The initial conditions are determined when specified final conditions are satisfied.

As a result of the difficulty associated with the integration of the optimizing equations obtained using the calculus of variations approach, numerical techniques for formulating and solving optimum aircraft performance problems have been developed. There are two techniques which are in active use today: The method of gradients and the method of dynamic programming. In the method of gradients the mixed boundary value problem is avoided.

> One commences with an initial solution which satisfies the boundary conditions and which has certain adjustable parameters depicting the functional forms of the control variables of the problem. These parameters are then adjusted, consonant with the boundary conditions, in directions which effect the greatest change in the payoff function, [the quantity to be optimized] that is, along the direction of the steepest descent. The criteria for adjusting the control variable parameters are updated from the subsequent solution, and the whole process is repeated until convergence is attained.*

The two point boundary value problem does not exist when dynamic programming is used since the boundary conditions are treated as constraints on the initial and final stages of the process.

> In this method the continuous process is reduced to a set of discrete processes or stages. . . . At each stage a search is conducted over the discrete control vector space of that stage in order to isolate the optimal policy for a given state subject to the fulfillment of the optimality principle for the subsequent stages. . . . The principle of optimality states that 'an optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.'†

A researcher investigating optimum aircraft performance has the choice of using either the new numerical methods especially suited for high speed computers or the traditional calculus of variations.

There are few published results on analog computer solutions to the optimizing Euler - Lagrange equations obtained using the calculus of variations. Analog computer solutions were investigated at

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*William G. Melbourne, "Space Flight Optimization," Celestial Mechanics and Space Flight Analysis, NASA SP-15, December 1962, p. 32.

†*Ibid.*

at Rand by Mengel (6) in 1951. Serious difficulties were encountered with the trial and error adjustment of the initial values of certain variables to meet specified end conditions. This adjustment is made very difficult by the lack of physical significance of the Lagrange multiplier variables used in the analysis.

Mengel studies two examples, the maximum range problem and the minimum time in climb problem. He first tries to obtain the maximum range for an aircraft accounting for the thrust and drag of the aircraft, but concludes that an analog computer solution of the optimizing Euler - Lagrange equations is not feasible. Instead of solving the Euler - Lagrange equations, he solves the equations of motion on the analog computer with an assumed flight path angle program. The flight path angle program is varied by trial and error until a maximum range is obtained.

Mengel then sets up the maximum range program for an aircraft without thrust. The final range determined by the analog computer becomes so sensitive to initial conditions that these conditions can not be determine accurately. In order to find the initial conditions exactly, he uses the best initial conditions he can determine from the analog computer solutions to the optimizing equations and substitutes these initial conditions along with an angle of attack program into the equations of motion. He then solves the equations of motion on the analog computer. The angle of attack program is varied by trial and error until a maximum range is obtained. In spite of these computer difficulties, he found that his approximate method could give maximum ranges within 0.2 per cent error of "exact" solutions obtained by means of the digital computers. The difference in computational time was appreciable however, being only minutes for the analog computer compared to hours for the digital computers.

The minimum time to climb problem solved by Mengel has been investigated by Kelley (7) and Miele (8) and they have obtained an analytical solution. It is interesting to note that by solving the general optimizing equations on the analog computer, Mengel discovered the discontinuous nature of the solution to this problem.

Present Investigation

Since solutions obtained using the methods of the calculus of variations remain close to the physics of a problem, improved or new techniques for integrating the system of Euler - Lagrange equations are of great interest. The assumptions and methods used to set up optimization problems greatly effect the complexity of the resulting Euler - Lagrange equations and it is important to pinpoint those assumptions which will simplify integration of the Euler - Lagrange equations without making the results unnecessarily restricted.

The objective of this study is to develop a fundamental and sufficiently simple approach to flight path optimizations, so that those optimizing Euler - Lagrange equations which can not be integrated analytically can at least be tractably integrated by means of an analog computer. The present investigation is applied to a study of optimum flight path trajectories for an aircraft in the atmosphere over a flat earth.

The Euler - Lagrange equations can be expressed in several forms. Miele (9), Leitmann (10) and other authors have obtained these equations using time as the independent variable. Kelley (7) favored using an unspecified parameter in his derivation of a general set of Euler - Lagrange equations but eventually chooses time for the independent variable when he discusses applications. Since this study is concerned with flight paths it was decided that time should be eliminated from the formulation and instead, the flight path length be used as the independent parameter. The constraint equations are conveniently expressed in terms of the flight path length.

Both the problem of Lagrange and the problem of Mayer are mathematically equivalent and it would be difficult to say that there is any distinct mathematical advantage in using the Lagrangian formulation over the Mayer formulation usually used. However, many problems can be formulated more naturally in the Lagrangian form and in some cases the introduction of new variable can be avoided by using the Lagrangian approach. For this reason the Lagrangian formulation is used here.

The dynamical subsidiary conditions are expressed in terms of an energy equation rather than the usual procedure of using the

the equations of motion. The use of an energy equation proves to be advantageous in recognizing the importance of certain assumptions.

The development of the basic equations used here will draw from the variational calculus as set forth by Bliss (3), Courant (11), and Forsyth (12) which has been updated for flight dynamics problems by Hestenes (4), Garfinkel (5), Miele (13), and others. The important deviations or similarities have been or will be pointed out.

CHAPTER II

DEVELOPMENT OF OPTIMIZING EQUATIONS

The Brachistochrone

The parallel between Bernoulli's brachistochrone problem and optimum airplane trajectories, allows the former to be used as a convenient starting point for a study of more complicated problems of interest. Bernoulli's brachistochrone problem may be stated as follows: a bead slides in a vertical plane from rest along a frictionless wire joining two points A and B. Find the shape of the wire so that the time from A to B is a minimum. The solution is easily obtained by following the usual procedure used in texts devoted to a study of the calculus of variations (14).

Taking the upper point A as the origin of coordinates and measuring y positive downward, the velocity v at a depth y is $\sqrt{2gy}$ and the time t from A to B is given by

$$t = \int_{s_1}^{s_2} \frac{ds}{v} = \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \sqrt{\frac{1+y'^2}{y}}\, dx \qquad (2.1)$$

where the prime denotes differentiation with respect to x. The arc length s has the value s_1 at A and the value s_2 at B. The range coordinate x has the value x_1 at A and the value x_2 at B. A necessary condition for t to obtain a minimum value is given by the Euler -

Lagrange equation. (See Appendix A.)

$$\frac{\partial}{\partial y}\sqrt{\frac{1+y'^2}{y}} - \frac{d}{dx}\frac{\partial}{\partial y'}\sqrt{\frac{1+y'^2}{y}} = 0 \tag{2.2}$$

The solution to this differential equation results in the following parametric equations which in turn describe the shape of the wire.

$$y = a(1+\cos 2\gamma) \tag{2.3}$$

$$x = a(\pi-2\gamma-\sin 2\gamma) \tag{2.4}$$

The constant a is determined from the location of the final point B. The parameter γ is the angle the wire makes with the horizontal. Equations (2.3) and (2.4) are the parametric equations of a cycloid. The cycloid is generated by tracing a point on a circle of radius a as the circle is rolled along the x - axis. The brachistochrone has the interesting feature that there are an infinite number of endpoints satisfying equations (2.3) and (2.4). Not only is the whole cycloid curve brachistochronic but every segment of it is also. This occurs because the velocity of the bead is a function of height only and not dependent on the path the bead takes. Five boundary conditions are necessary and sufficient to determine a complete solution. Three conditions have already been used in obtaining equations (2.3) and (2.4) (the coordinates of the initial point $x_1=y_1=0$ and the initial velocity $v_1=0$) and two more are needed as already noted (the coordinates of the final point x_2 and y_2). Neither the initial path angle γ nor the final velocity can be specified in addition to x_1, y_1, v_1, x_2, and y_2 without overconstraining the problem. If the initial flight path angle were specified, as might be the case for the equivalent problem of a glider in a dragless atmosphere, then one of the other conditions (perhaps x_2 or y_2) would have to be relinquished from specification.

<u>The Brachistochrone in a Resisting Medium</u>

The above method for setting up Bernoulli's brachistochrone problem has been used for years. However, this straight forward approach needs revision when the problem is generalized. For example, suppose the wire is not frictionless or perhaps it is immersed into a resisting medium or both. The relationship between the velocity and the coordinates can no longer be substituted directly into the integral as was done before. With dissipation, the kinetic plus potential energy are not constant. The velocity is no longer a simple function of height and must be obtained by integrating the equations of motion. Since the equations of motion are not integrable unless the path is known, the equations of motion themselves must be introduced into the formulation as constraints. This is accomplished by using Lagrange multipliers as discussed in Appendix A. The extended brachistochrone problem in a resisting medium may be stated as follows: minimize the integral,

$$t = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{v}\,dx \tag{2.5}$$

subject to the constraints (see figure 1 with T=0 and L=N)

$$m\dot{v} = \Sigma F_t \tag{2.6}$$

$$mv\dot{\gamma} = -mg\cos\gamma + N \tag{2.7}$$

$$y' = \tan\gamma \tag{2.8}$$

where m is the mass of the bead, the dot represents differentiation with respect to time, ΣF_t is the sum of the applied tangential forces (including gravity) and N is the normal force of constraint applied by the wire. The flight path angle explicitly contained in equation (2.7) is related to the other variables by means of equation (2.8). Equation (2.8) must be included as an additional constraint equation in order to avoid increasing the number of degrees of freedom. Equations (2.5) through (2.8) have derivatives with respect to two different variables (x and t). Since time appears as the "natural" independent variable in the equations of motion and since the independent variable used in the integral is quite flexible, time is often chosen as the independent variable. If time is used as the independent parameter, the above problem may be restated as follows: minimize the integral

$$t = \int_{t_1}^{t_2} dt \tag{2.9}$$

subject to the constraints

$$m\dot{v} = \Sigma F_t \tag{2.10}$$

$$mv\dot{\gamma} = -mg\cos\gamma + N \tag{2.11}$$

$$\dot{x} = v\cos\gamma \tag{2.12}$$

$$\dot{y} = v\sin\gamma \tag{2.13}$$

The last two constraints are now needed in order to define x and y. If the drag force is, at most, a function of all the dependent variables introduced so far (x,y,v,γ,N), then no more constraint equations are needed.

Introducing friction into the problem resulted in requiring four subsidiary conditions in addition to the integral. Instead of one Euler - Lagrange equation there are now five (one for each dependent variable) which must be solved together with the four equations of constraint. This gives a sufficient number of equations so that all of the dependent variables plus the Lagrange multiplier variables (one for each equation of constraint) can be solved as functions of time. From the solutions $x=x(t)$ and $y=y(t)$ the brachistochronic path between two points is obtained. The above method used to write the constraint equations is quite common among most authors (1,7,10,15) dealing with problems of this type.

The constraint equations can be written in a more compact form which reduces the number of Lagrange multiplier variables that are needed. Rather than write the independent equations of motion, equations (2.10) and (2.11), the dynamics of the bead can be conveniently expressed in terms of a single differential energy equation. (In three dimensions, a single energy equation may replace three equations of motion.) A change in kinetic energy is equal to the work done. Thus

$$mvdv = \Sigma F_t ds \tag{2.14}$$

In this case the arc length s appears as a "natural" independent variable and the problem may be restated in the following more compact form. Minimize the integral,

$$t = \int_{S_1}^{S_2} \frac{ds}{v} \tag{2.15}$$

subject to the constraints

$$mvdv = \Sigma F_t ds \tag{2.16}$$

$$x'^2 + y'^2 = 1 \tag{2.17}$$

where the prime, from now on, represents differentiation with respect to arc length. The second constraint is needed to define x and y. It replaces equations (2.12) and (2.13) in the previous formulation.

As long as F_t is only a function of the variables explicitly contained in equations (2.15 - 2.17), no further subsidiary conditions are needed. For example, suppose the bead experiences v^2 drag only, then the constraint equation (2.16) may be written as

$$mvdv = -mgdy-kv^2ds \tag{2.18}$$

The form of equation (2.18) suggests that this constraint equation could be introduced into the problem by the usual methods for handling non-holonomic constraints. This method has been used with success in dynamics problems and is discussed in Classical Dynamics books such as Goldstein (16), and Whittaker (17). However, this method is not completely general as explained in Appendix B, and can not be applied to problems with dissipative forces. The more general methods of Lagrange multipliers employed in Appendix A must be used. Because of this, the constraint (2.18) is written as follows,

$$mvv' = -mgy'-kv^2 \tag{2.19}$$

Consider a glider with drag proportional to v^2 following a flight path and a bead with the same type of drag sliding down a wire, both along brachistochronic paths. In the case of the bead, the wire exerts a normal force N on the bead as it slides. This force is a variable constraint force which depends on the bead's velocity and the shape of the wire. The magnitude of the force is equal to the sum of the normal component of gravity plus the normal inertia force. In the case of the glider, the normal force is replaced by a lift force created by the glider itself. The lift may be considered as a constraint force required to fly the optimum path. The two problems are identical as long as the lift of the glider is arbitrary. Equations (2.15 - 2.17) may be used to set up either problem as long as the drag is not a function of any new variables. The lift force (or normal force) does not enter into the formulation of the problem since it does no work. Once the optimum path is determined, the lift (or normal) force can be found separately from the normal equations of motion. The above observations are essential to the formulation of tractable optimizing equations.

<u>Airplane Flight Trajectories</u>

The following assumptions will be used in the analysis of flight trajectories.

1. The earth is flat with a constant acceleration of gravity.
2. The aircraft may be treated as a particle.
3. All motion is confined to a vertical plane.
4. Thrust is a specified function of velocity and altitude.

If the flight remains within the earth's atmosphere, the first assumption is valid. The second assumption implies that the angular accelerations about the airplane's axis are either not important or very small. This assumption is invalid if quite violent maneuvers are required to fly the optimum path. The third assumption will not be restrictive as long as some lateral condition is not imposed. (Maximum lateral range, for example). The fourth assumption is based on the fact that most aircraft fly according to some rated thrust program dictated by the engine manufacture. The take off and climb (an unsteady portion of the flight path of interest here) is usually performed at some set percentage of maximum thrust. If the thrust were not specified, then it would be optimized along with the flight trajectory. This adds to the complexity of the analysis. It is common in performance calculations to make an assumption of this type (18); to either assume that thrust is given and then determine an optimum flight path or to assume the flight path is specified and then determine the optimum thrust program.

Figure 2.1 depicts the forces and velocity of the aircraft.

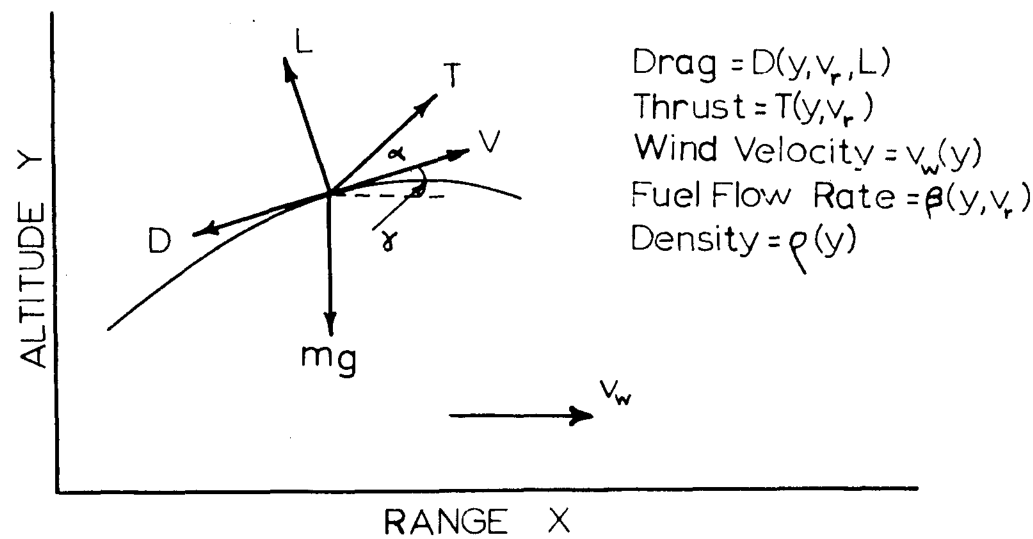

Figure 2.1

As with the brachistochrone, the energy equation and a definition of coordinates may be used as constraint equations for flight path optimization.

Energy Equation

$$mvv' + mgy' - T\cos\alpha + D = 0 \qquad (2.20)$$

Definition of Coordinates

$$x'^2 + y'^2 - 1 = 0 \qquad (2.21)$$

Additional equations are also needed.

Fuel Consumption

The mass of an aircraft is not constant and the mass flow rate may be expressed as a function of the velocity and altitude. If the mass flow rate is given by the function $\beta(y, v_r)$ then this function represents the additional constraint

$$\beta - m'v = 0 \qquad (2.22)$$

Wind Velocity

The lift, drag, thrust, and mass flow rate are functions of the relative velocity of the aircraft v_r. The relation between the inertial velocity v and the relative velocity v_r represents an additional constraint equation. If the wind velocity v_w is assumed to be in the direction of the x - axis and a function of height only, then the relation between v and v_r may be expressed in terms of the following functional form,

$$\Phi(y, v, v_r, y') = 0 \qquad (2.23)$$

Lift

If induced drag is to be included in the analysis, then lift must be determined and the resulting relation added as an additional equation of constraint. This relation is obtained from the normal equation of motion.

$$L - mv^2\gamma' - mg\cos\gamma + T\sin\alpha = 0 \tag{2.24}$$

Flight Path Angle

The flight path angle used in equation (2.24) is related to the variables. This condition is expressed by means of the following equation of constraint.

$$y' - \cos\gamma = 0 \tag{2.25}$$

Lift Coefficient

The lift developed by an aircraft is a function of the lift coefficient. The lift coefficient has definite maximum and minimum values determined from the geometry of the wings and flight conditions. In general, this requires that two more constraints must be introduced into the analysis. The first one is the definition of the lift coefficient. (The symbol S represents the planform area of the wing.)

$$C_L - \frac{L}{1/2\rho v_r^2 S} = 0 \tag{2.26}$$

The second one is an inequality constraint which expresses the limitations on the lift coefficient.

<u>Restriction on Lift Coefficient</u>

$$C_{L_{min}} \leq C_L \leq C_{L_{max}} \tag{2.27}$$

This inequality can be reduced to an equality if a new real variable ξ is introduced (19, p. 149) which is defined by the following equation.

$$(C_L - C_{L_{min}})\ (C_{L_{max}} - C_L\ ) - \xi^2 = 0 \tag{2.28}$$

There are 10 dependent variables in the above equations ($x, y, v, v_r, m, \gamma, \alpha, \xi, L$, and C_L). They are subject to eight equations of constraint. This leaves two variables which may be controlled for optimum performance. They may be, for example, the lift program $L = L(s)$ and the thrust deflection program $\alpha = \alpha(s)$. Once these variables are specified, the flight path is determined.

If there are no restrictions on the lift coefficient, then equations (2.26) and (2.28) may be omitted. If in addition, drag is not a function of lift (no induced drag) then equations (2.24) and (2.25) may be also omitted. The elimination of these four constraint equations greatly simplify the resulting analysis without an appreciable loss in generality. Induced drag is a major factor only at low speeds where high lift coefficients are used. The unconstrained lift coefficient will not introduce any error as long as the lift program needed to fly the optimum path falls within the capabilities of the aircraft. If the lift program does not remain within the vehicles capability, then the degree of divergence from these capabilities may give an indication of an area where lift coefficients should be improved.

Equations (2.24) through (2.28) will be omitted from the analysis and drag will be assumed to be independent of lift. This leaves six dependent variables $(x, y, v, v_r, m, \alpha)$ subject to four equations of constraint.

The optimum thrust deflection angle is easily determined if the integral quantity to be optimized does not contain α. The Euler - Lagrange equation α is written as follows: (See Appendix A.)

$$\lambda_1 T \sin\alpha = 0 \tag{2.29}$$

The Lagrange multiplier λ_1 can not equal zero since this would imply that the first equation of constraint is not needed. The energy equation is needed to determine the velocity of the aircraft. The thrust is arbitrary and non-zero. Thus equation (2.29) requires that the thrust be aligned with the flight path ($\sin\alpha = 0$). Eliminating equations (2.24) through (2.28) omitted the possibility of α being non-zero. Miele (1, p.77) has shown that when induced drag is considered, the angle α is non-zero and its magnitude may be easily determined. His result shows that for normal operating conditions the thrust deflection is very small.

With the assumptions listed so far, the general problem of obtaining optimum flight trajectories may be stated as follows: extremize the following integral

$$I = \int_{S_1}^{S_2} G(x, y, v, m, x', y', v', m')\,ds \tag{2.30}$$

subject to the constraints

$$\bar{m}vv' + \bar{m}gy' - T + D = 0 \tag{2.31}$$

$$x'^2 + y'^2 - 1 = 0 \tag{2.32}$$

$$\beta - m'v = 0 \tag{2.33}$$

$$\Phi(y,v,v_r,y') = 0 \tag{2.34}$$

where: $\bar{m} = m$ or $\bar{m} =$ constant, $T = T(y,v_r)$, $D = D(y,v_r)$, $\beta = \beta(y,v_r)$ By using the quantity $\bar{m}$ defined above, the mass of the aircraft may be assumed variable ($\bar{m} = m$) or the mass of the aircraft may be assumed constant ($\bar{m} = c$) without assuming that the fuel flow rate β is zero. There are five dependent variables (x,y,v,v_r,m) subject to four equations of constraint so that only one variable remains to be controlled for optimum performance. Note that this problem may be easily extended to three dimensions by adding the variable z to G and by changing the constraint equation (2.32) so that it becomes

$$x'^2 + y'^2 + z'^2 - 1 = 0 \tag{2.35}$$

Let: $$F = \lambda_1(\bar{m}vv' + \bar{m}gy' - T + D) + \lambda_2(x'^2 + y'^2 - 1) + \lambda_3(\beta - m'v) + \lambda_4 \Phi \tag{2.36}$$

and $q_i =$ the five variables (x,y,v,v_r,m) as i ranges from 1 to 5

Let: $$F_{q_i} = \frac{\partial F}{\partial q_i} - \frac{d}{ds}\frac{\partial F}{\partial q_i'} \tag{2.37}$$

and $$G_{q_i} = \frac{\partial G}{\partial q_i} - \frac{d}{ds}\frac{\partial G}{\partial q_i'} \tag{2.38}$$

Then the necessary condition that the integral I be an extremum is given by the Euler - Lagrange equations (See Appendix A).

$$G_{q_i} + F_{q_i} = 0 \tag{2.39}$$

Written out these equations become

x: $$G_x - \frac{d}{ds}(2\lambda_2 x') = 0 \tag{2.40}$$

y: $$G_y - \lambda_1\frac{\partial}{\partial y}(T-D) + \lambda_3\frac{\partial \beta}{\partial y} + \lambda_4\frac{\partial \Phi}{\partial y} - \frac{d}{ds}(\lambda_1\bar{m}g + \lambda_2 y' + \lambda_4\frac{\partial \Phi}{\partial y'}) = 0 \tag{2.41}$$

v: $$G_v + \lambda_1\bar{m}v' - \lambda_3 m' + \lambda_4\frac{\partial \Phi}{\partial v} - \frac{d}{ds}(\lambda_1\bar{m}v) = 0 \tag{2.42}$$

v_r: $$-\lambda_1\frac{\partial}{\partial v_r}(T-D) + \lambda_4\frac{\partial \Phi}{\partial v_r} + \lambda_3\frac{\partial \beta}{\partial v_r} = 0 \tag{2.43}$$

m: $$G_m + \lambda_1\frac{\partial \bar{m}}{\partial m}(vv' + gy') + \frac{d}{ds}(\lambda_3 v) = 0 \tag{2.44}$$

These equations have as their first integral (See Appendix A)

$$G - q_i'\frac{\partial G}{\partial q_i'} - q_i'\frac{\partial F}{\partial q_i'} = B \tag{2.45}$$

where B is a constant of integration.

Written out this becomes

$$G - q_i' \frac{\partial G}{\partial q_i'} - (2\lambda_2 x'^2 + 2\lambda_2 y'^2 + \lambda_1 \bar{m} g y' + \lambda_4 \frac{\partial \Phi}{\partial y'} y' + \lambda_1 \bar{m} v v' - \lambda_3 v m') = B \quad (2.46)$$

Equation (2.46) is simplified by using the constraint equations (2.31 - 2.34)

$$G - \frac{\partial G}{\partial q_i'} q_i' - 2\lambda_2 - \lambda_1 (T - D) - \lambda_4 \frac{\partial \Phi}{\partial y'} + \lambda_3 \beta = B \quad (2.47)$$

The transversality condition is given by (See Appendix A)

$$\left[B\,ds + \frac{\partial G}{\partial q_i'} dq_i + \frac{\partial F}{\partial q_i'} dq_i \right]_1^2 = 0 \quad (2.48)$$

or

$$\left[B\,ds + \frac{\partial G}{\partial q_i} dq_i + 2\lambda_2 x' dx + (\lambda_1 \bar{m} g + 2\lambda_2 y' + \lambda_4 \frac{\partial \Phi}{\partial y'}) dy + \lambda_1 \bar{m} v dv - \lambda_3 v dm \right]_1^2 = 0 \quad (2.49)$$

The five Euler - Lagrange equations (2.40) through (2.44) plus the four equations of constraint (2.31) through (2.34) contain nine unknowns ($\lambda_1, \lambda_2, \lambda_3, \lambda_4, x, y, v, v_r, m$) and all the unknowns may be found as functions of s. The first integral is not an independent equation (See Appendix A) and it may be used as a substitute for one of the Euler - Lagrange equations. The transversality condition is employed whenever some degree of freedom exists in choice of boundary values. Certain constants are evaluated in this way.

The solution to the above set of equations requires that the initial conditions be prescribed for each of the derivatives contained in equations (2.40) - (2.47). Since the Lagrange multipliers convey no apparent physical meaning, it is difficult to choose initial values for them. The elimination of any of the derivatives of the Lagrange variables aid in simplifying the solution. Equations (2.41) and (2.42) each contain the derivative of λ_1 and hence λ_1' may be eliminated by combining these two equations. The resulting loss in an Euler - Lagrange equation is regained by using the first integral expression in its place.

The variable λ_1' is eliminated between equation (2.41) and (2.42) by multiplying equation (2.41) by v, equation (2.42) by -g and adding. The resulting equation is given by (2.51) below. The variable λ_4 is expressed functionally from equation (2.43) and may be substituted into the other equations containing λ_4. If x' and y' are replaced by $\cos\gamma$, and $\sin\gamma$ in equations (2.40), (2.41), and (2.47) and if $(vv' + gy')$ replaced by m(T-D) in equation (2.44) and if m' is replaced by β/v in equation (2.42), then the optimizing equations for non-steady flight may be summarized as follows:

$$G_x - \frac{d}{ds}(2\lambda_2 \cos\gamma) = 0 \tag{2.50}$$

$$vG_y - gG_v - \lambda_1 v \frac{\partial}{\partial y}(T-D) + \frac{1}{\partial\Phi/\partial v_r}\left(v\frac{\partial\Phi}{\partial y} - g\frac{\partial\Phi}{\partial v}\right)\left[\lambda_1 \frac{\partial}{\partial v_r}(T-D) - \lambda_3 \frac{\partial\beta}{\partial v_r}\right] + \lambda_3\left(v\frac{\partial\beta}{\partial y} + g\frac{\beta}{v}\right) - v\frac{d}{ds}\left(2\lambda_2 \sin\gamma + \lambda_4 \frac{\partial\Phi}{\partial y'}\right) = 0 \tag{2.51}$$

$$G_m + \frac{\lambda_1}{m}\frac{\partial\bar{m}}{\partial m}(T-D) + \frac{d}{ds}(\lambda_3 v) = 0 \tag{2.52}$$

$$G - q_i'\frac{\partial G}{\partial q_i'} - 2\lambda_2 - \lambda_1(T-D) - \frac{1}{\partial\Phi/\partial v_r}\frac{\partial\Phi}{\partial y'}y'\left[\lambda_1\frac{\partial}{\partial v_r}(T-D) - \lambda_3\frac{\partial\beta}{\partial v_r}\right] + \lambda_3\beta = B \tag{2.53}$$

$$\left[Bds + \frac{\partial G}{\partial q_i'}dq_i + 2\lambda_2\cos\gamma dx + (\lambda_1 mg + 2\lambda_2\sin\gamma + \lambda_4\frac{\partial \Phi}{\partial y'})dy + \lambda_1\bar{m}vdv - \lambda_3 vdm\right]_1^2 = 0 \quad (2.54)$$

It will be convenient for the ensuing analysis to use a simplified relation for Φ. If the wind is parallel to the ground, then the exact relationship between the relative velocity v_r and the wind velocity is given by

$$v_r^2 = (v_w + v\cos\gamma)^2 + v^2\sin^2\gamma \quad (2.55)$$

or

$$v_r^2 = v_w^2 + v^2 + 2v_w v\cos\gamma \quad (2.56)$$

In order to simplify this expression, as a first approximation, the wind velocity is assumed small compared to the inertial velocity so that $(v_w/v)^2 = 0$. Thus

$$\frac{v_r}{v} = (1 + 2\frac{v_w}{v}\cos\gamma)^{1/2} \quad (2.57)$$

expanding equation (2.57) and dropping higher order terms involving (v_w/v) this reduces to

$$v_r = v + v_w\cos\gamma \quad (2.58)$$

As a second approximation the flight path angle can be assumed small [in equation (2.58) only] so that $\cos\gamma \doteq 1$ and the function Φ may be written as

$$\Phi = v_r - v - v_w = 0 \quad (2.59)$$

By using equation (2.59) and the following definitions

Def: $$\omega(y,v,v_r) = v\frac{\partial}{\partial y}(T-D) - g\frac{\partial}{\partial v_r}(T-D) + v\frac{\partial v_w}{\partial y}\frac{\partial}{\partial v_r}(T-D) \qquad (2.60)$$

Def: $$\psi(y,v,v_r) = g\frac{\partial \beta}{\partial v_r} - v\cdot\frac{\partial \beta}{\partial y} - g\frac{\beta}{v} - v\frac{\partial v_w}{\partial y}\frac{\partial \beta}{\partial v_r} \qquad (2.61)$$

Equations (2.50) - (2.54) may be written curtly as follows

$$G_x - \frac{d}{ds}(2\lambda_2 \cos\gamma) = 0 \qquad (2.62)$$

$$vG_y - gG_v - \lambda_1\omega - \lambda_3\psi - v\frac{d}{ds}(2\lambda_2 \sin\gamma) = 0 \qquad (2.63)$$

$$G_m + \frac{\lambda_1}{m}\frac{\partial \bar{m}}{\partial m}(T-D) + \frac{d}{ds}(\lambda_3 v) = 0 \qquad (2.64)$$

$$G - q_i\frac{\partial G}{\partial q_i'} - 2\lambda_2 - \lambda_1(T-D) + \lambda_3\beta = B \qquad (2.65)$$

$$\Big[Bds + \frac{\partial G}{\partial q_i'}dq_i + 2\lambda_2\cos\gamma\, dx + (\lambda_1\bar{m}g + 2\lambda_2\sin\gamma)dy + \lambda_1\bar{m}v\,dv - \lambda_3 v\,dm\Big]_1^2 = 0 \qquad (2.66)$$

These equations are further reduced for some specific optimum trajectories in the next chapter.

CHAPTER III

SOME OPTIMUM TRAJECTORY RELATIONS

An airplane is said to follow an optimum path if at the end of the trajectory some quantity associated with the flight is optimized. The shape of an optimum path depends on the quantity to be optimized and any conditions which may be imposed simultaneously. According to the assumptions of the last chapter, any integral quantity which contains the variables x, y, v, m and their first derivatives may be optimized. This optimization is carried out under various restrictions directly given by the constraint equations and indirectly given by the transversality condition.

It may be possible to find an optimum flight path of a given length, satisfying the constraint equations, which goes through a given initial and final point with a given initial and final mass and velocity. This represents a case in which none of the variables are "free" at their end points and the transversality condition is identically zero. If the value of certain variables are not specified at the final point or perhaps at the initial point or both, or if the final point lies along some curve, then the transversality condition will either directly or indirectly specify the best value for that variable at the terminal points. An instance is given by Bernoulli's brachistochrone problem with the final y coordinate free. The transversality condition requires that the brachistochronic curve must be perpendicular to a vertical line drawn through the final x coordinate. This condition in turn fixes the "best" final value for the y coordinate.

Solutions for the following problems will either be expressed in closed form or in a form convenient for analog computer solution. A discussion of some analog computer solutions and results will be left for the next chapter.

Minimum Time Flight Paths

The general equations on page 28 are specialized for minimum time flight paths by setting $G = \frac{1}{v}$. These equations then reduce to

$$2\lambda_2 \cos\gamma = A \tag{3.1}$$

$$\frac{g}{v^2} - \lambda_1 \omega - \lambda_3 \psi - A v \sec^2\gamma \, \gamma' = 0 \tag{3.2}$$

$$\lambda_1 \frac{\partial \bar{m}}{\partial m}(T - D) + m \frac{d}{ds}(\lambda_3 v) = 0 \tag{3.3}$$

$$\frac{1}{v} - A\sec\gamma - \lambda_1(T - D) + \lambda_3 \beta = B \tag{3.4}$$

$$\Big[B\,ds + 2\lambda_2 \cos\gamma\,dx + (\lambda_1 \bar{m} g + 2\lambda_2 \sin\gamma)dy + \lambda_1 \bar{m} v\,dv - \lambda_3 v\,dm\Big]_1^2 = 0 \tag{3.5}$$

where A is a constant of integration. Solving for λ_1 in equation (3.4) and substituting into equation (3.2) gives

$$A(T - D)\sec^2\gamma\, v\gamma' = \frac{g}{v^2}(T - D) - \omega\left[\frac{1}{v} - A\sec\gamma - B\right] - \lambda_3\left[\psi(T-D) + \beta\omega\right] \tag{3.6}$$

where $(T-D) \neq 0$. Equation (3.6) is simplified if $\lambda_3 = 0$. There are two ways to obtain this condition. If the mass flow is unimportant, equation (2.33) defining mass flow is left out of the formulation of chapter 2 by setting $\lambda_3 = 0$. Equivalently, $\lambda_3 = 0$ if the change in aircraft weight is negligible ($\bar{m}$ = constant) and the final mass of the aircraft is not specified. [From equation (3.3) $\lambda_3 v$ = constant. From the transversality condition $(\lambda_3 v)_2 = 0$. Since v is not continually zero, $\lambda_3 = 0$.] Leaving the final mass unspecified is not restrictive in this problem since a minimum time requirement should not restrict the fuel used. In fact if the final mass were specified, this condition could lead to an inconsistency. The fuel flow rate is constant in a rocket type aircraft and specifying the final mass would also specify the time of flight! If changes in airplane mass are important ($\bar{m} = m$), then the multiplier λ_3, is non-zero and can be determined from equations (3.3) and (3.4).

Four types of minimum time problems will be studied:

(1) minimum time to climb

(2) minimum time between two fixed points

(3) minimum time to intercept

(4) minimum time to range

The minimum time to climb flight path is of prime importance to an interceptor aircraft. The minimum time between two points flight path could be used by an air launched winged missile with some specified stationary ground target. The minimum time to intercept would allow an interceptor vehicle to rendezvous with an incoming missile as far away from a given base as possible. The minimum time to range flight path could be used by an attack type of aircraft which must zoom over a hostile target as quickly as possible once some border has been reached.

Minimum Time to Climb

A minimum time to climb flight path does not restrict the range flown or the length of the flight path. Because of this dx_2 and ds_2 are arbitrary and the transversality condition requires that $(2\lambda_2\cos\gamma)_2 = 0$ and $B = 0$. From equation (3.1), the first condition is discerned to be equivalent to setting $A = 0$. Thus either $2\lambda_2 = 0$ or $\cos\gamma = 0$. If $\cos\gamma = 0$ the airplane is in a vertical flight which represents a possible solution. The alternate conclusion, $\lambda_2 = 0$ implies that the corresponding constraint equation (2.32) is not needed. (Thus with $\lambda_2 = 0$, no initial or final conditions need be imposed on x.) Assuming $\lambda_2 = 0$ and substituting $A = B = 0$ into equation (3.6) results in the following minimum time to climb equation.

$$\frac{g}{V^2}(T - D) - \frac{\omega}{V} - \lambda_3\left[\psi(T - D) + \beta\omega\right] = 0 \tag{3.7}$$

If mass flow rate is important, then λ_3 may be determined by setting $\bar{m} = m$ in equation (3.3), substituting $\beta = m'v$ in equation (3.4) and adding the resulting equations.

$$\frac{d}{ds}(\lambda_3 mv) = -\frac{1}{V} \tag{3.8}$$

$$\lambda_3 = \frac{t_2 - t}{mv} \tag{3.9}$$

The constant of integration t_2 was evaluated in equation (3.9) by using the transversality condition $(\lambda_3)_2 = 0$. (Final mass is not specified.)

For a rocket type of aircraft β = constant and

$$m = m_1 + \beta t \tag{3.10}$$

$$m_2 = m_1 + \beta t_2 \tag{3.11}$$

therefore

$$t_2 - t = \frac{m_2 - m}{\beta} \tag{3.12}$$

If this result is substituted into equation (3.9) and equation (3.9) in turn substituted into equation (3.7), the latter equation reduces to (β = constant therefore $\psi = - \frac{g\beta}{V}$)

$$\frac{g}{V} (T - D) - \omega = 0 \tag{3.13}$$

This result is identical to equation (3.7) with $\lambda_3 = 0$. Using the definition for ω on page 28, equation (3.13) may be written out as follows

$$\frac{g}{V}(T - D) - V\frac{\partial}{\partial y}(T - D) + g\frac{\partial}{\partial V_r}(T - D) - V\frac{\partial V_w}{\partial y}\frac{\partial}{\partial V_r}(T - D) = 0 \tag{3.14}$$

Since T and D are functions of v_r and y only, once these functions are specified the solution to (3.14) will give the minimum time to climb velocity program as a function of height. The result is valid for either a rocket type of aircraft (β = constant) where mass changes are important ($\bar{m}$ = m) or for other types of aircraft (β variable) where mass changes are neglected ($\bar{m}$ = C). The solution to equation (3.14) is independent of any starting or ending conditions on y and v. The final velocity can not be left "open" since the transversality condition corresponding to this is oncompatible with equation (3.4). [Equation (3.4) written at the final point becomes $\frac{1}{v_2} - [\lambda_1(T-D)]_2 = 0$. The transversality condition corresponding to final velocity "open" is $(\lambda_1 mv)_2 = 0$. If $(\lambda_1)_2$ is zero then equation (3.4) can only be satisfied with infinite velocity at the final point. If $v_2 = 0$, then either (T-D) or λ_1 must be infinite at the final point.] Therefore the final velocity must be specified.

In order to fly the optimum path given by equation (3.14), the aircraft must initially be at a set of optimum conditions. At the end of the flight, the velocity is determined and can not be arbitrarily assigned. If an aircraft does not possess a set of optimum conditions, then a vertical dive or climb ($\cos\gamma = 0$) may be used to obtain them. This latter procedure has been extensively discussed by Miele (8). If the wind velocity is assumed to be zero, then equation (3.14) reduces to

$$\frac{\partial}{\partial v}\left[(T-D)v\right] = \frac{v}{g}\frac{\partial}{\partial y}\left[(T-D)v\right] \tag{3.15}$$

This equation has been obtained previously by Kelley (7, p. 65) and Miele (9, p. 29) by using the additional assumption that L = mg = constant. (This assumption replaces the normal equation of motion and requires that the flight path angle be small.) It is apparent that the $L = mg$ = constant assumption is not needed. In fact the lift force required to fly the optimum path is easily computed using the normal equation of motion (since the flight path is now known). Note that the $L = mg$ = constant assumption is completely incompatible with the vertical flight solution. This latter discrepancy has been previously pointed out by Kelley (7, p. 67).

Minimum Time Between Two Fixed Points

The optimizing equation for the minimum time between two points will be developed assuming that $\lambda_3 = 0$ (mass flow is unimportant). Since the length of the flight path is not fixed, the transversality condition requires that $B = 0$. If the final velocity is not specified, then the constant A can be determined in terms of final conditions. With $dv_2 = 0$, the transversality condition requires that $(\lambda_1 mv)_2 = 0$. This implies that either $v_2 = 0$ or $(\lambda_1)_2 = 0$. The first possibility is incompatible with equation (3.4). [Equation (3.4) written at the final point becomes $\frac{1}{v} - A\sec\gamma_2 - [\lambda_1(T-D)]_2 = 0$. With v_2 equal to zero the equation becomes indefinite.] The second possibility allows A to be evaluated from equation (3.4) as follows

$$\frac{1}{v_2} - A\sec\gamma_2 = 0 \qquad (3.16)$$

$$A = \frac{\cos\gamma 2}{v_2} \qquad (3.17)$$

Substituting $\lambda_3 = B = 0$, and equation (3.17) into equation (3.6) gives the following equation describing the minimum time path between two points.

$$(T-D)\sec^2\gamma v\gamma' = \left(\frac{V_2}{\cos\gamma_2}\right)\frac{g}{v^2}(T-D) - \left(\frac{V_2}{\cos\gamma_2}\right)\frac{\omega}{v} + \omega\sec\gamma \tag{3.18}$$

A discussion of the solution of this equation and the determination of optimum flight paths by the use of an analog computer is contained in the next chapter.

The solution to Bernoulli's brachistochrone problem may be obtained again using the more general formulation. Equation (3.6) can not be used since $T = D$. However if T-D, and λ_3 are set equal to zero in equation (3.2) ($\lambda_3 = 0$ since $\beta = 0$), the following result is obtained.

$$\frac{g}{v^2} - Av\sec^2\gamma\gamma' = 0 \tag{3.19}$$

By noting that $A = \frac{\cos\gamma}{v}$ from equation (3.4), the above equation may be written as

$$\gamma' = \frac{A^2 g}{\cos\gamma} \tag{3.20}$$

Using $\sin\gamma = y'$ and $\cos\gamma = x'$ this equation may be integrated to give

$$y = -\frac{1}{4A^2 g}(1 + \cos 2\gamma) + \text{constant} \tag{3.21}$$

and

$$x = \frac{1}{4A^2 g}(2\gamma + \sin 2\gamma) + \text{constant} \tag{3.22}$$

By setting $a = -\frac{1}{4\Lambda z_g}$ and applying the boundary conditions $v = 0$ at $x = 0$, $y = 0$, these equations reduce to those previously given on p.11. The final velocity can not be arbitrarily specified since the final velocity is determined from the geometry of the path as given by equation (3.4).

Minimum Time to Intercept

Suppose a missile traveling at a constant height h above the ground with a constant velocity u is to be intercepted in the least time. The location of the final range x_2 for both the interceptor and missile will depend on the time required to intercept the missile. If this final range can be determined, then a minimum time flight path can be computed from equation (3.18) of the previous section.

The final range is given by

$$x_2 = x_o \pm ut_2 \tag{3.23}$$

where x_o is the range of the missile when the interceptor is launched. The sign of the second term depends on the direction of the missile. The final time t_2 is to be a minimum.

If equation (3.23) is differentiated with respect to t_2,

$$dx_2 = \pm \; udt_2 \tag{3.24}$$

and dt_2 is replaced by ds_2/v_2, then equation (3.24) may be written as

$$\frac{dx_2}{ds_2} = \pm \; \frac{u}{v_2} \tag{3.25}$$

Equation (3.25) represents a condition which must be and, indeed, will be satisfied at the final point. It does not represent a constraint at the final point to be used in conjunction with the transversality condition. A minimum time flight path can not have the final time restricted.

A graphical procedure illustrated in Figure 3.1 may be used to find the minimum time to intercept flight path.

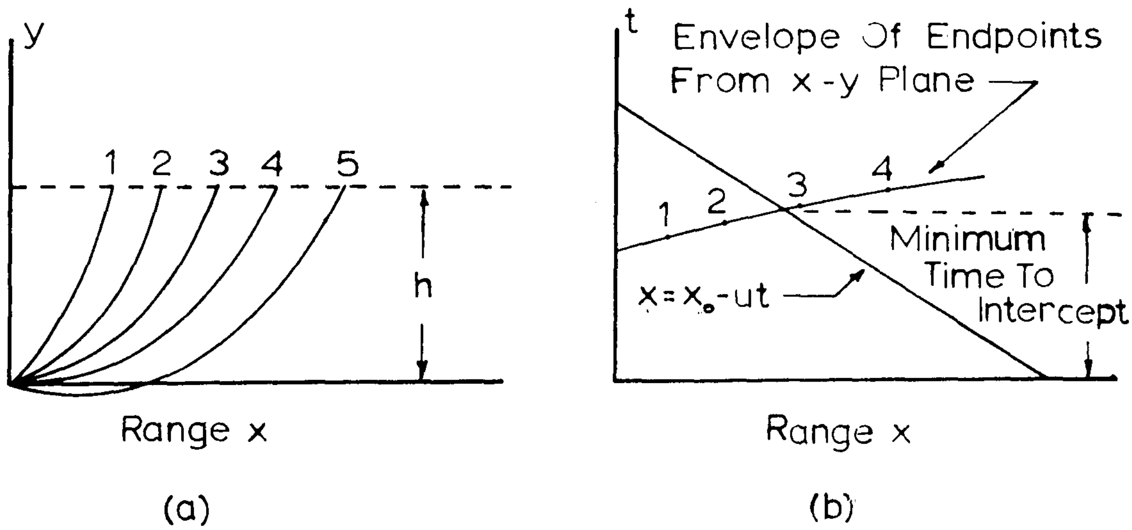

Figure 3.1

Figure 3.1(a) represents a family of optimum flight trajectories to a given height h which are calculated using equation (3.18). The envelope of the endpoints is transferred to the x - t plane shown in Figure 3.1(b). The intersection of this envelope with the time line of the missile represents the minimum intercept time. The corresponding flight path is given in the x-y plane.

Minimum Time to Range

The minimum time to range flight path has the final height unconstrained. The corresponding transversality condition is $(\lambda_1 \bar{m}g + 2\lambda_2 \sin\gamma)_2 = 0$. [$v_2 = 0$ is incompatible with equation (3.4).] Thus $\sin\gamma_2$ must be zero. [$(2\lambda_2)_2 = 0$ is incompatible with equation (3.1).] The condition $\sin\gamma_2 = 0$ requires that the aircraft be parallel to the x axis at the end of the flight. If the length of the flight path is unconstrained (B = 0) and fuel flow unimportant ($\lambda_3 = 0$), then the constant A in the optimizing equation (3.6) may be simply determined from equation (3.4) written as follows,

$$\frac{1}{v_2} - A\sec\gamma_2 = 0 \tag{3.26}$$

Since $\cos\gamma_2 = 1$, $A = 1/v_2$. Substituting $A = 1/v_2$ and $\lambda_3 = B = 0$ into equation (3.6) results in the following optimizing equation

$$(T-D)\sec^2\gamma\, v\gamma' = v_2 \frac{g}{v^2}(T-D) - \omega\left[\frac{v_2}{v} - \sec\gamma\right] \tag{3.27}$$

Maximum Range Flight Paths

When G is set equal to x' in the equations on page 28, the following relations are obtained.

$$2\lambda_2\cos\gamma = A \tag{3.28}$$

$$-\lambda_1\omega - \lambda_3\psi - Av\sec^2\gamma\,\gamma' = 0 \tag{3.29}$$

$$\lambda_1 \frac{\partial \bar{m}}{\partial m}(T - D) + m\frac{d}{ds}(\lambda_3 v) = 0 \tag{3.30}$$

$$-A\sec\gamma - \lambda_1(T - D) + \lambda_3\beta = 0 \tag{3.31}$$

$$\Big[Bds + (2\lambda_2\cos\gamma + 1)dx + (\lambda_1\bar{m}g + 2\lambda_2\sin\gamma)dy + \lambda_1\bar{m}vdv - \lambda_3 v\,dm\Big]_1^2 = 0 \tag{3.32}$$

The final value for range is unconstrained and the corresponding transversality condition requires that $(2\lambda_2\cos + 1)_2 = 0$. This result combined with equation (3.23) shows that $A = -1$. If equation (3.31) is solved for λ_1 and substituted along with $A = -1$ into equation (3.29), the following optimizing equation is obtained

$$(T - D)\sec^2\gamma\, v\gamma' = \omega\left[\sec\gamma - B\right] + \lambda_3\left[\psi(T - D) + \beta\omega\right] \tag{3.33}$$

Equation (3.33) may be further simplified for some cases of interest. Suppose a rocket type aircraft has a limited thrust duration and the range of the aircraft is to be maximized at burnout. In this case $dt_2 = ds_2/v_2$ is specified and the constant B is not zero. If mass flow may be neglected ($\lambda_3 = 0$) and if the final velocity is not specified then the constant B may be evaluated from equation (3.31). [The transversality condition for dv_2 unspecified is $(\lambda_1 mv)_2 = 0$. With thrust, v_2 will not be zero. Hence $(\lambda_1)_2 = 0$. This result combined with equation (3.31) shows that $B = \sec\gamma_2$.] The optimizing equation (3.33) in this case becomes

$$(T - D)\ \sec^2\gamma v\gamma' = \omega\,[\sec\gamma - \sec\gamma_2] \qquad (3.34)$$

As a second example, suppose that ds_2 is arbitrary (so that $B = 0$) and that $\bar{m} =$ constant. Equation (3.30) reduces to

$$\lambda_3 v = C \qquad (C = \text{constant}) \qquad (3.35)$$

If the final velocity is not specified, the multiplier λ_3 may be evaluated in terms of final conditions. The transversality condition requires that $(\lambda_1 mv)_2 = 0$. Since $v_2 = 0$ is incompatible with equation (3.35), $(\lambda_1)_2$ must be zero. Combining $(\lambda_1)_2 = 0$ with equation (3.31) gives

$$(\lambda_3)_2 = \frac{\sec\gamma_2}{\beta_2} \qquad (3.36)$$

Since $(\lambda_3)_2 = {}^{C}/v_2$ from equation (3.35), $C = v_2\sec\gamma_2/\beta_2$ and equation (3.35) may be rewritten as

$$\lambda_3 = -\ \frac{v_2\sec\gamma_2}{\beta_2 v} \qquad (3.37)$$

This equation is used in conjunction with the optimizing equation (3.33).

Minimum Fuel Consumption Flight Paths

When G is set equal to m' in the equations on page 28 they reduce to

$$2\lambda_2\cos\gamma = A \qquad (3.38)$$

$$-\lambda_1\omega - \lambda_3\psi - A\sec^2\gamma\, v\gamma' = 0 \tag{3.39}$$

$$\lambda_1\frac{\partial\bar{m}}{\partial m}(T-D) + m\frac{d}{ds}(\lambda_3 v) = 0 \tag{3.40}$$

$$-A\sec\gamma - \lambda_1(T-D) + \lambda_3\beta = B \tag{3.41}$$

$$\Big[Bds + 2\lambda_2\cos\gamma\, dx + (\lambda_1\bar{m}g + 2\lambda_2\sin\gamma)dy + \lambda_1\bar{m}v\,dv + (1-\lambda_3 v)dm\Big]_1^2 = 0 \tag{3.42}$$

Solving for λ_1 in equation (3.41) and substituting into equation (3.39) gives

$$A\sec^2\gamma(T-D)v\gamma' = \omega\Big[A\sec\gamma + B\Big] - \lambda_3\Big[(T-D)\psi + \beta\omega\Big] \tag{3.43}$$

Two simplified cases will be examined here,

(a) Minimum fuel to climb.

(b) Minimum fuel between two points.

Minimum Fuel to Climb

The first case does not restrict the range or arc length and the transversality condition requires that A = B = 0. Thus the optimizing equation becomes

$$\lambda_1\Big[(T-D)\psi + \beta\omega\Big] = 0 \tag{3.44}$$

If $\bar{m}$ is set equal to m in equation (3.40) and this equation combined with equation (3.41), the following result is obtained

$$\lambda_3 mv = \text{constant} \tag{3.45}$$

Since the final mass can not be specified, the transversality condition requires that this constant be non-zero. Consequently λ_3 is non-zero and may be divided out of eauation (3.44).

Suppose that there is zero wind, then equation (3.44) may be written as

$$(T-D)\left[g\frac{\partial \beta}{\partial v} - v\frac{\partial \beta}{\partial y} - g\frac{\beta}{v}\right] + \beta\left[v\frac{\partial}{\partial y}(T-D) - g\frac{\partial}{\partial v}(T-D)\right] = 0 \tag{3.46}$$

which is conveniently expressed as

$$\frac{\partial}{\partial v}\left[\frac{(T-D)v}{\beta}\right] = \frac{v}{g}\frac{\partial}{\partial y}\left[\frac{(T-D)v}{\beta}\right] \tag{3.47}$$

Previous investigators (8, p. 385) have obtained this equation using in addition to the assumptions used here the L = mg = constant assumption. This additional assumption is evidently not needed.

If an aircraft does not possess a set of optimum conditions (v and y) specified by equation (3.47), then a vertical dive or climb (an alternate solution deduced from the condition A = 0) may be used to obtain them. This same situation arose in the minimum time to climb flight paths.

Minimum Fuel Between Two Points

For the more general problem of minimum fuel between two points the multiplier λ_3 in equation (3.43) must be determined. Since the final mass can not be specified, the transversality condition requires that $[\lambda_3]_2 = \frac{1}{v_2}$. Combining this result with equation (3.45) gives

$$\lambda_3 = \frac{m_2}{mv} \tag{3.48}$$

If the mass flow rate is constant (β = constant), then equation (3.43) is further simplified (with β constant, $(T - D)\psi = -(T - D)\frac{g\beta}{v}$ to give

$$A\sec^2\gamma(T-D)v\gamma' = A\omega\sec\gamma - \frac{m_2}{mv}\beta\omega + \frac{m_2}{m}(T-D)\frac{\beta g}{v^2} \tag{3.49}$$

If the final velocity is not specified, $(\lambda)_2 = 0$ [v_2 can't be zero without violating the condition $(1 - \lambda_3 v)_2 = 0$ obtained from the transversality condition corresponding to the final mass unspecified] and equation (3.41) may be written as

$$- A\sec\gamma_2 + \frac{m_2\beta}{m_2 v_2} = 0 \tag{3.50}$$

Thus,

$$A = \beta\,\frac{\cos\gamma_2}{v_2} \tag{3.51}$$

When equation (3.51) is substituted into equation (3.49) the following optimizing equation is obtained

$$\sec^2\gamma(T-D)v\gamma' = \omega\sec\gamma + \frac{m_2}{m}\frac{V_2}{\cos\gamma_2}\left[\frac{(T-D)g}{v^2} - \frac{\omega}{v}\right] \tag{3.52}$$

It is interesting to note that if equations (3.9) and (3.12) are substituted into equation (3.6) equation (3.52) is obtained. This serves to verify the fact that the minimum time and the minimum fuel consumption flight paths between two points are identical for a rocket powered aircraft.

CHAPTER IV

SOME OPTIMUM TRAJECTORY SOLUTIONS

A great number of optimum flight paths can be investigated using the methods of chapter two. It is apparent that optimizing conditions can be obtained in closed form for only a few special problems. Bernoulli's brachistochrone, the minimum time to climb, and the minimum fuel to climb, solutions were expressible in such a form. For most problems, the non-linear differential optimizing equations must be integrated using some special technique. The analog computer offers a convenient way to handle these equations and a few examples are worked out in this chapter to demonstrate the procedure. All solutions in this chapter were obtained on the University of Arizona Donner model 3400 analog computer and Donner model 3735 function multipliers.

<u>The Brachistochrone</u>

It is convenient to start with Bernoulli's brachistochrone problem as a pilot example. The closed form solution to this problem is given by equations (3.21) and (3.22). These equations may be discarded, however, in favor of a solution generated by means of the analog computer.

Substituting $A = \frac{\cos\gamma}{v^2}$ into equation (3.20) gives

$$\gamma' = \frac{\cos\gamma}{v^2} \tag{4.1}$$

or $$\gamma' v = \dot{\gamma} = \frac{g\cos\gamma}{v} \tag{4.2}$$

Equation (4.2) represents an optimizing condition for this problem. Its solution in conjunction with the equation of motion in the tangential direction

$$\dot{v} = -\,g\sin\gamma \tag{4.3}$$

and the two coordinate defining equations

$$\dot{x} = v\cos\gamma \tag{4.4}$$

$$\dot{y} = v\sin\gamma \tag{4.5}$$

completely determine the flight path. The lift force required to fly this path is computed from the normal equation of motion

$$L = mv\dot{\gamma} + mg\cos\gamma \tag{4.6}$$

or $$L = 2mg\cos\gamma \tag{4.7}$$

These equations are in a form which can be easily solved on the analog computer. The quantities on the right side of equations (4.2) through (4.5), represented as voltages, are supplied to integrating amplifiers. The integrated results are supplied back through appropriate circuitry to generate the quantities themselves. Initial conditions (x_1,y_1,v_1, and γ_1) must be available for each of the integrating amplifiers. Since γ_1 is unknown, it is determined by varying a trial γ_1 until the flight path goes through the desired final point $(x_2 y_2)$. Thus a computer solution to even the simple brachistochrone problem involves a trial and error process. (It is not particularly difficult to "home in" on a final point in this case, however.)

The solution to the brachistochrone problem is an arc of a cycloid. This arc has the geometric properties shown in Figure 4.1. If v_1 is fixed, then every point in the plane below the dashed line may be reached by means of a cycloid curve of one cusp by adjusting the value of γ_1. Sufficiency proofs [see Forsyth (12, p. 43)] show that the cycloidal arc having its cusp at 0 with no other cusps in its range provides the curve of quickest descent.

Suppose a dragless and thrustless guided missile launched at point A (Figure 4.2) with a velocity v_1 was required to intercept the ground at some point B. For a given v_1 the points below $y = h + v_1^2/2g$ and to the right of the y axis are covered by a family of cycloids, generated by varying γ_1, as shown in Figure 4.2. One of these cycloids intercepts the point B. It is evident that if the point B were moved to D then the optimum path is physically restricted by the ground and an optimum path to point D can not be obtained using this analysis. In fact only points with range coordinates less than $\pi(h + \frac{v_1^2}{2g})$ have an optimum path given by a single arc of a cycloid which does not intercept the ground. Such a physical limitation, represented by the inequality constraints $y \geq 0$ may be reduced to equality constraints [See Hancock(19, p. 148)] and then introduced into the analysis of

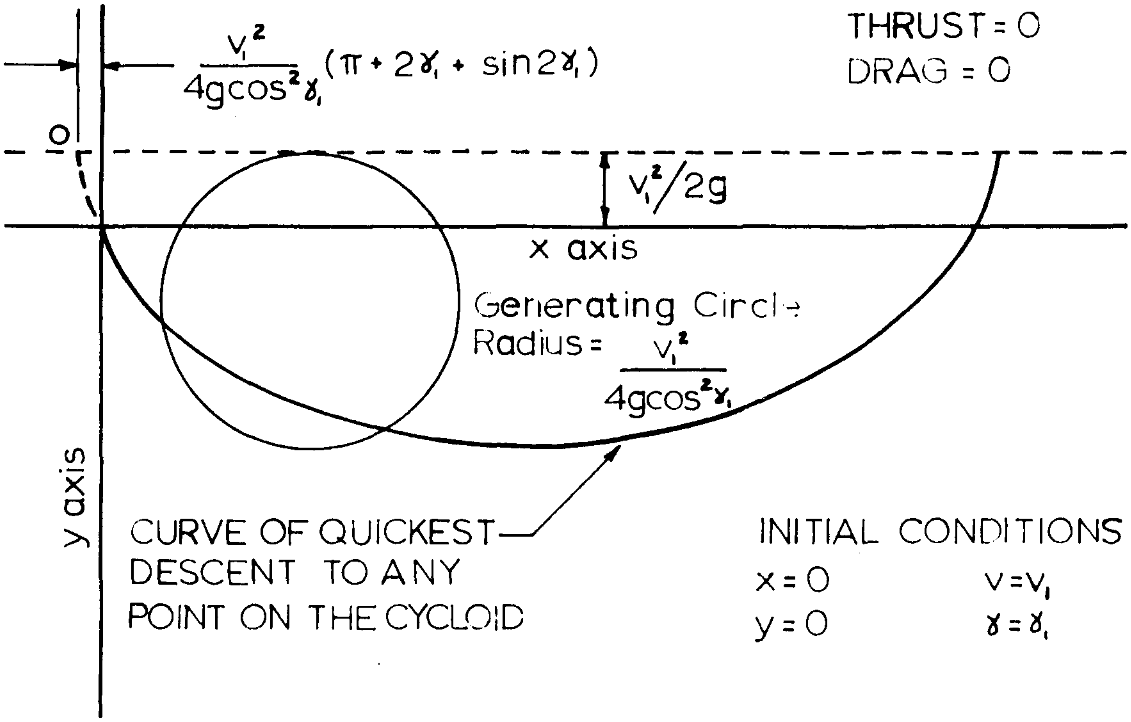

Figure 4.1 Geometric Properties Of The Brachistochrone

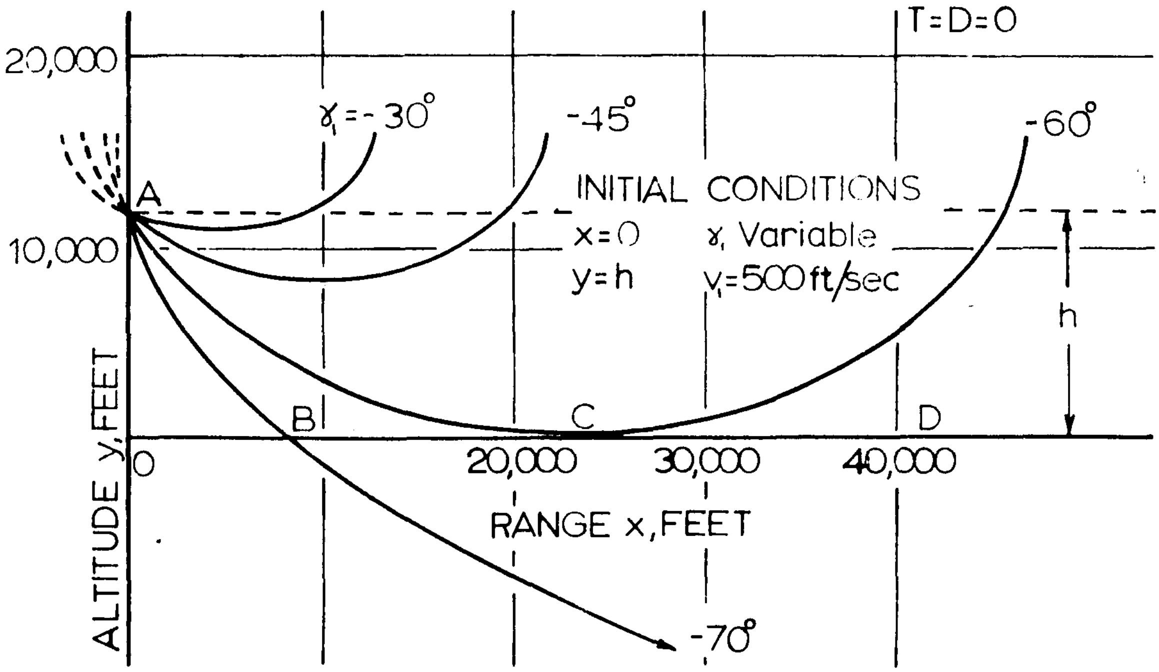

Figure 4.1 Brachistochrone With Variable γ_1

chapter II by the usual method of Lagrange multipliers. If such a procedure were used for this problem, it can be shown with the aid of the Weierstrass-Erdmann corner conditions (14, p. 31) that the optimum curve to D is given by the arcs AC and CD.

Figure 4.3 illustrates the effect of launching a vehicle along a brachistochronic path at a given angle with variable initial velocity. A family of cycloids similar to those in Figure 4.2 are obtained. If a family of cycloids similar to those in Figure 4.3 are superimposed on Figure 4.2, it is easy to demonstrate that any number of cycloids can be drawn between two points. This result can be verified by noting that the value of the constant A in the cycloid equations (3.21) and (3.22) is a function of the location of the final point (with the initial point at the origin). Since $A = \cos\gamma/v = \cos\gamma_1/v_1$, any combination of $\cos\gamma_1$ and v_1 which results in a given ratio are satisfactory initial condition for generating a family of cycloids through two points.

Figure 4.4 illustrates two brachistochronic solutions through two points. The upper flight path represents the minimum time curve for a missile launched at 10,000 feet and 500 ft/sec. Its target is a ground point 12,000 feet away. After a few trial runs the launch angle was determined to be -60°. The time of flight is 24 seconds. The lower flight path is for a missile launched at 10,000 feet and 300 ft/sec with the same ground target. Its launch angle must be higher, -70.5° and the time of flight is 30 seconds. The lift to weight ratio for each case is also illustrated in Figure 4.3. The

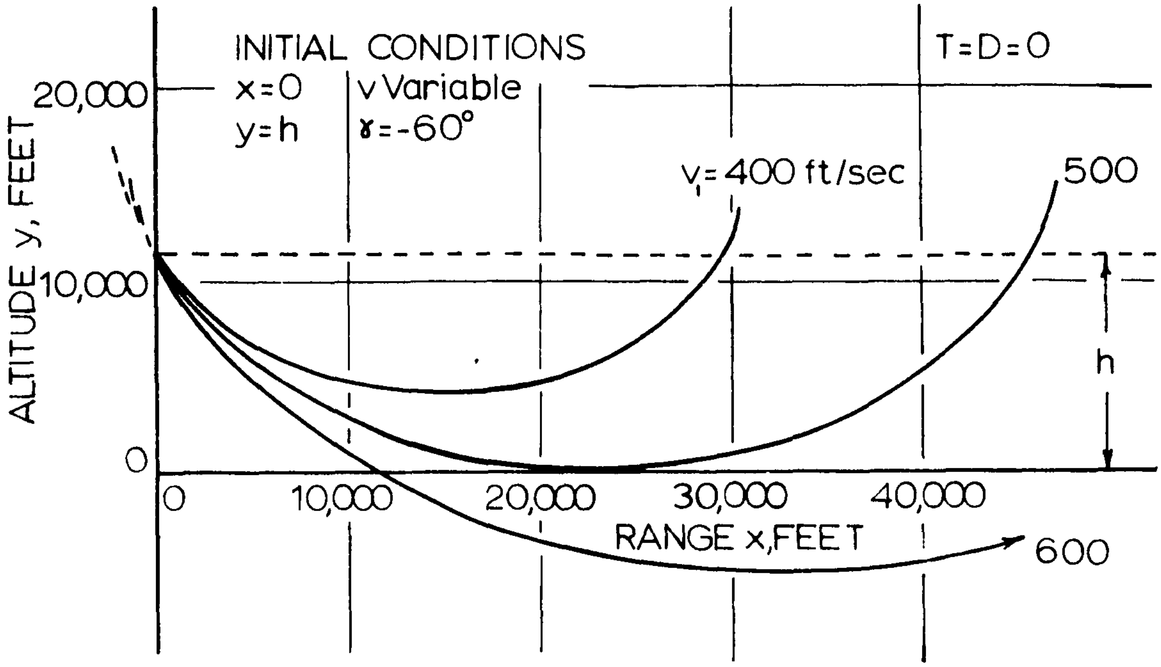

Figure 4.3 Brachistochrone with Variable v_1

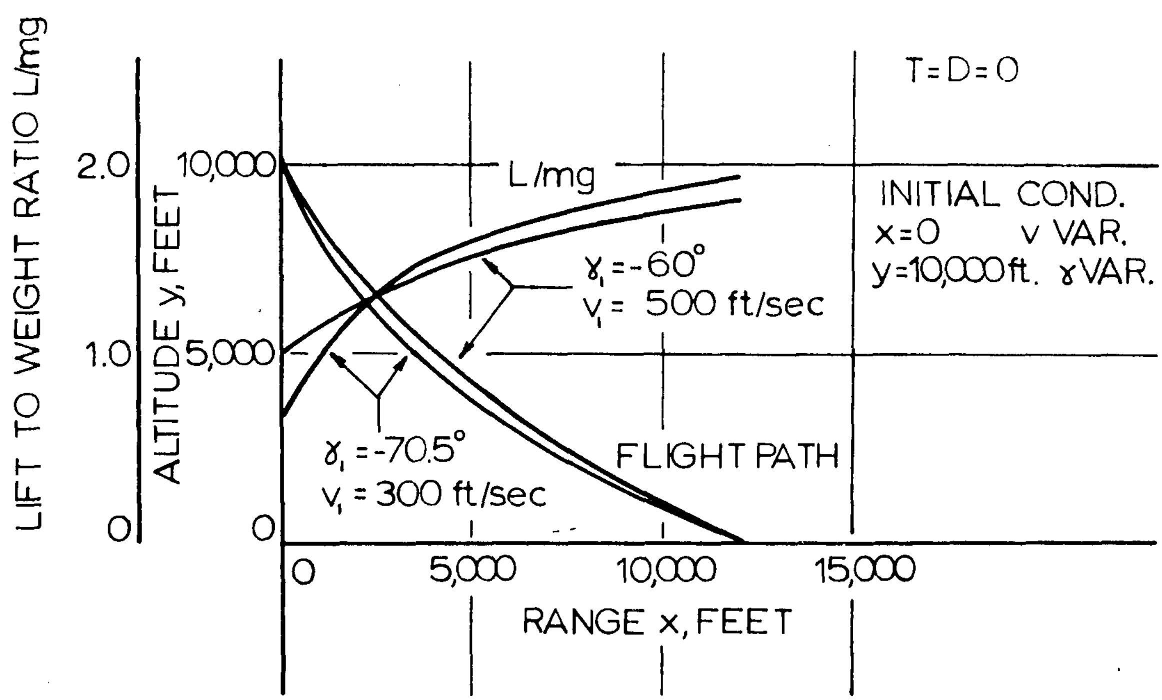

Figure 4.4 Minimum Time Between Two Points

vehicle with the lower initial launch velocity is characterized by a higher lift requirement at the latter part of the flight.

<u>The Brachistochrone with Drag</u>

Suppose the missile used in the previous example now experiences v^2 drag. If $\Phi = v_r - v$ (no wind) and $D = kv^2$ are substituted into equation (3.18), the optimizing condition may be written as follows

$$-kv^2 \sec^2\gamma\, v\gamma' = -\left(\frac{v_2}{\cos\gamma_2}\right)gk - \left(\frac{v_2}{\cos\gamma_2}\right)2gk + 2gkv\sec\gamma \tag{4.8}$$

or

$$\dot{\gamma} = 3g\left(\frac{v_2}{\cos\gamma_2}\right)\left(\frac{\cos\gamma}{v}\right)^2 - 2g\left(\frac{\cos\gamma}{v}\right) \tag{4.9}$$

The solution of this equation in conjunction with the equation of motion in the tangential direction

$$\dot{v} = -g\sin\gamma - \frac{k}{m}v^2 \tag{4.10}$$

and the two coordinate defining equations

$$\dot{x} = v\cos\gamma \tag{4.11}$$

$$\dot{y} = v\sin\gamma \tag{4.12}$$

completely determine the flight path. The lift force required to fly this path is computed from the normal equation of motion

$$L = mv\dot{\gamma} + mg\cos\gamma \tag{4.13}$$

Note that if $k = 0$, $\dfrac{v}{\cos\gamma} = \dfrac{v_2}{\cos\gamma_2}$, then equation (4.9) reduces to equation (4.2).

An analog computer solution to equations (4.9) - (4.13) require that initial values $(x_1, y_1, v_1, \gamma_1)$ be supplied to each of the integrating amplifiers and a value be given for the constant $(\dfrac{v_2}{\cos\gamma_2})$. Only three of these quantities are known, x_1, y_1, and v_1. The other two quantities must be determined on a trial basis. The requirement is that γ_1 and $(v_2/\cos\gamma_2)$ be chosen such that the final point $(x_2 y_2)$ is obtained with $v/\cos\gamma = v_2/\cos\gamma_2$ at that point. For a given set of trial conditions $(\gamma_1, v_2/\cos\gamma_2)$ there is only one final point for which the trajectory is brachistochronic. This situation is quite unlike the previous example where for a given trial condition (γ_1) there are an infinite number of endpoints for which the trajectory is brachistochronic.

Figure 4.5 illustrates the effect of varying the constant $(v_2/\cos\gamma_2)$. The curve labeled 1000 is obtained by using $(v_2/\cos\gamma_2) = (v_1/\cos\gamma_1)$. The dragless brachistochrone curve shown has $v/\cos\gamma = 1000$ all along its length. The effects of varying $(v_2/\cos\gamma_2)$ above and below 1000 are seen to greatly effect the shape of the curves and the location of their end points.

Figure 4.6 illustrates the effect of drag on the solution of the previous example problem. The aircraft is again launched at 10,000 feet with an initial velocity of 500 ft/sec. Three different values

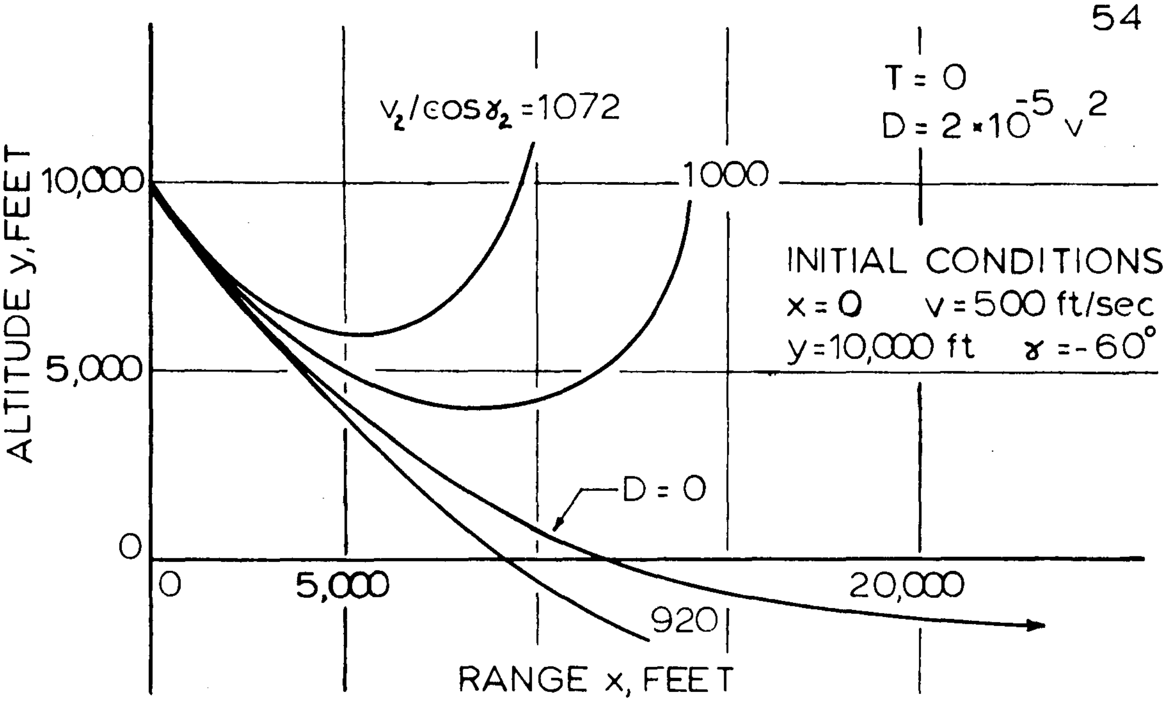

Figure 4.5 Brachistochrone With Drag

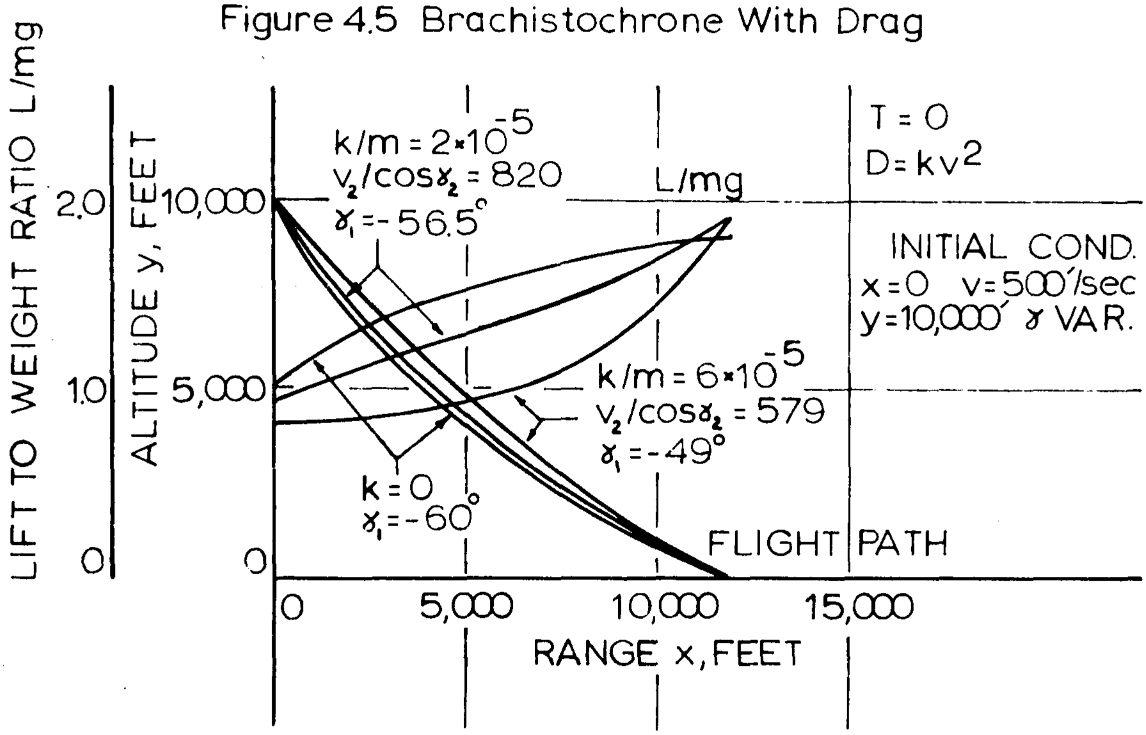

Figure 4.6 Minimum Time Between Two Points

for the drag constant are used.

(a) $k/m = 0$ This represents the dragless aircraft of the previous example. Its flight path is given by the lower curve.

(b) $k/m = 2\times10^{-5}$ This corresponds to an aircraft with a drag coefficient of 0.018, a wing loading of 33.3 lb/ft and a sea level density of .0023 slugs/ft. Its flight path is given by the middle curve.

(c) $k/m = 6\times10^{-5}$ This corresponds to an aircraft with a drag coefficient of 0.018, a wing loading of 11.1 lb/ft and a sea level density of .0023 slugs/ft.

In order to arrive at the final range of 12,000 feet, the initial launch angle must be decreased as k/m is increased. Even though end points are very sensitive to small changes in γ_1 and $v_2/\cos\gamma_2$ (several trials were needed to converge on the final point) the final shape of the optimizing curves are very similar. This result might have been predicted on the basis of Figure 4.5. The effect of drag is most noticeable beyond the minimum points of the curves.

Increasing k/m tends to flatten the flight path at the initial point with an increase in curvature toward the final point. This has the effect of decreasing the lift requirements over the major portion of the flight path. It is interesting to note that if these lift requirements are interpreted in terms of a lift coefficient (for an aircraft with a wing loading of 33.3) the maximum lift coefficient needed does not exceed 0.225. This result is entirely within the capabilities of most aircraft.

<u>The Brachistochrone with Thrust</u>

If $\Phi = v_r - v$ (no wind) and $T = T_1$ = constant are substituted into equation (3.18), the optimizing condition for a missile with constant thrust (and zero drag) is obtained.

$$T_1 \sec^2\gamma v \gamma' = \left(\frac{v_2}{\cos\gamma_2}\right)\frac{g}{v_2} T_1 \tag{4.14}$$

or

$$\dot{\gamma} = g\left(\frac{v_2}{\cos\gamma_2}\right)\frac{\cos\gamma^2}{v^2} \tag{4.15}$$

The solution of this equation in conjunction with the equation of motion in the tangential direction

$$\dot{v} = -g\sin\gamma + \frac{T_1}{mg} \tag{4.16}$$

and the two coordinate defining equations (4.11) and (4.12) completely determine the flight path. The lift force required to fly this path is computed from the normal equation of motion (4.13). Note that if $T_1 = 0$, $v/\cos\gamma = v_2/\cos\gamma_2$, then equation (4.14) reduces to equation (4.2).

The requirements for an analog computer solution to equations (4.14), (4.16), (4.11), (4.12) and (4.13) are identical to those outlined in the previous section.

The effects of varying $v_2/\cos\gamma_2$ are illustrated in Figure 4.7. The cycloid curve (labeled T = 0) has $v/\cos\gamma = 1000$ all along its length. The other brachistochronic curves have $v_2/\cos\gamma_2$ greater than this value. Increasing $v_2/\cos\gamma_2$ shortens the length of the flight path without appreciably changing the initial shape of these curves.

Figure 4.8 illustrates the effect of thrust on the solution to the first example problem. The aircraft is launched at 10,000 feet with an initial velocity of 500 ft/sec. The flight path for the thrustless and dragless aircraft of the first example is the lower curve. The upper curve is the flight path for the same aircraft with a constant thrust T = 1.55 mg. The final range of 12,000 ft. is specified for both cases. The effect of thrust is to decrease the initial launch angle and flatten the flight path in the vicinity of the final point. Adding thrust increases the lift requirements throughout most of the flight path. Lift is almost constant during the latter part of the flight.

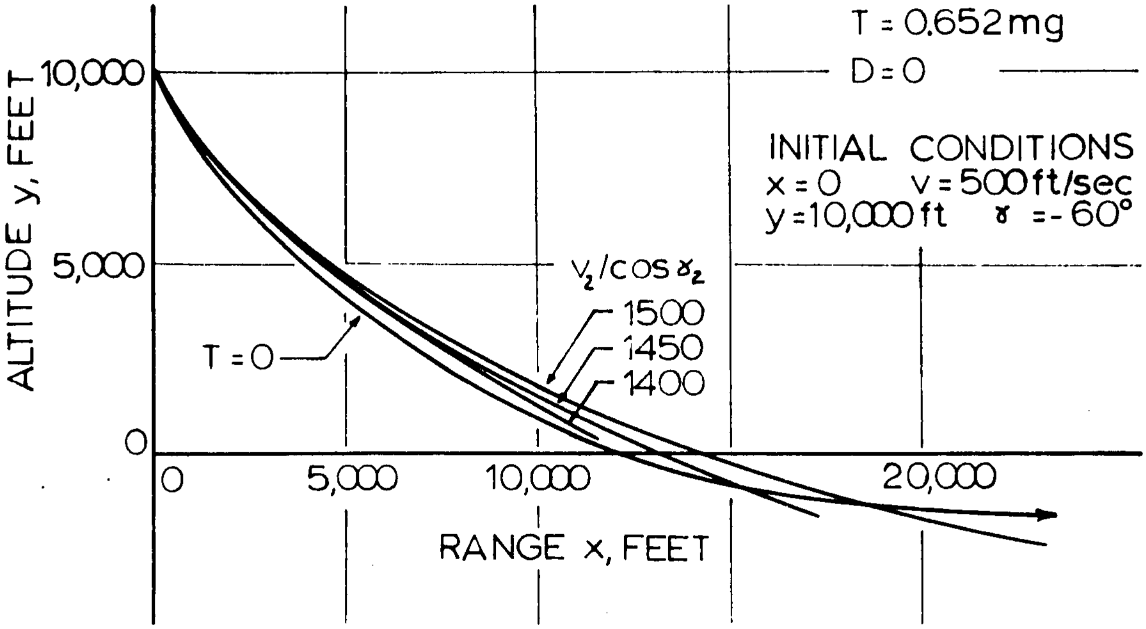

Figure 4.7 Brachistochrone With Thrust

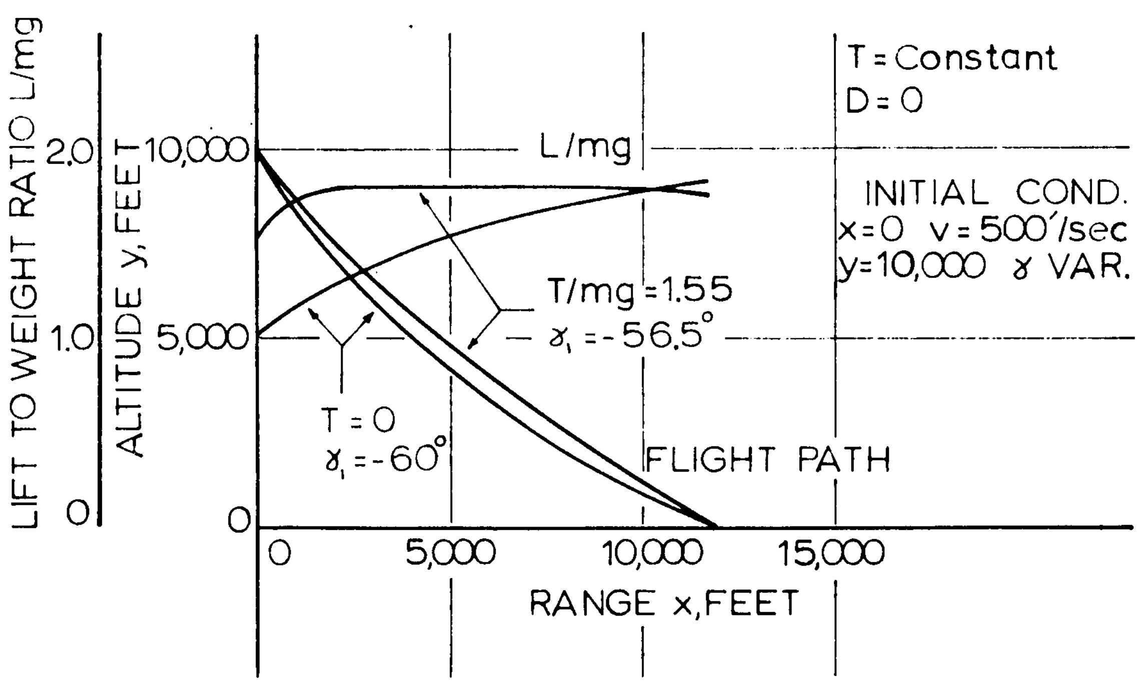

Figure 4.8 Minimum Time Between Two Points

CHAPTER V

DISCUSSION

In 1638 Galileo Galilei wrote ". . . the path of quickest descent from one point to another is not the shortest path, namely, a straight line, but the arc of a circle." (20 p. 239.) Fifty-eight years later John Bernoulli demonstrated that the path of quickest descent is not an arc of a circle but an arc of a cycloid. Galileo with his theorems on "naturally accelerated motion" was led by correct physical reasoning to an incorrect solution. Bernoulli, using the concept of varied paths, was led by correct mathematical analysis to a correct solution. However, the following physical principles are illustrated in both solutions.

(a) The quickest path is not the shortest path.

(b) The quickest path is concave upward.

These principles are learned from Bernoulli's solution, but had to be known in advance by Galileo for his solution.

Seldom does the practicing engineer obtain a "feeling" for a problem by using Galileo's process of astute physical reasoning. More often he learns the principles involved by examining particular solutions to a problem. These principles are then used to interpret the results of his ultimate goal, some very general problem. For example, a general problem in optimum flight dynamics would include all the control variables such as,

(a) Lift control

(b) Thrust control

(c) Roll control

(d) Pitch control

(e) Yaw control

and all the physical constraints such as,

(a) The ground

(b) Limits on lift

(c) Limits on thrust

(d) Limits on speed

(e) Limits on acceleration

A study of the solution to such a general problem at the onset would not necessarily represent the most efficient method for learning the principles involved (nor is it economical considering the difficulties in obtaining such a solution).

The objective of this study was to obtain a sufficiently simple, yet fundamental approach to flight path optimizations so that the optimizing equations could be readily studied using an analog computer. Chapters 2 and 3 represent the fulfillment of this objective.

Computer solutions allow the various forces associated with an aircraft to be studied separately or in combination and their effects analyzed. From the analysis, certain principles or trends can be determined. For example, Figures 4.6 and 4.8 show that thrust and drag, under normal operating conditions, only slightly change the shape of a brachistochronic curve (up to its minimum point). The effect is to decrease the average curvature of the flight path. It is concluded that the shape of the optimizing curve is not a strong function of tangential forces, but is most sensitive to the applied normal forces. This result strengthens the assumption, used in this analysis, that induced drag may be neglected.

The lift/weight ratios needed to fly the optimum flight paths of Figures 4.6 and 4.8 never exceed two. An aircraft with a wing loading of 30 capable of producing a lift coefficient of one can perform these maneuvers at sea level with speeds greater than 230 ft/sec. The decision of Chapter II to leave lift unconstrained will not invalidate the results of Chapter III for most high performance aircraft.

The curves in Chapter 4 represent only a fragment of the future work that can be done within the frame-work of this analysis. Future investigations can determine the nature of flight paths for optimizations other than brachistochronic. Future investigations can also study the effects of wind, atmospheric density changes, and mass changes on optimum flight paths. Work of this nature should help determine new assumptions that may ultimately simplify the more complex problems.

The result that brachistochronic flight paths are relatively independent of tangential forces can be used to approximate optimum thrust programs. This may be done by choosing a fixed flight path (from Figure 4.8, for example) and applying the optimizing procedures of Chapter II with thrust variable.

Equations (A.25) in Appendix A, used in Chapter II, represent only the necessary conditions for the integral (A.3) to be a maximum or minimum. The sufficiency conditions for optimum flight paths have not been investigated here and may be of questionable value (in view of the complexity of the "tests") for many physical problems. The nature of a problem generally indicates whether a maximum or minimum will be found. A maximum time or maximum fuel consumption flight path could not have much meaning.

A maximum range flight path with fuel unconstrained is infinite in length. The significance of the trajectory in this case might be clarified by examining sufficiency conditions.

CHAPTER VI

CONCLUSIONS

1. Optimum flight path problems may be naturally and simply formulated by using arc length as an independent parameter and an energy equation as a dynamical constraint. This procedure makes evident the important simplification which results from neglecting induced drag and limitations on lift. The additional assumption often made, $L = mg$, need not be used in order to make the analysis of optimum flight trajectories amenable to solution.

2. Several authors have restricted the flight path angle to small angles in order to obtain solutions to the minimum time to climb and the minimum fuel to climb problems. This restriction is not needed for solution if these problems are formulated using an energy equation in place of the equations of motion.

3. The optimizing equations developed in Chapter III lend themselves to analog computer solutions. An analog computer solution is easily interpreted in terms of optimum flight paths and requirements needed to fly these trajectories.

4. Analog computer solutions indicate that the location of trial optimum trajectory end points are very sensitive to trial initial conditions and constraints used in the optimizing equations.

These solutions also indicate that the general shape of a trial optimum trajectoy is not particularly sensitive to the initial conditions and constants. This latter result aids in obtaining accurate analog computer solutions.

5. Analog computer solutions to brachistochronic problems show that the shape of an optimum flight path is a strong function of the applied normal forces, but only weakly dependent on the applied tangential forces. Errors introduced by neglecting induced drag are small.

APPENDICES

APPENDIX A

An airplane is said to follow an optimum flight path if a result of following this path is to optimize some quantity associated with the trajectory. This quantity can usually be expressed in terms of an integral of the type

$$I = \int_{s_1}^{s_2} G(q_i, q_i')\,ds \qquad i = 1, \cdots n \qquad \text{(A.1)}$$

where G is not explicitly a function of the arc length s. Arc length is chosen here as the independent variable. The integral I is a functional and its value depends on the entire coarse of the argument function q_i. These functions may be subject to various constraints of the form

$$\phi_j(q_i, q_i') = 0 \qquad j = 1, \cdots m < n \qquad \text{(A.2)}$$

Usually further restrictions are imposed on the variables q_i at their end points. Functions q_i having proper end values and satisfying equation (A.2) must be determined so that I obtains a maximum or minimum value. If a variable q_i is not specified at an end point, then the best boundary value for that variable must be determined as part of the optimization.

The problem of optimizing the integral I subject to the constraints (A.2) may be expressed in terms of a single integral without subsidiary conditions by using Lagrange multiplier variables $\lambda_j(s)$. [See Courant (11, p. 167) and Forsyth (12, p. 389) for a discussion of Lagrange multipliers.] Let $F = \lambda_j \phi_j$, then the integral may be written as

$$I = \int_{s_1}^{s_2} (G + F)ds \tag{A.3}$$

subject to no constraints. In this case, the functions q_i plus the multiplier variables λ_j having proper end values must be determined so that the integral (A.3) is optimized.

<u>The First Variation</u>

The functional relations $q_i(s)$ and $\lambda_j(s)$ are not known, and the problem consists in finding among all the possible relations $q_i(s)$ and $\lambda_j(s)$ those that will make the integral (A.3) a maximum or minimum. It will be assumed that $q_i(s)$ and $\lambda_j(s)$ are single valued in the interval (s_1, s_2) and that F and G possess sufficient partial derivatives as required for the subsequent analysis. Any functions $q_i(s)$ and $\lambda_j(s)$ which allow the integral (A.3) to be determined are called admissible functions.

Let $q_i = q_i(s)$ and $\lambda_j = \lambda_j(s)$ be the equations for the admissible functions for which I is stationary. Then varied values for these functions may be represented as $q_i(s) + \epsilon r_i(s)$ and $\lambda_j(s) + \epsilon\mu_j(s)$ where ϵ is an arbitrary constant. The quantities $\mu_j(s)$ and n-m of the quantities $r_i(s)$ are arbitrary regular functions of s. (m of the r_i functions are not arbitrary because of the constraints) Variations of this type are known as weak variations, and will be the only type considered here.

The limits of integration may either be fixed or variable. If the limits are fixed, they remain unaffected for variations made on I. If the limits are variable to be determined from given data (to lie on a given curve or to be free) then the varied upper limit may be written as $s_2 + \epsilon w_2$ and the varied lower limit may be written as $s_1 + \epsilon w_1$ where w_1 and w_2 are finite quantities. Let

$$Q_i(s,\epsilon) = q_i(s) + \epsilon r_i(s) \tag{A.4}$$

$$\Lambda_j(s,\epsilon) = \lambda_j(s) + \epsilon\,\mu_j(s) \tag{A.5}$$

$$S_2(\epsilon) = s_2 + \epsilon w_2 \tag{A.6}$$

$$S_1(\epsilon) = s_1 + \epsilon w_1 \tag{A.7}$$

For these varied functions, with particular but arbitrary values, I is a function of ϵ and may be written as

$$I(\epsilon) = \int_{S_1}^{S_2} [G(Q_i, Q_i') + F(Q_i, \Lambda_j, Q_i')]\, ds \tag{A.8}$$

For I to be stationary it is necessary that $dI/d\epsilon = 0$ when $\epsilon = 0$

$$\frac{dI}{d\epsilon} = (G_2 + F_2)\frac{dS_2}{d\epsilon} - (G_1 + F_1)\frac{dS_1}{d\epsilon} + \int_{S_1}^{S_2} \frac{\partial}{\partial \epsilon}(G+F)\, ds \tag{A.9}$$

The integral may be written as follows

$$\int_{S_1}^{S_2} \left[\left(\frac{\partial G}{\partial Q_i} \frac{\partial Q_i}{\partial \epsilon} + \frac{\partial G}{\partial Q_i'} \frac{\partial Q_i'}{\partial \epsilon} \right) + \left(\frac{\partial F}{\partial Q_i} \frac{\partial Q_i}{\partial \epsilon} + \frac{\partial F}{\partial \Lambda_j} \frac{\partial \Lambda_j}{\partial \epsilon} + \frac{\partial F}{\partial Q_i'} \frac{\partial Q_i'}{\partial \epsilon} \right) \right] ds \tag{A.10}$$

or $$\int_{S_1}^{S_2} \left[\left(\frac{\partial G}{\partial Q_i} r_i + \frac{\partial G}{\partial Q_i'} r_i' \right) + \left(\frac{\partial F}{\partial Q_i} r_i + \frac{\partial F}{\partial Q_i'} r_i' \right) + \frac{\partial F}{\partial \Lambda_j} \mu_j \right] ds \tag{A.11}$$

Integrating the terms containing r_i by parts and collecting terms gives

$$r_i \frac{\partial}{\partial Q_i'}(G+F) \Big|_{S_1}^{S_2} + \int_{S_1}^{S_2} \left\{ \left[\frac{\partial}{\partial Q_i}(G+F) - \frac{d}{ds}\frac{\partial}{\partial Q_i'}(G+F) \right] r_i + \frac{\partial}{\partial \Lambda_j} \mu_j \right\} ds \tag{A.12}$$

The dependence of the first term on r_i may be eliminated as follows

$$Q_i(S_2\epsilon) = q_i(S_2) + \epsilon r_i(S_2) \tag{A.13}$$

Therefore $\left(\frac{dQ_i}{d\epsilon}\right)_2 = \left(\frac{dq_i}{dS}\right)_2 \frac{dS_2}{d\epsilon} + r_i(S_2) + \epsilon\left(\frac{dr_i}{dS}\right)_2 \frac{dS_2}{d\epsilon}$ (A.14)

$$\left(\frac{dQ_i}{d\epsilon}\right)_2 = r_i(S_2) + [q_i' + \epsilon r_i']_2 \frac{dS_2}{d\epsilon} \qquad \text{(A.15)}$$

$$\left(\frac{dQ_i}{d\epsilon}\right)_2 = r_i(S_2) + (Q_i')_2 \frac{dS_2}{d\epsilon} \qquad \text{(A.16)}$$

Thus $r\ (S_2) = \left(\frac{dQ_i}{d\epsilon}\right)_2 - (Q_i')_2 \frac{dS_2}{d\epsilon}$ (A.17)

Similarly $r_i(S_1) = \left(\frac{dQ_i}{d\epsilon}\right)_1 - (Q_i')_1 \frac{dS_1}{d\epsilon}$ (A.18)

Substituting (A.12), (A.17), and (A.18) into (A.9) gives

$$\frac{dI}{d\epsilon} = (G_2 + F_2)\frac{dS_2}{d\epsilon} - (G_1 + F_1)\frac{dS_1}{d\epsilon} + \left[\frac{\partial}{\partial Q_i'}(G+F)\right]_2 \left[\left(\frac{dQ_i}{d\epsilon}\right)_2 - (Q_i')_2 \frac{dS_2}{d\epsilon}\right]$$

$$- \left[\frac{\partial}{\partial Q_i'}(G+F)\right]_1 \left[\left(\frac{dQ_1}{d\epsilon}\right)_1 - (Q_i')_1 \frac{dS_1}{d\epsilon}\right] + \int_{S_1}^{S_2} \left\{\left[\frac{\partial}{\partial Q_i}(G+F) - \frac{d}{ds}\frac{\partial}{\partial Q_i'}(G+F)\right] r_i + \frac{\partial F}{\partial \Lambda_j}\mu_j\right\} ds$$

(A.19)

For the stationary value of I, $\frac{dI}{d\epsilon}$ is set $= 0$ when $\epsilon = 0$.

Setting $\epsilon = 0$ in (A.19) and rearranging gives

$$\left(\frac{dI}{d\epsilon}\right)_{\epsilon=0} = \left[(G+F) - q_i' \frac{\partial}{\partial q_i'}(G+F)\right]_2 \frac{ds_2}{d\epsilon} + \left[\frac{\partial}{\partial q_i'}(G+F)\right]_2 \left(\frac{dq_i}{d\epsilon}\right)_2$$

$$-\left[(G+F) - q_i' \frac{\partial}{\partial q_i'}(G+F)\right]_1 \frac{ds_1}{d\epsilon} - \left[\frac{\partial}{\partial q_i'}(G+F)\right]_1 \left(\frac{dq_i}{d\epsilon}\right)_1 \qquad \text{(A.20)}$$

$$+\int_{s_1}^{s_2} \left\{\left[\frac{\partial}{\partial q_i}(G+F) - \frac{d}{ds}\frac{\partial}{\partial q_i'}(G+F)\right] r_i + \frac{\partial F}{\partial \lambda_j} \mu_j \right\} ds$$

<u>The Euler - Lagrange Equations</u>

In order for I to have a maximum or minimum, it is necessary for the aggregate of terms on the right side of equations (A.20) to vanish. If they do not vanish, then they govern the value of $(dI/d\epsilon)_{\epsilon = 0}$ and it can be made positive or negative at choice. This result would exclude the possibility of a maximum or minimum.

The vanishing of the terms must occur under all possible conditions. Consider the following possibilities:

(a) A variation is made which is continuous throughout the range and vanishes at each extremity.

(b) A variation is made which is continuous throughout the range and does not vanish at the extremities but is subject to assigned external conditions at the limits.

Variations represented by (a) have r_i and μ_j zero at each limit so that

$$\int_{S_1}^{S_2}\left\{\left[\frac{\partial}{\partial q_i}(G+F)-\frac{d}{ds}\frac{\partial}{\partial q_i'}(G+F)\right]r_i+\frac{\partial F}{\partial \lambda_j}\mu_j\right\}ds \qquad \text{(A.21)}$$

The functions μ_j are arbitrary and n-m of the r_i functions are arbitrary. If λ_m of the multipliers are chosen so that the expression in brackets is zero, then

$$\frac{\partial}{\partial q_i}(G+F)-\frac{d}{ds}\frac{\partial}{\partial q_i'}(G+F)=0 \qquad i=1,\cdots m \qquad \text{(A.22)}$$

Now the remaining r_i are arbitrary and the integral can only be zero if

$$\frac{\partial}{\partial q_i}(G+F)-\frac{d}{ds}\frac{\partial}{\partial q_i'}(G+F)=0 \qquad i=m+1,\cdots n \qquad \text{(A.23)}$$

and

$$\frac{\partial F}{\partial \lambda_j}=0 \qquad i=1,\cdots m \qquad \text{(A.24)}$$

Equations (A.24) are the constraint equations (A.2) and equations (A.22) and (A.23) combined

$$\frac{\partial}{\partial q_i}(G+F) - \frac{d}{ds}\frac{\partial}{\partial q_i'}(G+F) = 0 \qquad i = 1, \cdots n \qquad \text{(A.25)}$$

are known as the Euler - Lagrange equations. These equations represent necessary conditions for the integral (A.3) to be a maximum or minimum. Sufficiency conditions are far more complicated. For additional information the reader is referred to the literature (3, 12).

The Transversality Condition

The variations represented by (b) of the previous section require that the limits be determined. As soon as this determination is achieved, equations (A.24) and (A.25) must apply and the vanishing of equation (A.20) requires only the additional condition

$$\left[(G+F) - q_i' \frac{\partial}{\partial q_i'}(G+F)\right]_2 ds_2 - \left[(G+F) - q_i' \frac{\partial}{\partial q_i'}(G+F)\right]_1 ds_1$$
$$+ \left[\frac{\partial}{\partial q_i'}(G+F)\right]_2 (dq_i)_2 - \left[\frac{\partial}{\partial q_i'}(G+F)\right]_1 (dq_i)_1 = 0 \qquad \text{(A.26)}$$

This condition known as the transversality condition, may be written in the following compact form.

$$\left\{\left[(G+F) - q_i' \frac{\partial}{\partial q_i'}(G+F)\right] ds + \frac{\partial}{\partial q_i'}(G+F)\, dq_i\right\}_1^2 = 0 \qquad \text{(A.27)}$$

It will be shown in the next section that with F and G independent of s, equation (A.27) may be written as follows

$$\left[B\,ds + \frac{\partial}{\partial q_i}(G + F)\,dq_i\right]_1^2 = 0 \qquad i=1,\cdots n \tag{A.28}$$

where B is a constant.

The First Integral

A mathematical consequence of the Euler - Lagrange equations is that

$$\frac{d}{ds}\left[(G+F) - q_i'\frac{\partial}{\partial q_i'}(G+F)\right] - \frac{\partial}{\partial s}(G+F) = 0 \tag{A.29}$$

This is demonstrated by performing the indicated operations on equation (A.29) and reducing it to (A.25)

$$\begin{aligned}\frac{\partial}{\partial s}(G+F) + \frac{\partial}{\partial q_i}(G+F)\frac{dq_i}{ds} + \frac{\partial}{\partial q_i'}(G+F)\frac{dq_i'}{ds} + \frac{\partial}{\partial \lambda_j}(G+F)\frac{d\lambda_j}{ds}\\ - \frac{d}{ds}\frac{\partial}{\partial q_i'}(G+F)q_i' - \frac{\partial}{\partial q_i'}(G+F)q_i'' - \frac{\partial}{\partial s}(G+F) = 0\end{aligned} \tag{A.30}$$

$$q_i'\left[\frac{\partial}{\partial q_i}(G+F) - \frac{d}{ds}\frac{\partial}{\partial q_i'}(G+F)\right] = 0 \tag{A.31}$$

If $q_i \neq 0$ then equation (A.31) is identical to equation (A.25). Since $F + G$ is explicitly independent of s, equation (A.29) reduces to

$$G - q_i' \frac{\partial G}{\partial q_i'} - q_i' \frac{\partial F}{\partial q_i'} = B \qquad i = 1, \cdots n \qquad (A.32)$$

where B is a constant. Equation (A.32) is the first integral of the Euler - Lagrange equations.

APPENDIX B

The energy equation used in the analysis of Chapter II may be written as follows

$$mvdv + mgdy + F_t ds = 0 \tag{B.1}$$

where F_t is the sum of the tangential forces. This equation is in a form identical to that used to express non-holonomic dynamical constraints in classical dynamics. [See Goldstein (16, p. 40) or Fox (14, p. 99).] It would be convenient to introduce equation (B.1) into the formulation of Chapter II by following the method used to handle non-holonomic constraints in dynamics texts. Unfortunately this method is not completely general and can not be used for constraints given by equation (B.1).

<u>Two Methods for Introducing Non-Holonomic Constraints</u>

Suppose the integral

$$I = \int_{S_1}^{S_2} G(q_i, q_i, s)\,ds \qquad i = 1, \cdots n \tag{B.2}$$

is subject to the constraints

$$P_i dq_i + Rds = 0 \qquad i = 1, \cdots n \tag{B.3}$$

where $P_i = P_i(q_i)$ and $R = R(q_i)$. The constraint is assumed non-integrable and for simplicity the limits of integration are assumed fixed.

Following the method outlined in Goldstein (16, p. 40) the following optimizing equations are obtained

$$G_{q_i} + \lambda P_i = 0 \qquad i = 1, \cdots n \tag{B.4}$$

Equation (B.4) is to be compared with the optimizing equations obtained using the methods of Appendix A.

Equation (B.3) may be considered as a functional relationship of the form

$$\phi(q_i, q_i', s) = P_i q_i' + R = 0 \tag{B.5}$$

In this case the optimizing equations are written

$$G_{q_i} + \lambda_1\left(\frac{\partial P_1}{\partial q_i} q_1' + \frac{\partial P_2}{\partial q_i} q_2' + \cdots + \frac{\partial R}{\partial q_i}\right) - \frac{d}{ds}\lambda_1 P_i = 0 \tag{B.6}$$

or, $$G_{q_i} + \lambda_1\left(\frac{\partial P_1}{\partial q_i} q_1' + \frac{\partial P_2}{\partial q_i} q_2' + \cdots + \frac{\partial R}{\partial q_i}\right) - P_i \lambda_1' - \lambda_1 \frac{\partial P_i}{\partial q_j} q_j' = 0 \tag{B.7}$$

$$G_{q_i} + \lambda_1\left[\left(\frac{\partial P_1}{\partial q_i} - \frac{\partial P_i}{\partial q_1}\right)q_1' + \left(\frac{\partial P_2}{\partial q_i} - \frac{\partial P_i}{\partial q_2}\right)q_2' + \cdots + \frac{\partial R}{\partial q_i}\right] - P_i \lambda_1' = 0 \tag{B.8}$$

Equation (B.8) evidently reduces to equation (B.4) when $R = 0$ and $P_i dq_i$ is a perfect differential. This, of course, contradicts the assumption that equation (B.3) is not integrable.

Limitations With the First Method

It is important to know under what circumstances equation (B.4) and equation (B.8) are equivalent. A close examination of the steps leading to the optimizing equation (B.4) is needed. The standard derivation of this equation [See Fox (14, p.99).] involves the important step of replacing the differential of the variables dq_i by their variations δq_i (where for convenience the Lagrange notation δ is used to denote a variation. The symbol δq_i is equivalent to ϵr_i used in Appendix A.) As already pointed out, this may be done when the equation of constraint is holonomic since in this case

$$P_i dq_i = 0 \tag{B.9}$$

may be written as

$$\phi(q_i) = 0 \tag{B.10}$$

so that

$$d\phi = \frac{\partial \phi}{\partial q_1} dq_1 + \frac{\partial \phi}{\partial q_2} dq_2 + \cdots + \frac{\partial \phi}{\partial q_n} dq_n = P_i dq_i \tag{B.11}$$

and (using Lagrange notation)

$$\delta\phi = \frac{\partial \phi}{\partial q_1} \delta q_1 + \frac{\partial \phi}{\partial q_2} \delta q_2 + \cdots + \frac{\partial \phi}{\partial q_n} \delta q_n = P_i \delta q_i \tag{B.12}$$

It is apparent that $\delta\phi$ can be obtained from $d\phi$ by replacing dq_i by δq_i.

If equation (B.9) is not holonomic then an equivalent equation (B.10) can not be found, and it appears questionable to replace the dq_i's by the δq_i's. In terms of dynamical systems, if the constraint equation (B.3) can be replaced by a system of forces which do no work (which will always be the case for rolling type of constraints used in dynamics) then virtual displacements (δ) can be made independent of any actual displacements and the constraint is not violated. If, however, the constraint can not be replaced by forces which do no work, then an independent displacement can not be made. The virtual relationship is then a function of how the virtual displacement is carried out.

A non-dissipative virtual displacement can not be made with equation (B.1) and as a consequence the assumptions of the first method are violated.

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