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A FREE BOUNDARY PROBLEM
FOR THE FLOW OF A LIQUID UNDER GRAVITY
THROUGH A PARTIALLY OBSTRUCTED ORIFICE

by

Vuryl Jess Klassen

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I hereby recommend that this dissertation prepared under my direction by Vuryl Jess Klassen entitled A Free Boundary Problem for the Flow of a Liquid Under Gravity Through a Partially Obstructed Orifice be accepted as fulfilling the dissertation requirement of the degree of Doctor of Philosophy

L. M. Milne-Thomson — Dissertation Director
5 April 1965 — Date

After inspection of the dissertation, the following members of the Final Examination Committee concur in its approval and recommend its acceptance:*

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*This approval and acceptance is contingent on the candidate's adequate performance and defense of this dissertation at the final oral examination. The inclusion of this sheet bound into the library copy of the dissertation is evidence of satisfactory performance at the final examination.

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ABSTRACT

Consider the sluice gate formed in the $z = x + iy$ plane by the lines

(i) $y = 0, \qquad -\infty < x < +\infty,$

(ii) $y = h, \qquad x \leq 0,$

(iii) $0 < b \leq y \leq h, \qquad x = 0.$

Let there be a steady two-dimensional irrotational motion of an inviscid incompressible fluid in the z-plane where the fluid which fills the channel ($0 < y < h$, $x \leq 0$) enters at infinity with a uniform velocity U. Assume gravity acts in the direction of the negative y-axis. Since the fluid will flow from the channel through the orifice of height b, a free streamline will form. We will assume the pressure is constant on this free streamline.

The problem under consideration is to find the free streamline and all the characteristics of the flow.

By the use of conformal mapping the flow can be mapped conformally onto the interior of a half disc where the complex potential of the corresponding flow is known. The problem of finding a mapping function which accomplishes this transformation is then reduced to the problem of solving a nonlinear integral equation.

An iterative technique is introduced, and utilising this procedure numerical solutions of the integral equation were found for various cases. Once a solution is known the entire corresponding flow can be determined.

The special case of flow without gravity is considered and it is shown that as h, the height of the channel, tends to infinity the solution tends to the solution of the classical problem of flow through an orifice.

1. INTRODUCTION

Free surface problems have been given considerable attention in the past. Many free surface problems have been solved in the absence of the gravitational field. The presence of such a field renders the problem much more difficult; indeed, very few exact solutions are known in this case.

The purpose of this paper is to find the free streamline and all other properties of the flow as an inviscid liquid under gravity flows from a sluice gate, where a sluice gate is defined and assumptions on the flow are stated in section 2.

The method of solution depends upon finding a one to one mapping function which maps the flow conformally onto the interior of a half disc where the boundaries correspond as given in section 2.

Since the complex potential is known for the corresponding flow in the half disc, by considering the velocity conditions the form of the mapping function can be obtained.

Using the form of the mapping function and the fact that the pressure is constant on the free streamline the problem of determining the desired transformation can be reduced to solving a nonlinear integral equation.

This method of reducing the problem to an integral equation was suggested by Dr. L. M. Milne-Thomson since he obtained an integral equation due to Nekrasov [4] when he considered a nonlinear theory of waves of constant form [3].

In section 5 the problem is simplified by neglecting gravity and in a limiting case it is shown that we have the problem of flow through an aperture [2].

An iterative technique for solving the integral equation is given in section 6. This method is similar to the method used by Bueckner [1] in solving Nekrasov's equation.

In section 7 the details are given how a specific problem can be solved numerically. Also included in this section are pictures of free streamlines which were obtained on a computer.

2. THE PROBLEM AND THE METHOD OF SOLUTION

Consider a semi-infinite rectangle $A_\infty ODA'_\infty$ in which the points A_∞, A'_∞ are infinitely distant and $A'_\infty D$ is at a height h vertically above the horizontal line $A_\infty O$. Extend the side $A_\infty O$ to B_∞ and take a point C on OD where the length of OC equals b. Now remove the portion OC, thus forming an orifice in the semi-infinite rectangle. We shall call the figure thus formed a sluice gate.

Assume gravity acts on the system.

Regard the sluice gate as being in the plane of a steady, two dimensional, irrotational motion where an incompressible inviscid fluid flows into the channel at $A'_\infty A_\infty$ with velocity U. We will assume the fluid is in contact with all the walls.

Since the fluid will flow out through the orifice OC, a free streamline $B'_\infty C$ will form. Upon this free streamline we will assume that the pressure is constant.

To fix a reference system let the point O correspond to the origin in the $z = x + iy$ plane. Let $y = 0$ correspond to the line $A_\infty O B_\infty$ and let $y = h$ on $A'_\infty D$.

Thus the sluice gate formed in the z-plane is given by the lines

$$\text{(i)} \quad y = 0, \qquad -\infty < x < \infty,$$

$$\text{(ii)} \quad y = h, \qquad x \leq 0,$$

$$\text{(iii)} \quad 0 < b \leq y \leq h, \qquad x = 0,$$

and it is illustrated in Figure 1.

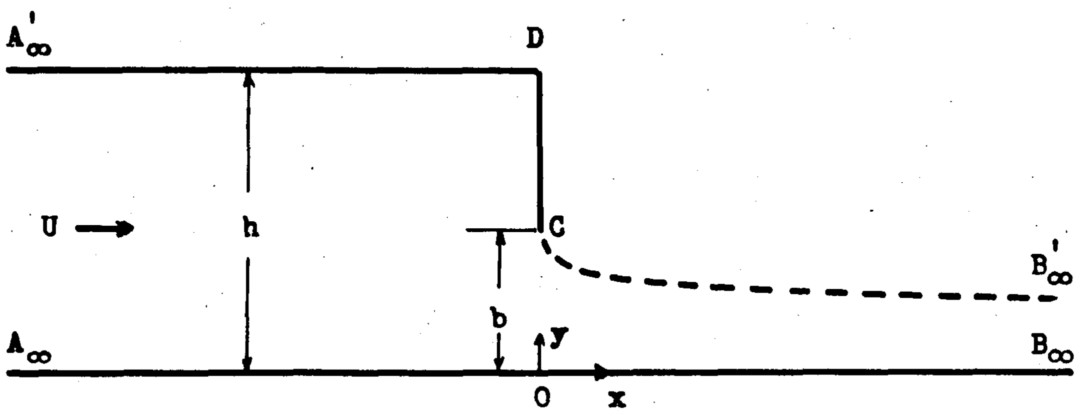

Figure 1

The z-plane

The problem under consideration is to find the equation of the free streamline and in doing this all other characteristics of the flow will be determined.

Since it is sufficient to determine the complex velocity in the z-plane let us conformally map the interior of the diagram in the z-plane onto the interior of the semicircle of radius one in the upper half Z-plane where $|Z| \leq 1$.

Let us map A, B, C, D on the points $Z=0$, $Z=1$, $Z=-1$, $Z=-a$ or A^*, B^*, C^*, D^* respectively where $0 < a < 1$.

The free streamline $B'_{\infty}C$ is mapped onto the circumference $|Z|=1$ where $\text{Im } Z \geq 0$.

The Z-plane is illustrated in Figure 2.

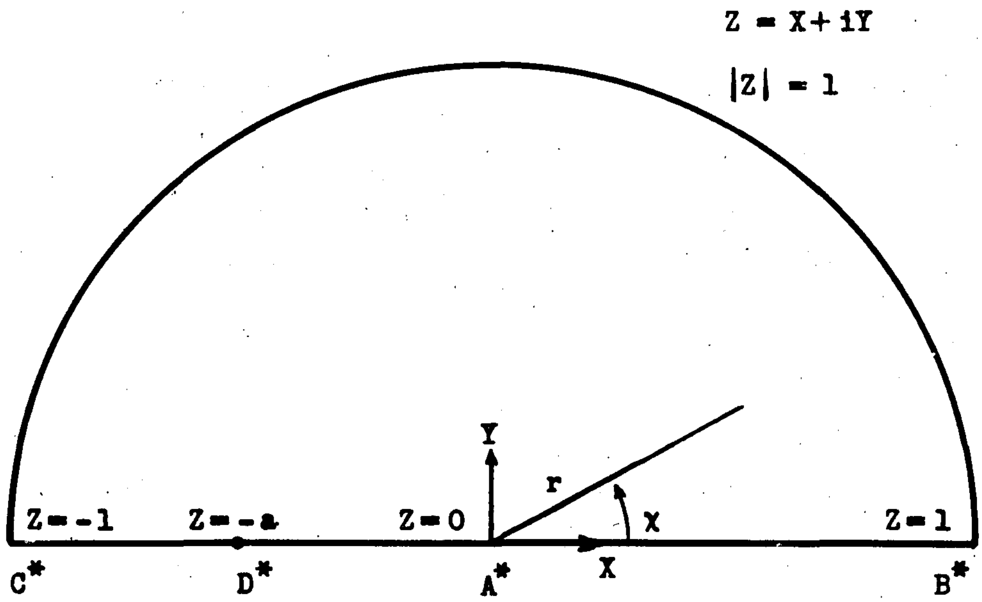

Figure 2

The Z-Plane

3. THE MAPPING FUNCTION

In the z-plane the flow is from a source at A_{∞} to a sink at B_{∞}. The only stagnation point is at D and throughout the fluid there is finite velocity.

In the Z-plane there is a source at A^* and a sink at B^*.

By the circle theorem [3] the complex potential of a sink of strength m at $Z = s > 1$, in the presence of a circular cylinder $|Z| \leq 1$, is given by

$$w = -m \ln Z + m \ln(Z - \frac{1}{s}) + m \ln(Z - s).$$

Now letting $s \rightarrow 1$ gives

$$w = -m \ln Z + 2m \ln(Z - 1).$$

With this complex potential the circle is a streamline and if $w = \phi + i\psi$ then

$$\text{on } A^*D^*C^*B^*, \qquad \psi = m\pi,$$
$$\text{on } A^*B^*, \qquad \psi = 2m\pi.$$

Now

$$\frac{dw}{dZ} = -\frac{m}{Z} + \frac{2m}{Z-1} = \frac{m(Z+1)}{Z(Z-1)}$$

and thus

$$u - iv = -\frac{dw}{dz} = -\frac{dw}{dZ}\frac{dZ}{dz} = -\frac{m(Z+1)}{Z(Z-1)}\frac{dZ}{dz} .$$

In the Z-plane there is infinite velocity at both A^* and B^*.

There is finite velocity everywhere in the z-plane, thus $\frac{dZ}{dz}$ must contain the factors Z and $Z-1$ in the numerator.

At C^* the velocity is zero in the Z-plane.

Since C is not a stagnation point in the z-plane the factor $Z+1$ must occur in the denominator of $\frac{dZ}{dz}$.

Now D is a stagnation point in the z-plane, and hence the numerator of $\frac{dZ}{dz}$ must vanish at $Z = -a$ and thus $\frac{dZ}{dz}$ must contain the factor $(Z+a)^d$ where $d > 0$.

Also, $\frac{dz}{dZ}$ must be holomorphic everywhere in a domain which contains the semicircle and its interior except at $Z = 0$, $Z = 1$, and $Z = -a$.

Thus the transformation from the Z-plane to the z-plane is given by

$$\frac{dz}{dZ} = \frac{(Z+1)h\,f(Z)}{Z(Z-1)(Z+a)^d}$$

where $d > 0$ and $f(Z)$ is holomorphic and non-zero inside and upon the semicircle.

The problem has now been reduced to finding d and the function $f(Z)$.

In the z-plane we have

$$u - iv = -\frac{dw}{dz} = -\frac{m(Z+a)^d}{hf(Z)} .$$

For z on $A_\infty B_\infty$, $v = 0$, and in the Z-plane $Z = r\epsilon(0,1)$.

Thus on $A_\infty B_\infty$

$$u - iv = u = -\frac{m(r+a)^d}{hf(r)}$$

and from this it follows that the coefficients in the expansion of f(Z) are all real.

Now for z on DC, $u = 0$, and $Z = -r\epsilon[-1,-a]$.

Since

$$u - iv = -iv = -\frac{m(-r+a)^d}{hf(-r)} ,$$

it follows that $(-r+a)^d$ must be pure imaginary and thus $d = \frac{1}{2}$.

Since f(Z) is holomorphic at $Z = 0$ let $f(Z) = \sum_{n=o}^{\infty} a_n Z^n$.

Thus we have

$$\frac{dz}{dZ} = \frac{(Z+1)h}{Z(Z-1)\sqrt{Z+a}} \sum_{n=o}^{\infty} a_n Z^n$$

$$= \frac{-a_o h}{Z\sqrt{Z+a}} + \frac{2a_o h}{(Z-1)\sqrt{Z+a}} + \frac{h}{\sqrt{Z+a}} \sum_{n=o}^{\infty} d_n Z^n$$

where

$$d_o = -a_1$$

and

$$d_n = -(2a_1 + 2a_2 + \ldots + 2a_n + a_{n+1})$$

where n is any positive integer.

Thus it follows that

$$z = -\frac{a_o h}{\sqrt{a}} \ln \frac{\sqrt{Z+a} - \sqrt{a}}{\sqrt{Z+a} + \sqrt{a}} + \frac{2a_o h}{\sqrt{a+1}} \ln \frac{\sqrt{Z+a} - \sqrt{a+1}}{\sqrt{Z+a} + \sqrt{a+1}}$$

$$-h\sqrt{Z+a} \sum_{n=o}^{\infty} c_n Z^n + C_1 + iC_2$$

where the c_n are linear combinations of the d_n and therefore the c_n are all real.

Separating z into real and imaginary parts gives

$$x = -\frac{a_o h}{\sqrt{a}} \ln \left| \frac{\sqrt{Z+a} - \sqrt{a}}{\sqrt{Z+a} + \sqrt{a}} \right| + \frac{2a_o h}{\sqrt{a+1}} \ln \left| \frac{\sqrt{Z+a} - \sqrt{a+1}}{\sqrt{Z+a} + \sqrt{a+1}} \right|$$

$$-h \operatorname{Re} \left\{ \sqrt{Z+a} \sum_{n=o}^{\infty} c_n Z^n \right\} + C_1,$$

$$y = -\frac{a_o h}{\sqrt{a}} \arg \frac{\sqrt{Z+a} - \sqrt{a}}{\sqrt{Z+a} + \sqrt{a}} + \frac{2a_o h}{\sqrt{a+1}} \arg \frac{\sqrt{Z+a} - \sqrt{a+1}}{\sqrt{Z+a} + \sqrt{a+1}}$$

$$-h \ \mathrm{Im} \left\{ \sqrt{Z+a} \sum_{n=o}^{\infty} c_n Z^n \right\} + C_2 .$$

Now on $A_\infty B_\infty$, $y = 0$, and on $A^* B^*$, $Z = r\varepsilon\ (0,1)$, thus

$$C_2 = -\frac{2a_o}{\sqrt{a+1}} \pi h.$$

On CD, $x = 0$, and in the Z-plane $Z = -r\varepsilon[-1,-a]$, and thus it follows that $C_1 = 0$.

We must now show that the boundary conditions are satisfied.

Recall

$$x = -\frac{a_o h}{\sqrt{a}} \ln \left| \frac{\sqrt{Z+a} - \sqrt{a}}{\sqrt{Z+a} + \sqrt{a}} \right| + \frac{2a_o h}{\sqrt{a+1}} \ln \left| \frac{\sqrt{Z+a} - \sqrt{a+1}}{\sqrt{Z+a} + \sqrt{a+1}} \right|$$

$$-h \ \mathrm{Re} \left\{ \sqrt{Z+a} \sum_{n=o}^{\infty} c_n Z^n \right\} ,$$

$$y = -\frac{a_o h}{\sqrt{a}} \arg \frac{\sqrt{Z+a} - \sqrt{a}}{\sqrt{Z+a} + \sqrt{a}} + \frac{2a_o h}{\sqrt{a+1}} \arg \frac{\sqrt{Z+a} - \sqrt{a+1}}{\sqrt{Z+a} + \sqrt{a+1}}$$

$$-h \ \mathrm{Im} \left\{ \sqrt{Z+a} \sum_{n=o}^{\infty} c_n Z^n \right\} - \frac{2a_o}{\sqrt{a+1}} \pi h ,$$

$$u - iv = -\frac{m\sqrt{Z+a}}{hf(Z)} .$$

On A^*B^*, $Z = r\epsilon(0,1)$, and hence on $A_\infty B_\infty$

$$x = -\frac{a_o h}{\sqrt{a}} \ln \left| \frac{\sqrt{r+a} - \sqrt{a}}{\sqrt{r+a} + \sqrt{a}} \right| + \frac{2a_o h}{\sqrt{a+1}} \ln \left| \frac{\sqrt{r+a} - \sqrt{a+1}}{\sqrt{r+a} + \sqrt{a+1}} \right|$$

$$-h\sqrt{r+a} \sum_{n=o}^{\infty} c_n r^n,$$

$$y = 0.$$

Now as $Z \to 0$ on A^*B^*, $x \to -\infty$ if $a_o < 0$, and as $Z \to 1$ on A^*B^*, $x \to \infty$ if $a_o < 0$.

On $A_\infty B_\infty$ we also have

$$u = -\frac{m\sqrt{r+a}}{hf(r)} ,$$

$$v = 0.$$

Thus we must have $f(r) < 0$ for all $r\epsilon[0,1]$.

On A^*D^*, $Z = -r\epsilon[-a,0)$. Hence we have

$$x = -\frac{a_o h}{\sqrt{a}} \ln \left| \frac{\sqrt{-r+a} - \sqrt{a}}{\sqrt{-r+a} + \sqrt{a}} \right| + \frac{2a_o h}{\sqrt{a+1}} \ln \left| \frac{\sqrt{-r+a} - \sqrt{a+1}}{\sqrt{-r+a} + \sqrt{a+1}} \right|$$

$$-h\sqrt{-r+a} \left\{ \sum_{n=o} c_n (-r)^n \right\},$$

$$y = -\frac{a_o h}{\sqrt{a}} \arg \frac{\sqrt{-r+a} - \sqrt{a}}{\sqrt{-r+a} + \sqrt{a}} + \frac{2a_o h}{\sqrt{a+1}} \arg \frac{\sqrt{-r+a} - \sqrt{a+1}}{\sqrt{-r+a} + \sqrt{a+1}} - \frac{2a_o \pi}{\sqrt{a+1}} h$$

$$= -\frac{a_o}{\sqrt{a}} \pi h .$$

Since the height of the channel is h, $-\frac{a_o}{\sqrt{a}} \pi = 1$.

As $Z \longrightarrow 0$ on A^*D^*, $x \rightarrow -\infty$ and when $Z = -a$, $x = 0$.

On $A'_\infty D$

$$u = -\frac{m\sqrt{-r+a}}{hf(-r)},$$

$$v = 0.$$

Thus we must have

$$f(-r) < 0$$

whenever $r \epsilon [0,a]$.

On D^*C^*, $Z = -r\varepsilon[-1,-a]$ and hence

$$x = -\frac{a_o h}{\sqrt{a}} \ln \left| \frac{\sqrt{-r+a} - \sqrt{a}}{\sqrt{-r+a} + \sqrt{a}} \right| + \frac{2a_o h}{\sqrt{a+1}} \ln \left| \frac{\sqrt{-r+a} - \sqrt{a+1}}{\sqrt{-r+a} + \sqrt{a+1}} \right| = 0,$$

$$y = -\frac{a_o h}{\sqrt{a}} \arg \frac{\sqrt{-r+a} - \sqrt{a}}{\sqrt{-r+a} + \sqrt{a}} + \frac{2a_o h}{\sqrt{a+1}} \arg \frac{\sqrt{-r+a} - \sqrt{a+1}}{\sqrt{-r+a} + \sqrt{a+1}} - \sqrt{r-a}\; h \sum_{n=o}^{\infty} c_n (-r)^n - \frac{2a_o \pi}{\sqrt{a+1}} h.$$

The velocity conditions on DC give

$$u = 0,$$

$$v = \frac{m\sqrt{r-a}}{hf(-r)},$$

and thus we must have

$$f(-r) < 0$$

for $r \varepsilon [a,1]$.

From the boundary conditions we see that

$$f(r) = \sum_{n=o}^{\infty} a_n r^n < 0$$

for all $r \varepsilon [-1,1]$.

4. THE INTEGRAL EQUATION

Let

$$u - iv = q^* e^{-i\sigma} = -\frac{m\sqrt{Z+a}}{hf(Z)}$$

where q^* is the speed and σ is the direction of flow at a point in the z-plane.

Let

$$e^{H_o(r,\chi)} = -f(re^{i\chi}).$$

On B^*C^*, $Z = e^{i\chi}$, $dZ = ie^{i\chi}d\chi$ where $0 \leq \chi \leq \pi$, and thus on the free streamline we have

$$\frac{dz}{d\chi} = \frac{h \cot \chi/2}{\sqrt{e^{i\chi} + a}} f(e^{i\chi})$$

$$= -\frac{h \cot \chi/2}{\sqrt{e^{i\chi} + a}} e^{H_o(1,\chi)}.$$

Let

$$H_o(r,\chi) = F_o(r,\chi) + iG_o(r,\chi),$$

$$re^{i\chi} + a = Q_o(r,\chi)\, e^{ig_o(r,\chi)},$$

$$q_o(r,\chi) = \frac{mQ_o^{\frac{1}{2}}(r,\chi)}{he^{F_o(r,\chi)}},$$

$$H_o(1,\chi) = H(\chi) = F(\chi) + iG(\chi),$$

$$e^{i\chi} + a = Q_o(1,\chi)e^{ig_o(1,\chi)} = Q(\chi)e^{ig(\chi)},$$

$$q_o(1,\chi) = q(\chi) = \frac{mQ^{\frac{1}{2}}(\chi)}{he^{F(\chi)}}.$$

Thus we have

$$\frac{dz}{d\chi} = -\frac{h \cot \chi/2}{Q^{\frac{1}{2}}(\chi)}\, e^{F(\chi)}\, e^{i[G(\chi) - g(\chi)/2]}.$$

Separating into real and imaginary parts gives

$$\frac{dx}{d\chi} = -\frac{h \cot \chi/2}{Q^{\frac{1}{2}}(\chi)}\, e^{F(\chi)} \cos\, [G(\chi) - g(\chi)/2],$$

$$\frac{dy}{d\chi} = -\frac{h \cot \chi/2}{Q^{\frac{1}{2}}(\chi)}\, e^{F(\chi)} \sin\, [G(\chi) - g(\chi)/2].$$

Now the complex velocity is

$$q^* e^{-i\sigma} = \frac{mQ_o^{\frac{1}{2}}(r,\chi)}{he^{F_o(r,\chi)}} e^{-i[G_o(r,\chi) - g_o(r,\chi)/2]} .$$

Thus the speed at any point is

$$q^* = q_o(r,\chi) = \frac{mQ_o^{\frac{1}{2}}(r,\chi)}{he^{F_o(r,\chi)}} ,$$

and the direction of flow is

$$\sigma = G_o(r,\chi) - g_o(r,\chi)/2.$$

On the free streamline we have

$$q^* = \frac{mQ^{\frac{1}{2}}(\chi)}{he^{F(\chi)}}$$

and

$$\sigma = G(\chi) - g(\chi)/2.$$

By Bernoulli's equation

$$\frac{p}{\rho} + \frac{1}{2} q^2 + gy = \text{constant}$$

on a streamline.

On the free surface the pressure is constant and thus on the free streamline we have

$$q(\chi)\,\frac{dq(\chi)}{d\chi} = -g\,\frac{dy}{d\chi}\,.$$

Hence

$$q(\chi)\,\frac{dq(\chi)}{d\chi} = g\,\frac{h\cot\chi/2}{Q^{\frac{1}{2}}(\chi)}\,e^{F(\chi)}\sin\,[G(\chi)-g(\chi)/2].$$

Multiplying by $q(\chi) = \dfrac{mQ^{\frac{1}{2}}(\chi)}{he^{F(\chi)}}$ gives

$$\frac{d}{d\chi}\left(\frac{m^3Q^{3/2}(\chi)}{h^3e^{3F(\chi)}}\right) = 3mg\,\cot\,(\chi)/2\,\sin\,[G(\chi)-g(\chi)/2].$$

Integrating with respect to χ gives

$$e^{-3F(\chi)} = \frac{3gh^3}{cm^2Q^{3/2}(\chi)}\left[c\int_0^{\chi}\cot t/2\,\sin\,[G(t)-g(t)/2]dt+1\right] \quad (1)$$

where c is a constant which must be determined.

Thus we want to find the functions $G(\chi)$ and $F(\chi)$ plus a constant c such that this equation is satisfied.

Taking the ln of equation (1) gives

$$-3F(\chi) = \ln \frac{3gh^3}{cm^2Q^{3/2}(\chi)} + \ln \left[1+c\int_0^{\chi} \cot t/2 \sin[G(t)-g(t)/2]dt\right] .$$

Differentiating with respect to χ and multiplying by $-1/3$ implies

$$\frac{d}{d\chi}[F(\chi)] = \frac{1}{2Q(\chi)}\frac{dQ(\chi)}{d\chi} - \frac{c \cot \chi/2 \sin[G(\chi)-g(\chi)/2]}{3\left(1+c\int_0^{\chi} \cot t/2 \sin[G(t)-g(t)/2]dt\right)} .$$

Let

$$H(\chi) = F(\chi)+iG(\chi) = \sum_{n=o}^{\infty} b_n(\cos n\chi + i \sin n\chi).$$

It can be shown that the b_n are real and hence

$$F(\chi) = \sum_{n=o}^{\infty} b_n \cos n\chi,$$

$$G(\chi) = \sum_{n=1}^{\infty} b_n \sin n\chi.$$

Now

$$\frac{d}{d\chi}[F(\chi)] = -\sum_{n=1}^{\infty} n\, b_n \sin n\chi$$

and

$$\sum_{m=1}^{\infty} b_m \sin m\chi = \frac{2}{\pi}\int_0^{\pi}\left(\sum_{m=1}^{\infty} \frac{\sin m\,\varepsilon \sin m\,\chi}{m}\right)\left(\sum_{n=1}^{\infty} n\, b_n \sin n\varepsilon\right) d\varepsilon.$$

Thus we have

$$G(\chi) = \frac{2}{\pi}\int_0^{\pi} N(\varepsilon,\chi)\left[-\frac{1}{2Q(\varepsilon)}\frac{dQ(\varepsilon)}{d\varepsilon} + \frac{c \cot \varepsilon/2 \sin[G(\varepsilon) - g(\varepsilon)/2]}{3\left(1 + c\int_0^{\varepsilon} \cot t/2 \sin[G(t) - g(t)/2]dt\right)}\right]d\varepsilon$$

where

$$N(\varepsilon,\chi) = \sum_{m=1}^{\infty} \frac{\sin m\varepsilon \sin m\chi}{m}.$$

Now if we can solve this nonlinear integral equation for $G(\chi)$ and a corresponding value of c, then with a given input flux, $H_0(r,\chi)$ is determined and from this all the characteristics of the flow can be found.

Thus the solution of our problem depends upon solving the above integral equation.

5. FLOWS WITHOUT GRAVITY

Now consider the case when $g = 0$. Bernoulli's equation implies that q is constant on the free streamline and since

$$q(\chi) = \frac{mQ^{\frac{1}{2}}(\chi)}{he^{F(\chi)}},$$

we have

$$q_1 = \frac{mQ^{\frac{1}{2}}(\chi)}{he^{F(\chi)}}$$

where $q_1 > 0$, since it is the velocity on the free streamline.

Now

$$e^{F(\chi)} = \frac{mQ^{\frac{1}{2}}(\chi)}{hq_1}$$

and thus

$$F(\chi) = \ln \frac{mQ^{\frac{1}{2}}(\chi)}{hq_1}.$$

Differentiating along the streamline gives

$$\frac{d}{d\chi} F(\chi) = \frac{1}{2Q(\chi)} \frac{dQ(\chi)}{d\chi} .$$

Thus the integral equation for the non-gravity case reduces to

$$G(\chi) = -\frac{2}{\pi} \int_0^{\pi} N(\varepsilon, \chi) \frac{1}{2Q(\varepsilon)} \frac{dQ(\varepsilon)}{d\varepsilon} d\varepsilon$$

where

$$N(\varepsilon, \chi) = \sum_{n=1}^{\infty} \frac{\sin n\varepsilon \sin n\chi}{n} .$$

Since $Q(\varepsilon)$ is fixed for a given value of a, $G(\chi)$ can be calculated.

Now when a is fixed

$$G(\chi) = \sum_{n=1}^{\infty} b_n \sin n\chi$$

is known and thus the b_n can be determined for $n \geq 1$.

For a specific problem U and h are given and thus m is fixed and we have

$$m = \frac{Uh}{\pi} .$$

We can now determine b_o since

$$b_o = \ln \frac{m\sqrt{a}}{hU}.$$

Now q_1 and $F(\chi)$ are known and we have

$$\frac{dy}{d\chi} = \frac{h \cot \chi/2}{-Q^{\frac{1}{2}}(\chi)} e^{F(\chi)} \sin [G(\chi) - g(\chi)/2]$$

$$= -\frac{m}{q_1} \cot \chi/2 \sin [G(\chi) - g(\chi)/2],$$

where m and q_1 are known constants.

When $\chi = \pi$, $y = y_c = b$ and thus

$$y - b = -\frac{m}{q_1} \int_{\pi}^{\chi} \cot \varepsilon/2 \sin[G(\varepsilon) - g(\varepsilon)/2] d\varepsilon$$

where $0 \leq \chi \leq \pi$.

The height of the free streamline at infinity y_B can be determined from the preceding equation or from the equation of continuity which gives

$$y_B = \frac{Uh}{q_1} .$$

Since

$$\frac{dx}{d\chi} = -\frac{m}{q_1}\cot \chi/2 \cos\,[G(\chi) - g(\chi)/2]$$

it follows that

$$x = -\frac{m}{q_1}\int_{\pi}^{\chi} \cot \varepsilon/2 \cos\,[G(\varepsilon) - g(\varepsilon)/2]d\varepsilon$$

where $0 \leq \chi \leq \pi$.

Thus we can solve for x and y and since $H_o(r,\chi)$ is also known all the properties of the flow can be determined.

If we consider the case when $g = 0$ and let the height of the channel tend to infinity then in the limit we should have the problem of flow through an aperture [2].

If b is fixed, as $h \longrightarrow \infty$, $a \longrightarrow o$ and hence it follows that $Q(\chi) \longrightarrow 1$, $g(\chi) \longrightarrow \chi$, and $G(\chi) \longrightarrow 0$.

We shall use the symbol $\doteq$ for "approximately equal to." Thus in the sequel $x \doteq y$ means that x is approximately equal to y.

If $a \doteq 0$ then $G \doteq 0$ and hence it follows that $e^{F(\chi)} \doteq e^{b_o}$. Let $he^{b_o} = K$.

The fluid speed on the free streamline is constant and from the above it is approximately $\frac{m}{K}$.

Thus

$$\frac{dx}{d\chi} \doteq -K \cot \chi/2 \cos \chi/2$$

$$\doteq -K(\csc \chi/2 - \sin \chi/2).$$

Integrating we have

$$x \doteq -2\, K[\ln \tan \chi/4 + \cos \chi/2] + \text{constant}.$$

When $\chi = \pi$ we have $x = 0$ and thus the constant is zero.

Hence

$$x \doteq -2\, K[\ln \tan \chi/4 + \cos \chi/2].$$

Now

$$\frac{dy}{d\chi} \doteq -K \cot \chi/2 \sin(-\chi/2)$$

$$\doteq K \cos (\chi/2)$$

thus

$$y \doteq 2K \sin(\chi/2) + \text{constant}.$$

When $\chi = 0$, $y = y_B$ = height of the free streamline at infinity and thus the constant equals y_B.

Therefore

$$y \doteq 2K \sin(\chi/2) + y_B.$$

Letting $\chi = \pi$ gives $y = y_c$ and thus

$$y_c \doteq 2K + y_B$$

giving

$$K \doteq \frac{y_c - y_B}{2}.$$

By the equation of continuity

$$m\pi = \text{flux entering channel} = \text{flux at B.}$$

Thus

$$m\pi = y_B\, q_B \doteq y_B \frac{m}{he^{b_o}} = m \frac{y_B}{he^{b_o}}\ .$$

This implies

$$bs^{b_o} \doteq \frac{y_B}{\pi} = \frac{\sigma b}{\pi}$$

where σ is the coefficient of contraction.

Since $y_c = b$ and $y_B = \sigma b$ it follows that

$$b = \sigma b + y_c - y_b.$$

But

$$y_c - y_B \doteq 2K = 2hs^{b_o} \doteq 2\,\frac{\sigma b}{\pi}$$

and thus

$$b \doteq \sigma b + 2\,\frac{\sigma b}{\pi} = \sigma\left(\frac{\pi b + 2b}{\pi}\right).$$

Hence it follows that

$$\sigma \doteq \frac{b\pi}{b\pi + 2b} = \frac{\pi}{\pi + 2}.$$

Thus we have the following approximations:

$$x = -\frac{2\sigma b}{\pi}\left[\ln \tan \frac{\chi}{4} + \cos \frac{\chi}{2}\right],$$

$$y = \frac{2\sigma b}{\pi}\left[\sin\left(\frac{\chi}{2}\right) + \frac{\pi}{2}\right],$$

$$\sigma = \frac{\pi}{\pi + 2} .$$

These equations agree with the results obtained for the flow through an aperture when the fluid is of infinite height [2].

6. NUMERICAL METHOD USED IN SOLVING THE NON-LINEAR INTEGRAL EQUATION

Consider the integral equation

$$G(\chi) = \frac{2}{\pi} \int_0^{\pi} N(\varepsilon,\chi) \left[-\frac{1}{2Q(\varepsilon)} \frac{dQ(\varepsilon)}{d\varepsilon} + \frac{c \cot \varepsilon/2 \sin[G(\varepsilon) - g(\varepsilon)/2]}{3\left(1+c\int_0^{\varepsilon} \cot t/2 \sin[G(t)-g(t)/2]dt\right)} \right] d\varepsilon$$

where

$$N(\varepsilon,\chi) = \sum_{m=1}^{\infty} \frac{\sin m\varepsilon \sin m\chi}{m}$$

$$Q(\varepsilon)e^{ig(\varepsilon)} = e^{i\varepsilon} + a.$$

We want to find a function $G(\chi)$ and a constant c such that the above equation is satisfied. With this in mind let

$$s(\chi) = c \cot \frac{\chi}{2} \sin [G(\chi) - g(\chi)/2].$$

Then

$$G(\chi) = \frac{2}{\pi} \int_0^{\pi} N(\varepsilon,\chi) \left[-\frac{1}{2Q(\varepsilon)} \frac{dQ(\varepsilon)}{d\varepsilon} + \frac{s(\varepsilon)}{3\left(1+\int_0^{\varepsilon} s(t)dt\right)} \right] d\varepsilon$$

and thus

$$s(\chi)=c\cot\frac{\chi}{2}\sin\left\{-\frac{g(\chi)}{2}+\frac{2}{\pi}\int_0^{\pi}N(\varepsilon,\chi)\left[-\frac{1}{2Q(\varepsilon)}\frac{dQ(\varepsilon)}{d\varepsilon}+\frac{s(\varepsilon)}{3\left(1+\int_0^{\varepsilon}s(t)dt\right)}\right]d\varepsilon\right\}.$$

Let

$$M[s(\chi)]=\cot\frac{\chi}{2}\sin\left\{-\frac{g(\chi)}{2}+\frac{2}{\pi}\int_0^{\pi}N(\varepsilon,\chi)\left[-\frac{1}{2Q(\varepsilon)}\frac{dQ(\varepsilon)}{d\varepsilon}+\frac{s(\varepsilon)}{3\left(1+\int_0^{\varepsilon}s(t)dt\right)}\right]d\varepsilon\right\}$$

and thus we have

$$s(\chi)=c\,M[s(\chi)].$$

Corresponding with an initial guess $s_o(\chi)$, $M[s_o(\chi)]$ can be calculated.

Following the numerical procedure as in [1], define c_o^* by using the method of least squares.

Thus minimize I where

$$I=\int_0^{\pi}(s_o(\chi)-c_o^*M[s_o(\chi)])^2d\chi.$$

If

$$\frac{\partial I}{\partial c_o^*}=0$$

then

$$\int_0^{\pi} (s_o(\chi) - c_o^* M[s_o(\chi)])\, M[s_o(\chi)]\, d\chi = 0$$

and thus we have

$$c_o^* = \frac{\int_0^{\pi} s_o(\chi)\, M[s_o(\chi)]\, d\chi}{\int_0^{\pi} (M[s_o(\chi)])^2\, d\chi} .$$

Now consider the iterative procedure

$$s_n(\chi) = c_{n-1}^* \, M[s_{n-1}(\chi)]$$

where

$$s_{n-1}(\chi) = c \cot \frac{\chi}{2} \sin [G_{n-1}(\chi) - \frac{g(\chi)}{2}],$$

$$c_{n-1}^* = \frac{\int_0^{\pi} s_{n-1}(\chi)\, M[s_{n-1}(\chi)] d\chi}{\int_0^{\pi} (M[s_{n-1}(\chi)])^2\, d\chi} ,$$

and n is any positive integer.

If at some step in the iterations we find an $s_n(X)$ and c_n^* such that $s_n(X) = s_{n-1}(X)$, and $c_n^* = c_{n-1}^*$, correct to say n significant figures, then the function $G(X) = G_n(X)$ and the constant $c = c_n^*$ form a solution to the integral equation with the corresponding accuracy.

7. NUMERICAL TREATMENT OF THE PROBLEM

We would like to find a numerical solution to the problem when given an input velocity U_o, the height of the channel h, and the width of the orifice b.

With this in mind choose a value for a. Using the iterative technique given in section 6 and starting with a properly chosen initial guess $s_o(\chi)$ we can find a $G^*(\chi)$ and a c^* which satisfy the integral equation.

To help determine a reasonable initial guess note that

$$1 + c \int_0^{s} \cot t/2 \sin \{G(t) - g(t)/2\}\, dt > 0.$$

and one would suspect that $\sin \{G(t) - g(t)/2\}$ would go from 0 to -1 as χ goes from 0 to π.

Now from $G^*(\chi)$ the b_n are determined for $n \geq 1$ and since

$$-\frac{a_o}{\sqrt{a}}\pi = 1$$

and

$$b_o = \ln(-a_o)$$

b_o is known.

Since

$$q(o) = -\frac{m\sqrt{1+a}}{h\sum_{i=o}^{\infty} a_i} = -\frac{U_o\sqrt{1+a}}{\pi\sum_{i=o}^{\infty} a_i}$$

the exit velocity is thus found and from the equation of continuity the exit height y_B can be obtained.

Now

$$\frac{b^*}{h} = -\int_0^{\pi} \frac{\cot \varepsilon/2}{Q^{\frac{1}{2}}(\varepsilon)} e^{F(\varepsilon)} \sin [G(\varepsilon) - g(\varepsilon)/2] d\varepsilon + \frac{y_B}{h}$$

and this determines the ratio $\frac{b^*}{h}$.

Since the ratio $\frac{b^*}{h}$ so obtained may not equal $\frac{b}{h}$ we may have to adjust a. As a tends to zero, $\frac{b^*}{h}$ also tends to zero. If a tends to unity, so does $\frac{b^*}{h}$. Thus by changing a one can find the value a^* which gives the deserved ratio $\frac{b}{h}$. Figure 3 shows a graph which was obtained numerically which relates a to $\frac{b}{h}$.

Now once a^* is determined we can solve the integral equation for a function $G^*(\chi)$ and a constant c^*. Let $F^*(\chi)$ and $q^*(\chi)$ correspond with this solution $G^*(\chi)$, c^*.

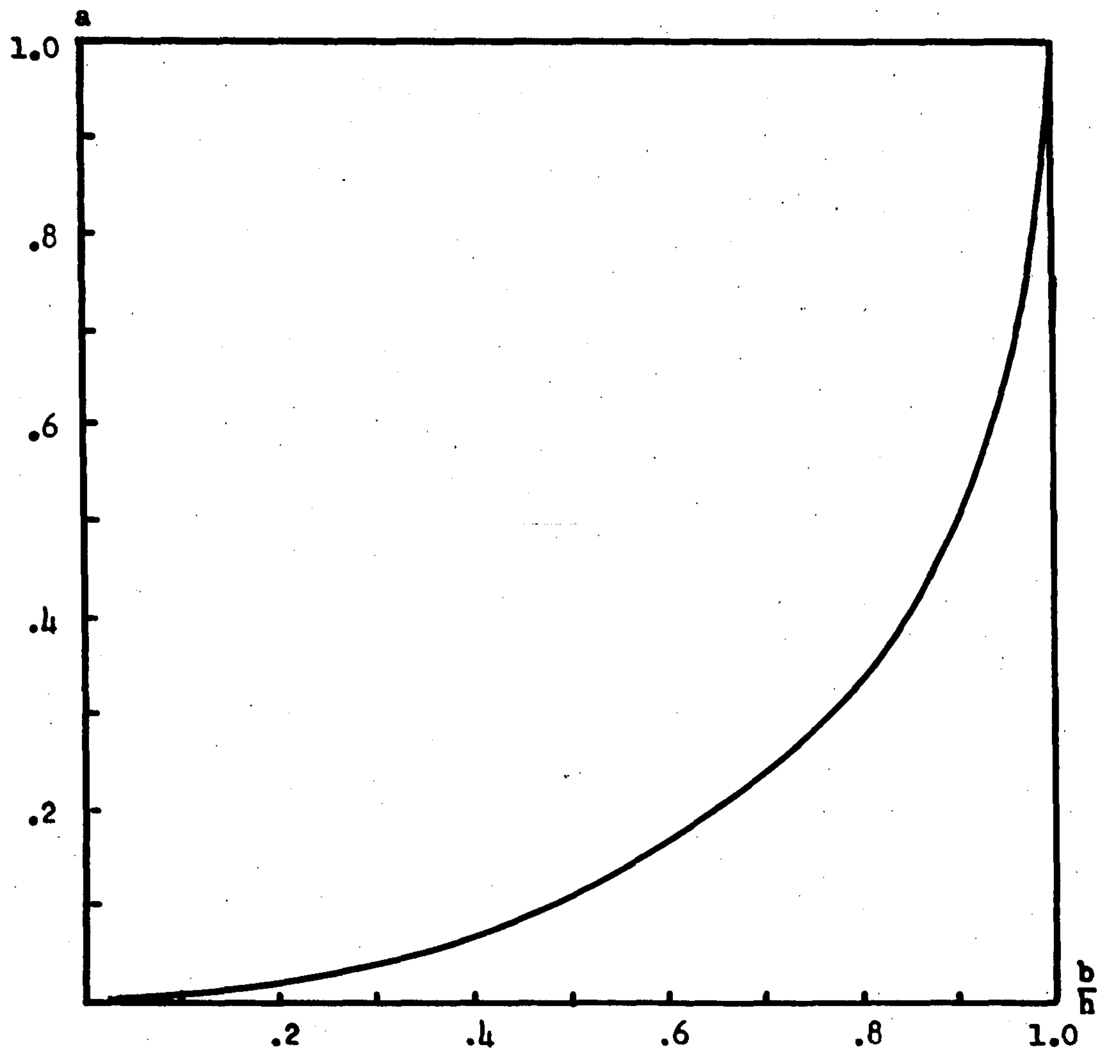

Figure 3

Graph of a to $\frac{b}{h}$.

Since

$$e^{-3F^*(\chi)} = \frac{3gh^3}{m^2Q^{3/2}(\chi)}\left[\int_0^{\chi}\cot t/2 \sin\,[G^*(t) - g(t)/2]dt + \frac{1}{c^*}\right]$$

it follows that

$$e^{-3F^*(0)} = \frac{3gh^3}{m^2Q^{3/2}(0)c^*} .$$

Now since

$$e^{F^*(\chi)} = \frac{mQ^{\frac{1}{2}}(\chi)}{hq^*(\chi)}$$

it follows that

$$c^* = \frac{3mg}{\left[q^*(0)\right]^3} .$$

But

$$q^*(0) = -\frac{m\sqrt{1+a}}{h\sum_{n=0}^{\infty} a_n}$$

where a and the a_n are known.

Hence using the above and the fact that $m\pi = U_o h$, we have

$$c^* = -\frac{3U_o hg\left(\sum_{i=o}^{\infty} a_i\right)^3 \pi^3}{U_o^3\left(\sqrt{1+a}\right)^3} = -\frac{3\left(\pi\sum_{i=o}^{\infty} a_i\right)^3}{\left(\sqrt{1+a}\right)^3}\frac{gh}{U_o^2}.$$

Hence we see that for a given solution $G^*(X)$ and c^* the ratio $\frac{gh}{U_o^2}$ is determined. Thus with this given solution we have an infinite family of flows depending on the choice of either U_o or h. Note, however, that if we assign one of these the other one is determined.

Suppose that we are given an input velocity U_o^{**} and a height h^{**}. Then we must find a solution $G^{**}(X)$ and c^{**} such that the integral equation is satisfied and such that

$$c^{**} = -\frac{3\left(\pi\sum a_i\right)^3 gh^{**}}{\left(\sqrt{1+a}\right)^3\left(U_o^{**}\right)^2}$$

Hence the ratio $\frac{gh^{**}}{\left(U_o^{**}\right)^2}$ is given and we note that

$$\frac{gh^{**}}{\left(U_o^{**}\right)^2} = -\frac{\left(\sqrt{1+a}\right)^3 c^{**}}{3\left(\pi\sum_{i=o}^{\infty} a_i\right)^3}, \qquad (2)$$

where $0\le X\le\pi$ and y_b = height of the free streamline at infinity.

Thus given an entrance velocity U_o, a height h, and an orifice width b, the free streamline can be found and since $H_o(r,\chi)$ is known, all the characteristics of the flow are determined, and thus the problem can be regarded as completely solved.

By considering various initial guesses for $s(\chi)$ it is possible to find new values of $G(\chi)$ and c until equation (2) is satisfied.

Once a solution is known, the exit velocity q(0) can be determined from c since

$$c = \frac{3mg}{q^3(0)} .$$

By the equation of continuity the height of the free streamline at infinity can be found. Since $F(\chi)$ is known, x and y can be determined and we have

$$x = -h\int_{\pi}^{\chi} \frac{\cot \varepsilon/2}{Q^{\frac{1}{2}}(\varepsilon)} e^{F(\varepsilon)} \cos[G(\varepsilon) - g(\varepsilon)/2]d\varepsilon$$

$$y = y_B - h\int_{0}^{\chi} \frac{\cot \varepsilon/2}{Q^{\frac{1}{2}}(\varepsilon)} e^{F(\varepsilon)} \sin[G(\varepsilon) - g(\varepsilon)/2]d\varepsilon.$$

Numerical calculations were carried out on the R-W 400. Curves can be displayed on a cathode ray tube. Figures 3, 4, and 5 show pictures of various flows thus obtained. The tables under the pictures list the plotted x and y values. Since the largest abscissa value had to be kept small to have a descriptive graph the last y value in each table is the calculated height of the free streamline at infinity.

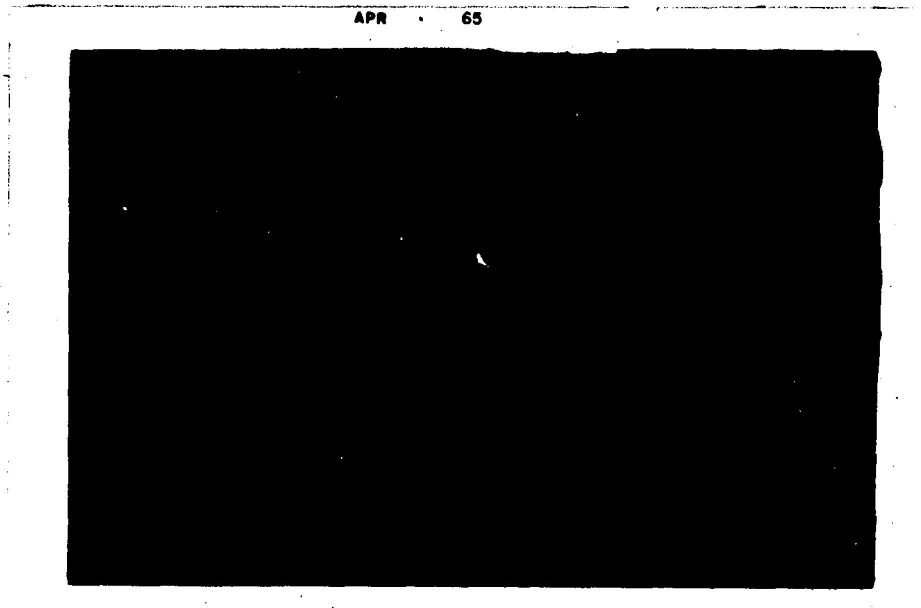

Figure 4

Free Streamline When $U_o = 1.573$ ft./sec., $h = 1.997$ ft., $b = .546$ ft.

$$\frac{U_o^2}{gh} = .039 \qquad \frac{b}{h} = .273$$

x	y
.000	.546
.026	.500
.061	.469
.101	.443
.207	.401
.306	.378
.408	.362
.508	.353
.617	.346
.700	.343
.870	.339
1.092	.336

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Figure 5

Free Streamline When $U_o = 2.739$ ft./sec., $h = 1.147$ ft., $b = .572$ ft.

$$\frac{U_o^2}{gh} = .204 \qquad \frac{b}{h} = .499$$

x	y
.000	.572
.008	.549
.027	.523
.051	.500
.111	.461
.177	.432
.312	.396
.496	.372
.685	.360
.772	.357
.949	.354
1.182	.351

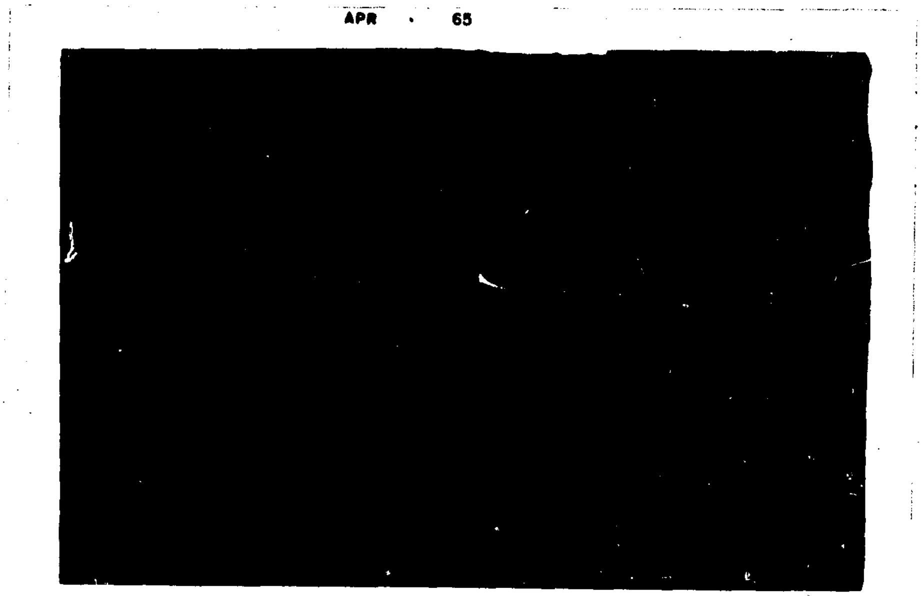

Figure 6

Free Streamline When $U_o = 4.394$ ft./sec., h= .715, b= .579 ft.

$$\frac{U_o^{\,2}}{gh} = .844 \qquad \frac{b}{h} = .810$$

x	y
.000	.579
.009	.561
.027	.544
.062	.521
.100	.504
.135	.491
.194	.474
.299	.453
.426	.437
.927	.413
1.133	.410
1.403	.408

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