TOPOLOGY OF TORUS-LIKE FIBER SPACES

by

Michael Henry Hall

A Dissertation Submitted to the Faculty of the

DEPARTMENT OF MATHEMATICS

In Partial Fulfillment of the Requirements
For the Degree of

DOCTOR OF PHILOSOPHY

In the Graduate College

THE UNIVERSITY OF ARIZONA

1966

THE UNIVERSITY OF ARIZONA

GRADUATE COLLEGE

I hereby recommend that this dissertation prepared under my direction by Michael Henry Hall entitled Topology of Torus-like Fiber Spaces be accepted as fulfilling the dissertation requirement of the degree of Doctor of Philosophy

Ronald Jacobowitz
Dissertation Director

May 3, 1966
Date

After inspection of the dissertation, the following members of the Final Examination Committee concur in its approval and recommend its acceptance:*

Ronald Jacobowitz	May 3, 1966
D. Trifan	May 4, 1966
Oma Hamara	May 4, '66
Donald E. Myers	5/4/66
Harry Cole	5/6/66

*This approval and acceptance is contingent on the candidate's adequate performance and defense of this dissertation at the final oral examination. The inclusion of this sheet bound into the library copy of the dissertation is evidence of satisfactory performance at the final examination.

STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfillment of requirements for an advanced degree at The University of Arizona and is deposited in the University Library to be made available to borrowers under rules of the Library.

Brief quotations from this dissertation are allowable without special permission, provided that accurate acknowledgment of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the head of the major department or the Dean of the Graduate College when in his judgment the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.

SIGNED: Michael Henry Hall

ACKNOWLEDGEMENTS

The author wishes to express his gratitude to several people: to Dr. Ronald Jacobowitz, for his advice and guidance during several years of study and during the preparation of this paper; to Dr. Harvey Cohn, who originally suggested this problem; to Dr. Rabe von Randow, who made available his notes from a course of lectures given by D. Puppe, on which the material of Section 11 is based; to Dr. Olga T. Todd, who communicated privately much of the number theoretic material of Section 14; and to his wife, Marti, whose typing of this paper was only a small part of her contribution.

This paper was prepared under the partial sponsorship of National Science Foundation Grant number G-7412.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

ABSTRACT

Let X be a topological space, $f:X \longrightarrow X$ a homeomorphism. Denoting by R_f the equivalence relation on $X \times I$ generated by requiring $(x,1) \equiv (f(x),0) \pmod{R_f}$, we may consider the quotient space $(X \times I)/R_f$, which we call the <u>torus-like space determined by X and f</u>, and denote by (X,f). Such a method of constructing topological spaces generalizes a problem arising in the study of Hilbert fundamental domains (Cohn (1)) in which the space X is a torus T, and f a homeomorphism f_A induced in a natural way on T by a 2x2 matrix A with integer entries and determinant ± 1. This situation suggests investigation of the relationship between algebraic properties of the matrix A and topological properties of (T,f_A).

In Chapter I, the point-set-topological properties of the spaces (X,f) are studied, and results giving sufficient conditions (expressed as conditions on the homeomorphism f) for the homeomorphism of these spaces are obtained. Specialized to the case of the Hilbert-domain problem, these conditions read:

1. If A, B are integrally similar matrices, then $(T,f_A) \approx (T,f_B)$.

2. For any matrix A, $(T,f_A) \approx (T,f_{A^{-1}})$.

The problem of finding necessary conditions for the homeomorphism of torus-like spaces seems to be much more difficult. Partial results are obtained, in Chapters II, III, and IV, in the nature of determination of the homotopy groups for path-connected torus-like

spaces, and of the singular homology groups for the Hilbert-domain case. Since the homotopy invariants are, a priori, topological invariants, homeomorphism of torus-like spaces necessarily entails isomorphism of these groups. Chapter V is devoted to a discussion of certain number-theoretic matters which enter into the classification problem for the Hilbert-domain spaces, and an indication that a more detailed study of the fundamental group of these spaces may yield useful information regarding the classification problem.

Chapter I. Torus-like spaces.

0. Introduction.

The following problem arises in the study of Hilbert fundamental domains (Cohn (1)). The torus, T, can be taken to be the quotient space of R^2, Euclidean 2-space, with respect to the equivalence relation E, where $(x,y) \equiv (x',y') \pmod{E}$ if and only if $x \equiv x' \pmod{1}$ and $y \equiv y' \pmod{1}$. We denote the equivalence class of (x,y) by $\{(x,y)\}$. Letting I be the closed unit interval, and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2x2 matrix with integer entries whose determinant is ± 1 (A is said to be unimodular), we can generate an equivalence relation Q on $T \times I$ by requiring $(\{(x,y)\},1) \equiv (\{(ax+by, cx+dy)\},0) \pmod{Q}$. What relationship exists between the algebraic classification of the matrix A (e.g., its similarity class) and the topological type of the quotient space $(T \times I)/Q$?

The problem can be generalized. If X is an arbitrary topological space, $f: X \longrightarrow X$ a homeomorphism, and I the closed unit interval, let Q be the equivalence relation on $X \times I$ generated by requiring $(x,1) \equiv (f(x),0) \pmod{Q}$. We can study the relationship between the map f and the topological type of the quotient space $(X \times I)/Q$. In Section 1, we show that the Hilbert-domain problem is a special case of the more general one, with X the two-dimensional torus, and f a homeomorphism arising naturally from the matrix A.

We will begin by studying the more general problem about which considerable information can be obtained; in Section 3, we specialize to the problem involving the matrix, where we obtain the results:

Theorem 3.1. If A, B are two matrices which are integrally similar (i.e., $A = PBP^{-1}$, where P is unimodular with integer entries), then the quotient spaces are homeomorphic;

Theorem 3.2. The quotient spaces obtained from a matrix A and from its inverse are homeomorphic.

The remainder of the paper is devoted to the investigation of the problem of necessary conditions for the homeomorphism of torus-like spaces. This problem is attacked by finding various homotopy invariants of those spaces. In Chapter II and Chapter III we determine the fundamental group and the higher homotopy groups for the more general problem; in Chapter IV the (integral) homology groups are computed for the Hilbert-domain problem.

In Chapter V the possibility of applying these results to obtain a classification of the spaces of the Hilbert-domain problem in terms of their defining matrices is discussed. Various number-theoretic considerations seem to be involved here, in addition to the algebraic and topological methods employed in the earlier chapters, and we have no conjecture at present regarding the solution of this classification problem.

1. Definition of torus-like spaces.

We shall need the following result from point-set topology.

Proposition 1.1. Let X be any topological space, R an equivalence relation on X and $k:X \longrightarrow X/R$ the canonical map. If $f:X \longrightarrow Y$ is a map such that $k(x) = k(x')$ implies $f(x) = f(x')$, then f induces a unique map $g:X/R \longrightarrow Y$ such that $gk = f$.

A proof may be found in Hilton and Wylie (8), page 28, Proposition 1.5.6.

As in the preceding section, we take the torus, T, to be R^2/E, and we let $q:R^2 \longrightarrow T$ be the canonical map. A 2x2 unimodular matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries represents the linear transformation $F_A:R^2 \longrightarrow R^2$ given by $F_A(x,y) = (ax+by, cx+dy)$. The transformation F_A is a continuous function from R^2 to R^2.

Theorem 1.2. The transformation F_A induces a map $f_A:T \longrightarrow T$ such that $f_A(q(x,y)) = q(ax+by, cx+dy)$. The map f_A is a homeomorphism and $(f_A)^{-1} = f_{A^{-1}}$.

Proof: Since a, b, c, d are integers, if $q(x,y) = q(x',y')$, then $qF_A(x,y) = qF_A(x',y')$. Thus, by Proposition 1.1, the map represented by the dotted arrow in the diagram

$$\begin{tikzcd} R^2 \arrow[r, "F_A"] \arrow[d, "q"] & R^2 \arrow[d, "q"] \\ T \arrow[r, dashed] & T \end{tikzcd}$$

exists and is unique. This is the map f_A. Since $f_A q = qF_A$, $f_A(q(x,y)) = q(ax+by,\ cx+dy)$.

The matrix A is unimodular, and therefore A^{-1} is unimodular and has integer entries. The linear transformation $F_{A^{-1}}: R^2 \longrightarrow R^2$ is the inverse of F_A. We have the diagram

$$\begin{tikzcd} R^2 \arrow[r, "F_A"] \arrow[d, "q"] & R^2 \arrow[r, "F_{A^{-1}} = (F_A)^{-1}"] \arrow[d, "q"] & R^2 \arrow[r, "F_A"] \arrow[d, "q"] & R^2 \arrow[d, "q"] \\ T \arrow[r, "f_A"] & T \arrow[r, "f_{A^{-1}}"] & T \arrow[r, "f_A"] & T \end{tikzcd}$$

in which each square is commutative. Since $F_{A^{-1}}F_A$ is the identity on R^2, by the uniqueness of the induced map $f_{A^{-1}}f_A$ is the identity on T; similarly $f_A f_{A^{-1}}$ is the identity on T. Thus f_A is invertible, and therefore a homeomorphism and $(f_A)^{-1} = f_{A^{-1}}$.

Q.E.D.

We can now restate the Hilbert-domain problem. Given a 2x2 unimodular matrix A with integer entries, there is associated a homeomorphism $f_A: T \longrightarrow T$. Let Q be the equivalence relation on TxI generated by requiring $(q(x,y),1) \equiv (f_A(q(x,y)),0) \pmod{Q}$. The quotient space $(T\times I)/Q$ is the space which was defined in the preceding section. Notice that if A is the identity matrix, then $(T\times I)/Q$ is the

3-dimensional torus. The generalization of this method of forming topological spaces that we shall consider is given by the following definition.

<u>Definition</u>. Let X be a topological space, $f: X \dashrightarrow X$ a homeomorphism, and let R_f be the equivalence relation on $X \times I$ generated by requiring that $(x,1) \equiv (f(x),0) \pmod{R_f}$. The quotient space $(X \times I)/R_f$ we shall call the <u>torus-like space determined by X and f</u>, and we shall write (X,f) instead of $(X \times I)/R_f$.

2. Point-set topological properties of torus-like spaces.

<u>Definition</u>. A map $f:X\longrightarrow Y$ is <u>closed</u> if $f(C)$ is closed for every closed set $C\subseteq X$.

<u>Lemma</u> 2.1. Let (X,f) be a torus-like space, and let $p:X\times I\longrightarrow(X,f)$ be the canonical map. Then p is a closed map.

Proof: Let $C\subseteq X\times I$ be closed. It is sufficient to show that $p^{-1}(p(C))$ is closed in $X\times I$. Let $p_1:X\times I\longrightarrow X$ be the projection. Then $p_1|(X\times\{1\})$ and $p_1|(X\times\{0\})$ are homeomorphisms. Also, $X\times\{1\}$ and $X\times\{0\}$ are closed in $X\times I$, since their complements are open. Thus, $C_1 = C\cap(X\times\{1\})$ and $C_o = C\cap(X\times\{0\})$ are both closed in $X\times I$ and therefore C_1 is closed in $X\times\{1\}$ and C_o is closed in $X\times\{0\}$. Thus, $p_1(C_1)$ and $p_1(C_o)$ are both closed in X, so $f(p_1(C_1))$ and $f^{-1}(p_1(C_o))$ are closed in X, since f is a homeomorphism. It follows that $f(p_1(C_1))\times\{0\}$ and $f^{-1}(p_1(C_o))\times\{1\}$ are both closed in $X\times I$. But

$$p^{-1}(p(C)) = C\cup f(p_1(C_1))\times\{0\}\cup f^{-1}(p_1(C_o))\times\{1\}, \text{ and}$$

since the finite union of closed sets is closed, $p^{-1}(p(C))$ is closed.

Q.E.D.

Let $\{Y_a\}_{a\in A}$ be a collection of topological spaces. We may topologize $Y = \bigcup_{a\in A} Y_a$ as follows: A subset $X\subseteq Y$ is open if and only if $X\cap Y_a$ is open in Y_a, for all $a\in A$. With this topology, Y is called the <u>topological sum</u> of the collection $\{Y_a\}_{a\in A}$.

Lemma 2.2. Let X be a topological space, $f:X \longrightarrow X$ a homeomorphism. Let $Y_1 = X \times [0,1]$, $Y_2 = X \times [2,3]$, and let Y be the topological sum of Y_1 and Y_2. Let R be the equivalence relation on Y generated by requiring that $(x,1) \equiv (f(x),2) \pmod R$. Define a map $G:Y \longrightarrow X \times [0,2]$ by

$$(G|Y_1)(x,t) = (x,t),$$

$$(G|Y_2)(x,t) = (f^{-1}(x),\ t-1).$$

Then G induces a unique map $g:Y/R \longrightarrow X \times [0,2]$, and g is a homeomorphism.

Proof: From the definition of G, $G(x,t) = G(x',t')$ if and only if $(x,t) \equiv (x',t') \pmod R$. By Proposition 1.1, g exists, and g must be injective. Furthermore, G is surjective, so g must be. Hence it will suffice to show that g is a closed map. Let $q:Y \longrightarrow Y/R$ be the canonical map. A subset $C \subseteq Y/R$ is closed if and only if $q^{-1}(C)$ is closed in Y, that is, if and only if $q^{-1}(C) \cap Y_1$ and $q^{-1}(C) \cap Y_2$ are closed in Y_1 and Y_2 respectively. Now, $G(Y_1)$ and $G(Y_2)$ are both closed in $X \times [0,2]$, and $G|Y_1$ and $G|Y_2$ are homeomorphisms. Therefore, if $C \subseteq Y/R$ is closed, both $G(q^{-1}(C) \cap Y_1)$ and $G(q^{-1}(C) \cap Y_2)$ are closed, and so $G(q^{-1}(C) \cap Y_1) \cup G(q^{-1}(C) \cap Y_2)$ is closed. But,

$$\begin{aligned} g(C) &= G(q^{-1}(C)) \\ &= G((q^{-1}(C) \cap Y_1) \cup (q^{-1}(C) \cap Y_2)) \\ &= G(q^{-1}(C) \cap Y_1) \cup G(q^{-1}(C) \cap Y_2), \quad \text{so } g(C) \text{ is closed.} \end{aligned}$$

Q.E.D.

Lemma 2.3. Lemma 2.2 remains true if we replace $[0,1]$ by $(0,1]$, $[2,3]$ by $[2,3)$, and $[0,2]$ by $(0,2)$ throughout.

We shall write "$X \cong Y$" in place of "X is homeomorphic to Y."

<u>Lemma</u> 2.4. Let $Y_1 = X \times [0,1]$, $Y_2 = X \times [2,3]$, $Y_3 = X \times [4,5]$, let Y be the topological sum of Y_1, Y_2, Y_3, and let $f,g:X \longrightarrow X$ be two homeomorphisms. Let R be the equivalence relation on Y generated by requiring that $(x,1) \equiv (f(x),2) \pmod R$, $(x,3) \equiv (g(x),4) \pmod R$. Then the map $H:Y \longrightarrow X \times [0,3]$ defined by

$$(H|Y_1)(x,t) = (x,t),$$
$$(H|Y_2)(x,t) = (f^{-1}(x),\ t-1),$$
$$(H|Y_3)(x,t) = (f^{-1}g^{-1}(x),\ t-2)$$

induces a unique map $h:Y/R \longrightarrow X \times [0,3]$, and h is a homeomorphism.

Proof: We have $H(x,t) = H(x',t')$ if and only if $(x,t) \equiv (x',t') \pmod R$, so that h exists, is unique, and is injective. That h is surjective follows from the surjectivity of H, and the proof that h is closed exactly parallels the final part of the proof of Lemma 2.3.

Q.E.D.

<u>Definition</u>. If R,S are two equivalence relations on a set X, we say that R is a <u>subrelation</u> of S if $R \subseteq S (\subseteq X \times X)$.

<u>Definition</u>. Let R,S be two equivalence relations on a set X, and suppose R is a subrelation of S. If $k:X \longrightarrow X/R$ (the set of equivalence classes) is the canonical function, then we may define an equivalence relation S/R on X/R as follows: $k(x) \equiv k(y) \pmod{S/R}$ if and only if $x \equiv y \pmod S$. This relation is well defined, since $R \subseteq S$.

We shall need the following result from point set topology; a proof may be found in Gaal (5), page 72, Theorem 1.

Proposition 2.5. Let R, S be equivalence relations on a topological space such that R is a subrelation of S. Then X/S and $(X/R)/(S/R)$ are homeomorphic, and a homeomorphism is obtained as follows. Let $k: X \longrightarrow X/S$, $m: X \longrightarrow X/R$ and $n: X/R \longrightarrow (X/R)/(S/R)$ be the canonical maps. We have the diagram

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "k"] & X/R \arrow[d, "n"] \\ X/S \arrow[r, dashed, "j"] & (X/R)/(S/R) \end{tikzcd} ,$$

where the map j exists by Proposition 1.1, and is the desired homeomorphism.

Lemma 2.6. Let X be a topological space, $f,g,h,j: X \longrightarrow X$ homeomorphisms such that $f = ghj$. Let $U_1 = X \times [0,1]$, $U_2 = X \times [2,3]$, $U_3 = X \times [4,5]$, and let U be the topological sum of U_1, U_2, U_3. Let S be the equivalence relation on U generated by requiring $(x,1) \equiv (j(x),2) \pmod S$, $(x,3) \equiv (h(x),4) \pmod S$, and $(x,5) \equiv (g(x),0) \pmod S$. Then $(X,f) \cong U/S$.

Proof: For convenience, we take $(X,f) = (X \times [0,3])/R_f'$, where R_f' is generated by requiring $(x,3) \equiv (f(x),0) \pmod{R_f'}$. Let R be the subrelation of S generated by requiring $(x,1) \equiv (j(x),2) \pmod R$ and $(x,3) \equiv (h(x),4) \pmod R$. Let $p: X \times [0,3] \longrightarrow (X,f)$, $q: U \longrightarrow U/R$, $r: U \longrightarrow U/S$, $s: U/R \longrightarrow (U/R)/(S/R)$ be the canonical maps, $k: U/S \longrightarrow (U/R)/(S/R)$ be the homeomorphism of Proposition 2.5, and let $M: U \longrightarrow X \times [0,3]$, $m: U/R \longrightarrow X \times [0,3]$ be the map defined in Lemma 2.4 for the homeomorphisms $j,h: X \longrightarrow X$, and its induced homeomorphism.

We have the diagram

$$\begin{tikzcd}
 & U \arrow[dl, "r"'] \arrow[d, "q"] \arrow[dr, "M"] & \\
U/S \arrow[dr, "k"'] & U/R \arrow[r, "m"] \arrow[d, "s"] & X\times[0,3] \arrow[d, "p"] \\
 & (U/R)/(S/R) \arrow[r, dashed] & (X,f)
\end{tikzcd}$$

and must show that the dotted arrow exists and is a homeomorphism. Suppose that $sq(u) = sq(u')$, but $q(u) \neq q(u')$. Then, if $u = (x,t)$, $u' = (x',t')$, we must have either $t = 5$, $t' = 0$ or $t = 0$, $t' = 5$. It clearly suffices to consider only the case $t = 5$, $t' = 0$. We have $sq(x,5) = sq(x',0)$, and $sq = kr$, so $kr(x,5) = kr(x',0)$; but k is a homeomorphism, so $r(x,5) = r(x',0)$, which occurs if and only if $x' = g(x)$. Thus we must check whether $pmq(x,5) = pmq(g(x),0)$, but since $mq = M$, we need only see if $pM(x,5) = pM(g(x),0)$. As defined in Lemma 2.4, $M(x,5) = (j^{-1}h^{-1}(x),3)$, and $M(g(x),0) = (g(x),0)$; $pM(x,5)$ is the R_f' equivalence class of $M(x,5) = (j^{-1}h^{-1}(x),3)$, which is the same as the equivalence class of $(fj^{-1}h^{-1}(x),0)$. However, since $f = ghj$, $fj^{-1}h^{-1} = g$, and hence $pmq(u) = pmq(u')$. Therefore, the map we need, say n, exists.

To show that n is injective, it suffices to show that if $pmq(u) = pmq(u')$, then $sq(u) = sq(u')$. We may assume $q(u) \neq q(u')$, but $pmq(u) = pmq(u')$. Then, since m is a homeomorphism, $mq(u) \neq mq(u')$. Thus, either $mq(u) = (x,3)$ and $mq(u') = (f(x),0)$, for some $x \in X$, or $mq(u) = (f(x),0)$, $mq(u') = (x,3)$ for some $x \in X$. It suffices to consider the case $mq(u) = (x,3)$, $mq(u') = (f(x),0)$. Since $mq = M$, $mq(u) = (x,3)$

implies $u = (hj(x),5)$; and $mq(u') = (f(x),0)$ implies $u' = (f(x),0) = (ghj(x),0) = (g(hj(x)),0)$. Therefore, $r(u) = r(u')$, so $kr(u) = kr(u')$; since $sq = kr$, $sq(u) = sq(u')$, and n is injective.

The map n is surjective since both m and p are, and we shall be through if we can show that it is closed. Let $C \subseteq (U/R)/(S/R)$ be closed. Then $n(C) = pm(s^{-1}(C))$. Now, $s^{-1}(C)$ is closed by the definition of the quotient topology, m is a homeomorphism and therefore closed, and p is closed by Lemma 2.1, and we are done.

Q.E.D.

Theorem 2.7. Let X be a topological space, $f,g,h:X \longrightarrow X$ homeomorphisms such that $f = ghg^{-1}$. Then $(X,f) \cong (X,h)$.

Proof: From Lemma 2.6, $(X,f) \cong U/S$, where U is the topological sum of U_1, U_2, U_3, with $U_1 = X \times [0,1]$, $U_2 = X \times [2,3]$, $U_3 = X \times [4,5]$, and S is generated by requiring $(x,1) \equiv (g^{-1}(u),2) \pmod S$, $(x,3) \equiv (h(x),4) \pmod S$, and $(x,5) \equiv (g(x),0) \pmod S$. We shall show that U/S is homeomorphic to (X,h). Let a, b be distinct abstract points and let $U_2^1 = X \times [2, 5/2] \times \{a\}$, $U_2^2 = X \times [5/2, 3] \times \{b\}$. Let W be the topological sum of U_1, U_2^1, U_2^2, U_3, and let T be the equivalence relation on W generated by requiring $(x, 5/2, a) \equiv (x, 5/2, b) \pmod T$. Then the map $A:W \longrightarrow U$ given by

$A|U_1$ inclusion on U_1

$A|U_3$ inclusion on U_3

$(A|U_2^1)(x,t,a) = (x,t)$

$(A|U_2^2)(x,t,b) = (x,t)$

induces a homeomorphism $\propto:W/T \longrightarrow U$. There is an equivalence relation K

on W induced by A and S by the rule $w \equiv w'(\text{mod } K)$ if and only if $A(w) \equiv A(w')(\text{mod } S)$, and T is a subrelation of K. Moreover, it is not difficult to show that $W/K \cong U/S$. K is generated by requiring

a) $(x, 5/2, a) \equiv (x, 5/2, b)(\text{mod } K)$

b) $(x,3,b) \equiv (h(x),4)(\text{mod } K)$

c) $(x,5) \equiv (g(x),0)(\text{mod } K)$

d) $(x,1) \equiv (g^{-1}(x),2,a)(\text{mod } K)$.

Let L be the subrelation of K generated by requirements c) and d) above. Then by Proposition 2.5, $W/K \cong (W/L)/(K/L)$. Now, let V be the topological sum of U_1, U_2^1, and U_3. The relation L is the identity relation on U_2^2, so let $L' = L|V$, and then W/L is homeomorphic to the topological sum of U_2^2 and V/L' (where these spaces are disjoint); for convenience we take W/L to be this topological sum. Let $p:V \longrightarrow V/L'$ be the canonical map. Then the equivalence relation K/L is generated by requiring

a') $p(x, 5/2, a) \equiv (x, 5/2, b)(\text{mod } K/L)$

b') $(x,3,b) \equiv p(h(x),4)(\text{mod } K/L)$.

Now, let $Q:V \longrightarrow X \times [0, 5/2]$ be defined by

$$(Q|U_2^1)(x,t,a) = (x,t)$$

$$(Q|U_1)(x,t) = (g^{-1}(x), t+1)$$

$$(Q|U_3)(x,t) = (x, t-4).$$

Then by Lemma 2.4, Q induces a homeomorphism $q:V/L' \longrightarrow X \times [0, 5/2]$. We may therefore take W/L to be the topological sum of U_2^2 and $X \times [0, 5/2]$ and the equivalence relation K/L is generated by requiring

a'') $qp(x, 5/2, a) = Q(x, 5/2, a) = (x, 5/2) \equiv (x, 5/2, b)(\text{mod } K/L)$

b'') $(x,3,b) \equiv qp(h(x),4) = Q(h(x),4) = (h(x),0)(\text{mod } K/L)$.

Let J be the subrelation of K generated by requirement a") above; then by Proposition 2.5, $(W/L)/(K/L) \approxeq ((W/L)/J)/((K/L)/J)$. By Lemma 2.2, $(W/L)/J \approxeq X\times[0,3]$. On $X\times[0,3]$, the equivalence relation corresponding to $((K/L)/J)$ is generated by requiring

b'") $(x,3) \equiv (h(x),0) \pmod{(K/L)/J}$, and therefore $((W/L)/J)/((K/L)/J) \approxeq (X,h)$. We thus have the string of homeomorphisms

$$(X,f) \approxeq U/S \approxeq W/K \approxeq (W/L)/(K/L) \approxeq ((W/L)/J)/((K/L)/J) \approxeq (X,h).$$

Q.E.D.

Theorem 2.8. Let X be a topological space, $f:X\longrightarrow X$ a homeomorphism. Then $(X,f) \approxeq (X,f^{-1})$.

Proof: Let $p:X\times I\longrightarrow(X,f)$, $q:X\times I\longrightarrow(X,f^{-1})$ be the canonical maps. Define $r:X\times I\longrightarrow X\times I$ by $r(x,t) = (x, 1-t)$. We have the diagram

$$\begin{tikzcd} X\times I \arrow[r, "r"] \arrow[d, "p"] & X\times I \arrow[d, "q"] \\ (X,f) \arrow[r, dashed] & (X,f^{-1}) \end{tikzcd}$$

and $qr(x,1) = q(x,0)$,

$$qr(f(x),0) = q(f(x),1) = q(f^{-1}f(x),0) = q(x,0).$$

Hence, Proposition 1.1 applies, and the map represented by the dotted arrow exists. It is certainly surjective, injectivity follows as in previous arguments, and , by Lemma 2.1, it is closed.

Q.E.D.

3. Applications to the case $X = T$.

If X is the torus, T, and f_A is the map induced by the matrix A as in Section 1, the results of Section 2 have convenient interpretations.

<u>Theorem</u> 3.1. Let A, B, C be 2x2 unimodular matrices with integer entries such that $A = CBC^{-1}$. Then $(T, f_A) \cong (T, f_B)$.

Proof: We have the diagram

$$\begin{tikzcd}
R^2 \arrow[r, "F_C"] \arrow[d] & R^2 \arrow[r, "F_B"] \arrow[d] & R^2 \arrow[r, "F_{C^{-1}}"] \arrow[d] & R^2 \arrow[d] \\
T \arrow[r, "f_C"'] & T \arrow[r, "f_B"'] & T \arrow[r, "(f_C)^{-1}"'] & T
\end{tikzcd}$$

since (cf. Theorem 1.2) $(F_C)^{-1} = F_{C^{-1}}$, $(f_C)^{-1} = f_{C^{-1}}$. Now, $F_C F_B F_{C^{-1}} = F_{CBC^{-1}} = F_A$, and by the uniqueness property in Proposition 1.1, we have $f_A = f_C f_B f_{C^{-1}}$. Now apply Theorem 2.7.

Q.E.D.

<u>Theorem</u> 3.2. Let A be a 2x2 unimodular matrix with integer entries. Then $(X, f_A) \cong (X, f_{A^{-1}})$.

Proof: $(f_A)^{-1} = f_{A^{-1}}$, so we may apply Theorem 2.8.

Q.E.D.

Chapter II. The fundamental group of (X,f).

4. Reduced free products of groups.

The results of this section are taken from Fox(4); proofs and further details may be found there.

Definition. Let $G^* = \{G_a\}_{a \in A}$ be any collection of groups, and let H be an arbitrary group. Suppose there is given a collection $F = \{f_a\}_{a \in A}$ of homomorphisms, where $f_a : G_a \longrightarrow H$. Then F is called a *homomorphism of G^* into H*.

Definition. Let $G^* = \{G_a\}_{a \in A}$ be a collection of groups. A *free product* of G^*, (H,F), is a pair consisting of a group H and a homomorphism $F : G^* \longrightarrow H$ such that given any homomorphism $E : G^* \longrightarrow K$, there exists a unique group homomorphism (in the usual sense) $d : H \longrightarrow K$ such that the diagram

$$\begin{tikzcd} G_a \arrow[rr, "e_a"] \arrow[dr, "f_a"'] & & K \\ & H \arrow[ur, "d"'] & \end{tikzcd}$$

is commutative for all $a \in A$. (Alternatively, we may drop uniqueness and require that the $f_a(G_a)$ generate H.)

Proposition 4.1. The free product exists and is unique up to isomorphism.

A proof may be found in Fox (4), pages 6 and 7.

We shall write $G_1 * G_2$ for the free product of the two groups G_1 G_2.

<u>Definition</u>. <u>The free group on a set</u> X is a pair (G,f), where G is a group and $f:X \longrightarrow G$ is a function such that given any function $h:X \longrightarrow H$, where H is a group, there is a unique group homomorphism $h^*:G \longrightarrow H$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & H \\ & G \arrow[ur, "h^*"'] & \end{tikzcd}$$

is commutative. The elements of X are called the <u>generators</u> of G. Note that the group G is the free product of $\{Y_x\}_{x \in X}$, where each Y_x is an infinite cyclic group.

<u>Proposition</u> 4.2. Any group G is a homomorphic image of some free group (Fox, <u>op.cit</u>., page 9).

<u>Definition</u>. Let X be the free group on $\{x_a\}_{a \in A}$, and let $R = \{r_b\}_{b \in B}$ be a non-empty subset of X. Let N be the smallest normal subgroup of X containing R, i.e., the intersection of all normal subgroups containing R. If G is any group isomorphic to X/N (written $G \cong X/N$), we shall say that $(\{x_a\} ; \{r_b\})$ is a <u>presentation</u> of G and shall write $G = (\{x_a\} ; \{r_b\})$. Thus a group isomorphic to G is obtained by imposing the relations $r_b = 1$, $b \in B$, on the free group X; for this reason, the elements r_b are called <u>relators</u>.

<u>Definition</u>. Consider the system of groups and homomorphisms

$$\begin{tikzcd}[row sep=small] & G_1 \\ G_o \arrow[ur, "f_1"'] \arrow[dr, "f_2"] & \\ & G_2 \end{tikzcd}$$

The <u>reduced</u> <u>free</u> <u>product</u> of this system is a group H and homomorphisms $h_1:G_1 \longrightarrow H$, $h_2:G_2 \longrightarrow H$ such that $h_1f_1 = h_2f_2$, with the property that given any group K and homomorphisms $k_1:G_1 \longrightarrow K$, $k_2:G_2 \longrightarrow K$ such that $k_1f_1 = k_2f_2$, there exists a unique homomorphism $m:H \longrightarrow K$ such that $mh_1 = k_1$, $mh_2 = k_2$, i.e., the diagram

$$\begin{tikzcd} & G_1 \arrow[dr, "h_1"] \arrow[drr, "k_1"] & & \\ G_o \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & H \arrow[r, "m"] & K \\ & G_2 \arrow[ur, "h_2"] \arrow[urr, "k_2"'] & & \end{tikzcd}$$

is commutative.

<u>Proposition</u> 4.3. The reduced free product of the system

$$\begin{tikzcd}[row sep=small] & G_1 \\ G_o \arrow[ur, "f_1"'] \arrow[dr, "f_2"] & \\ & G_2 \end{tikzcd}$$

exists and is unique up to isomorphism. Moreover, if G_o, G_1, G_2 have presentations

$$G_o = (\{x_a\} ; \{r_b\})$$
$$G_1 = (\{y_c\} ; \{s_d\})$$

$$G_2 = (\{z_m\} ; \{t_n\}),$$

then the reduced free product has the presentation

$$(\{y_c\} \cup \{z_m\} ; \{s_d\} \cup \{t_n\} \cup \{f_1(x_a)(f_2(x_a))^{-1}\}).$$

Proofs of the various statements of this proposition may be found in Fox (4), pages 15, 17, and 22. In this reference, our reduced free product is called the generalized free product.

5. The fundamental group of (X,f).

We shall need the following definitions.

<u>Definition</u>. Two maps $f_1, f_2: X \longrightarrow Y$ are said to be <u>homotopic</u>, $f_1 \simeq f_2$, if there is a map $H: X \times I \longrightarrow Y$ such that $H(x,0) = f_1(x)$, $H(x,1) = f_2(x)$. Then H is a <u>homotopy</u> from f_1 to f_2, or H <u>connects</u> f_1, f_2.

<u>Definition</u>. If $X_o \subseteq X$, $Y_o \subseteq Y$, and $f(X_o) \subseteq Y_o$, we write $f: X, X_o \longrightarrow Y, Y_o$. Then $f_1 \simeq f_2: X, X_o \longrightarrow Y, Y_o$ if there is a homotopy H connecting f_1, f_2 such that $H: X \times I, X_o \times I \longrightarrow Y, Y_o$.

<u>Definition</u>. If $f_1, f_2: X \longrightarrow Y$ agree on $A \subseteq X$, we say f_1 is <u>homotopic to</u> f_2 <u>rel A</u> (written $f_1 \simeq f_2$ rel A) if there is a homotopy H from f_1 to f_2 such that $H(a,t) = f_1(a)$ for all $a \in A$ and all $t \in I$.

The fundamental group of a path-connected space may be defined in the following manner: Let X be a topological space. A <u>path</u> in X is a map $w: I \longrightarrow X$. It is a path <u>from</u> $w(0)$ <u>to</u> $w(1)$. Two paths w, v in X are said to be <u>equivalent</u> if $w \simeq v: I \longrightarrow X$, rel $\dot{I}$. The equivalence class of w will be denoted by $[w]$. The <u>product</u> of two paths w, v, such that $w(1) = v(0)$, is the path $w*v: I \longrightarrow X$ defined by

$$\begin{aligned}(w*v)(t) &= w(2t), \ 0 \leq t \leq \tfrac{1}{2} \\ &= v(2t-1), \ \tfrac{1}{2} \leq t \leq 1 .\end{aligned}$$

This induces an associative multiplication in the set of equivalence classes of paths, by the rule $[w][v] = [w*v]$, when $w*v$ is defined. The <u>reverse</u> of a path w, $\overline{w}$, is the path defined by $\overline{w}(t) = w(1-t)$.

For any $x_o \in X$, a path w is a <u>loop</u> on x_o if $w(0) = w(1) = x_o$. Then the set of equivalence classes of loops on x_o (the base point) forms a group, denoted by $\pi_1(X,x_o)$, with $[w]^{-1} = [\bar{w}]$, and the identity is the class of the constant loop on x_o. Furthermore, if X is path-connected, these groups are isomorphic for all choices of the base point, and if w is a path from x_o to x_1, then $[w]$ induces an isomorphism $[w]_*: \pi_1(X,x_1) \longrightarrow \pi_1(X,x_o)$ by the rule $[w]_*([v]) = [w*v*\bar{w}]$; if u is a path from x_1 to x_2, then $[w]_*[u]_* = [w*u]$. If $f:X \longrightarrow Y$, then for any base point $x_o \in X$, f induces a map $f_*: \pi_1(X,x_o) \longrightarrow \pi_1(Y,f(x_o))$ given by $f_*([w]) = [fw]$.

If X is path-connected, we take the <u>fundamental group</u> of X, $\pi_1(X)$, to be a group to which all the $\pi_1(X,x_o)$ are isomorphic.

<u>Definition</u>. Two spaces, X, Y, are said to be <u>of the same homotopy type</u> or <u>homotopically equivalent</u> (written $X \simeq Y$) if there are maps $f:X \longrightarrow Y$, $g:Y \longrightarrow X$ such that $gf \simeq I_X$ (the identity map on X), and $fg \simeq I_Y$.

<u>Definition</u>. A property of topological spaces is said to be an <u>invariant of homotopy type</u> if whenever X has this property and $X \simeq Y$, then Y also has the property.

<u>Definition</u>. A subspace $X_o \subseteq X$ is a <u>deformation retract</u> of X if there is a homotopy $H:X \times I \longrightarrow X$, rel X_o, such that $H(x,0) = x$ and $H(x,1) \in X_o$, for all $x \in X$.

We shall use the following results concerning homotopy equivalence and the fundamental group.

Proposition 5.1. The fundamental group of a path-connected space is an invariant of homotopy type (Hilton and Wylie (8), page 235, Theorem 6.2.7).

Proposition 5.2. If X_0 is a deformation retract of X, then $X_0 \simeq X$. In particular, if X is path-connected, $\pi_1(X) \cong \pi_1(X_0)$ (Eilenberg and Steenrod (3), page 30, Lemma 11.7).

Proposition 5.3. If X,Y are path-connected spaces, then $\pi_1(X\times Y) \cong \pi_1(X)\times \pi_1(Y)$, the direct product of $\pi_1(X)$ and $\pi_1(Y)$ (Hilton and Wylie, op.cit., page 242, Theorem 6.4.1).

Corollary 5.4. The fundamental group of the torus is free abelian on two generators.

Corollary 5.5. For any path-connected space X, $\pi_1(X\times I) \cong \pi_1(X)$, $\pi_1(X\times(0,1)) \cong \pi_1(X)$.

The principal tool we shall use in the computation of the fundamental group of a torus-like space, (X,f) is the following proposition, known as van Kampen's theorem. The formulation we use is due to P. Olum (10), page 666, Corollary to Theorem II.

Proposition 5.6. Let X be a path-connected space, A, B path-connected subsets such that $A\cap B$ is path-connected and every point of x lies in the interior of A or of B. Let $x_0 \in A\cap B$. Then $\pi_1(X,x_0)$ is the reduced free product of the system free product of the system

$$\begin{tikzcd}[column sep=large]
 & \pi_1(A,x_o) \\
\pi_1(A\cap B,x_o) \arrow[ur, "j_1"] \arrow[dr, "j_2"'] & \\
 & \pi_1(B,x_o)
\end{tikzcd} ,$$

where j_1, j_2 are the maps induced by the injection maps of $A\cap B$ into A and B, respectively.

Now, let X be a path-connected space, $f:X\longrightarrow X$ a homeomorphism, and let $p:X\times I\longrightarrow(X,f)$ be the canonical map. Choose a point $x_o\in X$, and take $x_* = p(x_o, 1/2)$. This will be our base point in (X,f). Let $A' = X\times(0,1)$, $A = p(A')$. Let w be a path in X from $f(x_o)$ to x_o. Let v'_o be the path in $X\times I$ given by

$$\begin{aligned} v'_o(t) &= (f(x_o),\ 2t/3), & 0\le t\le 1/2 \\ &= (w(2t-1),\ t/3+1/6), & 1/2\le t\le 1, \end{aligned}$$

and v'_1 the path in $X\times I$ given by

$$v'_1(t) = (x_o,\ (t+1)/2) .$$

Then $v_o = pv'_o$ and $v_1 = p_1v'_1$ are paths in (X,f). Let $B' = v'_o(I)\cup v'_1(I)\cup X\times[0,\ 1/3]\cup X\times[2/3,1]$, and let $B = p(B') = v_o(I)\cup v_1(I)\cup p(X\times[0,\ 1/3]\cup X\times[2/3,1])$. Then $A\cap B = v_o((0,1])\cup v_1([0,1))\cup p(X\times(0,\ 1/3]\cup X\times[2/3,1))$, and A, B, and (X,f) satisfy the hypotheses of Proposition 5.6. We shall also find use for the paths u'_o, u'_1 in $X\times I$ defined by

$$u'_o(t) = (w(t),\ (t+2)/6) ,$$

$$u'_1(t) = (x_o,\ (t+3)/6) ,$$

and for $u_o = pu'_o$ and $u_1 = pu'_1$. Figure 1 shows the various paths in $X\times I$.

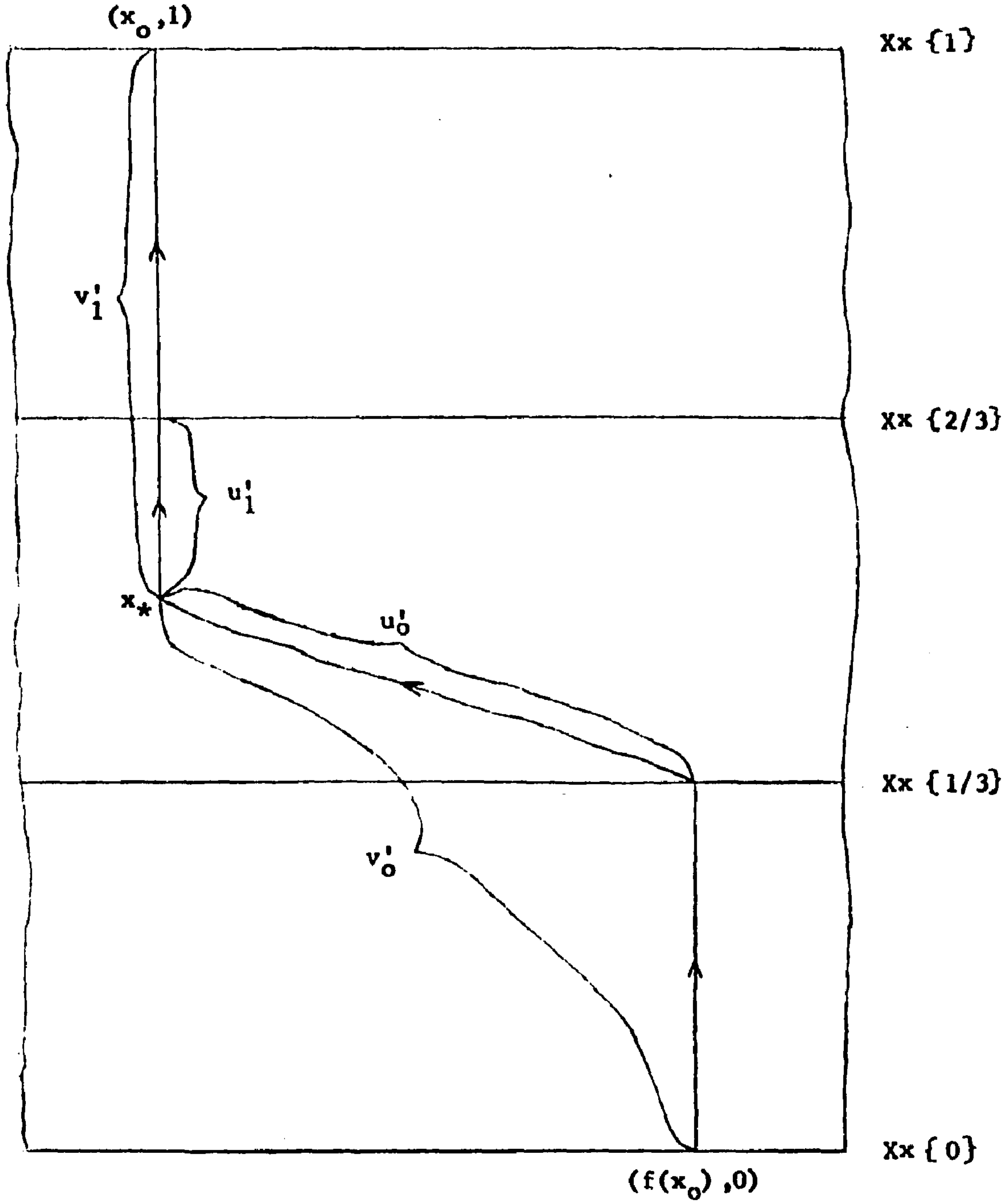

Figure 1. Paths in X×I.

Let $\pi_1(X,x_o) = (\{a_\alpha\};\ \{r_\beta\})$, where the a_α are equivalence classes of loops on x_o. Then

$$\pi_1(X,f(x_o)) = (\{[w]a_\alpha[\bar{w}]\};\ \{[w]_*(r_\beta)\}),$$

where $[w]_*: \pi_1(X,x_o) \longrightarrow \pi_1(X,f(x_o))$ is the isomorphism induced by $[w]$.

If $a_\alpha = [t_\alpha]$, there is for every $b \in I$ a loop $t_\alpha(b)$ in $X \times I$ on the point (x_o,b) defined by $t_\alpha(b)(u) = (t_\alpha(u),b)$, $u \in I$. We let $a'(b) = [t_\alpha(b)]$, $a_\alpha(b) = [pt_\alpha(b)]$. Similarly, we define $w'(b)$, $w(b)$, $r'_\beta(b)$, $r_\beta(b)$.

<u>Lemma</u> 5.7. $p(X\times\{\frac{1}{2}\})$ is a deformation retract of A.

Proof: On A', p is 1-1, thus a homeomorphism onto A. Let $q:A \longrightarrow A'$ be the inverse of this homeomorphism. Let $q':A\times I \longrightarrow A'\times I$ be the map defined by $q'(x,b) = (q(x),b)$ for $x \in A$. The homotopy $H' = A'\times I \longrightarrow A'$ defined by $H'((x,b),c) = (x,(1-c)b+\frac{1}{2}c)$ satisfies $H'((x,b),0) = (x,b)$, and $H'((x,b),1) \in X\times\{\frac{1}{2}\}$ for all $(x,b) \in A'$; moreover $H'((x,\frac{1}{2}),c) = (x,\frac{1}{2})$ for all $x \in X$. Hence $X\times\{\frac{1}{2}\}$ is a deformation retract of A', and H' is the retracting homotopy. We have the sequence of maps

$$A\times I \xrightarrow{q'} A'\times I \xrightarrow{H'} A' \xrightarrow{p} A,$$

and $H = pH'q':A\times I \longrightarrow A$ is the retracting homotopy we needed.

Q.E.D.

<u>Corollary</u> 5.8. $\pi_1(A,x_*) = (\{a_\alpha(\frac{1}{2})\};\ \{r_\beta(\frac{1}{2})\})$.

Proof: $p(X\times\{\frac{1}{2}\})$ is homeomorphic to X, and the isomorphism of the fundamental groups induced by the obvious homeomorphism takes a_α to

$a_\alpha(\frac{1}{2})$. Since $p(X \times \{\frac{1}{2}\})$ is a deformation retract of A and contains the base point x_*, we may take $\pi_1(A,x_*) = \pi_1(p(X\times\{\frac{1}{2}\}),x_*)$.

Q.E.D.

<u>Lemma</u> 5.9. $\pi_1(B,x_*) \cong \pi_1(X,x_o) * C$, where C is an infinite cyclic group.

Proof: Let $B_1 = v_o(I) \cup v_1(I)$, and let

$$B_2 = u_o(I) \cup u_1[\tfrac{1}{2},1] \cup p(X\times[0,1/3] \cup X\times[2/3,1]) .$$

Then B_1, B_2, and B satisfy the hypotheses of Proposition 5.6, with x_* as base point. Moreover, B_1 is homeomorphic to the circle, whose fundamental group is infinite cyclic. As the generator of $\pi_1(B_1,x_*)$ we take $[\bar{u}_o][u_o][v_1][v_o] = d$.

Since $B_1 \cap B_2 \cong I$, $\pi_1(B_1 \quad B_2,x_*) = \{1\}$, the trivial group.

All that remains is to compute $\pi_1(B_2,x_*)$. To do this, we show that $u_o(I) \cup p(X\times\{1/3\})$ is a deformation retract of B_2. We must define the proper homotopy $K: B_2\times I \longrightarrow B_2$, and we do this as follows:

For $0 \le t \le \frac{1}{2}$: $K(p(b,r),t) = p(b,r)$, if $0 \le r \le \frac{1}{2}$,

$K(p(b,r),t) = p(b,(1-2t)r+2t))$, if $\frac{1}{2} < r \le 1$,

For $\frac{1}{2} \le t \le 1$: $K(p(b,r),t) = p(b,(2-2t)r+(2t-1)/3)$, if $0 \le r \le 1/3$,

$K(p(b,r),t) = p(b,r)$, if $1/3 \le r \le \frac{1}{2}$

$K(p(b,r),t) = K(p(f(b),0),t)$, if $\frac{1}{2} < r \le 1$.

Then K is continuous since $p(b,1) = p(f(b),0)$; K is the identity on $u_o(I) \cup p(X\times\{1/3\})$ for all $t \in I$; and $K(B_2\times\{1\}) \subseteq u_o(I) \cup p(X\times\{1/3\})$. Thus $\pi_1(B_2,x_*) \cong \pi_1(u_o(I) \cup p(X\times\{1/3\}),x_*)$. Hence $\pi_1(B_2,x_*) \cong \pi_1(X,x_o)$, and as generators of this group we take all

$$[\bar{u}_o][w(1/3)]\,a_\alpha(1/3)\,[\bar{w}(1/3)][u_o] \, ,$$

with the corresponding relators. The lemma now follows from Proposition 5.6.

Q.E.D.

<u>Lemma</u> 5.10. $\pi_1(A\cap B, x_*) \cong \pi_1(X, x_o) * \pi_1(X, x_o)$.

Proof: Since $A \cap B \subseteq A$, p restricted to $p^{-1}(A\cap B)$ is a homeomorphism onto $A\cap B$. It is easy to see that $u_o(I) \cup u_1(I) \cup p(X\times\{1/3\}) \cup p(X\times\{2/3\})$ is a deformation retract of A B, and an application of Proposition 5.6 gives the result.

For later use, we note that the generators of the first copy of $\pi_1(X, x_o)$ can be taken to be all $[u_1]\, a_i(2/3)[\bar{u}_1]$ and those of the second copy of $\pi_1(X, x_o)$ can be taken to be all $[\bar{u}_o]\,[w(1/3)]\, a_i(1/3)[\bar{w}(1/3)]\,[u_o]$.

Q.E.D.

Now, let $i: A\cap B \to A$, $j: A\cap B \to B$ be the injections. We must determine the maps $i_*: \pi_1(A\cap B, x_*) \to \pi_1(A, x_*)$ and

$$j_*: \pi_1(A\cap B, x_*) \to \pi_1(B, x_*) .$$

It suffices to determine the effect of these maps on the generators of $\pi_1(A\cap B, x_*)$.

<u>Lemma</u> 5.11. $i_*([u_1]a_i(2/3)[\bar{u}_1]) = a_i(\tfrac{1}{2})$,

$$i_*([\bar{u}_o]\,[w(1/3)]\, a_i(1/3)[\bar{w}(1/3)]\,[u_o]) = a_i(\tfrac{1}{2}).$$

Proof: For a path, t, in A, define the path $H_1(t)$ as follows: t is a map $I \to A$, and we define a map $f_t: I^2 \to A\times I$ by $f_t(x,y) = (t(x), y)$. Let H be the homotopy defined in the proof of

Lemma 5.7. Then $Hf_t : I^2 \longrightarrow A$, and $H_1(t)$ is the path in A, $H_1(t) : I \longrightarrow A$, defined by $H_1(t)(x) = Hf_t(x,1)$. Then $H_1(t)(I) \subseteq p(X \times \frac{1}{2})$, and if t is a loop on x_*, so is $H_1(t)$, and $[H_1(t)] = [t]$. Now, $H_1(u_1) = e$, the constant loop on x_*, $H_1(u_o) = w(\frac{1}{2})$, $H_1(w(1/3)) = w(\frac{1}{2})$, and if $[t] = a_\alpha(2/3)$, $[H_1(t)] = a_\alpha(\frac{1}{2})$. The result follows immediately.

Q.E.D.

<u>Lemma</u> 5.12. Let $b_\alpha = [\overline{u}_o][w(1/3)]a_\alpha(1/3)[\overline{w}(1/3)][u_o]$. Then $j_*(b_\alpha) = b_\alpha$.

Proof: We may select a representative t_α of b_α such that $t_\alpha(I) \subseteq u_o(I) \cup p(X \times \{1/3\})$. Then $i_*(b_\alpha) = [t_\alpha] \in \pi_1(B, x_*)$. Since $t_\alpha(I) \subseteq B_2$, as defined in the proof of Lemma 5.9, we are done.

Q.E.D.

Let $[t_\alpha] = a_\alpha$. Then we write fa_α for $[ft_\alpha]$, where $f : X \longrightarrow X$ is the homeomorphism used in defining (X,f). We can express fa_α as some product of the $[w]a_\alpha[\overline{w}]$, since these elements generate $\pi_1(X, f(x_o))$.

<u>Lemma</u> 5.13. $j_*([u_1]a_\alpha(2/3)[\overline{u}_1]) = d[\overline{u}_o]fa_\alpha(1/3)[u_o]d^{-1}$, where d is as defined in the proof of Lemma 5.9.

Proof: Let $z_1 : I \longrightarrow B$ be defined by $z_1(t) = (x_o, (1-t)2/3+t)$, and let $z_2 : I \longrightarrow B$ be defined by $z_2(t) = (f(x_o), t/3)$. We then have $[u_1] = [v_1][\overline{z}_1]$, and $[\overline{v}_o] = [\overline{u}_o][\overline{z}_2]$. Now, in B,

$$[u_1] = [v_1][\overline{z}_1] = [\overline{u}_o][u_o][v_1][v_o][\overline{v}_o][\overline{z}_1] = d[\overline{v}_o][\overline{z}_1] = d[\overline{u}_o][\overline{z}_2][\overline{z}_1].$$

Therefore, $j_*([u_1]a_\alpha(2/3)[\overline{u}_1]) = d[\overline{u}_o][\overline{z}_2][\overline{z}_1]a_\alpha(2/3)[z_1][z_2][u_o]d^{-1}$.

Now, $[\bar{u}_o][\bar{z}_2][\bar{z}_1]\, a_\alpha(2/3)\, [z_1][z_2][u_o]$ has a representative, t_α, such that $t_\alpha(I) \subseteq B_2$, as defined in the proof of Lemma 5.9. Using the homotopy K defined in that lemma, for any path t such that $t(I) \subseteq B_2$, we define a path $K_1(t)$ in a manner similar to the definition of the path $H_1(t)$ in Lemma 5.11. From the definition of K, it follows that $[K_1(u_o)] = [u_o]$, $[K_1(z_1)] = [K_1(z_2)] = [e]$, where e is the constant loop on $p(f(x_o), 1/3)$, and if $[t_\alpha] = a_\alpha(2/3)$, then $[K_1(t_\alpha)] = fa_\alpha(1/3)$. The result now follows as in Lemma 5.11.

Q.E.D.

Before stating the main theorem of the section, we review our notation. We are considering a torus-like space (X,f), with X path-connected. We have $\pi_1(X,x_o) = (\{a_\alpha\}; \{r_\beta\})$, where each a_α is an equivalence class of loops on x_o. If $a_i = [t_i]$, and w is a path from $f(x_o)$ to x_o, we write fa_i for $[ft_i]$, and we can express this in terms of the a's as

$$fa_i = [w]\left(\prod_{k=1}^{n(i)} a_{\alpha(k,i)}\right)[\bar{w}],$$ where each $a_{\alpha(k,i)}$ is one of the given generators of $\pi_1(X,x_o)$

<u>Theorem</u> 5.14.

$\pi_1((X,f)) = (\{d\} \cup \{a_\alpha\};\ \{r_\beta\} \cup \{d(\prod_{k=1}^{n(i)} a_{\alpha(k,i)})d^{-1}a_i^{-1}\})$, where d is the class of loops defined in the proof of Lemma 5.9, and the indexing set for i is the same as the indexing set for α.

Proof: All that is necessary is to combine Proposition 5.6, Lemmas 5.7 - 5.13, and Proposition 4.3, together with the fact that in a group presentation, a relator of the form ab^{-1}, where a, b are generators, which corresponds to the relation $a = b$, allows us to drop the generator b and the relator ab^{-1} and replace b with a wherever b appears in a relator.

Q.E.D.

We shall apply the preceding result to compute the fundamental group of (T,f) when T is the torus and f is the homeomorphism induced by the unimodular matrix with integer entries $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By Corollary 5.4, the fundamental group of the torus is free abelian on two generators. Let $q:R^2 \longrightarrow T$ be the canonical map. Since $f(q(0,0)) = q(0,0)$, we use this point as our base point on the torus, to simplify calculations. Let $q(0,0) = p_o$. Let $x':I \longrightarrow R^2$ be defined by $x'(t) = (t,0)$, and let $y':I \longrightarrow R^2$ be defined by $y'(t) = (0,t)$. Then $x = qx'$ and $y = qy'$ are loops in T on p_o, and $X = [x]$ and $Y = [y]$ generate $\pi_1(T,p_o) = (\{X,Y\};\{XYX^{-1}Y^{-1}\})$. Now, $fx = fpx' = pFx'$, where $F:R^2 \longrightarrow R^2$ is the map induced by A. But $Fx' \simeq g_1:I \longrightarrow R^2$, rel $\dot{I}$, where g_1 is defined by

$$\begin{aligned} g_1(t) &= (2at,0), \text{ if } 0 \leq t \leq \tfrac{1}{2}, \\ &= (a,c(2t-1)), \text{ if } \tfrac{1}{2} \leq t \leq 1. \end{aligned}$$

Therefore $fx = pFx' \simeq pg_1:I \longrightarrow T$, rel $\dot{I}$, and so $[fx] = [pg_1]$, and it is easy to see that $[pg_1] = X^aY^c$. Similarly, $[fy] = X^bY^d$. We then have

<u>Theorem</u> 5.15. Let T be the torus, $f: T \longrightarrow T$ the homeomorphism induced by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, a unimodular matrix with integer entries. Then

$$\pi_1((T,f)) = (\{X, Y, Z\}; \{XYX^{-1}Y^{-1}, ZX^aY^cZ^{-1}X^{-1}, ZX^bY^dZ^{-1}Y^{-1}\}).$$

Proof: Apply Theorem 5.14.

Q.E.D.

Chapter III. Higher homotopy groups of torus-like spaces.

6. Absolute homotopy groups.

We shall need the following results from homotopy theory; for further details see Hilton (7) or Hu(8).

We consider Hilbert space H^∞ as the metric space consisting of all sequences of real numbers $u = (u_1, u_2, \ldots, u_n, \ldots)$ such that $\sum_{n=1}^{\infty} u_n^2$ converges, with the metric given by $d(u,v) = (\sum_{n=1}^{\infty} (u_n - v_n)^2)^{\frac{1}{2}}$.

The n-cube, I^n, is that subset of H^∞ given by

$$I^n = \{t \in H^\infty \mid 0 \leq t_i \leq 1,\ 1 \leq i \leq n,\ t_i = 0,\ i > n\}.$$

Thus the coordinates of points in I^n are $(t_1, \ldots, t_n, 0, \ldots)$; unless confusion might result, the string of zeros will generally be dropped. The boundary of I^n, written $\dot{I}^n$, consists of the points of I^n for which at least one t_i, $i \leq n$, is either 0 or 1.

Let Y be a path-connected space and let $y_o \in Y$. Suppose $f, g: I^n \longrightarrow Y$ such that $f(1, t_2, \ldots, t_n) = g(0, t_2, \ldots, t_n)$. Then we define $h: I^n \longrightarrow Y$ by

$$\begin{aligned} h(t_1, \ldots, t_n) &= f(2t_1, t_2, \ldots, t_n), \quad 0 \leq t_1 \leq \tfrac{1}{2}, \\ &= g(2t-1, t_2, \ldots, t_n), \quad \tfrac{1}{2} \leq t_1 \leq 1. \end{aligned}$$

Then h is continuous. We shall write $h = f*g$. Now, let $M_n(Y, y_o)$ be the set of all maps $I^n, \dot{I}^n \longrightarrow Y, y_o$. Then, if $f, g \in M_n(Y, y_o)$, $f*g$ is defined and $f*g \in M_n(Y, y_o)$, so that we have a multiplication defined in

$M_n(Y,y_o)$. Let $\pi_n(Y,y_o)$ be the set of all homotopy classes of elements of $M_n(Y,y_o)$ and let $[f]$ be the class of f.

Proposition 6.1. If $f,g \in M_n(Y,y_o)$, then $[f*g]$ depends only on $[f]$ and $[g]$ (Hilton (7), page 5).

The multiplication defined in $M_n(Y,y_o)$ thus induces a multiplication in $\pi_n(Y,y_o)$.

We define the subcube I_1^n of I^n to be the set of points $(t_1,\ldots,t_n)$ such that $a_i \leq t_i \leq b_i$, where $0 \leq a_i < b_i \leq 1$ for $i = 1,\ldots,n$. Then given $f \in M_n(Y,y_o)$, there exists $f' \in M_n(Y,y_o)$ such that $f' \in [f]$ and $f'(I^n - I_1^n) = y_o$. We shall say f' results from concentrating f on I_1^n. Now, let I_2^n be another subcube of I^n given by $a_i' \leq t_i \leq b_i'$, $i = 1,\ldots,n$, such that $b_1 \leq a_1'$ and the interiors of I_1^n and I_2^n are disjoint. We say I_1^n lies to the left of I_2^n. Then, if $f,g \in M_n(Y,y_o)$ and f', g' arise by concentrating f, g on I_1^n and I_2^n respectively, we can define a map $k \in M_n(Y,y_o)$ by $k \mid I_1^n = f'$, $k \mid I_2^n = g'$, $k(x) = y_o$ if $x \in I^n - (I_1^n \quad I_2^n)$. The proof of the following proposition is then obvious.

Proposition 6.2. $k \in [f*g]$.

Using this, one can obtain

Proposition 6.3. $[f*g] = [g*f]$ if $n > 1$ (Hilton, op.cit., page 7, Lemma 1.3).

Proposition 6.4. Under the operation of multiplication, the collection of homotopy classes, $\pi_n(Y,y_o)$ is a group. It is abelian if $n > 1$. (Hilton, op.cit., page 7, Theorem 1.4).

The group $\pi_n(Y,y_o)$ is called the n^{th} (<u>absolute</u>) <u>homotopy</u> <u>group</u> of Y,y_o. If $n = 1$, it is the fundamental group. We may also define a set $\pi_o(Y,y_o)$ by taking $\pi_o(Y,y_o)$ to be the set of path-components of Y. This set has no group structure, but it has a distinguished element, the path-component containing y_o. If Y is path-connected, $\pi_o(Y,y_o)$ has only one element, and we write $\pi_o(Y,y_o) = [0]$. Given a map $f:X,x_o \longrightarrow Y,y_o$, there is induced a homomorphism $f_*: \pi_n(X,x_o) \longrightarrow \pi_n(Y,y_o)$ given by $f_*[g] = [fg]$.

We remove the dependence on the base point, y_o, for path-connected spaces by

<u>Proposition</u> 6.5. For each $n > 0$, every path $w:I \longrightarrow X$ gives in a natural way an isomorphism $w_n: \pi_n(X,x_1) \longrightarrow \pi_n(X,x_o)$, where $x_o = w(0)$, $x_1 = w(1)$, which depends only on the homotopy class of the path w (relative to end points). If w is the degenerate path $w(I) = x_o$, then w_n is the identity automorphism. If w, v are paths with $v(0) = w(1)$, then $(wv)_n = w_n v_n$. Finally, for each path $w:I \longrightarrow X$ and each map $f:X \longrightarrow Y$, we have a commutative rectangle

$$\begin{tikzcd}
\pi_n(X,x_1) \arrow[r, "w_n"] \arrow[d, "f_*"] & \pi_n(X,x_o) \arrow[d, "f_*"] \\
\pi_n(Y,y_1) \arrow[r, "t_n"] & \pi_n(Y,y_o)
\end{tikzcd}$$

where $t = fw$, $y_o = f(x_o)$, $y_1 = f(x_1)$ (Hu (9), page 126, Theorem 14.1).

<u>Proposition</u> 6.6. The homotopy groups are invariants of homotopy type (Hilton, <u>op</u>.<u>cit</u>., page 15, Theorem 3.5).

7. Relative homotopy groups.

Let Y_o be a path-connected subspace of Y and let $y_o \in Y_o$. I^{n-1} is the face of I^n given by $t_n = 0$; let J^{n-1} be the union of the remaining (n-1)-faces of I^n, and let $M_n(Y,Y_o,y_o)$ be the set of all maps $f:I^n,I^{n-1},J^{n-1} \longrightarrow Y,Y_o,y_o$. Let $\pi_n(Y,Y_o,y_o)$ be the set of all homotopy classes of elements of $M_n(Y,Y_o,y_o)$, where the homotopies H are such that $H:I^n \times I, I^{n-1}\times I, J^{n-1}\times I \longrightarrow Y,Y_o,y_o$. If $n > 1$, we can define multiplication in $M_n(Y,Y_o,y_o)$ and this multiplication induces a multiplication in $\pi_n(Y,Y_o,y_o)$. With this multiplication $\pi_n(Y,Y_o,y_o)$ is a group and is abelian if $n \geq 3$. The procedure mimics that of the preceding section, except that we only consider subcubes I_1^n whose points $(t_1,\dots,t_n)$ satisfy $a_i \leq t_i \leq b_i$, $i = 1,\dots,n-1$, and $0 \leq t_n \leq b_n$. We then have

<u>Proposition</u> 7.1. Given $f \in M_n(Y,Y_o,y_o)$, there exists $f' \in [f]$ such that $f'(I^n - I_1^n) = y_o$ (Hilton (7), page 16, Lemma 4.1).

<u>Proposition</u> 7.2. Given $f,g \in M_n(Y,Y_o,y_o)$, $n \geq 2$. Suppose $f' \in [f]$, $g' \in [g]$ are concentrated on I_1^n, I_2^n respectively, where I_1^n lies to the left of I_2^n. We define $k \in M_n(Y,Y_o,y_o)$ in the same manner as for Proposition 6.2. Then $k \in [f*g]$ (Hilton, <u>op</u>.<u>cit</u>., page 16, Lemma 4.2).

<u>Proposition</u> 7.3. $[f*g] = [g*f]$ if $n > 2$ (Hilton, <u>op</u>.<u>cit</u>., page 16, Lemma 4.3).

<u>Proposition</u> 7.4. Under the operation of multiplication, $\pi_n(Y,Y_o,y_o)$ is a

group if $n \geqslant 2$, and is abelian if $n > 2$ (Hilton, op.cit., Theorem 4.4, page 17).

Although we cannot give $\pi_1(Y,Y_o,y_o)$ a group structure, it will sometimes be useful to consider the set $\pi_1(Y,Y_o,y_o)$. As in the case of the absolute homotopy groups, the relative homotopy groups are independent of base point (keeping the base point in Y_o) and the relative homotopy groups are invariants of (relative) homotopy type.

Finally, observe that $\pi_n(Y,y_o,y_o) = \pi_n(Y,y_o)$.

8. The exact homotopy sequence.

The injections $Y_o, y_o \longrightarrow Y, y_o$ and $Y, y_o, y_o \longrightarrow Y, Y_o, y_o$ induce maps $i: \pi_n(Y_o, y_o) \longrightarrow \pi_n(Y, y_o)$ and $j: \pi_n(Y, y_o) \longrightarrow \pi_n(Y, Y_o, y_o)$ which are homomorphisms for $n \geqslant 2$ (see Hilton (7), page 34).

We now define the boundary operator $d: \pi_n(Y, Y_o, y_o) \longrightarrow \pi_{n-1}(Y_o, y_o)$, $n \geqslant 1$. Let $f: I^n, I^{n-1}, J^{n-1} \longrightarrow Y, Y_o, y_o$ be a representative of a class in $\pi_n(Y, Y_o, y_o)$. If $n = 1$, $f(I^o)$ is a point of Y_o; we take $d([f])$ to be the path-component of $f(I^o)$, where $I^o = (0, 0, \ldots)$. If $n > 1$, $I^{n-1} \cap J^{n-1} = \dot{I}^{n-1}$, and so $f \mid I^{n-1}: I^{n-1}, \dot{I}^{n-1} \longrightarrow Y_o, y_o$. We take $d([f]) = [f \mid I^{n-1}] \in \pi_{n-1}(Y_o, y_o)$. This gives a homomorphism, if $n \geqslant 2$. Thus we have the sequence of homomorphisms:

$$\cdots \longrightarrow \pi_{n+1}(Y, Y_o) \xrightarrow{d} \pi_n(Y_o) \xrightarrow{i} \pi_n(Y) \xrightarrow{j} \pi_n(Y, Y_o) \xrightarrow{d} \cdots$$

which terminates in

$$\cdots \to \pi_2(Y, Y_o) \xrightarrow{d} \pi_1(Y_o) \xrightarrow{i} \pi_1(Y) \longrightarrow \pi_1(Y, Y_o) \longrightarrow 0$$

where the last two arrows represent functions, since $\pi_1(Y, Y_o)$ has no group structure, in general (we drop the base point for convenience). The sequence is called the <u>homotopy sequence</u> of the pair (Y, Y_o).

<u>Proposition</u> 8.1. The homotopy sequence of the pair (Y, Y_o) is exact.

A proof may be found in Hilton, <u>op.cit</u>., page 35, Theorem 2.1.

9. Fiber spaces.

We consider a given map $p:E \longrightarrow B$ of a space E, called the <u>total space</u>, into a space B, the <u>base space</u>.

<u>Definition</u>. Let X be a space, $f:X \longrightarrow B$ a given map. Then a map $f^*:X \longrightarrow E$ <u>covers</u> f if $f = pf^*$.

<u>Definition</u>. The map $p:E \longrightarrow B$ has the <u>covering homotopy property</u> for the space X if for every map $f^*:X \longrightarrow E$ and every homotopy $F:X \times I \longrightarrow B$ of the map $f = pf^*$, there is a homotopy $F^*:X \times I \longrightarrow E$ such that $F = pF^*$.

<u>Definition</u>. The map $p:E \longrightarrow B$ has the <u>polyhedral covering homotopy property</u> (abbreviated PCHP) if it has the covering homotopy property for every triangulable space X.

<u>Definition</u>. The map $p:E \longrightarrow B$ is a <u>fibering</u> if it has the PCHP. The space E is called a (Serre) <u>fiber space over the base space</u> B, with <u>projection</u> $p:E \longrightarrow B$. For each $b \in B$, $p^{-1}(b)$ is called the <u>fiber over</u> b.

<u>Definition</u>. A map $p:E \longrightarrow B$ has the <u>bundle property</u> if there is a space D such that for every point $b \in B$ there is an open neighborhood U of b and a homeomorphism $f_U:U \times D \rightarrow p^{-1}(U)$ such that $pf_U(t,d) = t$, $t \in U$, $d \in D$. The space E is called a <u>bundle space over the base space</u> B relative to the <u>projection</u> $p:E \longrightarrow B$. D is called the <u>director space</u>, the U's are the <u>decomposing neighborhoods</u>.

Proposition 9.1. Every bundle space E over B relative to $p:E \longrightarrow B$ is a fiber space over B relative to p (Hu (9), page 65, Theorem 4.1).

Proposition 9.2. If $f:X \longrightarrow Y$ is a fibering and $A = f^{-1}(B)$ and $f(x_o) = y_o \in B$ then the transformation $f_*: \pi_n(X,A,x_o) \longrightarrow \pi_n(Y,B,y_o)$ is one to one and onto for all $n > 0$ (Hu, op.cit., page 118, Property VI).

Now, let $p:E \rightarrow B$ be a fibering. Choose a point $b_o \in B$ such that $p^{-1}(b_o) = F$ is not empty. Choose a point e_o in F. Thus, we obtain a triple (E,F,e_o), whose homotopy sequence is

$$\longrightarrow \pi_{n+1}(E,F,e_o) \xrightarrow{d} \pi_n(F,e_o) \xrightarrow{i} \pi_n(E,e_o) \xrightarrow{i} \pi_n(E,F,e_o) \xrightarrow{d}$$

The projection $p:E \longrightarrow B$ defines a map $q:E,F,e_o \longrightarrow B,b_o,b_o$, and if $j:E,e_o,e_o \longrightarrow E,F,e_o$ is the injection, then $p = qj$. By Proposition 9.2, $q_*: \pi_n(E,F,e_o) \cong \pi_n(B,b_o,b_o) = \pi_n(B,b_o)$. Let $d_* = dq_*^{-1}$. Then we can construct the exact sequence:

$$\xrightarrow{p_*} \pi_{n+1}(B,b_o) \xrightarrow{d_*} \pi_n(F,e_o) \xrightarrow{i} \pi_n(E,e_o) \xrightarrow{p_*} \pi_n(B,b_o) \xrightarrow{d_*}$$

$$\cdots p_* \longrightarrow \pi_1(B,b_o) \xrightarrow{d_*} \pi_o(F,e_o) \xrightarrow{i_*} \pi_o(E,e_o),$$

which is the homotopy sequence of the fibering $p:E \longrightarrow B$ based at e_o.

10. Applications to torus-like spaces.

We shall compute the higher homotopy groups of the torus-like space (X,f) using fiber-space techniques. The following proposition will be useful.

<u>Proposition</u> 10.1. Let S^1 be the 1-sphere. Then $\pi_n(S^1) = 0$, $n > 1$.

Proof: This fact is well known; it is quickly proved by noticing that $h:R \longrightarrow S^1$ (R the real numbers) given by $h(x) = e^{ix}$, is a fibering, with discrete fibers F. Hence, for any $e_o \in F$, $\pi_n(F,e_o) = 0$, $n \geq 1$. Furthermore, $\pi_n(R,e_o) = 0$ for $n \geq 0$, since R is contractible, i.e., $\{e_o\}$ is a deformation retract of R. Now apply the exact sequence for the fibering to obtain

$$0 \longrightarrow \pi_n(S^1,b) \longrightarrow 0$$

for $n \geq 2$ and any base point $b \in S^1$.

Q.E.D.

<u>Lemma</u> 10.2. (X,f) is a fiber space over S^1, and X is homeomorphic to the fiber over any point of S^1.

Proof: Define $g:X \times I \longrightarrow I$ by $g(x,t) = t$. Let R be the equivalence relation on I generated by requiring $0 \equiv 1 \pmod R$ and let $h:I \longrightarrow I/R$ be the canonical map; then $I/R \approx S^1$.

We have the diagram

$$\begin{tikzcd} X\times I \arrow[r, "g"] \arrow[d, "p"'] & I \arrow[d, "h"] \\ (X,f) \arrow[r, "g'"] & I/R \cong S^1 \end{tikzcd}$$

where p is the canonical map. Since $p(x) = p(y)$ implies $hg(x) = hg(y)$, the map g' exists and is unique. This map is our candidate for the fibering. We may take g' to be given by $g'p(x,t) = e^{2\pi it}$. Then g' has the bundle property with director space X and decomposing neighborhoods homomorphic to $(0,1)$. This is obviously true for any point $g'(p(x,t))$ where $t \neq 0, 1$; and for $t = 0, 1$ it follows from an application of Lemma 2.3.

Q.E.D.

Let $x_o \in X$, and let $y_o \in (X,f)$ be given by $y_o = p(x_o,0)$. Then

<u>Theorem</u> 10.3. $\pi_n((X,f),y_o) \cong \pi_n(X,x_o)$, $n > 1$.

Proof: Let b_o S^1 be such that $b_o = g'p(x,0)$. Then $(g')^{-1}\{b_o\} = p(X\times\{0\})$ is the fiber over b_o. Now $p(X\times\{0\}) \cong X$, and, writing $Y = (X,f)$, the homotopy sequence of the fibering becomes

$$\cdots \longrightarrow \pi_{n+1}(S^1,b_o) \longrightarrow \pi_n(X,x_o) \longrightarrow \pi_n(Y,y_o) \longrightarrow \pi_n(S^1,b_o) \longrightarrow \cdots;$$

for $n > 1$, we have

$$0 \longrightarrow \pi_n(X,x_o) \longrightarrow \pi_n(Y,y_o) \longrightarrow 0 \ ,$$

which implies $\pi_n(X,x_o) \cong \pi_n(Y,y\ )$.

Q.E.D.

We now specialize to the case where X is the torus.

Lemma 10.4. Let T be the torus, and $x_0 \in T$ any point of T. Then $\pi_n(T, x_0) = 0$, $n > 1$.

Proof: We have $T = S^1 \times S^1$. Let $q: T \longrightarrow S^1$ be the natural projection (on, for example, the second coordinate space). Then $q: T \longrightarrow S^1$ is clearly a fibering with the fiber over b equal to S^1, for any b S^1. Then the homotopy sequence of the fibering reduces to

$$\longrightarrow 0 \longrightarrow \pi_n(T, x_0) \longrightarrow 0 \longrightarrow, \quad n \geq 2,$$

and so $\pi_n(T, x_0) = 0$, $n \geq 2$.

Q.E.D.

Corollary 10.5. If (X,f) is a torus-like space, where X is the torus, then, for any base point $y_0 \in (X,f)$, $\pi_n((X,f), y_0) = 0$, $n \geq 2$.

Proof: Combine Theorem 10.3 and Lemma 10.4.

Q.E.D.

Chapter IV. Homology groups of (T,f), with T the torus and f induced by a unimodular matrix.

11. Finite cell complexes.

We denote the singular chain complex of a space X by $S(X)$. $\widetilde{H}_n(X)$ is the n^{th} singular homology group of X; $H\ (X)$ is the n^{th} reduced singular homology group of X, which is obtained from the augmented singular complex. If $f:X \longrightarrow Y$, the induced map $H_p(X) \longrightarrow H_p(Y)$ will be denoted by $\underline{f}_p$. The singular complex is too large to be useful for the computation of the homology of a space X, and thus for a certain class of spaces called cell complexes we shall define a different chain complex and show that its homology is the same as that of $S(X)$. We first list a definition and several standard propositions of singular homology theory.

<u>Definition</u>. A space X is said to be <u>contractible</u> if there is a point $x_o \in X$ such that $\{x_o\}$ is a strong deformation retract of X.

<u>Proposition</u> 11.1. The reduced singular homology groups of a contractible space are null (i.e., the trivial group) (Hilton and Wylie (8), page 317, Theorem 8.1.14).

<u>Proposition</u> 11.2. If S^n is the n-sphere, then

$$\widetilde{H}_i(S^n) = 0, \quad i \neq n,$$

$$\widetilde{H}_n(S^n) = J, \quad \text{the integers}$$

(Eilenberg and Steenrod (3), page 46, Theorem 16.6).

Proposition 11.3. If U is a subset of X whose closure is contained in the interior of X_o, then the injection $i:X-U, X_o-U \rightarrow X, X_o$ induces an isomorphism $i_n: H_n(X-U, X_o-U) \cong H_n(X, X_o)$, for all n (Eilenberg and Steenrod, op.cit., page 199, Theorem 9.1).

Proposition 11.4. Let $X = X_1 \cup X_2 \cup \ldots \cup X_n$ be the union of disjoint sets each of which is closed (and therefore open) in X. Let $A_i \subseteq X_i$ and let $A = A_1 \cup A_2 \cup \ldots \cup A_n$. Let $i_j: X_j, A_j \rightarrow X, A$, $i = 1, \ldots, n$ be the inclusion maps. Then the homomorphisms $i_{j*}: H_q(X_j, A_j) \longrightarrow H_q(X, A)$ yield an injective representation of $H_q(X, A)$ as a direct sum (Eilenberg and Steenrod, op.cit., page 33, Theorem 3.2).

We are now ready to define cell complexes and compute their homology.

Definition. Let X be a T_1 topological space, and $A \subseteq X$. Then A is an n-cell if and only if A is homeomorphic to R^n, for some $n \geq 0$. (R^n is real n-dimensional Euclidean space.)

Definition. Let X be a T_1 topological space. X is said to be a cell complex if the following conditions are satisfied:

1. X is the disjoint union of finitely many cells, i.e.,

$X = \bigcup_{n=0}^{\infty} \bigcup_{i=1}^{a(n)} K_i^n$, where $K_1^n, \ldots, K_{a(n)}^n$ are the n-cells of X.

2. $\overline{K_i^q} - K_i^q \subseteq K^{q-1}$. (If $A \subseteq X$, $\overline{A}$ denotes the closure of A in X.)

3. For each K_i^q, there is a point $x_i^q \in K_i^q$ such that K^{q-1} is a deformation retract of $K^{q-1} \cup (K_i^q - x_i^q)$.

Definition. The dimension of the cell complex is the maximum n such that $a(n) \neq 0$. $K^p = \bigcup_{q \leq p} \bigcup_{i=1}^{a(q)} K_i^q$ is called the p-dimensional skeleton of X.

In order to compute the homology of a cell complex, we shall define a new chain complex, $S^c(X)$. We first must compute the homology of (K^p, K^{p-1}). Let

$L^p = K^{p-1} \bigcup_{i=1}^{a(n)} (K_i^p - x_i^p)$ for $p > 0$. The inclusion map $i: K^p, K^{p-1} \longrightarrow K^p, L^p$ induces a homomorphism $i_*: H_q(K^p, K^{p-1}) \longrightarrow H_q(K^p, L^p)$.

Proposition 11.5. i_* is an isomorphism for all q.

Proof: We need the following lemma.

Lemma 11.6. K^{q-1} is a deformation retract of L^q.

Proof: K^{q-1} is a deformation retract of $K^{q-1} \cup (K_i^q - x_i^q)$ for $i = 1, \ldots, a(q)$. Since $(K^{q-1} \cup (K_i^q - x_i^q)) \cap (K^{q-1} \cup (K_j^q - x_j^q)) = K^{q-1}$ for $i \neq j$, the homotopies defining K^{q-1} as a deformation retract of the $K^{q-1} \cup (K_i^q - x_i^q)$ agree on K^{q-1} and may be used to define the needed homotopy $H: L^q \times I \longrightarrow K^{q-1}$.

Q.E.D.

We have the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_i(K^{q-1}) \arrow[r] \arrow[d, "f_i"] & H_i(K^q) \arrow[r] \arrow[d, "g_i"] & H_i(K^q, K^{q-1}) \arrow[r] \arrow[d, "i_*"] & H_{i-1}(K^{q-1}) \arrow[r] \arrow[d, "f_{i-1}"] & H_{i-1}(K^q) \arrow[r] \arrow[d, "g_{i-1}"] & \cdots \\
\cdots \arrow[r] & H_i(L^q) \arrow[r] & H_i(K^q) \arrow[r] & H_i(K^q, L^q) \arrow[r] & H_{i-1}(L^q) \arrow[r] & H_{i-1}(K^q) \arrow[r] & \cdots
\end{tikzcd}$$

.

Both sequences are exact, g_i and g_{i-1} are isomorphisms since they are induced by the identity map, and f_i and f_{i-1} are isomorphisms by the preceding lemma. Hence, by the Five-lemma, i_* is an isomorphism.

Q.E.D.

Proposition 11.7. $H_q(K^p,K^{p-1}) = 0$, $q \neq p$

$H_p(K^p,K^{p-1}) \cong J \oplus J \oplus \ldots \oplus J$, $a(p)$ copies of the integers, J.

Proof: The case $p = 0$ is trivial, since K^{-1} is empty. Now let $p > 0$. Clearly, $K^{p-1} \subseteq L^p$; furthermore, K^{p-1} is closed in X, since $\overline{K^{p-1}} \subseteq K^{p-1}$, hence K^{p-1} is closed in K^p. We have $L^p = K^p - \{x_1^p,\ldots,x_{a(p)}^p\}$. Since X is a T_1 space, each $\{x_i^p\}$ is closed in X, so $\{x_1^p,\ldots,x_{a(p)}^p\}$ is closed in X; thus L^p is open in K^p, and the closure of K^{p-1} is contained in the interior of L^p. We may apply Proposition 11.3 and obtain

$$H_i(K^p,L^p) \cong H_i\left(\bigcup_{j=1}^{a(p)} K_j^p, \bigcup_{j=1}^{a(p)} (K_j^p - x_j^p)\right) .$$

However, $\overline{K_i^p} - K_i^p \subseteq K^{p-1}$, so $\overline{K_i^p} - K_i^p \not\subseteq \bigcup_{j=1}^{a(p)} K_j^p$. Therefore,

$\overline{K_i^p} \cap \bigcup_{j=1}^{a(p)} K_j^p = K_i^p$, and hence K_i^p is closed in $\bigcup_{j=1}^{a(p)} K_j^p$. By Proposition 11.4, $H_i(K^p,L^p) \cong \sum_{j=1}^{a(p)} H_i(K_j^p,K_j^p - x_j^p)$ (direct sum).

For $i = 1,\ldots,a(p)$, we have an exact sequence

$$\cdots \longrightarrow \tilde{H}_i(K_j^p) \longrightarrow H_i(K_j^p,K_j^p - x_j^p) \longrightarrow \tilde{H}_{i-1}(K_j^p - x_j^p) \longrightarrow \tilde{H}_{i-1}(K_j^p) \longrightarrow \cdots$$

and since K_j^p is homeomorphic to R^p, K_j^p is contractible, $\tilde{H}_i(K_j^p) = 0$ for all j, and $H_i(K^p, K_j^p - x_j^p) \cong H_{i-1}(K_j^p - x_j^p)$. Now, $K_j^p - x_j^p \approx S^{p-1}$, and by Proposition 11.2,

$$\begin{aligned} \tilde{H}_{i-1}(S^{p-1}) &= 0, \quad i-1 \neq p-1 \text{ or } i \neq p \\ &= J, \quad i-1 = p-1 \text{ or } i = p. \end{aligned}$$

The proposition now follows immediately.

Q.E.D.

We now can define $S_p^c(X)$. We take $S_p^c(X) = H_p(K^p, K^{p-1})$ and we define the boundary operator by use of the exact sequence. We have sequences

$$\cdots \longrightarrow H_p(K^p, K^{p-1}) \xrightarrow{d_p} H_{p-1}(K^{p-1}) \longrightarrow H_{p-1}(K^p) \longrightarrow \cdots \quad \text{and}$$

$$\cdots \longrightarrow H_{p-1}(K^{p-2}) \xrightarrow{i} H_{p-1}(K^{p-1}) \xrightarrow{r} H_{p-1}(K^{p-1}, K^{p-2}) \longrightarrow \cdots$$ which may be combined as

$$\begin{tikzcd} & & \arrow[d, "i"] & \\ {} \arrow[r] & H_p(K^p, K^{p-1}) \arrow[r, "d_p"] \arrow[dr, dashed, "d_p^c"'] & H_{p-1}(K^{p-1}) \arrow[r] \arrow[d, "r_{p-1}"] & {} \\ & & H_{p-1}(K^{p-1}, K^{p-2}) \arrow[d] & \\ & & {} & \end{tikzcd}$$

and we define $d_p^c : S_p^c(X) \longrightarrow S_{p-1}^c(X)$ by $d_p^c = r_{p-1} d_p$. That $d_{p-1}^c d_p^c = 0$

follows from the diagram

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & & \\
{} \arrow[r] & H_p(K^p,K^{p-1}) \arrow[r, "d_p"] \arrow[d, "d_p"] \arrow[dr, dashed, "d_p^c"] & H_{p-1}(K^{p-1}) \arrow[r] \arrow[d, "r_{p-1}"] & {} \arrow[d] & \\
{} \arrow[r] & H_{p-1}(K^{p-1}) \arrow[r, "r_{p-1}"] \arrow[d] & H_{p-1}(K^{p-1},K^{p-2}) \arrow[r, "d_{p-1}"] \arrow[d, "d_{p-1}"] \arrow[dr, dashed, "d_{p-1}^c"] & H_{p-2}(K^{p-2}) \arrow[r] \arrow[d, "r_{p-2}"] & {} \\
 & {} \arrow[r] & H_{p-2}(K^{p-2}) \arrow[r, "r_{p-2}"] \arrow[d] & H_{p-2}(K^{p-2},K^{p-3}) \arrow[r] \arrow[d] & {} \\
 & & {} & {} &
\end{tikzcd}$$

using the exactness of the middle row.

<u>Definition</u>. Given cell complexes X_1,X_2 and a map $f:X_1 \longrightarrow X_2$, f is said to be a <u>cellular</u> map if $f(K_1^q) \subseteq f(K_2^q)$ for all q, where K_1^q, K_2^q are the q-skeletons of X_1, X_2 respectively. Furthermore, a cellular map f induces a chain map f^c, $f_q^c = \underline{f}_q : S_q^c(X_1) = H_q(K_1^q,K_1^{q-1}) \rightarrow H_q(K_2^q,K_2^{q-1}) = S_q^c(X_2)$, since $f_q^c d_{q+1}^c = \underline{f}rd = rd\underline{f} = d_{q+1}^c f_{q+1}^c$.

<u>Proposition</u> 11.8. The inclusion $j_p:K^{p+1} \longrightarrow X$, $p \geq 0$, induces a homomorphism $\underline{j}_p:H_p(K^{p+1}) \longrightarrow H_p(X)$. Then $\underline{j}_p$ is an isomorphism for all p and if $f:X_1 \longrightarrow X_2$ is a cellular map, the diagram

$$\begin{tikzcd}
H_p(K_1^{p+1}) \arrow[r, "\underline{j}_p"] \arrow[d, "\underline{f}\,K_1^{p+1}"'] & H_p(X_1) \arrow[d, "\underline{f}"] \\
H_p(K_2^{p+1}) \arrow[r, "\underline{j}_p"] & H_p(X_2)
\end{tikzcd}$$

is commutative.

Proof: We proceed by induction on dim X.

<u>$\dim X = 0$ or 1</u>: Then $X = K^1 = K^2 = \ldots$, hence $H_0(K^1) = H_0(X)$, $H_1(K^2) = H_1(X)$, ..., $H_p(K^{p+1}) = H_p(X)$, etc.

<u>$\dim X = n \geq 1$</u>: For $p+1 < n$, we have the exact sequence

$$\cdots \longrightarrow H_{p+1}(K^n,K^{n-1}) \longrightarrow H_p(K^{n-1}) \xrightarrow{\underline{i}} H_p(K^n) \longrightarrow H_p(K^n,K^{n-1}) \longrightarrow \cdots$$

where $i:K^{n-1} \longrightarrow K^n$ is the injection. But since $p+1 < n$,

$H_{p+1}(K^n,K^{n-1}) = H_p(K^n,K^{n-1}) = 0$, by Proposition 11.7, so $\underline{i}$ is an isomorphism. But by the inductive hypothesis, $\underline{j}_p:H_p(K^{p+1}) \longrightarrow H_p(K^{n-1})$ is an isomorphism, taking $X = K^{n-1}$. So we have $H_p(K^{p+1}) \cong H_p(K^n) = H_p(X)$, since $X = K^n$, and the isomorphism is induced by the inclusion $j_p:K^{p+1} \longrightarrow X$. For $p+1 \geq n$, $X = K^{p+1}$, hence $H_p(K^{p+1}) = H_p(X)$.

Q.E.D.

<u>Proposition</u> 11.9. $H_p(X) = 0$ for $p > \dim X$.

Proof: If $\dim X = 0$, X is a finite set. The proposition then follows from Proposition 11.2 and Proposition 11.4. Suppose $\dim X = n > 1$. We have, for $i > n$, the exact sequence

$$\cdots \longrightarrow H_i(K^{n-1}) \longrightarrow H_i(K^n) \longrightarrow H_i(K^n,K^{n-1}) \longrightarrow \cdots \quad .$$

By the induction assumption $H_i(K^{n-1}) = 0$, and by Proposition 11.7, $H_i(K^n,K^{n-1}) = 0$.

Q.E.D.

Proposition 11.10. For every cell complex X there is an isomorphism $t_q : H_q(X) \longrightarrow H_q^c(X)$ for all q, such that if $f : X \longrightarrow Y$ is a cellular map, the diagram

$$\begin{tikzcd} H_q(X) \arrow[r, "t_q"] \arrow[d, "\underline{f}"] & H_q^c(X) \arrow[d, "f^c"] \\ H_q(Y) \arrow[r, "t_q"] & H_q^c(Y) \end{tikzcd}$$

is commutative.

Proof: By Proposition 11.8, there is an isomorphism $\underline{j}_q^{-1} : H_q(X) \longrightarrow H_q(K^{q+1})$. Consider the following commutative diagram.

$$\begin{tikzcd} & & {} \arrow[d] & & {} \arrow[d] & \\ & & H_{q+1}(K^{q+1},K^q) \arrow[d, "d_{q+1}"] \arrow[dr, dashed, "d^c_{q+1}"] & & H_{q-1}(K^{q-2}) \arrow[d] & \\ {} \arrow[r] & H_q(K^{q-1}) \arrow[r] & H_q(K^q) \arrow[r, "r_q"] \arrow[d, "\underline{i}"] & H_q(K^q,K^{q-1}) \arrow[r, "d_q"] \arrow[dr, dashed, "d^c_q"] & H_{q-1}(K^{q-1}) \arrow[r] \arrow[d, "r_{q-1}"] & {} \\ & & H_q(K^{q+1}) \arrow[d] & & H_{q-1}(K^{q-1},K^{q-2}) \arrow[d] & \\ & & H_q(K^{q+1},K^q) \arrow[d] & & {} & \\ & & {} & & & \end{tikzcd}$$

By Proposition 11.7, $H_q(K^{q+1},K^q) = 0$. By Proposition 11.9, $H_q(K^{q-1}) = H_{q-1}(K^{q-2}) = 0$. The diagram thus reduces to

$$\begin{tikzcd}
 & H_{q+1}(K^{q+1},K^q) \arrow[d, "d_{q+1}"] \arrow[dr, "d^c_{q+1}"] & & 0 \arrow[d] & \\
0 \arrow[r] & H_q(K^q) \arrow[r, "r_q"] \arrow[d, "\underline{i}"] & H_q(K^q,K^{q-1}) \arrow[r, "d_q"] \arrow[dr, "d^c_q"] & H_{q-1}(K^{q-1}) \arrow[r] \arrow[d, "r_{q-1}"] & {} \\
 & H_q(K^{q+1}) \arrow[d] & & H_{q-1}(K^{q-1},K^{q-2}) \arrow[d] & \\
 & 0 & & {} &
\end{tikzcd}$$

Now, r_q and r_{q-1} are monomorphisms, $\underline{i}$ is an epimorphism. We have

$$H_q(K^{q+1}) = \operatorname{Im} \underline{i} \cong H_q(K^q)/\operatorname{Ker} \underline{i} = H_q(K^q)/\operatorname{Im} d_{q+1}$$

$$\cong \operatorname{Im} r_q/r_q(\operatorname{Im} d_{q+1}) = \operatorname{Ker} d_q/\operatorname{Im} d^c_{q+1}$$

$$= \operatorname{Ker} d^c_q/\operatorname{Im} d^c_{q+1} = H^c_q(X) \quad .$$

The composition of $\underline{j}_q^{-1}$ and these isomorphisms gives $t_q: H_q(X) \approx H^c_q(X)$.

Finally, let f be a cellular map. Since f commutes with $\underline{j}_p^{-1}$, $\underline{i}$, and r, the diagram involving t_q is commutative.

Q.E.D.

12. Computation of homology of a complex by use of a surjective chain map.

It is well known that surjectivity or injectivity of a chain map does not necessarily imply the same properties for the induced homology homomorphisms. However, in the computation of the homology of a topological space which arises from another space by identification of certain subsets, it is frequently useful to compute from the complex associated with the original space. The following propositions justify this procedure.

<u>Theorem</u> 12.1. Let $f:C \longrightarrow D$ be a chain map such that $f_n:C_n \longrightarrow D_n$ is surjective for all n. We may then construct a new chain complex C', where $C'_n = C_n/\text{Ker } f_n$, such that $H_n(C') \cong H_n(D)$.

Proof: Since f_n is surjective, $D_n \cong C_n/\text{Ker } f_n$. Let $p_n:C_n \longrightarrow C_n/\text{Ker } f_n$ be the canonical homomorphism, and let $f'_n:C_n/\text{Ker } f_n \longrightarrow D_n$ be the unique isomorphism such that $f'_n p_n = f_n$. We shall write C'_n for $C_n/\text{Ker } f_n$, and thus have the diagram

$$\begin{tikzcd}
{} \arrow[d] & & {} \arrow[d] \\
C_n \arrow[rr, "f_n"] \arrow[dr, "p_n"'] \arrow[dd] & & D_n \arrow[dd] \\
& C'_n \arrow[ur, "f'_n"'] & \\
C_{n-1} \arrow[rr, "f_{n-1}"] \arrow[dr, "p_{n-1}"'] & & D_{n-1} \\
& C'_{n-1} \arrow[ur, "f'_{n-1}"'] &
\end{tikzcd} .$$

The next step is to define $d_n':C_n' \longrightarrow C_{n-1}'$. Let $x \in C_n'$, and choose any $c \in C_n$ such that $p_n(c) = x$. Define $d_n'(x) = p_{n-1}d_n(c)$. This definition is independent of the choice of c, for if also $p_n(c') = x$, then $f_n(c) = f_n(c')$ and hence $d_nf_n(c) = d_nf_n(c')$; but $d_nf_n = f_{n-1}d_n$, so $f_{n-1}d_n(c) = f_{n-1}d_n(c')$. Now, $f_{n-1} = f_{n-1}'p_{n-1}$, so $f_{n-1}'p_{n-1}d_n(c) = f_{n-1}'p_{n-1}d_n(c')$, but f_{n-1}' is an isomorphism, so $p_{n-1}d_n(c) = p_{n-1}d_n(c')$. Also, $d_{n-1}'d_n' = 0$: if $x \in C_n'$, choose $c \in C_n$ such that $p_n(c) = x$; then $d_n'(x) = p_{n-1}d_n(c)$, and since the boundary d_n' is independent of the choice of c, we have

$$d_{n-1}'(d_n'(x)) = p_{n-2}d_{n-1}(d_n(c)) = p_{n-2}d_{n-1}d_n(c) = 0 .$$

Thus, if we can show that $f':C' \longrightarrow D$ is a chain map, the proposition is proved. Let $x \in C_n'$; then $x = p_n(c)$ for some $c \in C_n$, so

$$\begin{aligned} d_nf_n'(x) = d_nf_n'p_n(c) = d_nf_n(c) = f_{n-1}d_n(c) &= f_{n-1}'p_{n-1}d_n(c) \\ &= f_{n-1}'d_n'(x) . \end{aligned}$$

Q.E.D.

<u>Theorem</u> 12.2. Let $B_n = d_{n+1}(C_{n+1})$. Then

$$H_n(D) \cong d_n^{-1}(\text{Ker } f_{n-1})/(B_n + \text{Ker } f_n).$$

Proof: Suppose $d_n'(x) = 0$, $x \in C_n'$. Then, for any $c \in C_n$ such that $p_n(c) = x$, we must have $p_{n-1}d_n(c) = 0$, or $d_n(c) \in \text{Ker } f_{n-1}$, hence $d_n(c) \in \text{Ker } f_{n-1}$ or $c \in d_n^{-1}$ $(\text{Ker } f_{n-1})$. Conversely, if $c \in d_n^{-1}(\text{Ker } f_{n-1})$, then $d_n'(p_n(c)) = 0$. Hence the n-cycles of C' are given by $p_n(d_n^{-1}(\text{Ker } f_{n-1})) = d_n^{-1}(\text{Ker } f_{n-1})/\text{Ker } f_n$.

Now suppose there is $y \in C_{n+1}'$ such that $x = d_{n+1}'(y)$ and let $c \in C_n$ be such that $p_n(c) = x$, $c' \in C_{n+1}$ such that $p_{n+1}(c') = y$. Then

$p_n d_{n+1}(c') = x = p_n(c)$, so $p_n(c - d_{n+1}(c')) = 0$, or $c-d_{n+1}(c') \in \mathrm{Ker}\ f_n$. Thus, $c \in B_n + \mathrm{Ker}\ f_n$. Now suppose $c \in C_n$, $c = d_{n+1}(c')+c''$, where $c'' \in \mathrm{Ker}\ f_n$. Then

$$\begin{aligned} p_n(c) &= p_n d_{n+1}(c') + p_n(c'') \\ &= p_n d_{n+1}(c') \\ &= d'_{n+1}(p_{n+1}(c')) \ . \end{aligned}$$

Hence the n-boundaries of C' are given by

$$p_n(B_n + \mathrm{Ker}\ f_n) = (B_n + \mathrm{Ker}\ f_n)/\mathrm{Ker}\ f_n.$$

Therefore,

$$H_n(D) \cong H_n(C') = (d_n^{-1}(\mathrm{Ker}\ f_{n-1})/\mathrm{Ker}\ f_n)/((B_n + \mathrm{Ker}\ f_n)/\mathrm{Ker}\ f_n)$$

$$\cong d_n^{-1}(\mathrm{Ker}\ f_{n-1})/(B_n + \mathrm{Ker}\ f_n) \ .$$

Q.E.D.

13. Homology groups of (T,f).

In this section, we consider only torus-like spaces given by (T,f), where T is the torus and f is induced by a unimodular matrix, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, whose entries are integers. Throughout this section,

$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, and $p: T\times I \longrightarrow (T,f)$ is the canonical map.

Let $F: R^2 \longrightarrow R^2$ be given by $F(x,y) = (ax+by, cx+dy)$. Then, as in Chapter I, F induces the map $f: T \longrightarrow T$.

Since T is path-connected, so are $T\times I$ and (T,f), hence the 0^{th} homology group is obvious (Hilton and Wylie (8), page 345, Theorem 8.8.2).

<u>Theorem</u> 13.1. $H_o((T,f)) = J$.

To determine the higher-dimensional homology groups of (T,f), we shall successively construct cellular decompositions for T, $T\times I$, and (T,f), and apply the methods of sections 11 and 12. First the torus: Let $q: R^2 \longrightarrow T$ be the canonical map, and let L_1 be the line $y = 0$, L_2 the line $x = 0$. Now, $T - (qF(L_1) \cup qF(L_2) \cup q(L_1) \cup q(L_2))$ is a disjoint union of 2-cells, $G_1, \ldots, G_e$, e some integer ≥ 1. As for a determination of the 0- and 1-cells, this depends on the entries of the matrix A. We first select the 0-cells. If $b = 0$, then $a = 1$, since A is unimodular; and thus $qF(L_1) = q(L_1)$. In this case (also in the case $a = 0$, $b = 1$), we take as the only 0-cell common to $q(L_1)$ and $qF(L_1)$ the point $q(0,0)$. If $ab \neq 0$, then $qF(L_1)$ intersects $q(L_1)$ in exactly $|a|$ points (including $q(0,0)$), and we take these points to be the 0-cells common to $q(L_1)$ and $qF(L_1)$. Similarly, we select 0-cells common to $qF(L_1)$ and $q(L_2)$, $qF(L_2)$ and $q(L_1)$, and $qF(L_2)$ and $q(L_2)$. This gives r distinct 0-cells, $H_1,\ldots,H_r$, where $r \geq 1$. Now, $q(L_1) - \{H_1,\ldots,H_r\}$ is the disjoint union of s 1-cells, $K_1,\ldots,K_s$, with $s \geq 1$; $q(L_2) - \{H_1,\ldots,H_r\}$ is the disjoint union of t 1-cells, $M_1,\ldots,M_t$; $qF(L_1) - \{H_1,\ldots,H_r\}$ is the disjoint union of u 1-cells, $N_1,\ldots,N_u$; and finally, $qF(L_2) - \{H_1,\ldots,H_r\}$ is the disjoint union of v 1-cells, $W_1,\ldots,W_v$.

It is easy to check that these 1-cells, together with the previously selected 0- and 2-cells, give us a cellular decomposition of the torus. Similarly, letting $G_i' = f^{-1}(G_i)$, $H_i' = f^{-1}(H_i)$, $K_i' = f^{-1}(K_i)$, $M_i' = f^{-1}(M_i)$, $N_i' = f^{-1}(N_i)$, and $W_i' = f^{-1}(W_i)$, we obtain another cellular decomposition of T.

Using these decompositions of T, we next construct a cellular decomposition of $T \times I$. For the single 3-cell we take $X = (T - (q(L_1) \cup q(L_2))) \times (0,1)$. As the 2-cells we take $Y_1 = (q(L_1) - q(0,0)) \times (0,1)$, $Y_2 = (q(L_2) - q(0,0)) \times (0,1)$, in addition to $G_1' \times \{1\},\ldots,G_e' \times \{1\}$, and $G_1 \times \{0\},\ldots,G_e \times \{0\}$; for simplicity, we shall, wherever possible, write G_i' instead of $G_i' \times \{1\}$, and G_i instead of $G_i \times \{0\}$. For 1-cells we take $Z = \{q(0,0)\} \times (0,1)$ together with all $K_i \times \{0\}$, $M_i \times \{0\}$, $N_i \times \{0\}$, $W_i \times \{0\}$ and $K_i' \times \{1\}$, $M_i' \times \{1\}$, $N_i' \times \{1\}$, $W_i' \times \{1\}$; for the 0-cells we take all $H_i \times \{0\}$ and all $H_i' \times \{1\}$; we will use the same notational simplification for the 0- and 1-cells as for the 2-cells.

This cellular decomposition for $T\times I$ in turn gives rise to one for (T,f), consisting of the p-images of the cells making up $T\times I$; we observe that the p-image of a "primed" cell of $T\times I$ is identical with that of the corresponding "unprimed" cell, e.g., $p(G_1') = p(G_1)$. We summarize the preceding discussion as

<u>Lemma</u> 13.2. Let $p: T\times I \longrightarrow (T,f)$ be the canonical map. Then

1) $\{X, Y_1, Y_2, G_1, \ldots, G_e, G_1', \ldots, G_e', Z, K_1, \ldots, K_s, K_1', \ldots, K_s', M_1, \ldots, M_t, M_1', \ldots, M_t', N_1, \ldots, N_u, N_1', \ldots, N_u', W_1, \ldots, W_v, W_1', \ldots, W_v', H_1, \ldots, H_r, H_1', \ldots, H_r'\}$, as defined above, is a cellular decomposition for $T\times I$;

2) $\{p(X), p(Y_1), p(Y_2), p(G_1), \ldots, p(G_e), p(Z), p(K_1), \ldots, p(K_s), p(M_1), \ldots, p(M_t), p(N_1), \ldots, p(N_u), p(W_1), \ldots, p(W_v), p(H_1), \ldots, p(H_r)\}$ is a cellular decomposition for (T,f);

3) p is a cellular map with respect to the decomposition in 1) and 2);

4) $p(G_i') = p(G_i), \quad 1 \leq i \leq e,$

$p(K_i') = p(K_i), \quad 1 \leq i \leq s,$

$p(M_i') = p(M_i), \quad 1 \leq i \leq t,$

$p(N_i') = p(N_i), \quad 1 \leq i \leq u,$

$p(W_i') = p(W_i), \quad 1 \leq i \leq v,$

$p(H_i') = p(H_i), \quad 1 \leq i \leq r.$

We shall first determine the homology homomorphisms induced by f and then, using this information, compute the homology groups of (T,f). It is well known (Hilton and Wylie (8), page 132) that the homology groups of the torus, T, are $H_2(T) = J$, $H_1(T) = J \oplus J$, $H_o(T) = J$; all other groups are trivial. Denote by f_o^*, f_1^*, f_2^* the homology isomorphisms induced by f.

<u>Lemma</u> 13.3. Let 1 be a generator of $H_2(T)$. Then

$$f_2^*(1) = 1, \text{ if } \Delta = 1,$$
$$= -1, \text{ if } \Delta = -1.$$

Proof: $T\times\{0\}$ is a deformation retract of $T\times I$, and so $H_2(T) \cong H_2(T\times I)$. We select generators of the groups of the cellular complex of $T\times I$ as follows: Let x be any generator of $S_3^c(T\times I)$, and choose generators g_i, g_i' of $S_2^c(T\times I)$ corresponding to G_i, G_i' such that

$$d_3^c(x) = \sum_{i=1}^{e} g_i - \sum_{i=1}^{e} g_i' \ .$$

These, of course, are not all the generators of $S_2^c(T\times I)$, but they are all we are interested in at the moment. Then $\sum_{i=1}^{e} g_i$ and $\sum_{i=1}^{e} g_i'$ represent the same generator of $H_2(T)$. Hence we may determine the homology homomorphism of f by use of the map $f':T\times\{1\} \longrightarrow T\times\{0\}$ given by $f'(t,1) = (f(t),0)$. This map is cellular with respect to the cellular with respect to the cellular decompositions of $T\times\{1\}$, $T\times\{0\}$ induced by the decomposition of $T\times I$. Now, $f'(G_i') = G_i$, and so $f_2^*(1) = 1$ or -1 according as $f_2'^c(g_i') = g_i$ or $-g_i$, and $f_2'(g_i') = g_i$ if and only if $f_1'^c(d_2^c(g_i')) = d_2^c(g_i)$.

We need only determine what $f_2'^c$ does to one g_i'. Select a g_i' such that $(q(0,0),1) \in \overline{G_i'}$. Now, $q^{-1}(G_i')$ (here we take the $G_i' \subset T$, instead of the $G_i' \subset T\times I$) is a disjoint union of open sets and we select any one

G_i'' such that $(0,0)\in\overline{G_i''}$. Then $f'(G_i') = qF(G_i'')\times\{0\}$, and $(0,0)\in\overline{F(G_i'')}$. Both $\overline{G_i''}$ and $\overline{F(G_i'')}$ are polygons whose boundaries are made up of line segments, and in each case precisely two of these segments intersect at $(0,0)$. For G_i'', let α' be the generator of $H_1^c(T\times I)$ corresponding to the 1-cell which is the q-image of the open-ended segment on the clockwise side of G_i'', β' the generator corresponding to the 1-cell which is the q-image of the open-ended segment on the counterclockwise side of G_i'', chosen so that $d_2^c\ (g_i') = \alpha'-\beta'+\ldots$; similarly choose α, β corresponding to the clockwise and counterclockwise side of $F(G_i'')$, so that $d_2^c(g_i) = \alpha-\beta+\ldots$

For the rest of the 1-cells, we choose generators in any way we like, requiring only $f_1'^c(n_i') = n_i$, $f_1'^c(m_i') = m_i$, $f_1'^c(k_i') = k_i$, $f_1'^c(w_i') = w_i$, where n_i' is the generator corresponding to N_i', etc. It is apparent that our previous choice satisfies this condition; in fact, we forced this condition, since we may choose generators h_i, h_i' corresponding to H_i, H_i' such that $f_0'^c(h_i') = h_i$.

If V is a finite dimensional vector space over R, an orientation to V is a function h from the set of bases of V to $\{1,-1\}$, such that if B, B' are two bases of V, and A is the matrix of the transformation which takes B to B', then $h(B)h(B')\det(A) > 0$. This coincides with the familiar notion of orientation in R^2 (see Godement (6), page 308, Remarque 4, for details). Hence if $\Delta = -1$, F reverses the sense of angles and if $\Delta = 1$, F preserves the sense of angles.

Now, if $\Delta = 1$, we have $f_1'^c(\alpha') = \alpha$, $f_1'^c(\beta') = \beta$, and hence $f_2'^c(g_i') = g_i$; if $\Delta = -1$, then $f_1'^c(\alpha') = \beta$, $f'^c(\beta') = \alpha$, and hence $f_2'^c(g_i') = -g_i$. Q.E.D.

We now choose generators for all the groups of the cellular complex, once and for all. In the plane, consider the segments $I_1 = \{(x,y): y = 0,\ 0 \leq x \leq 1\}$, $I_2 = \{(x,y): x = 0,\ 0 \leq y \leq 1\}$, $I_3 = F(I_1)$, $I_4 = F(I_2)$. Now, $I_1 - \{q^{-1}(H_1),\ldots,q^{-1}(H_r)\}$ is the disjoint union of open intervals $K_1^o,\ldots,K_s^o$, which are in 1-1 correspondence with the 1-cells, $K_1,\ldots,K_s$. For each of the H_i's involved select a generator h_i and for each K_i select a generator k_i such that if h_{i_1} corresponds to the endpoint of the K_i^o farthest from (0,0), and h_{i_2} corresponds to the endpoint of K_i^o closest to (0,0), then $d_1^c(k_i) = h_{i_1} - h_{i_2}$. Proceeding similarly, select generators m_i for the M_i's and generators h_i for the H_i's involved. At this stage, generators corresponding to all the H_i's will have been chosen. Now choose generators n_i, w_i corresponding to the N_i, W_i such that if h_{i_2} corresponds to the endpoint of, for example, N_i^o closest to (0,0), and h_{i_1} corresponds to the endpoint farthest from (0,0), then $d_1^c(n_i) = h_{i_1} - h_{i_2}$. Renumbering, if necessary, so that $H_1 = p(q(0,0) \times \{0\})$, select generators z corresponding to Z and h_1' corresponding to H_1' such that $d_1^c(z) = h_1' - h_1$. Then we proceed as before, selecting generators corresponding to the rest of the H_i', and to the K_i', M_i', N_i', and W_i', denoting these by h_i', k_i', m_i', n_i', and w_i', respectively.

For the 2-cells Y_1 and Y_2 select generators y_1 and y_2 such that

$$d_2^c(y_1) = \sum_{i=1}^{s} k_i - \sum_{i=1}^{u} n_i' \text{ and } d_2^c(y_2) = \sum_{i=1}^{t} m_i - \sum_{i=1}^{v} w_i' ,$$

generators g_i corresponding to G_i such that $\sum_{i=1}^{e} g_i$ is a cycle, and generators g_i' corresponding to G_i', x corresponding to X such that

$$d_3^c(x) = \sum_{i=1}^{e} g_i - \sum_{i=1}^{e} g_i' \quad .$$

<u>Theorem</u> 13.4. $H_3((T,f)) = J$, if $\Delta = 1$,

$= 0$, if $\Delta = -1$.

Proof: We recall the cellular decomposition of (T,f) given by Lemma 13.2, and that $p: T\times I \longrightarrow (T,f)$ is a cellular map with respect to this decomposition and that of $T\times I$. We then have a surjective chain map $p^c: S^c(T\times I) \longrightarrow S^c((T,f))$, and we may use the results of Section 12. In dimension 3, we have the following:

I. <u>$\Delta = 1$.</u> Then $d_4^c(S_4^c(T\times I)) = 0$, Ker $p_3^c = 0$, Ker $p_2^c =$ the group generated by all $g_i - g_i'$, $i = 1,\ldots,e$, by Lemma 13.3. $d_3^c(S_3^c(T\times I))$ is the group generated by $\sum_{i=1}^{e} g_i - \sum_{i=1}^{e} g_i'$. Hence $d_3^c(S_3^c(T\times I))$ Ker p_2^c is the group generated by $\sum_{i=1}^{e} g_i - \sum_{i=1}^{e} g_i'$, and $(d_3^c)^{-1}(\text{Ker } p_2^c) = S_3^c(T\times I) \cong J$. The result follows from Theorem 12.2.

II. <u>$\Delta = -1$.</u> The only thing changed is that now Ker $p_2^c =$ the group generated by all $g_i + g_i'$, by Lemma 13.3. Hence $d_3^c(S_3^c(T\times I)) \cap \text{Ker } p_2^c = 0$, and $(d_3^c)^{-1}(\text{Ker } p_2^c) = 0$.

Q.E.D.

<u>Lemma</u> 13.5. There is a basis for $H_2(T)$ such that the matrix of f_1^*, with respect to this basis, is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Proof: In the manner of the proof of Lemma 13.3, we may take

$k = \sum_{i=1}^{s} k_i$, $m = \sum_{i=1}^{t} m_i$ as representatives of a basis for $H_2(T)$.

Moreover, $\sum_{i=1}^{u} n_i' = n'$ and $\sum_{i=1}^{v} w_i' = w'$ are, respectively, representatives of the same basis elements. We determine the effect of f_1^* on the elements represented by k and m by considering $f_1'^c$ acting on n' and w'.

Now f_1' $(n') = \sum_{i=1}^{u} n_i$, and $f_1'^c(w') = \sum_{i=1}^{v} w_i$, but $\sum_{i=1}^{u} n_i$ is homologous to

ak + cm, for if we take the cellular decomposition of the plane given by taking q^{-1} of all G_i's, H_i's, K_i's, M_i's, N_i's, and W_i's, we find that the q^c image of the cellular 2-chain in the plane given by the sum of the generators (multiplied by 1) corresponding to those 2-cells inclosed in the triangle bounded by the segment from (0,0) to (a,c), the segment from (0,0) to (a,0), and the segment from (a,0) to (a,c), is a cellular 2-cycle whose boundary is

$ak + cm - \sum_{i=1}^{u} n_i$. Similarly, $\sum_{i=1}^{v} w_i$, is homologous to bk + dm.

Q.E.D.

<u>Theorem</u> 13.6. $H_2((T,f)) \cong J \oplus \text{Ker } (f_1^* - I_1)$, if $\Delta = 1$,

$$H_2((T,f)) \cong J_2 \oplus \text{Ker } (f_1^* - I_1), \text{ if } \Delta = -1,$$

where J_2 is the integers mod 2.

Proof: <u>$\Delta = 1$.</u> B_2 is generated by $\sum_{i=1}^{e} g_i - \sum_{i=1}^{e} g_i'$; Ker p_2^c is generated by all $g_i - g_i'$; Ker p_1^c is generated by $k_i - k_i'$, $m_i - m_i'$, $n_i - n_i'$, $w_i - w_i'$.

$(d_2^c)^{-1}(\text{Ker } p_1^c)$ is generated by $\sum_{i=1}^{e} g_i$, $\sum_{i=1}^{e} g_i'$, all $g_i - g_i'$, and all linear combinations $m(y_1+\alpha_1) + n(y_2+\alpha_2)$, such that α_1, α_2 are cellular 2-cycles not involving y_1, y_2 and satisfying the three conditions

$$\text{(i)} \quad p_1^c d_2^c(y_1+\alpha_1) = (a-1)\sum_{i=1}^{s} k_i + c\sum_{i=1}^{t} m_i,$$

$$\text{(ii)} \quad p_1^c d_2^c(y_2+\alpha_2) = b\sum_{i=1}^{s} k_i + (d-1)\sum_{i=1}^{t} m_i, \text{ and}$$

$$\text{(iii)} \quad \begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(That such α_1, α_2 exist is shown in the proof of Lemma 13.5; whether such m and n exist depends on the matrix.) Notice that if we fix α_1 and α_2 to be the 2-cycles used in Lemma 13.5, then all other 2-cycles α_1', α_2' satisfying these requirements differ from α_1, α_2 by elements of $(d_2^c)^{-1}(\text{Ker } p_2^c)$. Hence we may take as generators of $(d_2^c)^{-1}(\text{Ker } p_2^c)$, $\sum_{i=1}^{e} g_i$, $\sum_{i=1}^{e} g_i'$, all $g_i - g_i'$, and all elements $m(y_1+\alpha_1) + n(y_2+\alpha_2)$, where m and n satisfy (iii) and α_1, α_2 are these fixed 2-cycles. These last elements generate a direct summand of $(d_2^c)^{-1}(\text{Ker } p_1^c)$ whose intersection with $B_2 + \text{Ker } p_2^c$ is the trivial group. Therefore, this direct summand is isomorphic to a direct summand of $H_2((T,f))$ by Theorem 12.2. Lemma 13.3 implies this direct summand is isomorphic to $\text{Ker}(f_1^* - I_1)$. The quotient with respect to $B_2 + \text{Ker } p_2^c$ of the direct summand of $(d_2^c)^{-1}(\text{Ker } p_1^c)$ generated by $\sum_{i=1}^{e} g_i$, $\sum_{i=1}^{e} g_i'$, and all $g_i - g_i'$ is the group freely generated by the equivalence class of $\sum_{i=1}^{e} g_i$, and this class is not the zero class.

<u>$\Delta = -1$.</u> B_2 is generated by $\sum_{i=1}^{e} g_i - \sum_{i=1}^{e} g_i'$; Ker p_2^c by all elements of the form g_i+g_i'; Ker p_1^c by all elements k_i-k_i', m_i-m_i', n_i-n_i', and w_i-w_i'. Since $\Delta = -1$, $f_2^{!c}d_2^c(g_i') = -d_2^c(g_i)$ and therefore $(d_2^c)^{-1}(\text{Ker } p_1^c)$ is generated by $\sum_{i=1}^{e} g_i$, $\sum_{i=1}^{e} g_i'$, and elements $m(y_1+\alpha_1) + n(y_2+\alpha_2)$, as before. The elements $m(y_1+\alpha_1) + n(y_2+\alpha_2)$ as in the preceding case give us a direct summand of $H_2((T,f))$ isomorphic to Ker $(f_1^* - I_1)$; the other direct summand of $(d_2^c)^{-1}(\text{Ker } p_1^c)$ gives a group of order 2 generated by the class of $\sum_{i=1}^{e} g_i$.

Q.E.D.

<u>Theorem</u> 13.7. $H_1((T,f)) \cong J \oplus (H_1(T)/\text{Im } (f_1^* - I_1))$, where I_1 denotes the identity map of $H_1(T)$.

Proof: B_1 is generated by the boundaries of all g_i and g_i', together with the boundaries of y_1 and y_2; Ker p_1^c is generated by all k_i-k_i', m_i-m_i', n_i-n_i', and w_i-w_i'; Ker p_o^c is generated by all h_i-h_i'; $(d_1^c)^{-1}(\text{Ker } p_o^c)$ is generated by z, $\sum_{i=1}^{s} k_i$, $\sum_{i=1}^{s} k_i'$, $\sum^{t} m_i$, $\sum^{t} m_i'$, $\sum^{u} n_i$, $\sum^{u} n_i'$, $\sum^{v} w_i$, $\sum^{v} w_i'$, boundaries of g_i and g_i', and k_i-k_i', m_i-m_i', n_i-n_i', and w_i-w_i'. We now obtain equivalent sets of generators for B_1 + Ker p_1^c and $(d_1^c)^{-1}(\text{Ker } p_o^c)$ which simplify the calculation of the quotient group. First consider B_1 + Ker p_1^c. We have

$$d_2^c(y_1) = \sum_{i=1}^{s} k_i - \sum_{i=1}^{u} n_i', \quad d_2^c(y_2) = \sum_{i=1}^{t} m_i - \sum_{i=1}^{v} w_i' \quad .$$

Since $n_i - n_i' \in \mathrm{Ker}\ p_1^c$, $w_i - w_i' \in \mathrm{Ker}\ p_1^c$, we may use $\sum_{i=1}^{s} k_i - \sum_{i=1}^{u} n_i$, $\sum_{i=1}^{t} m_i - \sum_{i=1}^{v} w_i$ as generators in place of $d_2^c(y_1)$, $d_2^c(y_2)$. From the proof of Lemma 13.3, $\sum_{i=1}^{u} n$ differs from $a\sum_{i=1}^{s} k_i + c \sum m_i$ by the boundary of a 2-cycle involving only g_i's. Thus we may replace $\sum_{i=1}^{s} k_i - \sum_{i=1}^{u} n_i$ by $(a-1)\sum_{i=1}^{s} k_i + c\sum_{i=1}^{t} m_i$ as a generator of $B_1 + \mathrm{Ker}\ p_1^c$, and similarly we may use $b\sum_{i=1}^{s} k_i + (d-1)\sum_{i=1}^{t} m_i$ in place of $\sum_{i=1}^{t} m_i - \sum_{i=1}^{v} w_i$, so our generators of $B_1 + \mathrm{Ker}\ p_1^c$ are $(a-1)\sum_{i=1}^{s} k_i + c\sum_{i=1}^{t} m_i$, $b\sum_{i=1}^{s} k_i + (d-1)\sum_{i=1}^{t} m_i$, the boundaries of all g_i and g_i', and $k_i - k_i'$, $m_i - m_i'$, $n_i - n_i'$, $w_i - w_i'$. In the generators of $(d_1^c)^{-1}(\mathrm{Ker}\ p_o^c)$, we may dispense with $\sum_{i=1}^{s} k_i'$, $\sum_{i=1}^{t} m_i'$, $\sum_{i=1}^{u} n_i'$, and $\sum_{i=1}^{v} w_i'$, since we have, for example, both $\sum_{i=1}^{s} k_i$ and all $k_i - k_i'$. Furthermore, we need only use $\sum_{i=1}^{s} k_i$ and $\sum_{i=1}^{t} m_i$, since $\sum_{i=1}^{u} n_i$ differs from a linear combination of these elements by a sum of boundaries of g_i's, and these elements are contained in $(d_1^c)^{-1}(\mathrm{Ker}\ p_o^c)$; similarly for

$\sum_{i=1}^{v} w_i$. Hence a set of generators for $(d_1^c)^{-1}(\text{Ker } p_o^c)$ is z, all $\sum_{i=1}^{s} k_i$, $\sum_{i=1}^{t} m_i$, all $k_i - k_i'$, $m_i - m_i'$, $n_i - n_i'$, $w_i - w_i'$, and the boundaries of all g_i and g_i'. Since the intersection of the direct summand of $(d_1^c)^{-1}(\text{Ker } p_o^c)$ generated by z with $B_1 + \text{Ker } p_1^c$ is the trivial group, $(d_1^c)^{-1}(\text{Ker } p_o^c)/(B_1 + \text{Ker } p_1^c)$ is the direct sum of a group isomorphic to the subgroup generated by z (which in turn is isomorphic to J) and a group isomorphic to the quotient of the subgroup of $(d_1^c)^{-1}(\text{Ker } p_o^c)$ generated by the generators of $(d_1^c)(\text{Ker } p_o^c)$ other than z, with respect to $B_1 + \text{Ker } p_1^c$. The latter direct summand is isomorphic to the quotient of the group generated by $\sum_{i=1}^{s} k_i$ and $\sum_{i=1}^{t} m_i$ by the subgroup generated by $(a-1)\sum_{i=1}^{s} k_i + c\sum_{i=1}^{t} m_i$ and $b\sum_{i=1}^{s} k_i + (d-1)\sum_{i=1}^{t} m_i$. By Lemma 13.3, this is isomorphic to $H_1(T)/\text{Im } (f_1^* - I_1)$.

Q.E.D.

The actual computation of these homology groups can easily be carried out. Since $H_1(T) \cong J \oplus J$ and the transformation $f_1^* - I_1$ has the matrix $\begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix}$, the determination of $H_1(T)/\text{Im } (f_1^* - I_1)$ reduces to finding (by elementary transformations of matrices) the numbers which are called "invariant factors" or "elementary divisors" of this matrix. If m, n are these numbers, then $H_1(T)/\text{Im } (f_1^* - I_1) \cong J_m \oplus J_n$, where $J_m = J/mJ$. Also, $\text{Ker } (f_1^* - I_1)$ is a free group and it is easy to see that its rank is equal to the rank of the free part of $H_1(T)/\mathrm{Im}\,(f_1^* - I_1)$, i.e., to 0,1,2 according as neither, one, or both of the invariant factors m, n are zero. Thus the homology groups of (T,f) are completely determined by the determinant of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and the invariant factors of } A-I = \begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix}.$$

Chapter V. On finding necessary conditions for homeomorphism of torus-like spaces.

14. Some connections with algebraic number theory.

An interesting conjecture in the case of the Hilbert-domain problem is that integral similarity of matrices A and B is not only a sufficient condition for homeomorphism of the spaces (T, f_A) and (T, f_B), as shown in Theorem 3.1, but is also necessary. This conjecture is, unfortunately, false.

By Theorem 3.2, $(T, f_A) \cong (T, f_{A^{-1}})$, and thus if the above conjecture were true, we would have A similar to A^{-1} for every unimodular matrix A with integer entries. That this is not the case is shown as follows.

<u>Proposition</u> 14.1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a unimodular matrix with integer entries which is integrally similar to its inverse. Then

1) If $ad - bc = -1$, $A = A^{-1}$.

2) If $ad - bc = 1$, A is integrally similar to its transpose A^t.

Proof: 1) Since $ad - bc = -1$, $A^{-1} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$. Similarity of A and A^{-1} requires $a+d = -a-d$, or $a+d = 0$. Therefore,

$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, $A^{-1} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, and $A = A^{-1}$.

2) Since $ad - bc = 1$, $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. But

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A^{-1}.$$

If we let $R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then $A^{-1} = RA^tR^{-1}$. By assumption, there is an integral unimodular matrix P such that $A = PA^{-1}P^{-1}$. Therefore, $A = PA^{-1}P^{-1} = PRA^tR^{-1}P^{-1} = (PR)A^t(PR)^{-1}$.

Q.E.D.

Let us next recall the following results from algebraic number theory (see, e.g., Weiss (13)). A <u>quadratic field</u> F is a quadratic extension of the rationals Q. An <u>integer</u> in F is any element which satisfied a monic polynomial with rational integer coefficients; the set of integers in F forms a ring, R. An <u>integral basis</u> of F is a pair of integers of F which forms a basis for R as a module over the rational integers. A non-trivial finitely generated R-module contained in F is called a <u>fractional ideal</u> of F. Then any ideal of R (in the ring-theoretic sense) is a fractional ideal of F. A fractional ideal $\mathfrak{a}$ of F is called a <u>principal ideal</u> if $\mathfrak{a} = xR$, for some x F. Then the set I of fractional ideals forms an abelian group under multiplication, with R as the identity, and the set H of principal ideals forms a subgroup. The quotient group, I/H, is called the <u>ideal class group</u> of F; let us denote by cls $\mathfrak{a}$ the ideal class of the ideal $\mathfrak{a}$.

<u>Proposition</u> 14.2. (Taussky (11) and (12)) Let f be an irreducible quadratic polynomial with coefficients in Q, θ a complex root of f such

that $1, \theta$ form an integral basis of $Q(\theta)$, and let $\mathcal{M}(f,\theta)$ denote the set of all 2x2 matrices A with rational integer entries such that $f(A) = 0$. Then

1) Given $A \in \mathcal{M}(f,\theta)$, there is a characteristic vector α for A, relative to θ, whose components lie in $Q(\theta)$, and thus generate an ideal $\mathrm{Id}\,\alpha$ of $Q(\theta)$.

2) If β is any other vector with the same properties as α, then $\mathrm{cls}(\mathrm{Id}\,\alpha) = \mathrm{cls}(\mathrm{Id}\,\beta)$. In particular, each A determines in this way a unique ideal class $\mathcal{C}(A)$ in I/H (the ideal class group of $Q(\theta)$).

3) For $A, B \in \mathcal{M}(f,\theta)$, A and B are integrally similar if and only if $\mathcal{C}(A) = \mathcal{C}(B)$.

4) The mapping $\mathcal{C} : \mathcal{M}(f,\theta) \longrightarrow I/H$ is surjective.

5) $\mathcal{C}(A)$ has inverse $\mathcal{C}(A^t)$ in I/H (recall $A \in \mathcal{M}(f,\theta)$ if and only if $A^t \in \mathcal{M}(f,\theta)$, so that this statement is meaningful).

Combining the preceding two propositions, it is evident that if $ad - bc = 1$, the matrix is similar to its inverse only under rather involved number-theoretic conditions. In particular, the conjecture mentioned earlier is not true.

Additional information is available to us concerning the possible limitations in determining necessary conditions for homeomorphism of (T, f_A) and (T, f_B). For an arbitrary group π and an abelian group G, let us denote by $\mathcal{H}_n(\pi, G)$ the n^{th} homology group of π with coefficients in G, by $\mathcal{H}^n(\pi, G)$ the n^{th} cohomology group of π with coefficients in G. Eilenberg and MacLane ((2), Theorem I, page 482) have shown that if X is a path-connected topological space such that $\pi_n(X) = 0$ for $n > 1$, then the homology and cohomology groups of X are determined by the fundamental group $\pi_1(X)$; more precisely,

$$H_n(X,G) \cong \mathcal{H}_n(\pi_1(X),G),$$

$$H^n(X,G) \cong \mathcal{H}^n(\pi_1(X),G),$$

where $H_n(X,G)$ denotes the n^{th} singular homology group of X with coefficients in G, and $H^n(X,G)$ denotes the n^{th} singular cohomology group of X with coefficients in G.

By the results of Section 10, the spaces (T,f_A) satisfy the hypotheses of the Eilenberg-MacLane theorem. From the results of Section 13, it follows that the determinant of the matrix A and the invariant factors of A-I can be recovered from the group $\pi_1((T,f_B))$ for which we have an explicit presentation (Theorem 5.15). It seems likely that a more detailed study of this group might yield useful information concerning necessary conditions for homeomorphism of torus-like spaces (T,f_A).

REFERENCES

1. Harvey Cohn, A Numerical Study of Topological Features of Certain Hilbert Fundamental Domains (to appear).

2. Samuel Eilenberg and Saunders MacLane, Relations between homology and homotopy groups of spaces, Ann. of Math. 46 (1945), 480-509.

3. Samuel Eilenberg and Norman Steenrod, Foundations of Algebraic Topology, Princeton Univ. Press, Princeton, N. J., 1952.

4. Ralph H. Fox, Discrete Groups and Their Presentations, Lecture notes, Princeton Univ., Spring, 1955.

5. Steven A. Gaal, Point Set Topology, Academic Press, New York, 1964.

6. Roger Godement, Cours d'algèbre, Hermann, Paris, 1963.

7. P. J. Hilton, An Introduction to Homotopy Theory, Cambridge Univ. Press, Cambridge, 1961.

8. P. J. Hilton and S. Wylie, Homology Theory, Cambridge Univ. Press, Cambridge, 1960.

9. Sze-Tsen Hu, Homotopy Theory, Academic Press, New York, 1959.

10. Paul Olum, Non-abelian cohomology and van Kampen's theorem, Ann. of Math. 68 (1958), 658-668.

11. Olga Taussky, On a theorem of Latimer and MacDuffee, Canad. J. Math. 1 (1949), 300-302.

12. ____________, Classes of matrices and quadratic fields, Pacific J. Math. 1(1951), 127-132.

13. Edwin Weiss, Algebraic Number Theory, McGraw-Hill, New York, 1963.