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# ANALYSIS OF TIME-VARYING NETWORKS

by

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I hereby recommend that this dissertation prepared under my direction by Subrata K. Das

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## ABSTRACT

A method for analyzing linear time-varying systems has been developed in this dissertation. The systems under consideration are assumed to be stable in the sense that the solutions do not possess any singularity in the time-interval from zero to infinity and approach finite values after infinite time. Both the differential equation representation and the electrical network representation are dealt with.

First, several mathematical concepts that are useful are introduced. The discussions there center around the solution of the differential equations and the subject of approximation of functions.

Next, basic current-voltage inter-relationships for ordinary time-varying elements are given. This is followed by a discussion of a method of representation of the periodic time-varying systems. Analysis of the periodic systems is then considered. The results are obtained in the form of finite Fourier series. After this, a frequency converter system is described. This system is an outcome of the discussion of the method of analysis.

Time-varying systems with arbitrary time-variation are then examined. Both the basis functions (excitation-free response) and the particular integral (response due to a particular excitation) may be conveniently obtained. Starting

with a treatment of the differential equations with polynomial coefficients the method is later extended to cover equations with non-polynomial coefficients. Finally, a method of solving electrical networks is outlined. The procedure developed is suitable for computer programming. The answer is always obtained in a particular series form.

# CHAPTER 1

# INTRODUCTION

The subject of time-varying linear systems has engaged the interest and efforts of numerous investigators over a long period of time. Zadeh's article[1] summarizes the references in this area. Many authors have focussed attention at deriving specific criteria aimed at assuring stability.

Bongiorno[2] derived bounds on the time-varying parameters that are sufficient for stability. Darlington[3] also derived sufficient conditions for stability and some expressions of power in time-varying circuits. A more recent investigation by Kuh[4] studied the stability problem by employing the A-matrix of Bashkow[5] and Bryant[6]. Rohrer[7] deduced his stability criteria by considering the variation of stored energy in the system. Various time-domain stability criteria have also been described by Youla[8] and Desoer and Paige[9]. The question of stability of a special but important class of systems has been the subject of Sandberg's research[10]. Capacitors are the only time-varying components in his systems. The importance of his results stems from the fact that time-varying parametric capacitors are readily realized by reverse-biasing varactor diodes with a variable voltage source. In fact, the theory of parametric circuits is quite extensive[11-25].

Other researchers have concentrated their efforts on differing aspects of time-varying systems. McMahon[26], for instance, derived a synthesis technique for a class of time-varying systems based on transform techniques. Silverman and Meadows[27] usuing the concept of controllability demonstrated the use of time-varying networks to produce unilateral systems. Gagliardi[28] obtained a method of obtaining the differential equation of a time-varying network.

Turning attention to the analysis problem it is seen that a general method for solving a time-varying problem explicitely is not available. Gilbert[29] described a method of finding the response for various inputs in time-varying systems by approximating the weighting function in a convenient form, but the problem of finding the weighting function was unsolved. Kurth[30] treated the analysis of time-varying networks in the frequency domain. His method led to an infinite system of equations and lacked an error criterion. Applications of Mellin and Hankel transforms for solving time-varying systems was demonstrated by Gerardi[31]. However, the scope of his treatment was limited as it was necessary to develop a new transform for each type of system.

Numerical methods have often been used to solve time-varying system problems with limited success. The methods of Runge-Kutta, Milne and others[32,33] have found some applications since the advent of computers. However, a good error criterion is still lacking. Another disadvantage of the numerical methods is that they produce their results in the form of a set of values on a finite number of discrete points and the behavior of the solution beyond these points can only be surmised. Moreover, the system must be represented as an ordinary differential equation before any solution can be attempted. This is a drawback since deriving the differential equation from a network of time-varying resistors, inductors and capacitors is a problem in itself[28].

The method presented in this dissertation seeks to overcome as many as possible of these difficulties. A simple error criterion is formulated to ensure the accuracy of the results. The method, besides being applicable when the system is represented by an ordinary differential equation, can also be applied directly to electrical circuits which have the form of a ladder network with one component in each arm. The solutions are obtained over the time interval from zero to infinity. It is assumed that the systems under consideration have stable solutions in the sense that the solutions have no singularity in that interval and approach finite values after infinite time. This type of system is the one that is usually of most interest.

The material of the thesis is arranged as follows. The discussions in Chapter 2 are aimed at providing a background on which future developments proceed. Chapter 3 is mainly concerned with the periodic time-variation of systems whereas more general systems are described in Chapter 4. Chapter 5 summarizes the preceding material and gives recommendations for future works in this area.

# CHAPTER 2

# MATHEMATICAL BACKGROUND

In this chapter, an attempt is made to introduce some mathematical concepts related to the present research. More detailed treatment on many of the topics may be found in the works of various authors[34-42]. The ensuing discussion considers some properties of a linear differential equation.

## 2.1. THE LINEAR DIFFERENTIAL EQUATION

For a time-varying system, a general linear differential equation may be written in the form

$$\sum_{i=1}^{m} c_{i+1}(t)\frac{d^i v}{dt^i}+c_1(t)v = \sum_{i=1}^{n} d_{i+1}(t)\frac{d^i u_1}{dt^i}+d_1(t)u_1 \qquad (2.1)$$

where $c_i$ $(i=1,2,\ldots,m)$ and $d_i (i=1,2,\ldots,n)$ represent real valued functions of the independent variable t and v and $u_1$ may be considered as the output and the input of the system. Unless mentioned otherwise, our primary concern shall be the time interval, $0 \leq t \leq \infty$. Referring to equation (2.1), $u_1$ is a known function and it is assumed that its first n derivatives exist. Then, the right hand side of equation (2.1) may be represented by a single function, u. This leads to the following equation which will henceforth be our concern.

$$\sum_{i=1}^{m} c_{i+1}(t)\frac{d^i v}{dt^i} + c_1(t)v = u \qquad (2.2)$$

If $u(t) = 0$, $0 \leq t \leq \infty$ equation (2.2) takes the form of

$$\sum_{i=1}^{m} c_{i+1}(t)\frac{d^i v}{dt^i} + c_1(t)v = 0 \qquad (2.3)$$

and is referred to as the homogeneous differential equation. Otherwise, equation (2.2) will be termed a non-homogeneous equation.

Before proceeding any further the concept of linear independency needs to be described first. If $\alpha_i$ $(i=1,2,\ldots,m)$ are arbitrary constants the functions $w_i$ $(i=1,2,\ldots,m)$ are said to be linearly independent in a particular interval if in that interval $\sum_{i=1}^{m} \alpha_i w_i = 0$ implies that $\alpha_i = 0$ $(i=1,2,\ldots,m)$. Otherwise, the functions $w_i$ $(i=1,2,\ldots,m)$ are called linearly dependent functions. The following theorem regarding the solutions of equation (2.3) may now be stated.

Theorem 2.1. The differential equation (2.3) has m linearly independent solutions.

Writing the m solutions mentioned in the above theorem as $w_i$ $(i=1,2,\ldots,m)$ it is obvious that any linear combination of $w_i$, $\sum_{i=1}^{m} \beta_i w_i$ where the $\beta_i$ $(i=1,2,\ldots,m)$ are assumed to be real constants is also a solution of equation (2.3). The following theorem may be stated in this connection.

Theorem 2.2. If the function w represents any particular solution of the differential equation (2.3) it may be expressed as $w = \sum_{i=1}^{m} \beta_i w_i$. In this expression,

$\beta_i$ $(i=1,2,\ldots,m)$ represent arbitrary real constants.

This property implies that the set of functions $w_i$ $(i=1,2,\ldots,m)$ forms a linear space of functions and the basis for the space is $w_i$ $(i=1,2,\ldots,m)$. Thus in future discussions $w_i$ $(i=1,2,\ldots,m)$ will be referred to as the basis functions.

Certain topics in the theory of differential equations appear to be pertinent for discussion. The first of these deals with the method of reducing the order of a homogeneous differential equation when one basis function is known.

## 2.2. REDUCTION OF THE ORDER OF A LINEAR HOMOGENEOUS DIFFERENTIAL EQUATION

Consider the m-th order homogeneous differential equation

$$\sum_{i=1}^{m} c_{i+1}(t)\frac{d^i v}{dt^i}+c_1(t)v=0 \tag{2.3}$$

If one basis function $w_1$ of equation (2.3) is known the following method may be used to reduce the order of the differential equation. The ultimate motive for such reduction is to derive a second basis function. The second basis function, $w_2$, is obtained as $w_2=xw_1$ where x is an unknown function. Substitution of $w_2$ in equation (2.3) leads to

$$\sum_{i=1}^{m} c_{i+1}(t)\frac{d^i(xw_1)}{dt^i}+c_1(t)xw_1=0.$$

When the above equation is expanded by introducing

$$\frac{d(xw_1)}{dt} = x\frac{dw_1}{dt} + \frac{dx}{dt}w_1$$

$$\frac{d^2(xw_1)}{dt^2} = x\frac{d^2w_1}{dt^2} + 2\frac{dx}{dt}\frac{dw_1}{dt} + \frac{d^2x}{dt^2}w_1$$

..................................

$$\frac{d^m(xw_1)}{dt^m} = x\frac{d^mw_1}{dt^m} + \dots + \frac{d^mx}{dt^m}w_1$$

the coefficient of x is

$$\sum_{i=1}^{m} c_{i+1}(t)\frac{d^iw_1}{dt^i} + c_1(t)w_1$$

which clearly equals zero. Thus, if the designation $\frac{dx}{dt} = y$ is adopted the new differential equation in y

$$w_1\frac{d^{m-1}y}{dt^{m-1}} + \dots + \left[m\frac{d^{m-1}w_1}{dt^{m-1}} + (m-1)\frac{d^{m-2}w_1}{dt^{m-2}} + \dots + c_2(t)w_1\right]y = 0 \tag{2.4}$$

is one order less than equation (2.3). If $\gamma_1, \gamma_2, \dots, \gamma_{m-1}$ are the (m-1) basis functions of equation (2.4) the basis functions for the equation (2.3) are given as

$$w_1,\ w_1\int_{t_o}^{t}\gamma_i(t)\,dt \quad (i = 1,2,\dots,m-1) \tag{2.5}$$

where $t_o$ is an arbitrary point in the interval under consideration.

In Chapter 4 a method for finding $w_1$ will be developed. Once this is determined it will be possible to reduce the order of the differential equation to the form of equation (2.4). The basis function $\nu_1$ of equation (2.4) may then be obtained and the relation (2.5) may be used to determine a second basis function of equation (2.3). Next, with the help of $\nu_1$ and equation (2.4) the order of the equation may be further reduced and a third basis function obtained. This process may be continued till the solutions for all the basis functions have been arrived at.

Determination of all the basis functions will bring the investigation of the homogeneous differential equation (2.3) to an end. Following that the concern will be the non-homogeneous equation (2.2) with $u(t) \neq 0$ for some t in $[0,\infty]$.

## 2.3. THE NON-HOMOGENEOUS DIFFERENTIAL EQUATION

A solution to the non-homogeneous equation (2.2) is termed the particular integral, $w_p$. The particular integral may be determined from a knowledge of the basis functions $w_i$ $(i = 1,2,\ldots,m)$ of the corresponding homogeneous equation (2.3) and of the function u(t). The method begins by assuming that a solution for $w_p$ may be written as

$$w_p = \sum_{i=1}^{m} x_i w_i \tag{2.6}$$

where the functions $x_i$ are yet to be determined. If the $x_i$ are properly selected to satisfy the relations

$$\sum_{i=1}^{m} \frac{dx_i}{dt} w_i = 0$$

$$\sum_{i=1}^{m} \frac{dx_i}{dt} \frac{d^j w_i}{dt^j} = 0 \qquad (j=1,2,\ldots,m-1)$$

and

$$\sum_{i=1}^{m} \frac{dx_i}{dt} \frac{d^m w_i}{dt^m} = u \qquad (2.7)$$

it is easy to verify that the following equations will be valid :

$$w_p = \sum_{i=1}^{m} x_i w_i$$

$$\frac{d^j w_p}{dt^j} = \sum_{i=1}^{m} x_i \frac{d^j w_i}{dt^j} \qquad (j=1,2,\ldots,m-1)$$

and

$$\frac{d^m w_p}{dt^m} = \sum_{i=1}^{m} x_i \frac{d^m w_i}{dt^m} + u \qquad (2.8)$$

Equations (2.8) imply that

$$\sum_{i=1}^{m} c_{i+1}(t) \frac{d^i w_p}{dt^i} + c_1(t) w_p$$

$$= \sum_{j=1}^{m} x_j \left[ \sum_{i=1}^{m} c_{i+1}(t) \frac{d^i w_j}{dt} + c_1(t) w_j \right] + u$$

$$= u$$

Thus, if the functions $x_i$ $(i=1,2,\ldots,m)$ are found by solving equation (2.7) then $w_p$, as obtained from equation (2.6), would be the particular integral. The general solution w for the differential equation (2.2) may now be written as

$$v = v_p + \sum_{i=1}^{m} \beta_i v_i \tag{2.9}$$

where $\beta_i$ $(i=1,2,\ldots,m)$ are arbitrary real constants.

For the present, this concludes the discussion of differential equations. In the next two chapters, these will be referred to again in due context. The next subject that will be considered centers around certain properties of functions, specifically, the properties related to the theory of approximations.

## 2.4. SOME PROPERTIES OF FUNCTIONS

The first notion to be presented in this section is that of uniform convergence. The following definition is relevant.

Definition. An infinite sequence of functions $f_n$ converges uniformly on a set E to a function F if and only if for any $\epsilon > 0$ there is a number N such that for any $n \geq N$ and any $x \in E$, $|F(x)-f_n(x)| \leq \epsilon$.

The above concept of uniform convergence may be extended to introduce the idea of uniform approximation of a function F by a polynomial. This is accomplished by equating $f_n = \sum_{i=0}^{n} k_i x^i$ in the above definition, where the $k_i$ are real constants. The well-known Weierstrass theorem may be stated in this connection.

Theorem 2.3. Given $\epsilon>0$, a function F continuous on the closed interval [a,b] can be uniformly approximated to within $\epsilon$ on [a,b] by a polynomial.

The above theorem deals with the approximation of any continuous function including the case when the continuous function is the function identically zero in the finite interval [a,b]. An estimation of the error involved in the approximation is obtained by squaring the difference between the approximate and the true functions and integrating the result over the range of interest. The next theorem shows how a linear combination of several functions may approach zero in [a,b] in the sense that the error estimated as above decreases. That is, $\int_a^b [(\phi_1(\mathcal{T}) - \sum_{i=1}^{n} k_i \phi_{i+1}(\mathcal{T}))-0]^2 d\mathcal{T} = \int_a^b [\phi_1(\mathcal{T}) - \sum_{i=1}^{n} k_i \phi_{i+1}(\mathcal{T})]^2 d\mathcal{T}$ will decrease as n is increased.

Theorem 2.4. Let $\phi_i(\mathcal{T})$ $(i=1,2,\ldots,n+1)$ be linearly independent functions in [a,b] such that $\int_a^b \phi_i(\mathcal{T})\,\phi_j(\mathcal{T})\,d\mathcal{T}$ $(i,j=1,2,\ldots,n+1)$ exist and at least one $\int_a^b \phi_i(\mathcal{T})\,\phi_1(\mathcal{T})\,d\mathcal{T}$ $(i=2,3,\ldots,n+1)$ is non-zero. Then there exist constants $k_i$ $(i=1,2,\ldots,n)$ such that

$$I_1 = \int_a^b \phi_1^2(\mathcal{T})\,d\mathcal{T} > \int_a^b [\phi_1(\mathcal{T}) - \sum_{i=1}^{n} k_i \phi_{i+1}(\mathcal{T})]^2 d\mathcal{T} = I_2 \tag{2.10}$$

In order to prove the above theorem select

$$k = \psi^{-1}\psi_1 \tag{2.11}$$

where

$$k = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} \qquad \psi = \begin{bmatrix} \psi_{22} & \psi_{23} & \cdots & \psi_{2,n+1} \\ \psi_{32} & \psi_{33} & \cdots & \psi_{3,n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \psi_{n+1,2} & \psi_{n+1,3} & \cdots & \psi_{n+1,n+1} \end{bmatrix}$$

$$\psi_1 = \begin{bmatrix} \psi_{21} \\ \psi_{31} \\ \vdots \\ \psi_{n+1,1} \end{bmatrix} \qquad \psi_{ji} = \int_a^b \phi_j \, \phi_i \, d\tau$$

Equation (2.11) is a valid one if $\psi$ is non-singular. To show the property of non-singularity note that if $\psi$ is singular there exist non-zero constants $\alpha_i$ such that

$$0 = \sum_{i=1}^{n} \alpha_i \psi_{i+1,j+1} = \sum_{i=1}^{n} \alpha_i \int_a^b \phi_{i+1} \phi_{j+1} d\tau$$

$$= \int_a^b \Big( \sum_{i=1}^{n} \alpha_i \phi_{i+1} \Big) \phi_{j+1} d\tau \qquad (j = 1, 2, \ldots, n)$$

Since the above equation is valid for each $j$ it may also be summed over $j$ to yield

$$0 = \sum_{j=1}^{n} \alpha_j \int_a^b \Big( \sum_{i=1}^{n} \alpha_i \phi_{i+1} \Big) \phi_{j+1} d\tau$$

$$= \int_a^b \Big( \sum_{i=1}^{n} \alpha_i \phi_{i+1} \Big) \Big( \sum_{j=1}^{n} \alpha_j \phi_{j+1} \Big) d\tau$$

$$= \int_a^b \Big( \sum_{i=1}^{n} \alpha_i \phi_{i+1} \Big)^2 d\tau$$

This, of course, implies that $\sum_{i=1}^{n} \alpha_i \phi_{i+1} = 0$ which contradicts the assumption that the $\phi_i$ are linearly independent. Thus, the $\psi$ matrix is always non-singular and the validity of equation (2.11) is established. Also note from this equation and the statement of the theorem that at least one of the $k_i$ must be non-zero.

Rearranging equation (2.11) and premultiplying by k' where the prime stands for the transpose the following equation is obtained :

$$k' \Psi \, k = k' \Psi_1 \tag{2.12}$$

The left side of this equation may be expanded as follows :

$$k' \Psi \, k = \underline{k_1 \; k_2 \ldots k_n} \begin{bmatrix} \Psi_{22} & \Psi_{23} & \cdots \Psi_{2,n+1} \\ \Psi_{32} & \Psi_{33} & \cdots \Psi_{3,n+1} \\ \cdots & \cdots & \cdots \\ \Psi_{n+1,2} & \Psi_{n+1,3} & \cdots \Psi_{n+1,n+1} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

$$= \int_a^b \underline{k_1 \; k_2 \ldots k_n} \begin{bmatrix} \phi_2^2 & \phi_2\phi_3 \cdots \phi_2\phi_{n+1} \\ \phi_3\phi_2 & \phi_3^2 \cdots \phi_3\phi_{n+1} \\ \cdots & \cdots \\ \phi_{n+1}\phi_2 & \phi_{n+1}\phi_3 \cdots \phi_{n+1}^2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} d\mathcal{T}$$

$$= \int_a^b \underline{k_1 \; k_2 \ldots k_n} \begin{bmatrix} \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{n+1} \end{bmatrix} \underline{\phi_2 \;\; \phi_3 \cdots \phi_{n+1}} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} d\mathcal{T}$$

Since $\underline{\phi_2 \;\; \phi_3 \cdots \phi_{n+1}} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$ is a scalar it equals the transpose of itself. Thus,

$$k' \Psi \, k = \int_a^b \underline{k_1 \; k_2 \ldots k_n} \begin{bmatrix} \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{n+1} \end{bmatrix} \underline{k_1 \; k_2 \ldots k_n} \begin{bmatrix} \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{n+1} \end{bmatrix} d\mathcal{T}$$

$$=\int_a^b \left( \begin{pmatrix} k_1 & k_2 & \cdots & k_n \end{pmatrix} \begin{bmatrix} \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{n+1} \end{bmatrix} \right)^2 d\tau$$

$$=\int_a^b \left( \sum_{i=1}^{n} k_i \, \phi_{i+1} \right)^2 d\tau \tag{2.13}$$

The right hand side of equation (2.12), on the other hand, yields

$$k' \psi_1 = \sum_{i=1}^{n} k_i \, \psi_{i+1,1} \tag{2.14}$$

Thus, from equations (2.12), (2.13) and (2.14) the following equation may be written :

$$\sum_{i=1}^{n} k_i \, \psi_{i+1,1} = \int_a^b \left( \sum_{i=1}^{n} k_i \, \phi_{i+1} \right)^2 d\tau \tag{2.15}$$

From equation (2.10) it follows that

$$I_1 - I_2 = \int_a^b \left[ \phi_1^2 - \left( \phi_1 - \sum_{i=1}^{n} k_i \, \phi_{i+1} \right)^2 \right] d\tau$$

$$= \int_a^b \left( 2\phi_1 - \sum_{i=1}^{n} k_i \, \phi_{i+1} \right) \left( \sum_{i=1}^{n} k_i \, \phi_{i+1} \right) d\tau$$

$$= \int_a^b 2\phi_1 \left( \sum_{i=1}^{n} k_i \, \phi_{i+1} \right) - \int_a^b \left( \sum_{i=1}^{n} k_i \, \phi_{i+1} \right)^2 d\tau$$

$$= 2 \sum_{i=1}^{n} k_i \, \psi_{i+1,1} - \int_a^b \left( \sum_{i=1}^{n} k_i \, \phi_{i+1} \right)^2 d\tau \tag{2.16}$$

Finally, from equations (2.15) and (2.16), it may be seen that

$$I_1 - I_2 = \int_a^b \left( \sum_{i=1}^{n} k_i \, \phi_{i+1} \right)^2 d\tau > 0$$

which proves the theorem. In later discussions, the above theorem will be used in situations where the $\phi_i$ are polynomials.

The expression for k as given in equation (2.11) was assumed in proving the above theorem. It may be demonstrated that this choice of k produces the minimum estimate for $I_2$. Thus, $I_2$ reaches a minimum when the partial derivatives $\frac{\partial I_2}{\partial k_i} = 0$ $(i=1,2,\ldots,n)$. Computing the partial derivatives the following system of equations is obtained :

$$\sum_{i=1}^{n} k_i \, \psi_{j+1,i+1} = \psi_{j+1,1} \quad (j=1,2,\ldots,n) \tag{2.17}$$

where the $\psi_{ji}$ have the same meaning as that in the above theorem. It is asy to see that equation (2.17), when in matrix form is identical to equation (2.11).

This chapter has provided some mathematical background for future topics. In the next chapter, this background will be applied to a discussion of the properties of periodic systems.

# CHAPTER 3

# PERIODIC SYSTEMS

A linear time-varying system is usually represented by a differential equation or (in the case of an electrical network) by a circuit diagram. The system has a number of force-free solutions (basis functions), the number of them being equal to the order of the system and a solution (particular integral) produced due to the application of an excitation. In general, no method exists for obtaining exact solutions. Even if exact solutions are found they are often not expressible in convenient forms making it difficult to gain any insight into the nature of the system. In this dissertation, the aim is to derive a method of finding useful approximate solutions. To estimate the accuracy of the approximation an error criterion will be developed. In this chapter, after introducing a few topics basic to time-varying systems attention is focussed on periodic systems. In the next chapter, more general systems will be considered.

## 3.1. VOLTAGE-CURRENT RELATIONS IN TIME-VARYING CIRCUIT ELEMENTS

Time-varying components impose constraints on various variables of the system. In this section, the relations between voltage and current in the commoner two-terminal electrical circuit elements will be indicated. Let $i_r(t)$, $i_\ell(t)$ and $i_c(t)$ denote the currents flowing through the time-varying resistor $r(t)$, inductance $\ell(t)$ and capacitance $c(t)$ respectively. The voltages developed across these elements are given by the expressions

$$v_r(t) = r(t)i_r(t)$$
$$v_\ell(t) = \frac{d}{dt}[\ell(t)i(t)] = \ell(t)\frac{di_\ell(t)}{dt} + i_\ell(t)\frac{d\ell(t)}{dt}$$
$$v_c(t) = \frac{1}{c(t)}\int_{-\infty}^{t} i_c(t)\ dt \qquad (3.1)$$

On the other hand, if the voltages $v_r(t)$, $v_\ell(t)$ and $v_c(t)$ are applied across the resistor $r(t)$, inductance $\ell(t)$ and capacitance $c(t)$ respectively the currents flowing in them will be

$$i_r(t) = \frac{v_r(t)}{r(t)}$$
$$i_\ell(t) = \frac{1}{\ell(t)}\int_{-\infty}^{t} v_\ell(t)\ dt$$

and,
$$i_c(t) = \frac{d}{dt}[c(t)v_c(t)] = c(t)\frac{dv_c(t)}{dt} + v_c(t)\frac{dc(t)}{dt} \qquad (3.2)$$

These current-voltage inter-relationships will be used in future developments.

## 3.2. A REPRESENTATION OF A PERIODIC SYSTEM

In the differential equation (2.3), if the coefficients $c_i(t)$ $(i=1,2,\ldots,m+1)$ are assumed to be periodic functions of time (having period T) the system that the equation represents is referred to as a periodic system and the equation is termed a periodic differential equation. The right hand side of equation (2.3) is zero, implying that no outside excitation is impressed on the system. It is known from the Floquet-Poincare theory[38] that the periodic differential equation (2.3) possesses m linearly independent solutions, $w_i$, of the form

$$w_i(t) = e^{\rho_i(t)}\psi_i(t) \qquad (i=1,2,\ldots,m) \tag{3.3}$$

In this equation, each $\psi_i(t)$ is a periodic function of time of period T when the $e^{\rho_i T}$ $(i=1,2,\ldots,m)$ are distinct. The general solution for this excitation-free system is given as

$$w = \sum_{i=1}^{m}\beta_i w_i(t) \tag{3.4}$$

where the $\beta_i$ $(i=1,2,\ldots,m)$ are arbitrary real constants.

Now consider the network shown in Fig.3.1. In this, let $g_i(t)=\dfrac{-\rho_i}{\psi_i(t)}$ and $c_i(t)=\dfrac{1}{\psi_i(t)}$. Then, by inspection, the differential equation for this network is

$$\frac{d}{dt}\left(\frac{w_i}{\psi_i}\right) - \rho_i\left(\frac{w_i}{\psi_i}\right) = 0 \tag{3.5}$$

and the solution for this is evidently

$$w_i = e^{\rho_i(t)}\psi_i(t) \tag{3.6}$$

which by comparing with equation (3.3) is the same

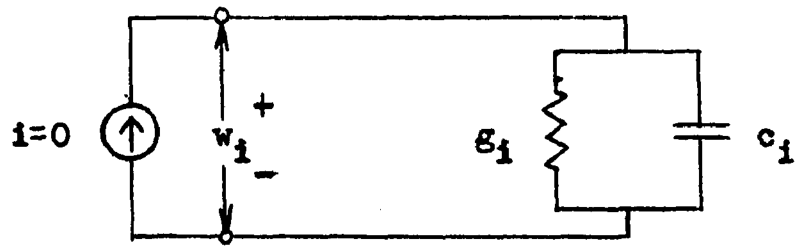



Fig.3.1 Network realizing one basis function

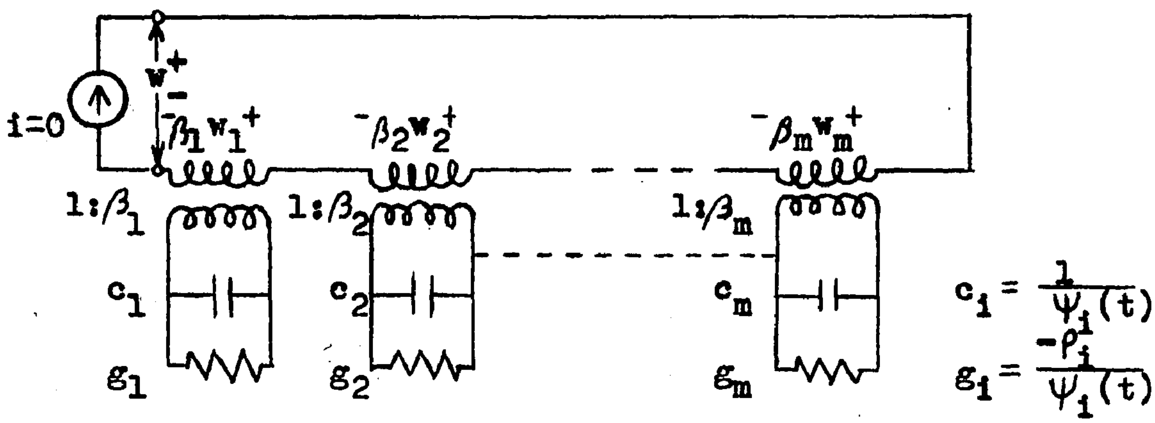



Fig.3.2 Network with transformers realizing all basis functions

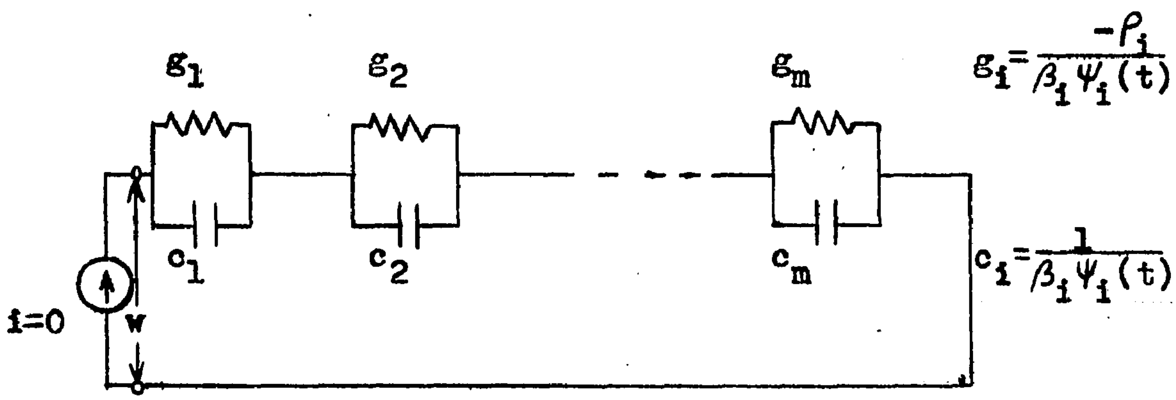



Fig.3.3 Network without transformers realizing all basis functions

expression as one of the m linearly independent solutions of the m-th order periodic differential equation. If several such networks are connected in series as shown in Fig.3.2 the overall network has the same solution as the general solution (3.4). Thus, it may be concluded that a periodic time-varying system can be represented in the form shown in Fig.3.2.

Another form without the transformers may be derived by noting that the network shown in Fig.3.1 with

$g_i = \frac{-\rho_i}{\beta_i \psi_i(t)}$ and $c_i = \frac{1}{\beta_i \psi_i(t)}$ may be represented by the

differential equation

$$\frac{d}{dt}\left(\frac{w_i}{\beta_i \psi_i}\right) - \rho_i\left(\frac{w_i}{\beta_i \psi_i}\right) = 0 \tag{3.7}$$

and the solution for it may be written as

$$w_i = \beta_i \; e^{\rho_i(t)} \psi_i(t) \tag{3.8}$$

Thus, the network shown in Fig.3.3, obtained by connecting several such networks in series, represents a general periodic time-varying system.

In the above discussions, the constants $\rho_i$ are, in general, complex numbers. When they are real-valued the system can be physically realized. It may also be noted that a dual case also exists where the basic network is a series combination of an inductance and a resistance. The representation for the overall system is formed by connecting a series of such networks in parallel.

The representation of a periodic system as described above is mainly of conceptual value. In the next section, however, more practical situations will be considered.

### 3.3. SOME RESULTS FOR PERIODIC SYSTEMS

The periodic system that will be considered below is assumed to be in a sinusoidal steady-state condition and under excitation by a periodic source. There are two sources of the frequency components present in the output of such a periodic system. The first source is the input frequency components that are transmitted to the output. Also, new frequencies are produced from the interaction of the input frequencies and the frequencies of time-variation of the components. This constitutes the second source of the output frequencies. As an example, if the input frequency is $\omega_1$ and the system under consideration has a periodically time-varying component of frequency $\omega_2$, the interaction of the two frequencies will produce frequencies of the form $m\omega_1 + n\omega_2$, where m and n are integers. In a practical system, selective networks are usually incorporated to prevent undesired frequencies appearing at the output. Examples of such systems may be found in parametric amplifier and frequency converter circuits. We may now outline a basic method of analysis.

The first step of the method is concerned with the finding of a suitable input $u_k$ to produce an output $v_k$, where

$$v_k = e^{jkt} \tag{3.9}$$

To implement this step, the following two cases are considered :

(1) For this case, it is assumed that the periodic system is represented in the form of the differential equation (2.2) where the coefficients $c_i(t)$ $(i=1,2,\dots,m+1)$ are periodic functions of time. To carry out the first step, the substitutions $v = v_k = e^{jkt}$ and $u = u_k$ will be made in equation (2.2). The following relation is found to be useful in this connection :

$$\frac{d^i v}{dt^i} = (jk)^i \; e^{jkt} \qquad (i=1,2,\dots,m) \tag{3.10}$$

Under the above substitutions the relation

$$u_k = \sum_{i=0}^{m} c_{i+1}(t) \; (jk)^i \; e^{jkt} \tag{3.11}$$

is obtained. Each of the coefficients $c_i(t)$, because of its periodic nature is expressible in the form of a Fourier series,

$$c_i(t) \; \sum_{r=0}^{\infty} c_{i,r} \; e^{jrt} \tag{3.12}$$

where the $c_{i,r}$ are complex constants. When equation (3.12) substituted in equation (3.11), the result is

$$u_k = \sum_{i=0}^{m} \sum_{r=0}^{\infty} c_{i+1,r} (jk)^i \; e^{j(k+r)t} \tag{3.13}$$

Equation (3.13) may be written in the form

$$u_k = \sum_{\ell=0}^{n} A_{\ell k} \; e^{j\ell t} \tag{3.14}$$

where the $A_{\ell k}$ are complex constants and where the number n is selected to include all the frequencies in the range of interest as dictated by the selective circuits.

(2) Consider the second method of representation of a periodic system, namely, a network. For convenience in computation, the network is first normalized as described in Appendix A.

Assuming first that the network is given in ladder form an output $v_k$ of the form shown in equation (3.9) is assumed. Various branch currents and voltages are then determined for this assumed output. This process is continued till the input has been found. This input is termed $u_k$. The example given in the next section illustrates this procedure. $u_k$ is obtained in the form of equation (3.14).

If the network is not a ladder with only one component in each arm, three alternate approaches may be tried :

(i) The procedure employed in dealing with a ladder network as described is applicable, i.e., $v_k$ as shown in equation (3.9) is assumed, and an attempt to relate the output and the input variables is made in this case.

(ii) Convert the network to the form of a ladder as described above or to a form where the statements made under (i) apply.

(iii) Find the differential equation for the network and proceed as in (i).

Now, it may be concluded that equation (3.14) represents the input, $u_k$, needed to produce an output, $v_k$, of the form given in equation (3.9) in the system under consideration. By letting k assume different values (not necessarily integers) it is possible to find the input necessary for an output $e^{jkt}$ with the assumed value of k. In order that suitable values of k may be selected it will be useful to determine the expected frequency components in the output. As described earlier, the expected frequency components may be found from a knowledge of the frequencies of time-variation of the components and of the input, and by considering the presence of selective networks. When the output frequencies of interest are determined, the input $u_k$ necessary to produce each of these frequencies seperately are determined by assigning these frequencies in turn to equation (3.14). Since the output is a linear combination of these frequencies the necessary input, u, to produce all these frequencies is also a linear combination of the $u_k$,

$$u=\sum_{k=0}^{p} B_k u_k \tag{3.15}$$

where $B_k$ $(k=0,1,\ldots,p)$ are complex constants to be determined and p obviously equals the number of frequency components present in the output. Remembering that u is periodic and that only a finite range of frequencies is of interest the following may be written :

$$u=\sum_{\ell=0}^{s} E_\ell \, e^{j\ell t} \tag{3.16}$$

where s is finite and $E_\ell$ are known complex constants. To find the unknown constants $B_k$ $(k=0,1,\ldots,p)$, equations (3.15) and (3.16) are equated. Then the output of the system is given by

$$v=\sum_{k=0}^{p} B_k v_k=\sum_{k=0}^{p} B_k \, e^{jkt} \tag{3.17}$$

This completes the description of the basic method for solving a periodic system. The procedure will now be illustrated by an example.

### 3.4. EXAMPLE

The network to be considered is shown in Fig.3.4. This is a simplified frequency converter circuit with an input frequency of 1 radian/sec. and an output frequency of 2 radians/sec. The time-varying component is the capacitance, $c(t) = 3 + 2\cos 3t$. A sinusoidal generator $u(t) = \sin t$ is connected to the input terminals 1-1'. An input filter resonating at the frequency of $v_1$ suppresses extraneous frequencies in the input circuit. Similarly,

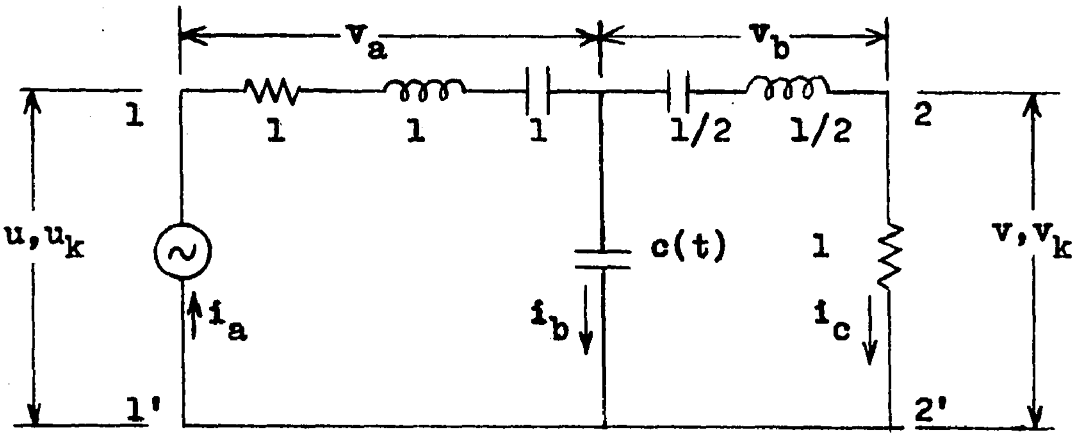



Fig.3.4 Time-varying frequency converter

the output filter which is resonant at the output frequency isolates the output from other frequencies produced due to interaction. The input resistance is included to represent the non-idealness of the generator and the output resistance represents the load. Various voltage and current variables have been labeled for convenience.

In analyzing this network it is assumed that corresponding to equation (3.9), a voltage

$$v_k = e^{jkt} \tag{3.18}$$

exists across terminals 2-2'. This means that the current flowing through the unity-valued resistor is

$$i_c = e^{jkt} \tag{3.19}$$

The voltage drop induced by this current across the output filter is

$$v_b = \frac{1}{2}\frac{di_3}{dt} + 2\int i_3\,dt$$

$$= j\left(\frac{k}{2} - \frac{2}{k}\right) e^{jkt} \tag{3.20}$$

Thus, the voltage appearing across the time-varying capacitance, c(t), is

$$v_b + v_k = e^{jkt}\left(1 + j\,\frac{k^2-4}{2k}\right) \tag{3.21}$$

The current through c(t) is then given by

$$i_b = \frac{d}{dt}\left[c\,(v_2 + v_k)\right]$$

$$= \frac{d}{dt}\Big[e^{jkt}(3 + j\,\frac{3k^2 - 12}{2k}) + e^{j(k+3)t}(1 + j\,\frac{k^2 - 4}{2k})$$

$$+ e^{j(k-3)t}(1 + j\,\frac{k^2 - 4}{2k})\Big]$$

$$= e^{jkt}(\frac{12 - 3k^2}{2} - j3k) + e^{j(k + 3)t}\Big[\frac{(4 - k^2)(k + 3)}{2k}$$

$$+ j(k + 3)\Big] + e^{j(k - 3)t}\Big[\frac{(4-k^2)(k-3)}{2k} + j(k-3)\Big] \tag{3.22}$$

Next, the input current is calculated from (3.19) and (3.22) as

$$i_a = i_b + i_c = e^{jkt}(\frac{14-3k^2}{2} + j3k) + e^{j(k+3)t}\Big[\frac{(4-k^2)(k+3)}{2k}$$

$$+j(k+3)\Big] + e^{j(k-3)t}\Big[\frac{(4-k^2)(k-3)}{2k} + j(k-3)\Big] \tag{3.23}$$

The input current $i_a$, on flowing through the input filter and resistor induces a voltage

$$v_a = \frac{di_a}{dt} + \int i_a\, dt + i_a \tag{3.24}$$

The expression for $i_a$ is introduced in the above equation from (3.23) and the resulting relation simplified to yield

$$v_a = e^{jkt}(10 - \frac{9}{2}k^2 - j\frac{3k^4 - 23k^2 - 14}{2k})$$

$$+ e^{j(k+3)t}(\frac{-3k^3 - 15k^2 - 12k + 12}{2k} - j\frac{k^4 + 6k^3 + 2k^2 - 30k - 32}{2k})$$

$$+ e^{j(k-3)t}(\frac{-3k^3 + 15k^2 - 12k - 12}{2k} - j\frac{k^4 - 6k^3 + 2k^2 + 30k - 32}{2k}) \tag{3.25}$$

The first step of the analysis method may be completed by noting that the input $u_k$ needed to produce an output $v_k = e^{jkt}$ is

$$
\begin{aligned}
u_k &= v_a + v_b + v_k \\
&= e^{jkt}\left(11 - \frac{9}{2}k^2 + j\frac{-3k^4+24k^2-18}{2k}\right) \\
&+ e^{j(k+3)t}\left(\frac{-3k^3-15k^2-12k+12}{2k} + j\frac{-k^4-6k^3-2k^2+30k+32}{2k}\right) \\
&+ e^{j(k-3)t}\left(\frac{-3k^3+15k^2-12k-12}{2k} - j\frac{-k^4+6k^3-2k^2-30k+32}{2k}\right)
\end{aligned}
\tag{3.26}
$$

Next, the expected frequency components in the output are determined. Interaction of the input frequency of 1 rad./sec. and of the frequency of 3 radians/sec. associated with the periodically time-varying capacitance produces frequencies of 1,2,3,... radians/sec. at the output. Thus, in the equation (3.26) it is necessary to substitute k=1,2,3,... and -1,-2,-3,... successively. This leads to the following equations :

$$u_1 = e^{jt}\left(\frac{13}{2} + j\frac{3}{2}\right) + e^{j4t}\left(-9 + j\frac{53}{2}\right) + e^{-j2t}\left(-6 + j\frac{5}{2}\right)$$

$$u_2 = e^{j2t}\left(-7 + j\frac{5}{2}\right) + e^{j5t}(-24 + j5) + e^{-jt}(-j1)$$

$$u_3 = e^{j3t}\left(-\frac{59}{2} - j\frac{15}{2}\right) + e^{j6t}\left(-40 - j\frac{139}{6}\right) + \left(1 + j\frac{5}{6}\right)$$

$$u_4 = e^{j4t}\left(-61 - j\frac{201}{4}\right) + e^{j7t}\left(-\frac{117}{2} - j65\right) + e^{jt}\left(-\frac{3}{2} + j1\right)$$

..........................................

$$u_{-1} = e^{-jt}\left(\frac{13}{2}-j\frac{3}{2}\right)+e^{j2t}\left(-6-j\frac{5}{2}\right)+e^{-j4t}\left(-9-j\frac{53}{2}\right)$$

$$u_{-2} = e^{-j2t}\left(-7-j\frac{15}{2}\right)+e^{jt}(j1)+e^{-j5t}(-24-j5)$$

$$u_{-3} = e^{-j3t}\left(-\frac{59}{2}+j\frac{15}{2}\right)+\left(1-j\frac{5}{6}\right)+e^{-j6t}\left(-40+j\frac{139}{6}\right)$$

$$u_{-4} = e^{-j4t}\left(-61+j\frac{201}{4}\right)+e^{-jt}\left(-\frac{3}{2}-j1\right)+e^{-j7t}\left(-\frac{117}{2}+j65\right)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (3.27)$$

Two cases of approximations will be considered below for comparison.

(1) As a first approximation, let us assume that the output contains only the desired frequency of 2 rad./sec, that is,

$$v = A_2\cos 2t + B_2\sin 2t$$

$$= e^{j2t}\left(\frac{A_2}{2}-j\frac{B_2}{2}\right) + e^{-j2t}\left(\frac{A_2}{2}+j\frac{B_2}{2}\right) \qquad (3.28)$$

where the constants $A_2$ and $B_2$ are to be determined. Upon substitution from equation (3.18) it is possible to write

$$v = v_2\left(\frac{A_2}{2}-j\frac{B_2}{2}\right) + v_{-2}\left(\frac{A_2}{2}+j\frac{B_2}{2}\right) \qquad (3.29)$$

Remembering that the outputs $v_2$ and $v_{-2}$ are produced by applying $u_2$ and $u_{-2}$ at the input respectively, the input, u, may be written as

$$u = u_2\left(\frac{A_2}{2}-j\frac{B_2}{2}\right) + u_{-2}\left(\frac{A_2}{2}+j\frac{B_2}{2}\right) \qquad (3.30)$$

On the other hand, the given input

$$u = \sin t = -\frac{j}{2}\, e^{jt} + \frac{j}{2}\, e^{-jt} \qquad (3.31)$$

Coefficients of $e^{jt}$ (or $e^{-jt}$) are equated in equations (3.30) and (3.31) yielding $A_2=-1$ and $B_2=0$. The approximation in this case is obviously too crude and the solution (3.28) with $A_2$ and $B_2$ as shown above is not acceptable. However, this case illustrates the fact that when a certain combination of several frequency components, as given in equation (3.30), is applied as the input the output contains only the desired frequency as shown in equation (3.28). This fact has been explored in the next section to propose an interesting network.

(2) Returning to the problem of finding an acceptable solution a few more frequency components will now be included in the output. Specifically, it is assumed that the output,

$$v = \sum_{k=1}^{4} (A_k \cos kt + B_k \sin kt) \qquad (3.32)$$

The equation corresponding to equation (3.30) in the previous situation is

$$u = \sum_{k=1}^{4} \left(\frac{A_k}{2} - j\frac{B_k}{2}\right) u_k - \left(\frac{A_k}{2} - j\frac{B_k}{2}\right) u_{-k} \qquad (3.33)$$

Equations (3.31) and (3.33) are now equated as before and the higher frequency terms of 5 and 6 radians/sec. neglected to yield $A_1=-.0304$, $B_1=.1805$, $A_2=.1139$, $B_2=.0218$, $A_3=0$,

$B_3=0$, $A_4=.056$ and $B_4=.0327$. Obviously, greater accuracy in the results may be obtained by including a larger number of terms in the output. In fact, an indication of the accuracy with which the amplitude of a particular component has been determined is obtained by noting the change of the value of the amplitude when one more component term is included for consideration.

The preceding paragraphs serve to introduce a method for solving a periodic time-varying system. In the next section, a feedback network which is a consequence of the discussion in this section will be described.

### 3.5. A FEEDBACK SCHEME

The frequency converter circuit described in the last section is a typical one, first, in the sense that the input frequency reacts with the frequency of variation of the time-varying element, and second, in the sense that filters are used to suppress the undesired frequency components. Conceptually, however, a different form of a frequency converter may be visualized. The system that will be considered is an example of the versatility of the time-varying systems.

It was pointed out in the previous section that a single frequency may be obtained at the output of a periodic time-varying system by applying a particular combination of several frequencies to the input. The combination is found by first assuming an output of the desired frequency and then calculating the needed input. Thus, it is clear that such a combination exists for any periodic time-varying system irrespective of the presence of any frequency selective branches in the system. If the time-variation of the components can be expressed as a finite Fourier series, the number of frequency components needed at the input is also finite.

Consider the arrangement shown in Fig.3.5 in which the block marked "Periodic System" has time-varying components that are expressible in finite Fourier series. A certain frequency $f_1$ is assumed at the output (point 2). The particular combination of frequencies $f_i$ $(i=0,1,\ldots,n)$ that must be applied to the input (point 1) to produce the assumed output is found. (The ordering of the frequencies $f_i$ is arbitrary.) More specifically, suppose it is found that if the desired output is

$$v = B_1 \sin 2\pi f_1 t \tag{3.34}$$

the input at point 1 would have to be of the particular form

$$u = \sum_{i=0}^{n} A_i \sin (2\pi f_i t - \alpha_i) \tag{3.35}$$

where the $A_i$ and $\alpha_i$ represent the amplitudes and phases of the frequency components $f_i$ in the input and $B_1$ is the amplitude of the desired output frequency, $f_1$. The phase

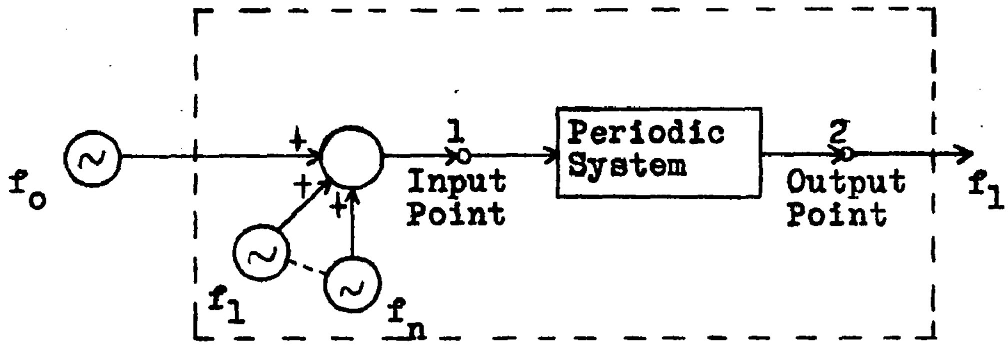



Fig.3.5 Frequency converter

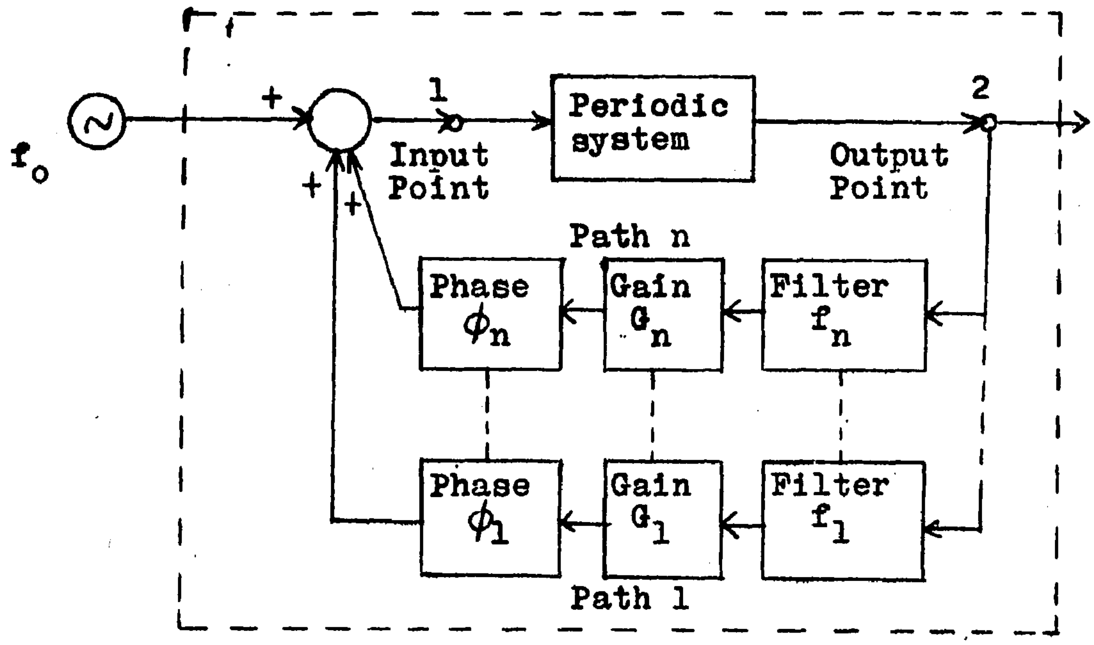



Fig.3.6 Frequency converter using feedback

angle of the output has been taken as zero without any loss of generality.

Now, consider the overall system shown in broken lines in Fig. 3.5. In this system, the frequency sources $f_i$ ($i=1,2,\ldots,n$) are incorporated. Let these and the outside excitation source $f_o$ be applied to point 1 in the suitable combination indicated in equation (3.35). As a result, the output of the overall system will consist of the single frequency $f_1$ as given in equation (3.34). Thus, the overall system shown in broken lines acts as an ideal frequency converter with a single input frequency $f_o$ and a single output frequency $f_1$.

A second form of the frequency converter system is shown in Fig. 3.6. The basic periodic system remains the same but the frequency sources $f_i$ ($i=1,2,\ldots,n$) are replaced by n feedback paths. Path i is designed to pass only the frequency $f_i$ and each path has adjustable gain ($G_i$) and phase ($\phi_i$) controls. To ensure the proper operation of the system it is desired to have the same input at point 1 (and hence the same output at point 2) as was present in the case of the system shown in Fig. 3.5. In other words, the input (at point 1) as in equation (3.35) and the output (at point 2) as in equation (3.34) are desired. To achieve this, the outside excitation source would have to be set to yield $A_o \sin(2\pi f_o t - \alpha_o)$ which is one of the terms in the right hand side of equation (3.35).

The other n terms of equation (3.35) consisting of the components $f_i$ $(i=1,2,\ldots,n)$ would have to be obtained through the n feedback paths of Fig. 3.6 by properly setting the gain $(G_i)$ and the phase $(\phi_i)$ controls as discussed below.

Since the frequency components $f_i$ $(i=1,2,\ldots,n)$ would have to be fed back the output must contain components of these frequencies. Thus, it is possible to state that the output of the system shown in Fig. 3.6 would be of the form

$$v=\sum_{i=1}^{n} B_i \sin(2\pi f_i t - \beta_i) + \Delta v, \; \beta_1 = 0 \qquad (3.36)$$

where $B_i$ $(i=1,2,\ldots,n)$ represent the amplitudes of the frequencies $f_i$ $(i=1,2,\ldots,n)$, $\beta_i$ $(i=1,2,\ldots,n)$ represent the phases associated with them and $\Delta v$ signifies the presence of other frequency components. No phase angle is associated with $f_1$ since this is taken as the reference as before. In order that the phase angles associated with the frequencies $f_i$ $(i=1,2,\ldots,n)$ in equation (3.35) be obtained correctly through the feedback of the output shown in equation (3.36) the phase controls $\phi_i$ would have to be set as follows :

$$\phi_1 = -\alpha_1, \quad \phi_i = \beta_i - \alpha_i \quad (i=2,3,\ldots,n)$$

In order that the amplitudes associated with the frequencies $f_i$ $(i=1,2,\ldots,n)$ in equation (3.35) be obtained correctly through the feedbask of the output shown in equation (3.36) the gain control $G_i$ would have to be set as follows :

$$G_i = \frac{A_i}{B_i} \qquad (i=1,2,\ldots,n)$$

Since it is desired to make the output in equation (3.36) approach that in equation (3.34) it is convenient to make $B_i \rightarrow 0$ $(i=2,3,\ldots,n)$ and $v \rightarrow 0$. Thus, it would be necessary to set $G_i \rightarrow \infty$ $(i=2,3,\ldots,n)$. The requirement of $v \rightarrow 0$ would be automatically satisfied as the input at point 1 approaches the input shown in equation (3.35).

Thus, under the above arrangement the performance of the system depicted in Fig.3.6 would approximate the performance of an ideal frequency converter.

The purpose of this chapter has been to introduce some basic concepts of time-varying systems and to present a method of analyzing periodic systems. More general systems will be considered in the next chapter and methods for analyzing them will be discussed.

## CHAPTER 4

## GENERAL SYSTEMS

In this chapter, methods for solving the general time-varying system with stable solutions will be developed. Stability, here, refers to the fact that the solutions should not possess any singularity in the interval $[0, \infty]$ and also that they approach a finite value as $t \to \infty$. Some simple examples of such solutions are $e^{-t}$, $\frac{\sin t}{t+1}$, $t\ e^{-t^2/2}$, etc. Some examples of non-stable solutions are $\frac{1}{t}$, $\cos t$, $e^t$, etc. The property of stability can often be established by using known techniques[2-4,8-10]. Two representations of the time-varying systems will be considered :

(1) a differential equation of the type

$$\sum_{i=1}^{m} c_{i+1}(t)\frac{d^i v}{dt^i} + c_1(t)v = u \tag{4.1}$$

and,

(2) a schematic drawing of a network with the time-variation of the components indicated explicitly.

The time interval of interest is denoted by $[t_a, t_b]$ where $t_a$ is finite and $t_b$ may be finite or infinite. Usually, the cases where $t_a=0$ and $t_b=\infty$ are of interest. At first, the differential equation representation will be considered. This will be followed by a treatment of the other representation.

The differential equation representation is too general to be treated effectively. Therefore, it is essential at first to impose several restrictions on it. Later, methods will be devised for eliminating these restrictions.

As a first restriction, let the homogeneous differential equation only be considered for the present. This is of the form

$$\sum_{i=1}^{m} c_{i+1}(t)\frac{d^i v}{dt^i} + c_1(t)v = 0 \tag{4.2}$$

As described in Chapter 2, equation (4.2) has the solution

$$w = \sum_{i=1}^{m} \beta_i w_i \tag{4.3}$$

where $w_i$ are the basis functions and $\beta_i$ are arbitrary real constants.

As a preliminary step to the investigation of the equation (4.2) it is necessary to inquire as to whether the basis functions are affected by multiplying the differential equation (4.2) by a certain function f which has either of the following properties,

(1) $f(t) \neq 0, \quad t_a \leq t \leq t_b$

(2) if $t_a \leq t_1, t_2, \ldots, t_n \leq t_b$ where n is finite, $f(t) \neq 0, \quad t_a \leq t < t_1, \ldots, \; t_n < t \leq t_b$ and $f(t_i)=0, (i=1,2,..,n)$.

In the first case, it is obvious that the basis functions of equation (4.2) and equation

$$f\left[\sum_{i=1}^{m} c_{i+1}(t)\frac{d^i v}{dt^i} + c_1(t)v\right]=0 \tag{4.4}$$

are same. In the second case, equations (4.2) and (4.4) have the same basis functions over the subintervals $[t_a, t_1)$, $(t_1, t_2), \ldots, (t_{n-1}, t_n), (t_n, t_b]$ since $f(t) \neq 0$ there.

The above discussion dealing with multiplying a differential equation by a function will be useful later. It is also useful for reducing the complexity of the coefficients $c_i$ $(i=1,2,\ldots,m+1)$ if there is any common factor f satisfying the above requirements. For example, the differential equation

$$e^{-t^2}\frac{d^2v}{dt^2} + e^{-t^2}(t+1)\frac{dv}{dt} + (e^{-t^2}\sin t)v = 0$$

may be simplified to a form

$$\frac{d^2v}{dt^2} + (t+1)\frac{dv}{dt} + (\sin t)v = 0$$

In the following section, the discussion on differential equations begins by considering first the equations with polynomial coefficients.

## 4.1. DIFFERENTIAL EQUATIONS WITH POLYNOMIAL COEFFICIENTS

The form of the differential equation to be considered in this section is

$$(c_{m+1,q+1}t^q + \ldots + c_{m+1,1})\frac{d^mv}{dt^m} + \ldots + (c_{1,q+1}t^q + \ldots + c_{11})v = 0$$

This may also be written as

$$\sum_{i=0}^{m-1} \frac{d^{m-i}v}{dt^{m-i}} \sum_{j=1}^{q+1} c_{m+1-i,j}\, t^{j-1} - v \sum_{j=1}^{q+1} c_{1j}\, t^{j-1} = 0 \qquad (4.5)$$

where $c_{i,j}$ are real constants. The differential equation (4.5) in the given form must be solved over the interval $[0,\infty]$. It will be convenient to convert equation (4.5) to a form where the solution over a finite interval will suffice. Specifically, the following transformation of variable will be used :

$$t = \frac{1 + \mathcal{T}}{1 - \mathcal{T}} \qquad (4.6)$$

where $\mathcal{T}$ is the new independent variable.

The transformation given in equation (4.6) has several useful properties. The plot $\mathcal{T}$ vs. t in Fig. 4.1 illustrates that any function in the t-domain over the interval $[0,\infty]$ is transformed to a new function in the $\mathcal{T}$-domain over the interval $[-1,1]$. The transformation is one-to-one[39] in the sense that if $\mathcal{T}_1$ is a point in the $\mathcal{T}$-domain corresponding to a point $t_1$ in the t-domain and $\mathcal{T}_2$ is a point in the $\mathcal{T}$-domain corresponding to a point $t_2$ in the t-domain then $t_1 \neq t_2$ always implies that $\mathcal{T}_1 \neq \mathcal{T}_2$. The transformation is also said to be "onto" which implies that every point of the $\mathcal{T}$-domain is the image of some point in the t-domain. Moreover, with the points $t_1$, $t_2$, $\mathcal{T}_1$ and $\mathcal{T}_2$ as indicated in the above discussion $t_1 \geq t_2$ implies that $\mathcal{T}_1 \geq \mathcal{T}_2$ and vice versa. The numbers and the orders of the

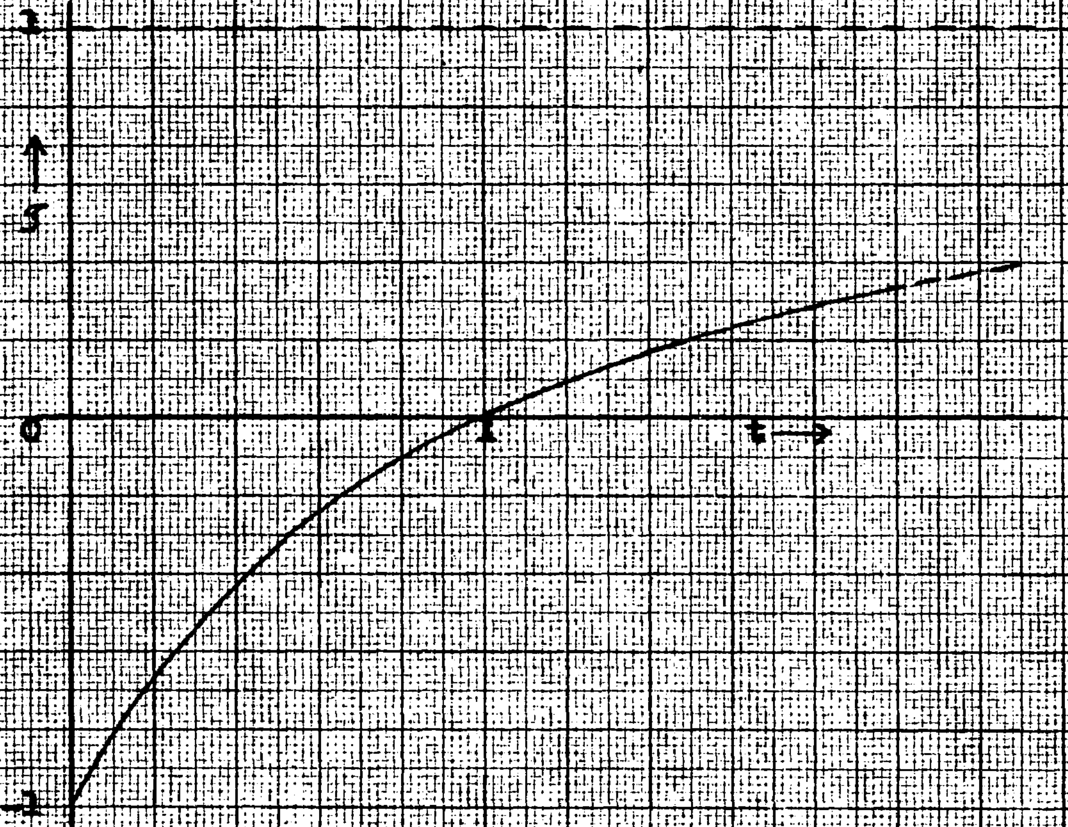



Fig.4.1 Relation between t and ς variables

singularities in the transformed function are the same as those in the original function. If the original function does not possess any singularity in the interval $[0,\infty]$ the transformed function will also show an absence of these in $[-1,1]$.

The above properties provide assurance that if the transformation (4.6) is applied to the differential equation (4.5) the resulting differential equation need be solved over the finite interval $[-1,1]$ only. Moreover, when this solution is obtained in the $\mathcal{T}$-domain the solution in the t-domain valid over $[0,\infty]$ may be obtained by the inverse transformation

$$\mathcal{T}=\frac{t-1}{t+1} \tag{4.7}$$

and there will be no reason for any ambiguity.

In order to implement the transformation (4.6) on the equation (4.5) the following substitutions are needed :

$$\frac{dv}{dt}=\frac{dv}{d\mathcal{T}}\;\frac{d\mathcal{T}}{dt}=0.5\;(1-\mathcal{T})^2\;\frac{dv}{d\mathcal{T}}$$

$$\frac{d^2v}{dt^2}=\frac{d}{dt}\left(\frac{dv}{dt}\right)=\frac{d}{d\mathcal{T}}\left(\frac{dv}{dt}\right)\frac{d\mathcal{T}}{dt}=0.25(1-\mathcal{T})^4\,\frac{d^2v}{d\mathcal{T}^2}-0.5(1-\mathcal{T})^3\,\frac{dv}{d\mathcal{T}}$$

..................................................

and, in general,

$$\frac{d^mv}{dt^m}=d_{m+1,m+1}(1-\mathcal{T})^{2m}\,\frac{d^mv}{d\mathcal{T}^m}+d_{m+1,m}(1-\mathcal{T})^{2m-1}\,\frac{d^{m-1}v}{d\mathcal{T}^{m-1}}$$

$$+\dots+d_{m+1,2}(1-\mathcal{T})^{m+1}\,\frac{dv}{d\mathcal{T}}$$

or, in a more compact form,

$$\frac{d^m v}{dt^m} = \sum_{j=1}^{m} d_{m+1,j+1}(1-\tau)^{m+j} \frac{d^j v}{d\tau^j} \tag{4.8}$$

where the constants $d_{ij}$ may be calculated as follows :

Assume that the expression corresponding to $\frac{d^{i-1}v}{dt^{i-1}}$ is

$$\frac{d^{i-1}v}{dt^{i-1}} = d_{i,i}(1-\tau)^{2i-2} \frac{d^{i-1}v}{d\tau^{i-1}} + d_{i,i-1}(1-\tau)^{2i-3} \frac{d^{i-2}v}{d\tau^{i-2}} + \dots + d_{i,3}(1-\tau)^{i+1} \frac{d^2v}{d\tau^2} - d_{i,2}(1-\tau)^{i} \frac{dv}{d\tau}$$

Then, it is clear that

$$\frac{d^i v}{dt^i} = \frac{d}{dt}\left(\frac{d^{i-1}v}{dt^{i-1}}\right) = \frac{d}{d\tau}\left(\frac{d^{i-1}v}{d\tau^{i-1}}\right)\frac{d\tau}{dt} = .5(1-\tau)^2 \frac{d}{d\tau}\left(\frac{d^{i-1}v}{d\tau^{i-1}}\right)$$

$$= .5(1-\tau)^2 \Big[d_{ii}(1-\tau)^{2i-2} \frac{d^i v}{d\tau^i} + \Big\{-(2i-2)d_{ii} + d_{i,i-1}\Big\}(1-\tau)^{2i-3} \frac{d^{i-1}v}{d\tau^{i-1}} + \dots + \Big\{-(i+1)d_{i,3} + d_{i,2}\Big\}(1-\tau)^{i} \frac{d^2v}{d\tau^2} + (-i d_{i,2})(1-\tau)^{i-1} \frac{dv}{d\tau}\Big] \tag{4.9}$$

On the other hand, from equation (4.8) it is also possible to write

$$\frac{d^i v}{dt^i} = d_{i+1,i+1}(1-\tau)^{2i} \frac{d^i v}{d\tau^i} + d_{i+1,i}(1-\tau)^{2i-1} \frac{d^{i-1}v}{d\tau^{i-1}} + \dots + d_{i+1,3}(1-\tau)^{i+2} \frac{d^2v}{d\tau^2} + d_{i+1,2}(1-\tau)^{i+1} \frac{dv}{d\tau} \tag{4.10}$$

Comparing equations (4.9) and (4.10) the recurrence relations for the constants $d_{ij}$ may be developed as

$$d_{i+1,i+1} = .5\, d_{i,i}$$

$$d_{i+1,j} = .5\left[-(i+j-2)d_{i,j} + d_{i,j-1}\right] \quad (j=3,4,\ldots,i)$$

$$d_{i+1,2} = .5\left[-i\, d_{i,2}\right]$$

$$(i = 3,4,\ldots,m) \tag{4.11}$$

It is known that $d_{22}=0.5$, $d_{33}=0.25$ and $d_{32}=-0.5$. The last two constants and the recurrence relationships (4.11) make it possible to find any $d_{ij}$. For example, taking i=3,

$$\frac{d^3v}{dt^3} = d_{44}(1-\mathcal{T})^6\frac{d^3v}{d\mathcal{T}^3} - d_{43}(1-\mathcal{T})^5\frac{d^2v}{d\mathcal{T}^2} - d_{42}(1-\mathcal{T})^4\frac{dv}{d\mathcal{T}}$$

where from equations (4.11), $d_{44}=.5\times .25=.125$, $d_{43} = .5\left[-(3+3-2)\times .25+(-.5)\right]=-.75$ and $d_{42}=.5\left[-3\times(-.5)\right]=.75$.

It should be remembered that after the transformation (4.6) is applied to equation (4.5) the function v in the resulting differential equation becomes a function of $\mathcal{T}$. To carry out the transformation in detail, the relations (4.6) and (4.8) are substituted in equation (4.5) to yield

$$\sum_{i=0}^{m-1}\left[\sum_{k=1}^{m-i} d_{m-i+1,k+1}(1-\mathcal{T})^{m-i+k}\frac{d^kv}{d\mathcal{T}^k}\right]\left[\sum_{j=1}^{q+1} c_{m+1-i,j}\times \left(\frac{1+\mathcal{T}}{1-\mathcal{T}}\right)^{j-1}\right] + v\sum_{j=1}^{q+1} c_{1j}\left(\frac{1+\mathcal{T}}{1-\mathcal{T}}\right)^{j-1} = 0 \tag{4.12}$$

The above equation may be rearranged by collecting terms of the same orders of $\frac{dv}{d\mathcal{T}}$ as follows :

$$\sum_{i=0}^{m-1} \frac{d^{m-i}v}{d\tau^{m-i}} \sum_{k=1}^{i+1} \sum_{j=1}^{q+1} \frac{c_{m-k+2,q-j+2}\, d_{m-k+2,m-i+1} (1+\tau)^{q-j+1}}{(1-\tau)^{q-2m+i+k-j}}$$

$$+ v \sum_{j=1}^{q+1} c_{1j} \left(\frac{1+\tau}{1-\tau}\right)^{j-1} = 0 \qquad (4.13)$$

The coefficients of the various $\frac{d^i v}{d\tau^i}$ and of v in the above equation have non-unity denominators. To facilitate further development it is necessary to convert these denominators to unity. This is done by noting the highest exponent (say, s) of $(1-\tau)$ in the coefficients of all the $\frac{d^i v}{d\tau^i}$ and of v and multiplying equation (4.13) by $(1-\tau)$ raised to that exponent. Two points need to be discussed in this connection.

First of all, if $c_{1,q+1} \neq 0$, then $s=q$, but if $c_{1,q+1}=0$ then $s<q$. Depending on whether or not some of the $c_{ij}$ $(i=1,2,\ldots,m+1 \; ; \; j=1,2,\ldots,q+1)$ are zero, $0 \leq s \leq q$. For example, if $q>2$ and $c_{1,q+1}=c_{1,q}=c_{1,q-1}=0$ but $c_{1j}(j=1,2,\ldots,q-2)$ and $c_{2,q+1}$ are non-zero, $s=q-2$.

Secondly, multiplying equation (4.13) by $(1-\tau)^s$ is justified by the preceding discussion on multiplying a differential equation by a function, f. In this case, $f(\tau)=(1-\tau)^s$ which reaches zero at $\tau=1$ and thus corresponds to $t=\infty$ in the time domain. Thus, the validity of the final solution may be questioned at $t=\infty$. But, if $t \to \infty$ the solution will remain valid.

Bearing these points in mind, equation (4.13) is multiplied by $(1-\mathcal{J})^s$. After simplification, the equation takes the following form :

$$\sum_{i=0}^{m-1} \frac{d^{m-i}v}{d\mathcal{J}^{m-i}} \sum_{j=1}^{p+1} b_{m+1-i,j}\,\mathcal{J}^{j-1} + v\sum_{j=1}^{p+1} b_{1j}\,\mathcal{J}^{j-1} = 0 \tag{4.14}$$

where $p = 2m+s$ and $b_{ij}$ are real constants.

Equation (4.14) is an ordinary linear differential equation of the same order of equation (4.5). If $w_i(\mathcal{J})$ $(i=1,2,\ldots,m)$, $-1 \leq \mathcal{J} \leq 1$ are the m basis functions of equation (4.14) then the basis functions of equation (4.5) will be given by applying relation (4.7) as $w_i(\frac{t-1}{t+1})$, $(i=1,2,\ldots,m)$, $0 \leq t \leq \infty$.

For obtaining a basis function of equation (4.14) over $[-1,1]$ an approximation of the form

$$w(\mathcal{J}) = 1 - \sum_{\substack{\ell=1 \\ \ell \neq \ell_d}}^{n-1} k_\ell \mathcal{J}^\ell \qquad -1 \leq \mathcal{J} \leq 1 \tag{4.15}$$

will be attempted for the basis function. The meaning of $\ell_d$ is made clear below. The real constants $k_\ell$ and n will have to be determined to satisfy certain error criteria which are also described later.

To explain the term $\ell_d$, suppose the substitution $v = \mathcal{J}^\ell$ $(\ell = 0,1,\ldots,n-1)$ is made in the left hand side of equation (4.14). Call the corresponding expressions, $\Phi_\ell$ $(\ell = 1,2,\ldots,n)$. A general expression for $\Phi_\ell$ may be given as

$$\phi_{\ell}=\sum_{j=1}^{n-p} a_{\ell,j}\,\mathcal{T}^{j-1} \qquad (\ell=1,2,\ldots,n)$$

where $a_{\ell j}=\frac{\ell!}{0!}b_{\ j}+\frac{\ell!}{1!}b_{\ell-1,j-1}+\frac{\ell!}{2!}b_{\ell-2,j-2}+\ldots$ with $b_{\ell j}=0$ for $\ell<1$ and/or for $p+2\leqslant j\leqslant 1$. Next, it is of interest to see if the $\phi_{\ell}\,(\ell=1,2,\ldots,n)$ form a linearly independent set of functions. It is evident that the subset $\phi_{\ell}\,(\ell=m,m+1,\ldots,n)$ is always linearly independent. If the whole set $\phi_{\ell}$ $(\ell=1,2,\ldots,n)$ is linearly independent it can be said that $\ell_d$ consists of a null set and equation (4.15) may be written as

$$w(\mathcal{T})=1-\sum_{\ell=1}^{n-1}k_{\ell}\,\mathcal{T}^{\ell} \tag{4.16}$$

In most cases, the above situation holds true but it is possible to construct examples where the subset $\phi_{\ell}$ $(\ell=1,2,\ldots,m)$ is linearly dependent. Enough numbers of $\phi_{\ell}$ $(1\leq\ell\leqslant m)$ would have to be removed in such a case to make the remaining set linearly independent. Now, $\ell_d$ is defined in the sense that $\phi_{\ell}\,(\ell=1,2,\ldots,m)$ is linearly dependent, but $\phi_{\ell}\,(\ell=1,2,\ldots m;\ \ell\neq\ell_d)$ is linearly independent. As an example, consider the following differential equation in the $\mathcal{T}$-domain,

$$\left(\frac{1}{2}-\frac{\mathcal{T}}{2}-\frac{\mathcal{T}^2}{2}+\frac{\mathcal{T}^3}{2}\right)\frac{d^2v}{d\mathcal{T}^2}+(1-\mathcal{T}^2)\frac{dv}{d\mathcal{T}}+(1+\mathcal{T})v=0$$

Taking $n=4$, we substitute $1,\mathcal{T},\mathcal{T}^2$ and $\mathcal{T}^3$ successively in the left hand side of the above equation and obtain $\phi_1=1+\mathcal{T}$,

$\phi_2 = 1+\mathcal{T}$, $\phi_3 = 1+\mathcal{T}$ and $\phi_4 = 3\mathcal{T} - 2\mathcal{T}^3 + \mathcal{T}^4$. It is obvious that $\ell_d = 2,3$ in the above case.

The question of finding an error criterion and devising a method of obtaining $k_\ell$ and n of equation (4.15) satisfying this criterion will now be investigated. If w of equation (4.15) is to be an approximation of a basis function of the differential equation (4.14) it must satisfy this equation. Substitution of w from equation (4.15) and simplification in terms of $\phi_i$ yields the relation

$$\phi_1 - \sum_{\substack{\ell=1 \\ \ell \neq \ell_d}}^{n-1} k_\ell \, \phi_{\ell+1} \cong 0 \tag{4.17}$$

which must be satisfied by proper selection of various parameters. In this connection, Theorem 2.4 will be recalled. The linearly independent functions referred to there are the functions $\phi_\ell$ ($\ell = 1,2,\ldots,n$ ; $\ell \neq \ell_d$). The interval $[a,b]$ will be taken as $[-1,1]$ . The integrals $\int_{-1}^{1} \phi_i \, \phi_j \, d\mathcal{T}$ ($i,j = 1,2,\ldots,n$; $i,j \neq \ell_d$) referred to in the theorem always exist since $\phi_\ell$ ($\ell = 1,2,\ldots,n$; $\ell \neq \ell_d$) are polynomials. Hence the theorem can be applied to obtain the constants $k_\ell$ ($\ell = 1,2,\ldots,n-1$). Specifically, if the notation $\psi_{ij} = \int_{-1}^{1} \phi_i \, \phi_j \, d\mathcal{T}$ ($i,j = 1,2,\ldots,n$) is used the constants $k_\ell$ will be given by

$$\begin{bmatrix} k_1 \\ k_2 \\ \cdot\cdot \\ k_{n-1} \end{bmatrix} = \begin{bmatrix} \Psi_{22} & \Psi_{23} & \cdots & \Psi_{2,n} \\ \Psi_{32} & \Psi_{33} & \cdots & \Psi_{3,n} \\ \cdots & \cdots & \cdots & \cdots \\ \Psi_{n2} & \Psi_{n3} & & \Psi_{n,n} \end{bmatrix}^{-1} \begin{bmatrix} \Psi_{21} \\ \Psi_{31} \\ \cdots \\ \Psi_{n1} \end{bmatrix} \qquad (4.18)$$

The next step is to determine whether $w(\mathcal{T})$ as given in equation (4.15) with the above values of $k_\ell$ may be called an approximate basis function of the differential equation (4.14) over [-1,1] in some sense. To test the validity of $w(\mathcal{T})$ the normalized least square error criterion will be used. First, let us consider a short description of the various aspects of the (unnormalized) least square error criterion. Let f be the function to be approximated in [a,b] in the $\mathcal{T}$-domain and $f_1$ be the approximation in this interval. The goal is to minimize the error given by

$$\int_a^b \left[f_1(\mathcal{T}) - f(\mathcal{T})\right]^2 d\mathcal{T}$$

The advantages of the least square error criterion are well-known. For example, it is usually the easiest for the purpose of computation.

Before describing the disadvantages of this criterion the specific form that it takes in the present situation will be outlined. The interval of interest is [-1,1]. The approximation $f_1$ is obtained by substituting w from equation (4.15) in the $\mathcal{T}$-domain differential equation (4.14) whereas the desired function to be approximated is $f = 0$. However, when equation (4.15) is substituted in equation (4.14) the result is the left hand side of equation (4.17). Thus,

$$f_1 = \phi_1 - \sum_{\substack{\ell=1 \\ \ell \neq \ell_d}}^{n-1} k_\ell \, \phi_{\ell+1}$$

and the error is

$$\int_{-1}^{1} [\phi_1 - \sum_{\substack{\ell=1 \\ \ell \neq \ell_d}}^{n-1} k_\ell \, \phi_{\ell+1}]^2 d\tau = I_2$$

Reference to Theorem 2.4 clearly shows that as n is increased the error decreases.

The major disadvantage of the (unnormalized) least square error criterion can be gathered from the following consideration. Suppose the original differential equation (4.5) is multiplied by a constant $K \neq 0$. This should not affect the basis functions and the error criterion in any way. The validity of this statement will now be tested.

Multiplying the differential equation (4.5) by a non-zero K also obviously implies multiplying the transformed differential equation (4.14) by K. In turn, this implies that the new $\phi$ functions are $\phi_i' = K \phi_i$ $(i = 1, 2, \ldots, n)$. Thus, when the $\psi_{ij} = \int_{-1}^{1} \phi_i \, \phi_j d\tau$ are formed the new values $\psi_{ij}' = K^2 \psi_{ij}$ $(i, j = 1, 2, \ldots, n)$. Since the $k_i$ are calculated by usuing equation (4.18) it is evident that

$$\begin{bmatrix} k_1' \\ k_2' \\ \cdot\cdot \\ k_{n-1}' \end{bmatrix} = \begin{bmatrix} K^2\psi_{22} & K^2\psi_{23}\cdots K^2\psi_{2n} \\ K^2\psi_{32} & K^2\psi_{33}\cdots K^2\psi_{3n} \\ \cdots\cdots & \cdots\cdots \\ K^2\psi_{n2} & K^2\psi_{n3}\cdots K^2\psi_{nn} \end{bmatrix}^{-1} \begin{bmatrix} K^2\psi_{21} \\ K^2\psi_{31} \\ \cdots\cdots \\ K^2\psi_{n1} \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \cdot\cdot \\ k_n \end{bmatrix}$$

As $k_i$ $(i=1,2,\ldots,n-1)$ remain invariant the basis functions are not affected.

However, when the error is formed as

$$I_2' = \int_{-1}^{1}\left[\phi_1' - \sum_{\ell=1}^{n-1} k_\ell\, \phi_{\ell+1}'\right]^2 d\tau$$

$$= K^2\int_{-1}^{1}\left[\phi_1 - \sum_{\ell=1}^{n-1} k_\ell\, \phi_{\ell+1}\right]^2 d\tau$$

$$= K^2 I_2$$

it is clear that the error changes.

To correct this situation, the normalized error is defined as

$$E_{norm} = \frac{I_2}{I_1}$$

where $I_1 = \int_{-1}^{1} \phi_1^2\, d\tau$ as mentioned in **Theorem** 2.4. It is obvious that $E_{norm}$ remains invariant when the differential equation (4.5) is multiplied by a non-zero K. This indicates that $E_{norm}$ is a satisfactory error criterion. By making n sufficiently large $E_{norm}$ will be brought below a certain prespecified level. When a $w(\tau)$ satisfying the error criterion has been obtained, $w(\frac{t-1}{t+1})$ is a basis function of the time-domain differential equation (4.5). Thus, substitution of $w(\frac{t-1}{t+1})$ in equation (4.5) will approximately make the resulting expression equal to zero over $[0,\infty]$. This fact may be used as a second checking procedure.

The whole procedure as discussed above can now be summarized in block diagram form as shown in Fig.4.2. A computer program usuing FORTRAN language has been written which encompasses the above procedure and also involves some methods to be described later in this chapter. The program is given in Appendix B and is explained by means of flow diagrams in Appendix C. Several examples will now be given to illustrate the method. The first of these will be discussed in detail, and references will be given to the appropriate portions of the computer program. As indicated in Fig.4.2, if the $E_{norm}$ stays above the prespecified limit after usuing a reasonable number of n, the system is probably unstable in the sense described at the beginning of this chapter.

## 4.2. EXAMPLE 1[38]

It is desired to find a basis function for the following second-order time-varying homogeneous differential equation :

$$(t^2+3t+2)\frac{d^2v}{dt^2} + (t^2+4t+2)\frac{dv}{dt} + tv = 0 \tag{4.19}$$

The differential (4.19) is in the form of equation (4.5).

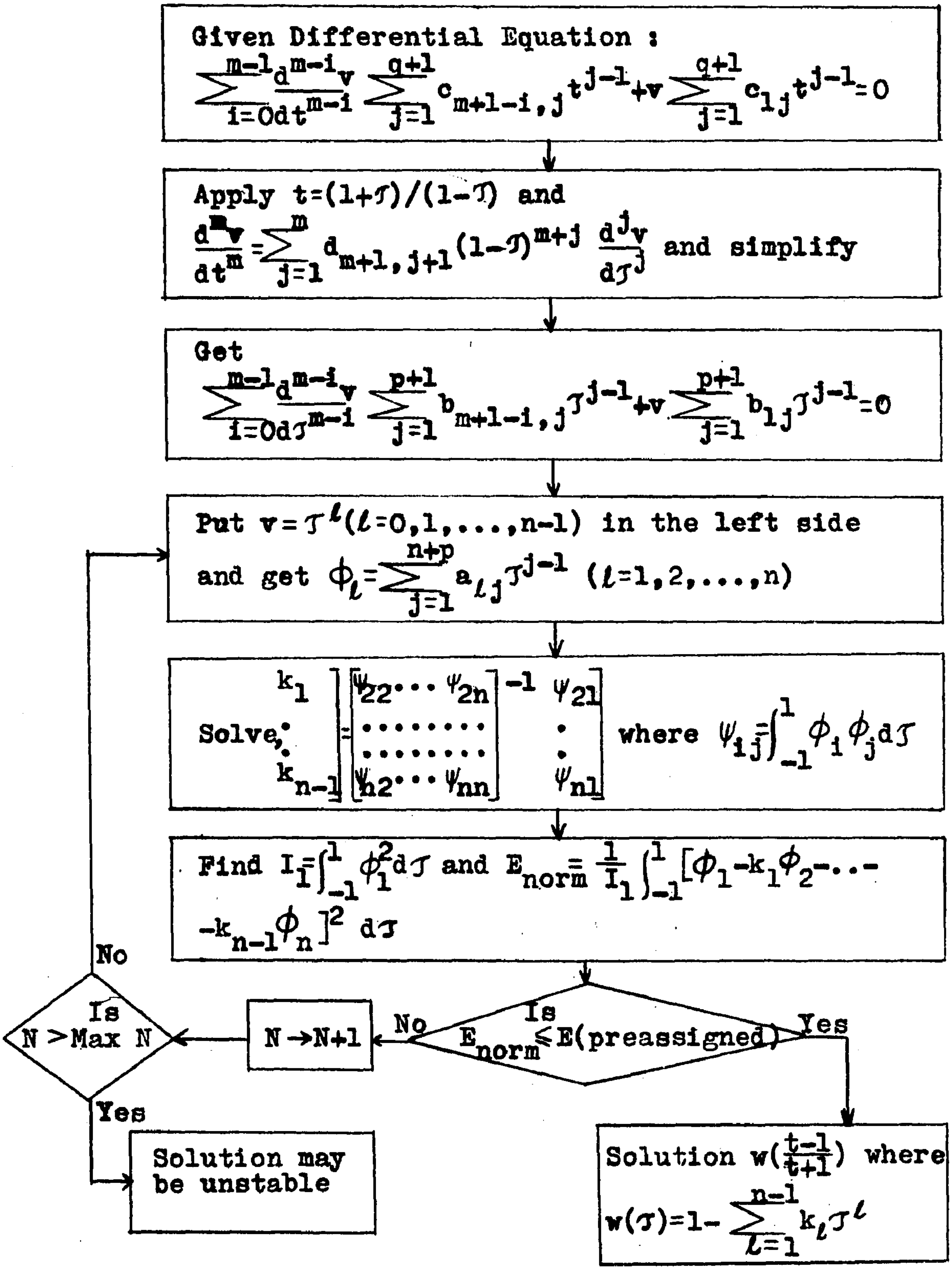



Fig.4.2 Block Diagram Representation of the Method of Solution

Referring to the block diagram in Fig. 4.2 the first step is to apply the transformation $t=\frac{1+\zeta}{1-\zeta}$. The actual operations regarding the transformation and simplification of the resulting expression are carried out by the SIMPL subroutine of the computer program described in Appendices B and C. As a result, the following differential equation in the $\zeta$-domain is obtained :

$$(.5\zeta^4-3\zeta^3+6\zeta^2-5\zeta+1.5)\frac{d^2v}{d\zeta^2} - (1.5\zeta^3-4.5\zeta^2+2.5\zeta+.5)\frac{dv}{d\zeta} + (\zeta+1)v=0 \qquad (4.20)$$

This corresponds to equation (4.14). The next step is to substitute $v=\zeta^{\ell}$ ($\ell=0,1,\ldots,n-1$) successively in the left hand side of equation (4.20) and label the corresponding expressions $\phi_\ell$ ($\ell=1,2,\ldots,n$). The actual operations necessary for this are done by the GETA subroutine. The maximum value that n could assume was MAXN = 9. This value was selected after taking into account the capability of the matrix inversion subroutine used in the program. Representative values of $\phi_\ell$ are $\phi_1=1+\zeta$, $\phi_2=.5+3.5\zeta-3.5\zeta^2+1.5\zeta^3$, $\phi_3=3-9\zeta+18\zeta^2-14\zeta^3+4\zeta^4$, etc. For the purpose of normalization of error the initial error $I_1$ may be calculated as $I_1=\int_{-1}^{1}\phi_1^2\,d\zeta=2.667$. As a next step, $\psi_{ij}=\int_{-1}^{1}\phi_i\,\phi_j\,d\zeta$ are found. Representative values are $\psi_{21}=1.6$, $\psi_{22}=16.076$, $\psi_{23}=-78.4$, etc. Taking $n=3$ first the constants $k_i$ are computed as given in equation

(4.18) to be $k_1 = .852$ and $k_2 = .154$. The normalized error $E_{norm}$ is found to be .025. The pre-assigned value (called as E in the program) for allowable normalized error is .006. This value has been determined after assessing the results of several test runs. For the present example, when $n=4$, $E_{norm} = 2.02 \times 10^{-4} < .006$, and $k_1 = .752$, $k_2 = .204$, and $k_3 = .032$. Thus, the basis function for the equation (4.19) is given by

$$w_1 = 1 - .752\left(\frac{t-1}{t+1}\right) - .204\left(\frac{t-1}{t+1}\right)^2 - .032\left(\frac{t-1}{t+1}\right)^3 \tag{4.21}$$

In the present case, it is possible to compare the approximate basis function obtained above with the exact basis function. It may be verified by direct substitution in equation (4.19) that $\beta_1 e^{-t} + \beta_2 \frac{1}{t+2}$ is an exact basis function where $\beta_1$ and $\beta_2$ are arbitrary real constants. The expression in equation (4.21) is equated with the exact basis function at two points ($t=0$ and $t=1$ arbitrarily selected) to evaluate $\beta_1$ and $\beta_2$. Thus, at $t=0$, $\beta_1 + \frac{\beta_2}{2} = 1.5799$ and at $t=1$, $.368\,\beta_1 + \frac{\beta_2}{3} = 1$. These are solved to yield $\beta_1 = .1784$ and $\beta_2 = 2.803$. For comparison, both the approximate basis function as given in equation (4.21) and the exact basis function, $.1784\ e^{-t} + 2.803\ \frac{1}{t+2}$ are now plotted as shown in Fig. 4.3. It is found that they agree quite closely.

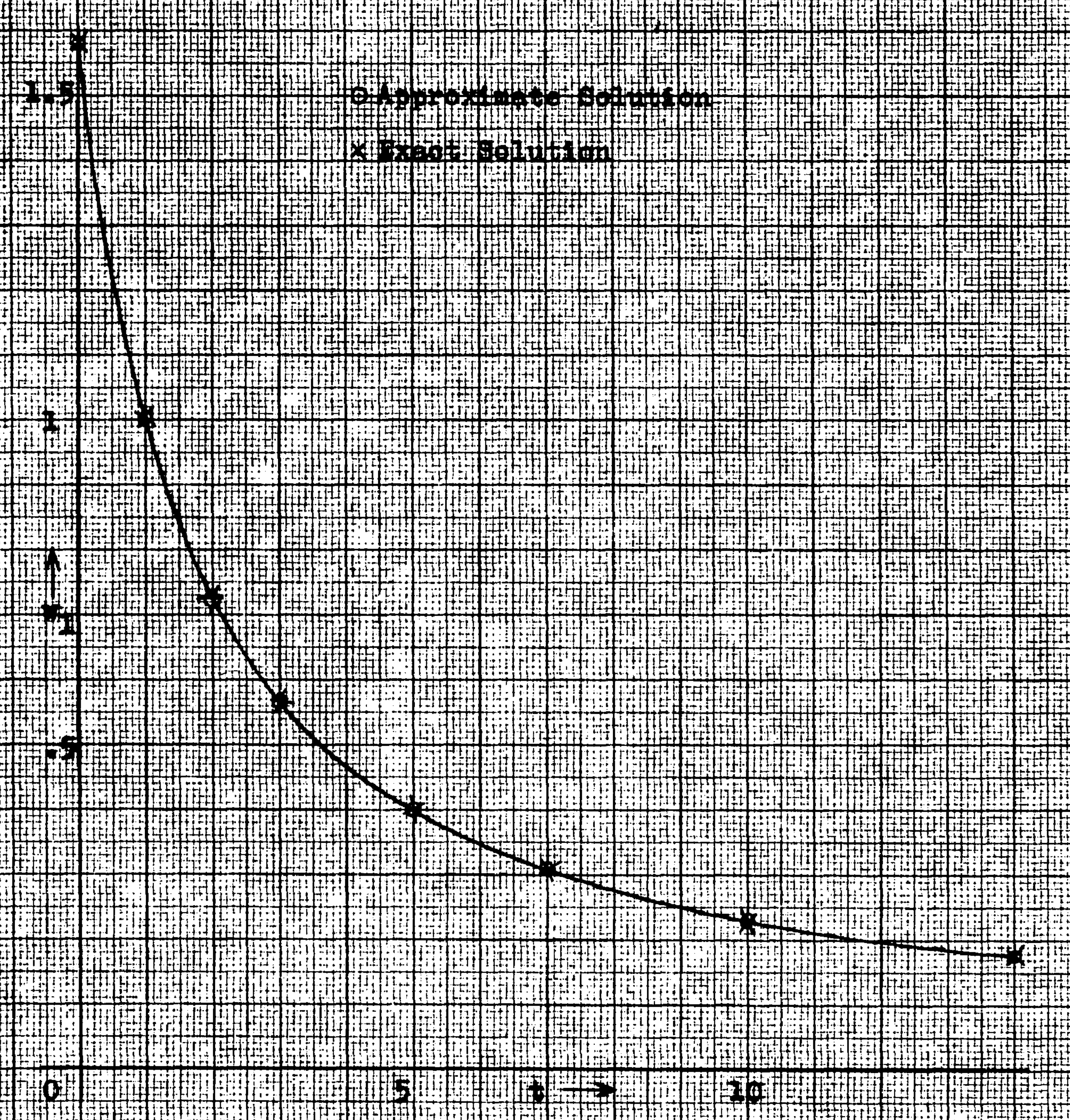



Fig.4.3 Solution Example 1

EXAMPLE 2

This example will serve to demonstrate that quite a high order differential equation may be successfully solved. The case that will be considered is

$$(.1t+.1)\frac{d^5v}{dt^5}+(1.5t+2)\frac{d^4v}{dt^4}+(8.5t+14.5)\frac{d^3v}{dt^3}+(22.5t+48)\frac{d^2v}{dt^2}$$
$$+(27.4t+72.4)\frac{dv}{dt}+(12t+39.4)v=0$$

After the transformation $t=\frac{1+\tau}{1-\tau}$ is applied the resulting equation is

$$(.0063\tau^{10}-.0625\tau^9+.2813\tau^8-.75\tau^7+1.31\tau^6-1.57\tau^5$$
$$+1.31\tau^4-.75\tau^3+.28\tau^2-.06\tau-.0063)\frac{d^5v}{d\tau^5}-(.0983\tau^9$$
$$-.656\tau^8+1.87\tau^7-2.62\tau^6+1.31\tau^5+1.31\tau^4-2.62\tau^3$$
$$+1.87\tau^2-.66\tau+.09)\frac{d^4v}{d\tau^4}+(.37\tau^8-1.5\tau^7+2.12\tau^6$$
$$-2.25\tau^5+5.62\tau^4-11\tau^3+10.87\tau^2-5.25\tau-1)\frac{d^3v}{d\tau^3}$$
$$+(.375\tau^7-.375\tau^6+.75\tau^5-13.12\tau^3+29.62\tau^2-24\tau$$
$$+6.75)\frac{d^2v}{d\tau^2}+(27.4\tau^2-54.8\tau+27.4)\frac{dv}{d\tau}+(-27.4\tau+51.4)v=0$$

Some representative values of $\phi_\ell$ are $\phi_1=51.4-27.4\tau$ and $\phi_2=27.4-3.4\tau$. The initial error, $I_1$ may be obtained as $I_1=\int_{-1}^{1}\phi_1^2\,d\tau=5.784\times10^3$. The next step is to obtain $\psi_{ij}$. Some representative values of these are $\psi_{21}=2.879\times10^3$, $\psi_{31}=1.2596\times10^3$. With $n=6$, $E_{norm}$ is found to be $1.04796\times10^{-6}$. The $k_\ell$ are $k_1=3.6087$, $k_2=-4.0567$, $k_3=1.0819$, $k_4=.3445$ and $k_5=.0348$. Thus, the basis function is given by the expression $w_1=1-\sum_{\ell=1}^{5}k_\ell\left(\frac{t-1}{t+1}\right)^\ell$

where the values of $k_\ell$ have been obtained above. The function is plotted in Fig. 4.4.

EXAMPLE 3[40]

The third example is designed to show how an unstable solution may be detected. The differential equation is given as

$$t^2 \frac{d^2v}{dt^2} + 4t\frac{dv}{dt} + 2v = 0$$

The $\tau$-domain equation is obtained after the transformation $t=\frac{1+\tau}{1-\tau}$ is applied and is

$$(.25\,\tau^4-.5\,\tau^2+.25)\frac{d^2v}{d\tau^2}+(.5\,\tau^3-1.5\,\tau^2-.5\,\tau+1.5)\frac{dv}{d\tau}+2v=0$$

The first three $\phi$ functions are obtained as

$$\phi_1=2,\ \phi_2=1.5+1.5\,\tau-1.5\,\tau^2+.5\,\tau^3,\ \phi_3=.5+3\tau-3\tau^3 +1.5\,\tau^4$$

The integral for $I_1$ is found to be $I_1=\int_{-1}^{1}\phi_1^2\,d\tau=8$. Some representative values of $\psi_{ij}$ are $\psi_{21}=4$, $\psi_{22}=4.571$. Next, values of $E_{norm}$ are calculated for various values of n. But even when $n=9$, $E_{norm}=.4708$ indicating that the basis function is likely to be unstable.

In this example, a direct verification of the above statement is possible since $\frac{\beta_1}{t}+\frac{\beta_2}{t^2}$ is an exact basis function as may be verified by substitution in the differential equation. Thus, as $t\rightarrow 0$ the basis function is unstable.

In the above sections a method of analyzing the homogeneous differential equation with polynomial

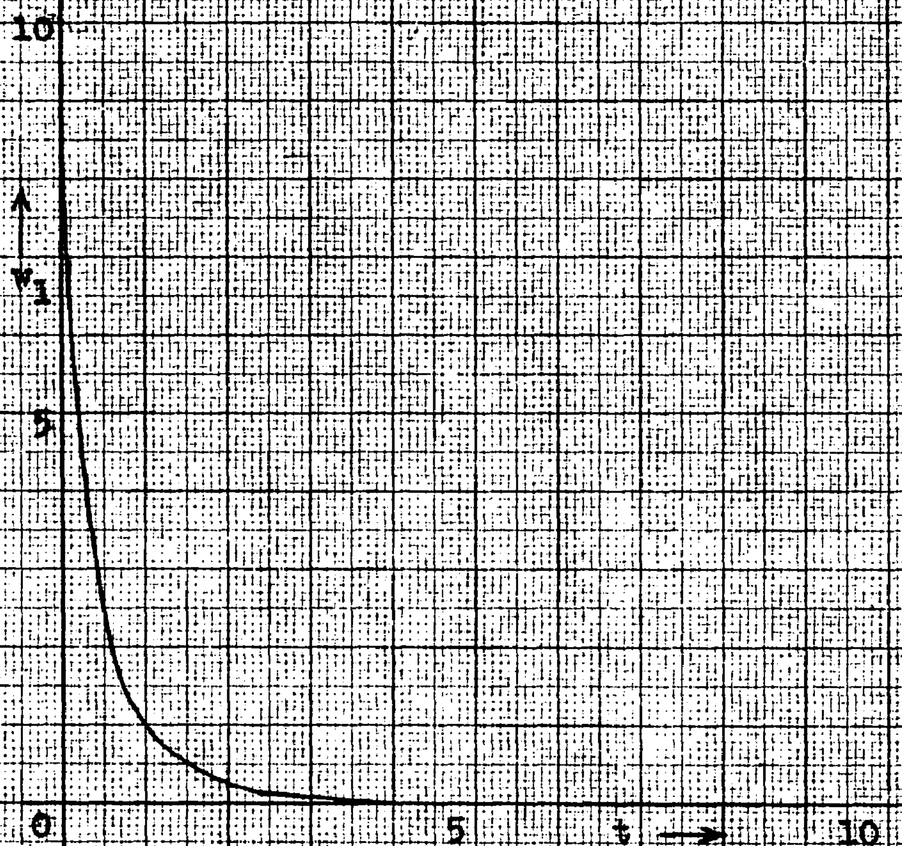



Fig.4.4 Solution of Example 2

coefficients has been developed. With some modifications the method may be used to cover more general cases as described below.

## 4.3. DIFFERENTIAL EQUATION WITH COEFFICIENTS THAT ARE ARBITRARY FUNCTIONS OF TIME

The expression for the general homogeneous equation is given in equation (4.2). To develop a method for solving this, the transformation $t = \frac{1+\tau}{1-\tau}$ is applied as before on equation (4.2). The result is

$$\sum_{i=0}^{m-1} c_{m+1-i}\left(\frac{1+\tau}{1-\tau}\right) \sum_{k=1}^{m-i} d_{m-i+1,k+1}(1-\tau)^{m-i+k} \frac{d^k v}{d\tau^k} + c_1\left(\frac{1+\tau}{1-\tau}\right)v = 0 \qquad (4.22)$$

where $d_{ij}$ have the same significance as that given in section 4.1. On collecting terms of the same orders of $\frac{dv}{d\tau}$ equation (4.22) can be rewritten as

$$\sum_{i=0}^{m-1} \frac{d^{m-i}v}{d\tau^{m-i}} \sum_{k=1}^{i+1} \frac{c_{m-k+2}\left(\frac{1+\tau}{1-\tau}\right)d_{m-k+2,m-i+1}}{(1-\tau)^{-2m+i+k-1}} + c_1\left(\frac{1+\tau}{1-\tau}\right)v = 0 \qquad (4.23)$$

As in section 4.1, equation (4.23) is multiplied by $(1-\tau)^s$ where $s$ $(s \geq 0)$ is the highest exponent of $(1-\tau)$ in the denominator of the coefficients of all the $\frac{d^i v}{d\tau^i}$ and of $v$. The resulting equation is then simplified to yield

$$\sum_{i=0}^{m-1} b_{m+1-i}(\tau)\frac{d^{m-i}v}{d\tau^{m-i}} + b_1(\tau)v = 0 \qquad (4.24)$$

which corresponds to equation (4.14) in Section 4.1.

The procedure to be followed now will depend on the nature of the functions $b_i(\tau)$ $(i=1,2,\ldots,m+1)$, specifically it will depend on whether they are continuous in $[-1,1]$ or whether they possess any discontinuities in this interval. The two cases are discussed below :

(1) If all the $b_i(\tau)$ $(i=1,2,\ldots,m+1)$ are continuous in $[-1,1]$ it is clear that by usuing Theorem 2.3 they can be approximated by polynomials, that is,

$$b_i(\tau) = \sum_{j\,1}^{p+1} b_{ij}\,\tau^{\,j-1} \tag{4.25}$$

Upon substituting equation (4.25) in equation (4.24) the resulting equation may be seen to be identical to equation (4.14) in Section 4.1. The procedure for solution from that point onwards remains the same.

(2) If some or all the $b_i(\tau)$ $(i=1,2,\ldots,m+1)$ are discontinuous in $[-1,1]$, the discontinuity may be of two types[41]. The first type refers to a point at which a function may be defined but is not continuous. Step functions, square waves etc. are examples of functions having such discontinuities. This type may be referred to as jump discontinuity.

If $b_i(\tau)$ of equation (4.24) have such jump discontinuities the interval $[-1,1]$ is broken into subintervals as shown below so that over each of these subintervals the functions $b_i(\tau)$ $(i=1,2,\ldots,m)$ are continuous. These subintervals will be named $[\tau_0,\tau_1)$,

$(\tau_1,\tau_2),\ldots,(\tau_{n-1},\tau_n]$ with $\tau_0=-1$ and $\tau_n=1$. Considering a typical subinterval $(\tau_{j-1},\tau_j)$ the functions $b_i(\tau)$ may be approximated there by polynomials. The procedure for solution to be followed from this point on is identical with the procedure described in Section 4.1 after equation (4.14) with the difference that $\psi_{ij}$ and the normalized error are calculated by usuing $\tau_{j-1}$ and $\tau_j$ as the lower and upper limits of integration instead of the previous -1 to 1 limits. The basis functions obtained will be valid over the subinterval $t_{j-1}=\frac{1+\tau_{j-1}}{1-\tau_{j-1}}$ to $t_j=\frac{1+\tau_j}{1-\tau_j}$. The procedure will have to be repeated over all the subintervals $[\tau_0,\tau_1)$, $(\tau_1,\tau_2),\ldots,(\tau_{n-1},\tau_n]$.

An alternate method of solving the problem where the $b_i(\tau)$ possess jump discontinuities is to smooth out the jumps and thereby make $b_i(\tau)$ continuous over $[-1,1]$. Then the procedure as described before for continuous $b_i(\tau)$ could be used.

The second type of discontinuity in functions refers to a point at which a function is not defined. An example of such a function is $f(\tau)=\frac{1}{\tau}$ which has a discontinuity at $\tau=0$. If all or some $b_i(\tau)$ in equation (4.24) have such discontinuities it may be possible to multiply equation (4.24) by a function $g(\tau)$ so that $gb_i(\tau)$ are continuous in $[-1,1]$. As an example, consider the case when $b_1(\tau)=\frac{b_1'(\tau)}{\tau}$ where $b_1'(\tau)$ is continuous in

$[-1,1]$. All the other $b_i(\tau)$ are assumed to be continuous in that interval. In such a case, obviously $g(\tau) = \tau$ removes the singularity.

When multiplying the differential equation by $g(\tau)$ it should be remembered that as discussed at the beginning of this chapter the solution will not be valid where $g(\tau) = 0$. After the $b_i$ coefficients have been made continuous the method described under (1) for continuous coefficients is applied.

The discussion of the method of obtaining one basis function of an n-th order equation is now complete. The means employed to obtain the remaining (n-1) basis functions is treated next.

## 4.4. FINDING THE OTHER BASIS FUNCTIONS

In Section 2.2, it was pointed out that if a basis function is known the order of the differential equation may be reduced by one and a second basis function obtained. This reduction of order may be carried out either in the $\tau$-domain or in the t-domain differential equation. However, the $\tau$-domain one yields more readily to this process and will be used in this investigation. A FORTRAN program for such a reduction is given in Appendix B. The flow diagram explaining this program is given in Appendix C.

A drawback of the method of reduction described above stems from the fact that the coefficients of the reduced differential equation are formed by employing derivatives upto an order (m-1) of the basis function, $w_1$. Thus, not only must $w_1$ be an appropriate representation of the basis function, but its (m-1) order derivatives must also represent the (m-1) order derivatives of the basis function. Thus, more accurate (and consequently higher order) approximation of $w_1$ must be made.

The above disadvantage may be overcome as described below. It was pointed out in Section 2.2 that the second basis function is obtained as $w_2 = w_1 \int_{\tau_0}^{\tau_1} \gamma_1(\tau)\, d\tau$ where $\gamma_1(\tau)$ is a basis function of the reduced differential equation in the $\tau$-domain. The function $\gamma_1(\tau)$ is realized by an approximation of the form, $\gamma_1 = 1 - \sum_{\ell=1}^{n} k_{\ell,1}\, \tau^{\ell}$. Noting that $w_1$ has been determined in the form $w_1 = 1 - \sum_{\ell=1}^{n} k_{\ell}\, \tau^{\ell}$ it is apparent that the form of $w_2$ is $w_2 = w_1 \int_{\tau_0}^{\tau} \gamma_1(\tau) d\tau$

$$= \tau - \sum_{\ell=1}^{n'} k'_{\ell}\, \tau^{\ell+1} \tag{4.26}$$

A more direct method for finding the second basis function may now be stated. Assume $w_2$ in the form of equation (4.26) and substitute this in the original $\tau$-domain differential equation (4.14). The following relation must be satisfied,

$$\phi_2 - \sum_{\ell=1}^{n'-1} k'_{\ell}\, \phi_{\ell+2} \cong 0$$

It is now clear that by re-naming the functions of the above equation from $\phi_i$ to $\phi_{i-1}$ $(i = 2, 3, \ldots, n'+1)$ the equation is identical with equation (4.17). Thus, the succeeding steps are the same as the ones described in connection with that equation. An example will serve to illustrate the process.

EXAMPLE 4

The second basis function for Example 1 given in Section 4.2 will be found. The first steps leading to the functions $\phi_i$ are identical to the steps in that example except that $I_1=\int_{-1}^{1}\phi_2^2\,d\tau=16.076$. This change is the result of the re-naming of the various $\phi_i$ functions. Also, due to this re-naming, $\psi_{ij}=\int_{-1}^{1}\phi_{i+1}\,\phi_{j+1}\,d\tau$. Taking these $\psi_{ij}$ and proceeding as described in Section 4.1 the second basis function may be obtained as

$$w_2=\frac{t-1}{t+1}-.192\left(\frac{t-1}{t+1}\right)^2-.525\left(\frac{t-1}{t+1}\right)^3-.341\left(\frac{t-1}{t+1}\right)^4-.085\left(\frac{t-1}{t+1}\right)^5$$

The $n=5$ solution, as written above, brings the error to $E_{norm}=1.86\times10^{-3}<.006$.

The above method for finding the second basis function may now be extended to obtain the remaining basis functions. Thus, for an m-th order **differential** equation (in the $\tau$-domain) the i-th basis function will be assumed to be

$$w_i=\tau^{i-1}-\sum_{\ell=1}^{n'}k_\ell\;\tau^{\ell+i-1}\qquad(i=1,2,\ldots,m)$$

This is substituted in equation (4.14) implying that the following relation needs to be satisfied :

$$\phi_i - \sum_{\ell=1}^{n'-1} k'_\ell \ \phi_{\ell+i} \cong 0$$

If the above $\phi$ functions are re-named from $\phi_{j+i}$ to $\phi_{j+1}$ $(j = 0,1,\ldots,n'-1)$ the equation is identical to equation (4.17) and the latter steps are similar.

To this point, the techniques for obtaining the basis functions have been developed. The next topic to be dealt with is the particular integral for a non-homogeneous differential equation.

## 4.5. THE NON-HOMOGENEOUS DIFFERENTIAL EQUATION

The general non-homogeneous equation is given in equation (4.1). Let us consider the particular integral, $w_p$, for this equation. The classical method for finding $w_p$ has been described in Section 2.3. It suffers from the disadvantage that (m-1) order derivatives of the basis functions are needed, making it mandatory that the basis functions be very accurately approximated. A second method more readily suited to our purpose will be described below.

Let the transform $t = \frac{1+\tau}{1-\tau}$ be applied to the differential equation (4.1). The result is

$$\sum_{i=0}^{m-1} c_{m+1-i}\left(\frac{1+\tau}{1-\tau}\right) \sum_{k=1}^{m-i} d_{m-i+1,k+1}(1-\tau)^{m-i+k} \ \frac{d^k v}{d\tau^k}$$

$$+ \ c_1\left(\frac{1+\tau}{1-\tau}\right)v = 0 \qquad (4.27)$$

where $d_{ij}$ have the same meaning as described in Section 4.1.

On collecting terms of the same orders of $\frac{dv}{d\tau}$ equation (4.27) may be re-written as

$$\sum_{i=0}^{m-1} \frac{d^{m-i}v}{d\tau^{m-i}} \sum_{k=1}^{i+1} \frac{c_{m-k+2}(\frac{1+\tau}{1-\tau})d_{m-k+2,m-i+1}}{(1-\tau)^{-2m+i+k-1}}$$

$$+ c_1(\frac{1+\tau}{1-\tau})v = u(\frac{1+\tau}{1-\tau}) \tag{4.28}$$

Next, equation (4.28) is multiplied by $(1-\tau)^s$ where s ($s \geq 0$) is the highest exponent of $(1-\tau)$ in the denominator of the coefficients of all the $\frac{d^i v}{d\tau^i}$, v and u. The resulting equation is then simplified to yield

$$\sum_{i=0}^{m-1} b_{m+1-i}(\tau)\frac{d^{m-i}v}{d\tau^{m-i}} + b_1(\tau)v = u(\frac{1+\tau}{1-\tau})(1-\tau)^s \tag{4.29}$$

which corresponds to equation (4.24).

If the functions $b_i(\tau)$ ($i=1,2,\ldots,m+1$) and $u(\frac{1+\tau}{1-\tau})(1-\tau)^s$ are polynomials in $\tau$ we can proceed directly. If they are continuous functions in $[-1,1]$ they are first approximated by polynomials. If some or all of them are discontinuous the procedure to be followed is similar to the one described in Section 4.3. Thus, in all these cases it may be assumed that

$$gb_i(\tau) = \sum_{j=1}^{p+1} b_{i,j}\tau^{j-1} \qquad (i = 1,2,\ldots,m+1)$$

and, $$gu(\frac{1+\tau}{1-\tau})(1-\tau)^s = \sum_{j=1}^{p+1} u_j\tau^{j-1} \tag{4.30}$$

where for continuous and first type of discontinuous functions $g = 1$, and for second type of discontinuity g is such as to make the representation (4.30) possible (see Section 4.3). When equation (4.30) is substituted in equation (4.29) the result is

$$\sum_{i=0}^{m-1} \frac{d^{m-i}v}{d\tau^{m-i}} \sum_{j=1}^{p+1} b_{m+1-i,j}\,\tau^{j-1} + v \sum_{j=1}^{p+1} b_{1j}\,\tau^{j-1} = \sum_{j=1}^{p+1} u_j \tau^{j-1} \tag{4.31}$$

which corresponds to equation (4.14). To obtain a particular integral of equation (4.31) over [-1,1], an approximation of the form

$$w_p(\tau) = 1 - \sum_{\substack{\ell=1 \\ \ell \neq \ell_d}}^{n-1} k_\ell \tau^\ell \qquad -1 \leqslant \tau < 1 \tag{4.32}$$

will be undertaken. This equation corresponds to equation (4.15) and the meaning of the various symbols used are the same as used there with one important exception. To explain this, it may be noted that when the substitution $v = \tau^\ell$ ($\ell = 0,1,\ldots,n-1$) are made in the left hand side of equation (4.31) the resulting expressions are termed $\phi_\ell$ ($\ell = 1,2,\ldots,n$). Unlike in Section 4.3 it is necessary to have

$$\phi_1 - \sum_{\substack{\ell=1 \\ \ell \neq \ell_d}}^{n-1} k_\ell \ \phi_{\ell+1} \cong \sum_{j=1}^{p+1} u_j \tau^{j-1} \tag{4.33}$$

Assuming that $\phi_\ell \neq \sum_{j=1}^{p+1} u_j \tau^{j-1}$ ($\ell = 1,2,\ldots,n$) equation (4.33) implies that it is desired to have

$$\left(\phi_1 - \sum_{j=1}^{p+1} u_j \tau^{j-1}\right) - \sum_{\substack{\ell=1 \\ \ell \neq \ell_d}}^{n-1} k_\ell \phi_{\ell+1} \cong 0 \tag{4.34}$$

If we define

$$\phi_1' = \phi_1 - \sum_{j=1}^{p+1} u_j \ \tau^{j-1} \tag{4.35}$$

equation (4.34) is identical with equation (4.17) and the later steps are carried out in the same manner as described in connection with that equation. Referring back to equation (4.32), equations (4.34) and (4.35) imply that while testing $\phi$ functions for linear independency to determine $\ell_d$ the functions under consideration should be $\phi_1', \phi_\ell (\ell = 2, 3, \ldots, n)$.

The case when

$$\phi_\ell = \sum_{j=1}^{p+1} u_j \mathcal{T}^{j-1} \qquad 1 \leqslant \ell \leqslant n \qquad (4.36)$$

was excluded in the above discussion. But when equation (4.36) is valid

$$w_p = \mathcal{T}^{\ell-1} \qquad 1 \leqslant \ell \leqslant n$$

and there is no need to proceed any further.

In the above sections, the differential equation representation was discussed. Means of obtaining the basis functions and the particular integral have been described. The following section will consider the case when the system is an electrical network.

## 4.6. NETWORK REPRESENTATION

The form of the network that will be considered here is the ladder network with one component in each arm. A discussion of the other forms of representation in the periodic case was presented in Section 3.3. The same comments are applicable here with obvious modifications.

To solve the ladder network an output $v_n = (\frac{t-1}{t+1})^n$ is assumed. Then it is necessary to find the input $u_n(t)$ needed at the input terminals to produce this output. The transformation $t = \frac{1+\tau}{1-\tau}$ is applied on $u_n(t)$ and the resulting expression denoted as $\phi_{n+1}$. Expressions of $\phi_1$, $\phi_2$,...,etc, may be obtained by letting $n = 0,1,\ldots$,etc. successively. These $\phi$ functions have the same significance as the $\phi$ functions described in connection with the solution of a differential equation. Thus, the procedure to be followed henceforth is identical with the procedure set down in Sections 4.1,4.3,4.4 and 4.5.

An example will now be given to illustrate the case.

EXAMPLE 5

Consider the network shown in Fig. 4.5. It is desired to find the excitation-free response of the network. Various component values have been indicated in the figure. Assuming $v = v_n = (\frac{t-1}{t+1})^n$ the following relations are evident.

$$i_1 = \frac{2n}{(t+1)^2} \left(\frac{t-1}{t+1}\right)^{n-1}$$

$$v_1 = \frac{2n}{(t+1)} \left(\frac{t-1}{t+1}\right)^{n-1}$$

$$i_2 = \frac{2n}{(t+1)} \left(\frac{t-1}{t+1}\right)^{n-1} + \left(\frac{t-1}{t+1}\right)^{n}$$

$$i_{c_2} = t\left[\frac{4n(n-1)}{(t+1)^3} \left(\frac{t-1}{t+1}\right)^{n-2}\right] + \left[\frac{2n}{(t+1)} \left(\frac{t-1}{t+1}\right)^{n-1} + \left(\frac{t-1}{t+1}\right)^{n}\right]$$

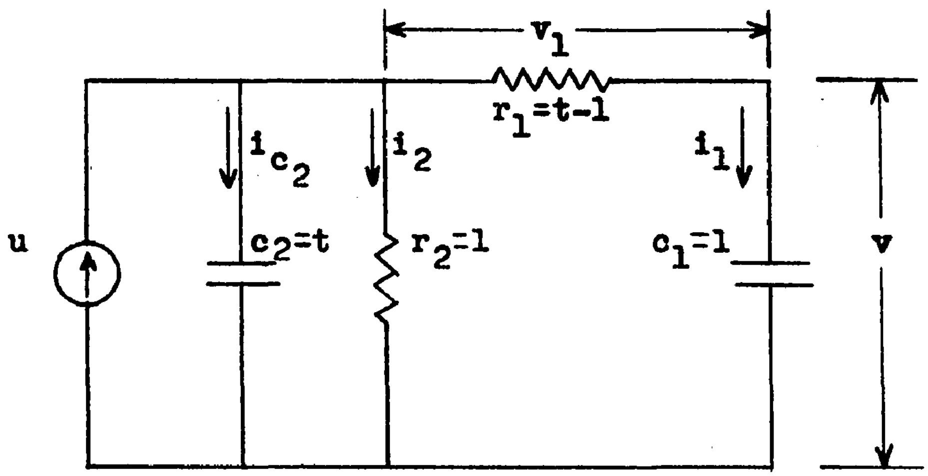



Fig.4.5 Time-varying Ladder Network

and finally,

$$u_n = \frac{2n}{(t+1)^2}\left(\frac{t-1}{t+1}\right)^{n-1} + \frac{2n}{(t+1)}\left(\frac{t-1}{t+1}\right)^{n-1} + \left(\frac{t-1}{t+1}\right)^n$$
$$+t\left[\frac{4n(n-1)}{(t+1)^3}\left(\frac{t-1}{t+1}\right)^{n-2}\right] + \left[\frac{2n}{(t+1)}\left(\frac{t-1}{t+1}\right)^{n-1} + \left(\frac{t-1}{t+1}\right)^n\right]$$

The transformation $t=\frac{1+\mathcal{T}}{1-\mathcal{T}}$ is next applied and the resulting expression termed $\phi_{n+1}$. Thus,

$$\phi_{n-1} = .5n(1-\mathcal{T})^2\mathcal{T}^{n-1} + 2n(1-\mathcal{T})\mathcal{T}^{n-1} + 2\mathcal{T}^n$$
$$+.5n(n-1)(1-\mathcal{T})^2(1+\mathcal{T})\mathcal{T}^{n-2}$$

Substituting $n=0,1,\ldots,$etc. successively the expressions for $\phi_1, \phi_2, \ldots,$etc. may be obtained. For the excitation-free network the following relation must be satisfied :

$$\phi_1 - \sum_{i=1}^{n} k_i \phi_{i+1} \cong 0$$

Values of the various $k_i$ and n are determined to satisfy the error criterion described in Section 4.1. Thus, the $\mathcal{T}$-domain solution is obtained as $1 - \sum_{i=1}^{n} k_i \mathcal{T}^i$. When the inverse transformation $\mathcal{T} = \frac{t-1}{t+1}$ is applied to the $\mathcal{T}$-domain solution the final answer is obtained as

$$1 - .749\left(\frac{t-1}{t+1}\right) - .169\left(\frac{t-1}{t+1}\right)^2 - .048\left(\frac{t-1}{t+1}\right)^3$$

## CHAPTER 5

## CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK

The purpose of this dissertation has been to find a convenient method of solving time-varying system problems. Both the excitation-free response and the response due to a particular excitation may be obtained. The method is applicable to an arbitrary high order and general type of system. The solution is valid over the whole time interval $[0, \infty]$ (in some cases, a finite number of points on this interval are possibly excluded). Because of the fixed form of the solution a computer program for plotting the result directly may be easily incorporated. This is believed to be of help when an understanding of the nature of a time-varying system is desired.

Future work in this area may be directed towards finding a method for solving an unstable time-varying system. The unstable nature may arise from the fact that the solution has singularities in the interval $[0, \infty]$. The system may also be said to be unstable if the solution does not approach a finite value after infinite time.

Another possible area for work may be in the field of general time-varying electrical networks. If the input needed to produce an assumed output can be determined for a network the method presented in this dissertation is directly applicable. Scope for more work exists for the networks where such an input cannot be determined. The difficult process of finding the differential equation for such a network and then solving it should preferably be avoided.

## APPENDIX A

In Chapter 3, it was mentioned that normalization of a periodically time-varying system may be made to arrive at convenient numbers for the various parameters. It is well-known that two types of normalization are used in time-invariant systems, frequency normalization and impedence normalization. Correspondingly, two types of normalization may be carried out for a periodically time-varying network under sinusoidal steady-state condition. These are named frequency normalization and magnitude normalization. (The term "impedance normalization" is avoided because impedance, as defined in a time-invariant system, has no meaning for the time-varying case). The method of normalization may be described as follows.

If the unnormalized inductance, resistance and capacitance are written respectively as

$$\ell = \sum_n L_n e^{j\omega_n t}$$

$$r = \sum_n R_n e^{j\omega_n t}$$

and

$$c = \sum_n C_n e^{j\omega_n t} \tag{A.1}$$

the normalized version of them will be

$$\ell = \sum_n \frac{L_n \omega_o}{r_o} e^{j\omega_n t/\omega_0}$$

$$r = \sum_n \frac{R_n}{r_o} \; e^{j\omega_n t/\omega_o}$$

and
$$c = \sum_n C_n \, \omega_o r_o \; e^{j\omega_n t/\omega_o} \qquad (A.2)$$

where $\omega_o$ and $r_o$ are the frequency and magnitude normalizing constants. Justification of the relations in equation (A.2) may be seen by considering the voltage-current relations in them. As an example, if $i$ and $v$ represent the current through and the voltage across a capacitance $c$ the relation is $i = \frac{d}{dt}(cv)$. Considering first the unnormalized form let the voltage be $v = e^{j\omega_v t}$. Then, taking $c$ as in equation (A.1) it follows that

$$i = \frac{d}{dt}\left(\sum_n C_n \; e^{j\omega_n t}\right) e^{j\omega_v t}$$
$$= \sum_n jC_n(\omega_n + \omega_v)\; e^{j(\omega_n + \omega_v)t} \qquad (A.3)$$

Next, the normalized case is considered with $v = e^{j\omega_v t/\omega_o}$ and $c$ as given in equation (A.2). This leads to

$$i = \frac{d}{dt}\left(\sum_n C_n \, \omega_o r_o \; e^{j\omega_n t/\omega_o}\right) e^{j\omega_v t/\omega_o}$$

$$= \sum_n jC_n r_o(\omega_n + \omega_v)\; e^{j(\omega_n + \omega_v)t/\omega_o} \qquad (A.4)$$

Now, since the frequency and the magnitude normalizing constants are known to be $\omega_o$ and $r_o$ respectively, the result obtained in equation (A.4) may be denormalized and it will agree with the result shown in equation (A.3). Similar verifications may be made with inductances and resistances.

## APPENDIX B

Various subroutines and the two main programs for finding a basis function and for reduction of order are given below. The FORTRAN language for IBM 7072 has been used.

```
(1)  SUBROUTINE BINOM (H,N,A)
     DIMENSION A(20)
     A(1)=1.0
     IF (N) 10,12,13
  10 STOP
  13 M=N+1
     DO 11 I=2, M
     G=M-I+1
     E=I-1
  11 A(I)=((H*A(I-1))*G)/E
  12 RETURN
     END

(2)  SUBROUTINE POLML (A,L,B,M,C,N)
     DIMENSION A(20),B(20),C(40)
     N=L+M-1
     DO 10 I=1,N
  10 C(I)=0.0
     DO 31 I=1,L
     K=I
     CP=A(I)
     DO 30 J=1,M
     C(K)=C(K)+CP*B(J)
  30 K=K+1
  31 CONTINUE
     RETURN
     END

(3)  SUBROUTINE CLECT (IQ,IR,C,IK)
     DIMENSION A(20),B(20), C(40)
     IF (IQ) 10,10,11
  11 IF (IR) 12,12,13
  13 H=1.0
     CALL BINOM (H,IQ,A)
     H=-1.0
     CALL BINOM (H,IR,B)
     IQ1=IQ+1
```

```
      IR1=IR+1
      CALL POLML (A,IQ1,B,IR1,C,IK)
      GO TO 16
   10 IF (IR) 14,14,15
   15 H=-1.0
      CALL BINOM (H.IR,C)
      IK=IR+1
      GO TO 16
   12 H=1.0
      CALL BINOM (H,IQ,C)
      IK=IQ+1
      GO TO 16
   14 C(1)=1.0
      IK=1
   16 RETURN
      END

  (4) SUBROUTINE LINE (M,IQ,IR,C,G,IK)
      DIMENSION G(10),C(5,10),CA(40)
      IQ1=IQ+1
      IF (IQ) 11,11,12
   12 IF (IR) 13,13,14
   11 IF (IR) 15,15,16
   13 IQR1=IQ+1
      GO TO 17
   15 IQR1=1
      GO TO 17
   16 IQR1=IR+1
      GO TO 17
   14 IQR1=IQ+IR+1
   17 DO 18 J=1,IQR1
   18 G(J)=0
      DO 10 I=1,IQ1
      IQV=IQ+1-I
      IRV=IR-1+I
      CALL CLECT (IQV,IRV,CA,IK)
      DO 10 J=1,IK
      G(J)=C(M+1,IQV+1)*CA(J)+G(J)
   10 CONTINUE
      RETURN
      END

  (5) SUBROUTINE SIMPL (M,IQ,IR,C,D,B)
      DIMENSION G(10),B(15,20),D(4,9),C(5,10)
      DO 10 K=1,M
      MK=M+1-K
      DO 10 I=K,M
      IQR2=IQ+IR+3-I-K
      M2I=M+2-I
      IRI=IR+2-I-K
      CALL LINE (MK,IQ,IRI,C,G,IK)
```

```
      IF (MK-M) 13,12,12
   12 DO 14 J=1,IQR2
   14 B(M2I,J)=D(MK+1,M2I)*G(J)
      GO TO 10
   13 DO 15 J=1,IQR2
   15 B(M2I,J)=D(MK+1,M2I)*G(J)+B(M2I,J)
   10 CONTINUE
      IF (IR-2*M) 17,17,18
   17 IQR3=IQ+1
      GO TO 19
   18 IQR3=IR+IQ-2*M+1
   19 MK=0
      IRI=IR-2*M
      CALL LINE (MK,IQ,IRI,C,G,IK)
      DO 16 L=1,IQR3
   16 B(1,L)=G(L)
      RETURN
      END

  (6) SUBROUTINE FACTL (I,P)
      IF (I) 13,14,15
   13 PRINT 16
   16 FORMAT (13H NO FACTORIAL)
      STOP
   14 P=1.0
      GO TO 17
   15 K=1
      DO 18 JA=1,I
   18 K=K*JA
      P=K
   17 RETURN
      END

  (7) SUBROUTINE GETA (N,IQR1,B,A)
      DIMENSION A(15,20), B(15,20)
      IRA=IQR1+N-1
      DO 42 I=1,N
      IM1=I-1
      CALL FACTL (IM1,R)
      DO 42 J=1,IRA
      IC=I
      JC=J
   53 IF (JC-IQR1) 54,54,52
   54 IF (IC-IQR1) 51,51,52
   52 B(IC,JC)=0.0
      JC=JC-1
      IC=IC-1
      IF (JC) 51,51,56
   56 IF (IC) 51,51,53
   51 IF (I-2) 33,34,34
   34 IF (J-1) 33,33,35
```

```
33 A(I,J)=R*B(I,J)
   GO TO 42
35 A(I,J)=R*B(I,J)
   IF (I-J) 37,37,38
37 IA=I-1
   GO TO 39
38 IA=J-1
39 DO 40 IB=1,IA
   CALL FACTL (IB,Q)
   IIB=I-IB
   JIB=J-IB
40 A(I,J)=A(I,J)+(R/Q)*B(IIB,JIB)
42 CONTINUE
   RETURN
   END
```

```
(8) SUBROUTINE NTGRL (C,M,PSI,II,JJ)
    DIMENSION PSI(15,20),S(40),C(40)
    PSI(II,JJ)=0
    DO 10 I=1,M,2
 10 S(I)=0
    DO 11 I=1,M,2
    P=I
    S(I)=(2.0*C(I))/P+S(I)
    PSI(II,JJ)=S(I)
 11 S(I+2)=S(I)
    RETURN
    END
```

(9) SUBROUTINE SOLVE (M,W)
This is a standard program for solving simultaneous equations and hence is not given here.

```
(10) SUBROUTINE DXDT (MM1,N,R)
     DIMENSION R(10,10)
     N2=N+2
     DO 12 I=1,MM1
     DO 12 J=2,N2
  12 R(I+1,J-1)=0.0
     DO 10 I=1,MM1
     N2I=N+2-I
     IF (N2I-1) 13,13,14
  14 DO 11 J=2,N2I
     RA=J-1
  11 R(I+1,J-1)=RA*R(I,J)
  10 CONTINUE
     RETURN
     END
```

```
(11) SUBROUTINE REDUC (M,N,IQ1,C,W,E)
     DIMENSION C(10,10),W(15,16),E(10,20),R(10,10),
```

```
 1WS(20),CS(20),ES(40)
  NM1=N-1
  M1=M+1
  NQU=N+IQ1-1
  DO 19 I=1,M
  DO 19 J=1,NQU
19 E(I,J)=0
  R(1,1)=1.0
  DO 14 J=2,N
14 R (1,J)=-W(J-1,N)
  MM1=M-1
  CALL DXDT (MM1,NM1,R)
  DO 22 I=1,M
22 PRINT 23, (R(I,J), J=1,N)
23 FORMAT (1X, 3H R= , 10F10.4)
  MMO=0
  NEW=N
16 INI=MMO+2
  DO 15 J=1,NEW
15 WS(J)=R(MMO+1,J)
  CALL FACTL(MMO,RA)
  DO 10 I=INI,M1
  IMO=I-1
  CALL FACTL (IMO,Q)
  NEI=I-MMO-1
  CALL FACTL (NEI,S)
  P=Q/(RA*S)
  DO 11 J=1,IQ1
11 CS(J)=P*C(I,J)
  CALL POLML(WS,NEW,CS,IQ1,ES,INDX)
  DO 13 J=1,INDX
13 E(NEI,J)=ES(J)+E(NEI,J)
10 CONTINUE
  MMO=MMO+1
  NEW=NEW-1
  IF (MMO-MM1) 16,16,18
18 DO 20 I=1,M
20 PRINT 21, (E(I,J), J=1,NQU)
21 FORMAT (1X, 3H E= , 10F10.4)
  RETURN
  END
```

(12) Main program for finding a basis function.

```
  DIMENSION C(5,10),D(4,9),B(15,20),A(15,20),
 1PSI(15,20),W(15,16),AX(20),BX(20),CX(40)
  READ 39,M,IQ,IR,MAXN,N,E
39 FORMAT (5I10,1F10.0)
  M1=M+1
  IQ1=IQ+1
  DO 66 I=1,M1
```

```
 66 READ 30, (C(I,J), J=1,IQ1)
 30 FORMAT (8F10.0)
    DO 67 I=2,M1
 67 READ 30, (D(I,J), J=2,M1)
    ITOU=IQ+IR+1
    CALL SIMPL(M,IQ,IR,C,D,B)
    MXN1=MAXN-1
    MXII=MAXN+ITOU-1
    CALL GETA(MAXN,ITOU,B,A)
    DO 62 J=1,ITOU
 62 AX(J)=B(1,J)
    CALL POLML (AX,ITOU,AX,ITOU,CX,ICX)
    CALL NTGRL (CX,ICX,B,N,N)
    ERRIN=B(N,N)
    PRINT 63, ERRIN
 63 FORMAT (1X, 15H INITIAL ERROR= 1PE17.8)
    DO 12 II=2,MAXN
    DO 10 JJ=1,MAXN
    DO 11 J=1,MXII
    AX(J)=A(II,J)
 11 BX(J)=A(JJ,J)
    CALL POLML (AX,MXII,BX,MXII,CX,MII)
 10 CALL NTGRL (CX,MII,PSI,II,JJ)
 12 CONTINUE
 18 PRINT 38,N
 38 FORMAT (1X, 3H N= 1I2)
    NII=N+ITOU-1
    NM1=N-1
    N1=N+1
    DO 15 I=2,N
    DO 14 J=2,N
 14 W(I-1,J-1)=PSI(I,J)
 15 W(I-1,N)=PSI(I,1)
    CALL SOLVE (NM1,W)
    DO 20 J=1,NII
    B(N+1,J)=A(1,J)
    DO 16 I=2,N
 16 B(N+1,J)=B(N+1,J)-W(I-1,N)*A(I,J)
 20 AX(J)=B(N+1,J)
    CALL POLML(AX,NII,AX,NII,CX,MII)
    CALL NTGRL(CX,MII,B,N1,N1)
    IF (B(N1,N1)-B(N,N)) 98,98,99
 99 PRINT 100
100 FORMAT (26H CHECK WHY ERROR INCREASES)
    GO TO 96
 98 ENORM=B(N1,N1)/ERRIN
    PRINT 36,ENORM
 36 FORMAT (1X,18H NORMALIZED ERROR= 1PE17.8)
    IF (ENORM-E) 97,104,104
104 IF (N1-MAXN) 105,105,17
105 N=N1
```

```
      GO TO 18
   97 PRINT 607
  607 FORMAT (17H SOLUTION VECTORS)
      DO 704 I=1,NM1
  704 PRINT 606, I, W(I,N)
  606 FORMAT(1X,I2,5X,1P6E17.8/(8X,1P6E17.8))
      GO TO 96
   17 PRINT 37
   37 FORMAT (17H MAY NOT CONVERGE)
   96 STOP
      END
```

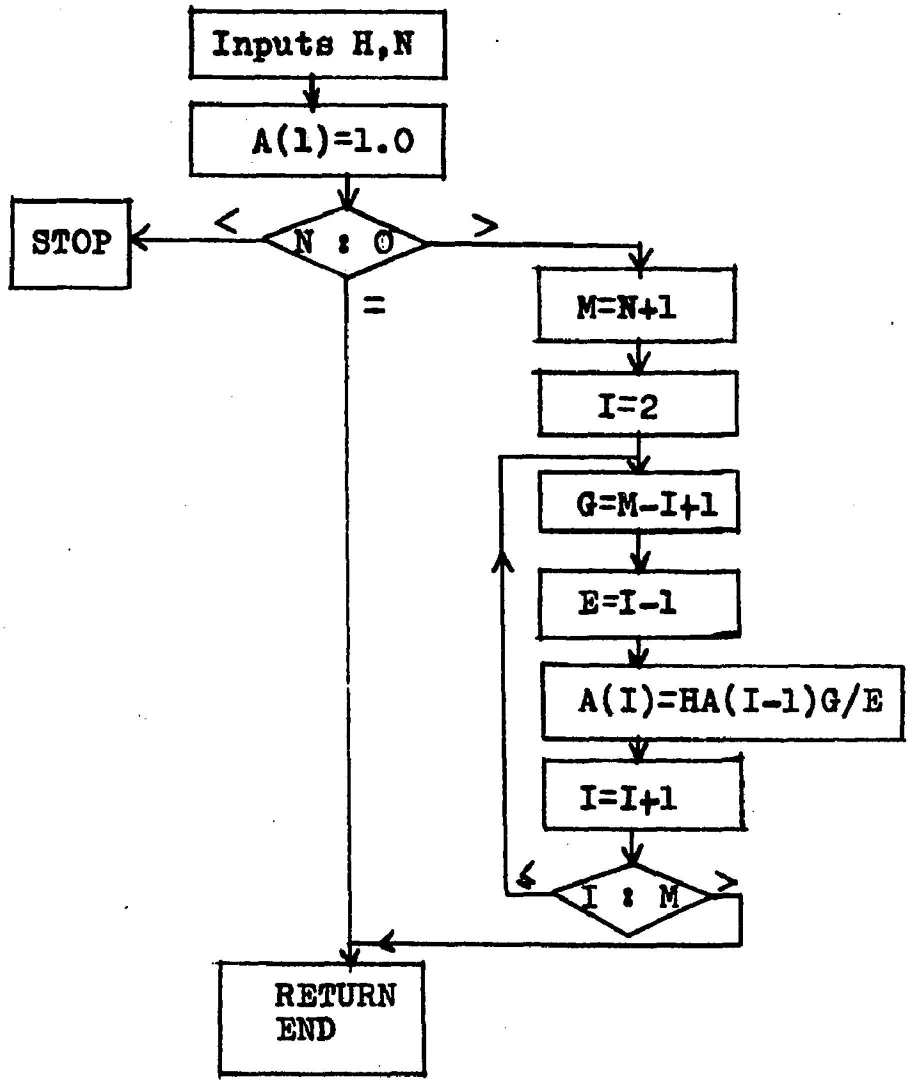
(13) Main program for reduction of order

```
      DIMENSION C(10,10),W(15,16),E(10,20)
      READ 10,M,N,IQ1
   10 FORMAT (3I10)
      M1=M+1
      DO 11 I=1,M1
   11 READ 12, (C(I,J),J=1,IQ1)
   12 FORMAT (8F10.0)
      READ 12,(W(I-1,N), I=2,N)
      CALL REDUC (M,N,IQ1,C,W,E)
      END
```

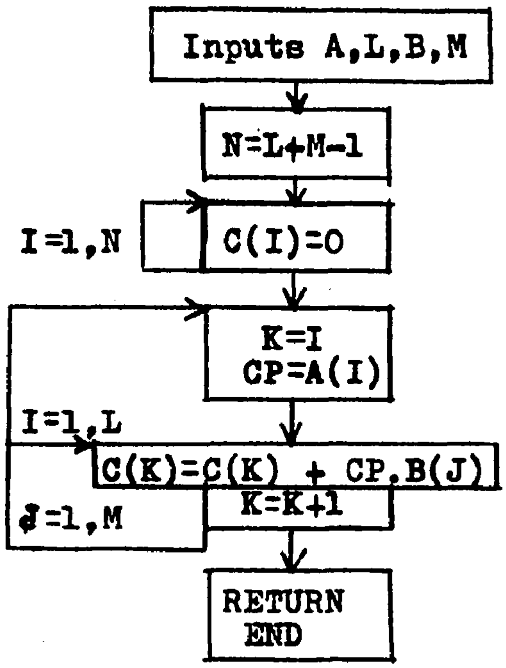
## APPENDIX C

Detailed flow diagrams for the various subroutines and the main program will be given below.

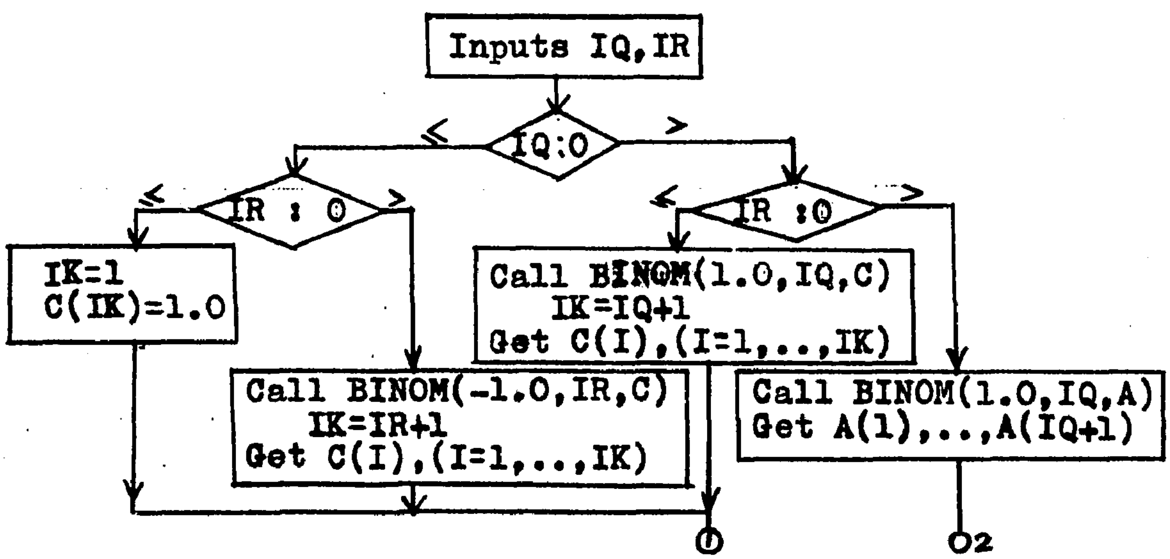
(1) Subroutine BINOM, Inputs H,N, outputs A(I) (I=1,2,...,N+1), purpose: to find the binomial coefficients of $(1+HX)^N$ and term them A(I), H = -1 or 1.



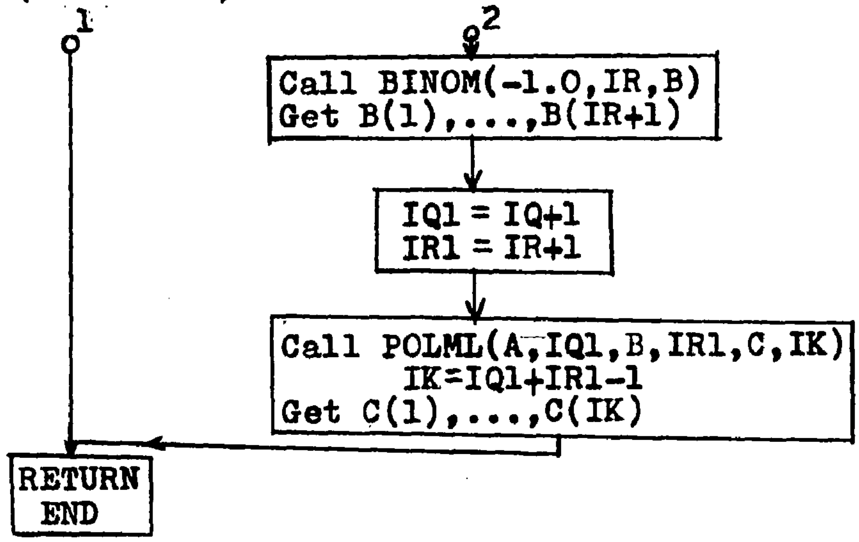
(2) Subroutine POLML. Inputs A,L,B,M,outputs C,N.
Purpose : To multiply two polynomials of degrees (L-1) and (M-1).



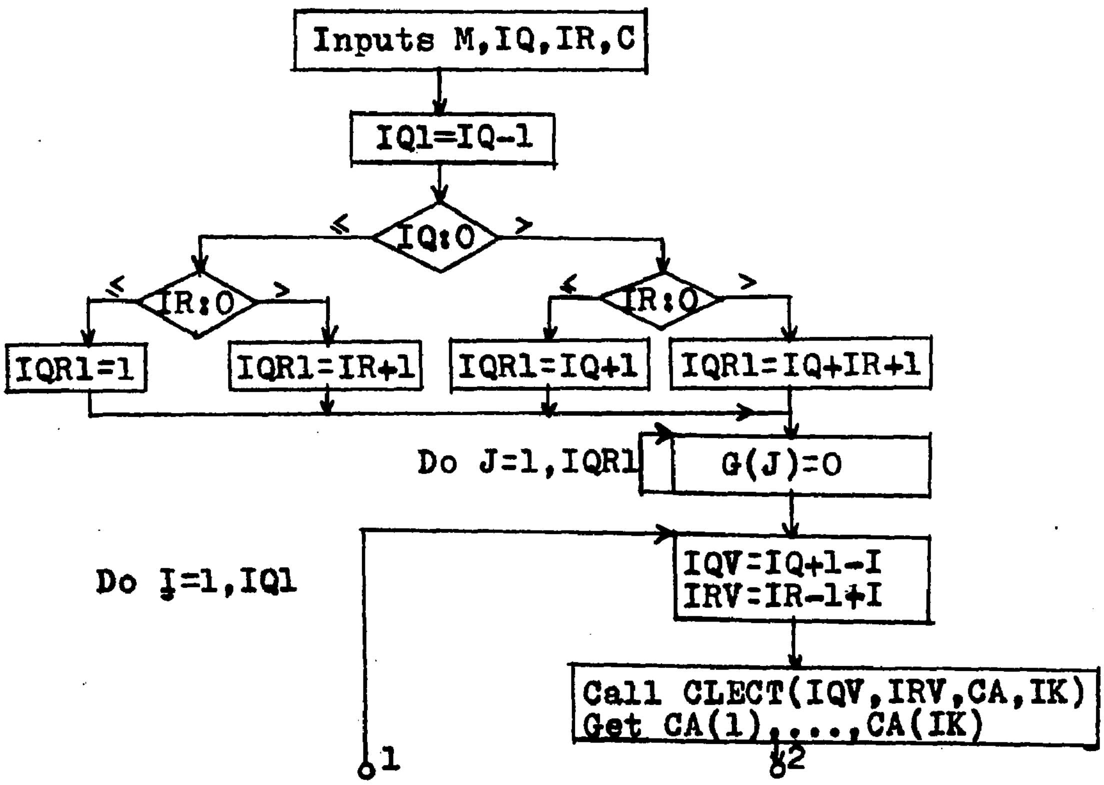

(3) Subroutine CLECT. Inputs IQ,IR, outputs C(I).
Purpose : To form the product $(1+X)^{IQ}(1-X)^{IR}$ subject to the condition that if IQ $\leq 0$, $(1+X)^{IQ}=1$ and if IR $\leq 0$, $(1-X)^{IR}=1$.



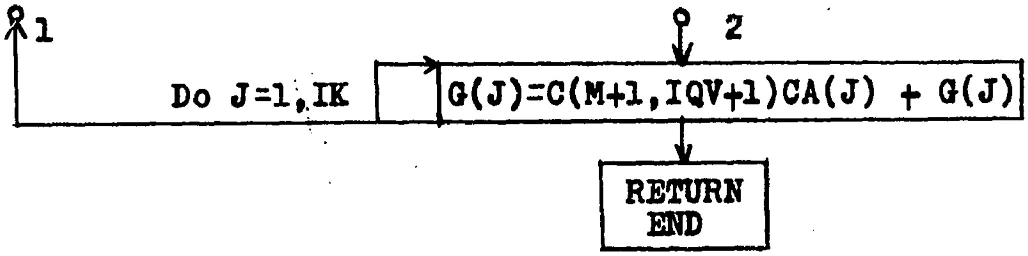
Subroutine CLECT (continued)



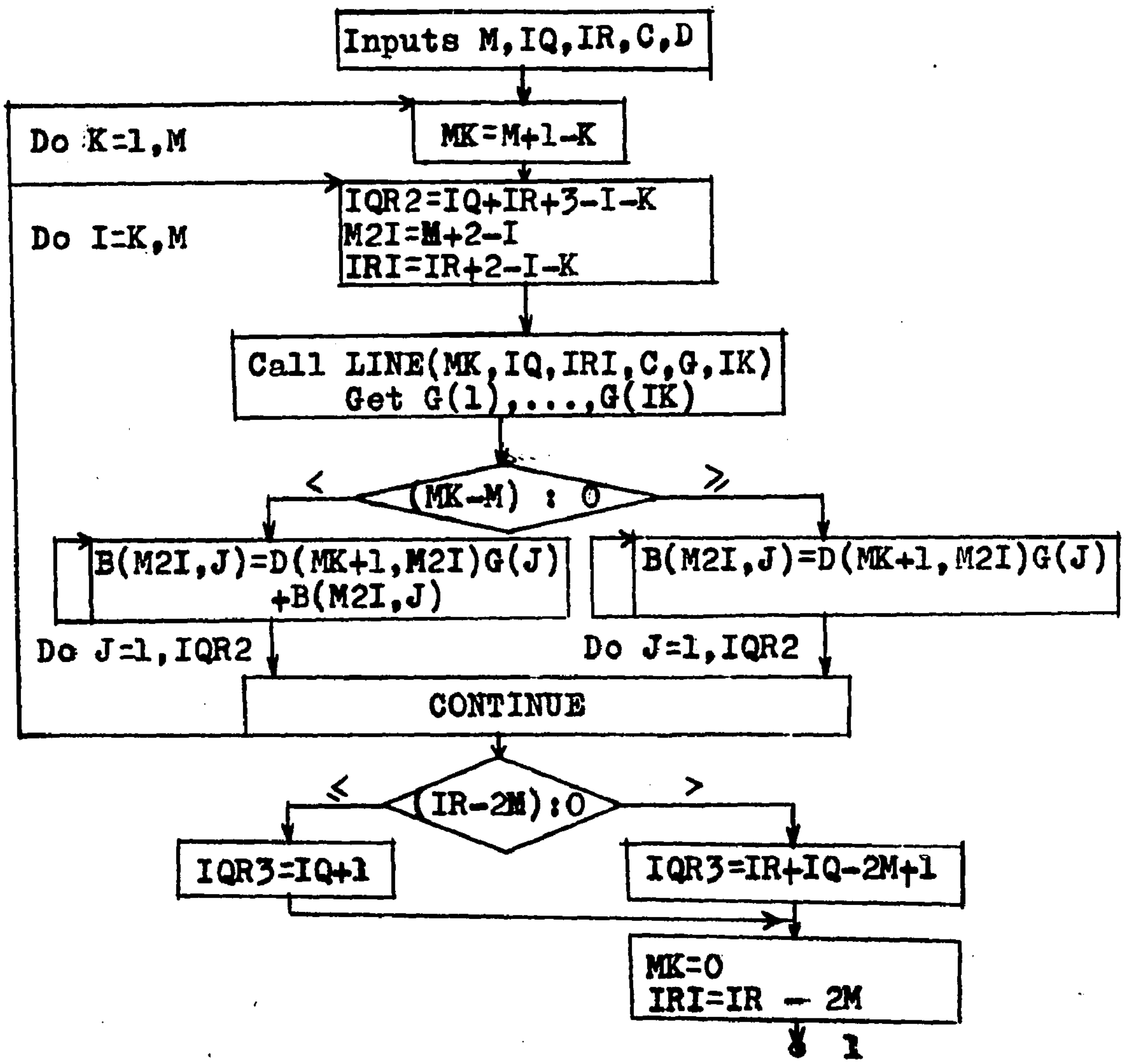

(4) Subroutine LINE(M,IQ,IR,C,G,IK). Inputs M, IQ, IR, C, outputs G(1),..,G(IK). Purpose; To form $G(1)+G(2)\mathcal{J}+..+G(IK)\mathcal{J}^{IK-1}=C(M+1,IQ+1)(1+\mathcal{J})^{IQ}(1-\mathcal{J})^{IR}+C(M+1,IQ)\cdot(1+\mathcal{J})^{IQ-1}(1-\mathcal{J})^{IR+1}+..+C(M+1,1)(1-\mathcal{J})^{IQ+IR}$.



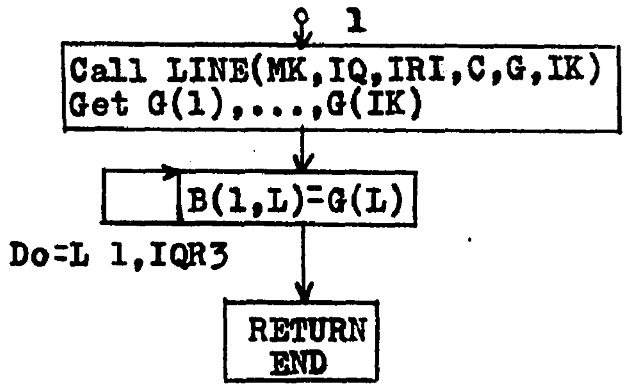
Subroutine LINE (continued)



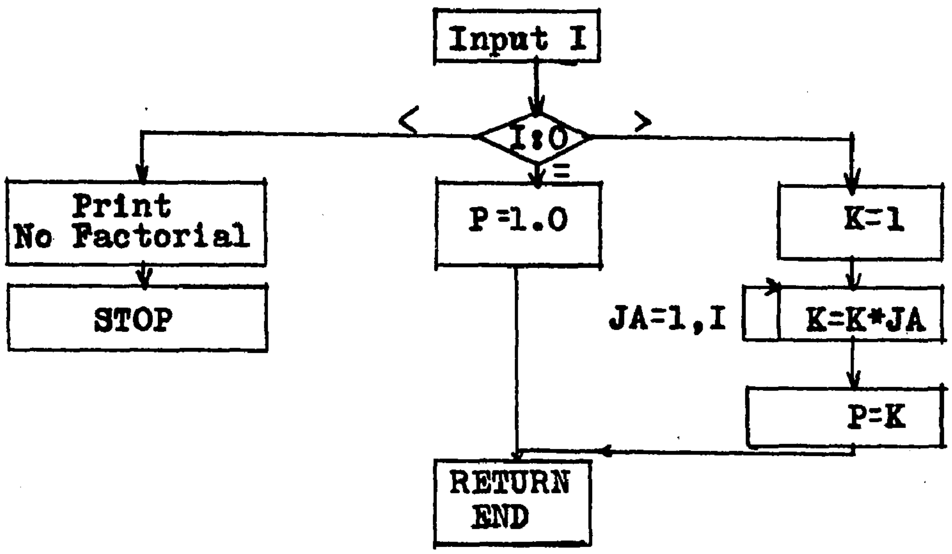

(5) Subroutine SIMPL (M,IQ,IR,C,D,B). Inputs M, IQ, IR, C, D, outputs B(I,J). Purpose: To apply the transformation $t=\frac{1+\zeta}{1-\zeta}$ to the t-domain differential equation and simplify.



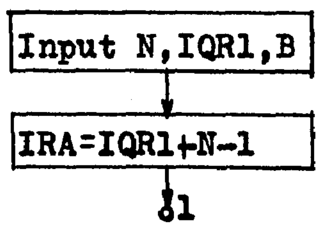
Subroutine SIMPL (continued)



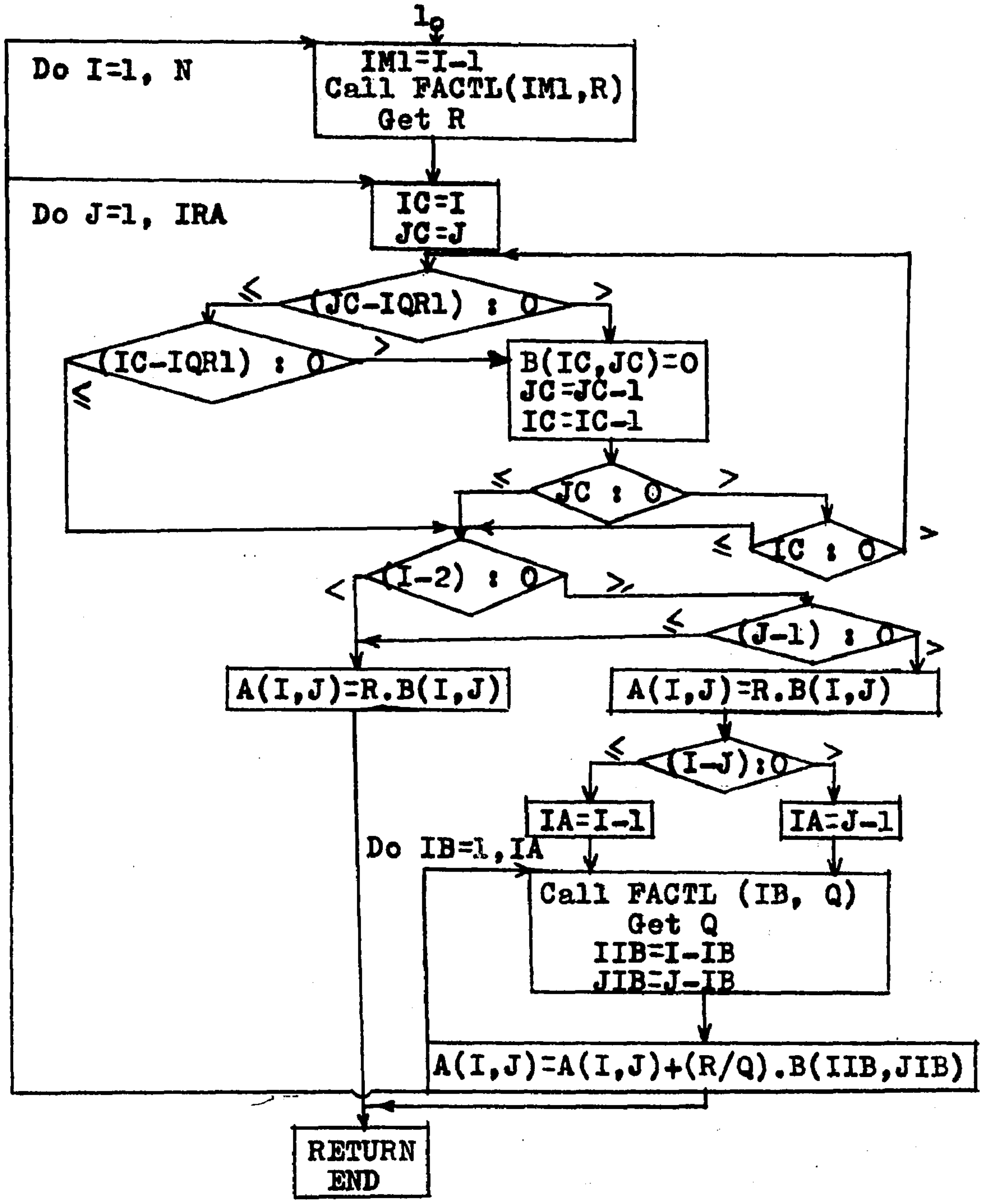

(6) Subroutine FACTL (I,P). Input I, output P. Purpose: To find P = Factorial I.




(7) Subroutine GETA (N,IQR1,B,A). Inputs N,IQR1,B, output A. Purpose: To find the $\phi$-functions from the $\mathcal{J}$-domain differential equation.

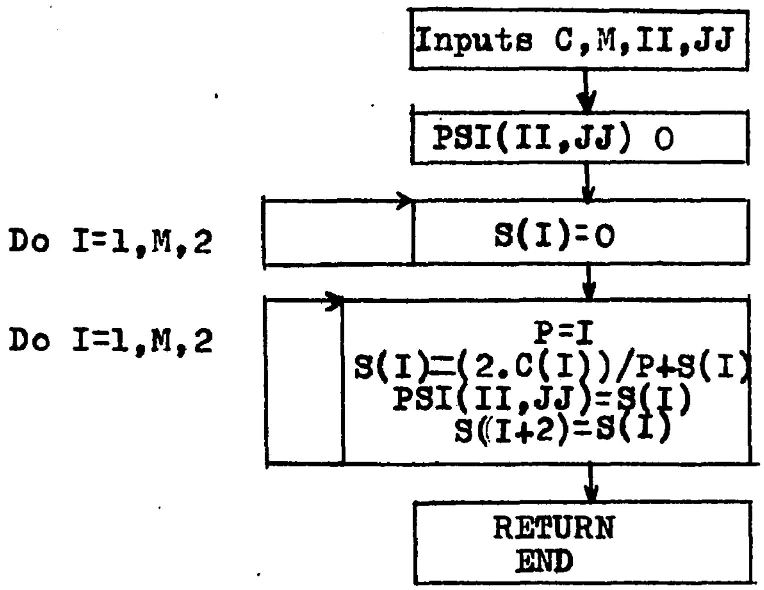


Subroutine GETA (continued)




(8) Subroutine NTGRL (C,M,PSI,II,JJ). Inputs C, M, II,JJ, output PSI. Purpose : To find $PSI(II,JJ)=\int_{-1}^{1}(C(1)+C(2)\zeta+\ldots+C(M)\zeta^{M-1})d\zeta$.

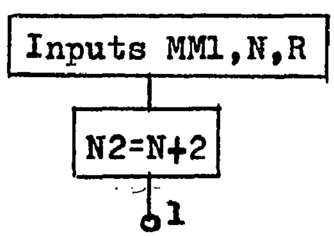
Subroutine NTGRL (continued)



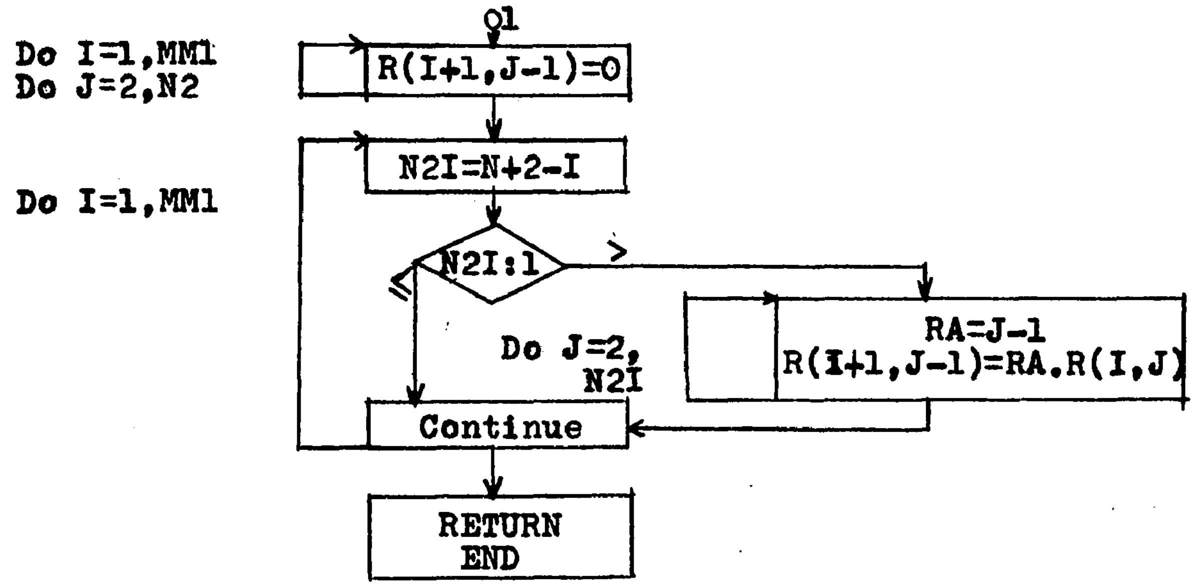

(9) Subroutine SOLVE (M,W). Inputs M,W, output W. Purpose : To solve M simultaneous equations with W(I,J), (I=1,2,...,M; J=1,2,...,M+1) coefficients and express the results as W(I,M+1), (I=1,2,...,M).

This is a standard program and hence will not be described here.

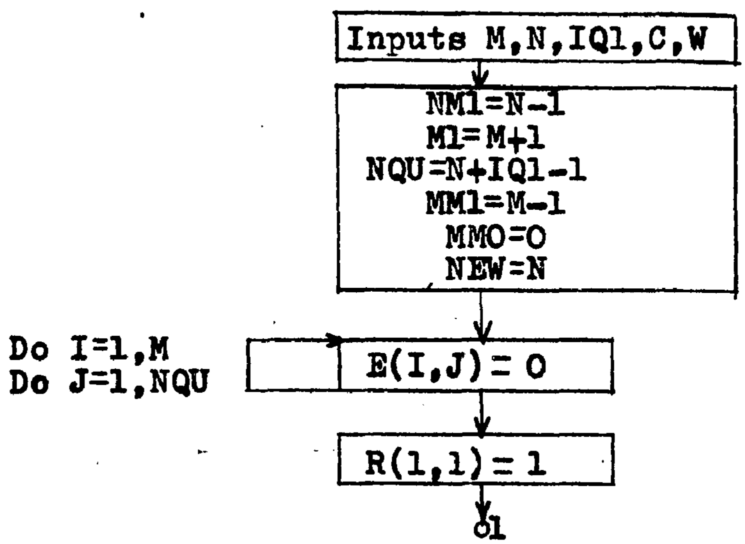
(10) Subroutine DXDT (MM1,N,R). Inputs MM1,N,R, output R. Purpose : To differentiate an N-th degree polynomial with coefficients R(1,J), (J=1,2,...,N+1) upto an order MM1 and express the results as R(I,J), (I=2,..,MM1+1; J=1,...,N+1-MM1).



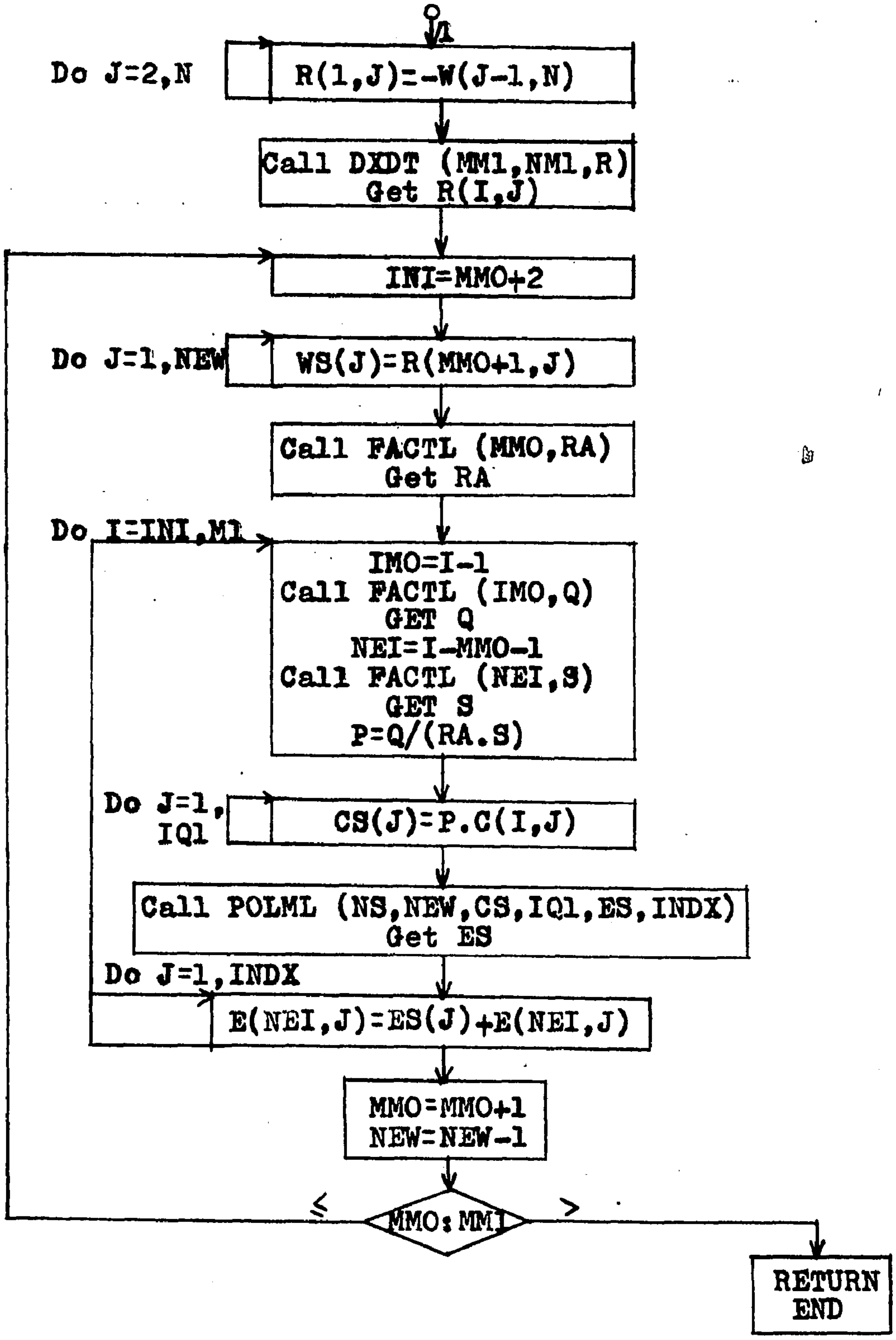
Subroutine DXDT (continued)




(11) Subroutine REDUC (M,N,IQ1,C,W,E). Inputs M,N, IQ1,C,W, output E. Purpose : To reduce the order of the differential equation (in $\mathcal{J}$) by one if a solution of order (N-1) is known. Coefficients of the original equation, C(I,J), (I=1,2,...,M+1; J=1,2,...,IQ1). Coefficients of the reduced equation, E(I,J), (I=1,..,M);(J=1,..,IQ1+N-1).

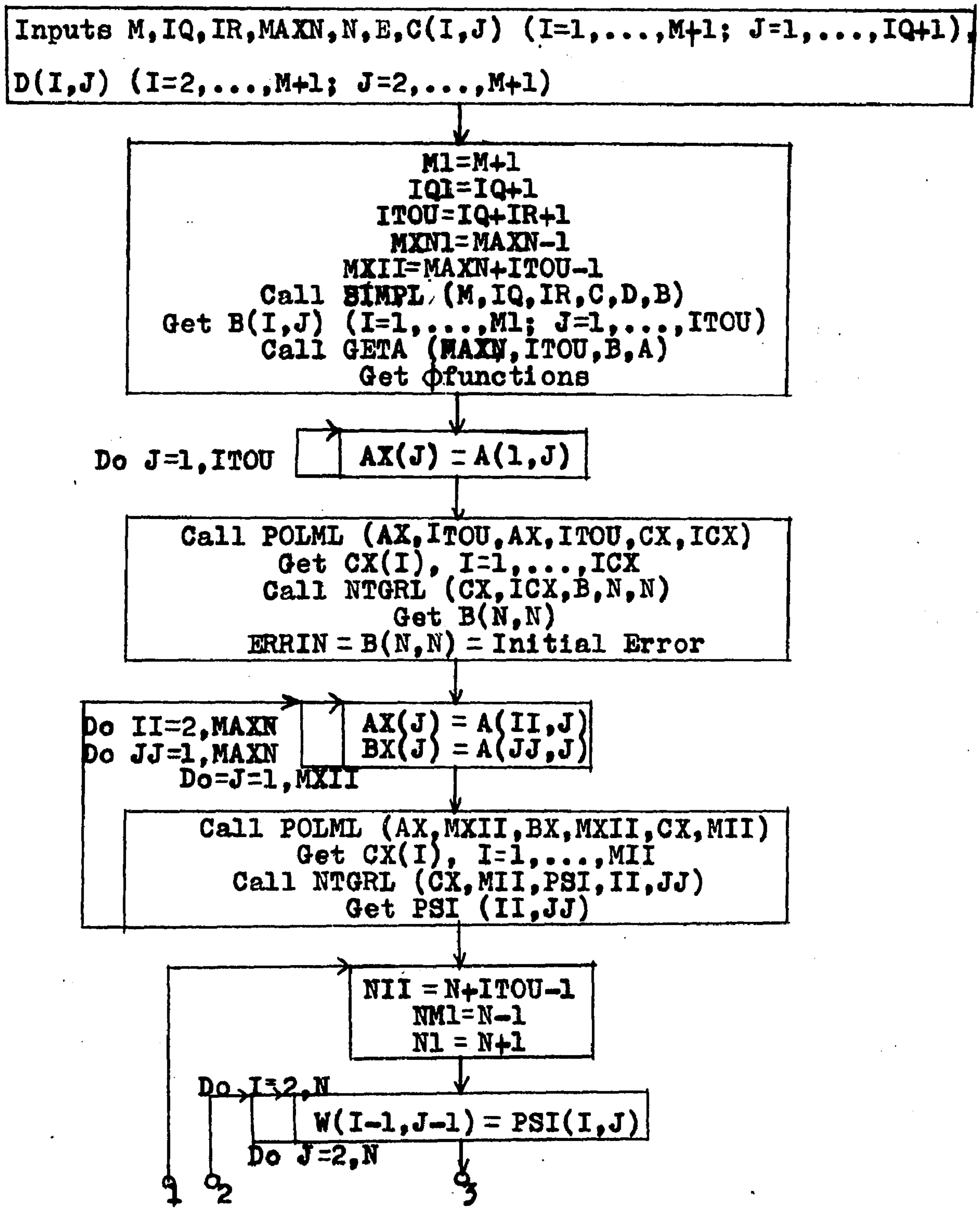


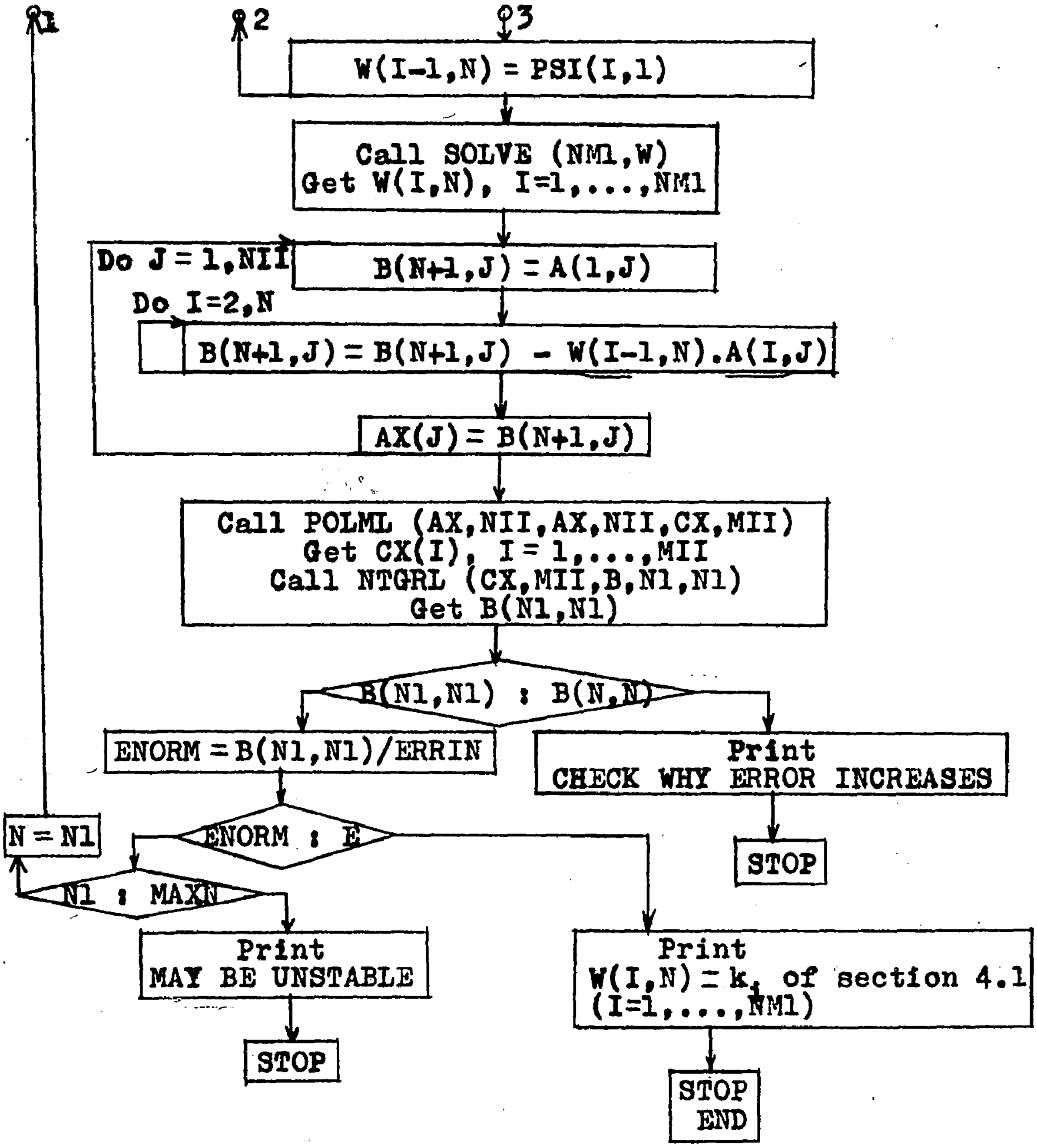
Subroutine REDUC (continued)



(12) Main program for obtaining a basis function.

Inputs M,IQ,IR,MAXN,N,E,C,D, output W. Purpose : To obtain a basis function.






(13) Main program for reduction of order. Inputs M,N,IQ1,C,W, output E. Purpose : To reduce the order of the differential equation with the help of a known basis function.

```
Inputs M,N,IQ1, C(I,J) (I = 1,...,M+1; J = 1,...,IQ1),
W(I-1,N) (I = 2,...,N)
            |
            v
Call REDUC (M,N,IQ1,C,W,E)
  Get E(I,J), (I=1,...,M)
        (J=1,...,N+IQ1-1)
            |
            v
      PRINT E(I,J)
            |
            v
           END
```

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