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by

Charles Wayne Swartz

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I hereby recommend that this dissertation prepared under my direction by Charles Wayne Swartz entitled An Operational Calculus Based on the Laplace Transform be accepted as fulfilling the dissertation requirement of the degree of Doctor of Philosophy

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ABSTRACT

In this paper a generalized function calculus is constructed by using a method of J. Mikusinski (Fund. Math., 1948) where the generalized functions are defined as equivalence classes of "weakly" convergent sequences. The Laplace transform is defined on the space G of generalized functions and the class of functions having a Laplace transform in the classical sense is imbedded in G in such a way that the classical Laplace transform coincides with the generalized transform defined on G.

After the generalized functions have been introduced, some of the elementary operations such as translation are defined in G, and the classical formulas for such operations are extended to G. The Dirac delta "function" is then introduced into the calculus. Differentiation and integration are defined in G and, in particular, it is shown that every generalized function is indefinitely differentiable. Convergence of the generalized functions is defined as suggested by Mikusinski and many of the familiar limit interchanges of distribution theory are established.

It is shown that the space G satisfies a certain completeness property with respect to sequential convergence and this property is

used to show that the Laplace transform maps the elements of G onto the class of all functions which are analytic in some half plane $Rz > a \geq 0$. In section 6 the transforms of distributional derivatives of continuous functions of exponential order are characterized and an inversion formula for such functions is presented.

Convolution of any two elements of G is defined in section 7. We solve a division problem with respect to convolution and then give some applications of convolution equations to linear differential equations with constant coefficients. It is shown that any differential operator with constant coefficients always has an elementary solution. Finally, the operation of multiplying a generalized function of finite order by a sufficiently well-behaved function is defined. Generalized functions which depend upon a parameter are considered and many of the limit interchanges often used in the operational calculus are justified.

In the final section, G is compared with some of the familiar generalized function spaces. The space of tempered distributions is imbedded in G in such a way that the algebraic structure and sequential convergence is preserved. Recently Weston (Proc. Royal Soc. London, 1959) and Miller (Arch. Rat. Mech. Anal., 1963) have constructed generalized function spaces by employing the Laplace transform. These spaces are briefly compared with G.

Introduction

We owe to Mikusinski [6] a very general method of defining generalized functions. His method is to view the generalized functions as equivalence classes of "weakly" convergent sequences. In particular, Mikusinski assumes three spaces, U, V, and W to be given, where W possesses a certain notion of sequential convergence. He also assumes that there is a mapping from $U \times V$ into W and denotes the mapping by (f,g), where $f \in U$, $g \in V$. A sequence $\{f_n\}$ in U is weakly convergent if for every g in V the sequence $\{(f_n,g)\}$ converges in W. Two weakly convergent sequences $\{f_n\}$ and $\{h_n\}$ in U are equivalent if for every g in V, $\lim(f_n,g) = \lim(h_n,g)$ in W. This relation is an equivalence relation and Mikusinski's generalized functions are defined as the equivalence classes determined by the relation.

G. Temple [9] has shown that by choosing the spaces U, V, and W appropriately the space of Schwartz distributions can be obtained by Mikusinski's method. Similarly, Lighthill [5] has constructed the space of tempered distributions.

In this paper we use Mikusinski's construction to extend the classical Laplace transform to a space of generalized functions. In our construction the space U is the space of perfect functions [10], V is the set $\{\exp(-zt)\}$ where z is treated as a parameter, and W is the set of all functions which are analytic in some half place $Rz > a \geq 0$. The mapping from $U \times V$ into W is the Laplace transform. The usual operations and formulas derived for the classical Laplace transform are extended to the space of generalized functions. Our construction also yields some very simple proofs of many of the well known theorems of distribution theory.

We first introduce the generalized functions and then establish some of the elementary properties concerning operations on the generalized functions. The Dirac delta "function" is introduced and the "usual" calculus of functions which have Laplace transforms is imbedded in the calculus of generalized functions. Convergence is defined and many of usual limit interchanges which are considered in distribution theory are justified. The class of functions which are Laplace transforms of generalized functions is characterized, and a characterization of functions which are transforms of distributional derivatives of continuous functions of exponential order is also presented. Convolution and multiplication by a suitably well-behaved function are also defined. Generalized functions depending on a parameter are discussed and some of the limit interchanges often used in the operational calculus are justified.

The class of functions which have classical Laplace transforms is imbedded in the space G of generalized functions in such a way that the Laplace transform defined on G and the classical Laplace transform coincide. Since these functions must satisfy certain growth properties, the space G bears no obvious resemblance to the space of Schwartz distributions [8]. However, the space of tempered distributions with positive support [8], which is constructed from functions with restricted growth properties, is imbedded in G in a natural way which preserves both the algebraic structure and sequential convergence (Theorem 10.1).

Recently, Weston [10] and Miller [7] have constructed generalized function spaces by employing the Laplace transform. Weston's space consists of the distributional derivatives of continuous functions of exponential order [11] and is therefore imbedded in G very naturally. Weston's space suffers from a certain lack of "completeness" in that a sequence of Laplace transforms of generalized functions may converge but there may be no generalized function whose Laplace transform is the limit function. Theorem 4.4 shows that the space G is "complete" in this sense and therefore may be viewed as the "completion" of Weston's space. Miller has constructed several generalized function spaces by completing the space $L_1[0,\infty)$ with respect to various norms and by employing inductive and projective limit topologies. These spaces are compared with G in section 10. It should be noted that Miller has largely neglected developing the calculus of his generalized function spaces whereas we have developed the calculus of G quite extensively; our methods are also of a much simpler nature than those used by Miller.

Some terminology and notation will now be introduced. Denote by A, the class of all functions $f(z)$ which are analytic in some half plane $Rz > a \geq 0$. This half plane may depend on the function. A sequence $\{f_n(z)\}$ of functions in A converges to $f(z)$ in A if there exists a half plane in which the $f_n(z)$ and $f(z)$ are analytic and $\lim f_n(z) = f(z)$ for all z in this half plane, where the convergence is uniform on compact subsets. The convergence will be denoted by $\lim f_n(z) = f(z)$.

The <u>support</u> of a continuous function g is the closure of the set $\{t:g(t) \neq 0\}$. A complex valued function $p(t)$ defined on $(-\infty,\infty)$ will be called <u>perfect</u> if:

(i) it has support contained in $[0,\infty)$,

(ii) it is infinitely differentiable

(iii) $p^{(k)}(0) = 0$ for all $k \geq 0$, and

(iv) $p^{(k)}(t)$ is of exponential order for every $k \geq 0$.

Every perfect function $p(t)$ has a Laplace transform $\int_0^\infty p(t) \exp(-zt)dt$ which determines an element of A [2].

<u>Definition 1.1</u>

A sequence $\{p_n(t)\}$ of perfect functions is called <u>fundamental</u> if the sequence $\{L\{p_n\}(z)\}$ converges in A, where $L\{p_n\}$ denotes the Laplace transform of p_n. Two fundamental sequences $\{p_n(t)\}$ and $\{q_n(t)\}$ <u>are equivalent</u> if $\lim L\{p_n\}(z) = \lim L\{q_n\}(z) = f(z)$.

It is easily seen that the relation defined above is an equivalence relation and the equivalence classes determined by it are called <u>generalized functions</u>. The space of all generalized functions will be denoted by G. If $\{p_n(t)\}$ is a fundamental sequence, the generalized function determined by $\{p_n(t)\}$ will be denoted by $[p_n(t)]$.

Throughout the remainder of this paper ordinary functions will be denoted by small letters whereas, with the single exception of the Dirac delta "function" $\delta(t)$, elements of G will be denoted by capital letters.

Finally, if $X(t) = [x_n(t)]$ is an element of G, the Laplace transform $L\{X(t)\}$ of $X(t)$ is defined by

$$(1) \qquad L\{X(t)\} = \lim L\{x_n(t)\}.$$

It follows easily from the definition of equivalence that the defining equation (1) is independent of the sequence representing $X(t)$. Moreover, $L\{X(t)\}$ is an element of A. Note that the same symbol L has been used to denote the classical Laplace transform and the transform defined on elements of G. In view of the convention regarding small and capital letters that is being used this should present no confusion.

2

Elementary Operations

In this section some of the elementary operations linking the generalized functions and their transforms will be derived.

Let $X(t) = [x_n(t)]$, $Y(t) = [y_n(t)]$ belong to G, and let c be a nonzero complex number.

<u>Definition 2.1</u>. The sum and scalar product are defined by

$$X(t) + Y(t) = [x_n(t) + y_n(t)], \quad cX(t) = [cx_n(t)].$$

It is clear that these definitions make sense and are independent of the sequences defining $X(t)$ and $Y(t)$. Moreover,

$$L\{cX(t)\} = cL\{X(t)\}$$

and

$$L\{X(t) + Y(t)\} = L\{X(t)\} + L\{Y(t)\}.$$

Similarly, we define

<u>Definition 2.2</u>

$$X(ct) = [x_n(ct)] \text{ and } X(t-d) = [x_n(t-d)], \text{ where } d \geq 0.$$

If $x_n(t)$ is of exponential order, i.e., if $x_n(t) = O(\exp(at))$, then we have $x_n(ct) = O(\exp(act))$ and $x_n(t-d) = O(\exp(at))$ so that each function in Definition 2.2 is perfect, and since

$$L\{x_n(ct)\}(z) = 1/c\ L\{x_n(t)\}(z/c)$$

$$L\{x_n(t-d)\}\ (z) = \exp(-dz)\ L\{x_n(t)\}(z),$$

it follows that the operations in Definition 2.2 are independent of the sequence representing $X(t)$. From this, we also have

$$L\{X(ct)\}(z) = 1/c\ L\{X(t)\}(z/c)$$

and

$$L\{X(t-d)\}(z) = \exp(-dz)\ L\{X(t)\}(z).$$

Proceeding in the same manner if $L\{X(t)\} = f$, where f is analytic in $Rz > a$, we obtain:

$$L\{\exp(ct)\ X(t)\}(z) = f(z-c),\ Rz > a + c,$$

and

$$L\{(-t)^n\ X(t)\}(z) = f^{(n)}(z)\ ,\ n = 0,1,\cdots\ .$$

3

The Delta Function

<u>Definition 3.1</u>

A sequence $\{\delta_n(t)\}$ of perfect functions will be called a δ-sequence if:

(i) $\delta_n(t) \geq 0$,

(ii) the support of $\delta_n(t)$ is contained in $[0,1/n]$, and

(iii) $\int_0^\infty \delta_n(t)dt = 1$.

<u>Example 1</u>.

Let

$$\delta_n(t) = \begin{cases} c_n \exp(-1/t + 1/(t-1/n)) & 0 \leq t \leq 1/n \\ 0 & t > 1/n \end{cases}$$

where c_n is such that $\int_0^\infty \delta_n(t)dt = 1$. Then $\{\delta_n(t)\}$ is a δ-sequence.

<u>Proposition 3.1</u>.

If $\{\delta_n(t)\}$ is a δ-sequence, then $\lim L\{\delta_n(t)\} = 1$.

Proof:

Let $\varepsilon > 0$. Each $\delta_n(t)$ is bounded so that $L\{\delta_n(t)\}$ converges for $Rz > 0$. Let K be a compact subset of $Rz > 0$, and let $M = \sup\{|z|: z \in K\}$. Then

$$|\exp(-zt) - 1| \le \sum_{j=1}^{\infty} |zt|^j/j! \le \sum_{j=1}^{\infty} (Mt)^j/j! = \exp(Mt) - 1 = h(t).$$

Since $h(t)$ is continuous and $h(0) = 0$, there exists $d > 0$ such that $|h(t)| < \varepsilon$, whenever, $0 \le t \le d$. Choose $N > 0$ such that $1/N < d$. Then for $n \ge N$, $z \in K$

$$\left| \int_0^{\infty} \delta_n(t)\exp(-zt)dt - 1 \right| \le \int_0^{1/n} \delta_n(t) \left|\exp(-zt) - 1\right| dt$$

$$\le \int_0^{1/n} \delta_n(t)\, h(t)dt < \varepsilon.$$

Thus $\lim L\{\delta_n(t)\} = 1$.

From Proposition 3.1, it follows that all δ-sequences are equivalent and therefore determine a generalized function which will be denoted by $\delta(t)$. Also from Proposition 3.1, we have $L\{\delta(t)\} = 1$, a property usually attributed to the Dirac delta function. More of the usual properties of the delta function will be derived in Sections 4, 7 and 8.

The following proposition will prove useful.

<u>Proposition 3.2</u>.

Let $\{\delta_n(t)\}$ be a δ-sequence and $f(t)$ a continuous function on $[0,\infty)$. Then

$$\lim f*\delta_n(t) = \lim \int_0^t f(t-s)\ \delta_n(s)ds = f(t),$$

where the convergence is uniform on compact subsets of $[0,\infty)$.

Proof:

Extend $f(t)$ to $(-\infty,\infty)$ by defining $f(t) = f(0)$ for $t < 0$. Let $\varepsilon > 0$ and M be a compact subset of $[0,\infty)$. Let b be such that $[0,b] \supseteq M$. Since $f(t)$ is uniformly continuous on $[-1,b]$, there exists $d > 0$ such that $|f(t-s) - f(s)| < \varepsilon$ whenever $|t| < d$ and $t-s,\ s \in [-1,b]$. Now choose $N > 0$ such that $1/N < d$, then for $n \geq N$ and $s \in M$

$$|f*\delta_n(s) - f(s)| \leq \int_0^{1/n} |f(s-t) - f(s)|\delta_n(t)dt < \varepsilon.$$

4

Derivatives, Integrals, and Convergence

Let the kth derivative and the kth iterated integral of a function $x(t)$ be denoted by $x^{(k)}(t)$ and $x^{(-k)}(t)$, respectively. The following proposition is immediate.

Proposition 4.1.

If $\{x_n(t)\}$ is a fundamental sequence such that $\lim L\{x_n(t)\} = g$, then $\{x_n^{(k)}(t)\}$ is a fundamental sequence and $\lim L\{x_n^{(k)}(t)\}(z) = z^k g(z)$ for $k = 0, \pm 1, \cdots$.

Proof:

The functions $x_n^{(k)}(t)$ are perfect and since $L\{x_n^{(k)}(t)\}(z) = z^k g(z)$, the conclusion follows.

Definition 4.1.

Let $X(t) = [\overline{x_n(t)}]$ belong to G. The distributional derivative of $X(t)$, $DX(t)$, is defined by $DX(t) = [\overline{x_n^{(1)}(t)}]$ and the indefinite integral, $D^{-1}X(t)$, is defined by $D^{-1}X(t) = [\overline{x_n^{(-1)}(t)}]$. Higher order derivatives and integrals are defined as usual.

From Proposition 4.1 it follows that the derivative and integral are independent of the sequence representing $X(t)$ and that $L\{D^{(k)}X(t)\}(z) = z^k L\{X(t)\}$. Furthermore, Theorem 4.1.

If $X(t) \varepsilon G$, then $X(t)$ has derivatives of all orders.

Example 4.1.

The identity $L\{D^k\delta(t)\}(z) = z^k$ follows from Proposition 4.1 and Proposition 3.1.

Convergence in G is introduced as suggested by Mikusinski [6], that is, the convergence is defined as a "weak" convergence with respect to the spaces involved.

Definition 4.2.

Let $X_n(t) \varepsilon G$ $(n = 1,2,\cdots)$. The sequence $\{X_n(t)\}$ converges to $X(t) \varepsilon G$ if $\lim L\{X_n(t)\} = L\{X(t)\}$. (Convergence with respect to a continuous parameter is defined analogously in section 9.) We write $\lim X_n(t) = X(t)$.

Proposition 4.2.

If $\lim X_n(t) = X(t)$, $\lim Y_n(t) = Y(t)$, and c is a real number, then $\lim cX_n(t) = cX(t)$ and $\lim (X_n(t) + Y_n(t)) = X(t) + Y(t)$.

The following theorem plays a central role in distribution theory.

Theorem 4.2

If $\lim X_n(t) = X(t)$, then $\lim D^k X_n(t) = D^k X(t)$ for $k = 0, \pm 1, \cdots$.

Proof: $\lim L\{D^k X_n(t)\} = z^k \lim L\{X_n(t)\} = z^k \; L\{X(t)\} = L\{D^k X(t)\}$.

The following theorem will be used later.

Theorem 4.3.

If the $X_n(t) \in G$ $(n = 1,2,\cdots)$ are such that $\lim L\{X_n(t)\}$ exists in A, then there exists $X(t) \in G$, such that $\lim X_n(t) = X(t)$.

Proof: Let $X_n(t) = [x_{nm}(t)]$. Then $\lim_m L\{x_{nm}(t)\} = L\{X_n(t)\}$. By hypothesis the sequence $L\{X_n(t)\}$ converges to a function g which is analytic in a half plane $Rz > a \geq 0$. The double sequence $\{L\{x_{nm}(t)\}\}$ has a subsequence which converges to g, uniformly on compact subsets of $Rz > a$. Let this subsequence be denoted $\{L\{y_k(t)\}\}$. Then $X(t) = [y_k(t)]$ is such that $L\{X(t)\} = g$.

Theorem 4.3 could be interpreted as "completeness" theorem. That is, the set of all functions analytic in some fixed half plane form a complete metric space under the Fréchet metric

$$d(f,g) = \sum_{1}^{\infty} (1/2^n) \; d_n(f,g)/(1 + d_n(f,g)),$$

where f and g are analytic in the half plane, K_n is an expanding sequence of compact sets whose union is the half plane, and $d_n(f,g) = \max\{|f(z) - g(z)|: z \in K_n\}$.

Convergence in this metric is precisely the convergence defined in A. Thus if $X_n(t)$ is a sequence of elements in G such that the $L\{X_n(t)\}$ are analytic in some half plane $Rz > a$ and are a Cauchy sequence in the Fréchet metric, then Theorem 4.3 guarantees that there is an element in G whose transform is $\lim L\{X_n(t)\}$. This matter will be discussed later in Section 10.

The following theorem characterizes $L\{G\}$, the image of G under L.

Theorem 4.4.

Let $f(z)$ be analytic in some half plane $Rz > a \geq 0$. Then there is an element $X(t) \in G$ such that $L\{X(t)\} = f$.

Proof: First note that any polynomial is the transform of an element of G so that from Theorem 4.3 it is enough to show that f is the limit (in the sense of convergence in A) of a sequence of polynomials.

Let D_n be an expanding sequence of closed, bounded rectangles such that $\bigcup_{n=1}^{\infty} D_n$ is the half plane $Rz > a$. Using a result in

[3, p. 303, Theorem 16.6.4], for every $n \geq 0$ there is a sequence $\{p_{nk}(z)\}$ of polynomials such that

$$\lim_{k} \max_{D_n} |p_{nk}(z) - f(z)| = 0.$$

Thus for every n there exists $k(n)$ such that

$$\max_{D_n} |p_{nk(n)}(z) - f(z)| < 1/n.$$

The sequence $\{p_{nk(n)}(z)\}$ is then such that

$$\lim_{n} p_{nk(n)}(z) = f(z).$$

Finally using the convergence defined, another interpretation of the distributional derivative may be given.

Theorem 4.5.

Let $h > 0$ and $X(t) \in G$. Then

$$\lim_{h \to o} [X(t-h) - X(t)]/(-h) = DX(t).$$

Proof: $$L\{[X(t-h) - X(t)]/(-h) - DX(t)\}(z) =$$

$$L\{X(t)\}(z) \cdot [(-1/h)\{\exp(-zh) - 1\} - z] =$$

$$L\{X(t)\}(z) \sum_{n=2}^{\infty} (-z)^n h^{n-1}/n!$$

and the series in the last term converges to 0 in A.

The formula above might be interpreted as a left-hand derivative in the classical sense.

5

Ordinary Functions

In this section the usual calculus of functions having Laplace transforms is imbedded in G. Functions of exponential order are considered first. Let E denote the set of functions defined on $[0,\infty)$ which are locally integrable and of exponential order. If $x(t) \in E$ and $\{\delta_n(t)\}$ is a δ-sequence, the function

$$x*\delta_n(t) = \int_0^t x(t-s)\delta_n(s)\, ds$$

is a perfect function provided that it is agreed that $x*\delta_n(t) = 0$ if $t < 0$. Moreover, $L\{x*\delta_n(t)\} = L\{x(t)\} \cdot L\{\delta_n(t)\}$ so that $\lim L\{x*\delta_n(t)\} = L\{x(t)\}$, and thus $\{x*\delta_n(t)\}$ is a fundamental sequence with $L\{[x*\delta_n(t)]\} = L\{x(t)\}$.

<u>Proposition 5.1</u>.

The correspondence $x(t) \to [x*\delta_n(t)]$ defines a mapping from E into G which preserves sums, scalar products, and the Laplace transforms in the two spaces agree. The element $[x*\delta_n(t)] \in G$ will be denoted by $x(t)^*$.

If $x(t) \in E$ such that $x^{(1)}(t) \in E$, then classically $L\{x^{(1)}(t)\}\ (z) = zL\{x(t)\}\ (z) - x(0)$. However, in the case of generalized functions

$$(1) \quad L\{D(x(t)^*)\} = \lim L\{(x*\delta_n(t))^{(1)}\} = z \lim L\{x*\delta_n(t)\} = z\, L\{x(t)^*\}$$

and

$$(2) \quad L\{(x^{(1)}(t))^*\} = L\{x^{(1)}(t)\} = zL\{x(t)\} - x(0) = zL\{x(t)^*\} - x(0).$$

From (1), (2), and Proposition 3.1,

$$(3) \qquad D(x(t)^*) = (x^{(1)}(t))^* + x(0) \cdot \delta(t),$$

the usual formula connecting the classical and distributional derivatives [4].

Locally integrable functions which are not in E but which do have Laplace transforms can also be imbedded in G. That is, the imbedding above will be extended to include these functions. A locally integrable function $x(t)$ such that $L\{x(t)\}$ converges in $Rz > a$ has the property that $x^{(-1)}(t) = o(\exp(at))$ [2]. $x^{(-1)}(t)^*$ is therefore well defined from the above considerations and also

$$L\{x^{(-1)}(t)^*\}\,(z) = L\{x^{(-1)}(t)\}\,(z) = 1/z\; L\{x(t)\}\,(z).$$

The imbedding above may be extended by defining $x(t) \to D(x^{(-1)}(t)*)$ since if $x(t)$ is of exponential order $x^{(-1)}(0) = 0$ and from (3) $D(x^{(-1)}(t)*) = ((x^{(-1)}(t))^{(1)})^* + x^{(-1)}(0) \cdot \delta(t) = x(t)^*$. This extended correspondence will again be denoted by $x(t) \to x(t)^*$ and the extension is easily seen to preserve sums and scalar products. The Laplace transforms in the two spaces again coincide. Thus the class of functions having Laplace transforms in the classical sense is imbedded in G such that the Laplace transform is invariant.

In this context Definition 1.1 can be given the following interpretation.

Proposition 5.1.

If $X(t) = [x_n(t)] \in G$, then $\lim\limits_n x_n(t)* = X(t)$.

Proof: $\lim\limits_n L\{x_n(t)*\} = \lim\limits_n L\{x_n(t)\} = L\{X(t)\}$.

The following formula is often used in operational calculus.

Example 5.1.

Let $y(t) = 1$ on $[0,\infty)$ and let $p_n(t) = -n[y(t-1/n)-y(t)]$. Then $p_n(t)*$ converges to $\delta(t)$ as $n \to \infty$, for $L\{p_n(t)*\} = (1/z) \cdot (-n \exp(-z/n)-1)$ converges to 1 in A.

The theorem which will now be presented relates convergence in E with convergence in G. It will play an important role in future developments.

Theorem 5.1.

Let $f_n(t)$ be defined on $[0,\infty)$ such that for some integer $r \geq 0$, $f_n^{(-r)}(t)$ is continuous, for some $c > 0$ $|f_n^{(-r)}(t)| \leq M \exp(ct)$ for every n, and $f_n^{(-r)}(t)$ converges uniformly on compact intervals of $[0,\infty)$ to a continuous function $h(t)$. Then

$$\lim L\{f_n^{(-r)}(t)\} = L\{h(t)\}.$$

Proof: Let $\varepsilon > 0$. Let M be a compact subset of $Rz > c$ and set $b = \inf\{x : z = x + iy \varepsilon M\}$. Then $c < b$ and for $z = x + iy\varepsilon M$

$$(1)\quad |L\{h(t) - f_n^{(-r)}(t)\}(z)| \leq \int_0^\infty |h(t) - f_n^{(-r)}(t)|\exp(-xt)dt$$

$$\leq \int_0^\infty |h(t) - f_n^{(-r)}(t)|\exp(-bt)dt$$

$$\leq \int_0^B |h(t) - f_n^{(-r)}(t)\ \exp(-bt)dt$$

$$+ 2M\int_B^\infty \exp(c - b)t\ dt.$$

Now B can be chosen so large that the last term in (1) is $< \varepsilon/2$, and then with such a B fixed the first term can be made $< \varepsilon/2$ for n sufficiently large by virtue of the uniform convergence of $f_n^{(-r)}(t)$

to $h(t)$ on compact intervals. It then follows that

$$\lim L\{f_n^{(-r)}(t)\} = L\{h(t)\}.$$

Corollary 5.1.

If the $f_n(t)$ are perfect and as in Theorem 5.1, then $\{f_n(t)\}$ is fundamental, $[f_n^{(-r)}(t)] = h(t)^*$, and

$$L\{[f_n(t)]\}(z) = z^r L\{h(t)\}(z).$$

Proof: Since $\lim L\{f_n(t)\} = z^r L\{h(t)\}$, $\{f_n(t)\}$ is a fundamental sequence and $L\{[f_n(t)]\}(z) = z^r L\{h(t)\}(z)$. From the conclusion of Theorem 5.1, $[f_n^{(-r)}(t)] = h(t)^*$.

Corollary 5.2.

If the $f_n(t) \varepsilon E$ and are as in Theorem 5.1, then $\lim f_n(t)^* = D^r(h(t)^*)$.

Proof: From Theorem 5.1, $\lim(f_n^{(-r)}(t))^* = h(t)^*$ so that differentiating r times yields $\lim f_n(t)^* = D^r(h(t)^*)$.

6

Generalized Functions of Finite Order

In this section generalized functions which are derivatives of continuous functions will be studied.

Definition 6.1.

An element $X(t)$ of G is of finite order if there is a continuous function $h(t)$ of exponential order such that $X(t) = D^r h(t)*$ for some $r \geq 0$. The smallest such r will be called the order of $X(t)$.

The following definition has been given by Korevarr [4].

Definition 6.2.

A fundamental sequence $\{x_m(t)\}$ is said to be of exponential type (N) if $|x_n^{(-N)}(t)| \leq M \exp(ct)$ for every n and some $c \geq 0$ An element $X(t) \in G$ is of exponential type (N) if $X(t) = [x_m(t)]$, where $\{x_n(t)\}$ is of exponential type (N).

Remark: If $\{x_n(t)\}$ is of exponential type (N), then it is also of exponential type $(N + p)$ for any $p \geq 0$.

The following theorem with some modifications in the statement and proof is again due to Korevarr [4]. We include this for the sake of completeness as it will be used in section 8.

Theorem 6.1

If $\{x_n(t)\}$ is of exponential type (N), then $x_n^{(-N-1)}(t)$ converges uniformly on compact subintervals of $[0,\infty)$ to a continuous function $y(t) \varepsilon E$.

Proof: Let $y_n(t) = x_n^{(-N-1)}(t)$ and note that

(1) $$|y_n(t)| \leq t\, M \exp(ct) \leq M \exp((c+1)t).$$

and

(2) $$|y_n(t_1) - y_n(t_2)| = |t_2 - t_1||x_n^{(-N)}(s)|$$

$$\leq |t_2 - t_1| M \exp(cs),$$

where

$$t_1 < s < t_2.$$

From (2) it follows that $y_n(t)$ is an equi-continuous family on any bounded subinterval of $[0,\infty)$, and from (1) $\{y_n(t)\}$

is also uniformly bounded over any such interval. Thus by Ascoli's theorem, there is a subsequence $\{y_{n1}(t)\}$, which converges uniformly on $[0,1]$. By applying the same argument to the sequence $\{y_{n1}(t)\}$, there is a subsequence $\{y_{n2}(t)\}$ which converges uniformly on $[0,2]$. Thus in general for every m, there exists a subsequence $\{y_{nm}(t)\}$ which converges uniformly on $[0,m]$. The diagonal sequence $\{y_{nn}(t)\}$ then converges uniformly over any compact interval in $[0,\infty)$. Let the limit function be $y(t)$ and observe that $y(t)$ is continuous and of exponential order.

Actually the full sequence $y_n(t)$ converges, for if this were not the case there would exist $\varepsilon > 0$, $B > 0$, $n_k \to \infty$ such that

(3) $$\sup\{|y_{n_k}(t) - y(t)| : t \in [0,B]\} \geq \varepsilon$$

From the first part of the proof since the $y_{n_k}(t)$ are uniformly bounded and equi-continuous on finite intervals, $y_{n_k}(t)$ has a subsequence, say $h_j(t)$, which converges uniformly on compact intervals to a continuous function $h(t)$. From (3),

(4) $$h(t) \neq y(t)$$

and from Theorem 5.1,

$$\lim L\{h_j(t)\} = L\{h(t)\}$$

and

$$\lim L\{y_{nn}(t)\} = L\{y(t)\}$$

so that

$$\lim L\{h_j^{(N+1)}(t)\} = z^{N+1} L\{h(t)\}$$

and

$$\lim L\{x_{nn}(t)\} = z^{N+1} L\{y(t)\}$$

From (4), $\lim L\{h_j(t)\} \neq \lim L\{x_{nn}(t)\}$. This contradicts the fact that $\{x_n(t)\}$ is a fundamental sequence. Therefore, the sequence $\{y_n(t)\}$ converges, uniformly on compact intervals.

Corollary 6.1.

If $\{x_n(t)\}$ is of exponential type (N), then $[x_n(t)]$ is of finite order $\leq N + 1$.

Proof: From Theorem 6.1, and Theorem 5.1,

$$\lim L\{y_n(t)\} = L\{y(t)\}$$

so that

$$\lim L\{x_n(t)\} = z^{N+1} L\{y(t)\}.$$

Hence

$$[x_n(t)] = D^{N+1} y(t)^*.$$

Theorem 6.2

If $X(t)$ is of finite order r, then it is of exponential type (r) .

Proof: Suppose $X(t) = D^r h(t)*$, $|h(t)| \leq M \exp(ct)$. Then

$$|h*\delta_n(t)| \leq \int_0^t |h(s)|\delta_n(t-s)ds \leq M \exp(ct) \int_0^t \delta_n(t-s)ds \leq M \exp(ct).$$

Therefore, since $X(t) = D^r h(t)* = [(h*\delta_n(t))^{(r)}]$, $X(t)$ is of exponential type (r) .

Corollary 6.1 and Theorem 6.2, thus characterize generalized functions of finite order. Another characterization in terms of the transforms will now be presented.

<u>Theorem 6.3.</u>

If $f(z)$ is analytic in $Rz > a \geq 0$ and $f(z) = 0(z^n)$, then $f(z)$ is the Laplace transform of an element of G of order $\leq n + 2$.

Proof: Let $g(z) = z^{-n-2}f(z)$. Then since $g(z) = 0(z^{-2})$, $h(t) = 1/2\pi \exp(at) \int_{-\infty}^{\infty} g(a + iy) \exp(iyt)dy$ is continuous, of exponential order, and $L\{h(t)\} = g$ [2]. Thus

$$L\{D^{n+2}h(t)*\}\ (z) = z^{n+2}g(z) = f(z).$$

<u>Theorem 6.4.</u>

If $X(t) \varepsilon G$ is of finite order r, then $L\{X(t)\}\ (z) = 0(z^{r-1})$.

Proof: If $X(t) = D^r h(t)*$, where $h(t) \varepsilon E$, then

$$L\{X(t)\}\ (z) = z^r L\ h(t)\ (z)$$

which implies

$$L\{X(t)\}\ (z) = 0(z^{r-1}).$$

The following example shows that not all elements of G are of finite order.

Example 6.1.

The series $\sum_{0}^{\infty} D^n\delta(t)/n!$ converges in G since for every k,

$$L\{\sum_{0}^{k} D^n(t)/n!\}(z) = \sum_{0}^{k} z^n/n!$$

and this series converges in A to $\exp(z)$. From Theorem 4.3, the series $\sum_{0}^{\infty} D^n\delta(t)/n!$ defines an element in G and from Theorem 6.4, this element is not of finite order.

The following result might be interpreted as an inversion formula for transforms of elements of finite order.

Theorem 6.5.

Let $X(t) = D^r h(t)^* \in G$, where $h(t)$ is continuous and $h(t) = O(\exp(ct))$. Let $L\{h(t)\} = f$. Then the sequence

$$f_n(t) = (1/2\pi)\exp(ct)\int_{-n}^{n} f(c+iy)\exp(iyt)dy$$

is such that

$$\lim D^r f_n(t)^* = X(t).$$

Proof: The integral $(1/2\pi)\exp(ct)\int_{-\infty}^{\infty} f(c+iy)\exp(iyt)dy$

converges absolutely and uniformly on compact subintervals of $[0,\infty)$

to h(t) [2]. Thus the sequence $f_n(t)$ converges to h(t), uniformly on compact intervals. Moreover,

$$|f_n(t)| \leq (1/2\pi)\exp(ct) \int_{-\infty}^{\infty} |f(c+iy)|dy = M\exp(ct).$$

Thus, from Theorem 5.1, $\lim L\{f_n(t)\} = L\{h(t)\}$ so that

$$(5) \qquad \lim f_n(t)^* = h(t)^*.$$

Differentiating (5) gives $\lim D^r(f_n(t)^*) = X(t)$.

Proposition 6.1.

Let g(z) be analytic in Rz > a and $g(z) = O(z^p)$. If $r = p + 2$, the function $x(t) = (1/2\pi i) \int_{c-i\infty}^{c+i\infty} z^{-r} g(z)\exp(zt)dz$ $(c > a)$ is such that $L\{D^r x(t)^*\} = g$.

Proof: x(t) is well defined, continuous, of exponential order, and such that $L\{x(t)\}(z) = z^{-r}g(z)$. Therefore, $L\{D^r x(t)^*\}(z) = g(z)$.

7

Convolution

Convolution of any two generalized functions will now be defined. Due to the fact that the convolution considered for Laplace transforms is an integral over a finite interval very general results can be derived.

Definition 7.1.

Let $X(t) = [x_n(t)]$ and $Y(t) = [y_n(t)]$ belong to G. The convolution of $X(t)$ and $Y(t)$, $X*Y(t)$, is defined by

$$\left[x_n*y_n(t)\right] = \left[\int_0^t x_n(t - s)y_n(s)ds\right].$$

Since each $x_n*y_n(t)$ is perfect and

$$\lim L\{x_n*y_n(t)\} = \lim L\{x_n(t)\}\cdot L\{y_n(t)\} = L\{X(t)\}\cdot L\{Y(t)\},\ \{x_n*y_n(t)\}$$

is a fundamental sequence, the convolution is independent of the sequences defining $X(t)$ and $Y(t)$, and finally

$$L\{X * Y(t)\} = L\{X(t)\} \cdot L\{Y(t)\}.$$

From Definition 7.1, and the usual properties of the convolution of functions, G is a commutative algebra over the complex field with identity $(\delta(t))$ and no zero divisors, where the multiplicative operation is convolution. Even such formula as

$$D^k X * Y(t) = (D^k X) * Y(t) = X * D^k Y(t)$$

hold for arbitrary $X(t), Y(t) \in G$.

Concerning convergence, the following result holds.

<u>Theorem 7.1</u>.

If $\lim X_n(t) = X(t)$ and $\lim Y_n(t) = Y(t)$, then

$$\lim X_n * Y_n(t) = X * Y(t).$$

Proof: Since $\lim L\{X_n * Y_n(t)\} = \lim L\{X_n(t)\} \cdot L\{Y_n(t)\} = L\{X(t)\} \cdot L\{Y(t)\} = L\{X * Y(t)\}$, the conclusion follows.

Using the fact that $D^p\delta(t) * X(t) = D^p X(t)$ and Theorem 7.1, the result of Theorem 4.2, follows readily.

Some applications of the convolution product will now be considered. We consider the convolution equation

$$A * X(t) = B(t), \tag{1}$$

where $A(t)$ and $B(t)$ are arbitrary elements of G. We show that this equation has a solution $X(t)$ under appropriate conditions.

Theorem 7.2

If $1/L\{A(t)\}$ is analytic in some half plane $Rz > a$, then (1) has a unique solution $X(t)$.

Proof: The function $f = L\{B(t)\}/L\{A(t)\}$ is analytic in some half plane. From Theorem 4.3, there exists $X(t)$ in G such that $L\{X(t)\} = f$. The desired solution of (1) is then $X(t)$. That $X(t)$ is unique follows from the fact that G has no zero divisors.

Now let P be a linear differential operator with constant coefficients. The equation

$$(2) \qquad P\{X(t)\} = B(t)$$

can be interpreted as a convolution equation of the form (1) by setting $A(t) = \sum_{j=o}^{n} a_j D^j \delta(t)$, where $P(x) = \sum_{j=o}^{n} a_j x^j$. Thus since $1/L\{A(t)\}$ is a rational function, from Theorem 7.2, it follows that (2) always has a solution for any $B(t)$ in G. In particular, any such differential operator has an elementary solution, that is, the equation $P\{X(t)\} = \delta(t)$ always has a solution. We also note that if $B(t)$ is of order m, then from Theorem 6.4 $L\{B(t)\} = O(z^{m-1})$. Hence

$$L\{B(t)\}/L\{A(t)\} = O(z^{m-n-1}),$$

and from Theorem 6.3, $X(t)$ is of order $\leq (m - n + 1)$.

Using this same idea we can consider differential operators of "infinite order", $P = \sum_{j=o}^{\infty} a_j D^j$. That is, the equation

$$P\{X(t)\} = B(t) \tag{3}$$

would correspond to the convolution equation (1) with $A(t) = \sum_{j=o}^{\infty} a_j D^j \delta(t)$, provided this series converges. We note that from Theorem 7.1, equation (1) actually has meaning. From Theorem 7.2, this equation will have a solution $X(t)$ if $1/L\{A(t)\}$ is analytic in some half plane. In particular, we note that equation (3) has a solution if $a_j = 1/j!$. This operator would correspond to the "differential operator" e^D which often appears in the Heaviside operational calculus.

8

Multiplication by a Smooth Function

In this section multiplication of a suitably well-behaved function by a generalized function of finite order will be defined. This multiplication corresponds roughly to the convolution of two Schwartz distributions since the Laplace transform of the product of two functions is given by a complex convolution formula. For this reason no attempt at a general definition of such a product will be made and only generalized functions of finite order will be considered. The functions considered will be infinitely differentiable and such that every derivative is of exponential order. Any such function will be called a <u>smooth function</u>.

Several lemmas will be needed for the definition.

<u>Lemma 8.1</u>.

Let $f_n(t)$ be continuous and $|f_n(t)| \leq M \exp(at)$ for $t \geq 0$ and suppose that $f_n(t)$ converges uniformly on compact subsets of $[0,\infty)$ to $f(t)$. Let $g(t)$ be continuous and $|g(t)| \leq M \exp(bt)$. Then

$$\lim L\{g(t)f_n(t)\} = L\{g(t) \cdot f(t)\}.$$

Proof: Since $|f_n(t)g(t)| \leq M \exp((a + b)t)$ and $f_n(t)g(t)$ converges to $f(t)g(t)$, uniformly on compact subsets of $[0,\infty)$, the lemma follows immediately from Theorem 5.1.

Lemma 8.2.

If $f_n(t)$ is of exponential type (N) and $a(t)$ is a smooth function, the sequence $\{a(t) \cdot f_n(t)\}$ is fundamental.

Proof: The proof goes by induction on (N). For $N = 0$, from Theorem 6.1, $f_n^{(-1)}(t)$ converges to a function $f(t)$, where the convergence is uniform on compact subsets. Since

$$(1) \qquad a(t)f_n(t) = (a(t)f_n^{(-1)}(t))^{(1)} - a^{(1)}(t)f_n^{(-1)}(t),$$

it follows from Lemma 8.1 and Proposition 4.1 that the sequence $\{a(t)f_n(t)\}$ is fundamental.

Suppose the lemma holds for $N < k$ and let $\{f_n(t)\}$ be exponential type (k). $f_n^{(-1)}(t)$ is then of exponential type $(k - 1)$. From the induction hypothesis $\{a(t)f_n^{(-1)}(t)\}$ and $\{a^{(1)}(t)f_n^{(-1)}(t)\}$ are fundamental, so that from (1) it follows that $\{a(t)f_n(t)\}$ is fundamental.

Lemma 8.3.

If $X(t) = [x_n(t)] = [y_n(t)]$, where $\{x_n(t)\}$ and $\{y_n(t)\}$ are exponential type and if $a(t)$ is smooth,

$$\lim L\{a(t)(x_n(t) - y_n(t))\} = 0.$$

Proof: By the remark following Definition 6.2, it may be assumed that $\{x_n(t)\}$ and $\{y_n(t)\}$ are both of exponential type (N). The proof again goes by induction on N .

Let $f_n(t) = x_n(t) - y_n(t)$. For $N = 0$, from Theorem 6.1, $f_n^{(-1)}(t)$ converges to 0, uniformly on compact subsets, and $|f_n^{(-1)}(t)| \leq M \exp(bt)$. From Lemma 8.1,

$$\lim L\{a(t)f_n^{(-1)}(t)\} = 0$$

and

$$\lim L\{a^{(1)}(t)f_n^{(-1)}(t)\} = 0.$$

Hence, from equation (1), $\lim L\{a(t) \cdot f_n(t)\} = 0$.

Suppose the result holds for $N < k$ and assume $\{f_n(t)\}$ is of exponential type (k). Then $f_n^{(-1)}(t)$ is of exponential type $(k - 1)$. From the induction hypothesis

$$\lim L\{a(t)f_n^{(-1)}(t)\} = 0$$

and

$$\lim L\{a^{(1)}(t)f_n^{(-1)}(t)\} = 0,$$

so that from equation (1)

$$\lim L\{a(t)f_n(t)\} = 0.$$

The following definition can now be made.

Definition 8.1.

Let $X(t) = [x_n(t)] \in G$, where $\{x_n(t)\}$ is of exponential type (N), and let $a(t)$ be smooth. The product $a(t) \cdot X(t)$ is defined as $a(t) \cdot X(t) = [a(t)x_n(t)]$.

From Lemma 8.2, the sequence $\{a(t)x_n(t)\}$ is fundamental and from Lemma 8.3, the product in Definition 8.1. is well defined.

Remark: If $X(t) = h(t)^* \in G$, where $h(t) \in E$, then the classical Laplace transform of $a(t) \cdot h(t)$ coincides with the generalized Laplace transform $L\{a(t) \cdot X(t)\}$. Indeed, $h(t)^* = [h * \delta_n(t)]$ and from Proposition 3.2, $h * \delta_n(t)$ converges to $h(t)$, uniformly on compact subsets. Therefore, $a(t) \cdot (h * \delta_n(t))$ converges to $a(t) \cdot h(t)$, uniformly on compact subsets, and from Theorem 5.1,

$$\lim L\{a(t) \cdot (h * \delta_n(t))\} = L\{a(t) \cdot h(t)\} = L\{a(t)X(t)\}.$$

Some of the usual identities will now be established.

Proposition 8.1.

Let $a(t)$ be smooth and $X(t) \varepsilon G$ be of finite order. Then $D(a(t) \cdot X(t)) = a^{(1)}(t)X(t) + a(t)DX(t)$.

Proof: Let $X(t) = [x_n(t)]$, where $\{x_n(t)\}$ is of exponential type. Then

$$D(a(t)X(t)) = \left[(a(t)x_n(t))^{(1)}\right] = \left[a^{(1)}(t)x_n(t)\right] + \left[a(t)x_n^{(1)}(t)\right]$$

$$= a^{(1)}(t)X(t) + a(t)DX(t).$$

A generalization of this formula is

$$\text{(2)} \qquad a(t)D^k X(t) = \sum_{j=o}^{k} (-1)^j \binom{k}{j} D^{k-j}(a^{(j)}(t)X(t)).$$

Proposition 8.2.

If $a(t)$ is smooth, then $a(t)\delta(t) = a(0)\delta(t)$.

Proof: Let $\delta_n(t)$ be δ-sequence. Then

$$\lim L\{(a(t) - a(0))\delta_n(t)\} = 0$$

since

$$|L\{(a(t) - a(0))\delta_n(t)\}(z)| \leq \int_o^{1/n} |a(t) - a(0)| \exp(-xt)\delta_n(t)dt$$

$$\leq \max_{o\leq t\leq 1/n} |a(t) - a(0)| = o(1) \quad \text{as} \quad n \to \infty .$$

In a similar fashion it can be shown that $a(t)\delta(t-n) = a(n)$. This identity can be used to illustrate that general limit interchanges are not possible with respect to multiplication. For $\lim \delta(t-n) = 0$, but if $a(t) = t$ the sequence $a(t)\delta(t-n) = n$ will not converge. The following does hold however.

<u>Proposition 8.3</u>.

Let $X_n(t) = D^k h_n(t)^* \in G \quad n \geq 0$, where $h_n(t) \in E$, and suppose $\lim h_n(t) = h_0(t)$, where the convergence is uniform on compact subsets, and $|h_n(t)| \leq M \exp(at)$. Let $a_n(t) (n \geq 0)$ be smooth and such that for $0 \leq j \leq k$, $\lim a_n^{(j)}(t) = a_0^{(j)}(t)$, where the convergence is uniform on compact subsets, and

$$|a_n^{(j)}(t)| \leq M \exp(at).$$

Then

$$\lim a_n(t) X_n(t) = a_0(t) X_0(t).$$

Proof: From equation (2),

$$a_n(t) X_n(t) = \sum_{j=o}^{k} (-1)^j \binom{k}{j} D^{(k-j)}(a_n^{(j)}(t) \cdot h_n(t)^*).$$

From Theorem 5.1, $\lim a_n^{(j)}(t) \cdot h_n(t)^* = a_0^{(j)}(t) \cdot h_0(t)^* \; (0 \leq j \leq k)$.

Therefore,

$$\lim a_n(t)X_n(t) = \sum_{j=o}^{k} (-1)^j \binom{k}{j} D^{k-j}(a_o^{(j)}(t) \cdot h_o(t)*) = a_o(t) \cdot X_o(t).$$

9

Generalized Functions Depending on a Parameter

In this section generalized functions which depend on a parameter will be considered and some of the limit interchanges which are common in distribution theory will be established. Throughout this section, let S be a subset of the complex numbers such that $c \varepsilon S$ is a limit point of S. Suppose that for every $b \varepsilon S$, $X(b,t) \varepsilon G$. Then convergence is defined by $\lim_{b \to c} X(b,t) = X(c,t)$ if $\lim_{b \to c} L\{X(b,t)\} = L\{X(c,t)\}$.

<u>Definition 9.1</u>.

The derivative of $X(b,t)$ with respect to the parameter b at $b = c$ is defined by

$$\lim_{b \to c} 1/(b - c)(X(b,t) - X(c,t)) = D_1X(c,t),$$

provided the limit exists.

When parameters are involved the distributional derivative will be denoted by $D_2X(b,t)$.

<u>Proposition 9.1</u>.

If $D_1X(c,t)$ exists, then $D_{21}X(c,t)$ exists and $D_{21}X(c,t) = D_{12}X(c,t)$.

Proof: Since

$$\lim_{b \to c} L\{1/(b-c)(D_2X(b,t) - D_2X(c,t)\}(z) = zL\{D_1X(c,t)\}(z) = L\{D_{12}X(c,t)\}(z),$$

the proposition follows.

<u>Proposition 9.2</u>.

If $D_1X(c,t)$ exists, then

$$\frac{\partial}{\partial b} L\{X(c,t)\}(z) = L\{D_1X(c,t)\}(z).$$

Proof: Since

$$L\{D_1X(c,t)\}(z) = \lim_{b \to c} L\{1/(b-c)(X(b,t) - X(c,t))\}(z)$$

$$= \lim_{b \to c} \{1/(b-c)\ L\{X(b,t)\}(z) - L\{X(c,t)\}(z)\} = \frac{\partial}{\partial b} L\{X(c,t)\}(z),$$

the conclusion follows.

<u>Remark</u>: Limit interchanges such as the ones given above have been used formally to solve classical partial differential equations by Laplace transform methods. The following example shows that such methods can also be employed in solving partial differential equations in generalized functions. We find a generalized function $Y(b,t)$

depending on the parameter b which is such that:

(1) $$D_{11}Y(b,t) = D_{22}Y(b,t) \quad (b \geq 0)$$

(2) $$Y(0,t) = F(t),$$

where $F(t)$ is a given generalized function. Proceeding in the usual fashion, we obtain

$$L\{Y(b,t)\}(z) = \exp(-bz)\ L\{F(t)\}(z) = L\{F(t - b)\}(z).$$

Hence, $Y(b,t) = F(t-b)$ is the required solution.

Finally, concerning convolution, the following holds,

<u>Proposition 9.3</u>.

If $D_1X(c,t)$ exists, then for every $Y(t) \in G$,

$$D_1(X(c,t) * Y(t)) = Y(t) * D_1X(c,t)$$

Proof: Since

$$\lim_{b \to c} L\{1/(b-c)(X(b,t) * Y(t) - X(c,t) * Y(t))\} =$$

$$L\{Y(t)\}\ L\{D_1X(c,t)\} = L\{Y(t) * D_1X(c,t)\},$$

$D_1(x(c,t) * Y(t))$ exists and is equal to $Y(t) * D_1X(c,t)$.

10

Comparisons with Other Generalized Function Spaces

In this final section, G is compared with other spaces of generalized functions. Of course, the most familiar of the generalized function spaces is Schwartz's space (D') [8]. Any locally integrable function is imbedded in (D') in a natural way so that a function such as $\exp(t^2)$ is included in (D'), whereas it is not imbedded in G since it grows too rapidly at ∞. Example 6.1 shows that there are elements in G which are not distributions in the sense of Schwartz since the series

$$\sum_{0}^{\infty} 1/k!\ D^k\delta(t)$$

does not converge in (D').

If S' denotes the space of tempered distributions and D' the space of distributions with positive support, then the space $S' \cap D'$ can be imbedded in G in a natural way. An element $T \in S' \cap D'$ has a representation in the form

$$T = D^k(1 + t^2)^m f(t), \tag{1}$$

where $f(t)$ is a bounded continuous function which vanishes for $t < 0$.

The Laplace transform of such an element is defined by

$$F(z) = T \cdot \exp(-zt) = z^k \int_0^\infty (1 + t^2)^m f(t)dt \quad [1]$$

which is in agreement with our definition. This shows that the correspondence

$$T \to D^k((1 + t^2)^m f(t))^* = T(t)$$

defines a mapping from $S' \cap D'$ into G, that is, the correspondence is independent of the representation (1) of T. This mapping clearly preserves sums, scalar products, and differentiation. Concerning convergence in the spaces, we have:

<u>Theorem 10.1</u>

Let $T_j \in S' \cap D'$ $(j \geq 1)$ be such that T_j converges to 0 in S'. Then $T_j(t)$ converges to 0 in G.

Proof: Since T_j converges to 0 in S', there exists k and m such that

$$T_j = D^k(1 + t^2)^m f_j(t),$$

where

$$\lim_j \sup_t |f_j(t)| = 0.$$

Therefore the sequence $\{(1 + t^2)^m f_j(t)\}$ converges to 0, uniformly on compact subsets, and, moreover, there are positive constants M and a such that $|(1 + t^2)^m f_j(t)| \leq M \exp(at)$. Thus by applying Theorem 5.1 and differentiating k times, it follows that the sequence $T_j(t)$ converges to 0 in G.

That $S' \cap D'$ does not exhaust G is easily seen by considering the element $(\exp(t))^*$ in G. Example 6.1 furnishes an example of a series which converges in G but not in S'.

Recently Weston [10] has constructed an operational calculus by using the Laplace transform. In his calculus perfect operators play the role of generalized functions. In [11], Weston showed that any perfect operator is the derivative (in the distributional sense) of a continuous function of exponential order so that the perfect operators can be identified with the elements of G which are of finite order. This identification or imbedding preserves sums, scalar products, and differentiation. Example 6.1 shows that there are elements in G which are not identified with any perfect operator. Weston's space also lacks a certain "completeness" property in that the Laplace transforms of a sequence of perfect operators may converge (in the sense of convergence in Λ) but there may be no perfect operator whose Laplace transform is the limit function. For example, exp(z) is not the transform of a perfect operator. From Theorem 4.3, we see that G does not suffer from this defect.

Miller [7] has also constructed several spaces of generalized functions by using the Laplace transform. We compare them very briefly with G. First the space of transforms of elements of X_F^+ may contain functions which are not analytic or only analytic in some bounded domain and therefore X_F^+ bears little resemblance to G. However, for any seat F, G will contain elements which are not in X_F; for if $a \in F$, then (exp(at))* is in G but not in X_F.

The space X_w^+ could be derived from the procedure we have employed if all functions considered were required to be analytic in the fixed half plane $Rz > w$. The topology thus imposed would be exactly the compact open topology which is the same topology that Miller has put on X_w^+. The space G is, of course, more general than X_w^+.

Finally, the space X is isomorphic to G in the algebraic sense. This follows from Theorem 4.4 and III in [7]. The topology which is imposed on X is not very manageable, whereas the convergence in G is quite simple.

In conclusion it should be noted that Miller has used some very powerful tools to construct the spaces considered above while the construction of G is based on a simple, elegant idea of Mikusinski.

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