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# A GENERAL APPROACH TO THE BOUND STATES OF SEVERAL NUCLEONS

by

Richard Kent Cooper

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GRADUATE COLLEGE

I hereby recommend that this dissertation prepared under my direction by Richard Kent Cooper entitled A General Approach to the Bound States of Several Nucleons be accepted as fulfilling the dissertation requirement of the degree of Doctor of Philosophy

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SIGNED: Richard Kent Cooper

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## ABSTRACT

The equations of motion of a system of N nucleons are written, using the most general velocity independent, parity conserving, two-body potential between pairs of nucleons. The spin and isospin dependences are eliminated by established techniques involving use of the theory of the group of permutations of N objects. The center of mass coordinates and orientational dependences are eliminated by re-expressing the operator for the kinetic energy in terms of suitable variables, and using techniques of angular momentum theory.

The equations for all possible three-nucleon problems are written in detail, i.e., the parameters appearing in the N-nucleon equations are given explicitly, and the coupled differential equations for the amplitudes of the various states of spin, isospin, and angular momentum are displayed.

# 1. INTRODUCTION

It is generally the case, when attempting to solve a complex problem of physics or mathematics, that the initial efforts are rather specialized approaches to restricted forms of the problem. As increasingly more successful treatments are developed, a general approach gradually evolves which is applicable to more general formulations of the problem. In the case of the problem of several interacting nucleons, the work to present [1-9]* has largely concentrated on calculations of the properties of the ground states of specific nuclei, predominantly the triton and the alpha particle. The approach presented in this work is one of a more general nature, in that the equations of motion of the system of nucleons are written not just for the ground state, but for all the bound states. Further, this more general approach furnishes a completely systematic means of attacking the problem, something that has been lacking in previous considerations.

As might be expected in generalizing an attack on a problem, the more general method may lack some of the complete

*Numbers in brackets refer to references as entered in the List of References.

detail that a specialized treatment of a restricted form of the problem can achieve. Such is the case for the method presented here, in that noncentral forces, namely the tensor force and the spin orbit force, both of which are known to play some role in nuclear interactions, have so far been excluded. Electromagnetic interactions have also been excluded. It is anticipated that these forms of interaction may be treated as perturbations, or, if that is not possible, at least the magnitude of their effects will be so obtained. In any case the treatment given here will provide an advanced standpoint from which to base a further attack on the problem. In this treatment, all nucleon masses are assumed to be equal, and non-relativistic quantum mechanics is assumed to be valid for the description of the system.

In outline, the treatment of the N-nucleon problem given here involves the elimination of the spin and isospin dependences through the use of the properties of the group of permutations of N objects (i.e., the symmetric group $S_N$). The resulting set of coupled differential equations with the spatial coordinates as independent variables are then reduced to another set of differential equations involving only "intrinsic" variables (that is, variables not depending on the location or orientation of the system) by expressing the operator for the kinetic energy in a form that exhibits the role of the angular momentum operators and then using the properties of the angular momentum wave functions in order to eliminate the dependences on spatial orientation. To illustrate the detailed workings of the approach, the equations of motion for all possible three-nucleon problems are written.

## 2. THE SCHRÖDINGER EQUATION

The system of N nucleons is described by a wave function $\Psi$ which is a function of the spatial coordinates $\underline{r}_i$, $i = 1 \ldots N$ (cartesian components $x_{i\alpha}$, $\alpha = 1, 2, 3$) of the N particles, as well as their spin and isotopic spin coordinates. The wave function satisfies the time independent Schrödinger equation (H is the Hamiltonian)

$$H\Psi = (T + V)\Psi = E\Psi, \tag{1}$$

where $T = \sum_{i=1}^{N} \sum_{\alpha=1}^{3} \left(\frac{-\hbar^2}{2M}\right)\frac{\partial^2}{\partial x_{i\alpha}^2}$ is the kinetic energy operator, V is the potential from which the nuclear forces are derived, and E is the total energy of the system. Assuming that nuclear forces are charge independent, velocity independent and parity conserving, the most general nuclear potential one can write between two nucleons, i and j, say, is

$$V_{ij}^{G} = V_c(r_{ij}) + V_\sigma(r_{ij})\sigma_i \cdot \sigma_j + V_\tau(r_{ij})\tau_i \cdot \tau_j$$

$$+ V_{\sigma\tau}(r_{ij})\sigma_i \cdot \sigma_j \tau_i \cdot \tau_j + [V_T(r_{ij}) + V_T'(r_{ij})\tau_i \cdot \tau_j]S_{ij},$$

where $S_{ij} = \frac{3\sigma_i \cdot r_{ij}\sigma_j \cdot r_{ij}}{r_{ij}^2} - \sigma_i \cdot \sigma_j$ is the tensor potential, which will not be considered further in this paper.

Thus, considering only two-body forces and, as has been stated, neglecting tensor forces and electromagnetic interactions, the most general potential which we can use in the Schrödinger equation is

$$V = \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} V_{ij} , \qquad (2)$$

where $V_{ij} = V_{ij}^G - [V_T(r_{ij}) + V_T'(r_{ij})\tau_i \cdot \tau_j]S_{ij}$ .

With this potential the constants of the motion which we shall use are the following: (1) the square of the total orbital angular momentum of the center of mass and that about the center of mass and their respective z-components; (2) the square of the total spin and its z-component; and (3) the square of the total isotopic spin and its 3-component.

The system of N nucleons must obey the Pauli exclusion principle because the nucleons have spin $\frac{1}{2}$ and are therefore fermions. The wave function as a result must be antisymmetric with respect to an interchange of two of the nucleons; that is, if all the coordinates of two of the nucleons are interchanged, the wave function must change sign. Using the theory of the group of permutations of N objects, $S_N$, Mahmoud and Cooper [10] have shown that the wave function for a system of N nucleons having total spin s' and total isospin t' can be written

$$\Psi(s',t',\underline{r}) = \sum_{c'\alpha'\beta'\gamma'} \frac{1}{\sqrt{\bar{c}'}} \Gamma(a'b'\bar{c}';\alpha'\beta'\gamma')$$

$$\psi_{a'\alpha'}(s)\psi_{b'\beta'}(t)\psi_{c'\gamma'}(\underline{r}) \qquad (3)$$

where $\psi_{a'\alpha'}(s)$ is a normalized eigenfunction of the total spin angular momentum having transformation properties under permutations of the nucleons the same as the $\alpha'$ basis vector of the a' irreducible representation of the group $S_N$. $\psi_{b'\beta'}$ has the same significance for isospin. $\psi_{c'\gamma'}(\underline{r})$ is a function of the spatial coordinates ($\underline{r}$ is an abbreviation for $\underline{r}_1 \; . \; . \; . \; \underline{r}_N$) transforming under permutations in the same way as the $\gamma'$ basis vector of the c' irreducible representation of $S_N$. Table I gives the $\psi_{a'\alpha'}(s)$ for three and four nucleons. The coefficients $\Gamma(a'b'c';\alpha'\beta'\gamma')$ are discussed at length in [10], and are tabulated there for three and four nucleons.

In a similar manner the potential is written in reference [10] as

$$V = \sum_{abc} G(abc) \sum_{\alpha\beta\gamma} \frac{1}{\sqrt{c}} \Gamma(abc;\alpha\beta\gamma) V_{a\alpha}(s) V_{b\beta}(t) V_{c\gamma}(\underline{r}). \quad (4)$$

The $V_{a\alpha}(s)$ contain spin operators, the $V_{b\beta}(t)$ contain isospin operators, and the $V_{c\gamma}(\underline{r})$ contain spatial coordinates in the form of interparticle distances. Table II lists the $V_{a\alpha}$ for three and four nucleons. Values of the coefficients G(abc) for three and four nucleon problems are given in Table III.

When the potential (4) operates on the wave function (3) various changes in the permutation symmetry are induced in the components of the wave function, and the resultant expression must be re-expanded in terms of the basis $\psi_{a'\alpha'}$ and $\psi_{b'\beta'}$. Such a procedure is followed in [10], and the spin and isospin functions are then eliminated yielding the set of coupled differential equations

$$(T-E)\psi_{c''\gamma''}(\underline{r}) + \sum_{abc} G(abc) \sum_{c'\gamma\gamma'} [a'b'c'']^{\frac{1}{2}} \Delta\begin{pmatrix} a & b & c \\ a' & b' & \bar{c}' \\ a' & b' & \bar{c}'' \end{pmatrix}$$

$$\times \lambda(aa'a')\lambda(bb'b') \Gamma(cc'c'';\gamma\gamma'\gamma'')$$

$$\times V_{c\gamma}(\underline{r})\psi_{c'\gamma'}(\underline{r}) = 0 \ . \quad (5)$$

Expressions for the so-called recoupling coefficients, $\Delta\begin{pmatrix} a & b & c \\ a' & b' & \bar{c}' \\ a' & b' & \bar{c}'' \end{pmatrix}$, and their values are given in [10]. The $\lambda(aa'a')$ are defined by the equation

$$\sum_{\alpha\alpha'} \Gamma(aa'a'';\alpha\alpha'\alpha'')V_{a\alpha}(s)\psi_{a'\alpha'}(s) = \lambda(aa'a')\psi_{a''\alpha''}\delta_{a'a''} \quad (6)$$

and are given in Table IV for the three and four nucleon cases.

Several authors have used the theory of the symmetric group to construct antisymmetric wave functions. Derrick and Blatt [4] use such a method both to construct antisymmetric wave functions and to determine which eigenstates of spin and orbital angular momentum can be present in the ground state of the triton. Derrick [5] then calculates the matrix elements of the Hamiltonian between the various eigenstates of spin, isospin, and orbital angular momenta, thereby obtaining sixteen coupled differential equations for the spatial functions involved in the ground state of the triton. The equations involve only the separations between the particles. A similar procedure for the triton, but without the introduction of Euler angles, is given by Cohen and Willis [8]. Cohen [7] also calculates the matrix elements for the alpha particle. While these papers approach the problem in a straightforward manner, all the steps involved in classifying the wave functions and calculating the matrix elements would have to be carried out all over again if the neutron-to-proton ratio were changed, whereas in the method described above one would need only to use a different set of the tabulated recoupling coefficients. Thus a considerable saving in effort is effected through the use of the recoupling coefficients.

TABLE I. SPIN WAVE FUNCTIONS

a. Three Nucleons

$$\psi_{11}(s) = \frac{1}{\sqrt{3}}[\alpha_1\alpha_2\beta_3 + \alpha_1\beta_2\alpha_3 + \beta_1\alpha_2\alpha_3]$$

$$\psi_{21}(s) = \sqrt{\tfrac{2}{3}}[\alpha_1\alpha_2\beta_3 - \tfrac{1}{2}(\alpha_1\beta_2\alpha_3 + \beta_1\alpha_2\alpha_3)]$$

$$\psi_{22}(s) = \sqrt{\tfrac{1}{2}}[\alpha_1\beta_2\alpha_3 - \beta_1\alpha_2\alpha_3]$$

b. Four Nucleons

$$\psi_{11}(s) = \frac{1}{\sqrt{6}}[\alpha_1\alpha_2\beta_3\beta_4 + \alpha_1\beta_2\alpha_3\beta_4 + \alpha_1\beta_2\beta_3\alpha_4 + \beta_1\alpha_2\alpha_3\beta_4 + \beta_1\alpha_2\beta_3\alpha_4 + \beta_1\beta_2\alpha_3\alpha_4]$$

$$\psi_{31}(s) = \frac{1}{\sqrt{6}}[\alpha_1\alpha_2\beta_3\beta_4 + \alpha_1\beta_2\alpha_3\beta_4 - \alpha_1\beta_2\beta_3\alpha_4 + \beta_1\alpha_2\alpha_3\beta_4 - \beta_1\alpha_2\beta_3\alpha_4 - \beta_1\beta_2\alpha_3\alpha_4]$$

$$\psi_{32}(s) = \frac{1}{\sqrt{3}}[\alpha_1\alpha_2\beta_3\beta_4 - \beta_1\beta_2\alpha_3\alpha_4 - \tfrac{1}{2}(\alpha_1\beta_2\alpha_3\beta_4 + \beta_1\alpha_2\alpha_3\beta_4 - \alpha_1\beta_2\beta_3\alpha_4 - \beta_1\alpha_2\beta_3\alpha_4)]$$

$$\psi_{33}(s) = \tfrac{1}{2}[\alpha_1\beta_2\alpha_3\beta_4 - \beta_1\alpha_2\alpha_3\beta_4 + \alpha_1\beta_2\beta_3\alpha_4 - \beta_1\alpha_2\beta_3\alpha_4]$$

$$\psi_{21}(s) = \frac{1}{\sqrt{3}}[\alpha_1\alpha_2\beta_3\beta_4 + \beta_1\beta_2\alpha_3\alpha_4 - \tfrac{1}{2}(\alpha_1\beta_2\alpha_3\beta_4 + \alpha_1\beta_2\beta_3\alpha_4 + \beta_1\alpha_2\alpha_3\beta_4 + \beta_1\alpha_2\beta_3\alpha_4)]$$

$$\psi_{22}(s) = \tfrac{1}{2}[\alpha_1\beta_2\alpha_3\beta_4 - \alpha_1\beta_2\beta_3\alpha_4 - \beta_1\alpha_2\alpha_3\beta_4 + \beta_1\alpha_2\beta_3\alpha_4]$$

## TABLE II. POTENTIAL FUNCTIONS

For $V_{a\alpha}(s)$, $\theta_{ij} = \underline{\sigma}_i \cdot \underline{\sigma}_j$; for $V_{a\alpha}(t)$, $\theta_{ij} = \underline{\tau}_i \cdot \underline{\tau}_j$; for $V_{a\alpha}(\underline{r})$, $\theta_{ij} = V(r_{ij})$.

a. Three Nucleons

$$V_{11} = \frac{1}{\sqrt{3}}[\theta_{12} + \theta_{13} + \theta_{23}]$$

$$V_{21} = \sqrt{\frac{2}{3}}[\theta_{12} - \tfrac{1}{2}(\theta_{13} + \theta_{23})]$$

$$V_{22} = \sqrt{\tfrac{1}{2}}[\theta_{13} - \theta_{23}]$$

b. Four Nucleons

$$V_{11} = \frac{1}{\sqrt{6}}[\theta_{12} + \theta_{13} + \theta_{14} + \theta_{23} + \theta_{24} + \theta_{34}]$$

$$V_{31} = \frac{1}{\sqrt{6}}[\theta_{12} + \theta_{13} + \theta_{23} - (\theta_{14} + \theta_{24} + \theta_{34})]$$

$$V_{32} = \frac{1}{\sqrt{3}}[\theta_{12} - \theta_{34} - \tfrac{1}{2}(\theta_{13} + \theta_{23} - \theta_{14} - \theta_{24})]$$

$$V_{33} = \tfrac{1}{2}[\theta_{13} - \theta_{23} + \theta_{14} - \theta_{24}]$$

$$V_{21} = \frac{1}{\sqrt{3}}[\theta_{12} + \theta_{34} - \tfrac{1}{2}(\theta_{13} + \theta_{14} + \theta_{23} + \theta_{24})]$$

$$V_{22} = \tfrac{1}{2}[(\theta_{13} - \theta_{14}) - (\theta_{23} - \theta_{24})]$$

TABLE III. G(abc)

a. Three Nucleons

$$G(111) = \frac{1}{\sqrt{3}} \qquad G(122) = G(212) = G(221) = G(222) = \sqrt{\frac{2}{3}}$$

b. Four Nucleons

$$G(111) = \frac{1}{\sqrt{6}} \qquad G(122) = G(212) = G(221) = G(222) = \frac{1}{\sqrt{3}}$$

$$G(133) = G(313) = G(331) = \frac{1}{\sqrt{2}}$$

$$G(233) = G(323) = G(332) = 1$$

TABLE IV. $\lambda(aa'a')$

a. Three Nucleons

| a \ a' | 1 | 2 | $\bar{1}$ |
|---|---|---|---|
| 1 | $\sqrt{3}$ | $-\sqrt{3}$ | $-3\sqrt{3}$ |
| 2 | 0 | $2\sqrt{3}$ | 0 |

b. Four Nucleons

| a \ a' | 1 | 3 | 2 | $\bar{3}$ | 1 |
|---|---|---|---|---|---|
| 1 | $\sqrt{6}$ | $-(1/3)\sqrt{6}$ | $-\sqrt{6}$ | $-(5/3)\sqrt{6}$ | $-3\sqrt{6}$ |
| 3 | 0 | 4 | 0 | $-4$ | 0 |
| 2 | 0 | $4/\sqrt{3}$ | $2\sqrt{6}$ | $-4/\sqrt{3}$ | 0 |

## 3. THE KINETIC ENERGY OPERATOR

You boil it in sawdust: You salt it in glue
You condense it with locusts and tape
Still keeping one principal object in view--
To preserve its symmetrical shape.

Fit the Fifth, The Hunting of the Snark
Lewis Carroll

In order to retain the usefulness of the symmetry considerations used in [10], any re-expression of the kinetic energy operator must retain the essential symmetry of that operator. If it can be written in such a way as to exhibit the dependence on the angular momentum operators, use can be made of the considerable body of techniques developed for dealing with angular momentum. F. Villars [3] has derived such an expression for the kinetic energy of a classical system of N particles of equal mass. While it is not difficult to extend his analysis to the case of a quantum system, the resulting parameters represent an unfortunate separation of terms insofar as practical calculations are concerned. A method which is computationally simpler is used by G. Derrick [5] in his treatment of the triton ground state. What follows is a generalization to N bodies of Derrick's treatment of the three body kinetic energy operator.

The kinetic energy operator $T = \frac{-\hbar^2}{2M} \sum_{i=1}^{N} \sum_{\alpha=1}^{3} \frac{\partial^2}{\partial x_{i\alpha}^2}$ is proportional to the Laplacian operator in a 3N dimensional space. If a coordinate transformation is performed in this space, then the Laplacian becomes, according to tensor analysis,

$$\nabla^2 = \sum_{i,j=1}^{3N} \frac{1}{\sqrt{g}} \frac{\partial}{\partial X^i} \left(\sqrt{g}\, g^{ik} \frac{\partial}{\partial X^k}\right) , \qquad (7)$$

where the $X^i$ are the new coordinates, $g^{ik} = \sum_{j\gamma} \frac{\partial X^i}{\partial x_{j\gamma}} \frac{\partial X^k}{\partial x_{j\gamma}}$ and $g = \frac{1}{\det g^{ik}}$. The coordinate transformation which we choose to carry out is one in which three of the new coordinates are the coordinates of the center of mass, three are taken to be the Euler angles $\theta_s$, s = 1, 2, 3*, specifying the orientation of the principal axes of the tensor of inertia of the system, and the remaining 3N-6 coordinates $\xi_\sigma$, $\sigma$ = 1, 2 . . . 3N-6, are "intrinsic" coordinates, chosen in some symmetric way. For the case of three and four particles, the intrinsic coordinates can be chosen to be the

---

*More common notation in the literature is $\alpha$, $\beta$, $\gamma$ for $\theta_1$, $\theta_2$, $\theta_3$, respectively. Several sets of Euler angles have been used by various authors; the choice in this paper follows M. E. Rose [11], namely, $\theta_1$ and $\theta_2$ are the azimuthal and polar angles, respectively, of the body-fixed z axis.

interparticle distances, since for these two cases they are 3N-6 in number. For five and more nucleons some other choice must be made since the number of interparticle distances $\frac{N(N-1)}{2}$ is greater than 3N-6. Thus we have (in an obvious notation)

$$X^1 = x_{c.m.} = \frac{1}{N} \sum_{j=1}^{N} x_{j1} , \qquad X^2 = y_{c.m.} = \frac{1}{N} \sum_{j=1}^{N} x_{j2} ,$$

$$X^3 = z_{c.m.} = \frac{1}{N} \sum_{j=1}^{N} x_{j3} , \qquad X^4 = \theta_1 , \qquad X^5 = \theta_2 ,$$

$$X^6 = \theta_3 , \qquad \text{and} \quad X^{6+\sigma} = \xi_\sigma .$$

The relation between the old and new coordinates is contained in the set of equations

$$x_{i\alpha} = (\underline{X}_{c.m.})_\alpha + \sum_{A=1}^{3} R_{A\alpha}(\theta_s) x'_{iA}(\xi_\sigma) , \qquad (8)$$

where the $R_{A\alpha}(\theta_s)$ are the elements of the orthogonal rotation matrix relating coordinates in a cartesian system at the center of mass with axes parallel to the original axes to the coordinates $x'_{iA}$ in a cartesian system at the center of mass with axes along the principal axes of the tensor of inertia. The following notation will be employed henceforth: (1) The coordinate system whose axes lie along the principal axes are known as the body-fixed axes, and components taken with respect to these axes carry subscripts denoted by capital latin letters; thus $x'_{iA}$ is the component along the A axis of the position vector (with respect to the center of mass) of the $i^{th}$ particle. (2) Components with respect to the original system of coordinates (called space-fixed) carry lower case greek subscripts. (3) Angular momentum, being an axial vector, is denoted in Villars' article in two ways. In order to facilitate references to his paper, we shall employ his notation, in which the components of an axial vector are denoted by a circumflex over the subscript, or are designated by double index notation, namely $L_{\alpha\beta} = x_\alpha p_\beta - x_\beta p_\alpha$. Thus $L_{12} = L_{\hat{3}}$, $L_{23} = L_{\hat{1}}$, and $L_{31} = L_{\hat{2}}$, and, in general $AB = \hat{C}$ (A, B, C cyclic).

Now as is pointed out in [5] for the case N = 3, and as is true for all $N \geqq 3$, the metric tensor $g^{ik}$ can be written as

$$g^{ik} = (SM\tilde{S})^{ik}\,, \qquad (\tilde{S} = \text{transpose of } S) \quad (9)$$

where S and M are 3N by 3N matrices. S is a function only of the Euler angles and is given by

$$S = \begin{bmatrix} I(3) & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & I(3N-6) \end{bmatrix}\,,$$

where I(3N-6) is a 3N-6 by 3N-6 identity matrix, I(3) is a 3 by 3 identity matrix, and $\theta$ (elements $\theta_{s,\hat{A}}$) is given by

$$\theta = \frac{1}{\sin\theta_2}\begin{bmatrix} -\cos\theta_3 & \sin\theta_3 & 0 \\ \sin\theta_2\sin\theta_3 & \sin\theta_2\cos\theta_3 & 0 \\ \cos\theta_2\cos\theta_3 & -\sin\theta_3\cos\theta_2 & \sin\theta_2 \end{bmatrix} \tag{10}$$

The (symmetric) matrix M is a function only of the intrinsic coordinates, and is given by

$$M = \left[\begin{array}{c|ccc|ccc}
\frac{1}{N}I(3) & & 0 & & & 0 & \\
\hline
 & \frac{\lambda_{xx}}{(\lambda_{yy}-\lambda_{zz})^2} & 0 & 0 & & & \\
0 & 0 & \frac{\lambda_{yy}}{(\lambda_{zz}-\lambda_{xx})^2} & 0 & & (\text{SYMMETRIC}) & \\
 & 0 & 0 & \frac{\lambda_{zz}}{(\lambda_{xx}-\lambda_{yy})^2} & & & \\
\hline
 & \sum_{i\alpha} M_{i\alpha,\hat{1}}\frac{\partial\xi_1}{\partial x_{i\alpha}} & \sum_{i\alpha} M_{i\alpha,\hat{2}}\frac{\partial\xi_1}{\partial x_{i\alpha}} & \sum_{i\alpha} M_{i\alpha,\hat{3}}\frac{\partial\xi_1}{\partial x_{i\alpha}} & \sum_{i\alpha}\frac{\partial\xi_1}{\partial x_{i\alpha}}\frac{\partial\xi_1}{\partial x_{i\alpha}} & \sum_{i\alpha}\frac{\partial\xi_1}{\partial x_{i\alpha}}\frac{\partial\xi_2}{\partial x_{i\alpha}} & \cdot \\
 & \sum_{i\alpha} M_{i\alpha,\hat{1}}\frac{\partial\xi_2}{\partial x_{i\alpha}} & \sum_{i\alpha} M_{i\alpha,\hat{2}}\frac{\partial\xi_2}{\partial x_{i\alpha}} & \sum M_{i\alpha,\hat{3}}\frac{\partial\xi_2}{\partial x_{i\alpha}} & \sum_{i\alpha}\frac{\partial\xi_2}{\partial x_{i\alpha}}\frac{\partial\xi_1}{\partial x_{i\alpha}} & \sum_{i\alpha}\frac{\partial\xi_2}{\partial x_{i\alpha}}\frac{\partial\xi_2}{\partial x_{i\alpha}} & \cdot \\
0 & \sum_{i\alpha} M_{i\alpha,\hat{1}}\frac{\partial\xi_3}{\partial x_{i\alpha}} & \sum_{i\alpha} M_{i\alpha,\hat{2}}\frac{\partial\xi_3}{\partial x_{i\alpha}} & \sum_{i\alpha} M_{i\alpha,\hat{3}}\frac{\partial\xi_3}{\partial x_{i\alpha}} & \sum_{i\alpha}\frac{\partial\xi_3}{\partial x_{i\alpha}}\frac{\partial\xi_1}{\partial x_{i\alpha}} & \sum_{i\alpha}\frac{\partial\xi_3}{\partial x_{i\alpha}}\frac{\partial\xi_2}{\partial x_{i\alpha}} & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right] \tag{11}$$

where $i_{xx}$, etc. are the principal moments of inertia per nucleon mass and

$$M_{i\alpha,AB} = \frac{1}{(\sum_i x'^2_{iA}) - (\sum_i x'^2_{iB})} [x'_{iA} R_{B\alpha} + x'_{iB} R_{A\alpha}] \ . \qquad (12)$$

While $M_{i\alpha,\hat{A}}$ contains Euler angles, the summation

$$\sum_{i\alpha} M_{i\alpha,\hat{A}} \frac{\partial \xi_\sigma}{\partial x_{i\alpha}} \equiv \eta_{\hat{A},\sigma}$$

is a function only of intrinsic variables.

Now it is easily seen that $\det g^{ik} = (\det S)^2 \det M^{ik}$, so that $\sqrt{g} = \sin \theta_2 \, m^{\frac{1}{2}}$ where $m = 1/\det M^{ik}$. Using this relation, plus the fact (see Villars [3]) that

$$L_{\hat{A}} = \sum_{s=1}^{3} \theta_{s,\hat{A}} \left(-i\hbar \frac{\partial}{\partial \theta_s}\right) ,$$

or in matrix notation,

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \tilde{\theta} \begin{pmatrix} -i\hbar\, \partial/\partial\theta_1 \\ -i\hbar\, \partial/\partial\theta_2 \\ -i\hbar\, \partial/\partial\theta_3 \end{pmatrix} , \quad \text{(body-fixed components)} \qquad (13)$$

the Laplacian becomes

$$\nabla^2 = \frac{1}{N}\nabla^2_{c.m.} + \sum_{A=1}^{3} I_{\hat{A}}\left(\frac{L_{\hat{A}}}{-i\hbar}\right)^2 + 2\sum_{A=1}^{3}\sum_{\sigma=1}^{3N-6}\sum_{i\alpha} M_{i\alpha,\hat{A}}\frac{\partial\xi_\sigma}{\partial x_{i\alpha}}$$

$$X \frac{L_{\hat{A}}}{(-i\hbar)}\frac{\partial}{\partial\xi_\sigma} + \sum_{A=1}^{3}\sum_{\sigma=1}^{3N-6} m^{-\frac{1}{2}}\left(\frac{\partial}{\partial\xi_\sigma} m^{\frac{1}{2}}\sum_{i\alpha} M_{i\alpha,\hat{A}}\frac{\partial\xi_\sigma}{\partial x_{i\alpha}}\right)$$

$$X \frac{L_{\hat{A}}}{(-i\hbar)} + \sum_{\sigma,\tau=1}^{3N-6} m^{-\frac{1}{2}}\frac{\partial}{\partial\xi_\sigma} m^{\frac{1}{2}} D_{\sigma\tau}\frac{\partial}{\partial\xi_\tau} \quad , \tag{14}$$

where $I_{\hat{1}} = \frac{i_{xx}}{(i_{yy}-i_{zz})^2}$ and cyclic permutations, and $D_{\sigma\tau} = \sum_{i\alpha}\frac{\partial\xi_\sigma}{\partial x_{i\alpha}}\frac{\partial\xi_\tau}{\partial x_{i\alpha}}$ . Thus the operator for the kinetic energy of a system of N particles is, finally,

$$T = \frac{P^2_{c.m.}}{2MN} + \frac{1}{2M}\sum_{\sigma,\tau=1}^{3N-6} m^{-\frac{1}{2}}\pi_\sigma m^{\frac{1}{2}} D_{\sigma\tau}\pi_\tau + \frac{1}{2M}\sum_{A=1}^{3} I_{\hat{A}} L^2_{\hat{A}}$$

$$+ \frac{1}{M}\sum_{A=1}^{3}\sum_{\sigma=1}^{3N-6}\eta_{\hat{A},\sigma}L_{\hat{A}}\pi_\sigma + (-i\hbar)\frac{1}{2M}\sum_{A}\sum_{\sigma} m^{-\frac{1}{2}} \tag{15}$$

$$X\left(\frac{\partial}{\partial\xi_\sigma} m^{\frac{1}{2}}\eta_{\hat{A},\sigma}\right)L_{\hat{A}} \quad ,$$

where $\underline{P}_{c.m.} = -i\hbar\underline{\nabla}_{c.m.}$ and $\pi_\sigma = -i\hbar\frac{\partial}{\partial\xi_\sigma}$ .

As far as the motion of the center of mass is concerned, the only place where the center-of-mass variables occur in the Hamiltonian is in the first term of the kinetic energy operator. The Hamiltonian separates into terms representing the kinetic energy associated with the motion of the center of mass, and the energy of the motion about the center of mass. That is,

$$H = \frac{P^2_{c.m.}}{2MN} + H' = \frac{P^2_{c.m.}}{2MN} + T' + V ,$$

where T' contains no center of mass variables. Thus the wave function may be written as a product of a center of mass wave function and a function depending on the particle coordinates relative to the center of mass. That is, in our expression for the wave function, we set

$$\psi_{c\gamma}(\underline{r}) = \Psi(\underline{R}_{c.m.})\psi'_{c\gamma}(\underline{r}'). \tag{16}$$

Putting this expression into the coupled equations (5), and carrying out the usual steps for separating variables, we obtain the equations for the $\psi'_{c\gamma}(\underline{r}')$:

$$(T'-E')\psi'_{c''\gamma''}(\underline{r}') + \sum_{abc} G(abc) \sum_{c'\gamma\gamma'} [a'b'c'']^{\frac{1}{2}}$$

$$\times \Delta\begin{pmatrix} a & b & \bar{c} \\ a' & b' & \bar{c}' \\ a' & b' & \bar{c}'' \end{pmatrix} \lambda(aa'a')\lambda(bb'b')\,\Gamma(cc'c'';\gamma\gamma'\gamma'') \tag{17}$$

$$\times V_{c\gamma}(\underline{r}')\psi'_{c'\gamma'}(\underline{r}') = 0 ,$$

where $E' = E - E_{c.m.}$, $E_{c.m.}$ being the energy associated with the center of mass motion.

## 4. ELIMINATION OF ANGULAR DEPENDENCES

If we now expand the wave function in terms of a product of a complete set of functions $\varphi_\alpha$ of the Euler angles and a set of functions of the intrinsic variables, we can separate out the dependences of the equations of motion on the Euler angles by re-expanding the terms $L_{\hat{A}}\varphi_\alpha$ and $(L_{\hat{A}})^2\varphi_\alpha$ as linear combinations of the $\varphi_\alpha$ and then using the orthogonality properties of the set.

A particularly useful set of functions to use are the functions $D^{(\ell)}_{m'm}(\theta_s)$, as defined in Rose [11]. These functions are eigenfunctions:

(1) of the square of total orbital angular momentum (with respect to the center of mass), with eigenvalue $\ell(\ell+1)\hbar^2$

(2) of the space-fixed z-component of the total angular momentum, with eigenvalue $-m'\hbar$

(3) of the body-fixed z-component of the total angular momentum, with eigenvalue $-m\hbar$.

The orthogonality property of these functions is given by

$$\int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 \sin\theta_2 \int_0^{2\pi} d\theta_3 \, D^{(j1)*}_{m_1' m_1} \, D^{(j2)}_{m_2' m_2} =$$

$$\frac{8\pi^2}{2j_1+1} \delta_{m_1' m_1} \delta_{m_2' m_2} \delta_{j_1 j_2} \, . \tag{18}$$

Since the square of the total angular momentum and its z-component are constants of the motion, we may write, for an energy eigenstate with angular momentum $\ell$ and projection m',

$$\psi'_{c\gamma}(\underline{r}') = \sum_{m=-\ell}^{\ell} D^{(\ell)}_{-m'm}(\theta_s)\varphi^{c\gamma}_m(\xi_\sigma) \, . \tag{19}$$

In order to calculate the expansions of $L_{\hat{\Lambda}} D^{(\ell)}_{m'm}$ and $(L_{\hat{\Lambda}})^2 D^{\ell}_{m'm}$ in terms of the same kind of functions, we use the following method, found, for instance, in Davydov [12]: The function $D^{(j)}_{m'm}(\theta_s)$ is defined as the matrix element of the rotation operator

$$D(\theta_s) = e^{-i\theta_1 J_z} \, e^{-i\theta_2 J_y} \, e^{-i\theta_3 J_z} \tag{20}$$

between states having the same value of the total angular momentum (quantum number j) and having magnetic quantum numbers m' and m. That is,

$$D^{(j)}_{m'm}(\theta_s) = (jm'|D(\theta_s)|jm) \, . \tag{21}$$

The J's in the exponentials are operators for the various space-fixed components of the angular momentum, measured in units of $\hbar$. If we differentiate both sides of this equation with respect to $\theta_\ell$, we have

$$\frac{\partial}{\partial\theta_\ell} D^{(j)}_{m'm}(\theta_s) = -i[jm'|D(\theta_s)A_\ell|jm] \tag{22}$$

where
$$A_1 = e^{+i\theta_3 J_z}\, e^{i\theta_2 J_y}\, J_z\, e^{-i\theta_2 J_y}\, e^{-i\theta_3 J_z}$$

$$A_2 = e^{i\theta_3 J_z}\, J_y\, e^{-i\theta_3 J_z} \tag{23}$$

$$A_3 = J_z$$

Now the A's can be expressed as a linear combination of the J operators

$$A_\ell = \sum_\lambda q_{\lambda\ell}\, J_\lambda \ . \tag{24}$$

From the previous expressions for the A's we can determine that the $q_{\lambda\ell}$'s form the elements of the nonsingular matrix

$$[q] = \begin{pmatrix} -\sin\theta_2\cos\theta_3 & \sin\theta_3 & 0 \\ \sin\theta_2\sin\theta_3 & \cos\theta_3 & 0 \\ \cos\theta_2 & 0 & 1 \end{pmatrix} . \tag{25}$$

The matrix inverse to [q] is

$$[q^{-1}] = \begin{bmatrix} -\csc\theta_2 \cos\theta_3 & \csc\theta_2 \sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ \cot\theta_2 \cos\theta_3 & -\cot\theta_2 \sin\theta_3 & 1 \end{bmatrix} . \qquad (26)$$

Introducing a complete set of angular momentum eigenfunctions into equation (22) we have

$$\frac{\partial}{\partial\theta_\ell} D^{(j)}_{m'm} = -i \sum_{m'',\lambda} D^{(j)}_{m'm''} (jm''|J_\lambda|jm) q_{\lambda\ell} . \qquad (27)$$

Multiplying both sides of this equation by $q^{-1}_{\ell\mu}$ and summing over $\ell$, we obtain, since $\sum_\ell q_{\lambda\ell}\, q^{-1}_{\ell\mu} = \delta_{\lambda\mu}$ ,

$$\sum_\ell q^{-1}_{\ell\mu} \frac{\partial}{\partial\theta_\ell} D^{(j)}_{m'm} = -i \sum_{m''} D^{(j)}_{m'm''} (jm''|J_\mu|jm) . \qquad (28)$$

The matrix elements on the right hand side are easily calculated from ordinary angular momentum theory, while we see that if we denote $\sum_\ell q^{-1}_{\ell\mu} \frac{\partial}{\partial\theta_\ell}$ by $K_\mu$, then we have

$$\begin{pmatrix} K_1 \\ \\ K_2 \\ \\ K_3 \end{pmatrix} = \begin{pmatrix} -\csc\theta_2\cos\theta_3 & \sin\theta_3 & \cot\theta_2\cos\theta_3 \\ \\ \csc\theta_2\sin\theta_3 & \cos\theta_3 & -\cot\theta_2\sin\theta_3 \\ \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial\theta_1} \\ \\ \frac{\partial}{\partial\theta_2} \\ \\ \frac{\partial}{\partial\theta_3} \end{pmatrix} . \tag{29}$$

Comparison of this equation with the equation (13) for the body-fixed components of the angular momentum shows that

$$\begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \frac{1}{(-i\hbar)} \begin{pmatrix} L_{\hat{1}} \\ L_{\hat{2}} \\ L_{\hat{3}} \end{pmatrix} .$$

Thus we have determined that

$$L_{\hat{A}} D^{(\ell)}_{m'm}(\theta_s) = -\sum_{m''} D^{(\ell)}_{m'm''} \, (\ell m''|L_\alpha|\ell m) \ , \tag{30}$$

where A and $\alpha$ have the same numerical value (i.e., 1, 2, 3).

We may calculate, then, as an example, that the operator for the body-fixed x-component of the angular momentum yields the following result when applied to $D^{(\ell)}_{m'm}$ :

$$L_1 D^{(\ell)}_{m'm}(\theta_s) = -\sum_{m''} D^{(\ell)}_{m'm''}(\ell m''|L_x|\ell m)$$

$$= -\sum_{m''} D^{(\ell)}_{m'm''}(\ell m''|\tfrac{1}{2}(L_+ + L_-)|\ell m)$$

$$= -\sum_{m''} D^{(\ell)}_{m'm''}\{\hbar/2\sqrt{(\ell-m)(\ell+m+1)}\ \delta_{m''m+1} \tag{31}$$

$$+ \hbar/2\sqrt{(\ell+m)(\ell-m+1)}\,\delta_{m''m-1}\}\ ,$$

$$L_1 D^{(\ell)}_{m'm} = -D^{(\ell)}_{m'm+1}\ \hbar/2\sqrt{(\ell-m)(\ell+m+1)}\ - D^{(\ell)}_{m'm-1}$$

$$X\ \hbar/2\sqrt{(\ell+m)(\ell-m+1)}\ .$$

Similar results are easily obtained for the other components of the angular momentum.

Thus when we insert (19) into (17), multiply by $D^{(\ell)*}_{-m'm''}$ and integrate over the space of the Euler angles, we obtain the set of equations for the $\varphi^{c\gamma}_m$,

$$\{\frac{1}{2M}\sum_{\sigma\tau} m^{-\frac{1}{2}}\pi_\sigma m^{\frac{1}{2}} D_{\sigma\tau}\pi_\tau + \frac{\hbar^2}{4M}[\ell(\ell+1)-m''^2]\,[\frac{i_{xx}}{(i_{yy}-i_{zz})^2}$$

$$+ \frac{i_{yy}}{(i_{zz}-i_{xx})^2}] + \frac{i_{zz}}{(i_{xx}-i_{yy})^2}\;\frac{m''^2\hbar^2}{2M} - \frac{m''\hbar}{M}$$

$$\times \sum_\sigma \eta_{\hat{3},\sigma}\pi_\sigma + \frac{i\hbar^2 m''}{2M}\sum_\sigma m^{-\frac{1}{2}}(\frac{\partial}{\partial\xi_\sigma} m^{\frac{1}{2}}\eta_{\hat{3},\sigma}) - E'\}\varphi_{m''}^{c''\gamma''}(\xi_\sigma)$$

$$+ [\sum_{abc} G(abc) \sum_{c'\gamma\gamma'} [a'b'c'']^{\frac{1}{2}}\,\Delta\begin{pmatrix} a & b & \bar{c} \\ a' & b' & \bar{c}' \\ a' & b' & \bar{c}'' \end{pmatrix}\lambda(aa'a')$$

$$\times \lambda(bb'b')\,\Gamma(cc'c'';\gamma\gamma'\gamma'')V_{c\gamma}(\xi_\sigma)]\varphi_{m''}^{c'\gamma'}(\xi_\sigma) \qquad (32)$$

$$+ \{\frac{i_{xx}}{(i_{yy}-i_{zz})^2} - \frac{i_{yy}}{(i_{zz}-i_{xx})^2}\}\,[\frac{1}{2M}\frac{\hbar^2}{4}]$$

$$\times [\sqrt{(\ell+m''+2)(\ell-m''-1)(\ell+m''+1)(\ell-m'')}\;\varphi_{m''+2}^{c''\gamma''}$$

$$+ \sqrt{(\ell-m''+2)(\ell+m''-1)(\ell-m''+1)(\ell+m'')}\;\varphi_{m''-2}^{c''\gamma''}]$$

$$- \frac{\hbar}{2M}\sqrt{(\ell+m''+1)(\ell-m'')}\;[\sum_\sigma(\eta_{\hat{1},\sigma}+i\eta_{\hat{2},\sigma})\pi_\sigma - \frac{i\hbar}{2}m^{-\frac{1}{2}}$$

$$\times (\frac{\partial}{\partial\xi_\sigma} m^{\frac{1}{2}}(\eta_{\hat{1},\sigma}+i\eta_{\hat{2},\sigma}))]\varphi_{m''+1}^{c''\gamma''} - \frac{\hbar}{2M}\sqrt{(\ell-m''+1)(\ell+m'')}$$

$$\times [\sum_\sigma(\eta_{\hat{1},\sigma}-i\eta_{\hat{2},\sigma})\pi_\sigma - \frac{i\hbar}{2}m^{-\frac{1}{2}}(\frac{\partial}{\partial\xi_\sigma} m^{\frac{1}{2}}(\eta_{\hat{1},\sigma}-i\eta_{\hat{2},\sigma}))]\varphi_{m''-1}^{c''\gamma''}$$

$$= 0$$

These equations represent the limit to which the Schrödinger equation can be reduced while retaining the symmetry under permutations, required by the formalism in which the particles are treated as being identical.

## 5. THE THREE-BODY EQUATIONS

Turning now to the specific problem of writing the equations for the three-body problem, considerable simplifications occur in the expression for the kinetic energy operator, primarily due to the fact that at any instant of time the positions of the three particles determine a triangle, a plane figure. For more than three nucleons, the equations (32) are fully as formidable as first glance would suggest. The geometry for three bodies is indicated in Figure 1. The body-fixed axes are labeled with capital latin letters. The body z-axis is taken normal to the plane in such a way that traversing the triangle from 1 to 2 to 3 amounts to a counterclockwise motion about the axis. The body x-axis is taken along the major axis of the triangle's inertia ellipse.

The intrinsic variables $\xi_\sigma$ are taken as $\xi_1 = s_1 = \sqrt{(x_2-x_3)^2 + (y_2-y_3)^2 + (z_2-z_3)^2}$ and cyclic permutations. Using these relations it is simple to calculate that the matrix whose elements are $D_{\sigma\tau}$ is

$$D = \begin{pmatrix} 2 & \cos\vartheta_3 & \cos\vartheta_2 \\ \cos\vartheta_3 & 2 & \cos\vartheta_1 \\ \cos\vartheta_2 & \cos\vartheta_1 & 2 \end{pmatrix},$$

where $\vartheta_1$ is the vertex angle at particle 1, etc., and by the law of cosines

$$\cos \vartheta_1 = \frac{s_2^2 + s_3^2 - s_1^2}{2s_2 s_3} \quad \text{and cyclic permutations.}$$

All the parameters occurring in the kinetic energy operator are listed in [5], and these results, all of which have been verified, will be introduced as needed.

Of particular interest is the fact that $\eta_{\hat{1},\sigma} = \eta_{\hat{2},\sigma} = 0$, and thus the equations (32) separate into two sets of equations, one set containing only even m", the other containing only odd values of m", because the terms that normally couple even and odd m" are zero when both $\eta_{\hat{1},\sigma}$ and $\eta_{\hat{2},\sigma}$ are zero.

The area of the triangle is

$$\Delta = \tfrac{1}{4}\sqrt{2s_1^2 s_2^2 + 2s_1^2 s_3^2 + 2s_2^2 s_3^2 - s_1^4 - s_2^4 - s_3^4} \;,$$

and a particularly useful set of relations is

$$i_{xx} i_{yy} = 4/3\ \Delta^2 \;, \qquad i_{xx} + i_{yy} = 1/3(s_1^2 + s_2^2 + s_3^2)$$

$$\equiv i_3 \;, \quad \text{and} \quad (i_{xx} - i_{yy}) = \sqrt{i_3^2 - \tfrac{16}{3}\Delta^2} \;.$$

With these relationships it is easily seen that

$$\frac{i_{xx}}{(i_{yy}-i_{zz})^2} + \frac{i_{yy}}{(i_{zz}-i_{xx})^2} = \frac{i_3}{4/3\ \Delta^2},$$

and

$$\frac{i_{xx}}{(i_{yy}-i_{zz})^2} - \frac{i_{yy}}{(i_{zz}-i_{xx})^2} = \frac{-\sqrt{i_3^2 - \frac{16}{3}\Delta^2}}{4/3\ \Delta^2}.$$

Lastly, $m^{\frac{1}{2}} = s_1 s_2 s_3$, and

$$\eta_{\hat{3},1} = \frac{12\Delta(s_2^2 - s_3^2)}{9\left[i_3^2 - \frac{16}{3}\Delta^2\right]^{1/2} s_1} \qquad \eta_{\hat{3},2} = \frac{12\Delta(s_3^2 - s_1^2)}{9\left[i_3^2 - \frac{16}{3}\Delta^2\right]^{1/2} s_2}$$

and

$$\eta_{\hat{3},3} = \frac{12\Delta(s_1^2 - s_2^2)}{9\left[i_3^2 - \frac{16}{3}\Delta^2\right]^{1/2} s_3}.$$

Before writing the final equations of motion, a slight modification of equation (32) must be discussed. It is clear from Table II that an expression of the form

$$\sum_{abc} G(abc) \sum_{\alpha\beta\gamma} \sqrt{\frac{1}{c}}\,\Gamma(abc;\alpha\beta\gamma) V_{a\alpha}(s) V_{b\beta}(t) V_{c\gamma}(\underline{r})$$

can only represent a potential $V = \sum_{i<j} V_{ij}$ in which every term $V_{ij}$ contains the product of spin and isospin operators.

But in the most general potential (2), terms containing only spin operators, terms containing only isospin operators, and terms containing neither are all present. These terms can be readily taken care of, for it is easily shown, at least for three and four nucleons, that the Majorana operator

$$V_M = \sum_{i<j} V_M(r_{ij})(1 + \underline{\sigma}_i \cdot \underline{\sigma}_j)(1 + \underline{\tau}_i \cdot \underline{\tau}_j) = \sum_{abc} G(abc) \sum_{\alpha\beta\gamma}$$

$$X \sqrt{\frac{1}{c}} \Gamma(abc;\alpha\beta\gamma) V'_{a\alpha}(s) V'_{b\beta}(t) \overset{M}{V}_{c\gamma}(r) \quad , \qquad (33)$$

where $\overset{M}{V}_{11}(r) = \dfrac{1}{\left[\frac{N(N-1)}{2}\right]^{\frac{1}{2}}} \sum_{i<j} V_M(r_{ij})$ etc., and

$$V'_{a\alpha} = V_{a\alpha} + \sqrt{\frac{N(N-1)}{2}}\, \delta_{a1} \ .$$

It is also easily shown that the operator

$$V_F = \sum_{i<j} V_F(r_{ij})(1 + \underline{\sigma}_i \cdot \underline{\sigma}_j)(1 - \underline{\tau}_i \cdot \underline{\tau}_j) = -\sum_{abc} G(abc) \sum_{\alpha\beta\gamma} \sqrt{\frac{1}{c}}$$

$$X \ \Gamma(abc;\alpha\beta\gamma) V'_{a\alpha}(s) V''_{b\beta}(t) \overset{F}{V}_{c\gamma}(r) \ , \qquad (34)$$

where $V''_{b\beta}(t) = V_{b\beta}(t) - \sqrt{\frac{N(N-1)}{2}}\, \delta_{b1}$. If we expand the left-hand sides of equations (33) and (34), we see that they both contain terms of the type found in the most general potential (equation (2)) but they each have a relationship between the various terms. By subtracting (34) from (33), adding a term of the form

$$V_{MF} = \sum_{i<j} V_{MF}(r_{ij})\sigma_i \cdot \sigma_j \tau_i \cdot \tau_j = \sum_{abc} G(abc) \sum_{\alpha\beta\gamma} \sqrt{\frac{1}{c}}$$

$$X \ \Gamma(abc;\alpha\beta\gamma) V_{a\alpha}(s) V_{b\beta}(t) \overset{MF}{V}_{c\gamma}(r) \ ,$$

and adding a term of the form $V_P = \sum_{i<j} V_P(r_{ij}) = \sqrt{\frac{N(N-1)}{2}}$ X $\overset{P}{V}_{11}(r)$, a general potential of the form (2) can be written. Another way of saying this is that if the terms of the potential (2) are broken up in the following manner:

$$V_c(r_{ij}) = V_P(r_{ij}) + V_M(r_{ij}) - V_F(r_{ij})$$

$$V_\sigma(r_{ij}) = V_M(r_{ij}) - V_F(r_{ij})$$

$$V_\tau(r_{ij}) = V_M(r_{ij}) + V_F(r_{ij})$$

$$V_{\sigma\tau}(r_{ij}) = V_{MF}(r_{ij}) + V_F(r_{ij}) + V_M(r_{ij}) \ ,$$

then the potential is expanded as

$$V = V_P + \sum_{abc} G(abc) \sum_{\alpha\beta\gamma} \sqrt{\frac{1}{c}} \Gamma(abc;\alpha\beta\gamma)[V_{a\alpha}(s) V_{b\beta}(t) \overset{MF}{V}_{c\gamma}(\underline{r})$$

$$+ V'_{a\alpha}(s) V'_{b\beta}(t) \overset{M}{V}_{c\gamma}(\underline{r}) + V'_{a\alpha}(s) V''_{b\beta}(t) \overset{F}{V}_{c\gamma}(\underline{r})] . \qquad (35)$$

Along with the $V'_{a\alpha}$ and $V''_{a\alpha}$ go $\lambda'(aa'a') = \lambda(aa'a') + \sqrt{\frac{N(N-1)}{2}}\,\delta_{a1}$ and $\lambda''(aa'a') = \lambda(aa'a') - \sqrt{\frac{N(N-1)}{2}}\,\delta_{a1}$.

With these changes, equations (32) become, for three nucleons,

$$\Big\{\frac{-\hbar^2}{2M}\Big[2\Big(\frac{\partial^2}{\partial s_1^2} + \text{c.p.}\Big) + \Big(2\cos\vartheta_3 \frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2} + \text{c.p.}\Big)$$

$$+ 4\Big(\frac{1}{s_1}\frac{\partial}{\partial s_1} + \text{c.p.}\Big)\Big] + \frac{\hbar^2}{4M}(\ell(\ell+1)-m''^2)\frac{3}{4}i_3\Delta^{-2}$$

$$+ \frac{i_3}{(i_3^2 - \frac{16}{3}\Delta^2)}\,\frac{m''^2\hbar^2}{2M} - \frac{m''\hbar}{M}\,\frac{12\Delta}{9\left[i_3^2 - \frac{16}{3}\Delta^2\right]^{1/2}}$$

$$\times \Big[\Big(s_2^2 - s_3^2\Big)\frac{\partial}{\partial s_1} + \text{c.p.}\Big)\Big] + \frac{i\hbar^2 m''}{2M}\sum_\sigma \frac{1}{s_1 s_2 s_3}\Big(\frac{\partial}{\partial s_\sigma} s_1 s_2 s_3 \eta_{\hat{3},\sigma}\Big)$$

$$+ V_P(s_\sigma) - E'\Big\}\varphi_{m''}^{c''\gamma''}(s_\sigma) - \frac{\sqrt{i_3^2 - \frac{16}{3}\Delta^2}}{4/3\;\Delta^2}\,\frac{\hbar^2}{8M}$$

$$\times \Big\{\sqrt{(\ell+m''+2)(\ell-m''-1)(\ell+m''+1)(\ell-m'')}\;\varphi_{m''+2}^{c''\gamma''}(s_\sigma)$$

$$+ \sqrt{(\ell-m''+2)(\ell+m''-1)(\ell-m''+1)(\ell+m'')}\;\varphi_{m''-2}^{c''\gamma''}\Big\}$$

$$+ \Big[\sum_{abc} G(abc) \sum_{c'\gamma\gamma'} [a'b'c'']^{\frac{1}{2}}\Delta\begin{pmatrix} a & b & c \\ a' & b' & \bar{c}' \\ a' & b' & \bar{c}'' \end{pmatrix}\{\lambda\lambda V_{c\gamma}\}$$

$$\times \Gamma(cc'c'';\gamma\gamma'\gamma'')\varphi_{m''}^{c'\gamma'}(s_\sigma)\Big] = 0 \quad . \qquad (36)$$

where c.p. stand for cyclic permutations of 1, 2, 3 and

$\{\lambda\lambda V_{c\gamma}\}$ stands for the expression $\{\lambda(aa'a')\lambda(bb'b')\overset{MF}{V}_{c\gamma}(s_\sigma)$ $+ \lambda'(aa'a')\lambda'(bb'b')\overset{M}{V}_{c\gamma}(s_\sigma) + \lambda'(aa'a')\lambda''(bb'b')\overset{F}{V}_{c\gamma}(s_\sigma)\}$.

These equations so far have not specified the identity of the nucleons, that is, they describe all three-nucleon problems. We will now write down the equations for all possible states of a three-nucleon system; first, however, we will need to introduce some abbreviated notation. Having written the explicit equations (36), we shall now retreat to the practicality of a shorthand symbolism, which, upon comparison with (36), is obvious.

$$(T(\ell,m'') + V_P(s_\sigma) - E')\varphi_{m''}^{c''\gamma''}(s_\sigma) - f(s_\sigma,\ell,m'')\varphi_{m''+2}^{c''\gamma''}$$
$$- f(s_\sigma,\ell,-m'')\varphi_{m''-2}^{c''\gamma''} + \sum_{abc} G(abc) \sum_{c'\gamma\gamma'} [a'b'c'']^{\frac{1}{2}}\Delta\begin{pmatrix} a & b & c \\ a' & b' & \bar{c}' \\ a' & b' & \bar{c}'' \end{pmatrix}$$
$$\times \{\lambda\lambda V_{c\gamma}\}\Gamma(cc'c'';\gamma\gamma'\gamma'')\varphi_{m''}^{c'\gamma'}(s_\sigma) = 0. \qquad (36')$$

Now the simplest case is the one for which the total spin of the system is 3/2 and the total isospin is 3/2. In this case the spatial wave function itself is antisymmetric with respect to an interchange of any two nucleons. The isospin 3/2 ($t = 3/2$) state corresponds either to a trineutron, a triproton, a triton (in a $t = 3/2$ state), or a Helium three nucleus (in a t = 3/2 state). None of these states of three nucleons is known to exist in a bound state, but this does not imply that it is useless to write the equations. Although the primary concern of this study is with bound states, nothing in the analysis has precluded its application to scattering. Thus the equations that follow describe the scattering of three nucleons in the various spin and isospin states. For spin 3/2 and isospin 3/2 we have the equation

$$(T(\ell,m'') + V_P(s_\sigma) - E')\varphi^{\overline{1}1}_{m''}(s_\sigma) - f(s_\sigma,\ell,m'')\varphi^{\overline{1}1}_{m''+2}$$

$$- f(s_\sigma,\ell,-m'')\varphi^{\overline{1}1}_{m''-2} + [\sqrt{3}\,\overset{MF}{V}_{11}(s_\sigma) + 4\sqrt{3}\,\overset{M}{V}_{11}(s_\sigma)]\varphi^{\overline{1}1}_{m''} = 0.$$

The next case of interest concerns the same nuclei as the previous one, but this time in a spin $\frac{1}{2}$ state. The governing equations are

$$(T(\ell,m'') + V_P(s_\sigma) - E' - \sqrt{3}\,\overset{MF}{V}_{11} - \sqrt{6}\,\overset{MF}{V}_{21} - 2\sqrt{6}\,\overset{M}{V}_{21})\varphi^{21}_{m''}$$

$$-f(s_\sigma,\ell,m'')\varphi^{21}_{m''+2} - f(s_\sigma,\ell,-m'')\varphi^{21}_{m''-2}$$

$$+ (\sqrt{6}\,\overset{MF}{V}_{22} + 2\sqrt{6}\,\overset{M}{V}_{22})\varphi^{22}_{m''} = 0,$$

and

$$(T(\ell,m'') + V_P - E' - \sqrt{3}\,\overset{MF}{V}_{11} + \sqrt{6}\,\overset{MF}{V}_{21} + 2\sqrt{6}\,\overset{M}{V}_{21})\varphi^{22}_{m''}$$

$$-f\varphi^{22}_{m''+2} - f\varphi^{22}_{m''-2} + (\sqrt{6}\,\overset{MF}{V}_{22} + 2\sqrt{6}\,\overset{M}{V}_{22})\varphi^{21}_{m''} = 0.$$

The third possibility is $s = 3/2$ and $t = \frac{1}{2}$, which describes either the triton or the helium three nucleus in a state with spin 3/2. The equations are

$$\{T(\ell,m'') + V_P - E' - \sqrt{3}\,\overset{MF}{V}_{11} - 4\sqrt{3}\,\overset{F}{V}_{11} - \sqrt{6}\,\overset{MF}{V}_{21} - 2\sqrt{6}\,\overset{M}{V}_{21}$$

$$-2\sqrt{6}\,\overset{F}{V}_{21}\}\varphi^{21}_{m''} - f\varphi^{21}_{m''+2} - f\varphi^{21}_{m''-2}$$

$$+(\sqrt{6}\,\overset{MF}{V}_{22} + 2\sqrt{6}\,\overset{M}{V}_{22} + 2\sqrt{6}\,\overset{F}{V}_{21})\varphi^{22}_{m''} = 0,$$

and

$$\{T(\ell,m'') + V_P - E' - \sqrt{3}\,\overset{MF}{V}_{11} - 4\sqrt{3}\,\overset{F}{V}_{11} + \sqrt{6}\,\overset{MF}{V}_{21} + 2\sqrt{6}\,\overset{M}{V}_{21}$$

$$+2\sqrt{6}\,\overset{F}{V}_{21}\}\varphi^{22}_{m''} - f\varphi^{22}_{m''+2} - f\varphi^{22}_{m''-2}$$

$$+(\sqrt{6}\,\overset{MF}{V}_{22} + 2\sqrt{6}\,\overset{M}{V}_{22} + 2\sqrt{6}\,\overset{F}{V}_{21})\varphi^{21}_{m''} = 0.$$

The last possibility is $s = \frac{1}{2}$ and $t = \frac{1}{2}$, which describes the same nuclei as the previous case, but in a state with spin $\frac{1}{2}$. The equations are greater in number than in any of the previous cases, and, unfortunately perhaps, these equations describe the ground state of the triton and the helium three nucleus. They are:

$$\{T(\ell,m'') + V_P - E' + \sqrt{3}\,\overset{MF}{V}_{11} - 4\sqrt{3}(\overset{MF}{V}_{11} + \overset{M}{V}_{11} + \overset{F}{V}_{11})\}\,\varphi^{11}_{m''}$$

$$- f\varphi^{11}_{m''+2} - f\varphi^{11}_{m''-2} - 2\sqrt{3}\,\overset{F}{V}_{21}\,\varphi^{21}_{m''} - 2\sqrt{3}\,\overset{F}{V}_{22}\,\varphi^{22}_{m''} = 0,$$

$$\{T(\ell,m'') + V_P - E' + \sqrt{3}\,\overset{MF}{V}_{11} + 4\sqrt{3}(\overset{MF}{V}_{11} + \overset{M}{V}_{11} + \overset{F}{V}_{11})\}\,\varphi^{\bar{1}1}_{m''}$$

$$- f\varphi^{\bar{1}1}_{m''+2} - f\varphi^{\bar{1}1}_{m''-2} + (+2\sqrt{3}\,\overset{MF}{V}_{22} - 2\sqrt{3}\,\overset{F}{V}_{22})\,\varphi^{21}_{m''}$$

$$+(-2\sqrt{3}\,\overset{MF}{V}_{21} - 2\sqrt{3}\,\overset{F}{V}_{21})\,\varphi^{22}_{m''} = 0,$$

$$\{T(\ell,m'') + V_P - E' + \sqrt{3}\,\overset{MF}{V}_{11} - 2\sqrt{3}\,\overset{F}{V}_{21}\}\,\varphi^{21}_{m''} - f\varphi^{21}_{m''+2} - f\varphi^{21}_{m''-2}$$

$$+2\sqrt{3}\,\overset{F}{V}_{22}\,\varphi^{22}_{m''} = 0,$$

and

$$\{T(\ell,m'') + V_P - E' + \sqrt{3}\,\overset{MF}{V}_{11} + 2\sqrt{6}\,(\overset{MF}{V}_{21} + \overset{M}{V}_{21} + \overset{F}{V}_{21})\}\varphi^{22}_{m''}$$

$$-f\varphi^{22}_{m''+2} - f\varphi^{22}_{m''-2} + 2\sqrt{6}\,(\overset{MF}{V}_{22} + \overset{M}{V}_{22} + \overset{F}{V}_{22})\varphi^{21}_{m''}$$

$$-4\sqrt{3}\,\overset{MF}{V}_{21}\,\varphi^{\overline{1}1}_{m''} = 0.$$

These equations complete the set of equations describing all the possible states of three nucleons. While there seems to be no great hope of solving them analytically, with the use of computers it may be possible to perform calculations on the three-nucleon system.

The three-body problem is rather special in that the geometry is planar, and the kinetic energy operator simplifies considerably. In order to exhibit the kind of equation more likely to be encountered when working with greater numbers of nucleons, we write here an equation for the $s = 2$, $t = 1$ state of four nucleons. This state is one which is presumably a possible state for a lithium four, helium four, or hydrogen four nucleus. Again, since none of these nuclei is known to exist in a bound state with spin 2, the equations have applicability to the scattering of four nucleons.

$$(T(\ell,m'') + V_P - E')\varphi^{\overline{3}1}_{m''}(\xi_\sigma) - f(\xi_\sigma,\ell,m'')\varphi^{\overline{3}1}_{m''+2}(\xi_\sigma)$$

$$-f(\xi_\sigma,\ell,m'')\varphi^{\overline{3}1}_{m''-2}(\xi_\sigma) - \frac{\hbar}{2M}\sqrt{(\ell+m''+1)(\ell-m'')}$$

$$\mathrm{X}\ [\sum_\sigma(\eta_{\hat{1},\sigma}+i\eta_{\hat{2},\sigma})\pi_\sigma - i\hbar\tfrac{1}{2}m^{-\frac{1}{2}}(\frac{\partial}{\partial\xi_\sigma}\ m^{\frac{1}{2}}(\eta_{\hat{1},\sigma}+i\eta_{\hat{2},\sigma}))]\varphi^{\overline{3}1}_{m''+1}$$

$$-\frac{\hbar}{2M}\sqrt{(\ell-m''+1)(\ell+m'')}[\sum_\sigma(\eta_{\hat{1},\sigma}-i\eta_{\hat{2},\sigma})\pi_\sigma$$

$$-i\hbar\tfrac{1}{2}m^{-\frac{1}{2}}(\frac{\partial}{\partial\xi_\sigma}\ m^{\frac{1}{2}}(\eta_{\hat{1},\sigma}-i\eta_{\hat{2},\sigma}))]\varphi^{\overline{3}1}_{m''-1}$$

$$+\frac{4}{\sqrt{3}}[-\tfrac{1}{2}\overset{MF}{V}_{11} - \overset{M}{V}_{11} - \overset{F}{V}_{11} + \sqrt{2}\,\overset{MF}{V}_{31} + 2\sqrt{2}\,\overset{M}{V}_{31} + 2\sqrt{2}\,\overset{F}{V}_{31}]\varphi^{\overline{3}1}_{m''}$$

$$-\frac{2}{3}\sqrt{6}\,[\overset{MF}{V}_{32} + 2\overset{M}{V}_{32} + 2\overset{F}{V}_{32}]\varphi^{\overline{3}2}_{m''}$$

$$-\frac{2}{3}\sqrt{6}\,[\overset{MF}{V}_{33} + 2\overset{M}{V}_{33} + 2\overset{F}{V}_{33}]\varphi^{\overline{3}3}_{m''} = 0.$$

Equations similar in appearance exist for $\varphi^{\overline{3}2}_{m''}$ and $\varphi^{\overline{3}3}_{m''}$, but little is to be gained by writing them here.

It is clear from the coupled set of differential equations that result from the elimination of the spin, isospin, and the angular dependences, that the price one must pay for reducing the number of variables describing the state of the system is increased complication of the equations of motion. As we have seen, elimination of the spin and isospin dependences couples functions of different permutation symmetry, while elimination of the angular dependences couples functions of different projections of the angular momentum on the body-fixed z-axis. Further progress in the analytical solution of the problem seems to be an extremely difficult task at best. The equations (32) are exact (insofar as the initial assumptions are correct); they involve no approximations, and it may be the case that suitable approximations will allow further insight into the nature of the solutions of the equations.

The application of numerical techniques with high-speed computing machines will probably provide the most practical solution of the N-nucleon equations. Through their use, properties (such as the magnetic moment and the energy and its dependence on the angular momentum) of the bound states of N-nucleon systems may be calculated.

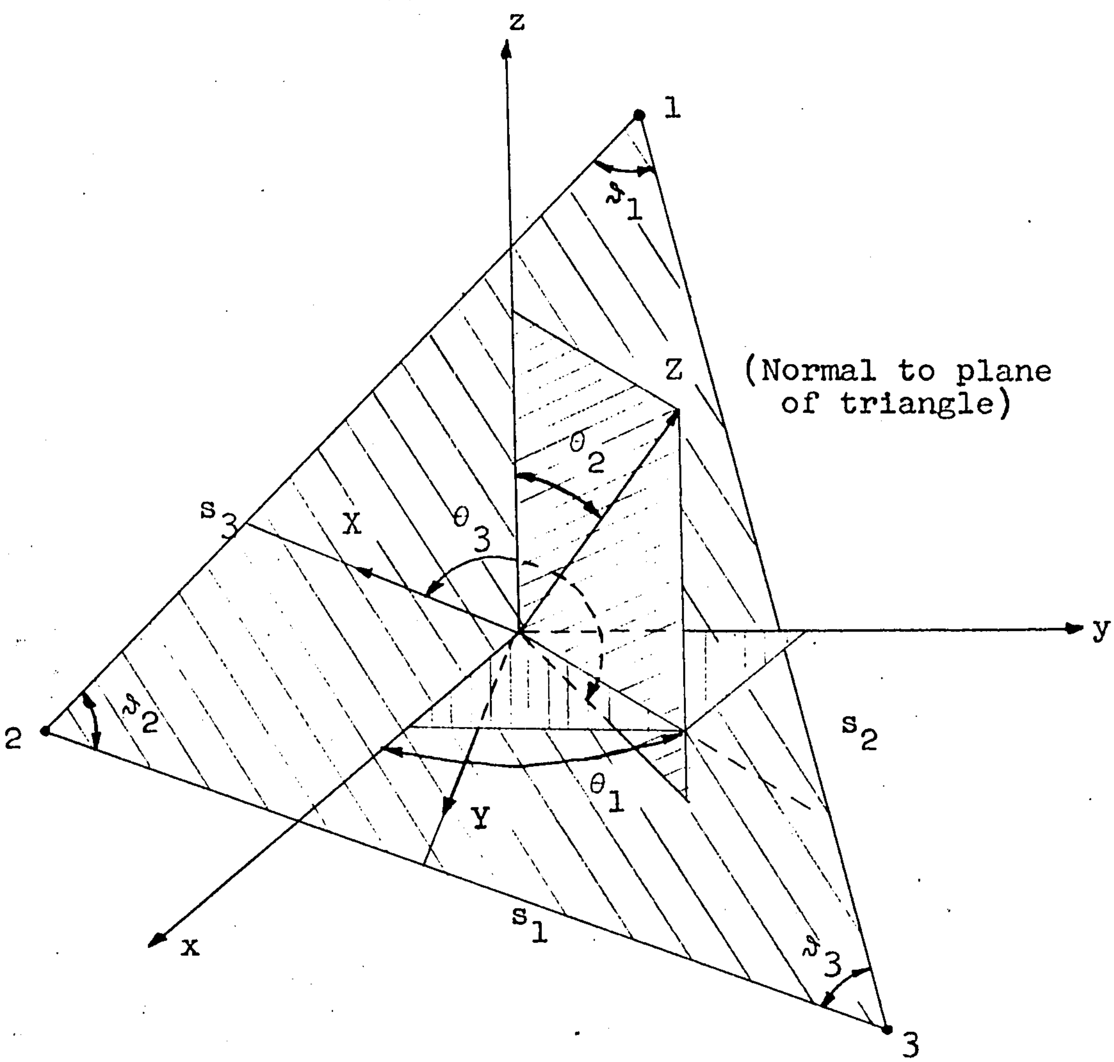


Figure 1. Geometry of the Three-Body Problem.

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