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1964

INTEGRATION OF ANALYTICAL AND CONVENTIONAL DESIGN TECHNIQUES FOR OPTIMUM CONTROL SYSTEMS

by

Rubert B. Streets, Jr.

A Dissertation Submitted to the Faculty of the

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In Partial Fulfillment of the Requirements
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DOCTOR OF PHILOSOPHY

In the Graduate College

THE UNIVERSITY OF ARIZONA

1963

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be accepted as fulfilling the dissertation requirement of the degree of Doctor of Philosophy

Andrew P. Sage — Dissertation Director — Nov. 13, 1963 — Date

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ABSTRACT

This study treats time-invariant control systems, with aperiodic and stationary random inputs, which are optimum in the sense that they minimize the expectation of some cost function. It is limited to the common case where the desired output is equal to the input signal and where there is no crosscorrelation between inputs. Wiener systems are the optimum linear systems for any inputs and a squared error cost function. Pugachev has shown that the optimum nonlinear system for wide-sense Gaussian inputs and an arbitrary cost function is a Wiener system plus a bias constant, which is zero if the criterion is an even function. The bias can be easily determined.

Wiener theory has three serious drawbacks: the mathematical sophistication which limits its audience; the algebraic complexity which limits its application to simple problems; and the difficulty of obtaining the required data. This study presents a new approach which eliminates the first two drawbacks. It develops a technique which yields a good approximation to the open-loop transfer function of Wiener systems by inspection, by using physical insight for the basic structural properties of optimum systems as shown on straight-line Bode plots. The cases treated are: input signal and noise; saturation constraints; load disturbances; and non-minimum-phase fixed elements.

In the following, the terms signal and noise refer to their normalized and spectrum factored energy or power spectral densities. Normalized means that the signal and noise cross at magnitude equals unity. Three new basic concepts are presented. When the signal is greater than the noise, the optimum open-loop system looks like the signal. A signal which is being constrained — transmitted noise when the noise is greater than the signal, or a signal under a saturation constraint above the saturation break frequency — has a minus one slope on a Bode plot. Any required breaks which are not in the given data are Butterworth.

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Chapter 1

INTRODUCTION

1.1 OPTIMUM SYSTEMS

Twenty years ago Wiener[23] developed the theory of optimum linear time-invariant systems for arbitrary stationary random inputs which minimize the mean of the square of the error. Various extensions of this approach have been contributed by other authors. These systems will be referred to as Wiener systems.

In a virtually unknown but very important paper Pugachev[18] has shown that the optimum nonlinear system for stationary Gaussian random inputs, which minimizes the mean of an arbitrary function of the error, is a Wiener system, plus perhaps a "bias" constant, m,

$$N(r(t)) = \int_{-\infty}^{t} h(t - \tau)\, r(\tau)\, d\tau + m$$

and that if the error criterion is even the bias constant is zero. (See also Sherman[19])

The bias constant can be easily determined by solving for β in the following expression,

$$\frac{\partial}{\partial \beta} \int_{-\infty}^{\infty} C(e) \exp\left[\frac{-(e - \beta)^2}{2\sigma^2}\right] de = 0$$

where C(e) is the arbitrary cost function, ξ is the mean, and σ^2 is the variance of the error of the Wiener system. Then the bias constant is $m = \xi - \beta$.

In summary, for Gaussian random inputs, a Wiener system is the best possible under ANY conditions.

1.2 NON-GAUSSIAN SIGNALS

The random inputs to control systems are generally approximately Gaussian. However, it is natural to wonder just how critical the Gaussian assumption is. A heuristic answer to this question can be obtained by considering zero-memory systems. These systems illustrate most of the principles which apply to infinite memory systems, but are considerably easier to work with. Consider the example of a signal and noise, each possessing a rectangular probability distribution. A comparison of the performance of the optimum linear system (for the mean-square error criterion) and of the optimum nonlinear system (for any even nondecreasing error criterion) yields the following result. The maximum performance loss measured in terms of RMS error, which is a function of the RMS noise to RMS signal ratio, is only 3.92% and it is usually much less. This indicates that signals must be very non-Gaussian in order for the optimum nonlinear system to perform significantly better than the optimum linear system.

Reference 24 presents a thorough investigation of performance

criteria for the optimum time response of control systems to step inputs. This report recommends ITAE (integrated time moment of absolute error) for optimization using analog computers and ITE^2 (integrated time moment of error-squared) for analytical investigations. The Wiener system for this case involves a narrow sense realizability constraint; therefore, it is the Butterworth system rather than the IE^2 system. The step responses of these systems are shown on pages 48-49, 79, and 84 of that report, and it can be seen that ITAE is only a little better than Butterworth and ITE^2. The transfer functions for these three criteria are shown on page 51 of that report, and it can be seen that the Butterworth and ITAE criteria yield very similar open-loop systems. The minimum mean-square or integral-square error criterion, which results in Butterworth systems in this case, yields systems which are optimum for their actual inputs rather than optimum for an artificial step input; therefore, it is a much more meaningful criterion. To summarize, the appropriate criterion for most control problems is the minimum mean-square or integral-square error criterion, and this yields Wiener systems.

1.3 SUMMARY

Wiener systems are the best possible time-invariant systems in nearly all cases. However, Wiener's powerful theory has not been the practical tool it should be because of two difficulties: its mathematical sophistication which limits its audience, and its

algebraic complexity which effectively limits its application to simple problems. The objective of this thesis is to remove both these difficulties.

This thesis develops a practical technique, which applies to most control problems, for the designing by inspection of an excellent approximation to the open-loop transfer function of optimum linear time-invariant (Wiener) control systems.

The approach used consists of designing Wiener systems by having insight for the basic structural properties of optimum systems and by visualizing or sketching straight line Bode plots.

There are four principle advantages to this technique. The computation required is easier than that of Wiener theory by a factor of from ten to infinity, depending on whether the problem is trivial or very complex. Unlike Wiener theory, it can be easily learned and used by any BSEE control engineer, instead of by only a few specialists. It gives the designer a great deal of physical insight for optimum systems. Unlike conventional theory, it yields the best possible system and focuses the designers attention on the fundamental aspects of the problem which yield ultimate performance limitations.

The one disadvantage of this technique is one which is inherent in Wiener theory. In practice, it is usually difficult to obtain records of typical signals, and to process them to obtain their spectral densities. However, it is reasonable to expect that the

latter problem will be solved soon. Then, the only difficulty facing a designer who wants to design the best possible system is the first problem.

Four new and important concepts are introduced in this thesis. See Section 2.1 for an explanation of notation and Section 3.1.2 for the statement of the problem. Concepts b. thru d. are excellent approximations, which represent basic structural properties of optimum systems.

a. The design of optimum systems by inspection concept. This requires the representation of factored power and energy spectra, Φ^+, on straight line Bode plots.

b. The matched filter and amplitude normalization concept. This states that the optimum open-loop transfer function, G, looks like the factored signal density spectrum, Φ^+, when the signal density spectrum is larger than the noise density spectrum, and when the signal and noise density spectra cross at magnitude equals one. That is

$$\text{for} \quad \Phi_s^+ > \Phi_n^+, \quad G \approx \Phi_s^+$$

$$\text{where} \quad \left|\Phi_s^+ (j\omega_c)\right| = \left|\Phi_n^+ (j\omega_c)\right| = 1$$

c. The -1 constraint concept. This states that the factored density spectra of signals which are to be constrained are bounded by a -1 slope on a Bode plot. There are two

principle cases.

(1) The factored transmitted noise density spectrum should be bounded by a -1 slope when the noise density spectrum is larger than the signal density spectrum. That is

for $\Phi_s^+ < \Phi_n^+$, slope $W\Phi_n^+ = -1$

(2) The factored density spectrum of a signal which is under a saturation constraint should be bounded by a -1 slope such that the constraint on its mean-square or integral-square value is satisfied. That is

for $\omega > 1/\rho$, slope $\Phi_i^+ = -1$

so that area $\Phi_i^+ = J$

d. The optimum Butterworth breaks concept. This states that any breaks (poles or zeros) in the optimum open-loop transfer function, which are not in the given data should be Butterworth.

It should be remarked that Pugachev's work, and the fact that Wiener systems are optimum in most situations are essentially new concepts.

Derivations (Sections 3.2.3, 3.3.3, 3.4.3, and 3.5.3) are for the specialist, and can be omitted without loss of continuity.

Chapter 2

BACKGROUND MATERIAL

2.1 NOTATION

2.1.1 BODE PLOTS

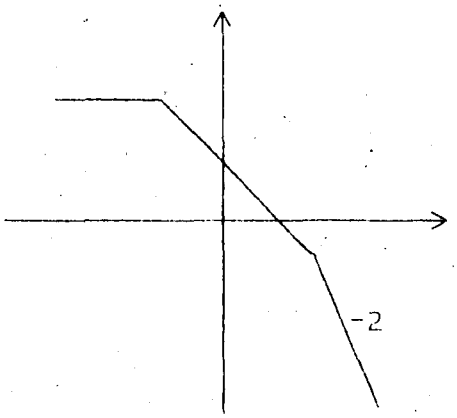


Figure 2.1-1 Typical Bode Plot

Figure 2.1-1 denotes the convention used for a straight line Bode magnitude plot. Axes are not marked. Horizontal axis is ω, in radians, at magnitude = 1. Vertical axis is magnitude at ω = 1. Scale is log-log, but the values marked are the actual values. Horizontal and vertical scales are always equal. -1, 0, +1 slopes are obvious and therefore, are not indicated. All other slopes are indicated.

2.1.2 CONTROL SYSTEMS

Figure 2.1-2 shows the block diagram and symbols used in this thesis.

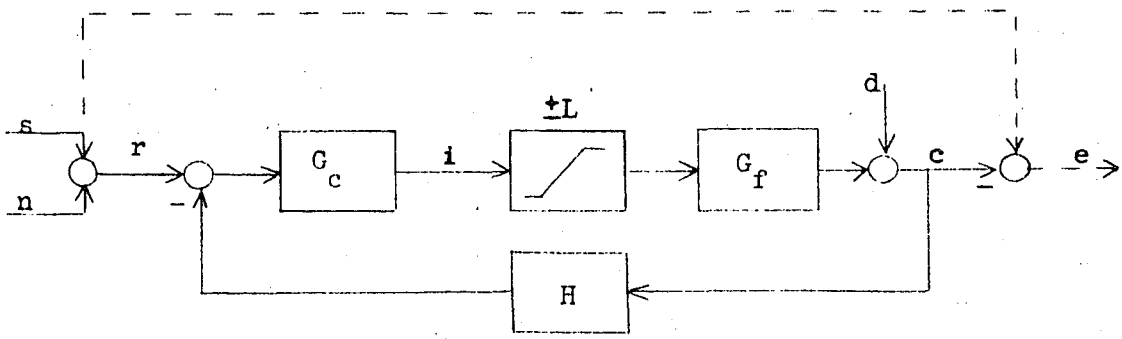


Figure 2.1-2 Basic Block Diagram

Lower-case letters usually denote time functions. Capital letters usually represent Laplace transforms of time functions.

Signals

s Input signal

n Input noise

d Load disturbance

i Input to saturating element

r System input

c System output

e System error

Transfer functions

W Closed-loop

G Open-loop, $G = G_c G_f$

G_f Fixed plant

G_c Compensation network

H Feedback elements (usually H = 1)

$$\frac{C}{R} = \frac{G}{1 + GH} = W$$

$$\frac{C}{D} = \frac{1}{1 + GH} = 1 - V \quad \text{(Note definition of V)} \text{ (See p.94)}$$

Saturating element

$\pm$ L Linear range of saturating element

J Allowable mean-square or integral-square value of saturating signal (usually $J = (L/2)^2$)

2.1.3 SCALE FACTORS

The two-sided Laplace transformation is used in all applications.

$$\mathcal{L}_B (f(t)) = \int_{-\infty}^{\infty} dt\, e^{st} f(t)$$

$$= \mathcal{L} (f(t); s) + \mathcal{L} (f(-t); -s)$$

Therefore, the power spectral density used in this work is 2π times that used by Korn[13], Lee[15], and Newton[17], but is the same as that used by Aseltine[1], Truxal[22], Chang[5], and others.

2.1.4 SPECTRAL DENSITIES

Φ Power spectral density of a random signal. Energy spectral density of an aperiodic signal. Subscripts **are the same as** signals in Section 2.1.2.

Φ^+ Superscript plus denotes spectrum factorization. Φ_x^+ consists of the square root of the gain, and the LHP (left half-plane) poles and zeros of Φ_{xx}. Φ_x^+ can be treated like a transfer function on a Bode plot.

$\oplus$ Denotes the addition of unfactored spectra.

$$\Phi_x^+ \oplus \Phi_y^+ \Rightarrow \Phi_{xx} + \Phi_{yy}$$

$< x^2 >$ Mean-square value of a random process x. Integral-square value of an aperiodic signal.

RMS Denotes $\sqrt{< x^2 >}$

I_n See Table 3.1-1 or Appendix E.2 of Newton[17] or Appendix D of Chang[5].

area Denotes the area under Φ_{xx}.

$$\text{area } \Phi_x^+ = < x^2 > = I_n = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} ds\, \Phi_{xx}(s)$$

$[\ ]_+$ Subscript plus denotes the realizability operation. For rational functions, make a partial fraction expansion and retain only the terms with poles in the left half-plane. Note

$$(X)^+ (X)^- = X, \qquad [X]_+ + [X]_- = X$$

2.1.5 GENERAL

λ Normalized complex frequency, $\lambda = s/v$.

ω_c Crossover frequency, $|G(j\omega_c)| = 1$, $|\Phi_s^+(j\omega_c)| = |\Phi_n^+(j\omega_c)|$.

slope Refers to Bode plots. Slope = k denotes a 20k db per decade slope.

$\mathcal{O}(\ )$ Order, $\mathcal{O}(A) = k$ if $\lim_{s \to \infty} A = Cs^k$; slope as $s \to \infty$

B_n n^{th} order Butterworth polynomial.

NMP Non-minimum phase.

T NMP time constant.

$1/\rho$ Saturation break frequency.

E_a Approximation error, $= \frac{e_a - e_o}{e_o}$ where

e_a is the RMS error of the system designed by inspection.

e_o is the RMS error of the Wiener system.

Unnecessary notation is usually omitted: + signs on block diagrams; the functional dependence on time, (t), and on the Laplace transform variable, (s); and labeling axes on Bode plots (See Section 2.1.1). Although s is used for both the input signal and for the Laplace transform variable (e.g. $\Phi_s^+(s)$) no confusion can result. The Γ and $\triangle$ used in the derivations are defined in Sections 5.4, 5.5, 4.4, and 7.3 of Newton's[17] book. All other notation is fairly standard in control theory and is explained as it is used.

2.2 SPECTRAL DENSITIES

Chapters 14 and 15 (only 35 pages) of Aseltine[1] are recommended as an excellent introduction to random signals, power spectral density, and related topics. Chapter 7 (43 pages) of Truxal[22] is also good. Because of the importance of these concepts the following simplified explanation is included.

Optimum systems are completely determined by the specification of their inputs and of their fixed elements. Therefore, a suitable description of their inputs, which are either random or aperiodic signals, is needed. Because the principle tool of the technique presented is that of Bode plots, a description in the frequency domain is required.

Random signals (Fig. 2.2-1) are signals whose future behavior cannot be predicted exactly, but certain properties of their future behavior can be described in terms of probability theory. (All random signals are assumed to be stationary and ergodic, that is time and ensemble averages are equal.) The property which is needed to describe this limited knowledge about a random signal is its power spectral density, Φ_{xx}. There are three different ways of interpreting a power spectral density.

(1) An experimental method for measuring a power spectral density, Φ_{xx}, is illustrated below.

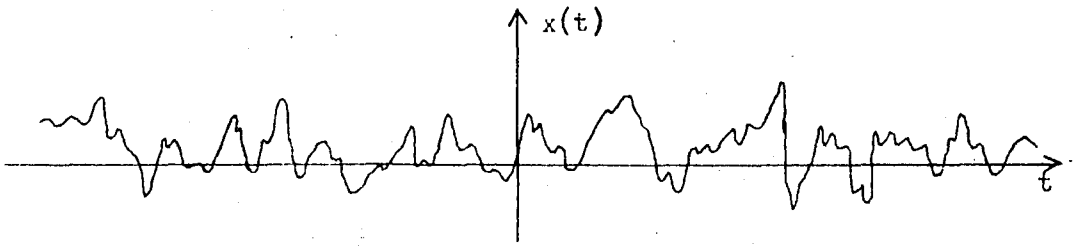
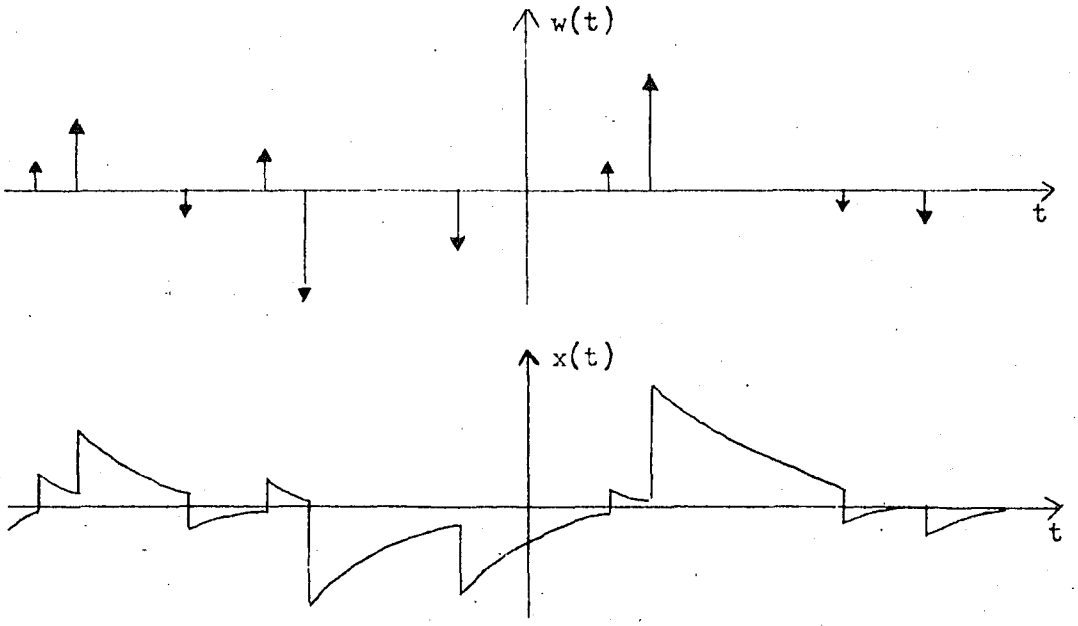
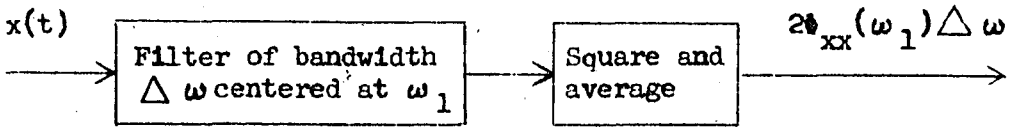


Figure 2.2-1 Typical Random Process



$$\Phi_{ww}(s) = 1 \qquad w(t) \rightarrow \boxed{\frac{1}{1+s}} \rightarrow x(t) \qquad \Phi_{xx}(s) = \frac{1}{1-s^2}$$

Figure 2.2-2 Conceptual Model of Random Process



The average power dissipated in a 1-ohm resister by those frequency components of a voltage x(t) lying in a band between ω and $\omega + d\omega$ is $2\,\Phi_{xx}(\omega)d\omega$. See Huskey and Korn[11], Chapter 5, Part 9 for details.

(2) The average value of the product of a random function of time with the same function displaced τ sec is called the autocorrelation function.

$$R_{xx}(\tau) = \langle x(t)\, x(t+\tau) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt\, x(t)\, x(t+\tau)$$

This is "shift, multiply and average". The autocorrelation function is qualitatively a measure of the regularity of a function. If the value τ sec from now is closely dependent upon the present value, $R(\tau)$ will in general be large. The two-sided Laplace transform of the autocorrelation function is defined to be the power spectral density.

$$\Phi_{xx}(s) = \int_{-\infty}^{\infty} d\tau\, e^{-s\tau}\, R_{xx}(\tau)$$

The mean-square value of the random signal is:

$$\langle x^2 \rangle = R_{xx}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, \Phi_{xx}(\omega)$$

(3) The following physical model is very helpful in visualizing a random signal. Briefly, a random signal can be regarded as being obtained by passing a white noise thru an appropriate shaping filter. A good example of white noise, $w(t)$, is a wave formed by impulses, whose "area" a_k is a random variable with zero mean and finite variance, and which occur at the random incidence times t_k determined by the state changes of a Poisson process with mean count rate α. See Figure 2.2-2.

$$w(t) = \sum_{k=1}^{\infty} a_k \, \delta(t - t_k)$$

Assume that the variance of a_k, $E(a_k^2)$, equals $1/\alpha$. Then

$$R_{ww}(\tau) = \delta(\tau)$$

$$\Phi_{ww}(s) = 1$$

White noise has no correlation between samples and has a power spectral density which is constant for all frequencies (analogy to white light). The output, $x(t)$, of the shaping filter with an impulse response, $y(t)$, to this white noise, $w(t)$, is

$$x(t) = \sum_{k=1}^{\infty} a_k \, y(t - t_k)$$

See Figure 2.2-2. This is called impulse noise, and it approximates a Gaussian random process if many pulses overlap (many small impulses very close together). The power spectral density is

$$\Phi_{xx}(s) = Y(s)\, Y(-s)$$

where $V(s)$ is the transfer function of the shaping filter. Therefore, a random signal with a power spectral density Φ_{xx} can be regarded as being the output of a filter with the transfer function $X(s) = \Phi_x^+(s)$ whose input is the white noise described above. See Korn[13] Section 18.11-5. For the special case where $a_k = 1$, see Lee[15] Chapter 8, Section 6, and Chapter 13, Section 7.

A very important property of power spectral densities is that the output power spectral density is related to the input power spectral density and the system transfer function by

$$\Phi_{oo}(s) = W(s)\, W(-s)\, \Phi_{ii}(s)$$

$$\Phi_o^+(s) = W(s)\, \Phi_i^+(s)$$

A very useful concept is that Φ_x^+ can be treated like a transfer function on a Bode plot.

For aperiodic signals the formal substitution in the mathematics of: aperiodic for random; energy for power (spectral density); and integral for mean, i.e.

$$\int_{-\infty}^{\infty} dt \quad \text{for} \quad \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt$$

is valid for our purposes. Therefore, an aperiodic signal, $y(t)$, has

an energy spectral density, $\Phi_{yy}(s)$, which is related to its Laplace transform, Y(s), by

$$\Phi_{yy}(s) = Y(s)\, Y(-s)$$

2.3 BODE PLOTS

It is assumed that the reader has had an undergraduate course in control systems, and understands how to design with Bode plots. Chapter 6 of Bower and Schultheiss[4] (47 pages) is an excellent introduction of this technique. Because straight line Bode plots are the principle tool used in this work, the following brief outline of basic points is presented. (See also Section 2.1.1)

For scale plots it is convenient to use linear graph paper, 5 divisions to the inch, and to use a horizontal and vertical scale of 1" = a factor of 10. The intermediate divisions represent factors of 1.6, 2.5, 4.0, and 6.3. For greater accuracy use the D and L scale of a slide rule. A very basic and important relation which will be used frequently in the next chapter is shown in Figure 2.3-1. A useful approximation for the closed-loop transfer function is shown in Figure 2.3-2.

The relationships between the Bode plot of an open-loop transfer function G and conventional specifications on its closed-loop behavior are shown in Figure 2.3-3 thru 2.3-5b. The relations

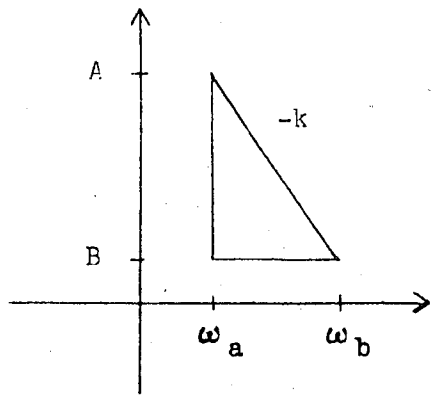
$$25^\circ \leq PM \leq 70^\circ$$

$$PM \approx 10 + 80\zeta \approx 70 - PO \approx \frac{60}{M_p}$$

$$\omega_c \approx \frac{PW}{1.5} \approx \frac{1}{T_D} \approx \frac{1.5}{T_R}$$

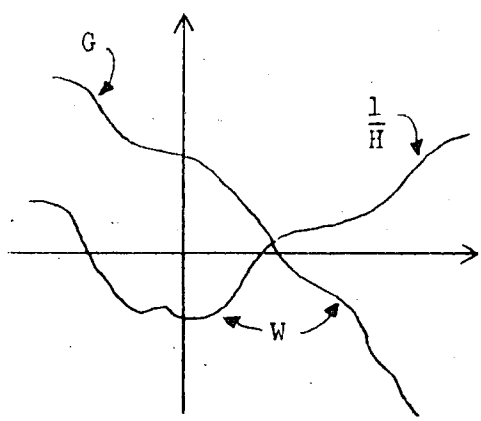
(Notation is defined on Figures 2.3-4,5)

are the author's. They are good approximations, and they are much more convenient than conventional relations. Like conventional relations they are strictly valid only for second order systems. See Chapter 2 of Woklovitch[24] and Gibson[8] for details. To briefly summarize: the low frequency asymmptote at $\omega = 1$ (in general, the larger the better) is inversely proportional to the steady state error and determines its character; the crossover frequency (in general, the larger the better) is inversely proportional to the time delay or speed of response; and the wider the -1 slope band at crossover (one decade is usually adequate) the better the transient response.



$$A(\omega_a)^k = B(\omega_b)^k$$

Figure 2.3-1 Basic Relation on Bode Plots



$$W = \frac{G}{1 + GH} = \frac{G/H}{1/H + G}$$

If $G \gg 1/H$ $W \approx 1/H$

If $G \ll 1/H$ $W \approx G$

Figure 2.3-2 Approximation for W

(1) The steady state error is inversely proportional to the "low frequency asymptote at $\omega = 1$."

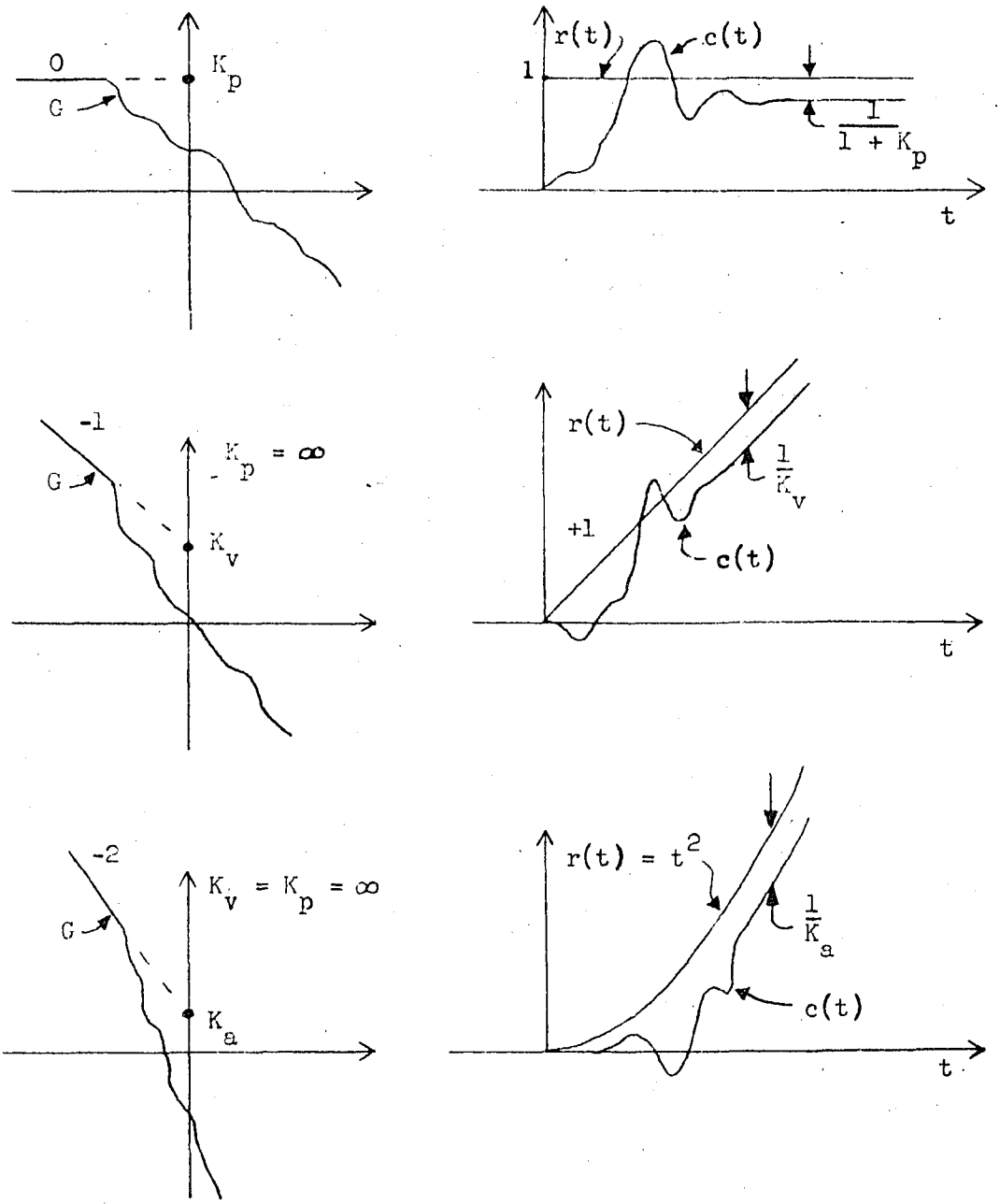
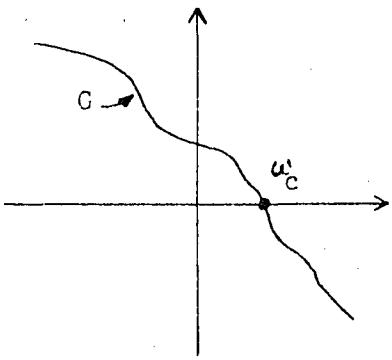
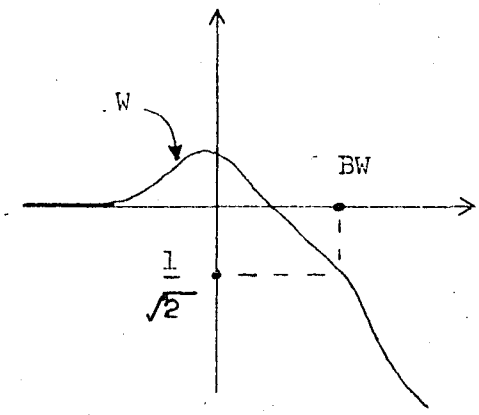


Figure 2.3-3 Bode Plots and Steady State Error

(2) The speed of response is inversely proportional to the "crossover frequency ω_c'."

$$\omega_c \approx \frac{BW}{1.5} \approx \frac{1}{T_D} \approx \frac{1.5}{T_R} \qquad 25^\circ \leq PM \leq 70^\circ$$





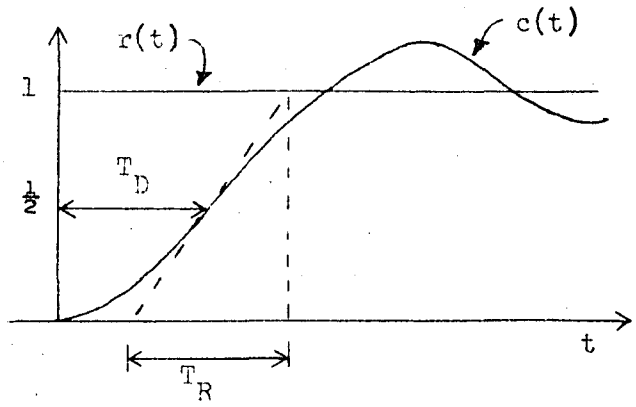


Figure 2.3-4 Bode Plots and Speed of Response

(3) The transient behavior is given by the "width of the -1 band at crossover" which determines the phase margin.

Phase margin may be determined on a slide rule by:

(a) Conventional relations (Ref. 4, pp. 83, 86)

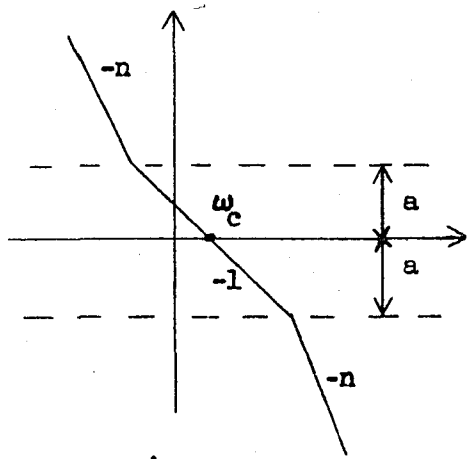
$$PM = 180^\circ + \sum \beta \text{ zeros} - \sum \beta \text{ poles}$$

$$\beta = \tan^{-1} \omega_c/\omega_1 \text{ for } (1 + s/\omega_1) \text{ terms}$$

$$= \tan^{-1} \frac{2\zeta(\omega_c/\omega_n)}{1 - (\omega_c/\omega_n)^2} \text{ for } \left[1 + 2\zeta\frac{s}{\omega_n} + \frac{s^2}{\omega_n^2}\right] \text{ terms}$$

$$\tan^{-1} (-x) = \pi - \tan^{-1} x$$

(b) -1 Band Algorithm



Read directly on slide rule!

$n = 2 \quad PM = \tan^{-1} a - \tan^{-1} 1/a$

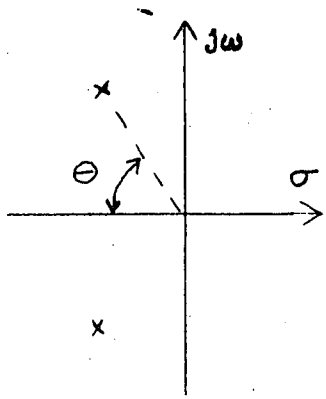
$n = 3 \quad \frac{PM + 90}{2} = \tan^{-1} a - \tan^{-1} 1/a$ (binomial)

Example: $PM = 40^\circ$, $n = 2$

slide rule { T 65|25 ; L .669 } $a = .331$ decades

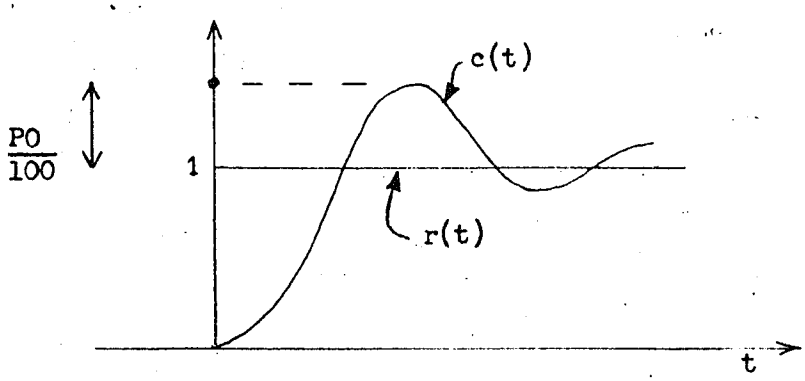
Figure 2.3-5a Bode Plots and Phase Margin

$$PM \approx 10 + 80\zeta \approx 70 - PO \approx \frac{60}{M_p} \qquad 25^\circ \leq PM \leq 70^\circ$$



$$\zeta = \cos\Theta$$

$$W = \frac{1}{1 + 2\zeta s/\omega_n + s^2/\omega_n^2}$$



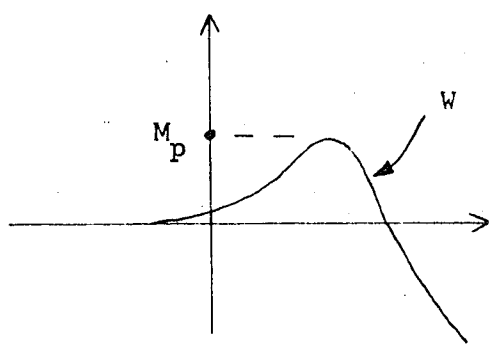


Figure 2.3-5b Phase Margin and Transient Behavior

Chapter 3

OPTIMUM CONTROL SYSTEMS BY INSPECTION

3.1 INTRODUCTION

3.1.1 BASIC POINTS

The purpose of this chapter is to develop a practical technique, which applies to most control problems, to design an excellent approximation to the open-loop transfer function of optimum linear time-invariant (Wiener) control systems by inspection.

The basic assumptions are outlined below. The standard assumptions of Wiener theory are used.

a. The control system is optimum in the sense of minimizing the mean-square or integral-square error. The systems are linear and time-invariant.

In addition the following assumptions, which are satisfied in most control problems, are made.

b. The desired output is the input signal.

c. There is no crosscorrelation between inputs.

The following "control problem" assumptions are necessary to avoid meaningless solutions.

d. The RMS error is finite, and is less than one half the RMS value of the signal.

e. $\mathcal{O}(W) \leqq \mathcal{O}(G_f) \leqq -1$

for $\omega > \omega_c$, slope $W \leqq -1$

The technique which is developed has the following characteristics.

a. The solutions are obtained graphically by using straight line Bode plots, either accurately plotted, roughly sketched or merely visualized. This, together with insight for the basic structural properties of optimum systems, replaces the usual algebraic manipulation of Wiener theory which becomes virtually impossible in nontrivial problems.

b. The input noise problem is divided into three separate problems by frequency ranges: $\Phi_s^+ \gg \Phi_n^+$, $\Phi_s^+ \approx \Phi_n^+$ and $\Phi_s^+ \ll \Phi_n^+$. This division of the problem into three separate problems by frequency ranges is analogous to the conventional technique for the design of Class A electronic amplifiers.

c. The various aspects of the problem are considered separately. It is assumed that for control problems (assumption d) that

there exists little interaction between the solutions for input noise, saturation constraints, load disturbances, and NMP (non-minimum phase) fixed elements.

The order in which the various aspects of the problem are presented is based upon tutorial considerations. In practice the order of importance is usually:

a. Saturation constraints (always present).

b. Load disturbances.

c. Input white noise.

d. Input colored noise.

e. NMP fixed elements.

3.1.2 STATEMENT OF THE PROBLEM

Given: Φ_{ss}, Φ_{nn}, Φ_{dd}

Several sets of G_f, $\pm L$, and $\langle i^2 \rangle \leq J$

The block diagram of Figure 3.1-1.

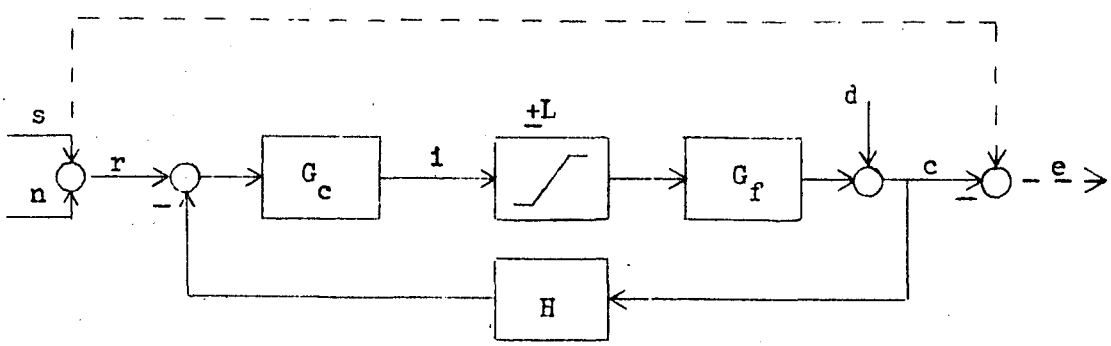


Figure 3.1-1 Basic Block Diagram

Find: Optimum G_c and H to minimize $< e^2 >$ with all $< i^2 > \leq J$.
Usually $J = (L/2)^2$.

Most control problems can be transformed into this standard form.

Except in Section 3.5, H = 1. No generality is lost as the conversion to $H \neq 1$ can easily be made. See Chapter 8 of Bower and Schultheiss[4].

One problem constraining n signals is treated as n problems each constraining one signal.

3.1.3 SIGNAL SPECTRA

The basic data required for inputs is the factored power or energy density spectra, Φ_x^+. This can be obtained as follows.

If the input is an aperiodic signal then Φ_x^+ is the Laplace transform of x(t). If the input is a random combination of aperiodic signals $X_i(s)$, each with a probability of occurrence of P_i, then the energy spectral density is given by

$$\Phi_{xx} = \sum P_i \, X_i(s) \, X_i(-s)$$

and it is factored as shown below.

If the input is a random signal, then Φ_x^+ is the Laplace transform of the autocorrelation function, $R_{xx}(\tau)$, for positive τ.

If the power spectral density Φ_{xx} is given by another author, carefully check basic definitions and convert to the system used here. See Table 3.1-4. Then to obtain Φ_x^+ from Φ_{xx}, factor as shown below.

$$(a + bs)(a - bs) = a^2 - b^2s^2$$

$$(a + bs + cs^2)(a - bs + cs^2) = a^2 - (b^2 - 2ac)s^2 + c^2s^4$$

$$(a + bs + cs^2 + ds^3)(a - bs + cs^2 - ds^3) = a^2 - (b^2 - 2ac)s^2 + (c^2 - 2bd)s^4 - d^2s^6$$

Example

If $\Phi_{xx} = 1/(1 + s^4)$, then $a = 1$, $c = 1$, $b^2 - 2ac = 0$ and $\Phi_x^+ = 1/(1 + \sqrt{2}s + s^2)$

The pattern is obvious. This convenient method of factoring is valid because the power density spectrum is symmetrical with both axes.

Normalization is an extremely useful tool, and the control engineer should be proficient in its use. Frequency normalization is explained in Sections 2.5 and 4.6 of Newton[17], and the reader is referred to that presentation for details. The use of λ for the complex variable instead of s denotes frequency normalization. Amplitude normalization applies only in determining the optimum system, and not in determining the error. The signal and noise spectra are normalized so that the crossover frequency occurs at amplitude equals unity.

$$W = \frac{1}{(\Phi_{ss} + \Phi_{nn})^+}\left[\frac{\Phi_{ss}}{(\Phi_{ss} + \Phi_{nn})^-}\right]_+$$

$$= \frac{1}{(\Phi_{ss}/k^2 + \Phi_{nn}/k^2)^+}\left[\frac{\Phi_{ss}/k^2}{(\Phi_{ss}/k^2 + \Phi_{nn}/k^2)^-}\right]_+$$

where $k^2 = \left|\Phi_{nn}(j\,\omega_c)\right| = \left|\Phi_{ss}(j\omega_c)\right|$

The usefullness of this procedure will become apparent in Sections 3.2 and 3.3.

3.1.4 INTEGRAL TABLES

Mean-square or integral-square values must be calculated analytically. A short table of integrals is given in Table 3.1-1. For a more complete table see Appendix E.2 of Newton[17] or Appendix D of Chang[5].

$$< x^2 > = I_n$$

I_1 and I_2 are simple and easy to learn. I_3 and I_4 can be used, but I_n for $n \geq 5$ is something to be avoided. The following approximation technique can be used to reduce the order of I_n.

The mean-square or integral-square value of a signal is the area under the density spectra, Φ_{xx}. This can be visualized by making a straight line Bode plot of Φ_x^+. There are two important points, when thinking about "spectrum area". First, a spectrum must

Given: $$I_n = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} ds\,\Phi_{ss}(s)$$

$$\Phi_x^+ = \frac{c_0 + c_1 s + \cdots + c_{n-1}s^{n-1}}{d_0 + d_1 s + \cdots + d_n s^n}$$

Then:

$$I_1 = \frac{c_0^2}{2d_0 d_1}$$

$$I_2 = \frac{c_1^2 d_0 + c_0^2 d_2}{2d_0 d_1 d_2}$$

$$I_3 = \frac{c_2^2 d_0 d_1 + (c_1^2 - 2c_0 c_2)\, d_0 d_3 + c_0^2 d_2 d_3}{2d_0 d_3(-d_0 d_3 + d_1 d_2)}$$

Table 3.1-1 A Short Table of Integrals.

Given: $$\Phi_x^+ = \frac{c_0 + \cdots + c_{n-1}(s/b)^{n-1}}{B_n(s/b)}$$

Then:

$$I_1 = \frac{b}{2}\, c_0^2$$

$$I_2 = \frac{b}{2}\,\frac{(c_0^2 + c_1^2)}{\sqrt{2}}$$

$$I_3 = \frac{b}{2}\,\frac{2(c_0^2 + c_2^2) + (c_1^2 - 2c_0 c_2)}{3}$$

$$I_4 = \frac{b}{2}\,\frac{(1+\sqrt{2})(c_0^2 + c_3^2) + (c_1^2 - 2c_0 c_2) + (c_2^2 - 2c_1 c_3)}{2\sqrt{2+\sqrt{2}}}$$

Table 3.1-2 A Short Table of Integrals with a Butterworth Denominater.

be bounded by $A/(1 + s/v)$ for some A and v in order to have a finite $< x^2 >$. Second, for the spectra shown in Figure 3.1-2 the area of Φ_x^+ is $A^2 v$ times the area of Φ_y^+. The "spectrum area" is concentrated towards the upper right. The second point leads to the useful approximation technique shown in Figure 3.1-3. Often Φ has a Butterworth denominator. Then the expression for I_n can be simplified as shown in Table 3.1-2.

3.1.5 INTUITION

Intuitively one recognizes that for frequencies where the signal is much greater than the noise, the optimum closed-loop transfer function should equal the desired transfer function which equals unity, i.e. for $\Phi_s^+ \gg \Phi_n^+$, $W = 1$; and that for frequencies where the signal is much smaller than the noise, for $\Phi_s^+ \ll \Phi_n^+$, the optimum closed-loop transfer function, W, should attenuate the transmitted noise making its power finite.

Because Butterworth polynomials provide the "optimum" maximally flat frequency response, it is not very surprising that they are also optimum, or almost optimum, for the crossover region breaks necessary to provide adequate phase margin.

Consider the factored transmitted noise spectrum, $W \Phi_n^+$, and frequencies where the noise is much larger than the signal, $\Phi_s^+ \ll \Phi_n^+$. If $\mathcal{O}(W \Phi_n^+) \geq 0$, then the transmitted noise power is infinite. If

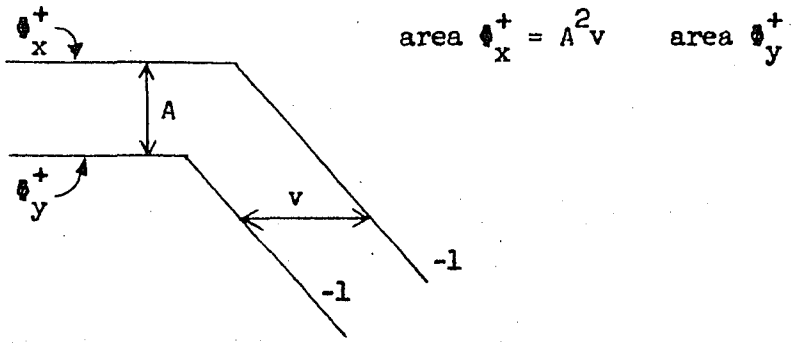
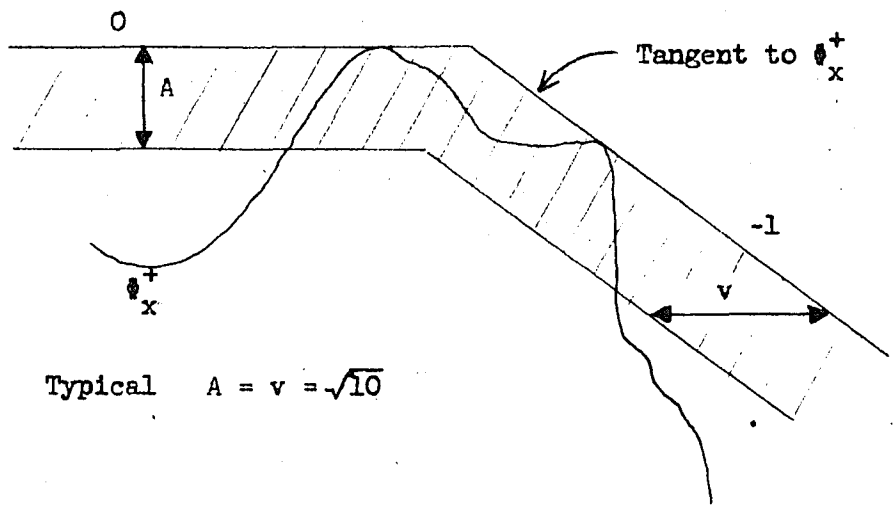


Figure 3.1-2 Example Illustrating Spectrum Area.



In the shaded area use the exact expression. In the remaining area make convenient approximations to reduce the order of I_n.

Figure 3.1-3 Approximation to Lower Order of I_n.

$\mathcal{O}(W\ \Phi_n^+)$ is reduced from 0 to -1, then the noise error is reduced to a finite value. Further reduction of $\mathcal{O}(W\ \Phi_n^+)$ reduces the noise error by only a finite amount. Assume that Φ_n^+ is monotone increasing. In this region therefore, $\mathcal{O}(\Phi_n^+) \geq 0$, and $\mathcal{O}(W) = \mathcal{O}(W\ \Phi_n^+) - \mathcal{O}(\Phi_n^+)$. Every reduction in $\mathcal{O}(W\ \Phi_n^+)$ increases the signal error by a finite amount. Therefore, the maximum gain appears to be in reducing the noise error from infinity to a finite value, while only increasing the signal error by a finite amount. That is, requiring that $\mathcal{O}(W\ \Phi_n^+) = -1$.

If one philosophically considers the technique of mathematical optimization one realizes that the answer is already contained in the statement of the problem. It may not be easy or even possible to obtain, but it is inevitably there. Therefore, a direct relationship between the solution for a class of problems and the given data is quite reasonable. The best known example of this is the maximum-power transfer theorem of circuit theory. This states that for a source to deliver the maximum power to a load, the load impedance should be the complex conjugate of the source impedance. In communication theory there is the matched filter concept. See Davenport and Root[7], pages 244-247. This states that to maximize the output signal-to-noise ratio, where the signal is a known function of time and the noise is white noise, the optimum weighting function has the form of the signal run backward in time.

Consider the case of white noise, $\Phi_{nn} = K$, a constant. It

has been often shown (i.e., Truxal[22] Eq.(8.29)) that the optimum unrealizable closed-loop transfer function is given by

$$W_u = \frac{\Phi_{ss}}{\Phi_{nn} + \Phi_{ss}}$$

It is obvious that dividing through by Φ_{nn} will not affect W, so this is done. The normalized Φ_{ss} is divided up into factors containing only LHP roots Φ_s^+, and factors containing only RHP roots Φ_s^-, each which the square root of the gain.

$$W_u = \frac{\Phi_s^+ \Phi_s^-}{1 + \Phi_s^+ \Phi_s^-}$$

If it is naïvely assumed that all that has to be done to make W realizable is to throw away factors containing RHP poles, i.e. Φ_s^-, the following result is obtained.

$$W = \frac{\Phi_s^+}{1 + \Phi_s^+}$$

but

$$W = \frac{G}{1 + G}$$

therefore: $G \approx \Phi_s^+$

But only for $\Phi_s^+ > 1$, because for $\Phi_s^+ < 1$ the $\angle(G\,\Phi_n^+) = -1$.

The preceding points which are intuitively reasonable are indeed valid. This will be demonstrated in the following sections.

3.1.6 CHAOS IN NOTATION

The purpose of this section is to enable the nonspecialist to refer to standard texts on this subject without getting horribly mixed up in notation and scale factors.

In statistical control theory there exist marvelous opportunities to make errors by factors of 2, 2π, $\sqrt{2\pi}$, and 4π.

Table 3.1-3 presents six different forms which are used for transform pairs. These are used randomly in three different situations:

a. System weighting functions - transfer functions.

b. Aperiodic signals - energy spectral densities.

c. Correlation functions - power spectral densities.

In addition four different forms are used for spectral densities. See Korn[13] Eq.(18.10.17). These are related by

$$\Gamma(f) = 2\gamma(f) = 2\pi\, G(\omega) = 4\pi\, g(\omega)$$

Table 3.1-4 presents the notation used in this thesis and in fourteen standard texts on the subject. Factors of 2 and 2π are added so that all notation in a column are equivalent. The relationship between the columns is given by the scale factor.

$f_1(t) = \int_{-\infty}^{\infty} F_1(f)\, e^{i\omega t} df$	$F_1(f) = \int_{-\infty}^{\infty} f_1(t)\, e^{-i\omega t} dt$
$f_2(t) = \frac{1}{2} \int_{-\infty}^{\infty} F_2(f)\, e^{i\omega t} df$	$F_2(f) = 2 \int_{-\infty}^{\infty} f_2(t)\, e^{-i\omega t} dt$
$f_3(t) = \int_{-\infty}^{\infty} F_3(\omega)\, e^{i\omega t} d\omega$	$F_3(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_3(t)\, e^{-i\omega t} dt$
$f_4(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_4(\omega)\, e^{i\omega t} d\omega$	$F_4(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_4(t)\, e^{-i\omega t} dt$
$f_5(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_5(\omega)\, e^{i\omega t} d\omega$	$F_5(\omega) = \int_{-\infty}^{\infty} f_5(t)\, e^{-i\omega t} dt$
$f_6(t) = \frac{1}{2} \int_{-\infty}^{\infty} F_6(\omega)\, e^{i\omega t} d\omega$	$F_6(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f_6(t)\, e^{-i\omega t} dt$

Table 3.1-3 Commonly Used Transform Pairs.

T = Time
E = Ensemble

		Power Density (Random) Power (Periodic)				Energy Density (Aperiodic)
		Two-sided		One-sided		Two-sided
		ω	f	ω	f	ω
Scale Factor →	Ave.	$\frac{1}{2\pi}$	1	$\frac{1}{\pi}$	2	$\frac{1}{2\pi}$
Streets	T	$\frac{\Phi(s)}{2\pi}$				$\frac{\Phi(s)}{2\pi}$
Korn[13]	T	$g(\omega)$	$\gamma(\nu)$	$G(\omega)$	$\Gamma(\nu)$	
	E	$\mathcal{G}(\omega)$		$G(\omega)$		
Lee[15]	T, E	$\Phi(\omega)$				$\Phi(\omega)$
Newton, Gould and Kaiser[17]	T, E	$\Phi(s)$				$\frac{I(s)}{2\pi}$
Middleton[16]	T				$\mathcal{W}^{(j)}(f)$	$\frac{\lvert S^{(j)}(f)\rvert^2}{2\pi}$
	E				$\mathcal{W}(f)$	
Chang[5]	T, E	$\frac{\Phi(s)}{2\pi}$				$\frac{A(s)A(-s)}{2\pi}$
Bendat[2]	E	$\frac{G(\omega)}{2}$	$\frac{\mathcal{G}(f)}{2}$			
Davenport and Root[7]	T		$\mathcal{S}(f)$			
	E		$S(f)$			
Laning and Battin[14]	E			$G(\omega)$		$\frac{E(\omega,x)}{2}$
Solodovnikov[21]	T, E	$\frac{S(\omega)}{2\pi}$				
Smith[20]	T	$\frac{\phi(\lambda)}{2\pi}$				
Blackman and Tukey[3]	T		$P(f)$			
Aseltine[1]; Truxal[22]	T	$\frac{\Phi(j\omega)}{2\pi}$				
Horowitz[10]	E					$\frac{E\,M(s)N(-s)}{2\pi}$

Table 3.1-4 The Chaos in Notation for Signal Spectra

3.2 INPUT WHITE NOISE

3.2.1 INTRODUCTION

This is an important and common case. In control problems the noise is usually assumed to be white, often because of the lack of any better data. The white noise case also clearly illustrates the basic "matched filter" concept.

3.2.2 ALGORITHMS

a. Control Problem Rule

Φ_s^+ is a "low pass" function.

$$\frac{\Phi_s^+(0)}{\Phi_n^+(0)} \geq 10$$

b. White Noise Theorem

Given: $\Phi_{ss} = \frac{N}{D}$, $\Phi_{nn} = 1$

Then: $G = \frac{(N + D)^+ - D^+}{D^+}$

c. White Noise Algorithm

Normalize so that $\left| \Phi_s^+(j\omega_c) \right| = \left| \Phi_n^+(j\omega_c) \right| = 1$. Then

$\Phi_s^+ \gg \Phi_n^+ \qquad G \approx \Phi_s^+$

$\Phi_s^+ \approx \Phi_n^+$ add Butterworth zeros at $\omega = \omega_c/\sqrt{2}$ so that for

$\Phi_s^+ \ll \Phi_n^+$ slope G $\Phi_n^+ = -1$

3.2.3 DERIVATIONS

a. Control Problem Rule

The rule that Φ_s^+ is a "low pass" function simply recognizes that this is a characteristic of almost all control systems and their signal inputs. It is not a necessary part of the theory, but it does simplify the presentation. The rule that $\Phi_s^+(0) \geq 10\, \Phi_n^+(0)$ is a result of the control problem assumption that the RMS error $\leq$ 1/2 RMS value of the signal. This rule is justified in the section on error analysis.

b. White Noise Theorem

Given: $\Phi_{ss} = \frac{N}{D}$, $\Phi_{nn} = 1$ (normalized)

Then: $G = \frac{(N + D)^+ - D^+}{D^+}$

Proof: $W = \frac{1}{(\Phi_{nn} + \Phi_{ss})^+} \left[\frac{\Phi_{ss}}{(\Phi_{nn} + \Phi_{ss})^-} \right]_+$

$= \frac{D^+}{(D + N)^+} \left[\frac{N}{D^+(D + N)^-} \right]_+$

$= \frac{D^+}{(D + N)^+} \left[\frac{(D + N)^+ - D^+}{D^+} + \frac{(D + N)^- - D^-}{(D + N)^-} \right]_+$

$(D + N) - D^+(D + N)^- + D^+(D + N)^- - D = N$

(cont) Given: $\mathscr{O}(D) = 2n, \quad \mathscr{O}(N) \leq 2(n-1)$

Then: $\mathscr{O}(D^+) = \mathscr{O}((D+N)^+) = n$

$$\mathscr{O}((D+N)^+ - D^+) = \mathscr{O}((D+N)^- - D^-) = n-1$$

$$W = \frac{(D+N)^+ - D^+}{(D+N)^+} \quad , \quad G = \frac{W}{1-W} \quad , \quad \therefore \; G = \frac{(N+D)^+ - D^+}{D^+}$$

Note. This is a very useful result by itself. It eliminates two operations, $[\;]_+$ and G from W, which are difficult on nontrivial problems. Also, on nontrivial problems it yields the answer in a better form.

Example. The following example with six arbitrary parameters illustrates the validity of the partial fraction expansion.

Given: $\Phi_{ss} = \dfrac{c_0^2 - c_1^2 s^2 + c_2^2 s^4}{d_0^2 - d_1^2 s^2 + d_2^2 s^4 - d_3^2 s^6}$, $\quad \Phi_{nn} = 1$

$$\triangle = \frac{(c_0^2 + d_0^2) - (c_1^2 + d_1^2)\, s^2 + (c_2^2 + d_2^2)\, s^4 - d_3^2 s^6}{d_0^2 - d_1^2 s^2 + d_2^2 s^4 - d_3^2 s^6}$$

$$= \frac{(e_0^2) - (e_1^2 - 2e_0e_2)\, s^2 + (e_2^2 - 2e_1d_3)\, s^4 - d_3^2 s^6}{(d_0^2) - (D_1^2 - 2d_0D_2)\, s^2 + (D_2^2 - 2D_1d_3)\, s^4 - d_3^2 s^6}$$

$$c_0^2 = e_0^2 - d_0^2$$

$$c_1^2 = e_1^2 - 2e_0e_2 - D_1^2 + 2d_0D_2$$

$$= (e_1^2 - D_1^2) - (e_0 - d_0)(e_2 + D_2) - (e_0 + d_0)(e_2 - D_2)$$

$$c_2^2 = e_2^2 - 2e_1d_3 - D_2^2 + 2D_1d_3$$

$$= (e_2^2 - D_2^2) - 2d_3(e_1 - D_1)$$

$$\left[\frac{\Gamma}{\Delta}\right]_+ = \left[\frac{c_0^2 - c_1^2 s^2 + c_2^2 s^4}{(d_0 + D_1 s + D_2 s^2 + d_3 s^3)(e_0 - e_1 s + e_2 s^2 - e_3 s^3)}\right]_+$$

$$= \left[\frac{(e_0 - d_0) + (e_1 - D_1)\,s + (e_2 - D_2)\,s^2}{d_0 + D_1 s + D_2 s^2 + d_3 s^3} + \right.$$

$$\left. + \frac{(e_0 - d_0) - (e_1 - D_1)s + (e_2 - D_2)s^2}{e_0 - e_1 s + e_2 s^2 - d_3 s^3}\right]_+$$

s^0	$+(e_0 - d_0)(e_0 + d_0)$			$= c_0^2$
s^1	$-(e_0 - d_0)(e_1 - D_1)$	$+ (e_1 - D_1)(e_0 - d_0)$		$= 0$
s^2	$+(e_0 - d_0)(e_2 + D_2)$	$- (e_1 - D_1)(e_1 + D_1)$	$+ (e_2 - D_2)(e_0 + d_0)$	$= -C_1^2$
s^3	$-(e_0 - d_0)(0)$	$+ (e_1 - D_1)(e_2 - D_2)$	$- (e_2 - D_2)(e_1 - D_1)$	$= 0$
s^4		$- (e_1 - D_1)(2d_3)$	$+ (e_2 - D_2)(e_2 + D_2)$	$= c_2^2$
s^5			$- (e_2 - D_2)(0)$	$= 0$

c. <u>White Noise Algorithm</u>

(1) Matched Filter Rule

Given: $\Phi_s^+ \gg \Phi_n^+$, then $N \gg D$ and

$$G = \frac{(N + D)^+ - D^+}{D^+} \approx \frac{N^+ - D^+}{D^+} \approx \frac{N^+}{D^+} = \Phi_s^+$$

(2) Approximation

$B_n(\lambda)$ = n^{th} order Butterworth polynomial

$$1 + (-\lambda^2)^n \triangleq B_n(\lambda)\, B_n(-\lambda)$$

$$B_{n-1}(\sqrt{2}\,\lambda) \approx B_n(\lambda) - \lambda^n$$

Examples

n = 1	(1)	both
n = 2	$(1 + \lambda/.707)$	both
n = 3	$(1 + 2(.707)\lambda/.707 + (\lambda/.707)^2)$	both
n = 4	$(1 + \lambda/.707)(1 + 2(.500)\lambda/.707 + (\lambda/.707)^2)$	$B_3(\sqrt{2}\,\lambda)$
	$(1 + \lambda/.667)(1 + 2(.422)\lambda/.757 + (\lambda/.757)^2)$	$B_4(\lambda) - \lambda^4$

(3) Theorem

Given: $\Phi_{ss} = \dfrac{1}{(-\lambda^2)^n}$, $\Phi_{nn} = 1$, $\lambda = s/\omega_c$

Then:
$$G \approx \frac{B_{n-1}(\sqrt{2}\,\lambda)}{\lambda^n}$$

Proof: White noise theorem; approximation.

(4) The actual justification of the white noise algorithm rests on two facts. First, it is reasonable. This is shown below. Second, it works. This is demonstrated by the examples and the error analysis.

An arbitrary Φ_{ss} can be approximated as follows.

$$\Phi_{ss} = \frac{1}{(-\lambda^2)^{n_k}} \qquad \omega_k < \omega < \omega_{k+1}$$

$$\Phi_s^+ \approx \Phi_n^+$$

$$G \approx \frac{B_{n-1}(\sqrt{2}\,\lambda)}{\lambda^n} \qquad \omega \approx \omega_c$$

$\therefore$ Add Butterworth zeros at $\omega = \omega_c/\sqrt{2}$

so that G has a -1 slope.

$$\Phi_s^+ \gg \Phi_n^+$$

$$G \approx \frac{B_{n-1}(\sqrt{2}\,\lambda)}{\lambda^{n_k}} = \frac{1}{\lambda^{n_k}} = \Phi_s^+, \quad \omega_k < \omega < \omega_{k+1}$$

This is true for all k ($\omega_k < \omega_c/\sqrt{2}$)

$$\therefore G = \Phi_s^+ \qquad \omega < \omega_c/\sqrt{2}$$

This is verified by the Matched Filter Rule

$$\Phi_s^+ \ll \Phi_n^+$$

$$G\,\Phi_n^+ \approx \frac{B_{n-1}(\sqrt{2}\,\lambda)}{\lambda^{n_k}} = \frac{1}{\lambda} \qquad \omega_k < \omega < \omega_{k+1}$$

This is true for all k ($\omega_k > \omega_c/\sqrt{2}$)

$\therefore$ slope of $G\,\Phi_n^+ = -1 \quad \omega > \omega_c/\sqrt{2}$

3.2.4 EXAMPLES

A simple numerical example is completely worked out in Figure 3.2-1 to illustrate the algorithm. Following this is Table 3.2-1 which presents seven typical examples with literal coefficients. This table is a "do it yourself kit" which enables the reader to practice using the white noise algorithm.

Table 3.2-1 also demonstrates that no generality is lost by amplitude and frequency normalization. Divide both density spectrums by the noise density spectrum. Define v as the dominate break frequency, and define a normalized complex frequency $\lambda = s/v$. Also define a normalized signal-to-noise factor, b. Table 3.2-2 shows the result of normalizing the examples of Table 3.2-1.

3.2.5 ERROR ANALYSIS

It is important to know just how much performance loss is caused by using the approximate system instead of the exact system.

The RMS value (rather than the mean-square value) of the error is used as the standard of comparison for two reasons.

a. It has the same dimensions as the error, and hence presents a fairer comparison.

b. RMS values have an intuitive significance for electrical engineers.

Given:

$$\Phi_{ss} = \frac{10^4}{-s^2(10^2 - s^2)}$$

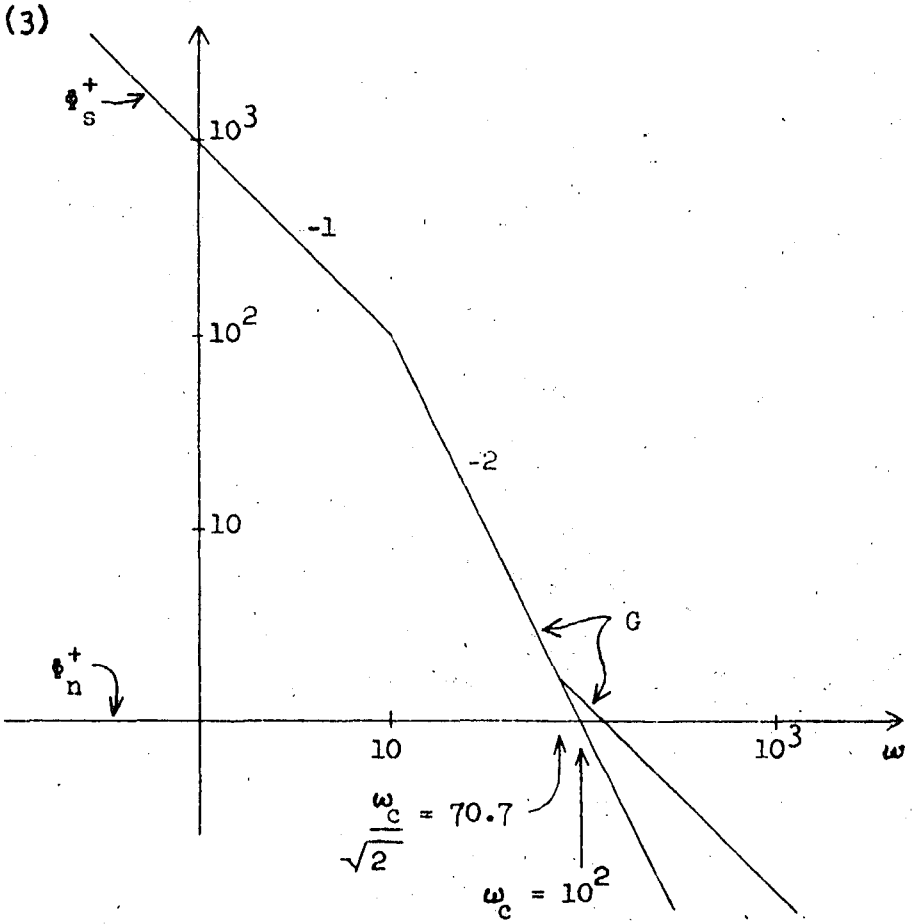
$$\Phi_{nn} = \frac{1}{10^4}$$

Find: G

Solution: (1) $\Phi_{ss} = \dfrac{10^6}{-s^2(1 - s^2/10^2)}$ $\qquad \Phi_{nn} = 1$

(2) $\Phi_s^+ = \dfrac{10^3}{s(1 + s/10)}$ $\qquad \Phi_n^+ = 1$

(3)



$$G = \frac{10^3\,(1 + s/70.7)}{s(1 + s/10)}$$

Figure 3.2-1 White Noise Example (All steps shown).

	Φ_{ss}	Φ_{nn}	Optimum open loop transfer function, G	
			By Inspection	Exact
A	$\frac{d^2}{-s^2}$	n^2	$(d/n)\frac{1}{s}$	$(d/n)\frac{1}{s}$
B	$\frac{d^2}{1-s^2/v^2}$		$\frac{(d/n)\ 1}{1+s/v}$ $\quad d/n > 10$	$\frac{\sqrt{1+(d/n)^2}-1}{1+s/v}$
C	$\frac{d^2}{s^4}$		$\frac{(d/n)(1+s/\alpha)}{s^2}$ $\alpha = \frac{\sqrt{d/n}}{\sqrt{2}}$	$\frac{(d/n)(1+s/\beta)}{s^2}$ $\beta = \frac{\sqrt{d/n}}{\sqrt{2}}$
D	$\frac{d^2}{-s^2(1-s^2/v^2)}$		$\frac{(d/n)(1+s/\alpha)}{s(1+s/v)}$ $\alpha = \frac{v\sqrt{d/nv}}{\sqrt{2}} \quad v < \frac{d/n}{\sqrt{2}}$ $\alpha = v \quad v > \frac{d/n}{\sqrt{2}}$	$\frac{(d/n)(1+s/\beta)}{s(1+s/v)}$ $\beta = \frac{v}{2}(1+\sqrt{1+2d/nv})$

Table 3.2-1a White Noise Examples

	Φ_{ss}	Φ_{nn}	Optimum open loop transfer function, G	
			By Inspection	Exact
E	$\dfrac{d^2}{1 + s^4/v^4}$ $(\zeta = .707)$	n^2	$\dfrac{(d/n)(1 + s/\alpha)}{(1 + \sqrt{2}\, s/v + s^2/v^2)}$ $\alpha = v\dfrac{\sqrt{d/n}}{\sqrt{2}} \quad d/n > 10$	$\dfrac{(\sqrt{1 + (d/n)^2} - 1)(1 + s/\beta)}{(1 + \sqrt{2}\, s/v + s^2/v^2)}$ $\beta = \dfrac{v(1 + (1 + (d/n)^2)^{\frac{1}{4}})}{\sqrt{2}}$
F	$\dfrac{d^2}{(1 - s^2/v^2)^2}$ $(\zeta = 1)$		$\dfrac{(d/n)(1 + s/\alpha)}{(1 + s/v)^2}$ $\alpha = v\dfrac{\sqrt{d/n}}{\sqrt{2}} \quad d/n > 10$	$\dfrac{(\sqrt{1 + (d/n)^2} - 1)(1 + s/\beta)}{(1 + s/v)^2}$ $\beta = v\,\dfrac{(\sqrt{1 + c} + \sqrt{2})}{\sqrt{2}}$ $c = \sqrt{1 + (d/n)^2}$
G	$\dfrac{d^2(1 - s^2/v^2)}{s^4}$		$\dfrac{(d/n)(1 + s/\alpha)}{s^2}$ $\alpha = v \quad v < \dfrac{\sqrt{d/n}}{\sqrt{2}}$ $\alpha = \dfrac{\sqrt{d/n}}{\sqrt{2}} \quad v > \dfrac{\sqrt{d/n}}{\sqrt{2}}$	$\dfrac{(d/n)(1 + s/\beta)}{s^2}$ $\beta = \dfrac{v}{\sqrt{1 + \dfrac{2v^2}{d/n}}}$

Table 3.2-1b White Noise Examples

	Φ_{ss}	Φ_{nn}
A	$\frac{b^2}{-\lambda^2}$	
B	$\frac{b^2}{1 - \lambda^2}$	
C	$\frac{b^2}{\lambda^4}$	
D	$\frac{b^2}{-\lambda^2(1 - \lambda^2)}$	1
E	$\frac{b^2}{1 + \lambda^4}$	
F	$\frac{b^2}{(1 - \lambda^2)^2}$	
G	$\frac{b^2(1 - \lambda^2)}{\lambda^4}$	

Table 3.2-2 Normalized White Noise Examples

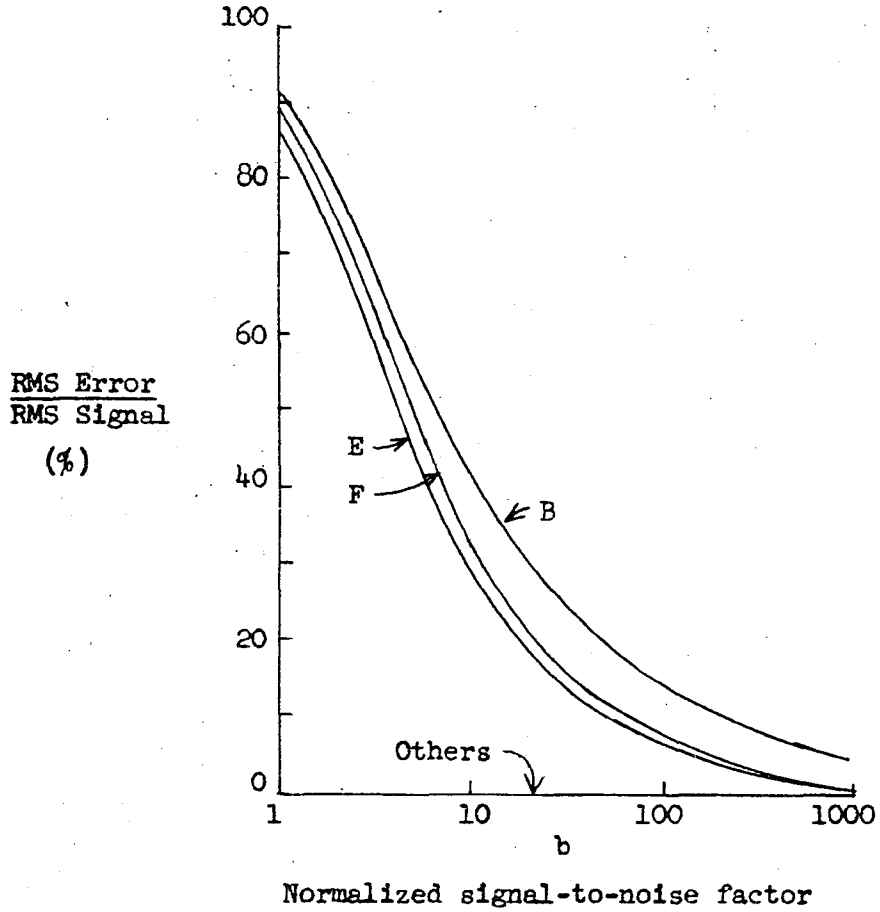


Figure 3.2-2 Data for the Control Problem Rule

The seven examples of the preceding section are presented in normalized form in Table 3.2-2.

Figure 3.2-2 presents the ratio of the RMS error to the RMS value of the signal, as a function of the normalized signal-to-noise factor, b. Control problem assumption d requires that the RMS error $\leq \frac{1}{2}$ the RMS value of the signal. Figure 3.2-2 shows that for examples B, E, and F, the normalized signal-to-noise factor, b, should be greater than 10. This leads to the following rule:

$$\frac{\Phi_s^+(0)}{\Phi_n^+(0)} \geq 10$$

The appropriate criteria for evaluating the approximate system is as follows:

$$E_a = \frac{e_a - e_0}{e_0} \times 100$$

Where e_a = RMS error of the system designed by inspection

e_0 = RMS error of the Wiener system

E_a = "Approximation Error", or the increase in RMS error caused by using the approximate system.

Figure 3.2-3 presents the approximation error E_a as a function of the normalized signal-to-noise factor, b, for the seven examples subject to the control problem rule. It can be seen from Figure 3.2-3 that the approximation error is less than 4%. In other words, the approximate system is an excellent approximation.

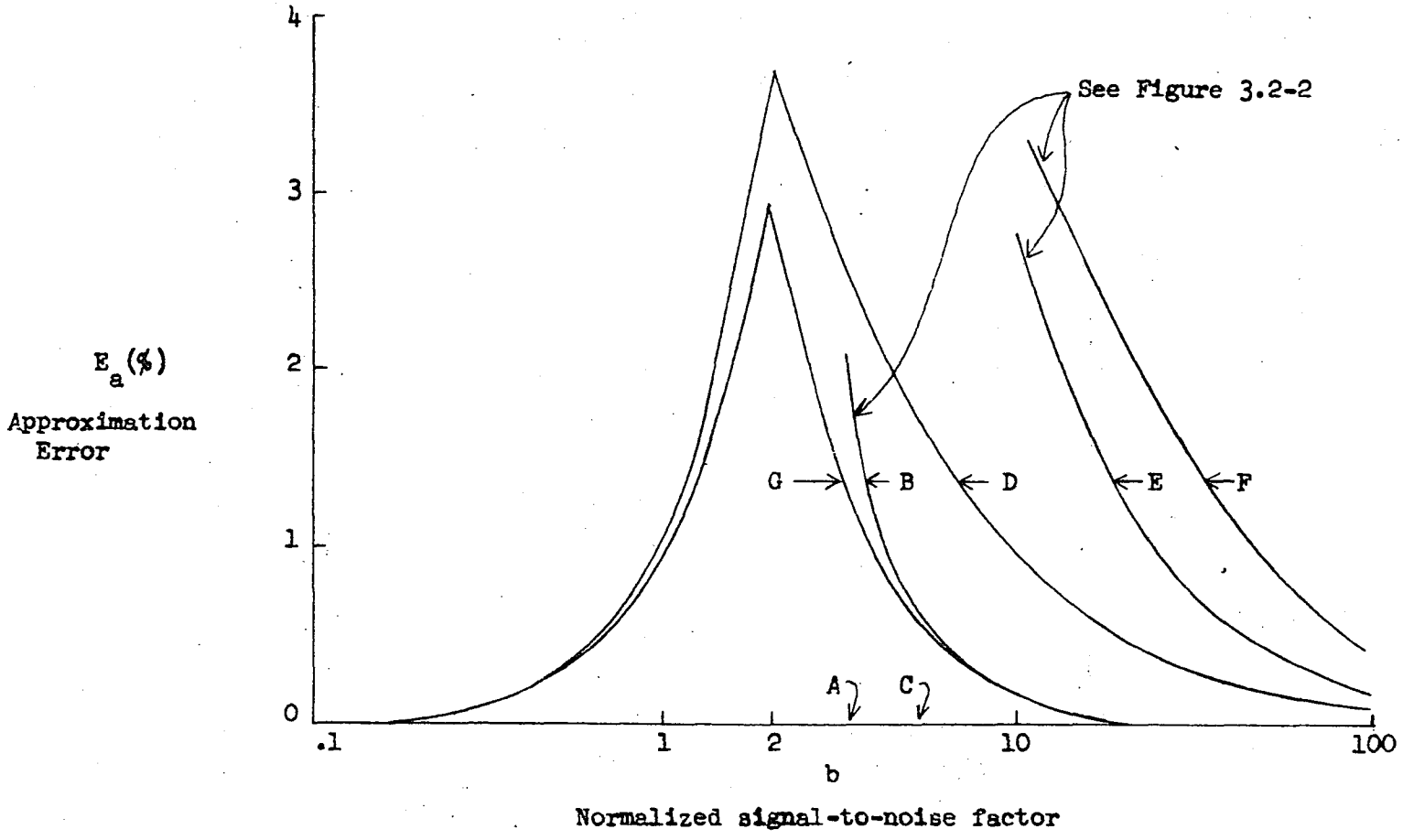


Figure 3.2-3 Error Analysis for White Noise Examples

3.3 INPUT COLORED NOISE

3.3.1 INTRODUCTION

In the preceding section it was shown that for white noise:

$$\Phi_s^+ \gg \Phi_n^+ \quad G \approx \Phi_s^+$$

$$\Phi_s^+ \ll \Phi_n^+ \quad \text{slope } G\ \Phi_n^+ = -1$$

In this section these results are extended to include the colored noise case.

3.3.2 ALGORITHMS

a. Realizability Rule

To satisfy $\mathcal{O}(W) \leq \mathcal{O}(G_f) = -n \leq -1$, add a noise, $\Phi_n^+ = (ks)^{n-1}$, where k is arbitrary. To also satisfy $\omega_c \leq b$, then k is determined by $\Phi_s^+(b) = \Phi_n^+(b)$.

b. Colored Noise Rule

$\omega > \omega_c$ Φ_n^+ is monotone increasing

$\omega < \omega_c$ $\Phi_n^+ < \Phi_s^+$, $\Phi_n^+(0)$ is finite.

c. Lemma

Given: $\Phi_s^+ = \frac{1}{\lambda^k}$, $\Phi_n^+ = \lambda^m$; $k \geq 1$, $m \geq 0$

Then:

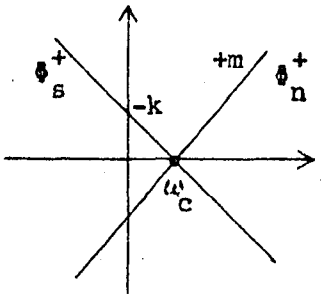
$$G = \frac{{}^{k}B_{k+m}}{B_{k+m}^{m+1}}$$

where ${}^{k}B_{k+m}$ denotes the first k terms, and

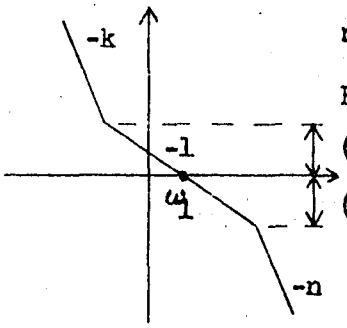
B_{k+m}^{m+1} denotes the last m+1 terms of the $k+m^{th}$ order normalized Butterworth polynomial. (See pp. 55-56)

d. Approximation

Given:



Then:



$n = m+1$, $\omega_1 = \omega_c 2^{(k-n)/2}$

Butterworth breaks

(k+1)/10 decades

(n+1)/10 decades

(See pp. 60-63)

e. Colored Noise Algorithm

Normalize so that $\left|\Phi_s^+(j\omega_c)\right| = \left|\Phi_n^+(j\omega_c)\right| = 1$. Then $\Phi_s^+ \gg \Phi_n^+$ $G = \Phi_s^+$

$$\Phi_s^+ \approx \Phi_n^+ \qquad G = \frac{k B_{k+m}}{B_{k+m}^{m+1}} \qquad \begin{array}{l} \text{slope } \Phi_s^+(\omega_c) = -k \leq -1 \\ \text{slope } \Phi_n^+(\omega_c) = m \geq 0 \end{array}$$

$$\Phi_s^+ \ll \Phi_n^+ \qquad \text{slope } G\,\Phi_n^+ = -1.$$

3.3.3 DERIVATION

a. Realizability Rule

The realizability constraint $\mathcal{O}(W) \leq \mathcal{O}(G_f) \leq -1$ is always present. Occasionally there is also the bandwidth constraint $\omega_c \leq b$. This constraint would be present, for example, if the dead time approximation

$$e^{-sT} = \frac{1 - Ts/2}{1 + Ts/2} \qquad \text{for } \omega_c \leq \frac{2}{T}$$

is used in G_f.

The reader is referred to Chapter 2 of Kaiser's[12] report for a presentation of "Narrow Sense Realizability", and to Chapter 8 of Newton's[17] book for a presentation of "Bandwidth Limitation". Both of these depend upon Chapters 4 and 5 of Newton's book.

It is straightforward to show that both these procedures consist of adding a noise $\Phi_{nn} = (-k^2 s^2)^{n-1}$ to the usual filter problem. In the narrow sense realizability problem $n = -\mathcal{O}(G_f)$, and k is arbitrary. In the bandwidth limitation problem n is arbitrary and k can be determined by $\Phi_{ss}(b) = \Phi_{nn}(b)$. The

equations for setting the bandwidth are unnecessary because the bandwidth is obvious and more meaningful on a Bode plot.

It is usually better to neglect these somewhat arbitrary constraints, because it is always necessary to design for a saturation constraint, and this can usually uniquely handle both of these constraints at the same time. The main purpose in presenting this rule is to show that the colored noise rule, Φ_n^+ is monotone increasing for $\omega > \omega_c$, is a reasonable assumption.

b. Colored Noise Rule

The rule that Φ_n^+ is monotone increasing for $\Phi_n^+ > \Phi_s^+$ can always be satisfied by using the realizability rule. The slope of the added noise is ≥ 0 (usually $>$), and the noise can be made to go thru ω_c by proper choice of k. This rule is necessary to eliminate cases which violate the control problem assumption that for $\omega > \omega_c$ the slope $W \leq -1$. The actual rule is that for $\Phi_s^+ < \Phi_n^+$ the slope of transmitted noise $W \Phi_n^+ = -1$ (almost always). If $\mathcal{O}(\Phi_n^+) = -2$ the $\mathcal{O}(W) = +1$.

The rule that for $\omega < \omega_c$, $\Phi_n^+ < \Phi_s^+$ simply avoids "bandpass" control systems. The rule that $\Phi_n^+(0)$ is finite is necessary for two reasons. If $\Phi_n^+(0) = \infty$ and $\Phi_s^+(0)$ is finite, a "bandpass" control system results. If both $\Phi_n^+(0)$ and $\Phi_s^+(0) = \infty$ then the error is infinite.

c. Butterworth Coefficients

The coefficients for n = 1, 2, and 3 are used quite often and are automatically memorized.

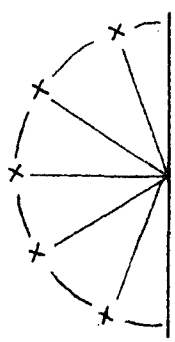
B_1 : (1, 1)

B_2 : $(1, \sqrt{2}, 1)$

B_3 : (1, 2, 2, 1)

The coefficients for n > 3 are very simple to calculate. All one has to remember is that the roots are uniformly spaced on the unit circle, and composed of the factors $(1 + \lambda)$ and $(1 + 2\zeta\lambda + \lambda^2)$, where ζ is the cosine of the angle from the negative real axis to the root. For example, the complete calculation for B_5, using only a slide rule, is shown below.

B_5 $10\overline{)360}^{36}$ 72° $2(.309) = .618$
36° $2(.809) = 1.618$



1	1.618	1
1	.618	1
1	1.618	1
	.618	1.000
		1
1	2.236	3
1	1	
1	2.236	3
	1	2.236

B_5 (1 3.236 5.236, 5.236, 3.236, 1)

For the purpose of reference the Butterworth coefficients for n = 0 to 7 are presented in Table 3.3-1. The last portion of the table is included because it is useful for the specialist.

n	$B_n(\lambda)$							
0	1							
1	1	1						
2	1	$\sqrt{2}$	1					
3	1	2	2	1				
4	1	2.613	3.414	2.613	1			
5	1	3.236	5.236	5.236	3.236	1		
6	1	3.864	7.464	9.141	7.464	3.864	1	
7	1	4.494	10.103	14.606	14.606	10.103	4.494	1
4	1	$\sqrt{2}\,\alpha$	α^2	$\sqrt{2}\,\alpha$	1			
5	1	$\sqrt{2}\,\beta$	β^2	β^2	$\sqrt{2}\,\beta$	1		
6	1	$\sqrt{2}\,\gamma$	γ^2	$\sqrt{2}(\gamma^2-1)$	γ^2	$\sqrt{2}\,\gamma$	1	

$$\alpha^2 = 2 + \sqrt{2}$$

$$\beta^2 = 3 + \sqrt{5}\ , \quad \sqrt{2}\beta = 1 + \sqrt{5}$$

$$\gamma^2 = 4 + \sqrt{12}$$

Table 3.3-1 Table of Butterworth Coefficients.

d. Lemma

Given: $\Phi_s^+ = \dfrac{1}{\lambda^k}$, $\Phi_n^+ = \lambda^m$, $k \geq 1$, $m \geq 0$

Then: $G = \dfrac{{}^{k}B_{k+m}}{B_{k+m}^{m+1}}$, $W = \dfrac{{}^{k}B_{k+m}}{B_{k+m}}$

where: ${}^{k}B_{k+m}$ denotes the first k terms, and

B_{k+m}^{m+1} denotes the last m+1 terms of the $k+m^{th}$ order normalized Butterworth polynomial.

Proof:

$\Gamma = \Phi_{ss}$, $\Delta = \Phi_{ss} + \Phi_{nn} = \dfrac{B_{k+m}(\lambda)\, B_{k+m}(-\lambda)}{(\lambda)^k\, (-\lambda)^k}$

$$\left[\frac{\Gamma}{\Delta}\right]_+ = \left[\frac{1}{\lambda^k\, B_{k+m}(-\lambda)}\right]_+$$

$$= \left[\frac{A_0 + A_1\lambda + \ldots + A_{k-1}\lambda^{k-1}}{\lambda^k} + \ldots\right]_+$$

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left.\frac{1}{B_{k+m}(-\lambda)}\right|_{\lambda = 0}$$

$$= \frac{1}{n!}\frac{d^n}{d\lambda^n}\left.\frac{B_{k+m}(\lambda)}{1 - (-\lambda^2)^{k+m}}\right|_{\lambda = 0}$$

Note if $v(0) = 1$, $v^{(k)}(0) = 0 \quad k > 0$

then $\dfrac{d^n}{d\lambda^n}\left.\dfrac{u}{v}\right|_{\lambda = 0} = u^{(n)}(0)$

$$A_n = \frac{1}{n!} B^{(n)}_{k+m}(0) \qquad B_{k+m}(\lambda) = \sum_{j=0}^{k+m} a_j \lambda^j$$

$$B^{(n)}_{k+m}(\lambda) = \sum_{j=n}^{k+m} j(j-1) \cdots (j-n+1)\, a_j\, \lambda^{j-n}$$

$$B^{(n)}_{k+m}(0) = n!\, a_n \qquad \therefore A_n = a_n$$

$$\left[\frac{\Gamma}{\Delta^-}\right]_+ = \frac{k_{B_{k+m}}}{\lambda^k} \qquad W = \frac{1}{\Delta^+}\left[\frac{\Gamma}{\Delta^-}\right]_+ = \frac{k_{B_{k+m}}}{B_{k+m}} \qquad C = \frac{W}{1-W}$$

This Lemma is an important result. It is the general solution to five superficially different problems.

(1) Noise Filtering - Special Cases

$$\Phi_s^+ = \frac{A}{s^k}, \quad \Phi_n^+ = s^m$$

(2) Saturation Constraint - Special Cases (Ref. 17)

$$\Phi_s^+ = \frac{A}{s^k}, \quad \Phi_n^+ = 0, \quad G_s = s^{k+m} \quad (= 1/G_f)$$

(3) Bandwidth Limitation (Ref. 17)

$$v(\varepsilon) = \frac{A}{s^k}, \quad \mathcal{O}(W) = -(m+1), \quad W_f = s^m$$

(4) Narrow Sense Realizability (Ref. 12)

$$v(\varepsilon) = \frac{A}{s^k}, \quad \mathcal{O}(G_f) = -(m+1), \quad G_r = s^m$$

(5) General Case - Crossover Region Approximation

$$\Phi_s^+ = \frac{A}{s^k}, \quad \Phi_n^+ = s^m \quad \text{for } \omega \approx \omega_c$$

This last case is the important one.

This Lemma represents optimum systems under many conditions. These will be referred to as "standard" optimum systems. However, in this concise equation form it is a little hard to interpret. Therefore, the frequency normalized optimum G for k = 1, 2, 3 and m = 0, 1, 2, 3 are presented in Table 3.3-2 and Figures 3.3-1, 2, and 3.

Note the low phase margin (PM) when both k > 1 and m > 0. For example k = 2, m = 1 yields

$$G = \frac{.5\,(1 + \lambda/.5)}{\lambda^2(1 + \lambda/2)}$$

where there exists only 6/10 of a decade of -1 slope at crossover, and the PM = 36.9°. If one had added an artificial noise $\Phi_n^+ = \lambda$ to the problem specifications, for example to obtain narrow sense realizability, one might blame the low PM on the minimization of the artificial transmitted noise which is not present in the real world, and therefore, use a larger PM. The fallacy of this reasoning is demonstrated by the following example.

Given: $\Phi_s^+ = \frac{1}{\lambda^2}$, $\Phi_n^+ = 0$, $\mathcal{O}(G) = -2$

k	m	G
1	0	$\frac{1}{\lambda}$
	1	$\frac{(.7071)}{\lambda\,(1 + \lambda/1.414)}$
	2	$\frac{(.5000)}{\lambda\,(1 + 2(.7071)\,\lambda/1.414 + (\lambda/1.414)^2)}$
	3	$\frac{(.3827)}{\lambda\,(1 + \lambda/1.500)(1 + 2(.4218)\,\lambda/1.321 + (\lambda/1.321)^2)}$
2	0	$\frac{(1 + \lambda/.7071)}{\lambda^2}$
	1	$\frac{(.5000)(1 + \lambda/.5000)}{\lambda^2\quad(1 + \lambda/2.000)}$
	2	$\frac{(.2929)(1 + \lambda/.3827)}{\lambda^2(1 + 2(.7071)\,\lambda/1.848 + (\lambda/1.848)^2)}$
	3	$\frac{(.1910)(1 + \lambda/.3090)}{\lambda^2(1 + \lambda/1.929)(1 + 2(.3967)\,\lambda/1.648 + (\lambda/1.648)^2)}$
3	0	$\frac{(1 + 2(.7071)\,\lambda/.7071 + (\lambda/.7071)^2)}{\lambda^3}$
	1	$\frac{(.3828)(1 + 2(.7071)\,\lambda/.5412 + (\lambda/.5412)^2)}{\lambda^3(1 + \lambda/2.613)}$
	2	$\frac{(.1910)(1 + 2(.7071)\,\lambda/.4370 + (\lambda/.4370)^2)}{\lambda^3\quad(1 + 2(.7071)\,\lambda/2.288 + (\lambda/2.288)^2)}$
	3	$\frac{(.1094)(1 + 2(.7071)\lambda/.3660 + (\lambda/.3660)^2)}{\lambda^3(1 + \lambda/2.345)(1 + 2(.3846)\,\lambda/1.975 + (\lambda/1.975)^2)}$

Table 3.3-2 Table of Standard Optimum Systems.

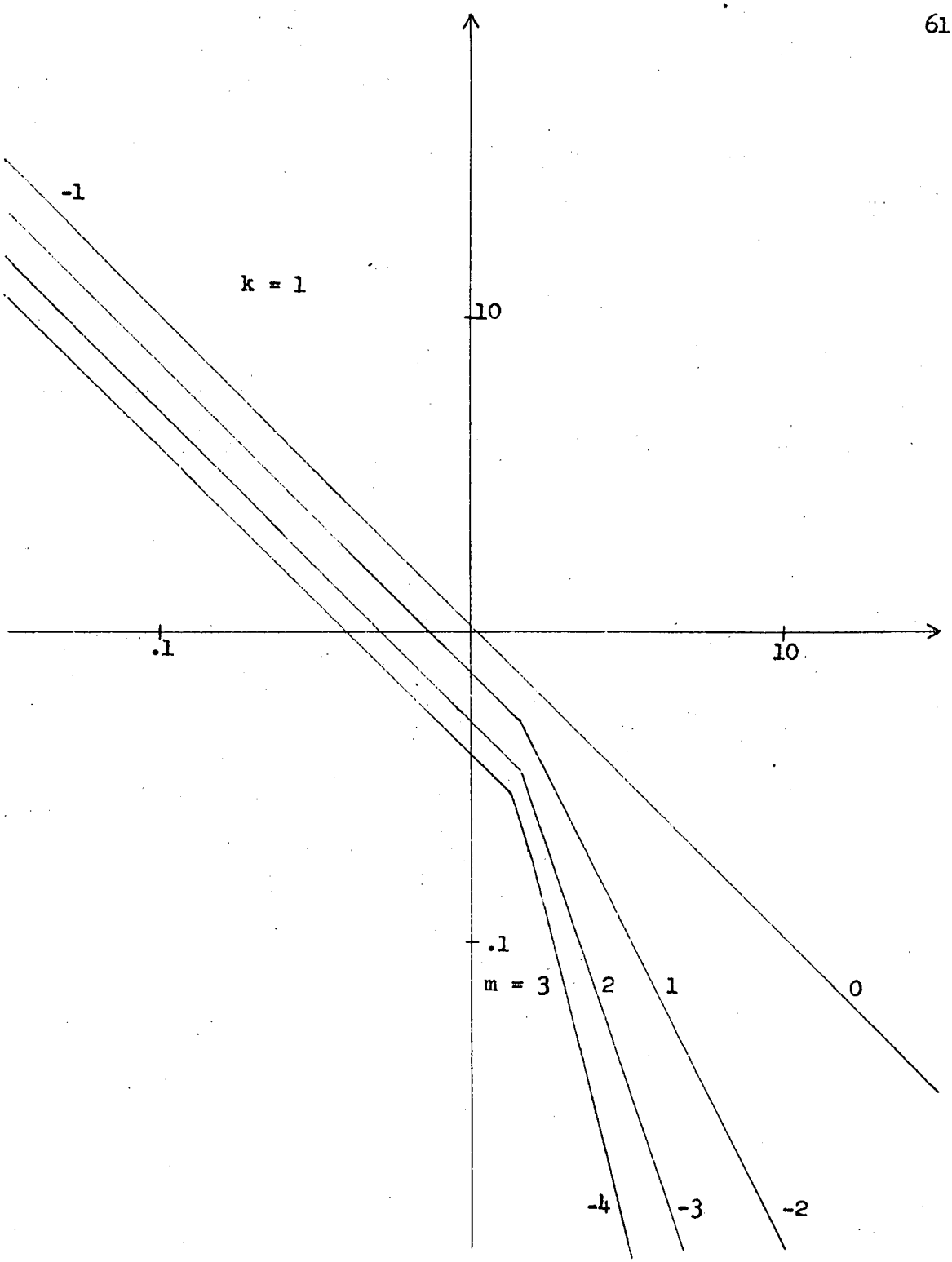


Figure 3.3-1 Standard Optimum Systems, Zero Position Error.

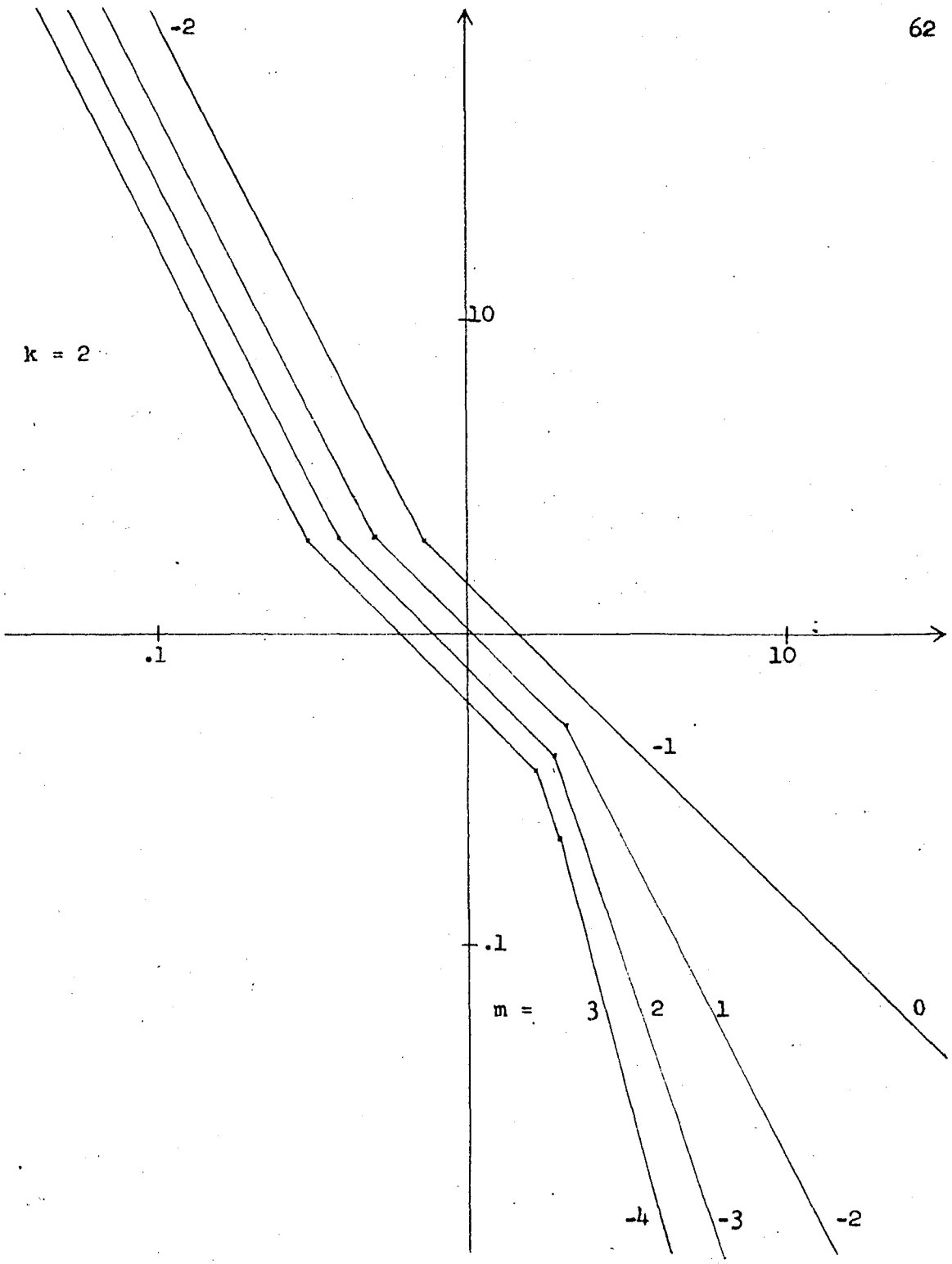


Figure 3.3-2 Standard Optimum Systems, Zero Velocity Error.

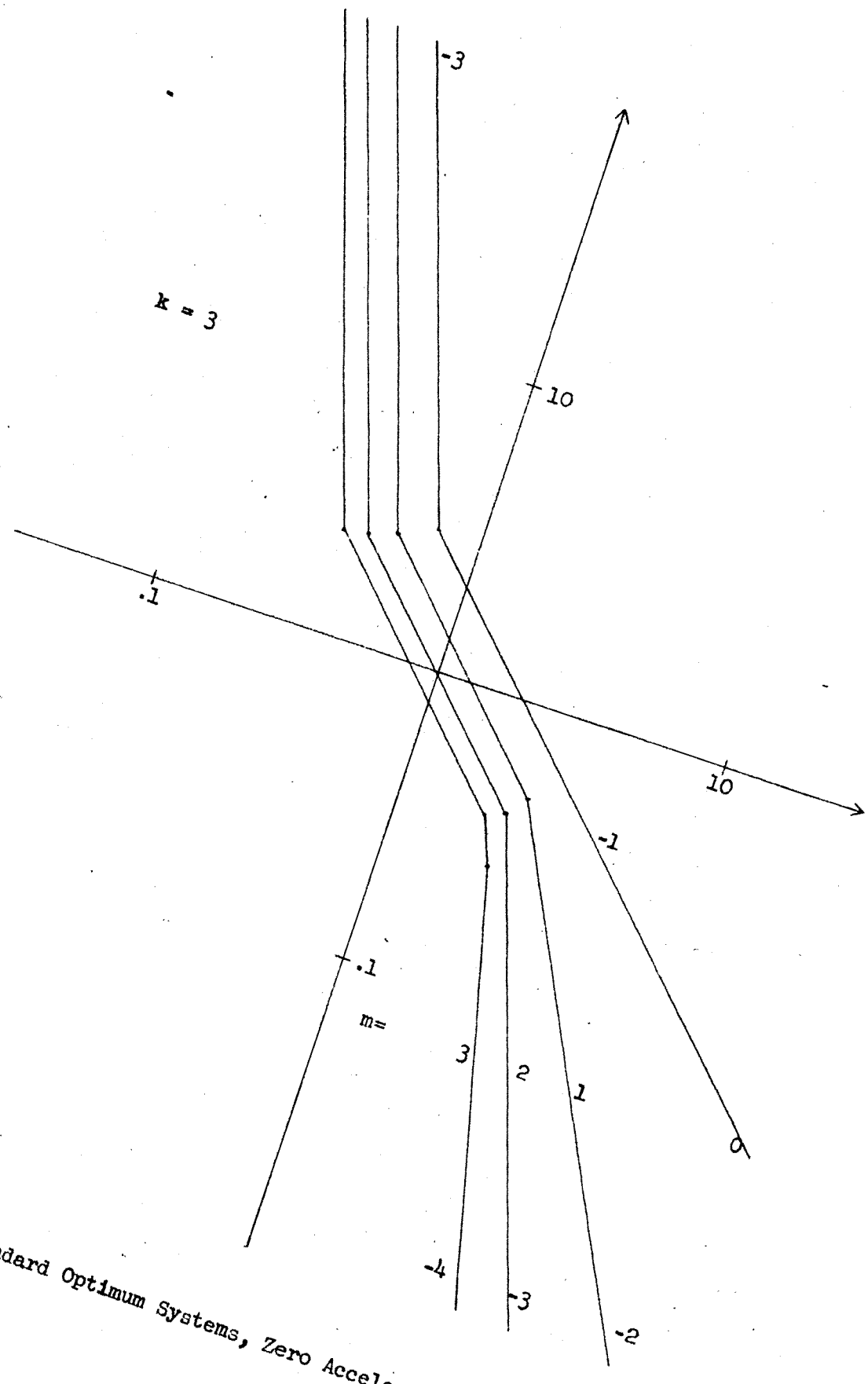


Figure 3.3-3 Standard Optimum Systems, Zero Acceleration Error.

Find: Optimum G

Solution: By symmetry, the maximum PM (with minimum length of -1 slope) is obtained with

$$G = \frac{1/\alpha\,(1 + \alpha\lambda)}{\lambda^2\,(1 + \lambda/\alpha)}$$

and the problem is to find the optimum α

$$(1 - W)\,\Phi_s^+ = \frac{\alpha + \lambda + \alpha\lambda^2}{1 + \alpha\lambda + \alpha\lambda^2 + \lambda^3}$$

$$\sigma_e^2 = \frac{\alpha^2 - \alpha + 1}{2(\alpha - 1)}$$

$$\frac{d}{d\alpha}\,\sigma_e^2 = 0 \text{ yields } \alpha = 2$$

The low PM, and poor step-function response, is an inherent characteristic of zero velocity error systems. Graham and Lathrop[9] present similar results for the ITAE criterion.

Table 3.3-3 presents some useful data for the specialist on the standard optimum system given by the Lemma. For example, data from Part I is used in Section 3.6 to determine the performance loss due to the presence of nonminimum phase fixed elements. Part II can be used to determine the (usually very high) price paid for narrow-sense realizability, or a specific G. This illustrates that the best procedure is to design for a saturation constraint, and add an additional break at $\omega \geq 10\,\omega_c$ in order to achieve narrow-sense

Given: $\Phi_s^+ = \frac{1}{\lambda^k}$, $W = \frac{{}^kB_{k+m}(s/b)}{B_{k+m}(s/b)}$

I. $\Phi_n^+ = \lambda^m$, $\Phi_e^+ = (1-W)\ \Phi_s^+ \oplus W\ \Phi_n^+$

$\frac{<e^2>}{b}$		m			
		0	1	2	3
k	1	1.000	1.414	2.000	2.613
	2	1.414	3.000	6.309	11.708
	3	2.000	6.309	17.944	--

II. For $\omega > 3\ \omega_c$, slope $G_f = -n$; $\Phi_{e_s}^+ = (1-W)\ \Phi_s^+$

$\frac{<e_s^2>}{b}$		n			
		1	2	3	4
k	1	.500	1.061	1.667	2.287
	2	.354	1.500	3.943	8.196
	3	.333	2.366	8.972	--

Note: Values are squared error, not RMS error.

Table 3.3-3 Squared Error for Standard Optimum Systems.

realizability.

Consider: $k = 1$, $m = 2$

$$G = \frac{1/2}{\lambda(1 + (.71)\ \lambda/\sqrt{2} + (\lambda/\sqrt{2})^2)}$$

This requires complex poles which are not easy to synthesize with RC networks. The use of Binomial roots in G in place of Butterworth roots usually results in only a small increase in RMS error. In this case

$$G = \frac{1/2}{\lambda(1 + \lambda/\sqrt{2})^2}$$

and the increase in RMS error is only 1.83 percent.

e. Approximation

The justification for the approximation is to plot the results and to compare it with the exact system as shown in Figure 3.3-1, 2, and 3. The agreement is good. The main purpose of the approximation is to assist the engineer in visualizing what the equation means and what optimum systems look like in the crossover region.

f. Colored Noise Algorithm

The actual justification of the colored noise algorithm rests on three facts. First, it is similar to the white noise

algorithm. Second, it is reasonable. This is shown below. Third, it works. This is demonstrated by the examples and the error analysis.

In the following plausibility argument Φ_n^+ is assumed to be monotone increasing for all ω. Note examples A, H, and I which demonstrate that this restriction can be relaxed. An arbitrary Φ_s^+ and Φ_n^+ can be approximated as follows:

$$\Phi_s^+ = \frac{1}{\lambda^{k_j}}, \quad \Phi_n^+ = \lambda^{m_j}, \quad \omega_j < \omega < \omega_{j+1}$$
$$k_j \geq 1, \quad m_j \geq 0$$

$$\Phi_s^+ \approx \Phi_n^+$$

$$G \approx \frac{k B_{k+m}}{B_{k+m}^{m+1}} \qquad \text{slope } \Phi_s^+(\omega_c) = -k$$
$$\text{slope } \Phi_n^+(\omega_c) = m$$

$$\Phi_s^+ \gg \Phi_n^+$$

$$G \approx \frac{1}{\lambda^{k_j}} = \Phi_s^+ \qquad \omega_j < \omega < \omega_{j+1}$$

This is true for all j $(\omega_j < \omega_c)$

$$\therefore G \approx \Phi_s^+$$

$$\Phi_s^+ \ll \Phi_n^+$$

$$G\,\Phi_n^+ \approx \frac{1}{\lambda^{m_{j+1}}}\,(\lambda^{m_j}) = \frac{1}{\lambda} \qquad \omega_j < \omega < \omega_{j+1}$$

This is true for all j $(\omega_j < \omega_c)$

$\therefore$ slope of $G\,\Phi_n^+ = -1$

3.3.4 EXAMPLES

Nine examples are presented in Figure 3.3-4. This figure should be studied carefully as "one picture is worth a thousand words". The same examples are tabulated in Table 3.3-4 to enable the reader to practice using the colored noise algorithm.

3.3.5 ERROR ANALYSIS

The theory of error analysis is explained in the preceding section. Figure 3.3-5 presents the approximation error E_a for the nine colored noise examples of Table 3.3-4. Figure 3.3-5 demonstrates that if the closest noise break frequency is one half decade or more away from the crossover frequency ω_c, then the approximation error is negligible.

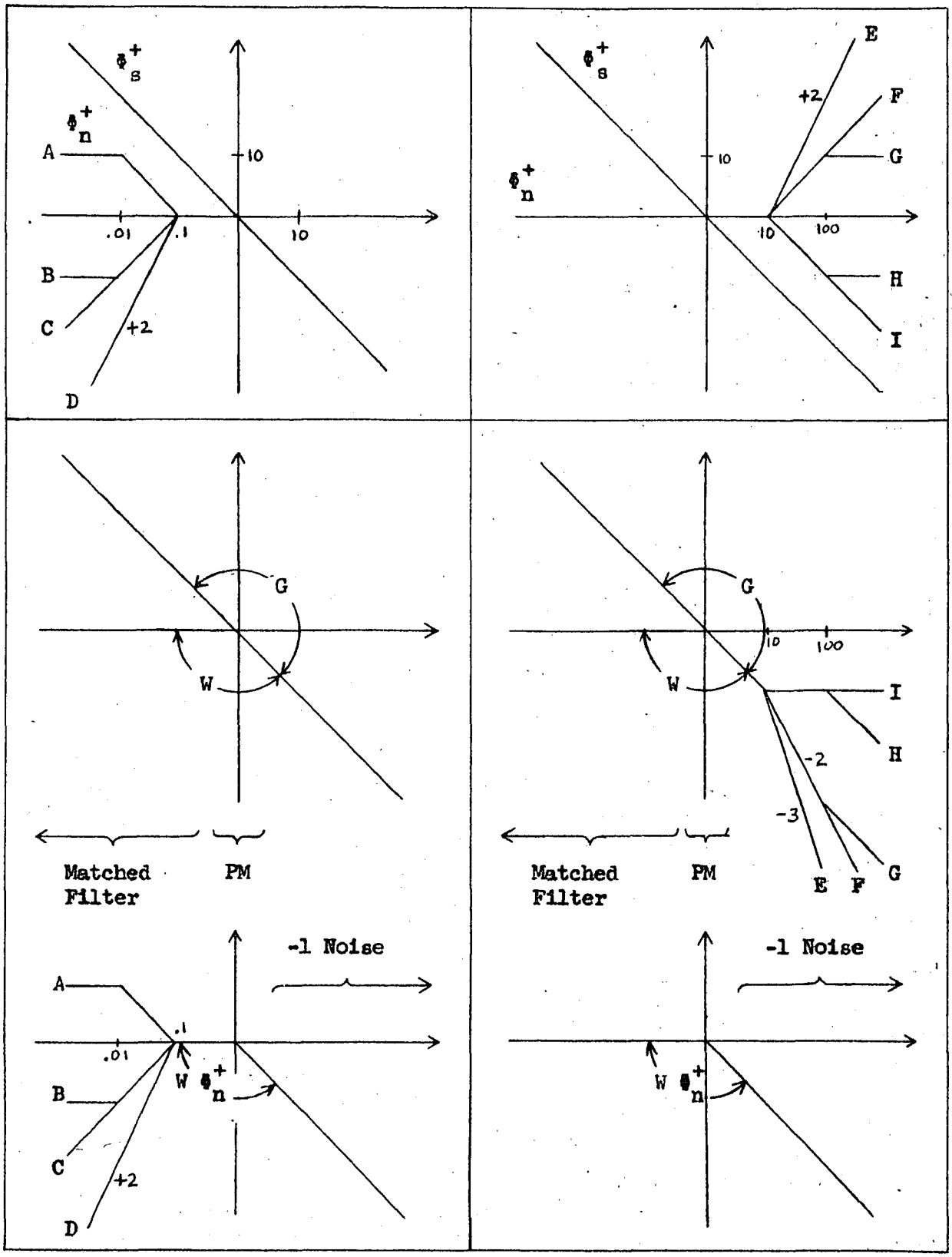


Figure 3.3-4 Colored Noise Examples (α = 1/10)

	Φ_{ss}	Φ_{nn}	G	
			Approx. ($\alpha < 1$)	Exact
A	$\frac{1}{-\lambda^2}$	$\frac{10^2(\alpha^2 - \lambda^2)}{(\alpha^2 - 10^2\lambda^2)}$	$\frac{1}{\lambda}$	$\frac{\alpha + 10\lambda}{\lambda(b - 10 + 10\lambda)}$ $b = 10\sqrt{1 + \alpha/5 + \alpha^2}$
B		$\frac{(\alpha^2 - 10^2\lambda^2)}{10^2(\alpha^2 - \lambda^2)}$		$\frac{10\ (\alpha + \lambda)}{\lambda(b - 10 + 10\lambda)}$ $b = 10\sqrt{1 + 2\alpha + \alpha^2/10^2}$
C		$\frac{-\lambda^2}{\alpha^2 - \lambda^2}$		$\frac{\alpha + \lambda}{\lambda(b - 1 + \lambda)}$ $b = \sqrt{1 + 2\alpha}$
D		$\frac{\lambda^4}{\alpha^4 + \lambda^4}$		$\frac{(\alpha^2 + \sqrt{2}\,\alpha\lambda + \lambda^2)}{\lambda((b - \sqrt{2}\,\alpha) + (c - 1)\lambda + \lambda^2)}$ $b^2 = 2\alpha^2 c, \quad c^2 = 1 + 2b$

Table 3.3-4a Colored Noise Examples ($\alpha < 1$)

	Φ_{ss}	Φ_{nn}	G	
			Approx. ($\alpha < 1$)	Exact
E	$\frac{1}{-\lambda^2}$	$1 + \alpha^4\lambda^4$	$\frac{1}{\lambda(1 + \sqrt{2}\,\alpha\lambda + \alpha^2\lambda^2)}$	$\frac{1}{\lambda(b + c\lambda + \alpha^2\lambda^2)}$ $b^2 = 1 + 2c, \quad c^2 = 2\alpha^2 b$
F		$1 - \alpha^2\lambda^2$	$\frac{1}{\lambda(1 + \alpha\lambda)}$	$\frac{1}{\lambda(b + \alpha\lambda)}$ $b = \sqrt{1 + 2\alpha}$
G		$\frac{10^2(1 - \alpha^2\lambda^2)}{(10^2 - \alpha^2\lambda^2)}$	$\frac{(10 + \alpha\lambda)}{10\lambda(1 + \alpha\lambda)}$	$\frac{(10 + \alpha\lambda)}{\lambda(b - \alpha + 10\alpha\lambda)}$ $b = 10\sqrt{1 + 2\alpha + \alpha^2/10^2}$
H		$\frac{(10^2 - \alpha^2\lambda^2)}{10^2(1 - \alpha^2\lambda^2)}$	$\frac{10(1 + \alpha\lambda)}{\lambda(10 + \alpha\lambda)}$	$\frac{10(1 + \alpha\lambda)}{\lambda(b - \alpha + 10\alpha\lambda)}$ $b = 10\sqrt{1 + \alpha/5 + \alpha^2}$
I		$\frac{1}{1 - \alpha^2\lambda^2}$	$\frac{1 + \alpha\lambda}{\lambda}$	$\frac{1 + \alpha\lambda}{\lambda(b - \alpha)}$ $b = \sqrt{1 + \alpha^2}$

Table 3.3-4b Colored Noise Examples ($\alpha < 1$)

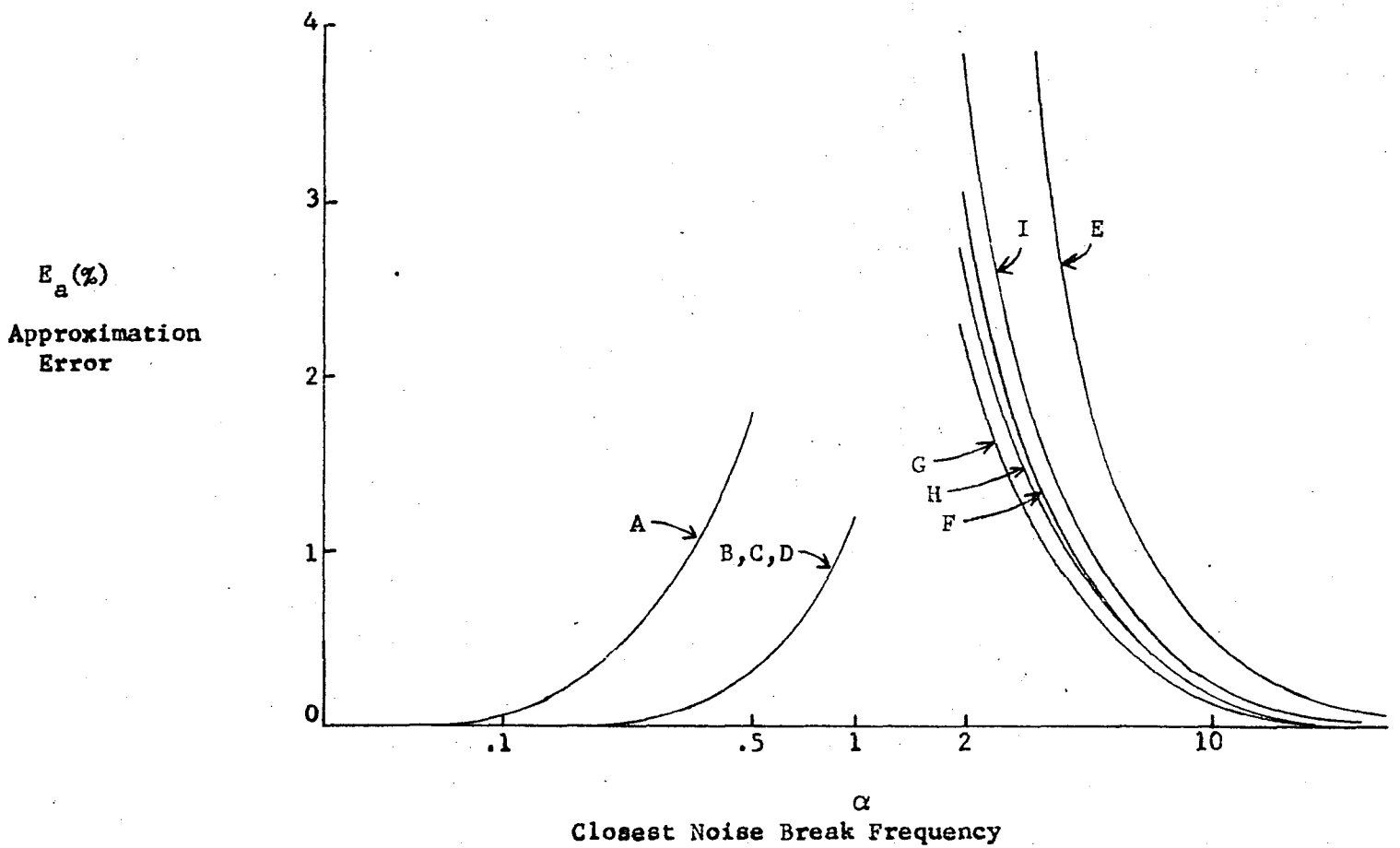


Figure 3.3-5 Error Analysis for Colored Noise Examples.

3.4 SATURATION CONSTRAINTS

3.4.1 INTRODUCTION

This is the most important case in this chapter, because almost all physical control systems will possess only a finite linear operating region.

It is important to note that the systems designed here are not the best system with a saturation constraint, they are the best linear system with a saturation constraint. In using this theory we are admitting that we are willing to pay the price, often high, in order to remain in the nice, comfortable linear world.

The number of available fixed plants, each with its own saturation limit, is small. Therefore, to minimize $< e^2 >$ with $< i^2 > = J_1$, is a better problem formulation than to minimize $< i^2 >$ with $< e^2 > = J_2$. Consequently only the former is considered here.

Note that if there is no noise, $\Phi_n^+ = 0$, a common case, and no saturation constraint, then the optimum $G_c = \infty$, which is unrealizable.

In designing for a saturation constraint the fixed plant, G_f, is selected so that its input is the signal which is being constrained, and the actual fixed plant is from that point on immaterial. A problem constraining n signals is treated as n problems each

constraining one signal. One constraint will be the critical one, and this one yields the solution. The others may be neglected.

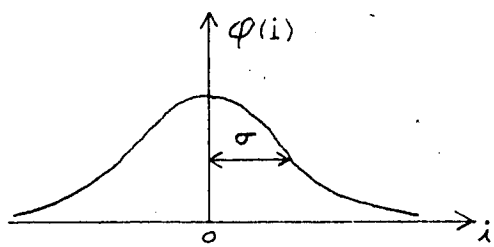
3.4.2 ALGORITHMS

a. Saturation Rule

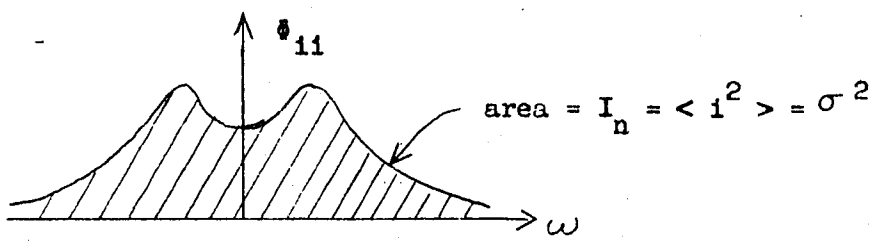
$$\frac{\phi_s^+(0)}{G_f(0)} < \infty$$

b. Saturation Constraint

Assume i(t) is a Gaussian signal with zero mean. The probability distribution of i(t) is shown below. 68.3% of the area of $\varphi(i)$ is between $\pm\sigma$ . . σ is the standard deviation.



This is related to its power spectral density by



$\pm L$ = Linear range of saturating element. J = Allowable mean-square or integral-square value of saturating signal. Let P_s be the probability of saturation. Then for

J	P_s
$(L/1.5)^2$	13.36%
$(L/2.0)^2$	4.45%
$(L/2.5)^2$	1.24%

(See Korn[13] page 900)

$J = (L/2)^2$ is a good choice. Note that this is very arbitrary.

c. Saturation Algorithm

The use of common sense and the fact that the spectrum of the signal being constrained is bounded by a -1 slope moving to the left will solve these problems. For tutorial purposes the following cookbook recipe is included.

(1) One n constraint problem goes to n one constraint problems. One constraint dominates.

(2) Check Saturation Rule.

(3) Assume no constraint. Find G_n; W_n from G_n.

(4) Sketch $\frac{\Phi_i^+}{W_s} = \frac{W_n \Phi_r^+}{G_f}$

(5) "-1 slope constraint moving to the left". (See p. 79)

Find W_s, $\mathcal{O}(W_s) = \mathcal{O}(G_f)$

with Butterworth break at $1/\rho$ such that

for $\omega > 1/\rho$ slope $\Phi_i^+ = -1$, $1/\rho$ = saturation break

and area $\Phi_i^+ = J$. Find $\rho(J)$.

Do graphically, approximate freely. (See p. 81)

(6) Method I - Easier (for $1/\rho > 10\omega_c$, for an approximate

solution).

Sketch Φ_s^+, Φ_n^+ and $\frac{k\Phi_r^+}{G_f}$ with k such that $1/\rho$ is a break frequency. Use Colored Noise Algorithm.

(6) Method II - More accurate.

$W = W_n W_s$, W_n from (3), W_s from (5). Find G from W, and add breaks to make G proportional to Φ_s for $\omega < \omega_c$.

Both Methods

(7) In the real world, make complete analog computer simulation, and vary J in $W_s(J)$ to find optimum J.

3.4.3 DERIVATION

This derivation is based on Newton's[17] book.

$$W = \frac{1}{\Delta^+}\left[\frac{\Gamma}{\Delta^-}\right]_+ \qquad (5.4\text{-}28)$$

For the filtering problem, where G_f is minimum phase, and, of course, $G_i = 1$ and $\Phi_{nd} = 0$.

$$\Gamma = \Phi_{ss} \qquad (5.5\text{-}6;\ 4.4\text{-}6,7)$$

$$\Delta = \Phi_{ss} + \Phi_{nn} \qquad (5.5\text{-}4;\ 4.4\text{-}9)$$

For the saturation constraint problem, where G_f is minimum phase, let $G_f = 1$ in equations (7.3- 18,19) and then let $G_s = 1/G_f$. This results in:

$$\Gamma = \Phi_{ss}$$

$$\triangle = \left[1 + \frac{k^2}{G_f(s)\, G_f(-s)}\right]\left[\Phi_{ss} + \Phi_{nn}\right]$$

$$= \Phi_{ss} + \left[\Phi_{nn} + \frac{k^2\left[\Phi_{ss} + \Phi_{nn}\right]}{G_f(s)\, G_f(-s)}\right]$$

$$= \Phi_{ss} + \Phi_{n_1 n_1} + \Phi_{n_2 n_2}$$

where k is determined by

$$\langle i^2 \rangle = \text{area } \Phi_i^+ = J$$

$$\Phi_i^+ = W\,\frac{1}{G_f}\,(\Phi_{ss} + \Phi_{nn})^+ = \frac{W\,\Phi_{n_2}^+}{k}$$

The problem is now theoretically solved by the solution to the input colored noise problem of Section 3.3.2. However, in this form it is inconvenient to apply the required amplitude normalization and to determine k. Therefore, further investigation of this problem is required.

The justification of the saturation algorithm is twofold. First, it is reasonable, which is shown below. Second, it works, which is demonstrated by the examples given in the next section.

Consider the case where there is no constraint, k = 0. This is an input colored noise problem. Consider the case where there is no input noise, $\Phi_{nn} = 0$. This is also an input colored noise problem where

$$\Phi_n^+ = \frac{K\,\Phi_r^+}{G_f}$$

Therefore, for

$$\omega > 1/\rho \quad \text{slope } W\,\Phi_n^+ = -1$$

where $1/\rho$ is the saturation break frequency (for $\Phi_n^+ = 0$, $1/\rho = \omega_c$). But

$$\Phi_i^+ = \frac{W\,\Phi_r^+}{G_f} = \frac{W\,\Phi_n^+}{K}$$

And consequently for

$$\omega > 1/\rho \quad \text{slope } \Phi_i^+ = -1$$

That is, the saturation constraint consists of bounding Φ_i^+ by a "-1 slope moving to the left" until area $\Phi_i^+ = J$ is satisfied. The saturation break is, of course Butterworth, and other breaks are added, when necessary, to cancel breaks in Φ_i^+ for $\omega > 1/\rho$.

Combining the above characteristics in a simple manner, it is assumed that

$$W = W_n W_s$$

where W_n is from (3) and

W_s is a correction factor to provide

$\omega > 1/\rho$ slope $\Phi_i^+ = -1$

and area $\Phi_i^+ = J$

Note that for $\omega < \omega_c$, $G \approx \Phi_s^+$ still holds. Step (5) in the cookbook recipe should now be obvious. Figure 3.4-1 shows the exact solution to a problem involving input noise and a saturation constraint. It demonstrates that the "-1 slope moving to the left" is still valid in the general case.

When calculating the area Φ_i^+, keep the analytical expression

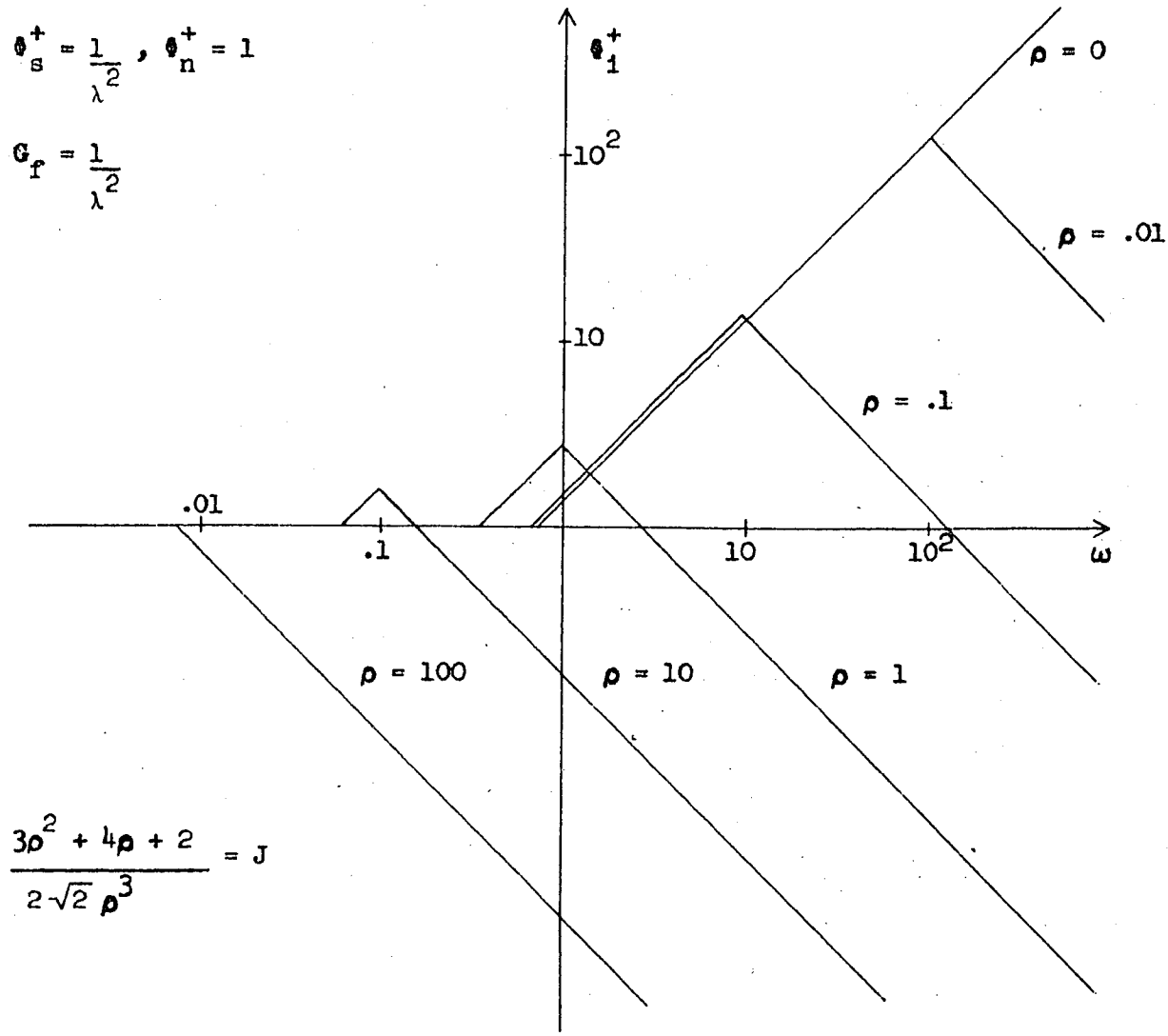


Figure 3.4-1 Example Illustrating the -1 Slope Moving to the Left

SIMPLE in order to reduce the order of I_n. (See Section 3.1.4.)

APPROXIMATE FREELY, using straight line Bode plots as a guide. See Figure 3.4-2 for one of the "tricks of the trade". $< i^2 >$ = area $\Phi_i^+ = J$, but the choice of J is rather arbitrary, so any approximation errors are unimportant.

Method II is more accurate because the basic rule is for

$\omega > \omega_c$ slope W $\Phi_n^+ = -1$

The approximation that for $W < 1$, $G \approx W$ is very good for $W \ll 1$, but is only approximate when $W \approx 1$. However G is more convenient to work with because it avoids the $G = W/(1 - W)$ operation.

The Saturation Rule is obtained as follows:

$$< i^2 > = \text{area } \Phi_i^+ < \infty \text{ , } \Phi_i^+ = W \frac{1}{G_f} (\Phi_{ss} + \Phi_{nn})^+$$

But $W(0) = 1$, $\Phi_n^+(0) < \infty$

Therefore $\dfrac{\Phi_s^+}{G_f}(0) < \infty$

This gives the Saturation Rule.

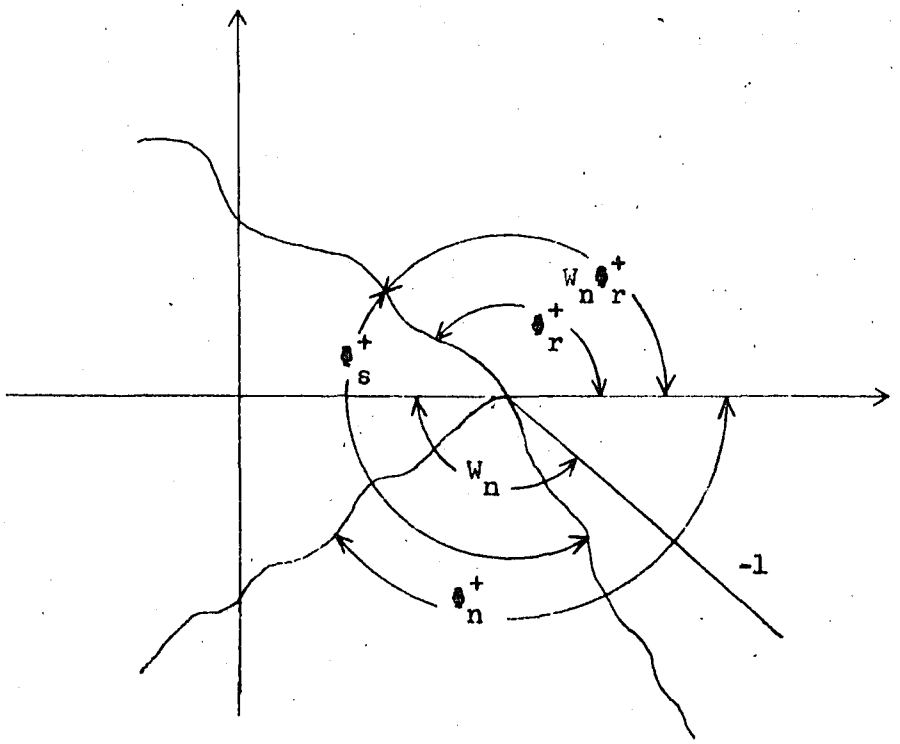
The selection of J is so arbitrary, that as a last step, the designer should perform a complete analog computer simulation, including all non-linearities, and vary ρ in $W_s(\rho)$ in order to determine just how much saturation can really be tollerated.

3.4.4 EXAMPLES

Figures 3.4-3 thru 3.4-8 present six examples employing both

Given: Φ_s^+ , Φ_n^+

Find: $W_n \Phi_r^+$ approximately



Extension to Φ_i^+ and to colored noise for $\omega > \omega_c$ is obvious.

Figure 3.4-2 Approximations for Saturating Signal

methods to illustrate the technique and to enable the reader to practice using the algorithm. For tutorial reasons these examples are presented in great detail. When one has become familiar with the technique most of the steps illustrated need not be written down. If further practice is desired, examples given in the paper by Coales[6] can be used. Coales presents 104 cases which do not require interpolation. Note that only 32 of these satisfy the control problem assumption that RMS error $\leq$ 1/2 the RMS value of the signal. To avoid interpolation choose one of Coales cases and work backward to obtain a problem specification.

No error analysis is included in this section for two reasons. First, the theory is very similar to the colored noise case, where approximation errors have been shown to be negligible. Second, the reader can pick his own examples from Coales[6] and compare the solution by inspection with the exact solution.

3.5 LOAD DISTURBANCES

3.5.1 INTRODUCTION

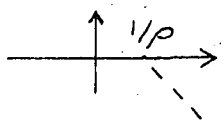
This is a common and very practical case in the design of control systems. The two basic reasons feedback is used are to reduce fixed plant variations and to reduce load disturbances.

This is the only section in which it is assumed the $H \neq 1$, because this is the only problem where the extra degree of freedom

Given: $\boxed{\Phi_s^+ = \frac{1}{\lambda}\ ,\ \Phi_n^+ = 1,\ \ G_f = \frac{1}{\lambda}\ ,\ \ J}$

Solution:

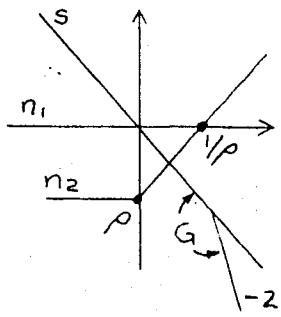
(4) $\dfrac{W_n \Phi_r^+}{G_f} = \left(\dfrac{1}{1+\lambda}\right)\left(\dfrac{\lambda}{1}\right)\left(\dfrac{1+\lambda}{\lambda}\right) = 1$

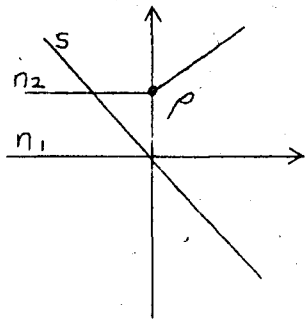


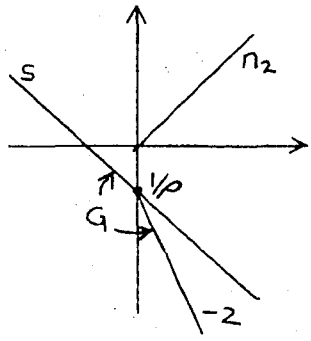
(5) $\langle i^2 \rangle = \dfrac{(1)^2}{2(1)(\rho)} = J$ $\qquad \boxed{\rho = \dfrac{1}{2J}}$

Method I.

(6) $\rho < 1$ $\qquad \dfrac{k\,\Phi_r}{G_f}$ $\qquad \rho > 1$







$\boxed{\rho < 1 \qquad G = \dfrac{1}{\lambda(1+\rho\lambda)}}$

$\boxed{\rho > 1 \qquad G = \dfrac{1/\rho}{\lambda(1+\lambda)}}$

Method II.

(6) $W = \left(\dfrac{1}{1+\lambda}\right)\left(\dfrac{1}{1+\rho\lambda}\right)$ $\qquad G = \dfrac{1}{1 + (1+\rho)\lambda + \rho\lambda^2 - 1}$

$$\boxed{G = \frac{1/(1+\rho)}{\lambda(1 + \rho/(1+\rho)\lambda)}}$$

Exact Solution

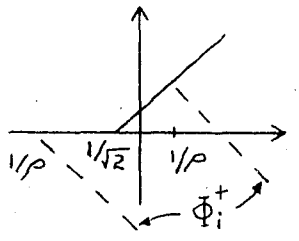
Given by Method II.

Figure 3.4-3 Example 1, Saturation Constraints

Given: $\boxed{\Phi_s^+ = \frac{1}{\lambda^2}\ ,\ \Phi_n^+ = 1,\ \ G_f = \frac{1}{\lambda^2}\ ,\ \ J}$

Solution:

(4) $$\frac{W_n \Phi_r^+}{G_f} = \frac{1 + \sqrt{2}\,\lambda}{1 + \sqrt{2}\,\lambda + \lambda^2}\ \ \frac{\lambda^2}{1}\ \ \frac{1 + \sqrt{2}\,\lambda + \lambda^2}{\lambda^2} = 1 + \sqrt{2}\,\lambda$$



$\rho < \sqrt{2}$

$$\Phi_i^+ = \frac{1 + \sqrt{2}\,\lambda}{1 + \sqrt{2}\,\rho\lambda + \rho^2\lambda^2}$$

$$\langle i^2 \rangle = \frac{(1)^2(\rho^2) + (\sqrt{2})^2(1)}{2(1)(\sqrt{2}\,\rho)(\rho^2)} = \boxed{\frac{2 + \rho^2}{2\sqrt{2}\,\rho^3} = J}$$

$\boxed{\rho < 1}$

$\rho > \sqrt{2}$

$$\Phi_i^+ = \frac{1 + \sqrt{2}\,\rho\lambda}{1 + \sqrt{2}\,\rho\lambda + \rho^2\lambda^2}$$

check $\rho = 1$

$J = \frac{3}{2\sqrt{2}}$ both

$$\langle i^2 \rangle = \frac{(1)^2(\rho^2) + (\sqrt{2}\,\rho)^2(1)}{2(1)(\sqrt{2}\,\rho)(\rho^2)} = \boxed{\frac{3}{2\sqrt{2}\,\rho} = J}$$

$\boxed{\rho > 1}$

Method I

(6) $$\frac{k\Phi_r^+}{G_f} = k\lambda^2\ \ \frac{(1 + \sqrt{2}\,\lambda + \lambda^2)}{\lambda^2} = (1 + \sqrt{2}\,\lambda + \lambda^2)\,k$$

$\boxed{\rho < 1}$

$$G = \frac{1 + \sqrt{2}\,\lambda}{\lambda^2(1 + \sqrt{2}\,\rho\lambda + \rho^2\lambda^2)}$$

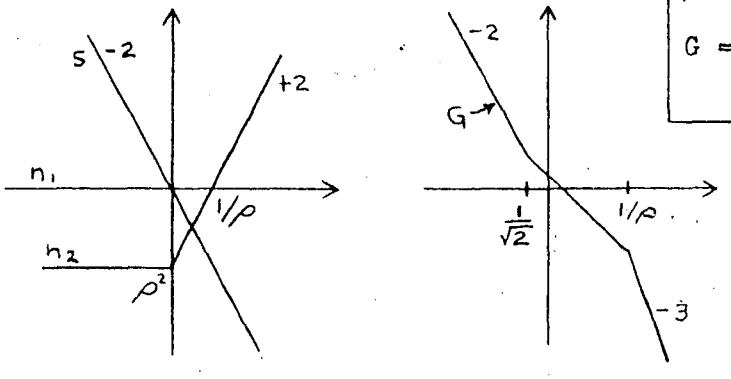
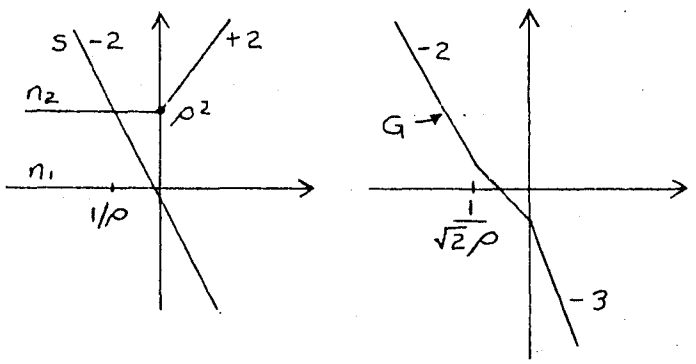


Figure 3.4-4a Example 2, Saturation Constraints



$$\rho > 1$$

$$G = \frac{1 + \sqrt{2}\,\rho\lambda}{\rho^2\lambda^2\,(1 + \sqrt{2}\,\lambda + \lambda^2)}$$

Method II

(6) $$W = \frac{1 + \sqrt{2}\,\lambda}{1 + \sqrt{2}\,\lambda + \lambda^2} \qquad \frac{1}{1 + \sqrt{2}\,\rho\lambda + \rho^2\lambda^2}$$

$$\begin{array}{ccc} 1 & \sqrt{2} & 1 \\ \hline 1 & \sqrt{2}\,\rho & \rho^2 \end{array}$$

$$\begin{array}{ccc} \sqrt{2} & 2\rho^2 & \sqrt{2}\,\rho^2 \\ & 1 & \sqrt{2}\,\rho \quad \rho^2 \end{array}$$

$$K_v = \infty \longrightarrow \frac{-1 \quad -\sqrt{2}\,(\rho + 1)}{(\rho + 1)^2\lambda^2 + \sqrt{2}\,\rho(\rho + 1)\lambda^3 + \rho^2\lambda^4}$$

$$G = \frac{1 + \sqrt{2}\,(\rho + 1)\lambda}{(\rho + 1)^2\lambda^2 \left[1 + \frac{\sqrt{2}\,\rho}{\rho + 1}\,\lambda + \frac{\rho^2}{(\rho + 1)^2}\,\lambda^2\right]}$$

Exact Solution

G as given by Method II

$$\frac{3\rho^2 + 4\rho + 2}{2\sqrt{2}\,\rho^3} = J$$

Figure 3.4-4b Example 2, Saturation Constraints

Given:

$$\Phi_{ss} = \frac{4000}{1 - (s/.2)^2}, \quad \Phi_{nn} = \Phi_{dd} = 0 \qquad \langle e^2 \rangle \leq (.2)^2$$

$$G_f = \frac{40}{s^2(1 + s/100)}, \quad i_2 = \frac{10^{-3}}{1 + s/100}\, i_1 \qquad \langle i_1^2 \rangle \leq (30)^2, \quad \langle i_2^2 \rangle \leq (.03)^2$$

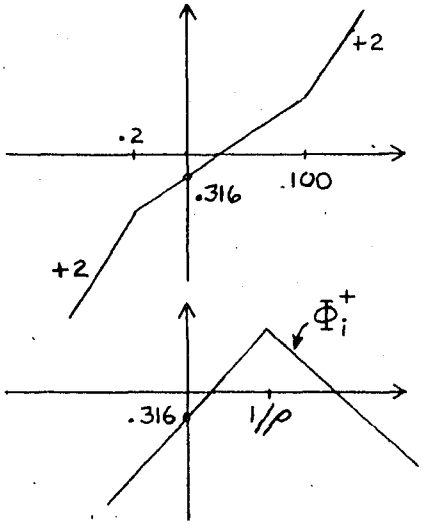
Solution:

(1) Let $i_3 = 10^{-3} i_1$

$\langle i_2^2 \rangle << \langle i_3^2 \rangle = 10^{-6}(30)^2 = (.03)^2$

Design for i_1 constraint only.

(4) $$\frac{W_n \Phi_r^+}{G_f} = (1)\,\frac{s^2(1 + s/100)}{40}\,\frac{63.3}{(1 + s/.2)} = \frac{1.58\, s^2(1 + s/100)}{(1 + s/.2)}$$



Neglect break at .2

area $<< 2(.2)(.1) << 900$

$$\frac{\Phi_i^+}{W_s} = .316\, s(1 + s/100)$$

Assume $1/\rho < 100$

$$\Phi_i^+ = \frac{0 + .316\,\lambda}{1 + \sqrt{2}\,\rho\lambda + \rho^2\lambda^2}$$

$$\langle i^2 \rangle = \frac{(0)^2(\rho^2) + (.316)^2(1)}{2(1)(\sqrt{2}\,\rho)(\rho^2)} = 900$$

$$\frac{.1}{2\sqrt{2}\,\rho^3} = 900 \qquad \frac{1}{\rho^3} = 25{,}450 \qquad \boxed{\frac{1}{\rho} = 29.4}$$

Method II

$$W = \frac{1}{(1 + s/100)(1 + 2\,s/29.4 + (s/29.4)^2)}$$

1	.0481	.001155	
-1	.01	.000481	.00001155
	.0581	.001636	.00001155
.0581	(1	.0281	.000199)
	1	2 s/71	$s^2/71^2$

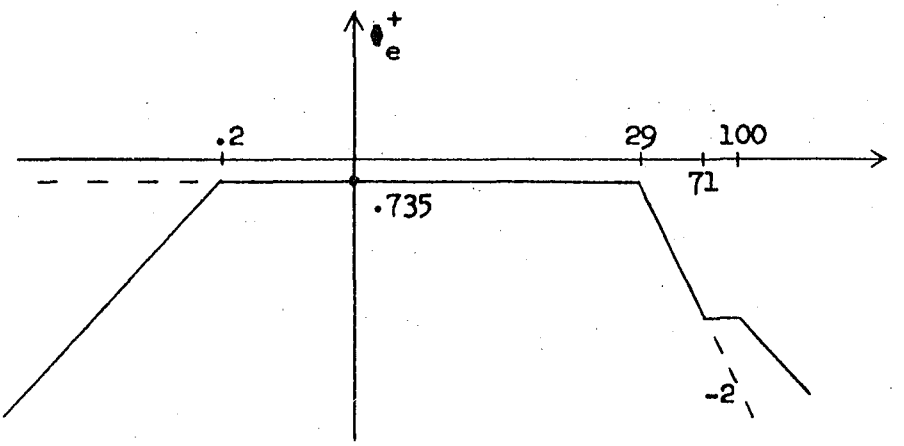
$$\boxed{G = \frac{17.2}{s(1 + s/71)^2}}$$

Figure 3.4-5a Example 3, Saturation Constraints

$$\Phi_e^+ = (1 - W)\,\Phi_s^+$$

$$= \frac{.0581\, s(1 + s/71)^2}{(1 + \sqrt{2}s/29.4 + (s/29.4)^2)(1 + s/100)} \quad \frac{63.3}{(1 + s/.2)} \qquad 3.68$$

$$\frac{.735 + 0s}{1 + \sqrt{2}\, s/29.4 + (s/29.4)^2}$$



$$\langle i^2 \rangle \approx \frac{(.735)^2\,(1/29.4^2) + (0)^2\,(1)}{2(1)(\sqrt{2}/29.4)(1/29.4)^2} = \frac{.540\,(29.4)}{2.83} = 5.62$$

$$\langle i^2 \rangle \approx (2.37)^2 \gg (.2)^2$$

Specifications are incompatible

Exact solution

Newton[17] pages 205-213. Agreement is excellent.

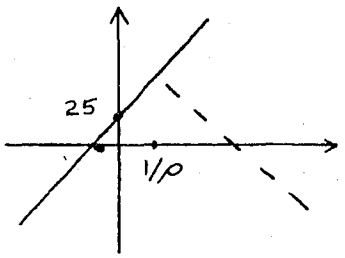
Figure 3.4-5b Example 3, Saturation Constraints

Given: $\boxed{\Phi_{ss} = \dfrac{625}{1 - s^2}, \quad \Phi_{nn} = 1, \quad G_f = \dfrac{1}{s(1+s)}, \quad J = \dfrac{(25)^2}{2} \times 10^8}$

Solution:

(4) $\dfrac{W_n \Phi_r^+}{G_f} = \dfrac{1}{1 + s/25} \; \dfrac{s(1+s)}{1} \; \dfrac{25(1 + s/25)}{1+s} = 25s$

(5)



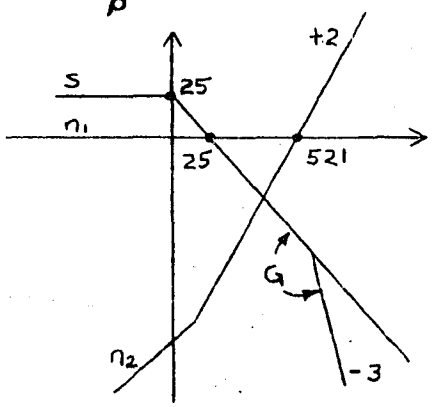
$$\Phi_1^+ = \frac{0 + 25s}{1 + \sqrt{2}\,\rho s + \rho^2 s^2}$$

$$\langle i^2 \rangle = \frac{(25)^2}{2\sqrt{2}\,\rho^3} = \frac{(25)^2}{2} \times 10^8$$

$$\frac{1}{\rho^3} = 141.4 \times 10^6 \qquad \boxed{\frac{1}{\rho} = 521}$$

(6)

Method I. $\dfrac{1}{\rho} \gg 10\omega_c$



$$\frac{k\,\Phi_r^+}{G_f} = k\,s(1+s)\,\frac{25\,(1 + s/25)}{1+s}$$

$$= 25\,k\,s(1 + s/25)$$

$$\boxed{G = \frac{25}{(1+s)(1 + \sqrt{2}\,s/521 + (s/521)^2)}}$$

Exact Solution

Coales[6] page 756.

$\alpha = \beta = \rho = 1, \quad a = K = N = 1 \qquad \mu^2 = \dfrac{\langle s^2 \rangle}{\langle i^2 \rangle}$

$v = .040, \quad \mu = 10^{-4}$

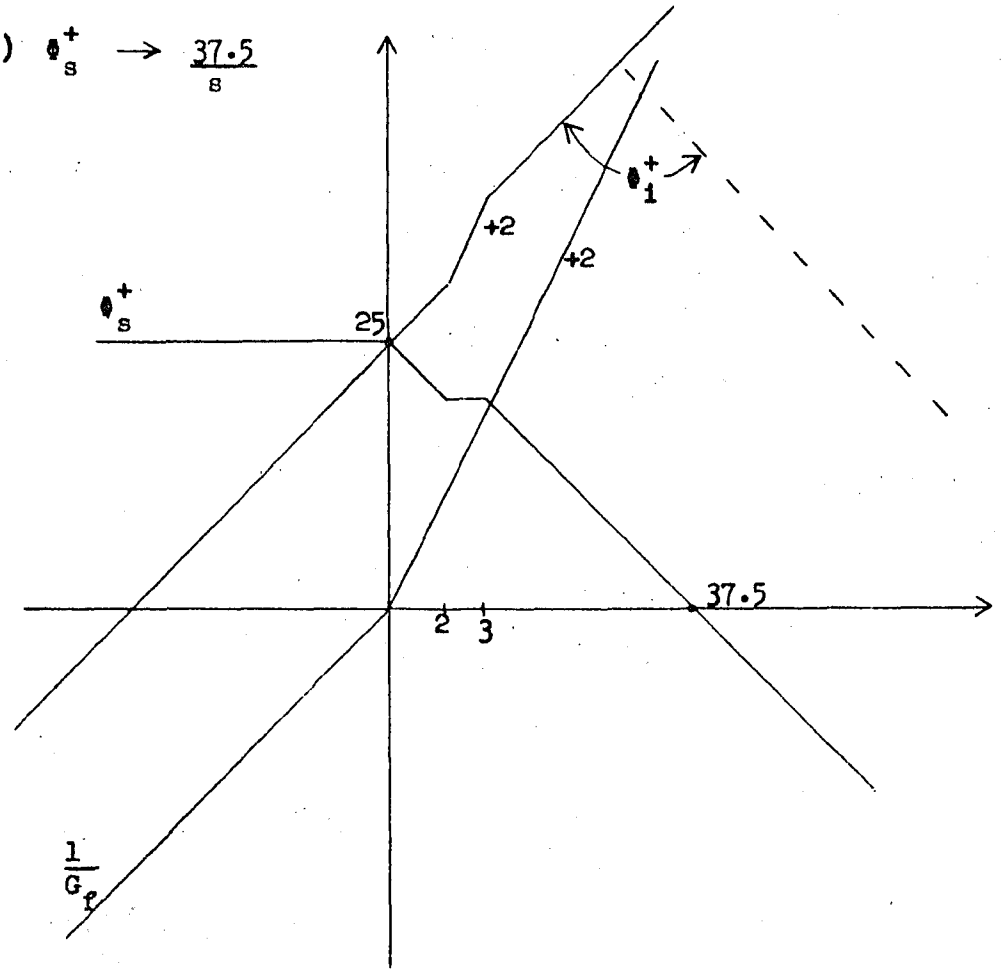
$$G = \frac{22.5}{(1+s)(1 + \sqrt{2}\,s/554 + (s/554)^2)}$$

Figure 3.4-6 Example 4, Saturation Constraints

Given: $\Phi_s^+ = \dfrac{25(1 + s/2)}{(1 + s)(1 + s/3)}$, $\Phi_n^+ = 0$, $G_f = \dfrac{1}{s(1 + s)}$, $J = 4.1 \times 10^8$

Solution:

(4) $\Phi_s^+ \rightarrow \dfrac{37.5}{s}$



(5) $\Phi_1^+ \approx \dfrac{0 + 37.5\ s}{1 + \sqrt{2}\,\rho s + \rho^2 s^2}$

$$\langle i^2 \rangle = \frac{(37.5)^2}{2\sqrt{2}\,\rho^3} = 4.1 \times 10^8 \qquad \frac{1}{\rho^3} = \frac{11.6 \times 10^8}{1.41 \times 10^3} = 8.24 \times 10^5$$

$$\frac{1}{\rho} = 93.8$$

Method II

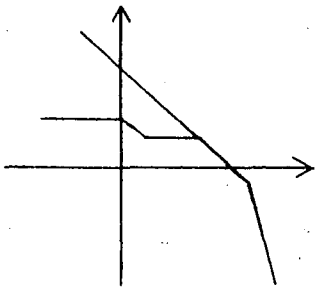
(6) $W = \dfrac{1}{1 + \sqrt{2}\ s/93.8 + (s/93.8)^2}$

Figure 3.4-7a Example 5, Saturation Constraints

$$G = \frac{93.8/\sqrt{2}}{s\,(1 + s/\sqrt{2}\,93.8)} = \frac{66.3}{s\,(1 + s/133)}$$

$$= \frac{66.3}{(1 + s)(1 + s/133)} \cdot \frac{2 + s}{3 + s}$$

$$\boxed{G = \frac{44.2\,(1 + s/2)}{(1 + s)(1 + s/3)(1 + s/133)}}$$



Exact Solution

Coales[6] page 757

$\beta = 3,\ \alpha = 2,\ \rho = 1,\quad a = K = 1,\quad N = 0$

$v = 0,\ \mu = 10^{-3}$

$$G = \frac{44.9\,(1 + s/2.02)}{(1 + s)(1 + s/3)(1 + s/133)}$$

Figure 3.4-7b Example 5, Saturation Constraints

Given: $\boxed{\Phi_{ss} = \dfrac{b^2(1 - \lambda^2/\alpha^2)}{\lambda^4(1 - \lambda^2/\beta^2)}\ ,\ \Phi_{nn} = 1,\ G_f = \dfrac{K}{\lambda(1 + \lambda/\gamma)}\ ,\ J}$

Solution:

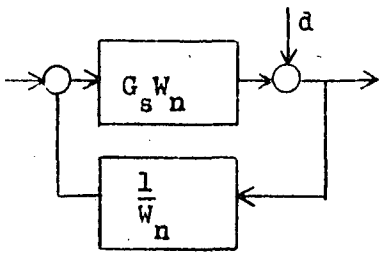
None

Specifications are incompatible $\dfrac{\Phi_s^+}{G_f}(0) = \infty$

Figure 3.4-8 Example 6, Saturation Constraints

can be uniquely utilized. Here, there are two independent specifications, (1) C/R, optimum transmission of a signal in the presence of noise, and (2) C/D, the optimum reduction of a load disturbance; there are two degrees of freedom in specifying the system, (1) G_c, and (2) H. The problem of "degrees of freedom" of a system and realizability of system specifications is an important topic of Horowitz's[10] book. For example see Section 6.1.

Conceptually, the optimum solution with a saturation constraint is given approximately by



where

W_n = optimum W with no saturation constraint.

$W_s = \frac{G_s}{1 + G_s}$ saturation constraint factor

If there is no saturation constraint then $G_s = \infty$. The need for a saturation constraint is obvious.

3.5.2 ALGORITHMS

a. H = 1 Method - Nonoptimum

Let $\Phi_{s'}^{+} = \Phi_s^{+} \oplus \Phi_n^{+}$ and design as before.

b. H ≠ 1 Method - Optimum

(1) Use Saturation Constraint Algorithm

V: $\Phi_{s'}^{+} = \Phi_{d}^{+}$, $\Phi_{n}^{+} = 0$, G_f, J

W: Φ_{s}^{+} , Φ_{n}^{+} , G_f, J

Conceptually: V is optimum system to transmit the load disturbance to the output.

(2) $\Phi_{i}^{+} = \frac{W}{G_f} \Phi_{r}^{+} \oplus \frac{V}{G_f} \Phi_{d}^{+}$

(3) $H = \frac{V}{W}$, $G = \frac{W}{1 - V} = G_c G_f$

3.5.3 DERIVATIONS

The transform of the output signal is

$$C = \frac{G_c G_f}{1 + G_c G_f H} R + \frac{1}{1 + G_c G_f H} D$$

If there is no saturation constraint, then the optimum solution obviously is

$$G_c = \infty \text{ , } H = \frac{1}{W_n} \text{ and } C = W_n R$$

The conceptual discussion in the introduction is demonstrated as follows:

$$W = \frac{G}{1 + GH} = \frac{G_s W_n}{1 + G_s W_n (1/W_n)} = \frac{G_s}{1 + G_s} W_n = W_s W_n$$

$$1 - V = \frac{1}{1 + GH} = \frac{1}{1 + G_s}$$

$$V = \frac{G_s}{1 + G_s} = W_s$$

$$\phi_i^+ = \frac{1}{G_f} (W \phi_r^+ \oplus V \phi_d^+)$$

$$= \frac{1}{G_s} (W_s W_n (\phi_s^+ \oplus \phi_n^+) \oplus W_s \phi_d^+)$$

The H=1 Method is derived as follows:

$$E = S - \frac{G}{1 + GH} (S + N) - \frac{1}{1 + GH} D$$

$$= \left[\frac{1 + G(H-1)}{1 + GH}\right] S - \frac{G}{1 + GH} N - \frac{1}{1 + GH} D$$

If $H = 1$, then

$$E = \frac{1}{1 + G} (S - D) - \frac{G}{1 + G} N$$

$$= (1 - W) (S - D) - WN$$

$$\phi_e^+ = (1 - W) (\phi_s^+ \oplus \phi_d^+) \oplus W \phi_n^+$$

The $H \neq 1$ Method, which is optimum, is based on Chang[5], pages 63-73. His exact solution is outlined below.

$$Y^+ = \left[1 + \frac{k^2}{G_f(s)\, G_f(-s)}\right]^+$$

$$\frac{C}{R}(s) = W = \frac{1}{Y^+ \phi_r^+} \left[\frac{\phi_{ss}}{Y^- \phi_r^-}\right]_+$$

$$\frac{C}{D}(s) = 1 - V, \quad V = \frac{1}{Y^+ \phi_d^+} \left[\frac{\phi_d^+}{Y^-}\right]_+$$

$$\phi_i^+ = \frac{W}{G_f} \phi_r^+ \oplus \frac{V}{G_f} \phi_d^+$$

$$H = \frac{V}{W} \qquad G = \frac{W}{1 - V} = G_c G_f$$

$$\phi_e^+ = (1 - W)\,\phi_s^+ \oplus W\,\phi_n^+ \oplus (1 - V)\,\phi_d^+$$

Intuitively, this is reasonable. The optimum way to force C/D = 1 - V to be zero over as large a frequency range as possible, coincides with the optimum way to force V to be unity over as large a frequency range as possible. This last problem has already been solved.

3.5.4 EXAMPLES

Three examples are presented. Example 1 illustrates when designing for $H \neq 1$ is advantageous. Example 2 illustrates again design with general parameters. Example 3 represents a practical situation, a problem with numbers.

3.6 NMP FIXED ELEMENTS

3.6.1 INTRODUCTION

NMP (non-minimum phase) fixed elements are important because they introduce a definite performance limitation when they are present.

There are three common types of non-minimum phase functions.

$1 - Ts$ RHP zero

e^{-Ts} dead time

$e^{-\sqrt{Ts}}$ distributed lag

Given: $\boxed{\Phi_s^+ = \frac{1}{\lambda}\ ,\ \Phi_n^+ = 1,\ \Phi_d^+ = \frac{d}{\lambda}\ ,\ G_f = \frac{1}{\lambda}\ ,\ J}$

Exact Solution

$H \neq 1$	$H = 1$
$G = \frac{1}{k\lambda(1+\lambda)}$	$G = \frac{1}{\lambda(k + 1/c + k/c\lambda)}$
$H = 1 + \lambda$	$H = 1$
$\frac{1}{k} = \frac{2J}{c}$	$c = \sqrt{1 + d^2}$
$< e^2 > = 1 + \frac{kc}{2}$	$< e^2 > = c\left(1 + \frac{kc}{2}\right)$

RMS error using H = 1

$E = ((1 + d)^{\frac{1}{4}} - 1) \times 100$

d	E(%)
10	217
1	18.9
.1	.25

Rule: If $\Phi_s^+ \gg \Phi_d^+$, then H = 1 can be used

Figure 3.5-1 Example 1, Load Disturbances

Given: $\phi_s^+ = \frac{A}{s}$, $\phi_n^+ = 0$, $\phi_d^+ = \frac{B/n}{s(1 + s/v)}$, $v = \frac{n}{M}$

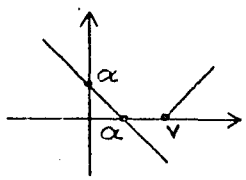
$$G_f = \frac{K_T/n}{s(1 + s/v)}, \quad J$$

Solution by Inspection

[W] $$\frac{k\phi_s^+}{G_f} = k \frac{n\, s(1 + s/v)}{K_T} \frac{A}{s} = \frac{A\,(1 + s/v)}{\alpha}$$

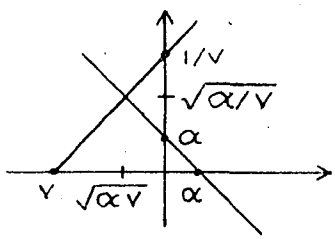
$$\alpha = \frac{K_T}{kn}$$

let $\phi_s^+ = \frac{\alpha}{s}$, $\phi_n^+ = (1 + s/v)$



$v > \alpha \qquad W = \frac{1}{(1 + s/\alpha)(1 + s/v)}$

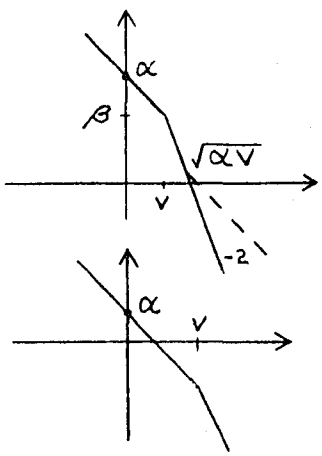
$\sqrt{\frac{\alpha}{v}}\sqrt{\alpha v} = \alpha(1) \qquad \sqrt{\frac{\alpha}{v}}\frac{1}{\sqrt{\alpha v}} = \frac{1}{v}(1)$



$v < \alpha \qquad W = \frac{1}{1 + \sqrt{2}\, s/\sqrt{\alpha v} + s^2/\alpha v}$

[V] $$\frac{k\phi_d^+}{G_f} = k \frac{n}{K_T} \frac{B}{n} = \frac{kB}{K_T}$$

normalized $\phi_d^+ = \frac{B}{ns(1 + s/v)} \qquad \frac{K_T}{kB} = \frac{\alpha}{s(1 + s/v)}, \quad \phi_n^+ = 1$



$\alpha(1) = \beta(v) \qquad (\alpha/v)(v)^2 = (1)(\omega)^2$

$\beta = \alpha/v \qquad \omega = \sqrt{\alpha v}$

$v > \alpha \qquad V = \frac{1}{1 + s/\alpha}$

$1 - V = \frac{s/\alpha}{1 + s/\alpha}$

$v < \alpha \qquad G_v = \frac{\alpha(1 + \sqrt{2}\, s/\sqrt{\alpha v})}{s(1 + s/v)}$

Figure 3.5-2a Example 2, Load Disturbances

$$V \qquad v < \alpha \qquad V = \frac{1 + \sqrt{2}\, s/\sqrt{\alpha v}}{1 + \left(\frac{\sqrt{2}}{\sqrt{\alpha v}} + \frac{1}{\alpha}\right) s + \frac{s^2}{\alpha v}}$$

$$1 - V = \frac{s/\alpha\,(1 + s/v)}{1 + \sqrt{2}\, s/\sqrt{\alpha v} + s^2/\alpha v}$$

$$\phi_i^+ = \frac{W}{G_f}\,\phi_r^+ \oplus \frac{V}{G_f}\,\phi_d^+$$

$v > \alpha$

$$\phi_i^+ = \frac{1}{(1 + s/\alpha)(1 + s/v)}\,\frac{ns(1 + s/v)}{K_T}\,\frac{A}{s} \oplus \frac{1}{(1 + s/\alpha)}\,\frac{B}{K_T}$$

$$\langle i^2 \rangle = \boxed{\frac{A^2 n^2 + B^2}{K_T^2}\,\frac{\alpha}{2} = J, \quad v > \alpha}$$

$v < \alpha$

$$\phi_i^+ = \frac{(1 + s/v)}{(1 + \sqrt{2}\, s/\sqrt{\alpha v} + s^2/\alpha v)}\,\frac{nA}{K_T} \oplus \frac{1 + \sqrt{2}\, s/\sqrt{\alpha v}}{1 + \sqrt{2}\, s/\sqrt{\alpha v} + s^2/\alpha v}\,\frac{B}{K_T}$$

$$\langle i^2 \rangle = \left[\frac{nA}{K_T}\right]^2 \frac{1/\alpha v + 1/v^2}{2\sqrt{\frac{2}{\alpha v}}\,\frac{1}{\alpha v}} + \left[\frac{B}{K_T}\right]^2 \frac{1/\alpha v + 2/\alpha v}{2\sqrt{\frac{2}{\alpha v}}\,\frac{1}{\alpha v}}$$

$$\boxed{= \frac{\sqrt{\alpha v}}{2\sqrt{2}\,K_T^2}\left[n^2 A^2\left[1 + \frac{\alpha}{v}\right] + 3B^2\right] = J, \; v < \alpha}$$

$$H = \frac{V}{W} \qquad v > \alpha \qquad H = 1 + s/v$$

$$v < \alpha \qquad H = 1 + \sqrt{2}\, s/\sqrt{\alpha v}$$

$$G = \frac{W}{1 - V} \qquad v > \alpha \qquad G = \frac{\alpha}{s(1 + s/v)}$$

$$v < \alpha \qquad G = \frac{\alpha}{s(1 + s/v)}$$

$$\boxed{G = \frac{\alpha}{s(1 + s/v)}, \quad H = 1 + bs}$$

$$\boxed{v > \alpha \quad b = \frac{1}{v}; \quad v < \alpha \quad b = \sqrt{\frac{2}{\alpha v}}}$$

Figure 3.5-2b Example 2, Load Disturbances

Exact Solution

(1) Chang[5] pages 74-76

(2) $\omega_0 = \omega_1$, in order to obtain an analytical answer for constraint.

(3) $G = \dfrac{\alpha}{s(1 + s/v)}$

$H = 1 + (b-1/\alpha)s \qquad b = \sqrt{\dfrac{1}{\alpha^2} + \dfrac{2}{\alpha v}}$

$$\frac{1}{2bK_T^2}\left[A^2n^2(1 + \alpha/v) + B^2(1 + \alpha v(b - 1/\alpha)^2)\right] = J$$

Figure 3.5-2c Example 2, Load Disturbances

Given: $\boxed{\Phi_s^+ = \frac{10(1 + s/1)}{s(1 + s/.1)}\ ,\ \Phi_n^+ = 0,\ \Phi_d^+ = \frac{10}{s(1 + s/10)}}$

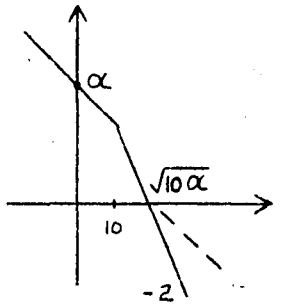
$\boxed{G_f = \frac{1}{s(1 + s/10)}\ ,\ \langle i^2 \rangle \leq (90)^2}$

Solution by Inspection

I Set Constraint

V $\frac{k\Phi_d^+}{G_f} = 10k$ let $\alpha = 1/k$ normalized $\Phi_d^+ = \frac{\alpha}{s(1 + s/10)}\ ,\ \Phi_n^+ = 1$

2 cases



$\alpha > 10 \quad V = \frac{1}{1 + s/\sqrt{10\alpha}}$

$\Phi_{i_d}^+ = \frac{V}{G_f}\Phi_d^+ \approx \frac{10}{1 + s/\sqrt{10\alpha}}$

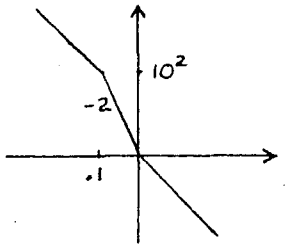
$\langle i_d^2 \rangle \approx 50\sqrt{10\alpha}$

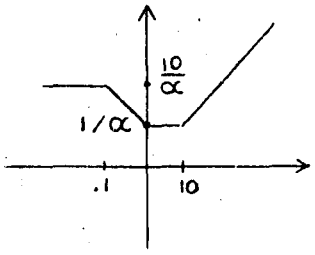
$\alpha < 10 \quad V \approx \frac{1}{1 + s/\alpha}$

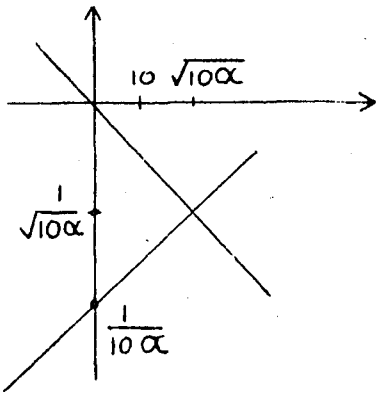
$\Phi_{i_d}^+ \approx \frac{10}{1 + s/\alpha}$

$\langle i_d^2 \rangle \approx 50\,\alpha$

W $\frac{k\Phi_s^+}{G_f} = \frac{1}{\alpha}\, s(1 + s/10)\, \frac{10(1 + s)}{s(1 + s/.1)} = \frac{10(1 + s)(1 + s/10)}{\alpha(1 + s/.1)}$







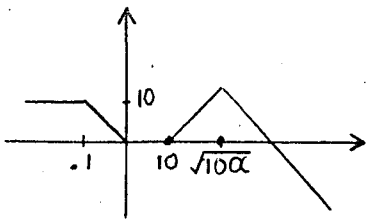
4 cases, start with smallest constraint

$\alpha > 10 \qquad W \approx \frac{1}{1 + \sqrt{2}\, s/\sqrt{10\alpha} + s^2/10\alpha}$

Figure 3.5-3a Example 3, Load Disturbances

$$\Phi_{i_s}^{+} = \frac{W\Phi_s^{+}}{G_f} \approx \frac{1}{1 + \sqrt{2}\, s/\sqrt{10\alpha} + s^2/10\alpha} \quad \frac{s(1 + s/10)}{1} \quad \frac{10(1 + s/1)}{s(1 + s/.1)}$$

$$\Phi_{i_s}^{+} \approx \frac{1 + s/10}{1 + \sqrt{2}\, s/\sqrt{10\alpha} + s^2/10\alpha}$$



$$\langle i_s^2 \rangle \approx \frac{1/10\alpha + 1/10^2}{2(1)\sqrt{\frac{2}{10\alpha}\frac{1}{10\alpha}}}$$

$$\approx \left(1 + \frac{\alpha}{10}\right)\frac{\sqrt{10\alpha}}{2\sqrt{2}}$$

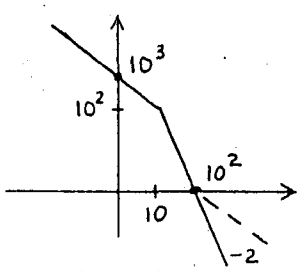
$\alpha > 10$

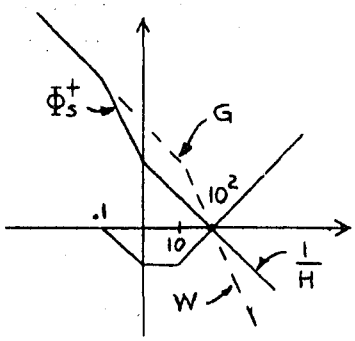
$$\langle i_s^2 \rangle \approx \left(50 + \frac{(1 + \alpha/10)}{2\sqrt{2}}\right)\sqrt{10\alpha} = F(\alpha) \approx (90)^2$$

$F(1) = (50.36)(3.16) = 159$
$F(10) = (50.71)(10) = 507$
$F(100) = (50.89)(31.6) = 1700$
$F(1000) = (85.70)(100) = 8570 = (92.6)^2$

Close enough! No need to consider other cases. $\alpha = 10^3$

II Find System





$$G_v = \frac{10^3(1 + s/70.7)}{s(1 + s/10)}$$

$$V = \frac{1 + s/70.7}{1 + \left(\frac{1}{70.7} + \frac{1}{10^3}\right)s + \frac{s^2}{100^2}}$$

$$1 - V = \frac{s(1 + s/10)}{10^3(1 + \sqrt{2}\, s/100 + s^2/100^2)}$$

$$W = \frac{1}{1 + \sqrt{2}\, s/100 + s^2/100^2}$$

$$H = \frac{V}{W} = \boxed{1 + s/70.7 = H}$$

$$G = \frac{W}{1 - V} = \frac{10^3}{s(1 + s/10)} \qquad \boxed{G_c = 10^3}$$

checks Note: $\omega < \omega_c$ $GH \neq \Phi_s^{+}$

Figure 3.5-3b Example 3, Load Disturbances

Exact Solution

(1) Chang[5] pages 74-76.

(2) Chang did not determine ω_k and 2ζ in terms of $< i^2 >$. It is not easy. Using $\alpha = 10^3$ yields $\omega_k = 10^2$ and $2\zeta = \sqrt{2.01}$.

(3) $G_c = 1000 \left[\frac{1 + s/.9874}{1 + s}\right]$

$H = (1 + s/71.4) \left[\frac{1 + s/.9989}{1 + s/.9874}\right]$

or $H_1 = \frac{1.426}{1000} \left[\frac{1 + s/.1008}{1 + s/.9874}\right]$

(4) Note advantages of Solution by Inspection.

a. Constraint can actually be evaluated.

b. It is easier than plugging numbers into the general exact solution (even if it were available).

c. Solution is in much better form.

d. The effect of the variation of a parameter is obvious graphically. Just try to interpret the effect of varying ω_0 (say from .1 to .01) in the exact solution!

Figure 3.5-3c Example 3, Load Disturbances

Because

$$(1 - Ts)G_1 = \frac{1 - Ts}{1 + Ts} \quad (1 + Ts)G_1$$

$$e^{-Ts} \approx \frac{1 - T/2s}{1 + T/2s} \qquad \omega_c \leq \frac{2}{T}$$

$$e^{-\sqrt{Ts}} \approx \frac{1}{1 + Ts} \qquad \omega_c \leq \frac{5}{T}$$

it is only necessary to design for an all-pass element.

$$\frac{1 - Ts}{1 + Ts}$$

and, in last two cases, a bandwidth constraint. The NMP break frequency is 1/T.

The difficulties caused by the presence of the all-pass element are obvious. It introduces 180° phase lag with no amplitude attenuation. Control systems must have less than 180° phase lag at, ω_c, crossover.

It is also obvious that it is only necessary to design for the lowest NMP break frequency. The other NMP fixed elements can simply be included in the expression for G as a last step.

A great deal more work can be done on the NMP problem than is presented here. However, these results are useful by themselves.

3.6.2 ALGORITHMS

a. NMP Rule

If $1/T < \omega_c$ a satisfactory design is unlikely.

b. NMP Algorithm

If $10\omega_c < 1/T$

then find optimum W_n or G_n for $T = 0$, and

$$W \approx \frac{1 - Ts}{1 + Ts} W_n \quad \text{or}$$

$$G \approx \frac{1 - Ts}{1 + Ts} G_n$$

3.6.3 DERIVATION

NMP Algorithm is obvious from Table 3.6-1. If $T \ll 1$ then $W = \frac{1 - T\lambda}{1 + T\lambda} W_n$ where W_n is the optimum W ignoring the NMP fixed elements. For $W \ll 1$, $G \approx W$, and therefore, $G \approx \frac{1 - T\lambda}{1 + T\lambda} G_n$. Figure 3.6-2 presents an error analysis which demonstrates that for the frequency range specified the approximation is excellent. NMP Rule is obvious from Table 3.6-2. Figure 3.6-1 clearly demonstrates that for $1/T \ll \omega_c$, and slope $\phi_s^+ < -1$, even the optimum system is highly undesirable.

3.6.4 EXAMPLES

Figures 3.6-3 and 3.6-4 present two examples.

$\phi_n^+ = 1, \quad G_f = \frac{1 - T\lambda}{1 + T\lambda}$	
$\phi_s^+ = \frac{1}{\lambda}$	$W = \left(\frac{1 - T\lambda}{1 + T\lambda}\right) \frac{1}{(1 + \lambda)}$ $< e^2 > = 1 + 2T$
$\phi_s^+ = \frac{1}{\lambda^2}$	$W = \left(\frac{1 - T\lambda}{1 + T\lambda}\right) \frac{1 + (\sqrt{2} + 2T)\lambda}{1 + \sqrt{2}\,\lambda + \lambda^2}$ $< e^2 > = \sqrt{2}\,(1 + 2\sqrt{2}\,T + 4T^2 + \sqrt{2}\,T^3)$
$\phi_s^+ = \frac{1}{\lambda^3}$	$W = \left(\frac{1 - T\lambda}{1 + T\lambda}\right) \frac{1 + 2(1 + T)\lambda + 2(1 + T)^2\lambda^2}{1 + 2\lambda + 2\lambda^2 + \lambda^3}$ $< e^2 > = 2\,(1 + 4T + 8T^2 + 8T^3 + 4T^4 + T^5)$

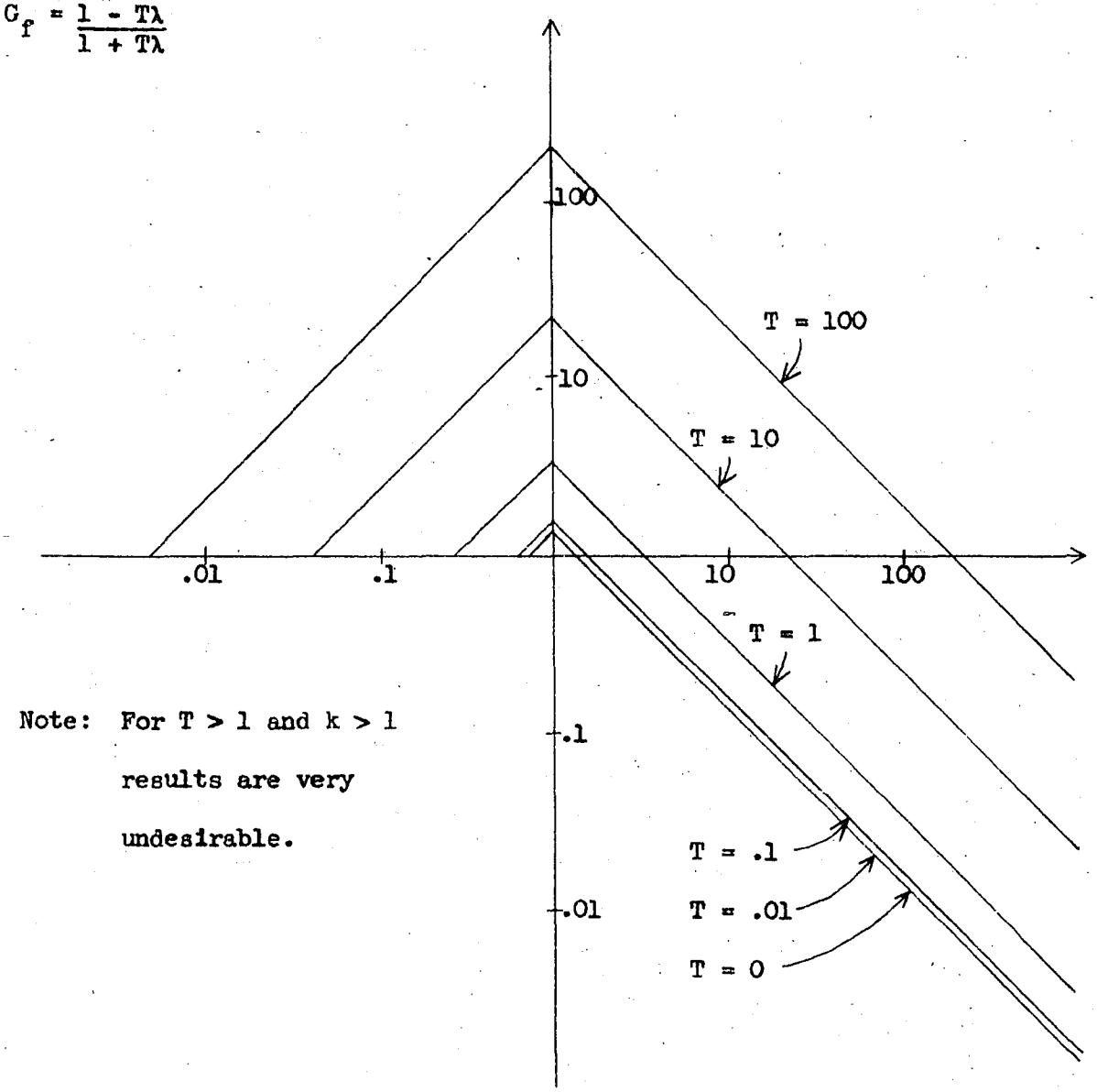
Table 3.6-1 Standard Optimum Systems with an All-pass NMP Fixed Plant

		$\left[\frac{< e^2 >}{< e^2 > T = 0}\right]^{1/2}$		
		k = 1	k = 2	k = 3
T	.01	1.010	1.014	1.020
	.1	1.095	1.151	1.220
	1	1.732	3.040	5.099
	10	4.583	42.936	385.80

Table 3.6-2 Ultimate Performance Limitation Due to an All-pass Fixed Plant

$$\Phi_s^+ = \frac{1}{\lambda^2}\ ,\ \Phi_n^+ = 1$$

$$G_f = \frac{1 - T\lambda}{1 + T\lambda}$$



Note: For T > 1 and k > 1 results are very undesirable.

Figure 3.6-1 Example of an Optimum W with an All-pass NMP Fixed Plant.

$$G_a = \left[\frac{1 - T\lambda}{1 + T\lambda}\right] G_n$$

$$\Phi_s^+ = \frac{1}{\lambda^k}, \quad \Phi_n^+ = 1$$

$$G_f = \frac{1 - T\lambda}{1 + T\lambda}$$

G_n = Optimum G with $G_f = 1$.

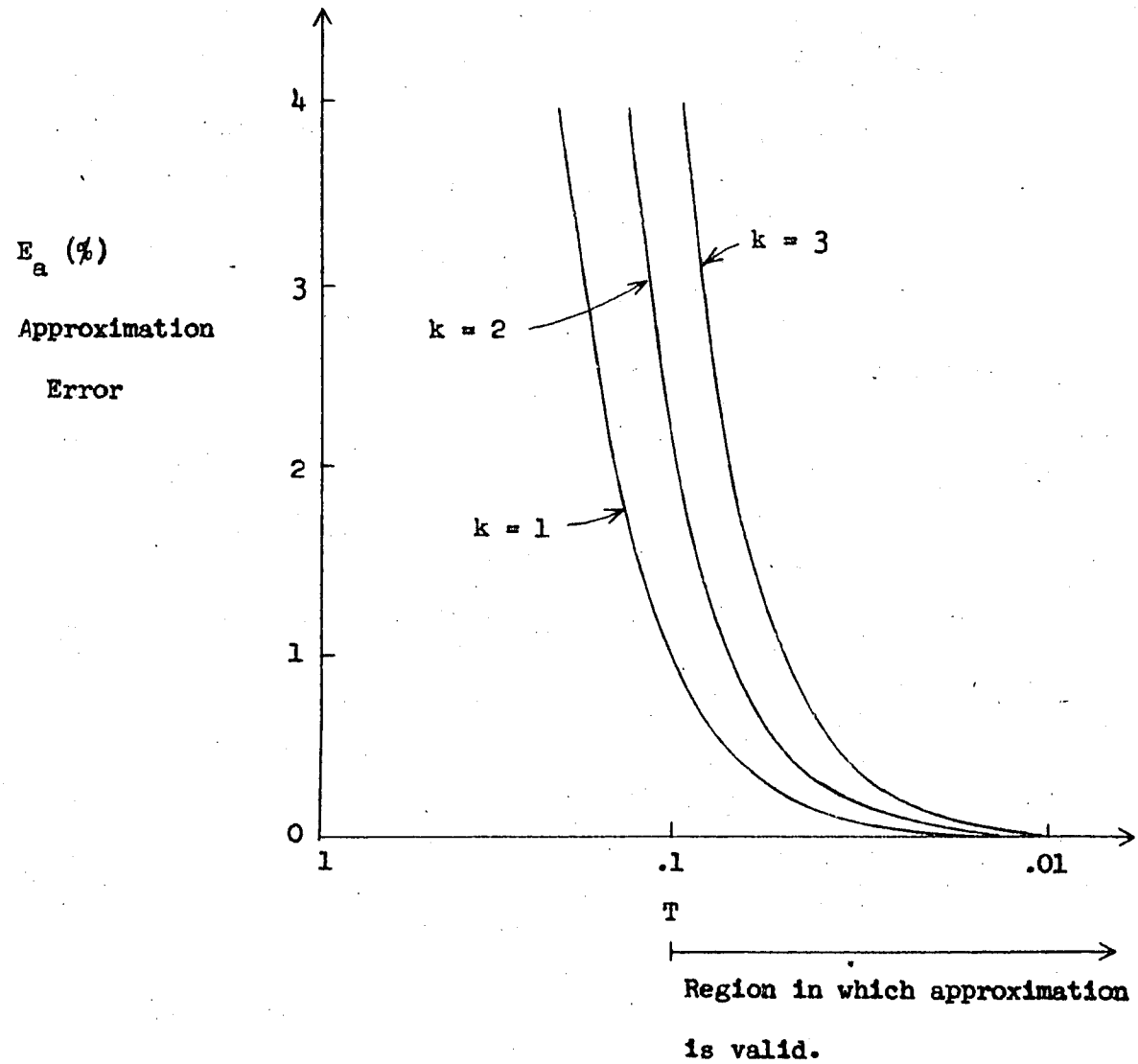


Figure 3.6-2 Error Analysis for Approximate System.

Given: $\boxed{\Phi_s^+ = \frac{A}{s}\ ,\ \Phi_n^+ = 0,\ \Phi_d^+ = \frac{B}{s}\ ,\ G_f = \frac{1 - s/z}{1 + s/z}\ ,\ J = \infty}$

Solution: $W = \frac{1 - s/z}{1 + s/z} = V \qquad 1 - V = \frac{2s/z}{1 + s/z}$

$H = \frac{V}{W} = \boxed{1 = H} \qquad G = \frac{W}{1 - V} = \boxed{\frac{z(1 - s/z)}{2s} = G}$

Exact Solution: Chang[5] pages 90-91

Figure 3.6-3 Example 1, NMP Fixed Plant

Given: $\boxed{\Phi_s^+ = \frac{1}{s}\ ,\ \Phi_n^+ = \Phi_d^+ = 0,\ G_f = \left(\frac{1 - Ts}{1 + Ts}\right)\frac{1}{s}\ ,\ J = 5}$

Solution: Assume $1/T > 10\omega_c$

$$G_f = \frac{1}{s}\ ,\quad \frac{k\Phi_s}{G_f} = k = \frac{1}{\alpha}$$

normalized $\Phi_s^+ = \frac{\alpha}{s}\ ,\ \Phi_n^+ = 1,\ G = \frac{\alpha}{s}\ ,\ \omega_c = \alpha$

$$\Phi_i^+ = \frac{W\Phi_s^+}{G_f} = \frac{1}{1 + s/\alpha}\ ,\quad \langle i^2 \rangle = \frac{\alpha}{2} = J = 5 \qquad \alpha = 10$$

$$\boxed{G = \frac{10}{s}\ \frac{1 - Ts}{1 + Ts} \qquad \frac{1}{T} > 100}$$

Exact Solution:

$$G = \frac{10(1 - Ts)}{s(1 + 20T + Ts)}$$

Figure 3.6-4 Example 2, NMP Fixed Plant

Chapter 4

DISCUSSION AND SUMMARY

4.1 EXAMPLES

Twenty seven examples which illustrate the technique (another eighteen examples illustrate specific points) have been presented. Most of these examples are analogous to the "2 x 4 = 8, and $\sqrt{25} = 5$ stage" of learning to use a slide rule. The real advantage of a slide rule appears only when solving less trivial problems such as $(2\pi)^{1.63} \ln 39.3/(27.3)^3$. The same applies here. These simple examples were used in order to simplify the tutorial presentation and to make feasible exact solutions for the purposes of error analysis. It is obvious that an increase in the number of break frequencies in the problem specification rapidly makes the exact solution unfeasible while the solution by inspection is unaffected.

Example 3 of Section 3.4, and Example 3 of Section 3.5 are simple but nontrivial practical problems which demonstrate the power of the technique presented. A fair evaluation requires the actual working out of the exact solution including all steps, and then comparing it with the solution by inspection. It is a convincing demonstration.

For tutorial reasons the examples are presented with all steps included just as the author first worked them out.

Practical control problems generally consist of sets of numbers rather than sets of general parameters. The technique presented is even easier to use with numbers than with general parameters. The use of general parameters requires more proficiency with Bode plots and sometimes the splitting of cases. However, most of the examples presented are in terms of general parameters in order to illustrate how to handle this less practical but more difficult case. Many of the examples used in textbooks on optimum control do not represent practical control problems. They are in terms of general parameters in order to illustrate complex algebraic manipulations, or they treat rare cases such as lagging filters in order to increase the small number of analytically tractable examples.

4.2 APPLICATIONS

Numerous applications which have not been mentioned become obvious when one is familiar with the technique presented. However, a few comments are appropriate.

Consider the practical problem of selecting a fixed plant, for example a motor. The fixed plant should look as much like the factored normalized signal density spectrum as possible. Any gain not supplied by the fixed plant must be supplied by the compensation, and that amplifies the input to the saturating element creating

saturation problems. Using the technique presented the trade-off between fixed plant gain and linear operating range is obvious.

Nonlinear and/or adaptive systems are almost always considerably more complex and expensive than optimum linear nonadaptive systems. The only justification for a nonlinear and/or adaptive system is that it performs considerably better than the optimum linear nonadaptive system for the same situation, and that the improvement in performance justifies the added complexity and cost. This comparison should always be made. It is easy to show on several examples presented in the literature on adaptive systems that a Wiener system is superior to the "optimum" adaptive system they present. A proficiency with Wiener systems is valuable even for those whose primary interest is in the newer and more glamorous aspects of control theory.

In a practical situation, a designer would probably work out numerous designs by inspection until he decided upon the final design. Then as a check solution a specialist could work out the exact solution to the point where he had a set of virtually unsolvable equations. At that point he substitutes the solution by inspection into the set of equations verifying that it is the correct solution. It is far easier to verify that the n roots of an n^{th} order equation satisfy it, than it is to obtain those n roots from the equation. Then the specialist could write up an impressive final report full of integral equations.

The design of Wiener systems for complex cases could be accomplished using a digital computer. However, programing and computer time cost money, and this approach yields no physical insight for the problem or the solution.

4.3 EXTENSIONS

One very important problem remains to be solved. There exists a very definite need for a convenient method of obtaining spectral densities. An analog computer setup would be very desirable. If this is not feasible the spectrum analyzer should be inexpensive enough that all universities and companies doing control work would buy one. It should be fast, say take fifteen minutes to obtain a spectral density from a magnetic tape record or the output of a black box with an input from a white noise generator. An accuracy of $\pm$ 5 percent would be more than adequate.

The other major extension which should be made is to publish a tutorial presentation on this technique so that it becomes available to a large number of control engineers who could use it.

There are several other extensions of the technique presented which could be made. Some elegance could be added to the derivations. This would somewhat extend the region of validity of this technique. Non-minimum-phase systems could use more work. The technique could be extended to some of the cases which were not treated here, such

as integrating and differentiating systems. A useful addition would be to make a completely parallel presentation of the pure continuous and pure discrete cases. See Yaglom[25] for an example of a parallel approach. The pure discrete case is usually sufficient to handle most sampled-data problems.

4.4 SUMMARY

This work presents a new approach to an old solution for a very broad category of practical control problems. It consists of the design by inspection of Wiener optimum control systems, and is a significant tutorial, and computational contribution to Wiener theory. This work demonstrates the feasibility and great power of this approach, but it lacks polish. The primary contribution of this work is the viewpoint presented. With this viewpoint, common sense will solve most problems. See again the technical summary in Section 1.3, the basic points in Section 3.1.1, the statement of the problem in Section 3.1.2, and the examples in Figure 3.3-4.

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