"Best Possible Inequalities for the Lowest Points of the Fundamental Domains of the Hilbert Modular Groups for $R(\sqrt{5})$ and $R(\sqrt{2})$"

by

Robert Henry DeVore

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I hereby recommend that this dissertation prepared under my direction by Robert H. DeVore

entitled The Lowest Points of the Fundamental Domains of the Best Possible Inequalities for Hilbert Modular Groups for $R(\sqrt{5})$ and $R(\sqrt{2})$

be accepted as fulfilling the dissertation requirement of the degree of Doctor of Philosophy

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After inspection of the dissertation, the following members of the Final Examination Committee concur in its approval and recommend its acceptance:*

M. S. Cheema	April 28, 1964
D. E. Myers	4/29/64
D. Tufan	April 28, 1964
L. J. Mordell	April 28, 1964

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BEST POSSIBIE INEQUALITIES FOR THE LOWEST POINTS OF THE FUNDAMENTAL DOMAINS FOR THE HILBERT MODULAR GROUPS FOR $R(\sqrt{5})$ AND $R(\sqrt{2})$

Robert Henry DeVore, Ph. D.

The University of Arizona, 1964

Director: Harvey Cohn

The principal result is $(\mathrm{Im}z)(\mathrm{Im}z') = yy' > \sqrt{5}/4 (.56)$ for (z,z') in the fundamental domain of the Hilbert Modular Group for $Q(5^{1/2})$. It is an improvement of the result $yy' \geq .54$ of Götzky and Siegel (Math. Ann., 1928). The new result is also best possible.

Using a result of H. Cohn (Amer. J. Math., 1960), namely $yy' \geq 4/10$, it suffices to show that for every (z,z') with $1/\varepsilon_o^2 \leq y'/y \leq \varepsilon_o^2$ and $4/10 \leq yy' \leq \sqrt{5}/4$ there exist $z_\nu = -1/(z - \nu)$ and $z_\nu' = -1/(z' - \nu')$ where ν and ν' are conjugate integers such that $y_\nu y'_\nu \geq \sqrt{5}/4$. A pair (z_c,z_c) was found for which $y_c y'_c = \sqrt{5}/4$, $y'_c/y_c = \varepsilon_o$ and $N|z_c| = N|z_c - 1| = N|z_c - \varepsilon_o| = N|z_c + 1 - \varepsilon_o| = 1$. The region whose boundary is $(x^2 + y_c^2)(x'^2 + y'^2_c) = 1$ and its translates to the lattice points generated by $(1,1)$ and $(\varepsilon_o,\varepsilon_o')$ in the x,x'-plane were shown to cover the plane. This and the fact that the inequality was shown to need

verification only on the lines $y'/y = \varepsilon_o^{\pm 1}$ between $yy' = \sqrt{5}/4$ and $yy' = 4/10$ yielded the desired result.

Attempts to prove similar inequalities for the field $Q(2^{1/2})$ are made, but they would not prove successful as the method now stands.

1. Introduction

The Hilbert modular group H for a quadratic field $Q(m^{\frac{1}{2}})$, (m square-free and greater than 1), is the group of transformations

$$(1) \qquad (z,z') \to T(z,z') = \left(\frac{\alpha z+\beta}{\gamma z+\delta} , \frac{\alpha' z'+\beta'}{\gamma' z'+\delta'} \right)$$

where α' is the conjugate of α, $\operatorname{Im} z = y > 0$, $\operatorname{Im} z' = y' > 0$, $\alpha\delta - \beta\gamma = \varepsilon$ (= totally positive unit), and the coefficients lie in the ring of quadratic integers in $Q(m^{\frac{1}{2}})$.

A fundamental domain R for the Hilbert modular group H is defined according to the general method of Siegel [7]. It is a set of points containing at least one transform of every point (z,z') with $\operatorname{Im} z > 0$ and $\operatorname{Im} z' > 0$. Also if any two points in R are equivalent under transformation (1) then they will lie on the boundary of R. (The traditional treatment for one complex variable can be found in Gunning [4].)

We restrict ourselves to quadratic fields $Q(m^{\frac{1}{2}})$, (m square-free and greater than 1), for which the class number is one [6] in order that the fundamental domain should have a "floor" above (say) $\operatorname{Im} z = \operatorname{Im} z' =$ constant.

The fundamental domain for H is found by a direct generalization of the method used in obtaining the fundamental domain for the rational case.

Little is known about the shape of the fundamental domain, but the most important feature is the lowest point because of its immediate application to quadratic forms (see section 6). (Here "lowest point" refers to the minimum value of $\mathrm{Imz} \cdot \mathrm{Imz}'$ for (z,z') in the fundamental domain.)

For a quadratic field $Q(m^{\frac{1}{2}})$ with discriminant d and class number one [6], Blumenthal [1] proved that the lowest point satisfied $\mathrm{Imz} \cdot \mathrm{Imz}' \geq \frac{1}{2d}$.

For $Q(5^{1/2})$ the best result for the lowest point up to now was $\mathrm{Im}\, z \cdot \mathrm{Im}\, z' \geq .54\ldots$ [3].

The principal new result is the discovery that the best possible result on the lowest point for $Q(5^{1/2})$ is
$\mathrm{Imz} \cdot \mathrm{Imz}' \geq 5^{1/2}/4 = .56\ldots$.
It occurs for

$$(2) \qquad z_c = \frac{1 + \sqrt{5}}{4} + i\,\frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$$

and

$$z_c' = \frac{1 - \sqrt{5}}{4} + i\,\frac{1}{2}\sqrt{\frac{5 + \sqrt{5}}{2}} \quad .$$

Special properties of $Q(5^{1/2})$ are used which are not present for $Q(2^{1/2})$.

2. One Dimensional (i. e. Rational) Results

For the rational case the Hilbert modular group is the group of transformations

$$z \to Tz = \frac{az + b}{cz + d} \tag{3}$$

where $Imz = y > 0$, $ad - bc = 1$ and the coefficients are rational integers. All transformations with $c = 0$ are of the form $T_\infty z = z + b$, and form the subgroup called H_∞ which leaves ∞ fixed.

We define as in [2] $N_{c,d}(z) = |cz + d|^2$ which is used in the identity

$$Im\ Tz = \frac{(ad - bc)\ Im\ z}{N_{c,d}(z)}. \tag{4}$$

We now define as in [2] the valence of z by the equation

$$v(z) = \sup(ImTz) \tag{5}$$

for $T \in H$. Thus $v(Tz) = v(z)$ for $T \in H$ since H is a group.

The fundamental domain is now constructed with the help of the following lemmas:

LEMMA 1. For any fixed z there are only a finite number of pairs (c,d) for which $N_{c,d}(z)$ is less than any preassigned constant K.

Proof. The inequalities

$$K \geq N_{c,d}(z) = |cz+d|^2 \geq c^2y^2 \tag{6}$$

limit c to a finite number of possibilities and for each of these the inequalities

$$K \geq N_{c,d}(z) = |cz+d|^2 \geq (cx+d)^2 \tag{7}$$

limit d.

LEMMA 2. For any z a T exists for which $v(z) = \mathrm{Im}(Tz)$. This follows easily from lemma 1.

We next define R_o to be the infinite intersection (see Figure 1)

$$R_o = \bigcap_{(c,d)=1,c>0} \{z | N_{c,d}(z) \geq 1\}. \tag{8}$$

LEMMA 3. A necessary and sufficient condition that $v(z) = \mathrm{Im}z$ is that $z \in R_o$.

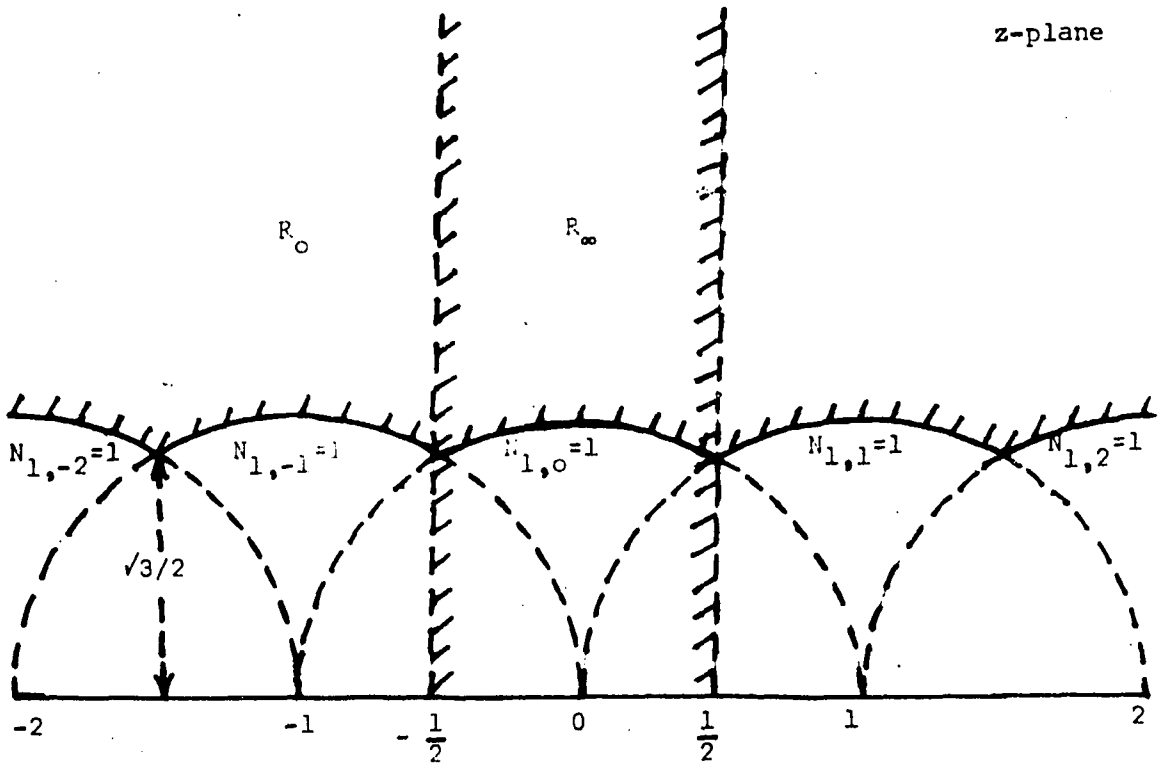

Figure 1

Fundamental Domain for the
One Variable Case ($R = R_o \cap R_\infty$)

Proof. If z is in R_o, then for any T, $\mathrm{Im}Tz \leq \mathrm{Im}z$ follows from equation (4). Conversely if $z \notin R_o$, then $N_{c,d}(z) < 1$ for some pair (c,d) with $(c,d) = 1$. Thus a T exists for which $v(z) \geq \mathrm{Im}Tz > \mathrm{Im}z$.

LEMMA 4. *If z_1 and z_2 are both in R_o and $z_1 = Tz_2$ then one or both of the following conditions hold:*

(9) $z_1 = T_\infty z_2$ *or*

(10) z_1 *and* z_2 *lie on the boundary of* R_o.

Proof. If $T \notin H_\infty$ then, since both z_2 and Tz_2 are in R_o, $\mathrm{Im}z_2 = \mathrm{Im}Tz_2$ by lemma (3). Hence $N_{c,d}(z_2) = 1$. Similar reasoning applied to z_1 shows $N_{c,d}(z_1) = 1$.

A fundamental domain for H can be taken as

(11) $$R_\infty : -1/2 \leq x \leq 1/2.$$

Then lemmas (1) through (4) combined yield the following:

THEOREM 1. *$R = R_o \cap R_\infty$ with boundary points identified under H is a fundamental domain for H.*

(Conventionally R is written R_o/H_∞ to indicate that R contains only one representative of each class of points in R_o under H_∞.)

3. The Floor for the One Dimensional Result.

Before discussing the floor it is desirable to refer to a well-known result [7]:

THEOREM 2 . A positive definite quadratic form in integral variables x,y with real (not necessarily integral) coefficients

$$\alpha x^2 + \beta xy + \gamma y^2$$

represents some number $\xi = G(x_o,y_o)$ satisfying

$$\xi \leq \sqrt{\frac{-\Delta}{3}} \quad ,\Delta = \beta^2 - 4\alpha\gamma \cdot \tag{12}$$

LEMMA 5. R_o can be expressed as

$$R_o = \bigcap_d \{z | N_{1,d}(z) \geq 1\}.$$

Proof. Define $F(c,d)$, a positive definite quadratic form with parameter z and variables c and d, to be

$$F(c,d) = N_{c,d}(z) = (cx+d)^2 + c^2y^2. \tag{13}$$

By theorem (2) there exists a pair (c,d) such that $0 < F(c,d) \leq \sqrt{\Delta/3} = 2y/\sqrt{3}$. c and d can be assumed relatively prime since otherwise their greatest common divisor could be factored out and the inequality would be improved. Thus for a fixed z there exists a $T \in H$ such that

(14) $$v(z) \geq \mathrm{Im}Tz = y/N_{c,d}(z) \geq \frac{\sqrt{3}}{2} = H_o \ .$$

Hence if $c>1/H_o$, i.e. if $c > 1$, then

(15) $$N_{c,d}(z) \geq c^2y^2 \geq c^2H_o^{\,2} > 1 \quad \text{for} \quad z \in R_o$$

and therefore $N_{c,d}(z) = 1$ can not be part of the boundary of R_o.

The floor of the fundamental domain is taken to be all points z for which $|z| = 1$ and $y \geq \sqrt{3}/2$.

4. Generalization to the Two Variable Case.

Let H_∞ be the subgroup of H leaving $z = \infty$ fixed. Then $T_\infty \in H_\infty$ is of the form $T_\infty z = \varepsilon z + \beta$ where β is a quadratic integer in $Q(m^{1/2})$ and ε is a totally positive unit. A basis for the integers in $Q(m^{1/2})$ is $[1,\omega]$ where $\omega = (d + d^{1/2})/2$ and

$$d = \begin{cases} m & \text{if } m \equiv 1(4) \\ 4m & \text{if } m \not\equiv 1(4) \end{cases} . \tag{16}$$

Two special subgroups of H_∞ are the translation subgroup ($\varepsilon = 1$) and the unit subgroup ($\beta = 0$) having fundamental domains S and E respectively where

$$S = \{(z,z') \mid |x-x'| < d^{1/2}/2,\ |\omega x' - \omega' x| < d^{1/2}/2\} \tag{17}$$

(the parallelogram determined by $(1,1)$ and (ω,ω') with center at the origin) and

$$E = \{(z,z') \mid 1/\varepsilon_+ \le y'/y \le \varepsilon_+\}$$

where ε_+ is the fundamental totally positive unit. R_∞, the fundamental domain for H_∞, is defined to be $S \cap E$ (with boundary points identified under H_∞).

Let $N(\alpha y) = (\alpha y)(\alpha' y')$, $S(\alpha z) = \alpha z + \alpha' z'$ and

$N_{\gamma,\delta}(z) = N|\gamma z + \delta|^2 = |\gamma z + \delta|^2 |\gamma' z' + \delta'|^2$.

$N_{\gamma\delta}(z) \equiv N_{\gamma_0\delta_0}(z)$ if γ, δ and γ_0, δ_0 have a unit as a factor of proportionality. Define $v(z) = \sup N(\mathrm{Im} Tz)$ for $T \in H$. Then

$$N(\mathrm{Im} Tz) = N(\mathrm{Im} z)/N_{\gamma,\delta}(z). \tag{18}$$

The inequalities in lemma 1 for the one variable case are replaced by

$$N_{\gamma,\delta}(z) \geq \begin{cases} N(\gamma)^2 \ N(\mathrm{Im} z)^2 \\ (\gamma y)^2 \ (|\delta'| - |\gamma' x'|)^2 \\ (\gamma' y')^2 \ (|\delta| - |\gamma x|)^2 \end{cases} \tag{19}$$

Also R_o in the one variable case is replaced by

$$R_o = \bigcap_{(\gamma,\delta)=1, \gamma \neq 0} \{z | N_{\gamma,\delta}(z) \geq 1\}. \tag{20}$$

Lemma 4 does not change going from the one variable to the two variable case.

THEOREM 3. The fundamental domain of H is $R = R_o \cap S \cap E$ identifying boundary points by H.

5. The Floor for the Two Variable Case.

The positive definite quadratic form in lemma 5 is replaced by

$$(21)\quad F(\gamma,\delta)=S|\gamma z+\delta|^2=(\gamma x+\delta)^2+\gamma^2y^2+(\gamma'x'+\delta')^2+\delta'^2y'^2 .$$

If γ and δ are replaced by $\gamma = \eta_1 + \eta_2\,\omega$ and $\delta=\eta_3+\eta_4\omega$, $F(\gamma,\delta)$ becomes a positive definite form in the rational integers η_1, η_2, η_3 and η_4. Using a result of Korkine and Zolatareff [5], we find there exists an integral quadruple $(\eta_1,\eta_2,\eta_3,\eta_4)$ such that

$$(22)\qquad 0 < F(\gamma,\delta) \le 2^{1/2}(dyy')^{1/2} \quad \text{where} \quad d=(\omega-\omega')^2$$

Since the geometric mean is less than or equal to the arithmetic mean,

$$(23)\qquad 2\,N_{\gamma,\delta}^{1/2}(z) \le S|\gamma z + \delta|^2 = F(\gamma,\delta).$$

Therefore for any z a pair (γ,δ) exists for which

$$(24)\qquad N(\mathrm{Im}Tz) = N(\mathrm{Im}z)/N_{\gamma,\delta}(z) \ge 2/d .$$

γ and δ can be assumed to be relatively prime, otherwise

a common factor could be factored out which would improve the inequality. Since the class number is assumed to be one there exists a pair (α,β) such that $Tz=(\alpha z+\beta)/\gamma z+\delta$ is a transformation ϵH and such that $N(\mathrm{Im}Tz) \geq 2/d$. In lemma 5 H_o is replaced by $H_o = H_o\{Q(m^{1/2})\} = \min V(z)$.

THEOREM 4. For the Hilbert modular group H in a quadratic field of discriminant d every z is equivalent to at least one Tz for which $N(\mathrm{Im}Tz) \geq 2/d$, i.e. $H_o\{Q(m^{1/2})\} \geq 2/d$.

LEMMA 5. R_o can now be replaced by

$$R_o = \bigcap_{(\gamma,\delta)=1,\ |N(\gamma)|\leq 1/H_o} \{z \mid N_{\gamma,\delta}(z) \geq 1\} . \tag{25}$$

Proof. If $|N(\gamma)| > 1/H_o$, then for all $z \in R_o$, $N(\mathrm{Im}\ z) = V(z) \geq H_o$, which implies $N_{\gamma,\delta}(z) > 1$. Thus $N_{\gamma,\delta}(z) = 1$ could not be part of the boundary of R_o if $|N(\gamma)| > 1/H_o$.

The floor of the fundamental domain R consists of points in $R \cap \partial R_o$. We say the floor is simple [2] if it is contained in a set of hypersurfaces $N_{\gamma,\delta}(z) = 1$ for which $|N(\gamma)| = 1$. It can be shown [3] that by a suitable choice of fundamental domain, the floor will consist of a single hypersurface $N|z| = 1$.

THEOREM 5. (Götzky) For $Q(\sqrt{5})$ the fundamental domain has a simple floor.

Proof. By theorem 4, $H_o \geq 2/5$. In $Q(\sqrt{5})$ there does not exist an integer γ for which $|N(\gamma)| = 2$. Thus the floor is simple.

Götzky showed that the floor of the fundamental domain can be taken as a single hypersurface, namely $|zz'|^2 = 1$. He used a difficult method which consisted essentially of approximating the x, x' cross-section of R by a parallelogram.

6. Applications to Quadratic Forms

Consider the following positive definite binary quadratic forms:

$$\alpha x^2 + \beta xy + \gamma y^2$$

(26)

$$\alpha' x'^2 + \beta' x'y' + \gamma' y'^2$$

where the coefficients are conjugate integers in $Q(m^{1/2})$, a real quadratic field of discriminant d and class number one.

We now define θ and θ' to be

$$\theta = \frac{-\beta + \sqrt{-\Delta}}{2\alpha}, \quad -\Delta = \beta^2 - 4\alpha\gamma < 0$$

(27)

$$\theta' = \frac{-\beta' + \sqrt{-\Delta'}}{2\alpha'}, \quad -\Delta' = \beta'^2 - 4\alpha'\gamma' < 0 .$$

Thus

(28)

$$\begin{cases} \mathrm{Im}\theta = \dfrac{\theta - \bar{\theta}}{2i} = \dfrac{\sqrt{\Delta}}{2\alpha} \\ \mathrm{Im}\theta' = \dfrac{\theta' - \bar{\theta}'}{2i} = \dfrac{\sqrt{\Delta'}}{2\alpha'} \end{cases} .$$

If (θ,θ') is in the fundamental domain (i.e., if the quadratic forms (26) are reduced) then

$$H_o \leq N(\mathrm{Im}\theta) = \frac{\sqrt{N(\Delta)}}{4\,N(\alpha)} \quad . \tag{29}$$

Squaring both sides of (29) and multiplying both sides by $16(N(\alpha))^2$ we obtain

$$N(\Delta) \geq (4H_o)^2 \quad \left(N(\alpha)\right)^2 \quad . \tag{30a}$$

Consider the (conjugate) positive definite binary quadratic forms (26) in $Q(m^{1/2})$ of discriminants $-\Delta$ and $-\Delta'$ in terms of equivalence under transformations

$$\begin{cases} x = \rho x_o + \sigma y_o \\ y = \lambda x_o + \mu y_o \end{cases} \qquad \begin{cases} x' = \rho' x'_o + \sigma' y'_o \\ y' = \lambda' x'_o + \mu' y'_o \end{cases}$$

with coefficients in $Q(m^{1/2})$ and determinants

$$\rho\mu - \lambda\sigma = \varepsilon > 0 \ , \quad \rho'\mu' - \sigma'\lambda' = \varepsilon' > 0.$$

Then α, for some equivalent form, will satisfy

$$N(\alpha) \leq \sqrt{N(\Delta)}/4H_o \ . \tag{30b}$$

7. Heuristic Derivation of Critical Value for $\sqrt{5}$ and attempt for $\sqrt{2}$

In the one variable fundamental domain the lowest point ($z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$) is a fixed point, z, satisfying the equation

$$z = Tz$$

where T is in the Hilbert group H. Here $Tz = \frac{(z-1)}{z}$.

For $Q(5^{1/2})$ we try fixed points hoping that, as in the one variable fundamental domain, the lowest point is a fixed point.

Set

$$z = \frac{\alpha z + \beta}{\gamma z + \delta} \tag{31}$$

where $\alpha\delta - \beta\gamma = \varepsilon$, a totally positive unit. Solving (31) for z and using $\alpha\delta - \beta\gamma = \varepsilon$ we obtain

$$\gamma z^2 + (\delta - \alpha)z - \beta = 0$$

$$z = \frac{(\alpha - \beta) + \sqrt{-\Delta}}{2\gamma}. \tag{32}$$

where

$$\begin{cases} \Delta = -(\delta - \alpha)^2 - 4\beta\gamma = 4\varepsilon - \tau^2 \\ \text{and} \quad \tau = \alpha + \delta . \end{cases}$$

In order that $\text{Im } z \neq 0$, $\text{Im } z' \neq 0$, we require

$$\Delta > 0 \quad , \quad \Delta' > 0$$

(33)

$$4\varepsilon > \tau^2, \quad 4\varepsilon' > \tau'^2 \ .$$

If (z, z') is taken to be in the fundamental domain we obtain

$$N(\text{Im } z) = \frac{\sqrt{N(\varepsilon - \tau^2/4)}}{N(\gamma)} \geq 4/10 (=2/d). \tag{34}$$

Since ε is a totally positive unit and the fundamental unit $\varepsilon_o = (1+\sqrt{5})/2$ has a negative norm, ε must be an even power of the fundamental unit. Thus ε can be incorporated into τ^2 and γ so that inequality (34) can be rewritten in the simpler form

$$N(\text{Im } z) = \frac{\sqrt{N(1 - \tau^2/4)}}{N(\gamma)} \geq 4/10 \ . \tag{35}$$

The following inequalities are a consequence of (35):

$$\begin{cases} |\tau| < 2 \\ |\tau'| < 2 \\ 0 \leq N(\tau) < 4. \end{cases} \tag{36}$$

These imply that τ is a unit or zero since $N(\tau)$ can not be equal to 2 or 3 (In $R(5^{1/2})$, 2 and 3 do not have factors). We try $\tau = \frac{1 + 5^{1/2}}{2}$ and we can make the choice $\alpha = \varepsilon_0$, $\beta = -1$, $\gamma = 1$, $\delta = 0$. It satisfies all the necesaary conditions and gives the value of z_c in section 1 above. For this value

$$(37) \qquad -1/z_c = z_c - \varepsilon_0$$

and

$$N(\text{Im } z_c) = \sqrt{5}/4 > 4/10.$$

When the procedure is carried out for $Q(2^{1/2})$ we replace (35) by

$$(38) \qquad N(\text{Im } z) \geq 1/4 (= 2/d).$$

We use an unpublished result of H. Cohn that the $\geq$ sign can be replaced by a $>$ sign in (38).

We next show that the only values of $N(\text{Im } z)$ for z a fixed point are $\geq 1/2$ or $\leq 1/4$. Specifically we replace (35) again by

(39) $$N(\text{Im } z) = \frac{\sqrt{N(1-\tau^2/4)}}{N(\gamma)} > \frac{1}{4}$$

Inequality (36) holds again, but this time

$$N(\tau) = 0, 1, 2, \quad \text{while} \quad N(\gamma) = 1, 2, 4, \cdots$$

We exclude the case where $N(\gamma) = 2$ and $N(\tau) = 1$, because then $\alpha\delta \equiv 0(\text{mod}\sqrt{2})$ (since $\alpha + \delta = \tau$) while $\beta\gamma \equiv 0(\text{mod}\sqrt{2})$ leading to a contradiction ($\alpha\delta - \beta\gamma = 1$). Thus the left hand side of (39) has no values $\geq 1/2$ or $\leq 1/4$.

Later on (§10 below), however, we show that a value of z_* exists for which

(40) $$1/4 < N(\text{Im } z_*) = .475 \cdots < 1/2 ,$$

hence the minimal valence is not produced by a fixed point.

8. The Main Result

The main new result, Theorem (below), is established by the following sequence of lemmas:

Lemma 6. *A zone, $(X^2 + a^2)(Y^2 + b^2) \leq 1$, is convex if $1/2 \leq ab \leq 1$.*

Proof. Let $X = ax$, $Y = by$ and $k = 1/ab$. We wish to show $(x^2 + 1)(y^2 + 1) = k^2$ is convex if $1 \leq k \leq 2$. Because of symmetry with respect to both axes we need only show that $D^2y \leq 0$ for $x \geq 0$, $y \geq 0$. Differentiating

$$(41) \qquad (x^2 + 1)(y^2 + 1) = k^2$$

once we obtain

$$(42) \qquad (x^2 + 1)y\, Dy + x(y^2 + 1) = 0.$$

Differentiating (42) once we obtain

$$(43) \qquad (x^2 + 1)\,[yD^2y + (Dy)^2] + 4xyDy + y^2 + 1 = 0.$$

Using equation (43) we see that $D^2y \leq 0$ is equivalent to

$$(44) \qquad (x^2 + 1)(Dy)^2 + 4xyDy + y^2 + 1 \geq 0.$$

Solving equation (42) for Dy and putting this value in equation (44) we obtain

$$(45) \qquad \frac{x^2(y^2 + 1)^2}{y^2(x^2 + 1)} - \frac{4x^2(y^2 + 1)}{x^2 + 1} + y^2 + 1 \geq 0.$$

After multiplying both sides of (45) by $y^2 (x^2 + 1)$ we obtain

$$(46) \qquad -2x^2y^2 + x^2 + y^2 \geq 0.$$

Let $1 + x^2 = \xi$ and $1 + y^2 = \eta$ $(\xi\eta = k^2)$. After making this change of variables and after simplifying, inequality (46) becomes

$$(47) \qquad -2\xi\eta + 3\xi + 3\eta - 4 \geq 0.$$

The minimum of the left side of (47), holding $\xi\eta$ constant, occurs for $\xi = \eta$. Putting $\xi = \eta$ in (47) we obtain

$$(48) \qquad -2\xi^2 + 6\xi - 4 \geq 0.$$

Since ξ and η are equal, $\xi = k$. Thus inequality (48) can be written

$$-2k^2 + 6k - 4 \geq 0. \tag{49}$$

This inequality is equivalent to

$$(k - 2)(k - 1) \leq 0, \tag{50}$$

from which the lemma follows .

Lemma 7. For every (z,z') with $(y,y') = (y_c,y'_c) = \left(\frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}, \frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}}\right)$ {or $(y,y') = (y'_c,y_c)$} there exists a pair of conjugate integers (ν, ν') such that

$$\text{Im}\left(\frac{-1}{z - \nu}\right) \cdot \text{Im}\left(\frac{-1}{z' - \nu'}\right) \geq \sqrt{5}/4 . \tag{51}$$

Proof. Inequality is the same as

$$\frac{y_c}{(x-\nu)^2 + y_c^2} \cdot \frac{y'_c}{(x'-\nu')^2 + y'^2_c} \geq \sqrt{5}/4 \tag{52}$$

which is equivalent to

$$[(x-\nu)^2 + y_c^2][(x'-\nu')^2 + y_c'^2] \leq 1 \tag{53}$$

because $y_c y_c' = \sqrt{5}/4$. Proving (17) is equivalent to showing that the zone $(x^2 + y_c^2)(x'^2 + y_c'^2) \leq 1$ and its translates to all the lattice points cover the x,x'-plane. Because the zones are convex, the fundamental parallelogram inside the zone in Figure 2 is entirely within the zone. The zones cover the plane because the parallelogram and its translates to all the lattice points cover the plane.

LEMMA 8. For any (x,x') and (y,y') such that $1/\varepsilon_0 \leq y'/y \leq \varepsilon_0$, $yy' = A$ where $4/10 \leq A \leq \sqrt{5}/4$,

$$(54) \qquad \frac{yy'}{(x^2 + y^2)(x'^2 + y'^2)} \geq \min \left\{ \frac{y_2 y_2'}{(x^2+y_2^2)(x'^2+y_2'^2)}, \frac{y_4 y_4'}{(x^2+y_4^2)(x'^2+y_4'^2)} \right\}$$

where $y'_2/y_2 = \varepsilon_0$, $y'_4/y_4 = 1/\varepsilon_0$, and $y_2 y_2' = y_4 y'_4 = A$.

Proof. Since $yy' = A$,

$$(55) \qquad \frac{yy'}{(x^2 + y^2)(x'^2 + y'^2)} = \frac{A}{(x^2 + y^2)(x'^2 + A^2/y^2)} = f(y) .$$

For y in the interval determined by $1/\varepsilon_0 \leq y'/y \leq \varepsilon_0$ and $yy' = A$, $f(y)$ takes on its minimum for y determined by $y'/y = \varepsilon_0$ and $yy' = A$, or for y determined by $y'/y = 1/\varepsilon_0$ and $yy' = A$. This follows from the fact that $D^2 f(y) < 0$.

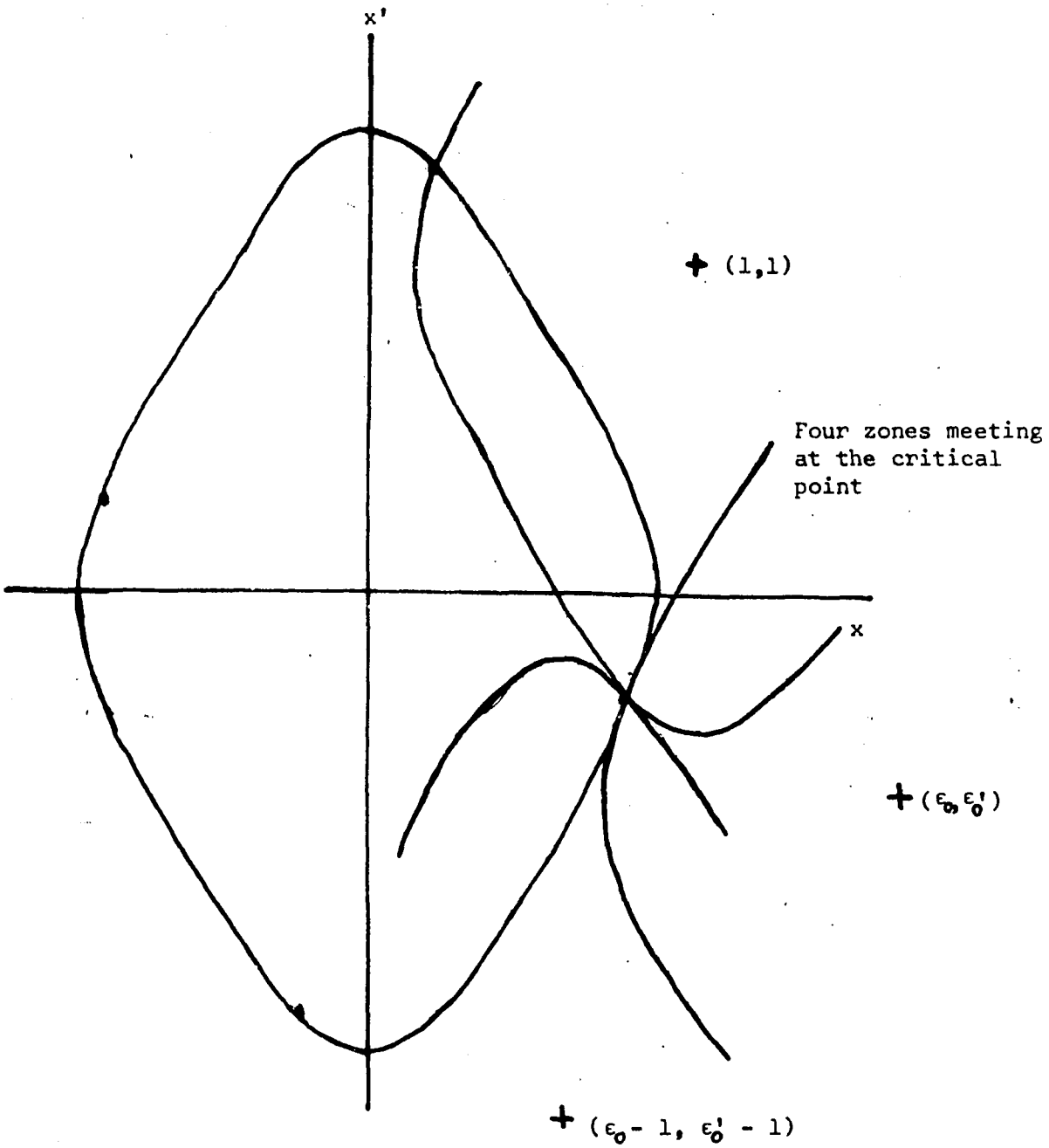

Figure 2

$$[(x-\nu)^2 + y_c^2]\,[(x'-\nu')^2 + y_c'^2] = 1$$

Lemma 9. For any (x,x'), $y'/y = \varepsilon_0$ (or $1/\varepsilon_0$) and $4/10 \leq yy'(=A) \leq \sqrt{5}/4$,

$$(56) \qquad \frac{yy'}{[(x-\nu)^2 + y^2][(x'-\nu')^2 + y'^2]} \geq \frac{y_c y'_c}{[(x-\nu)^2 + y_c^2][(x'-\nu')^2 + y'^2_c]}$$

where (ν,ν') is chosen as in lemma 7.

Proof. After performing cross multiplication and using the fact that $y'/y = \varepsilon_0$ we see that inequality (56) is equivalent to

$$(57) \qquad (x-\nu)^2 (x'-\nu')^2 \leq (yy')(y_c y'_c) = A\sqrt{5}/4$$

where $4/10 \leq A \leq \sqrt{5}/4$ and (ν,ν') is chosen so that $[(x-\nu)^2 + y_c^2]\,[(x'-\nu')^2 + y'^2_c] \leq 1$. Thus to prove inequality (57) it suffices to show that the system

$$(58) \qquad \begin{aligned} &(x-\nu)^2 (x'-\nu')^2 = A\sqrt{5}/4 \\ &[(x-\nu)^2 + y_c^2][(x'-\nu')^2 + y'^2_c] = 1 \end{aligned}$$

has no real solution for $4/10 \leq A \leq \sqrt{5}/4$. Clearly it suffices to show that the system has no real solution for the worst case, which occurs for $A = 4/10$. Substitution of

$(x'-\nu')^2$ from the first equation of (58) into the second equation leads to the following quadratic equation in $w = (x-\nu)^2$:

$$(59) \qquad w^2(4y'_c{}^2) + w(A\sqrt{5} - 11/4) + (A\sqrt{5}\, y_c^2) = 0.$$

The discriminant D of this equation satisfies the inequality

$$D = (A\sqrt{5} - 11/4)^2 - 5A\sqrt{5} \leq 0$$

if A lies between the two positive roots of $D = 0$, shown here:

$$(60) \qquad (21 - 8\sqrt{5})/4\sqrt{5} \leq A \leq (21 + 8\sqrt{5})/4\sqrt{5} .$$

By direct computation the lowest root has the value .359··· which is less than the value $A = 4/10$.

Blumenthal's improvable bound [1] for the lowest point $(1/2d = 1/10)$ is clearly much too small.

Lemma 10. For $\varepsilon_o \leq y'/y \leq \varepsilon_o^2$ {or $1/\varepsilon_o^2 \leq y'/y \leq 1/\varepsilon_o$} and $4/10 \leq yy' \leq \sqrt{5}/4$ we have

$$(61)\qquad \frac{yy'}{[(x-\nu)^2+y^2][(x'-\nu')^2+y'^2]} \geq \frac{y_3y_3'}{[(x^*-\nu^*)^2+y_3^2][(x^{*\prime}-\nu^{*\prime})^2+y_3'^2]}$$

where $y_3y_3' = yy'$, $y_3' = y_3$, $(\varepsilon_o x, \varepsilon_o' x') = (x^*, x^{*\prime})$ and $(\varepsilon_o y, \varepsilon_o' \nu') = (\nu^*, \nu^{*\prime})$.

Proof. The following inequality

$$(62)\qquad \frac{yy'}{[(x-\nu)^2+y^2][(x'-\nu')^2+y^2]} \geq \min \left\{ \begin{array}{l} \dfrac{y_1y_1'}{[(x-\nu)^2+y_1^2][(x'-\nu')^2+y_1'^2]} \\ \\ \dfrac{y_5y_5'}{[(x-\nu)^2+y_5^2][(x'-\nu')^2+y_5'^2]} \end{array} \right\}$$

where $4/10 \leq y_1y_1' \leq \sqrt{5}/4$, $y_1'/y_1 = \varepsilon_o^2$, $4/10 \leq y_5y_5' \leq \sqrt{5}/4$ and $y_5'/y_5 = 1/\varepsilon_o^2$ follows by the reasoning used in lemma 9. Suppose the minimum is

$$\frac{y_1y_1'}{[(x-\nu)^2+y_1^2][(x'-\nu')^2+y_1'^2]} \text{ or } \left\{ \frac{y_5y_5'}{[(x-\nu)^2+y_5^2][(x'-\nu')^2+y_5'^2]} \right\}.$$

Then the lemma follows from the equation

$$(63)\qquad \frac{y_1y_1'}{[(x-\nu)^2+y_1^2][(x'-\nu')^2+y_1'^2]} = \frac{y_3y_3'}{[(\varepsilon x-\varepsilon\nu)^2+y_3^2][(\varepsilon'x'-\varepsilon'\nu')^2+y_3'^2]}$$

with the following change of variables:

$$y_3 = \varepsilon_o y_1 \quad , \quad x^* = \varepsilon_o x \quad , \quad \nu^* = \varepsilon_o \nu ,$$

(64)

$$y'_3 = y'_1/\varepsilon_o \quad , \quad x^{*\prime} = \varepsilon'_o x' \quad , \quad \nu^{*\prime} = \varepsilon'_o \nu' \ .$$

The main result (below) is now seen to follow from lemmas 7 through 10 in the manner indicated in Figure 3.

THEOREM 6. For any (x,x') and any (y,y') satisfying $1/\varepsilon_o^2 \le y'/y \le \varepsilon_o^2$ and $4/10 \le yy' \le \sqrt{5}/4$ there exists a (ν,ν') such that

$$\frac{yy'}{[(x-\nu)^2 + y^2][(x'-\nu')^2 + y'^2]} \ge \sqrt{5}/4 \tag{65}$$

Proof. If $1/\varepsilon_o \le y'/y \le \varepsilon_o$ then the inequality follows from lemmas 7 through 9.

Lemma 10 reduces the case $\varepsilon_o \le y'/y \le \varepsilon_o^2$ {or $1/\varepsilon_o^2 \le y'/y \le 1/\varepsilon_o$} to the case where $1/\varepsilon_o \le y'/y \le \varepsilon_o$ but with (x,x') replaced by $(x^*,x^{*\prime})$, which doesn't present any difficulty, and with (ν,ν') replaced by $(\nu^*,\nu^{*\prime})$. The relation between (ν,ν') and $(\nu^*,\nu^{*\prime})$ is such that if (ν,ν') is a pair of integers than so is $(\nu^*,\nu^{*\prime})$ and conversely. Consequently the existence of a pair of integers $(\nu^*,\nu^{*\prime})$ always implies the existence of another pair of integers (ν,ν').

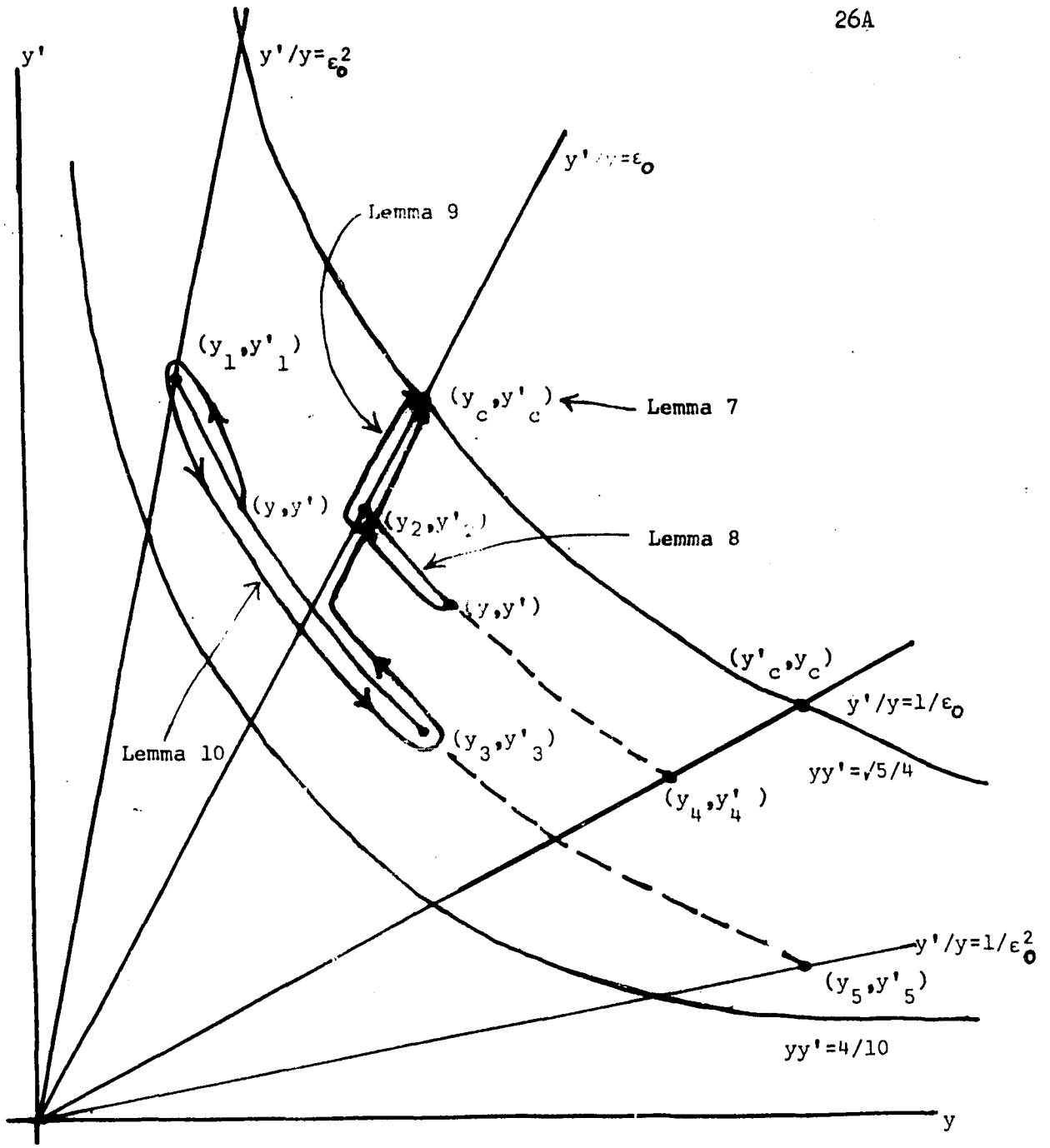

Figure 3

Use of Lemmas

9. Proof that the Main Result is Best Possible.

THEOREM 7. The constant $(\sqrt{5})/4$ is best possible.

Proof. It suffices to show that $N(|\gamma z_c+\delta|^2) \geq 1$ for γ and δ arbitrary integers in $Q(\sqrt{5})$ and

$$(66)\quad z_c = x_c + iy_c = \varepsilon_o/2 + i\,\frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}}\quad (-1/z_c = z_c - \varepsilon_o).$$

The proof amounts to observing that $N(|\gamma z_c+\delta|^2)$ can be thought of as the norm of an algebraic integer in the field $Q(z_c,\sqrt{5})$. Let $\xi_1 = \gamma z_c + \delta$. Then the conjugates of ξ_1 are $\xi_2 = \gamma' z_c' + \delta'$, $\xi_3 = \gamma \bar{z}_c + \delta$ and $\xi_4 = \gamma' \bar{z}_c' + \delta'$ where γ' is the conjugate of γ in $Q(\sqrt{5})$, $\bar{z}_c$ is the complex conjugate of z_c and z'_c is the conjugate of z_c in $Q(\sqrt{5})$. The inequality expresses the fact that the norm of an integer in $Q(z_c,\sqrt{5})$ is greater than or equal to one.

10. Failure of the Method for $Q(2^{1/2})$

For $Q(2^{1/2})$ the critical value will be seen to be produced by some unknown point which is not a fixed point. We conjecture its value to be produced by

$$z_1 = \frac{1+\sqrt{2}}{2} + i\lambda \tag{67}$$

$$z_1' = \frac{1-\sqrt{2}}{2} + i\lambda$$

where $\lambda^2 = \frac{-3+2\sqrt{6}}{4} = .475\cdots$. By the argument given in §7 above, all we need prove is that the valence of z_1 is λ^2 (which lies between 1/2 and 1/4). In fact we shall show that if $\gamma \neq 0$

$$N|\gamma z_1 + \delta|^2 > 1 \tag{68}$$

except for the following four cases:

$$N|z_1| = N|z_1 - 1| = N|z_1 - \sqrt{2}| = N|z_1 - (1+\sqrt{2})| = 1. \tag{69}$$

(or cases equivalent through multiplication of these values of $\gamma z_1 + \delta$ by a unit).

We note that

$$(70) \qquad N|\gamma z_1+\delta|^2 \geq N[\mathrm{Im}(\gamma z_1+\delta)]^2=N(\gamma)^2\,\lambda^4$$

Therefore if $N(\gamma) > \frac{1}{\lambda^2} = 2.15\cdots$ then equation (68) is automatically satisfied. This leaves for consideration only γ = unit or $\gamma = 2^{1/2}\cdot$ unit. By removing a unit factor from γ and δ we are left only with $\gamma = 2^{1/2}$ and $\gamma = 1$.

First consider $\gamma = 2^{1/2}$. From the following inequalities:

$$(71)\quad N|\sqrt{2}z_1+\delta|^2 = \left[\left(\frac{1+\sqrt{2}}{\sqrt{2}}+\delta\right)^2+2\lambda^2\right]\left[\left(\frac{1-\sqrt{2}}{-\sqrt{2}}+\delta'\right)^2+2\lambda^2\right]$$

$$\geq 4\lambda^4 + \frac{1}{4}N(1+\sqrt{2}+\sqrt{2}\delta)^2 + 2\lambda^2[\cdots]$$

$$\geq .9 + \frac{1}{4}N(w)^2$$

where $w = 1 + \sqrt{2} + \sqrt{2}\delta$ we conclude that, since $w \neq 0$ and δ is an integer, $N(w)^2 \geq 1$. Hence

$$(72) \qquad N|\sqrt{2}\,z_1+\delta|^2 \geq .9 + 1/4 > 1,$$

therefore (68) holds.

Next consider $\gamma = 1$. Here we take the following expansion:

$$(73) \quad N|z_1+\delta|^2 = \lambda^4 + \frac{1}{16} N(\psi)^2 + \lambda^2 \frac{S(\psi^2)}{4}$$

where S is the trace, and

$$(74) \qquad \psi = 1 + \sqrt{2} + 2\delta.$$

Hence $N|z_1 + \delta|^2 > 1$ unless

$$(75) \qquad N(\psi) < 4$$

and

$$(76) \qquad S(\psi^2) < 4/\lambda^2 = 8.50 \cdots .$$

From (75), as before, $\psi = 0$ or $\psi =$ unit since $N(\psi) = 2$ is impossible for (74). We consider for which units ψ is it possible that $S(\psi^2) < 8.50$. We note $\psi = 1$ is impossible by (74) while if $\psi = \sqrt{2} + 1$, $\psi^2 = 3 + 2\sqrt{2}$ and $S(\psi^2) = 6$. If ψ is a higher power of $\sqrt{2} + 1$ then $S(\psi^2) \geq S((3 + 2\sqrt{2})^2) = S(17 + 12\sqrt{2}) = 34$. Therefore using $\psi = \pm\sqrt{2} \pm 1$ in (74) we obtain four different values of δ which are shown in (74). This establishes the valence for (67).

If the method of proof for $Q(5^{1/2})$ were to extend to $Q(2^{1/2})$ using z_1 in place of z_c, then the following system (see lemma (9)) would have to have no real solutions:

$$(77)\qquad \begin{cases} (x-\nu)^2 \ (x'-\nu')^2 = A\lambda^2 \\ [(x-\nu)^2 + \lambda^2][(x'-\nu')^2 + \lambda^2] = 1 \end{cases}$$

for $2/d = 1/4 \leq A \leq \lambda^2$. We shall show that there is no real solution for $A = \lambda^2$ so that even if we could improve our initial estimate of $H_o(\geq 2/d)$ there would be no way of making this method work.

We set $A = \lambda^2$ and we eliminate $(x' - \nu')^2$ using the variable $w = (x-\nu)^2$. We then obtain from (77) the equation

$$(78)\qquad \lambda^2 w^2 + (2\lambda^4 - 1)w + \lambda^6 = 0$$

with discriminant $1 - 4\lambda^4 = \dfrac{12\sqrt{6} - 29}{4} > 0$.

Thus we have proved that an extension of the method for $Q(5^{1/2})$ to $Q(2^{1/2})$ could not be performed in routine fashion.

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SPECIAL DEFINITIONS

1. Discriminant of a quadratic field $Q(m^{1/2})$ ..... See equation 16.

2. Fixed point ..... z is a fixed point for a group of transformations H if $z=Tz$ for $T \in H$ and $T \neq$ identity transformation.

3. Floor ..... The floor of the fundamental domain is that part of ∂R which does not intersect ∂R_∞.

4. Fundamental domain ..... See page 1.

5. Fundamental totally positive unit. .. A fundamental totally positive unit is a unit ε_+ with the property that all totally positive units are of the form ε_+^n for some integer n. (For the fields $Q(2^{1/2})$ and $Q(5^{1/2})$ ε_+ is the square of the fundamental units $\varepsilon_o = \sqrt{2}+1$ and $\varepsilon_o = (1+\sqrt{5})/2$ respectively.)

6. Hilbert group ..... See page 1.

7. Simple floor ..... See page 11.

8. Totally positive ..... A quadratic integer α is totally positive, $\alpha \gg 0$, if $\alpha > 0$ and $\alpha' > 0$

9. Valence ..... See quation 5. This definition of valence is taken from [2]. Classically the term valence denoted an equivalence class under (say) the Hilbert group, but this meaning is no longer in active use.

10. Zone (in a lattice)..... For a function $F(x_1,x_2)$ defined in the x_1,x_2 plane, the zone of a lattice point (ν,ν') is the set of (x_1,x_2) for which

$$|F(x_1-\nu,x_2-\nu')| \leq |F(x_1-\mu,x_2-\mu')|$$

for any other (μ,μ') in the lattice.