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I hereby recommend that this dissertation prepared under my direction by Rollin T. Sandberg entitled On the Compatibility of Integrals be accepted as fulfilling the dissertation requirement of the degree of Doctor of Philosophy.

Berthold Schweizer
Dissertation Director

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ABSTRACT

Let f be a real finite-valued function defined on the interval $[a,b]$; and for $c > 0$ let $\sum_c{}_a^b f = \sum_{n=1+w(a/c)}^{w(b/c)} f(nc)$, where w is the greatest integer function. The uniform integral of f on $[a,b]$, denoted by $(U)\int_a^b f$, is given by the limit $\lim_{c\to 0} c \sum_c{}_a^b f$, provided this limit exists.

The purpose of this paper is to investigate the relationships between the uniform integral and the Riemann and Lebesgue integrals. It is easily shown that the uniform integral is a proper extension of the Riemann integral and furthermore that there are functions which are uniformly but not Lebesgue (resp., Lebesgue but not uniformly) integrable.

Using successively more complicated arguments it is proved that the uniform and Lebesgue integrals are compatible on the following classes of functions:

1) functions equal almost everywhere to Riemann integrable functions;

2) bounded functions;

3) all functions if $[a,b]$ does not contain the origin and if $0 \in [a,b]$ all functions whose oscillation at 0 is finite.

The argument in the proof of compatibility for bounded functions hinges on a (privately communicated) result of J. G. van der Corput, namely: If G is an open subset of $[a,b]$, $m(G)$ is the Lebesgue measure of G, and $\mu(G;c)$ the number of integral multiples of the positive number c contained in G, then for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $0 < c < \delta$ then $\int_0^1 ct\,\mu(G;ct)dt < (1+\varepsilon)m(G)$, where the integral is taken in the Lebesgue sense. The compatibility proof for unbounded functions depends on a generalization of van der Corput's lemma to a countable number of open sets. Another consequence of van der Corput's lemma is the fact that for any measurable subset A of $[a,b]$, $\lim_{c \to 0} \int_0^1 ct\,\mu(A;ct)dt$ exists and equals $m(A)$.

Lastly, the n-dimensional uniform integral is defined and the corresponding compatibility problem for bounded n-place functions is settled affirmatively.

TABLE OF CONTENTS

1. INTRODUCTION

In defining an integral it may be desirable for the sake of simplicity to use a regular or uniform partition of the domain of integration. In the case of the Riemann integral, for example, W. F. Osgood [7, p. 118], and more recently, A. Sklar [11], have considered a definition which employs a sequence of partitions P_n of the interval $[a,b]$ in which P_n consists of the points of the form $k/2^n$, where k is an integer such that $a \leq k/2^n \leq b$, together with the end points a and b. Furthermore, Sklar has shown that if the upper and lower Riemann sums for each P_n are defined in the usual way, then the upper and lower integrals, and hence the Riemann integral itself, can be defined sequentially.

Motivated by this characterization of the Riemann integral, Sklar has recently constructed a new integral, called the uniform integral. It is defined as follows: Let f be a real finite-valued (but possibly unbounded) function defined on the interval $[a,b]$; and, for $c > 0$ let

$$\sum_{c\,a}^{b} f = \sum_{n=1+w(a/c)}^{w(b/c)} f(nc),$$

where w is the greatest integer function. The uniform integral of f on $[a,b]$, denoted by $(U)\int_a^b f$ is given by the limit,

$$(U) \int_a^b f = \lim_{c \to 0} \; c \underset{c}{\sum}{}_a^b f,$$

provided the latter exists. Similarly, the upper and lower uniform integrals are given by

$$(U) \int_a^{\overline{b}} f = \limsup_{c \to 0} \; c \underset{c}{\sum}{}_a^b f \quad \text{and} \quad (U) \int_{\underline{a}}^b f = \liminf_{c \to 0} \; c \underset{c}{\sum}{}_a^b f,$$

respectively. It is clear that the uniform integral is a linear operator on the class of all uniformly integrable functions on $[a,b]$.

The purpose of this dissertation is to study the relationship of the uniform integral to the Riemann integral and the Lebesgue integral. It will be shown that for bounded functions the uniform integral is an extension of the Riemann integral and is compatible with the Lebesgue integral, in the sense that if f is both uniformly and Lebesgue integrable on $[a,b]$ then $(U) \int_a^b f = (L) \int_a^b f$. The proof of the first of these facts is simple and direct. The proof of compatibility, however, lies much deeper and depends crucially on a result of J. G. van der Corput.[1]

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[1] Privately communicated.

The body of this thesis begins with Section 3 which consists of various examples including those that distinguish the uniform integral from the Riemann and Lebesgue integrals. In Section 4 it is shown that the uniform integral is an extension of the Riemann integral. In Section 5 the uniform and Lebesgue integrals are shown to be compatible for functions that are equal almost everywhere to Riemann integrable functions.

Section 6 deals with the general compatibility proof for the uniform and Lebesgue integrals for bounded functions. The argument here hinges on the following lemma of van der Corput: If G is an open subset of $[a,b]$, $m(G)$ is the Lebesgue measure of G, and $\mu(G;c)$ the number of integral multiples of the positive number c contained in G, then for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $0 < c < \delta$ than $\int_0^1 ct\mu(G;ct)dt < (1+\varepsilon)m(G)$, where the integral is taken in the Lebesgue sense. The compatibility of the uniform and Lebesgue integrals follows readily from this lemma. Another consequence of van der Corput's lemma, interesting in its own right, is the fact that for any measurable subset A of $[a,b]$, $\lim_{c\to 0^+} \int_0^1 ct\mu(A;ct)dt$ exists and equals $m(A)$.

Section 8 is devoted to the compatibility problem for unbounded functions. With the aid of a generalization of van der Corput's lemma to a countable number of open sets it is shown that the uniform and Lebesgue integrals are compatible on any interval $[a,b]$ that does not contain the origin and if $0 \epsilon [a,b]$, for all functions whose oscillation at 0 is finite. Whether this last restriction can be removed is still an open question.

In Section 9 the n-dimensional uniform integral is defined and the corresponding compatibility problem for bounded n-place functions is settled in the affirmative. This extension to n-dimensions is straightforward since the results of Section 6 are readily generalized.

To avoid interrupting the continuity of the main argument, certain results that can be obtained routinely as well as several digressions are treated in three appendices which appear at the end.

2. SYMBOLS AND ABBREVIATIONS

Symbols

I: the set of integers

$w(x)$: the greatest integer in x

$\sum_{c\ a}^{\ \ b} f = \sum_{n=1+w(a/c)}^{w(b/c)} f(nc)$, where f is a real finite-valued function defined on $[a,b]$ and $c > 0$.

$m(A)$: the Lebesgue measure of the linear set A

$m_n(A)$: the Lebesgue measure of the n-dimensional set A

$m^*(A)$: the outer Lebesgue measure of the linear set A

$m_*(A)$: the inner Lebesgue measure of the linear set A

χ_A: the characteristic function of the set A

$\mu_c(A)$ or $\mu(A;c)$: the number of integral multiples of the positive number c contained in the bounded linear set A

Abbreviations

R - integrable for Riemann integrable.

L - integrable for finitely Lebesgue integrable.

U - integrable for finitely uniformly integrable.

3. EXAMPLES

The following examples show that neither of the uniform or Lebesgue integrals is an extension of the other and that the class of functions which are both uniformly and Lebesgue integrable is not the class of Riemann integrable functions.

1. Let f be an arbitrary function defined on the half-open interval $(0,b]$. Let g be the function defined by

$$g(x) = \begin{cases} f(x), & 0 < x \leq b, \\ 0, & x = 0, \\ -f(-x) & -b \leq x < 0. \end{cases}$$

Then $\underset{c}{c\sum}_{-b}^{b} g$ is equal to $c\,g(b)$ when b/c is an integer and 0 otherwise. Consequently $(U)\int_{-b}^{b} g$ exists and is equal to zero. Thus any odd function is U-integrable on any interval symmetric about the origin. It follows that there exist functions which are U-integrable but neither L-integrable nor R-integrable.

2. Example 1 depends on the peculiar relationship that exists between the uniform integral and the number 0 and is due to the fact that the integral multiples of any non-zero number are symmetric about 0. It is desirable to have an example free from

this "defect." Let $[a,b]$ be a given interval. By changing addition to multiplication in Vitali's proof of the existence of a non-measurable set, it can be shown (see Appendix II) that there exists a non-measurable subset H of $[a,b]$ such that if x and y are in H and $x \neq y$, then x/y is irrational. Let χ_H be the characteristic function of H. Then for any $c > 0$ at most one integral multiple of c can belong to H. Thus $\sum_{c}{}_{a}^{b} \chi_H$ is either 0 or 1; and hence $(U) \int_a^b \chi_H$ exists and is equal to 0. The function χ_H is U-integrable but not L-integrable. This same result can also be obtained by using a non-measurable Hamel basis instead of the set H (Appendix II).

3. Let χ_R be the characteristic function of the rationals in $[a,b]$. Then χ_R is L-integrable. However, $\sum_{c}{}_{a}^{b} \chi_R = w(b/c)-w(a/c)$ when c is rational and 0 otherwise; and since

$$\lim_{c \to 0} c[w(b/c) - w(a/c)] = b-a,$$

it follows that $(U) \overline{\int}_a^b \chi_R = b-a$ while $(U) \underline{\int}_a^b \chi_R = 0$, whence χ_R is not U-integrable. Thus there exist functions which are L-integrable but not U-integrable.

4. Let f be the function defined on $[0,2]$ by

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \text{ and } x \text{ rational,} \\ 0, & 0 \leq x \leq 1 \text{ and } x \text{ irrational,} \\ 0, & 1 < x \leq 2 \text{ and } x \text{ rational,} \\ 1, & 1 < x \leq 2 \text{ and } x \text{ irrational.} \end{cases}$$

Clearly, f is L-integrable; and the argument given in Example 3 shows that f is U-integrable, with $(U) \int_0^2 f = 1$. Thus there exist functions which are both U-integrable and L-integrable but not R-integrable. Granting the result of the next section, i.e., that the uniform integral is an extension of the Riemann integral, this example shows that the class of R-integrable functions is a proper subset of the class of functions which are both U-integrable and L-integrable.

4. COMPARISON WITH THE RIEMANN INTEGRAL

The purpose of this section is to consider the relationship between the uniform and the Riemann integral.

<u>Lemma 4.1</u>: If f is bounded on $[a,b]$ then

$$(1) \qquad (R)\underline{\int_a^b} f \leq (U)\underline{\int_a^b} f \leq (U)\overline{\int_a^b} f \leq (R)\overline{\int_a^b} f \,.$$

Proof: Let $k(c) = w(b/c) - w(a/c)$,

$$A_n = \{f(x) \mid x \in [c(w(a/c)+n-1),\, c(w(a/c)+n)] \cap [a,b]\},$$

$$M_n = \sup A_n \quad \text{and} \quad m_n = \inf A_n, \qquad n = 1, 2, \cdots, k(c).$$

Then $m_n \leq f[c(w(a/c)+n)] \leq M_n$, $\quad n = 1, 2, \cdots, k(c)$, and hence,

$$(2) \qquad c\sum_{n=1}^{k(c)} m_n \leq c \underset{c}{\sum}{}_{a}^{b} f \leq c\sum_{n=1}^{k(c)} M_n \,.$$

Now, each summation in (2) differs from the corresponding (lower or upper) Riemann sum by at most a term which is $o(1)$ - namely the contribution from the interval $[\,cw(b/c), b\,]$. Thus taking the lim inf of the first and second terms and the lim sup of the second and third yields to desired chain of inequalities (1).

Strict inequality can hold in all three places in (1). To see this replace "$1 < x \leq 2$" by "$1 < x \leq 3$" in Example 4. Then

$$0 = (R) \underline{\int_0^3} f < (U) \underline{\int_0^3} f = 1 < 2 = (U) \overline{\int_0^3} f < (R) \overline{\int_0^3} f = 3.$$

As a direct consequence of Lemma 4.1, we have

Theorem 4.1: If f is Riemann integrable on $[a,b]$ then f is uniformly integrable on $[a,b]$ and the two integrals have the same value.

Thus the uniform integral is an extension of the Riemann integral.

Corollary 4.1: If f is uniformly integrable on $[a,b]$ then

$$(3) \qquad (R) \underline{\int_a^b} f \leq (U) \int_a^b f \leq (R) \overline{\int_a^b} f .$$

Corollary 4.2: If f is bounded and both uniformly and Lebesgue integrable, then

$$(4) \qquad \left|(U)\int_a^b f - (L)\int_a^b f\right| \le (R)\int_a^{\overline{b}} f - (R)\int_{\underline{a}}^b f \,.$$

Proof: The inequality (3) remains valid when $(U)\int_a^b f$ is replaced by $(L)\int_a^b f$. From this the desired result follows by subtraction.

Note that the inequality (4) yields a bound for the difference of the uniform and Lebesgue integrals of f.

5. THE RESTRICTED COMPATIBILITY PROBLEM

In this section we show that the uniform and Lebesgue integrals are compatible on the class of functions which are equal almost everywhere to Riemann integrable functions. We begin with a simple lemma which we state without proof.

Lemma 5.1: If A is a linear set, c a real number, and $A_c = \{cx \mid x \in A\}$ then $m^*(A_c) = |c|\, m^*(A)$.

Lemma 5.2: If A is a set of Lebesgue measure zero, then there exists a real number $t \neq 0$ such that for all non-zero rational numbers r, $rt \notin A$.

Proof: Enumerate the rational numbers in a sequence $\{r_i\}$. For each integer i, let $A_i = \{r_i x \mid x \in A\}$. By Lemma 5.1 $m^*(A_i) = |r_i|\, m^*(A) = 0$. Consequently,

$$0 \leq m^* \left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m^* (A_i) = 0$$

and since $r_i t \in A_i$ implies $t \in A$, it follows that any non-zero number t in the complement of $\bigcup_{i=1}^{\infty} A_i$ has the desired property.

Lemma 5.3: If A is a set of the first category then there exists a real number $t \neq 0$ such that for all non-zero rational numbers r, $rt \notin A$.

Proof: If A is of the first category then so are the sets A_i, $i = 1, 2, \cdots$, defined above and $\bigcup_{i=1}^{\infty} A_i$. Since the real line is of second category, the complement of $\bigcup_{i=1}^{\infty} A_i$ is non-empty. Any non-zero number t in this complement has the desired property.

Theorem 5.1: If f is uniformly integrable on $[a,b]$ and equal almost everywhere to a Riemann integrable function g, then

$$(1) \qquad (U)\int_a^b f = (L)\int_a^b f = (R)\int_a^b g.$$

Proof: Let $h = f-g$, $S(c) = c \sum\limits_{c}{}_a^b h$, and $A = \{x \mid h(x) \neq 0\}$. Then $mA = 0$ and there exists a number $t \neq 0$ satisfying the conditions of Lemma 5.2. For this t and for all positive integers n, we have $S(|t|/n) = 0$ unless $a < 0 \leq b$, in which case $S(|t|/n) = (|t|/n)\, h(0)$. In either case, $\lim_{n \to \infty} S(|t|/n) = 0$. Since h, being the difference of two U-integrable functions, is U-integrable, this implies that $(U)\int_a^b h = 0$ and

$$(U)\int_a^b f = (U)\int_a^b g = (R)\int_a^b g.$$

Lastly,

$$(L) \int_a^b f = (L) \int_a^b g = (R) \int_a^b g,$$

and the theorem is proved.

Theorem 5.2: If f and g are both uniformly integrable on $[a,b]$ and differ only on a set of the first category then $(U) \int_a^b f = (U) \int_a^b g$.

Proof: This follows from Lemma 5.3 in the same way that Theorem 5.1 follows from Lemma 5.2.

Corollary 5.1: If f is uniformly integrable on $[a,b]$ and equal to a Riemann integrable function g except on a set of the first category then $(U) \int_a^b f = (R) \int_a^b g$.

6. THE GENERAL COMPATIBILITY PROBLEM

In this section we show that the Lebesgue and uniform integrals are compatible for bounded functions. We begin with several lemmas among which the crucial one is Lemma 6.5 which is due to van der Corput.

For any bounded set H and $c > 0$, let $\mu(H:c)$ or $\mu_c(H)$ be the number of integral multiples of c in H. It is evident that μ_c is an integral-valued (finite) measure on the class of all subsets of any bounded set and that $\mu_c(H)=\sum_{ca}^{b} \chi_H$ for any subset H of $(a,b]$.

Lemma 6.1: If G is an open set contained in $[a,b]$, then $\liminf_{c\to 0} c\,\mu_c(G) \geq mG$, where mG is the Lebesgue measure of G.

Proof: We have $G = \bigcup_{i=1}^{\infty}(a_i, b_i)$ where the intervals are pairwise disjoint. Since

$$\mu_c((a_i, b_i)) = w(b_i/c) - w(a_i/c) - \chi_I(b_i/c),$$

where I is the set of integers, we have

$$\lim_{c \to o} c\mu_o((a_i, b_i)) = b_i - a_i, \quad i = 1, 2, \cdots, \quad .$$

Therefore, for every n

$$\sum_{i=1}^{n} (b_i - a_i) = \lim_{c \to o} c\mu_c\left(\bigcup_{i=1}^{n} (a_i, b_i)\right) \leq \liminf_{c \to o} c\mu_c(G),$$

and hence $mG \leq \liminf_{c \to 0} \mu_c(G)$.

<u>Corollary 6.1:</u> If $\{G_n\}$ is a sequence of open sets contained in $(a,b]$, then $\liminf_{c \to o} \sum_{n=1}^{\infty} c\mu_c(G_n) \geq \sum_{n=1}^{\infty} mG_n$.

<u>Proof</u>: For any integer k, $\sum_{n=1}^{k} mG_n \leq \sum_{n=1}^{k} \liminf_{c \to o} c\mu_c(G_n)$ $\leq \liminf_{c \to o} \sum_{n=1}^{k} c\, \mu_c(G_n) \leq \liminf_{c \to o} \sum_{n=1}^{\infty} c\, \mu_c(G_n)$; and letting $k \to \infty$ yields the result.

<u>Lemma 6.2</u>: If H is any one of the four intervals with endpoints α, β where $a < \alpha < \beta \leq b$, then the Riemann integrals $\int_0^1 ct\, \mu\, (H;ct)dt$ and $\int_0^1 ct(w(\beta/ct) - w(\alpha/ct))dt$ exist. The integrals are equal whenever $\alpha, \beta \neq 0$. Moreover $\mu(H;ct)$ is a non-negative, measurable function of t .

Proof: As a function of t if $\beta \neq 0$, then $w(\beta/ct)$ is a step-function with discontinuities only at a sequence of points $\{t_n\}$, where $|t_n| > |t_{n+1}|$ and $\lim_{n\to\infty} t_n = 0$ (e.g., for $\beta > 0$, $w(\beta/ct)$ is left continuous and $t_n = \beta/cn$, where $n > 0$). Now although $w(\beta/ct)$ is unbounded in $(0,1]$, $ctw(\beta/ct)$ is bounded there, and is continuous whenever $w(\beta/ct)$ is, i.e., almost everywhere. It follows that $ctw(\beta/ct)$ is Riemann integrable. Similarly, the functions $ctw(\alpha/ct)$ and $ct\mu(H;ct)$ are Riemann integrable. Thus all the integrals in question exist. Their equality when $\alpha,\beta \neq 0$ follows from the fact that in this case $\mu(H;ct)$ and $\mu(H;ct)$ and $\mu((\alpha,\beta];ct) = w(\beta/ct) - w(\alpha/ct)$ are equal almost everywhere (for example, $\mu([\alpha,b];ct) - \mu((\alpha,\beta];ct) = \chi_I(\alpha/ct)$, etc.). The measurability of $\mu(H;ct)$ follows from that of $ct\,\mu(H;ct)$ and that of $1/ct$.

Lemma 6.3: If $a < \alpha < \beta \leq b$ then

$$\int_0^1 ct\,\mu\,((\alpha,\beta);ct)dt = \int_0^1 ct\,\mu\,((-\beta,\,-\alpha);ct)dt.$$

Proof: $\mu((\alpha,\beta);ct) = \mu((-\beta,\,-\alpha);ct)$.

Lemma 6.4: If H is an interval with endpoints α, β contained in $[a,b]$, then $\int_0^1 ct\,\mu\,(H;ct)dt < \beta - \alpha + c$.

Proof: Since $w(\beta/ct) \leq \beta/ct$ and $\alpha/ct - 1 < w(\alpha/ct)$, we have $\int_0^1 ct\,\mu\,(H;ct)dt \leq \int_0^1 ct\,\mu\,([\alpha,\beta];ct)dt = \int_0^1 ct[w(\beta/ct) - w(\alpha/ct) + \chi_I(\alpha/ct)]dt < \int_0^1 (\beta - \alpha + 2ct)dt = \beta - \alpha + c.$

Lemma 6.5. (van der Corput): If G is an open set contained in the interval $[a, b]$, then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < c < \delta$, then the Lebesgue integral $\int_0^1 ct\,\mu\,(G;ct)dt < (1 + \varepsilon)mG$.

Proof: It suffices to prove the lemma for $a \geq 0$. For if $b \leq 0$, the result follows from Lemma 6.3; and if $a < 0 < b$, we partition the interval $[a,b]$ into $[a,0) \cup [0,b]$ and use the additivity of the measures μ_c and m.

The set G, being open, may be represented in the form $G = \bigcup_{i=1}^{\infty} (a_i, b_i)$, where the (a_i, b_i), $i = 1, 2, \cdots$, are the component intervals of G. Now $\mu(G;ct) = \sum_{i=1}^{\infty} \mu((a_i, b_i);ct)$. by the countable additivity of μ. By Lemma 6.2 $\mu((a_i, b_i);ct)$ is a non-negative, measurable function of t for every i. Hence $\mu(G;ct)$ is a non-negative, measurable function of t; and since $\overline{G} \subset [a,b]$, $ct\,\mu(G;ct)$ is a bounded, non-negative, measurable function. Thus the Lebesgue integral

$$J(c) = \int_0^1 ct\mu\,(G;ct)dt$$

exists.

Now let $\varepsilon > 0$ be given, and set $\sigma = \frac{\varepsilon m G}{4}$, $P_1 = G \cap [0,\sigma]$ and

$$J_1(c) = \int_0^1 ct\mu(P_1;ct)dt.$$

Then by Lemma 6.4, we have

$$(1) \qquad J_1(c) \leq \int_0^1 ct\,\mu([0,\sigma];ct)dt < \sigma + c.$$

Next, since $\sum_{i=1}^{\infty} (b_i - a_i)$ converges, there is an $\alpha > 0$ such that the intervals of G of length less than α form a set of measure less than $\frac{1}{64b}\varepsilon^2 m^2(G)$. Let $(A_i, B_i) = (a_i, b_i) \cap (\sigma, b)$; Let P_2 (resp., P_3) be the union of all intervals (A_i, B_i) for which $B_i - A_i \geq \alpha$ (resp., $0 < B_i - A_i < \alpha$); and let

$$J_i(c) = \int_0^1 ct\mu\,(P_i;ct)dt, \qquad i = 2, 3.$$

Then

$$J(c) = J_1(c) + J_2(c) + J_3(c).$$

Furthermore, the number of intervals in P_2 is less than b/α and $m(P_3) < \frac{1}{64b}\varepsilon^2 m^2(G)$.

If (A_i, B_i) is an interval belonging to P_2, we have

$$\int_0^1 ct\mu\,((A_i,B_i);ct)dt < B_i - A_i + c.$$

Summing over the intervals in P_2 we obtain

$$(2)\quad J_2(c) = \int_0^1 ct\mu\,(\bigcup(A_i,B_i);ct)dt = \sum \int_0^1 ct\mu((A_i,B_i);ct)dt$$

$$< \sum (B_i-A_i) + \frac{b}{a}\,c \le mG + \frac{cb}{a}.$$

It remains to estimate $J_3(c)$ for c small. To this end let (A,B) be an interval belonging to P_3 and let $c < \frac{1}{4}\,\varepsilon\, mG$. Then, since $A \ge \frac{1}{4}\,\varepsilon mG$, we have $w(A/c) \ge 1$.

For $r > 0$, $w(r/tc) = k$ if and only if $\frac{r}{c(k+1)} < t \le \frac{r}{ck}$.
Thus $w(A/c) = N$ is equivalent to $\frac{A}{c(N+1)} < 1 \le \frac{A}{cN}$.

Consequently, upon setting $w(A/c) = N$, we obtain

$$(0,1) \le \bigcup_{k=N}^{\infty} \left[\frac{A}{c(k+1)}, \frac{A}{ck}\right] \subset \bigcup_{k=N}^{\infty} \left[\frac{B}{c(k+1)}, \frac{B}{ck}\right).$$

Next,

$$\int_0^1 ct\mu\,((A,B);ct)dt = \int_0^1 ct(w(B/ct) - w(A/ct))dt$$

$$\leq \sum_{k=N}^{\infty} \int_{\frac{A}{c(k+1)}}^{\frac{A}{ck}} ct(w(B/ct) - w(A/ct))dt$$

$$= \left(\sum_{k=N}^{\infty} \int_{\frac{A}{c(k+1)}}^{\frac{A}{ck}} ctw(B/ct)dt - \int_{\frac{A}{c(k+1)}}^{\frac{A}{ck}} ctw(A/ct)dt \right) = \sum_{k=N}^{\infty} \int_{\frac{A}{c(k+1)}}^{\frac{A}{ck}} ctw(B/ct)dt$$

$$- \sum_{k=N}^{\infty} \int_{\frac{A}{c(k+1)}}^{\frac{A}{ck}} ctw(A/ct)dt \leq \sum_{k=N}^{\infty} \int_{\frac{B}{c(k+1)}}^{\frac{B}{ck}} ctw(B/ct)dt - \sum_{k=N}^{\infty} \int_{\frac{A}{c(k+1)}}^{\frac{A}{ck}} ctw(A/ct)dt$$

$$= \sum_{k=N}^{\infty} \int_{\frac{B}{c(k+1)}}^{\frac{B}{ck}} ctk\ dt - \sum_{k=N}^{\infty} \int_{\frac{A}{c(k+1)}}^{\frac{A}{ck}} ctk\ dt = \frac{B^2 - A^2}{2c} \sum_{k=N}^{\infty} \left\{ \frac{2k+1}{k(k+1)^2} \right\}$$

$$< \frac{B^2 - A^2}{2c} \sum_{k=N}^{\infty} \frac{2k+2}{k(k+1)^2} = \frac{B^2 - A^2}{c} \sum_{k=N}^{\infty} \frac{1}{k(k+1)} = \frac{B^2 - A^2}{cN}.$$

Therefore

$$\int_0^1 ct\mu\ ((A,B);ct)dt \leq \frac{B^2 - A^2}{cN} = \frac{B^2 - A^2}{A} \cdot \frac{A}{c(N+1)} \cdot \frac{N+1}{N} < \frac{B^2 - A^2}{A}\ 2,$$

since $A/c(N + 1) < 1$ and $(N + 1)/N \leq 2$. But $(B^2 - A^2)/A = (B-A)(B+A)/A$ $\leq 2b(B-A)4/\varepsilon mG$ since $A \geq \varepsilon mG/4$ and $A < B \leq b$. Hence,

$$\int_0^1 ct\mu\ ((A,B);ct)dt < (B-A)16b/\varepsilon mG.$$

Now summing over the intervals in P_3 and applying Lebesgue's convergence theorem, we have

$$J_3(c) = \int_0^1 ct\,\mu(\bigcup(A_j,B_j);ct)dt$$

$$= \int_0^1 ct \sum (\ \mu((A_j,B_j);ct))dt = \sum \int_0^1 ct\,\mu((A_j,B_j);ct)dt$$

$$< \sum \frac{(B_j-A_j)16b}{\varepsilon mG} < \frac{16b}{\varepsilon mG}\,\frac{\varepsilon^2 m^2(G)}{64b} = \frac{1}{4}\varepsilon mG.$$

Combining (1), (2) and (3) yields

$$J(c) < (\frac{1}{4}\varepsilon mG + c) + (mG + cb/\alpha) + \frac{1}{4}\varepsilon mG < (1+\varepsilon)mG$$

for $c < \delta = \frac{1}{2}\varepsilon mG \min(\frac{1}{2}, \frac{\alpha}{(\alpha+b)})$.

<u>Lemma 6.6.</u> If G is an open set contained in $[a,b]$, then $\liminf_{c\to 0} c\,\mu(G;c) \leq mG$.

<u>Proof:</u> Suppose the contrary. Then there exist $\delta, \varepsilon > 0$ such that for $0 < c < \delta$, $c\,\mu(G;c) > (1+\varepsilon)mG$. Hence $\int_0^1 ct\,\mu(G;ct)dt > \int_0^1 (1+\varepsilon)mG\,dt = (1+\varepsilon)mG$. But this contradicts the preceding lemma.

Combining Lemma 6.1 and Lemma 6.6, we obtain

<u>Lemma 6.7.</u> If G is an open set contained in $[a,b]$, then $\liminf_{c\to 0} c\,\mu(G;c) = m(G)$.

<u>Theorem 6.1:</u> If f is a bounded function defined on the interval $[a,b]$ and if f is both uniformly and Lebesgue integrable, then $(U)\int_a^b f = (L)\int_a^b f$.

Proof: Let $M = \sup_x |f(x)|$. By Lusin's theorem, for every $\varepsilon>0$, there exists a function h which is continuous on (a,b) and an open subset G of $[a,b]$ such that $\{x|x \epsilon [a,b], f(x) \neq h(x)\} \subseteq G$, $m(G) < \varepsilon/4M$ and $\sup_x |h(x)| \leq M$. Let $s = f-h$. Then s is uniformly and Lebesgue integrable, $s(x) = 0$ on $[a,b]-G$ and $\sup_x |s(x)| \leq 2M$. Furthermore $\left|(L)\int_a^b s\right| < 2Mm(G) < \varepsilon/2$. Now let $A=G\cup\{a,b\}$ and let $r = 2M\chi_A$. Then we have

$$\left| c\underset{c}{\overset{b}{\sum}}{}_{a}\, s \right| \leq c\underset{c}{\overset{b}{\sum}}{}_{a}\, |s| \leq c\underset{c}{\overset{b}{\sum}}{}_{a}\, r = 2M\left(c\underset{c}{\overset{b}{\sum}}{}_{a}\, \chi_A\right).$$

Now,

$$\liminf_{c\to o} 2M\left(c\underset{c}{\overset{b}{\sum}}{}_{a}\, \chi_A\right) = 2M \liminf_{c\to o} c\underset{c}{\overset{b}{\sum}}{}_{a}\, \chi_G$$

$$= 2M \liminf_{c\to o} c\mu_c(G) = 2M\, m(G) < \varepsilon/2.$$

Hence $\left|(U)\int_a^b s\right| < \varepsilon/2$. Consequently for every $\varepsilon>0$ we have

$$\left|(U)\int_a^b f - (L)\int_a^b f\right| = \left|(U)\int_a^b f - (U)\int_a^b h + (L)\int_a^b h - (L)\int_a^b f\right|$$

$$\leq \left|(U)\int_a^b f - (U)\int_a^b h\right| + \left|(U)\int_a^b h - (L)\int_a^b f\right|.$$

But $(U)\int_a^b h = (L)\int_a^b h$, since h is continuous. Thus

$$\left|(U)\int_a^b f - (L)\int_a^b f\right| \leq \left|(U)\int_a^b f - (U)\int_a^b h\right| + \left|(L)\int_a^b h - (L)\int_a^b f\right|$$

$$= \left|(U)\int_a^b (f-h)\right| + \left|(L)\int_a^b (h-f)\right| = \left|(U)\int_a^b s\right| + \left|(L)\int_a^b (-s)\right| < \varepsilon.$$

Thus the theorem is proved and the uniform and Lebesgue integrals are compatible on the class of bounded functions.

7. ADDITIONAL RESULTS

Lemma 7.1: If G is an open subset of the interval $[a,b]$, then $\mu_c(G)$ is a measurable function of c and $\lim_{c\to 0} J(c) = \lim_{c\to 0} \int_0^1 ct\,\mu\,(G;ct)dt$ exists and is equal to $m(G)$.

Proof: The measurability of $\mu_c(G)$ follows from the argument used in Lemma 6.5. And from the conclusion of Lemma 6.5 we have that $\limsup_{c\to 0} J(c) \leq m(G)$.

From Lemma 6.1 it follows that for every $\varepsilon>0$, there exists a $\delta>0$ such that if $0 < c < \delta$ then $c\mu(G;c) > m(G) - \varepsilon$. Thus, for $0 < c < \delta$, $J(c) = \int_0^1 ct\mu(G;ct)dt > \int_0^1 (m(G)-\varepsilon)dt = m(G)-\varepsilon$. Hence, $\liminf_{c\to 0} J(c) \geq m(G)$ and the lemma is proved.

(Note: The inequality $\liminf_{c\to 0} J(c) > m(G)$ may also be obtained directly from Lemma 6.5 and a slightly modified form of Fatou's lemma. See Appendix III)

Lemma 7.2: If F is a closed subset of $[a,b]$, then $\mu_c(F)$ is a measurable function of c and $\lim_{c\to 0}\int_0^1 ct\mu(F;ct)dt$ exists and is equal to $m(F)$.

Proof: We have $F = [a,b] - G$, where G is an open subset of $[a,b]$. Hence $\mu_c(F) = \mu_c([a,b]) - \mu_c(G)$ is the difference of two measurable functions and thus measurable.

Next, using Lemma 6.2

$$\lim_{c\to 0}\int_0^1 ct\ \mu(F;ct)dt$$

$$=\lim_{c\to 0}\left[\int_0^1 ct\ \mu([a,b];ct)dt - \int_0^1 \mu(G;ct)dt\right]$$

$$=\lim_{c\to 0}\int_0^1 ct\ \mu((a,b);ct)dt - \lim_{c\to 0}\int_0^1 ct\ \mu(G;ct)dt$$

$$=(b-a) - m(G) = m(F).$$

Lemma 7.3: If H is a set of measure zero then $\mu_c(H)$ is a measurable function of c and $\mu_c(H) = 1$ or 0 for almost all c in $(0,1]$ depending on whether or not $0 \in H$.

Proof: Let $H_0 = H - \{0\}$; let $S = \{c \mid \mu_c(H_0) > 0,\ 0<c\}$ and for all integers n, let $V_n = \{c \mid nc \in H_0,\ 0 < c\}$. Clearly, $\bigcup_{n=-\infty}^{\infty} V_n = S$. Furthermore $V_0 = \phi$, and for $n \neq 0$ we have $V_n \subseteq (1/n)H_0 = \{x/n \mid x \in H_0\}$; hence, $m^*(V_n) \leq |1/n| m^*(H_0) = 0$. Therefore, $m(S) = 0$ and $\mu_c(H_0)$ is measurable since $\mu_c(H_0) = 0$ almost everywhere. If $0 \in H$, then since $\mu_c(\{0\}) = 1$ we have $\mu_c(H) = 1$ almost everywhere, whence $\mu_c(H)$ is measurable.

Theorem 7.1: If A is a measurable subset of $[a,b]$, then $\mu_c(A)$ is a measurable function of c and $\lim_{c\to 0}\int_0^1 ct\,\mu(A;ct)dt$ exists and equals $m(A)$.

Proof: First let $K = \bigcup_{i=1}^{\infty} F_i$ where the F_i are closed sets; i.e., K is an F_σ set. Letting $T_n = \bigcup_{j=1}^{n} F_j$, we have the partition $K = T_1 \cup \bigcup_{n=1}^{\infty} (T_{n+1}-T_n)$. Since for each n T_n is closed $\mu_c(T_{n+1}-T_n) = \mu_c(T_{n+1}) - \mu_c(T_n)$ is measurable as is $\mu_c(K)$. Since every measurable set is the union of an F_σ set and a set of measure zero, it follows that μ_c is measurable for all measurable subsets A of $[a,b]$.

Next, for any $\varepsilon > 0$ if F and G are, respectively, closed and open subsets of $[a,b]$ such that $F \subseteq A \subseteq G$, and $m(F) > m(A) - \varepsilon$, $m(G) < m(A) + \varepsilon$, then

$$m(F) \leq \liminf_{c\to o}\int_o^1 ct\,\mu\,(A;ct)dt \leq \limsup_{c\to o}\int_o^1 ct\mu(A;ct)dt \leq m(G).$$

It follows that the limit in question exists and equals $m(A)$.

Using Lemma 6.7 we can prove the following

<u>Theorem 7.2:</u> Let $G = \bigcup_{i=1}^{\infty} (a_i,b_i)$ be an open set with component intervals (a_i,b_i). Then

$$(1)\quad \liminf_{c\to o} c\sum_{i=1}^{\infty} [w(b_i/c)-w(a_i/c)] = m(G) \quad \text{and}$$

$$(2)\quad \liminf_{c\to o} c\sum_{i=1}^{\infty} [p(a_i/c)-p(b_i/c)] = 0,$$

where $p(x) = x-w(x)$ is the fractional part of x.

<u>Proof:</u> Let $G* = \bigcup_{i=1}^{\infty}(a_i,b_i]$. Then

$$\mu_c(G*) = \sum_{i=1}^{\infty}[w(b_i/c)-w(a_i/c)].$$

Now $m(G) = \liminf_{c\to o} c\,\mu_c\,(G) \leq \liminf_{c\to o} c\,\mu_c\,(G*)$.

But because of Lemma 6.2, the conclusion of Lemma 6.5 is valid for G* as well as for G. Therefore $m(G) \geq \liminf_{c\to o} c\mu_c(G*)$. This proves (1).

Taking the limit inferior as $c \to o$ in both members of the equation

$$c\sum_{i=1}^{\infty}[w(b_i/c)-w(a_i/c)] = \sum_{i=1}^{\infty}(b_i-a_i) + c\sum_{i=1}^{\infty}[p(a_i/c)-p(b_i/c)]$$

we obtain

$$m(G) = m(G) + \liminf_{c\to o} c\sum_{i=1}^{\infty}[p(a_i/c)-p(b_i/c)]$$

and hence (2).

8. THE COMPATIBILITY PROBLEM FOR UNBOUNDED FUNCTIONS

We now turn our attention to the compatibility problem for finite-valued unbounded functions.* Whereas, in section 6 we approximated bounded measurable functions by continuous functions and applied Luzin's theorem, here we approximate the unbounded measurable functions by semi-continuous functions and, in effect, use the Vitali-Caratheodory theorem [8, p.75]. For this, the following generalization of van der Corput's Lemma (6.5) is required.

<u>Lemma 8.1:</u> Let $\{G_n\}$ be a sequence of open subsets of $[a,b]$ such that $\sum_{n=1}^{\infty} m(G_n)$ converges and having the additional property that if $O \varepsilon [a,b]$, then there exists an open interval V containing O such that all but a finite number of sets $V \cap G_n$ are empty. Then for every sufficiently small $\varepsilon > 0$, there is a $\delta > 0$ such that $0 < c < \delta$ implies $J(c) = \sum_{n=1}^{\infty} \int_0^1 ct\, \mu(G_n; ct)dt$ $< (1+\varepsilon) \sum_{n=1}^{\infty} m(G_n)$.

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*We will also use the symbol $(U)\underline{\int}_a^b f$ to denote the lower uniform integral of the extended real-valued function f, where, as before, $(U)\underline{\int}_a^b f$ is defined to be $\liminf_{c \to 0} c \sum_c^b{}_a f$, whenever this limit is finite.

Proof: As before, we may reduce the proof of the lemma to the case, $a \geq 0$.

Let $S = \sum_{n=1}^{\infty} m(G_n)$; let $V = (\beta, \gamma)$; let K be the largest integer such that $V \cap G_n$ is non-empty; let $0 < \varepsilon < 4\gamma K/s$ and let $\sigma = S\varepsilon/4K$. Hence $\sigma < \gamma$ so that $[0,\sigma] \subset V$ and $[0,\sigma] \cap G_n$ is empty whenever $n > K$.

Furthermore,

$$(1) \quad J_1(c) = \sum_{n=1}^{\infty} \int_0^1 ct\, \mu([0,\sigma] \cap G_n; ct)\,dt = \sum_{n=1}^{K} \int_0^1 ct\, \mu([0,\sigma] \cap G_n; ct)\,dt$$

$$< K(\sigma + c) = \tfrac{1}{4}\varepsilon S + Kc.$$

Now let the double sequence of component intervals of the open sets $\{G_n\}$ be ordered in a simple sequence $\{(a_i, b_i)\}$. Then there is an $\alpha > 0$ such that the terms of $\{(a_i, b_i)\}$ disjoint from $[0,\sigma]$ and of length less than α form a set P_3 of measure less than $\varepsilon^2 S^2/64b$. Furthermore, the number of intervals in the remaining set P_2 does not exceed S/α. Therefore, summing over the intervals (A_i, B_i) of P_2 we have

$$(2) \quad J_2(c) = \sum \int_0^1 ct\, \mu((A_i, B_i); ct)\,dt < \sum (B_i - A_i + c) < S + cS/\alpha.$$

In considering the intervals (A_j, B_j) of P_3 with the exception of the replacement of $m(G)$ by S, we determine, as previously, that $\int_0^1 ct\, \mu((A_j, B_j); ct)\,dt \leq 16b(B_j - A_j)/S$ if $c < S\varepsilon/4$.

Therefore, if $c < S\varepsilon/4$, then

$$(3) \quad J_3(c) = \sum \int_0^1 ct\, \mu((A_j, B_j); ct)\,dt \leq \sum \frac{(B_j - A_j)16b}{S\varepsilon} < \frac{16b}{S\varepsilon} \cdot \frac{\varepsilon^2 S^2}{64b} = \tfrac{1}{4} S\varepsilon.$$

Combining equations (1), (2) and (3) we have $J(c) < (\frac{1}{4}S\varepsilon + Kc) + (S + cS/\alpha) + \frac{1}{4}\varepsilon S < (1+\varepsilon)S$ if $0 < c < \delta$ where $\delta = \frac{1}{2}\varepsilon S \min(\frac{1}{2}, \alpha/(\alpha K + S))$.

<u>Lemma 8.2:</u> If $\{G_n\}$ is a sequence of open subsets of $[a,b]$ satisfying the hypothesis of Lemma 8.1, then

$$\liminf_{c \to 0} \sum_{n=1}^{\infty} c\,\mu(G_n;c) = \sum_{n=1}^{\infty} m(G_n).$$

<u>Proof:</u> The corollary to Lemma 6.1 implies $\liminf_{c\to 0} \sum_{n=1}^{\infty} c\mu(G_n;c) \geq \sum_{n=1}^{\infty} m(G_n) = S$. If the strict inequality holds, then there exist $\delta, \varepsilon > 0$ such that if $0 < c < \delta$, then $c\sum_{n=1}^{\infty}\mu(G_n;c) > (1+\varepsilon)S$. Using Beppo Levi's theorem we have

$$\sum_{n=1}^{\infty}\int_0^1 ct\,\mu(G_n;ct)dt = \int_0^1 \sum_{n=1}^{\infty} ct\,\mu(G_n;ct)dt > \int_0^1 (1+\varepsilon)Sdt$$

$= (1+\varepsilon)S$, contradicting the previous lemma.

<u>Lemma 8.3:</u> If h is a non-negative Lebesgue integrable function on $[a,b]$ (i.e., $(L)\int_a^b h$ exists and is finite) such that the oscillation** of h at 0 is finite whenever $0 \varepsilon [a,b]$, then $(U)\underline{\int_a^b} h \leq (L)\int_a^b h$.

<u>Proof:</u> For all positive integers n and p define $E_n^p = \{x \mid x \varepsilon [a,b]$ and $\frac{n-1}{2^p} \leq h(x) < \frac{n}{2^p}\}$ and $r_p = 2^{-p}\sum_{n=1}^{\infty} n\chi_{E_n^p}$.

**The oscillation of h at 0 is $\lim_{\delta\to 0^+}\sup\{h(x) \mid x\varepsilon R(\delta)\} - \lim_{\delta\to 0^+}\inf\{h(x) \mid x\varepsilon R(\delta)\}$ where $R(\delta) = [a,b]\cap(-\delta,\delta)$.

Then for every $\varepsilon > 0$ there is a sufficiently large p such that

$$(L)\int_a^b h + \tfrac{1}{2}\varepsilon > (L)\int_a^b r_p = 2^{-p} \sum_{n=1}^{\infty} nm\,(E_n^p).$$

If $0 \varepsilon [a,b]$, then there exists an interval V containing 0 and a positive integer K such that $n > K$ implies $V \cap E_n^p = \phi$. For each n, let G_n^p be an open subset of $[a,b]$ such that $E_n^p \subseteq G_n^p$ and $m(G_n^p - E_n^p) < \varepsilon/n2^{n-p+1}$. Furthermore, by taking V to be a closed interval, we can select the G_n^p in such a manner that if $E_n^p \cap V = \phi$, then $G_n^p \cap V = \phi$ as well.

Thus,

$$2^{-p} \sum_{n=1}^{\infty} nm\,(G_n^p) < 2^{-p} \sum_{n=1}^{\infty} n(m(E_n^p) + \varepsilon/n2^{n-p+1}) = (L)\int_a^b r_p + \tfrac{1}{2}\varepsilon.$$

It follows that

$(L)\int_a^b h + \varepsilon > 2^{-p} \sum_{n=1}^{\infty} nm\,(G_n^p)$, whence the series $\sum_{n=1}^{\infty} nm\,(G_n^p)$ converges.

From the sequence $\{G_n^p\}$ we form the sequence $\{H_n^p\} = \{G_1^p, G_2^p, G_2^p, \dots, G_n^p, \dots, G_n^p, \dots\}$ in which each G_n^p occurs n times. Consequently $\sum_{n=1}^{\infty} \chi_{E_n^p} = \sum_{n=1}^{\infty} n\chi_{G_n^p}$. Furthermore, the sequence $\{H_n^p\}$ satisfies the hypotheses of Lemma 10.2, i.e., each H_n^p is an open subset of $[a,b]$, $\sum_{n=1}^{\infty} m(H_n^p)$ converges and $(H_n^p \cap V) = \phi$ for n sufficiently large.

Thus $$(U)\underline{\int_a^b} \sum_{n=1}^{\infty} n\chi_{G_n^p} = (U)\underline{\int_a^b} \sum_{n=1}^{\infty} \chi_{H_n^p}$$

$$= \liminf_{c\to 0} c \sum_{n=1}^{\infty} \mu(H_n^p;c) = \sum_{n=1}^{\infty} m(H_n^p) = \sum_{n=1}^{\infty} nm\,(G_n^p).$$

Hence,

$$(U)\underline{\int_a^b} h \leq (U)\underline{\int_a^b} 2^{-p} \sum_{n=1}^{\infty} n \chi_{G_n^p} = 2^{-p} \sum_{n=1}^{\infty} nm\,(G_n^p) < (L)\int_a^b h + \varepsilon.$$

Therefore

$$(U)\underline{\int_a^b} h \leq (L)\int_a^b h.$$

Corollary 8.1: If g is a function which is bounded below and Lebesgue integrable on $[a,b]$ and such that the oscillation at 0 is finite whenever $0 \in [a,b]$, then $(U)\underline{\int_a^b} g \leq (L)\int_a^b g$.

Proof: Let M be a lower bound for g. Then $g-M$ satisfies the hypotheses of the Lemma 8.3. Hence,

$$(U)\underline{\int_a^b}(g-M) \leq (L)\int_a^b (g-M) = (L)\int_a^b g - M(b-a) \quad \text{and}$$

$$(U)\underline{\int_a^b}(g-M) = (U)\underline{\int_a^b} g - (U)\underline{\int_a^b} M = (U)\underline{\int_a^b} g - M(b-a).$$

Thus $(U)\underline{\int_a^b} g \leq (L)\int_a^b g$.

Corollary 8.2: If g is bounded above and Lebesgue integrable on $[a,b]$ and if the oscillation of g at 0 is finite whenever $0 \in [a,b]$; then $(U)\int_a^{\overline{b}} g \geq (L)\int_a^b g$.

Proof: Since $-g$ satisfies the hypotheses of the lemma, we have $(U)\underline{\int_a^b}(-g) \leq (L)\int_a^b (-g)$. Thus $(L)\int_a^b g \leq -(U)\underline{\int_a^b}(-g)$ $= (U)\int_a^{\overline{b}} g$.

Corollary 8.3: If f is bounded and Lebesgue integrable $[a,b]$, then $(U)\underline{\int_a^b} f \leq (L)\int_a^b f \leq (U)\int_a^{\overline{b}} f$.

Theorem 8.1: If f is both Lebesgue and uniformly integrable on $[a,b]$ and, furthermore if the oscillation of f at 0 is finite whenever $0 \varepsilon [a,b]$, then $(U)\int_a^b f = (L)\int_a^b f$.

Proof: For each positive integer n, let

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \geq -n \\ -n & \text{if } f(x) < -n. \end{cases}$$

Each f_n satisfies the hypotheses of the previous lemma and since $f \leq f_n$ we have

$$(U)\int_a^b f \leq (U)\underline{\int}_a^b f_n \leq (L)\int_a^b f_n.$$

Since $f_n \leq |f|$ and $\lim_{n\to\infty} f_n = f$ we have $\lim_{n\to\infty} (L)\int_a^b f_n = (L)\int_a^b f$.

Hence, $(U)\int_a^b f \leq (L)\int_a^b f$.

Moreover, $-f$ satisfies the hypotheses of the theorem if f does, implying

$$(U)\int_a^b (-f) \leq (L)\int_a^b (-f) \quad \text{and} \quad (U)\int_a^b f \geq (L)\int_a^b f,$$

and the theorem follows.

As our final result, we shall prove a theorem analagous to Corollary 8.3, but with the roles of the uniform and Lebesgue integrals interchanged.

Lemma 8.4: If f is a finite, lower (upper) semi-continuous and Lebesgue integrable function on $[a,b]$ such that the oscillation of f at 0 is finite whenever $0 \varepsilon [a,b]$, then

$$(U)\underline{\int}_a^b f = (L)\int_a^b f \quad \left((U)\overline{\int}_a^b f = (L)\int_a^b f\right).$$

Proof: Suppose f is lower semi-continuous and finite; then f is bounded below. Hence, by Lemma 8.3, we have

$$(U)\int_a^b f \leq (L)\int_a^b f.$$

Furthermore, f is the limit of a non-decreasing sequence of continuous functions $\{f_n\}$. Therefore, for all n, $f_n \leq f$ and $(L)\int_a^b f_n = (U)\int_a^b f_n \leq (U)\underline{\int}_a^b f$.

Since $\lim_{n\to\infty} (L)\int_a^b f_n = (L)\int_a^b f$, we conclude $(L)\int_a^b f \leq (U)\underline{\int}_a^b f$ and the inequality follows.

The other part is proved similarly.

Remark: If we assume instead of Lebesgue integrability that $(L)\int_a^b f = +\infty$, then by Levi's theorem $\lim_{n\to\infty}(L)\int_a^b f_n = +\infty$ so $(U)\underline{\int}_a^b f$ cannot be finite either.

Corollary 8.4: If f is bounded and lower semi-continuous on $[a,b]$, then $(L)\int_a^b f = (R)\underline{\int}_a^b f = (U)\underline{\int}_a^b f$. Similarly, if f is bounded and upper semi-continuous, then $(L)\int_a^b f = (R)\overline{\int}_a^b f = (U)\overline{\int}_a^b f$.

Proof: If f is lower semi-continuous, the inequality $(L)\int_a^b f \leq (R)\underline{\int}_a^b f$ is proved as in lemma 4. (Also see [2,p.206]). The reverse inequality is well known.

Definition: The upper and lower Lebesgue integrals in the sense of Daniell [4,p.169] are given by:

$$(L)\overline{\int_a^b} f = \inf\left\{(L)\int_a^b u \mid u \in T\right\} \text{ where}$$

$T = \{u \mid u \geq f$ and u is an extended real valued function which is lower semi-continuous and bounded below$\}$; similarly,

$$(L)\underline{\int_a^b} f = \sup\left\{(L)\int_a^b u \mid u \in P\right\} \quad \text{where}$$

$P = \{u \mid u \leq f$ and u is an extended real valued function which is upper semi-continuous and bounded above$\}$.

<u>Theorem 8.2:</u> If f is bounded and uniformly integrable on $[a,b]$, then $(L)\underline{\int_a^b} f \leq (U)\int_a^b f \leq (L)\overline{\int_a^b} f$.

<u>Proof:</u> Let $S = \{u \mid u \geq f$, u is lower semi-continuous and bounded$\}$. We shall show $\inf\left\{(L)\int_a^b u \mid u \in T\right\}$ $= \inf\left\{(L)\int_a^b u \mid u \in S\right\}$. If the equation did not hold since $S \subseteq T$, there would exist a $t_o \in T$ such that for all $s \in S$ $(L)\int_a^b t_o < (L)\int_a^b s$. However, if f is bounded by M, then letting $s_o = \min(t_o, M)$ we have $s_o \in S$ and $(L)\int_a^b s_o < (L)\int_a^b t_o$, a contradiction.

By Corollary 8.4 $u \in S$ implies $(U)\underline{\int_a^b} u = (L)\int_a^b u$. Thus $(L)\overline{\int_a^b} f = \inf\left\{(U)\underline{\int_a^b} u \mid u \in S\right\} \geq (U)\int_a^b f$.

The other inequality follows similarly.

This concludes the section on the compatibility problem for unbounded functions. So far the problem has not been resolved completely.

9. THE N-DIMENSIONAL UNIFORM INTEGRAL

In considering a generalization of the uniform integral for n-place functions, we can use either a grid of n-dimensional squares or n-dimensional rectangles. It is more natural and useful to take the latter course.

Let $[a,b] = \mathop{X}_{i=1}^{n} [a_i,b_i]$ be the Cartesian product of the n intervals $[a_i,b_i]$. For all $i = 1,2,\dots,n$ let $0 < c_i$ and $c = (c_1,c_2,\dots,c_n)$; let $|c| = \max c_i$ and $\|c\| = \prod_{i=1}^{n} c_i$. If f is a real finite-valued n-place function defined on $[a,b]$ and $N_i = w(b_i/c_i) - w(a_i/c_i)$ for $i = 1,2,\dots,n$, denote

$$\sum_{k_n=1}^{N_n} \cdots \sum_{k_2=1}^{N_2} \sum_{k_1=1}^{N_1} f(c_1[w(a_i/c_i)+k_1],\dots,c_n[w(a_i/c_i)+k_n]) \text{ by}$$

${\sum_c}_a^b f$. The uniform integral of f on $[a,b]$, denoted by $(U_n)\int_a^b f$ is given by the limit, $\lim_{|c|\to 0} \|c\| {\sum_c}_a^b f$, provided the latter exists.

If the n-place function f is Riemann (R_n) integrable on $[a,b]$, then an argument similar to the one given in section 4, shows that, but for a certain term of the order $o(1)$, $c {\sum_c}_a^b f$ is a Riemann sum of norm $|c|$ and hence converges to the Riemann integral of f over $[a,b]$ as $|c| \to 0$. Thus the n-dimensional uniform integral is an extension of the n-dimensional Riemann integral.

By extending the arguments of chapter 6, we shall also show that for bounded functions the n-dimensional uniform (U_n) integral is compatible with the n-dimensional (L_n) integral.

Let $H \subseteq [a,b]$; let $0 < c_i$, $i = 1,\ldots,n$ and let $\mu(H;c)$ or $\mu_c(H)$ be the number of elements of the set $\{(c_1k_1,\ldots,c_nk_n) \mid k_i \epsilon I$ and $(c_1k_1,\ldots,c_nk_n) \epsilon H\}$. Again μ_c is an integral-valued measure on the class of all subsets of a given bounded set and $\mu_c(H;c) = \sum_c{}_a^b \chi_H$ for all $H \subseteq (a,b]$.

Next we extend the general compatibility theorem of section 6 by first generalizing the lemmas of that section.

Let the open subset G of $[a,b]$ be expressed as a countable partition $\bigcup_{j=1}^{\infty} \left\{ \mathop{X}_{i=1}^{n} (\alpha_i^j, \beta_i^j] \right\}$, [5, p.126]. Then

$$\mu_c(G;c) = \sum_{j=1}^{\infty} \prod_{i=1}^{n} \left[w(\beta_i^j / c_i) - w(\alpha_i^j / c_i) \right] .$$

<u>Lemma 9.1:</u> If G is an open set contained in $[a,b]$, then $\liminf_{|c| \to 0} \mu_c(G) \geq m_n(G)$, where $m_n(G)$ is the n-dimensional Lebesgue measure of G.

The proof is similar to that of Lemma 6.1.

<u>Lemma 9.2:</u> If $\mathop{X}_{i=1}^{n} (\alpha_i,\beta_i) \subseteq H \subseteq \mathop{X}_{i=1}^{n} [\alpha_i,\beta_i]$ and $\alpha_i,\beta_i \neq 0$ for all i, then the integrals $(R_n)\int_0^1 \|ct\|\, \mu(H;ct)dt$ and $\prod_{i=1}^{n} (R_1)\int_0^1 c_i u[w(\beta_i/c_i u) - w(\alpha_i/c_i u)]du$ exist and are equal.

(Here $ct = (c_1t_1,\ldots,c_nt_n)$ and $\mathop{X}\limits_{i=1}^{n} (0,1]$ is the interval of integration for the n-dimensional integral.)

Again the crux of the argument is that the integrands for any two H's are bounded, continuous almost everywhere and equal almost everywhere. In particular if $H = \mathop{X}\limits_{i=1}^{n} (\alpha_i \beta_i]$, then

$$(R_n)\int_0^1 \|ct\| \; \mu(H;ct)dt = (R_n)\int_0^1 \prod_{i=1}^{n} \left\{c_1 t_1 [w(\beta_i/c_i t_1) - w(\alpha_i/c_i t_i)]\right\} dt_i$$

$$= \prod_{i=1}^{n} (R_1)\int_0^1 c_i u[w(\beta_i/c_i u) - w(\alpha_i/c_i u)]du.$$

__Lemma 9.3:__ If $(\alpha,\beta) \subseteq [a,b]$, then if H is obtained by reflecting (α,β) about one or several of the coordinate axes, it follows that $\int_0^1 \|ct\| \; \mu((\alpha,\beta);ct)dt = \int_0^1 \|ct\| \; \mu(H;ct)dt$.

__Proof:__ The integrands are identical.

__Lemma 9.4:__ If $\mathop{X}\limits_{i=1}^{n} (\alpha_i,\beta_i) \subseteq H \subseteq \mathop{X}\limits_{i=1}^{n} [\alpha_i,\beta_i]$, then

$$\int_0^1 \|ct\| \; \mu(H;ct)dt < \prod_{i=1}^{n} (\beta_i - \alpha_i + c_i).$$

This result is immediate and implies

__Lemma 9.4':__ If $\mathop{X}\limits_{i=1}^{n} (\alpha_i \beta_i) \subseteq H \subseteq \mathop{X}\limits_{i=1}^{n} [\alpha_i,\beta_i]$, then $\int_0^1 \|ct\| \; \mu(H;ct)dt < m_n(H) + (2^n-1)|c|M$, where $M = \max((\beta_i-\alpha_i)^{n-1},1)$.

__Proof:__ The first term of the product $\prod_{i=1}^{n} ([\beta_i-\alpha_i]+c_i)$ is $m_n(H)$. The remaining 2^n-1 terms are all products consisting of 0 to n-1 $[\beta_i-\alpha_i]$'s and n to 1 c_j's, and are each bounded by $|c|M$.

Lemma 9.5: If G is an n-dimensional open subset of the interval $[a,b]$, then for every $\varepsilon > 0$, there is a $\delta > 0$ such that if $0 < |c| < \delta$, then the n-dimensional Lebesgue integral

$$J(c) = \int_0^1 \|ct\| \quad \mu(G; ct)dt < (1+\varepsilon)m_n(G).$$

Proof: As in Lemma 6.5, we can reduce the proof to the case where $a_i \geq 0$ for all i. For if $b_i \leq 0$ for some i, we apply Lemma 9.3; and if for some i $a_i < 0 < b_i$, then we partition $[a,b]$ into intervals that intersect the coordinate axes at most on the boundary of these intervals, apply Lemma 9.3 and use the additivity of μ_c and m_n.

The Lebesgue integral $J(c)$ exists and again we shall estimate it by expressing it as a sum of three integrals.

Let $\varepsilon > 0$ be given; let $\sigma = \varepsilon m_n(G)/4ns^{n-1}$; $s = \max b_i$, $M = \max(s^{n-1}, 1)$, $S_k = \left(\mathop{X}_{i=1}^{k-1} [a_i,b_i] x [0,\sigma] x \mathop{X}_{i=k+1}^{n} [a_i,b_i]\right)$ for $k = 1,\ldots,n$, $P_1 = \bigcup_{k=1}^{n} S_k$ and $J_1(c) = \int_0^1 \|ct\| \quad \mu(P_1; ct)dt.$

By Lemma 9.4' we have

$$(1) \quad J_1(c) \leq \sum_{k=1}^{n} \int_0^1 \|ct\| \quad \mu(S_k; ct)dt < n[\sigma s^{n-1} + (2^n-1)|c|M]$$

$$= \varepsilon m_n(G)/4 + n(2^n-1)|c|M.$$

Next there is an $\alpha > 0$ such that the intervals of G of measure less than α form a set of measure less than $\varepsilon m_n(G)\sigma^n/4(4s)^n$. Let $(A_j,B_j) = \mathop{X}_{i=1}^{n} (\alpha_i^j, \beta_i^j] \cap \mathop{X}_{i=1}^{n} (\sigma, b_i]$; let P_2 (resp., P_3) be the union of all intervals (A_j,B_j) for which $m_n(A_j,B_j) \geq \alpha$ (resp., $0 < m_n(A_j,B_j) < \alpha$) and let $J_i(c) = \int_0^1 \|ct\| \ \mu(P_i;\ ct)dt$ for $i = 2,3$. The number of intervals in P_2 is less than s^n/α and $M_n(P_3) < \varepsilon m_n(G)\sigma^n/4(4s)^n$.

Hence,

$$J_2(c) < m_n(G) + |c|\, s^n/\alpha. \tag{2}$$

If (A,B) is any component interval of P_3 and $w(A_i/c_i) = N_i$, if $|c| < \sigma$ we have $\int_0^1 \|ct\| \ \mu((A,B);ct)dt$

$$= \prod_{i=1}^{n} \sum_{k=N_i}^{\infty} \int_{A_i/c_i(k+1)}^{A_i/c_ik} c_i u[w(B_i/c_iu) - w(A_i/c_iu)]du$$

$$\leq \prod_{i=1}^{n} \frac{(B_i^2 - A_i^2)}{c_iN_i} \leq \prod_{i=1}^{n} \left\{\frac{2s(B_i - A_i)}{c_iN_i}\right\}$$

$$\leq (2s)^n \prod_{i=1}^{n} \frac{(B_i - A_i)}{c_iN_i} \leq (2s)^n \prod_{i=1}^{n} [2(B_i - A_i)/A_i]$$

$$\leq m_n(A,B)\ (4s)^n/\sigma^n.$$

Hence,

$$J_3(c) < m_n(P_3)(4s)^n/\sigma^n < \tfrac{1}{4}\varepsilon\, m_n(G). \tag{3}$$

Combining equations (1), (2) and (3) we have

$$J(c) < (\tfrac{1}{4}\varepsilon\, m_n(G) + n(2^n - 1)|c|M) + (m_n(G) + |c|\, s^n/\alpha) + \tfrac{1}{4}\varepsilon\, m_n(G)$$

$< (1+\varepsilon)m_n(G)$ for all $|c| < \delta$, where

$$\delta = \frac{\varepsilon m_n(G)}{2} \min\ (1/2ns^{n-1},\ \alpha/(\alpha n(2^n - 1)M + s^n)).$$

Lastly, arguing as in Section 6 we have

Lemma 9.6: If G is an n-dimensional open subset of $[a,b]$, then $\liminf_{c\to o} \int_o^1 \|ct\| \, \mu(G;ct)dt = m_n(G)$.

Theorem 9.1: If f is a bounded n-place function that is both Lebesgue and uniformly integrable on $[a,b]$, then

$$(U_n)\int_a^b f = (L_n)\int_a^b f.$$

Proof: The proof of Theorem 6.1 applies with only minor notational changes.

10. APPENDIX I

Theorem 10.1: For any real number a, $\lim_{c \to 0} w(a/c) = a$.

Proof: For $c \neq 0$, $w(a/c) = a/c - p(a/c)$, where $0 \leq p(a/c) < 1$. Hence

$$cw(a/c) = a - cp(a/c)$$

from which the result follows.

For $c \neq 0$ define

$$\underset{c}{\sum\nolimits_{a}^{b}} f = \begin{cases} \displaystyle\sum_{n=1}^{w(b/c)-w(a/c)} f(c[w(a/c)+n]) & w(b/c) > w(a/c) \\ 0 & w(b/c) = w(a/c) \\ -\underset{c}{\sum\nolimits_{b}^{a}} & w(b/c) < w(a/c) \end{cases}$$

Theorem 10.2: For any numbers a, b, c any positive d, and any function f defined on the appropriate intervals,

$$\underset{d}{\sum\nolimits_{a}^{b}} f + \underset{d}{\sum\nolimits_{b}^{c}} f = \underset{d}{\sum\nolimits_{a}^{c}} f.$$

<u>Proof</u>: If $w(a) < w(b) < w(c)$, then

$$\underset{d}{\sum}{}_{a}^{c} f = f(d[w(a/d) + 1]) + f(d[w(a/d)+2]) + \cdots + f(dw(c/d))$$

$$= [f(d[w(a/d) + 1]) + \cdots + f(dw(b/d))] + [f(d[w(b/a) + 1]) + \cdots$$

$$+ f(dw(c/d))] = \underset{d}{\sum}{}_{a}^{b} f + \underset{d}{\sum}{}_{b}^{c} f.$$

If $w(b) < w(a) < w(c)$

$$\underset{d}{\sum}{}_{a}^{b} f + \underset{d}{\sum}{}_{b}^{c} f = - \underset{d}{\sum}{}_{b}^{a} f + \underset{d}{\sum}{}_{b}^{c} f = - f(d[w(b/d) + 1]) - \cdots$$

$$- f(dw(a/d)) + f(d[w(b/d) + 1]) + \cdots + f(dw(c/d)) = f(dw(a/d)+1) + \cdots$$

$$+ f(dw(c/d)) = \underset{d}{\sum}{}_{a}^{c} f.$$

The other cases are proved in a similar manner.

The uniform integral, as we have defined it, is the right-hand limit, $\lim_{c \to 0^+} c \underset{c}{\sum}{}_{a}^{b} f$. Correspondingly, one can define a uniform integral by means of a left-hand limit or an ordinary two-sided limit. However, these three possible definitions all yield the same integral. This is shown by the following argument.

<u>Lemma 10.1</u>: If $c > 0$ and $a < b$, then ${\sum_c}_a^b f = \sum_{a<nc\leq b} f(nc)$, where the summation on the right is over all the integral multiples of c in $(a,b]$.

<u>Proof</u>: ${\sum_c}_a^b f = \sum_n f(c[w(a/c) + n])$ for $n=1, \cdots, w(b/c)-w(a/c)$. Now $w(a/c)\leq a/c<w(a/c) + 1$ and $c > 0$. Thus $cw(a/c)\leq a<c[w(a/c)+1]$. Similarly, $cw(b/c) \leq b < c[w(b/c) + 1]$.

<u>Lemma 10.2</u>: If $c < 0$ and $a < b$, then ${\sum_c}_a^b f = -\sum_{a\leq nc<b} f(nc)$.

<u>Proof</u>: ${\sum_c}_a^b f = -{\sum_c}_b^a f = -\sum_{n=1}^{k} f(c[w(b/c) + n])$, where $k = w(a/c) - w(b/c)$. Since $c < 0$ we have $cw(a/c) \geq a > c[w(a/c) + 1]$ and $cw(b/c) \geq b > c[w(b/c) + 1]$, from which the result follows as before.

<u>Lemma 10.3</u>: If $c > 0$, $a < b$, $S(c) = c{\sum_c}_a^b f$ and $s(-c) = -c{\sum_{-c}}_a^b f$, then $S(c) - S(-c) = c\{\chi_I(b/c)f(b) - \chi_I(a/c)f(a)\}$.

<u>Proof</u>: $S(c) - S(-c) = c \sum_{a<nc\leq b} f(nc) - c \sum_{a\leq nc<b} f(nc)$

$= c(\sum_{a<nc<b} f(nc) + \chi_I(b/c)f(b)) - c(\sum_{a<nc<b} f(nc) + \chi_I(a/c)f(a))$

$= c\{\chi_I(b/c)f(b) - \chi_I(a/c)f(a)\}$.

Theorem 10.3: The existence of any one of the three limits $\lim_{c\to 0} c\sum_{a}^{b} f$, $\lim_{c\to 0^+} c\sum_{a}^{b} f$ and $\lim_{c\to 0^-} c\sum_{a}^{b} f$ implies the existence of the other two and the equality of all three.

Proof: From the previous lemma, for $c > 0$ $|S(c) - S(-c)| \leq c(|f(b)| + |f(a)|)$. Thus $\lim_{c\to 0^+} S(c)$ exists if and only if $\lim_{c\to 0^+} S(-c) = \lim_{c\to 0^-} S(c)$ exists; and when they exist, then $\lim_{c\to 0} S(c)$ exists and all these limits are equal.

Theorem 10.4: The integral $(U)\int_a^b f$ exists if and only if $(U)\int_b^a f$ exists, and when they exist $(U)\int_a^b f = -(U)\int_b^a f$.

Proof: $(U)\int_a^b f = \lim_{c\to 0} c\sum_{a}^{b} f = \lim_{c\to 0} c(-\sum_{b}^{a} f)$ $= -\lim_{c\to 0} c\sum_{b}^{a} f = -(U)\int_a^b f$.

Theorem 10.5: If the left side exists, then

$$(U)\int_a^b f + (U)\int_b^c f = (U)\int_a^c f.$$

Proof:

i) $a \leq b \leq c$

$$(U)\int_a^b f + (U)\int_b^c f = \lim_{d\to 0} d \underset{d}{\sum}{}_a^b f + \lim_{d\to 0} d \underset{d}{\sum}{}_b^c f$$

$$= \lim_{d\to 0} d\left(\underset{d}{\sum}{}_a^b f + \underset{d}{\sum}{}_b^c f\right) = \lim_{d\to 0} d \underset{d}{\sum}{}_a^c f = (U)\int_a^c f.$$

ii) $b < a \leq c$

$$(U)\int_a^b f + (U)\int_b^c f = -\,(U)\int_b^a f + (U)\int_b^c f = \lim_{d\to 0} (-d) \underset{d}{\sum}{}_b^a f$$

$$+ \lim_{d\to 0} d \underset{d}{\sum}{}_b^c f = \lim_{d\to 0} d\left[-\underset{d}{\sum}{}_b^a f + \underset{d}{\sum}{}_b^c f\right]$$

$$= \lim_{d\to 0} d\left[\underset{d}{\sum}{}_a^b f + \underset{d}{\sum}{}_b^c f\right] = \lim_{d\to 0} d \underset{d}{\sum}{}_a^c f = (U)\int_a^c f.$$

The other cases are proved in a similar manner.

11. APPENDIX II

Theorem 11.1: Any interval $[a,b]$ contains a non-measurable subset H such that if x,y are in H and $x \neq y$, then x/y is irrational.

Proof: We first establish the theorem for an interval T which contains the point 1 and is of the form $T = T(s) = [1/s, s]$ where s is a rational number greater than 1.

Given $T = [1/s, s]$, let $W = [1/s^2, s^2]$; and for any x in T let $K(x) = \{xr | r \text{ rational and } xr \in W\}$.

The collection of distinct sets $K(x)$ forms a partition of W. First of all, any w in W may be expressed in the form $w = xr$ where x,r belong to T and r is rational. (Merely pick any rational r sufficiently close to $w^{\frac{1}{2}}$ and solve for x.) Thus $w \in K(x)$ and $W \subseteq \bigcup_x K(x)$. Since $K(x) \subseteq W$, it follows that $W = \bigcup_x K(x)$. Next if $z \in K(x) \cap K(y)$, then $z = xr_x = yr_y$ where r_x and r_y are rational. Consequently, for any $t \in K(y)$, we have $t = yr = x(r\,r_x/r_y)$ whence $t \in K(x)$ and $K(y) \subseteq K(x)$. Similarly $K(x) \subseteq K(y)$. Thus for any $x,y \in T$, the sets $K(x)$, $K(y)$ are either identical or disjoint.

Applying the Axiom of Choice, let A be a set obtained by selecting one element from each of the distinct sets $K(x)$;

let $\{r_k\}$ be the rationals of W enumerated in a sequence; let $A_k = \{xr_k | x \varepsilon A\}$; and let $m_*(A) = \alpha$, $m^*(A) = \beta$. Therefore, $m_*(A_k) = r_k m_*(A) = r_k\alpha$ and $m^*(A_k) = r_k m^*(A) = r_k\beta$.

The sets A_k are pairwise disjoint. To see this, suppose $n \neq m$ and $z \varepsilon A_n \cap A_m$. Then $z = x_n r_n = x_m r_m$, where $x_n, x_m \varepsilon A$. Thus x_n/x_m is rational, which implies $x_n = x_m$; but this cannot be since $r_n \neq r_m$.

Next, $T \subseteq \bigcup_{k=1}^{\infty} A_k$. For if $x \varepsilon T$, then since $T \subseteq W$, there exists an $x_o \varepsilon A$ such that $x \varepsilon K(x_o)$. Thus $x = x_o r_k$, where $x_o \varepsilon T$ and r_k is a rational number in W, i.e., $x \varepsilon A_k$. Now $s-1/s = m(T) \leq m^*(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} m^*(A_k) = \sum_{k=1}^{\infty} \beta r_k$. Hence $\beta > 0$. Since $A \subseteq T$ and each $r_k \varepsilon W$, we have $A_k \subset [1/s^3, s^3]$. Therefore, $s^3 - s^{-3} = m[s^{-3}, s^3] \geq m_*(\bigcup_{k=1}^{\infty} A_k) \geq \sum_{k=1}^{\infty} m_*(A_k) = \alpha \sum_{k=1}^{\infty} r_k$. Since the series $\sum_{k=1}^{\infty} r_k$ consists of all the rationals in W, it diverges to $+\infty$. Hence $\alpha = 0$.

Consequently, $m_*(A) = \alpha = 0 < \beta = m^*(A)$. Thus A is non-measurable and the theorem is proved for the intervals $T(s)$.

Now to extend the result, let $[a,b]$ be any given interval and $[c,d]$ any subinterval of $[a,b]$ such that c and d are rational and of the same sign, say positive. Let $r = \frac{c+d}{2}$ and let s_o be chosen so close to 1 that $\varepsilon = \max(s_o - 1, 1 - 1/s_o) = s_o - 1 < (d-c)/(d+c)$. Then we have $[1/s_o, s_o] \subset [1-\varepsilon, 1+\varepsilon]$ and

$[r/s_o, rs_o] \subset [r(1-\varepsilon), r(1+\varepsilon)] \subset [c,d]$. Finally, let A_o be the non-measurable subset of $T(s_o)$ constructed above and let $A = \{rz \mid z \in A_o\}$. The set A is a non-measurable subset of $[c,d] \subset [a,b]$; and if $x, y \in A$ and $x \neq y$, then $x = rz_1$, $y = rz_2$, where $z_1, z_2 \in A_o$. Hence $x/y = z_1/z_2$ is irrational. Since the same argument applies when c and d are negative, the theorem is proved.

Theorem 11.2: If H is a Hamel basis (for the real numbers over the rationals), then $(U)\int_a^b \chi_H$ exists and equals zero.

Proof: Since the members of H are linearly independent, H cannot contain distinct multiples of any positive number c. It follows that $\sum_c {}_a^b \chi_H$ equals o or 1 for all $c > 0$ and

$\lim_{c \to o} c \sum_c {}_a^b \chi_H$ exists and equals 0.

Moreover, there do exist non-measurable Hamel bases [10] so $[a,b] \cap H$ may be non-measurable as well.

12. APPENDIX III

A modified form of Fatou's Lemma

Let a be a limit point of the linear set S. Suppose for every $c \in S$, there is defined a non-negative summable function g_c with domain D. If $\liminf_{c \to a} g_c = g$, if g is measurable and if T is any measurable subset of D, then $\int_T g \leq \liminf_{c \to a} \int_T g_c$.

<u>Proof:</u> The equation $\liminf_{c \to a} g_c = g$ implies that for any sequence $\{c_n\}$ of distinct terms of S converging to a, we have $\liminf_{n \to \infty} g_{c_n} \geq g$.

Therefore, since g is measurable using Fatou's lemma, we have $\int_T g \leq \int_T \liminf_{n \to \infty} g_{c_n} \leq \liminf_{n \to \infty} \int_T g_{c_n}$. Hence, $\int_T g \leq \liminf_{c \to a} \int_T g_c$.

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