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SOLUTION OF MECHANICAL VIBRATION PROBLEMS WITH MIXED RESPONSE-EXCITATION INFORMATION

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James William Straight

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I hereby recommend that this dissertation prepared under my direction by James William Straight entitled Solution of Mechanical Vibration Problems with Mixed Response-Excitation Information be accepted as fulfilling the dissertation requirement of the degree of Doctor of Philosophy

Roger A. Anderson	January 30, 1967
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After inspection of the dissertation, the following members of the Final Examination Committee concur in its approval and recommend its acceptance:*

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SIGNED: J W Straight

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LIST OF SYMBOLS

EA	Longitudinal stiffness
EI	Flexural stiffness
$f(t)$	Externally applied force
f_ξ	Internal force at $x = \xi$
k	Spring constant
K_i	i th generalized stiffness
$[\ K\ ]$	Generalized stiffness matrix
$[k_R]$	Reduced stiffness matrix
m	Mass
M_i	i th generalized mass
$[\ M\ ]$	Generalized mass matrix
$[m_R]$	Reduced mass matrix
q	Generalized coordinate
U	Longitudinal displacement
U_g	Known longitudinal displacement
U_ℓ	Longitudinal displacement at $x = \ell$
W	Lateral deflection
W_g	Known lateral deflection
φ	Eigenvector; eigenfunction
$\overline{\varphi}$	Adjoint eigenvector; adjoint eigenfunction
$[\Phi]$	Transformation matrix

ω^2	Eigenvalue
ξ	Specified location
μ	Mass density
$(\ )'$	Derivative with respect to x
$(\dot{\ })$	Derivative with respect to t

ABSTRACT

Vibrations problems are considered in which one of the exciting forces is unknown but in which one of the response motions is known. As this is a first study of problems of this nature, only simple systems such as flexure of beams and longitudinal vibrations of bars are examined.

The basic method of solution involves recasting the equations of motion into a form in which the unknown exciting force does not appear and in which the known response motion appears as an exciting force. Since the coefficient matrices of these reduced equations are not symmetric, the usual method of uncoupling equations is not applicable. After the equations are transformed to new coordinates which are based upon the eigenvectors of the reduced equations, they are uncoupled through the use of the eigenvectors of a related set of equations, called the adjoint equations.

This method, which is successful for lateral vibrations of beams, is only partially successful for the longitudinal vibrations of bars. The equations of motion for systems of this type are divided into two sets, one of which can be uncoupled and one which cannot be uncoupled. The latter set of equations is transformed to a more efficient set of coordinates which yields a reasonably well behaved set of equations with little coupling.

The method presented is well adapted to computer techniques and a Fortran program is developed for the solution of the reduced equations.

The method is applied to lumped mass systems and is extended to two cases of continuous systems.

CHAPTER 1

INTRODUCTION

The study of vibrations may involve three or more distinct types of problems. In the usual vibrations problem, the system parameters and the forces applied to the systems are known. The solution involves the determination of the motion response of the system. Another type of problem is one in which the motion response and the exciting forces of the system are known and the system parameters are to be determined. As an example of this latter type of problem in the area of electronics, the electrical engineer is often called on to design a system which will give a desired output for a given input.

A third type of problem, which will be referred to in this paper as the inverse problem, is one in which the system parameters, some of the exciting forces, and part of the motion response of the system are known. The unknowns are the remaining forces and motions. This paper will be concerned with the solution of the inverse problem.

The idea for this study came from the unpublished analysis of a specific rocket dynamics problem made by the Jet Propulsion Laboratory. The torsional motion of the rocket structure resulting from the gimbaling motion of the rocket motor was measured at a point. The problem was to determine the unknown gimbaling motion of the rocket motor, representing the unknown torque input. Although an answer to this inverse problem was obtained using classical methods, the equations of motion were in an inconvenient form and were poorly behaved. The present general study was initiated to find more suitable methods. From a careful survey of the literature in the field of vibrations, it appears that no general studies have been published on the inverse problem.

In the classical method of analysis of the usual vibrations problem, the motion is described in terms of the normal modes of the system. This method is convenient in that the equations of motion are uncoupled and can be solved separately. Further, the method is efficient since the solution for the response of the system is often quite convergent with the mode number, and the number of equations can often be reduced. The purpose of this study is to yield methods for solving the inverse problem which are similarly convenient and efficient.

As this is a first study, the inverse problems considered are somewhat simplified. The systems considered are undamped and linear and are excited by an unknown force at one point with the motion response of one point known. Specifically, most consideration is given to the longitudinal vibrations of bars and the flexural vibrations of beams. The study begins with an examination of lumped mass systems and then expands the ideas to continuous systems. The methods proposed for lumped mass systems are adapted to the use of a computer. Because of the exploratory nature of the study, the approach has been to try out all the available ideas on example problems and to single out the most useful methods. It remains for further studies to generalize the results and to give general proofs.

CHAPTER 2

BASIC METHODS

Proposed methods for the solution of the inverse problem will be developed in this chapter. The specific examples used for illustration will be simple ones, those which may be easily solved without the aid of automatic digital or analog computing equipment. Chapter 4 will extend the methods of this chapter to complex systems which require such help.

2.1 Motion Given at the Point of Application of the Unknown Force

The simplest case of the inverse problem is one in which the point of known motion is the same as the point of application of the unknown force. This problem may be recognized as that of a moving constraint. Consider an n degree of freedom system for which x_1, x_2, ... x_n are the displacements of the n points. For simplicity, let x_n be the coordinate whose motion is known (and the coordinate at which the unknown force is applied). The equations of motion are

$$\begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & & \cdot \\ \cdot & & \ddots & \cdot \\ \cdot & & & \cdot \\ \cdot & & & 0 \\ 0 & \cdots & 0 & m_n \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \ddot{x}_n \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ k_{n1} & \cdots & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ f(t) \end{Bmatrix} \qquad (2\text{-}1)$$

where m_i is the mass associated with the i th coordinate and k_{ij} is the stiffness influence coefficient relating the i th and j th coordinates. f(t) is the unknown force. Equation 2-1 is a set of n equations in n unknowns. One of the unknowns, f(t), appears on the right-hand side of the equations. The number of equations and the number of unknowns can be reduced to n-1 by eliminating the last equation. The resulting equations, in which all of the unknown quantities appear on the left-hand side, can be written as

$$\left[\begin{array}{cccc|c} m_1 & 0 & \dots & 0 & 0 \\ 0 & m_2 & & \cdot & \cdot \\ \cdot & & \ddots & \cdot & \cdot \\ \cdot & & \ddots & 0 & \cdot \\ 0 & \dots & 0 & m_{n-1} & 0 \end{array}\right] \left\{\begin{array}{c} \ddot{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \ddot{x}_n \end{array}\right\}$$

$$+ \left[\begin{array}{cccc|c} k_{11} & k_{12} & \cdots & k_{1,n-1} & k_{1n} \\ k_{21} & k_{22} & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ k_{n-1,1} & & \cdots & k_{n-1,n-1} & k_{n-1,n} \end{array}\right] \left\{\begin{array}{c} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{array}\right\} \qquad (2\text{-}2)$$

Since the known coordinate is x_n, the terms containing it may be placed on the right-hand side of the equations, as follows.

$$\begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & & \cdot \\ \cdot & & \ddots & \cdot \\ \cdot & & \ddots & 0 \\ 0 & \cdots 0 & & m_{n-1} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \ddot{x}_{n-1} \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1,n-1} \\ k_{21} & k_{22} & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ k_{n-1,1} & \cdots & \cdots & k_{n-1,n-1} \end{bmatrix} \begin{Bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \end{Bmatrix}$$

$$= - \begin{bmatrix} 0 & k_{1,n} \\ 0 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & k_{n-1,n} \end{bmatrix} \begin{Bmatrix} \ddot{x}_n \\ x_n \end{Bmatrix} \qquad (2\text{-}3)$$

These equations, called the reduced equations, are n-1 equations in n-1 unknowns. The reduced mass matrix and the reduced stiffness matrix of Eqs. 2-3 are symmetric since the nth row and column of the matrices of Eqs. 2-1 were removed. These equations may be solved by classical methods.

Following the usual procedure, the eigenvalues and and eigenvectors are obtained from the homogeneous equations. The equations of motion are then transformed to new coordinates q using

$$\{x\} = [\varphi]\{q\}$$

where $[\varphi]$ is the matrix of eigenvectors (modal matrix). The transformed equations are then premultiplied by $[\varphi]^T$. The resulting equations are uncoupled because of the orthogonality property of the eigenvectors. Hence, the forced equations may then be solved individually. Once all of the motions are known, the nth equation of the original set of equations, Eqs. 2-1, may be solved for f(t), the exciting force.

The reduced equations, Eqs. 2-3, are the equations of motion for the original system with the nth point fixed and with the i th point forced by $-k_{i,n}x_n$. Therefore, the eigenvalues and eigenvectors of the reduced system can be interpreted as those of the system with the nth point fixed.

Example 1

Consider the three degree of freedom longitudinal vibration problem shown in Fig. 2.1.

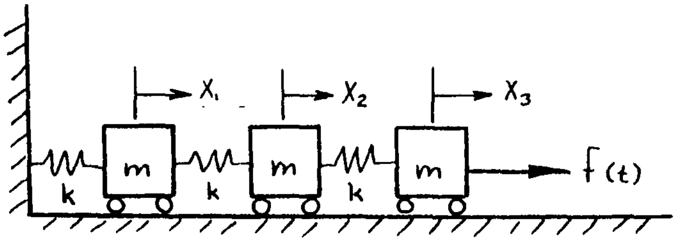

Fig. 2.1

Assume that $x_3 = x_3(t)$ is known and that $f(t)$ is unknown. The equations of motion are

$$m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ f(t) \end{Bmatrix}$$

The reduced equations can be obtained by dropping the last equation and putting terms involving the known coordinate on the right-hand side giving

$$m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + k \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = - \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} m\ \ddot{x}_3 \\ k\ x_3 \end{Bmatrix}$$

which are two equations in the two unknowns x_1 and x_2. Since the same row and column, the third, were removed, the matrices are symmetric and classical methods may be used to uncouple the equations. The uncoupled equations may be solved for the generalized coordinates. A transformation will yield x_1 and x_2. With x_1, x_2, and x_3 known, the last of the original equations

$$m\ddot{x}_3 - kx_2 + kx_3 = f(t)$$

may be solved for the unknown force, $f(t)$.

The reduced equations are the equations of motion for the system shown in Fig. 2.2.

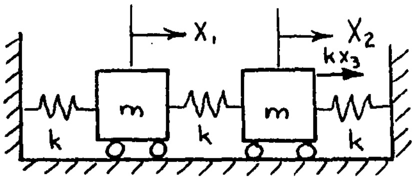

Fig. 2.2

Thus, the eigenvectors required for uncoupling the reduced equations of motion are those of the system with x_3 fixed.

2.2 Motion Given at a Point Other Than That of Application of the Unknown Force

The more general case of the inverse problem is one in which the point of known motion is not the same as the point of application of the unknown force. Consider again the system described in Sec. 2.1. The equations of motion are those of Sec. 2.1, written as

$$\begin{bmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & 0 \\ 0 & \dots & 0 & m_n \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \ddot{x}_n \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} & \dots & k_{1,n} \\ k_{21} & k_{22} & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ k_{n,1} & \dots & \dots & k_{n,n} \end{bmatrix} \begin{Bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ f(t) \end{Bmatrix} \tag{2-1}$$

Assume that the motion of some coordinate other than x_n is known. For convenience, assume that this known coordinate is x_{n-1}. As before, the unknown force may be removed by dropping the forced equation of motion (the nth equation), leading to

$$\left[\begin{array}{cccc|c} m_1 & 0 & \cdots & 0 & 0 \\ 0 & m_2 & & \cdot & \cdot \\ \cdot & & \ddots & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & 0 & \cdot \\ 0 & \cdots & 0 & m_{n-1} & 0 \end{array}\right] \left\{\begin{array}{c} \ddot{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \ddot{x}_n \end{array}\right\}$$

$$+ \left[\begin{array}{cccc|c} k_{11} & k_{12} & \cdots & k_{1,n-1} & k_{1n} \\ k_{21} & k_{22} & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ k_{n-1,1} & \cdots & \cdots & k_{n-1,n-1} & k_{n-1,n} \end{array}\right] \left\{\begin{array}{c} x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{array}\right\} \tag{2-2}$$

Since x_{n-1} is known, terms containing it may be placed on the right-hand side of the equations. Thus, the reduced equations may be written as

$$\left[\begin{array}{cccc|c} m_1 & 0 & \dots & 0 & 0 \\ 0 & m_2 & & \cdot & \cdot \\ \cdot & & \ddots & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & 0 & \cdot \\ 0 \dots 0 & & & m_{n-2} & \cdot \\ \hline 0 & \dots & \dots & \cdot & 0 \end{array}\right] \left\{\begin{array}{c} \ddot{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \ddot{x}_{n-2} \\ \hline \ddot{x}_n \end{array}\right\}$$

$$+ \left[\begin{array}{cccc|c} k_{11} & k_{12} & \cdot\cdot & k_{1,n-2} & k_{1,n} \\ k_{21} & k_{22} & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ k_{n-2,1} & & \cdots & k_{n-2,n-2} & k_{n-2,n} \\ \hline k_{n-1,1} & & \cdots & k_{n-1,n-2} & k_{n-1,n} \end{array}\right] \left\{\begin{array}{c} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-2} \\ \hline x_n \end{array}\right\}$$

$$= - \left[\begin{array}{cc} 0 & k_{1,n-1} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & \cdot \\ m_{n-1} & k_{n-1,n-1} \end{array}\right] \left\{\begin{array}{c} \ddot{x}_{n-1} \\ x_{n-1} \end{array}\right\} \tag{2-4}$$

Examination of these equations reveals that the reduced stiffness matrix is non-symmetric and the reduced mass matrix is singular. The reason for this condition is that in this case the column removed from the matrices was not the same as the row that was removed. In general, the inertia and stiffness matrices of the reduced equations will be non-symmetric if the point of known motion is not the same as the point of application of the unknown force. The shift from symmetric to non-symmetric matrices is the major distinction between the special case of Sec. 2.1 and the general case.

Usually, eigenvalues and eigenvectors for these reduced equations may be found. However, these eigenvectors do not exhibit the property of orthogonality with respect to the mass and stiffness matrices. Thus, they cannot be used directly to uncouple the reduced set of equations.

The treatment of non-symmetric systems outlined by Hildebrand (1)[1] and Halfman (2) utilizes the eigenvalues and eigenvectors of a new set of equations called the adjoint equations. A discussion of the method is given in Appendix A. The adjoint set of equations is defined

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[1] Numbers in parentheses refer to the list of references.

as the set of homogeneous reduced equations with the mass and stiffness matrices replaced by their transposes and the coordinate describing the motion of the forced point interchanged with the coordinate of the point where motion is known. If x_{n-1} is the known coordinate, the form of the adjoint equations is

$$\left[\begin{array}{cccc|c} m_1 & 0 & \ldots & 0 & 0 \\ 0 & m_2 & & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & 0 & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ 0 & \ldots & 0 & m_{n-2} & \cdot \\ \hline 0 & \ldots & \ldots & \ldots & 0 \end{array}\right] \left\{\begin{array}{c} \ddot{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \ddot{x}_{n-1} \end{array}\right\}$$

$$+ \left[\begin{array}{cccc|c} k_{11} & k_{21} & \cdots & k_{n-2,1} & k_{n-1,1} \\ k_{12} & k_{22} & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ k_{1,n-2} & \cdots & \cdots & k_{n-2,n-2} & k_{n-1,n-2} \\ \hline k_{1,n} & \cdots & \cdots & k_{n-2,n} & k_{n-1,n} \end{array}\right] \left\{\begin{array}{c} x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{array}\right\} \tag{2-5}$$

Mathematically, the eigenvectors of the reduced equations, Eqs. 2-4, and of the adjoint equations, Eqs. 2-5, may be found by classical methods, that is, by assuming a trial solution as

$$\{\ddot{x}\} = \{A\}\sin \omega t$$

where the A's are constants to be determined. This expression is substituted into the homogeneous reduced equations. For a nontrivial solution of the resulting homogeneous algebraic equations in the A's, the determinant of coefficients of the A's must vanish. The determinant is the characteristic equation and its solution yields the eigenvalues ω^2. These eigenvalues may be either positive or negative. Once the eigenvalues are known, the algebraic equations may be solved for the eigenvectors. Let $[\varphi]$ be the matrix of the eigenvectors of the reduced equations, Eqs. 2-4, and let $[\bar{\varphi}]$ be the matrix of the eigenvectors of the adjoint equations, Eqs. 2-5. These eigenvectors exhibit a property similar to orthogonality called bi-orthogonality. The biorthogonal relationships are

$$\left\lfloor \bar{\varphi}_{1i}\bar{\varphi}_{2i}\cdots\bar{\varphi}_{n-1,i} \right\rfloor \quad [m_R] \begin{Bmatrix} \varphi_{1,j} \\ \varphi_{2,j} \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{n-2,j} \\ \varphi_{n,j} \end{Bmatrix} \begin{matrix} = 0 \quad i \neq j \\ \\ \\ \\ \\ \\ = M_{ii} \quad i = j \end{matrix}$$

and

$$\left\lfloor \bar{\varphi}_{1i}\bar{\varphi}_{2i}\cdots\bar{\varphi}_{n-1,i} \right\rfloor \quad [k_R] \begin{Bmatrix} \varphi_{1,j} \\ \varphi_{2,j} \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{n-2,j} \\ \varphi_{n-j} \end{Bmatrix} \begin{matrix} = 0 \quad i \neq j \\ \\ \\ \\ \\ \\ = K_{ii} \quad i=j \end{matrix}$$

where $[m_R]$ is the non-symmetric reduced mass matrix and $[k_R]$ is the non-symmetric reduced stiffness matrix.

Thus,

$$[\bar{\varphi}]^T \; [m_R] \; [\varphi] = [\ulcorner M \lrcorner] \tag{2-6}$$

and

$$[\bar{\varphi}]^T \; [k_R] \; [\varphi] = [\ulcorner K \lrcorner] \tag{2-7}$$

where the diagonal matrix $[\ulcorner M \lrcorner]$ is defined as the generalized mass matrix and $[\ulcorner K \lrcorner]$ is defined as the generalized stiffness matrix. If the transformation

$$\{x\} = [\varphi]\,\{q\} \qquad (2\text{-}8)$$

is made in Eqs. 2-4 and these equations are premultiplied by $[\bar{\varphi}]^T$, the resulting equations will be uncoupled. Thus

$$[\bar{\varphi}]^T\,[m_R][\varphi]\{\ddot{q}\} + [\bar{\varphi}]^T\,[k_R][\varphi]\{q\} = [\bar{\varphi}]^T\,\{u\} \qquad (2\text{-}9)$$

where $\{u\}$ is the entire right-hand side of Eqs. 2-4. Applying the results of Eqs. 2-6 and 2-7 to Eq. 2-9 yields

$$[\ulcorner M \lrcorner]\{\ddot{q}\} + [\ulcorner K \lrcorner]\{q\} = [\bar{\varphi}\ ]^T\,\{u\}$$

These uncoupled equations may be solved individually.

The eigenvectors of the reduced equations may be better understood through a physical interpretation. The homogeneous part of the reduced set of equations, Eqs. 2-4, is simply the set of equations for the system with no external forces at x_1 through x_{n-1} and a node at x_{n-1}. Since the nth equation has been dropped, a force can exist at x_n. Thus, for positive eigenvalues, the eigenvectors are those of the system with a force-free node at the point of given motion and harmonic excitation at x_n. These could be found experimentally by applying a harmonic force at x_n and varying the frequency until a node appeared at x_{n-1}. The frequencies and mode shapes for which nodes appear are the eigenvalues and eigenvectors of the reduced equations. A similar interpretation is valid for negative eigenvalues. The only difference is that the excitation is exponential rather than harmonic.

The eigenvectors of the adjoint set of equations may be similarly interpreted. If, in the original homogeneous set of equations, Eqs. 2-1, the (n-1)th equation is removed and x_n set equal to zero, the result is

$$\left[\begin{array}{cccc|c} m_1 & 0 & \cdots & 0 & 0 \\ 0 & m_2 & & \cdot & \cdot \\ \cdot & & \ddots & \cdot & \cdot \\ \cdot & & & 0 & \cdot \\ 0 & \cdots & 0 & m_{n-2} & \cdot \\ \hline 0 & \cdots & \cdots & \cdots & 0 \end{array}\right] \begin{Bmatrix} \ddot{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \ddot{x}_{n-1} \end{Bmatrix}$$

$$+ \left[\begin{array}{cccc|c} k_{11} & k_{12} & \cdots & k_{1,n-2} & k_{1,n-1} \\ k_{21} & k_{22} & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ k_{n-2,1} & \cdots & \cdots & k_{n-2,n-2} & k_{n-2,n-1} \\ \hline k_{n,1} & \cdots & \cdots & k_{n,n-2} & k_{n,n-1} \end{array}\right] \begin{Bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix}$$

Since $m_{ij} = m_{ji}$ and $k_{ij} = k_{ji}$, these equations may also be written

$$\left[\begin{array}{cccc|c} m_1 & 0 & \dots & 0 & 0 \\ 0 & m_2 & & \cdot & \cdot \\ \cdot & & \ddots & \cdot & \cdot \\ \cdot & & \ddots & 0 & \cdot \\ 0 \dots 0 & & & m_{n-2} & \cdot \\ \hline 0 & \dots & \dots & \dots & 0 \end{array}\right] \begin{Bmatrix} \ddot{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \ddot{x}_{n-1} \end{Bmatrix}$$

$$+ \left[\begin{array}{cccc|c} k_{11} & k_{21} & \dots & k_{n-2,1} & k_{n-1,1} \\ k_{12} & k_{22} & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ k_{1,n-2} & \dots & & k_{n-2,n-2} & k_{n-1,n-2} \\ \hline k_{1,n} & \dots & \dots & k_{n-2,n} & k_{n-1,n} \end{array}\right] \begin{Bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} \tag{2-10}$$

These equations represent a system with a harmonic force applied at x_{n-1} (since the (n-1)th equation has been removed) and a node at x_n. Equations 2-10 are of exactly the same form as the adjoint equations, Eqs. 2-5. Hence, for positive eigenvalues, the eigenvectors of the adjoint equations are the shapes of the system with a harmonic force applied at the point of given motion and a force-free node at the point of application of the unknown force. For negative eigenvalues, the excitation is exponential rather than harmonic.

The equations developed so far have been for systems without inertial coupling. The form of the equations changes very little for inertially coupled systems. The reduced stiffness matrix is the same as shown in Eq. 2-4. The reduced mass matrix is altered to a form identical to the reduced stiffness matrix. The procedure used to uncouple the reduced equations remains unchanged.

Example 2

Consider the four-degree of freedom longitudinal vibration problem illustrated by Fig. 2.3.

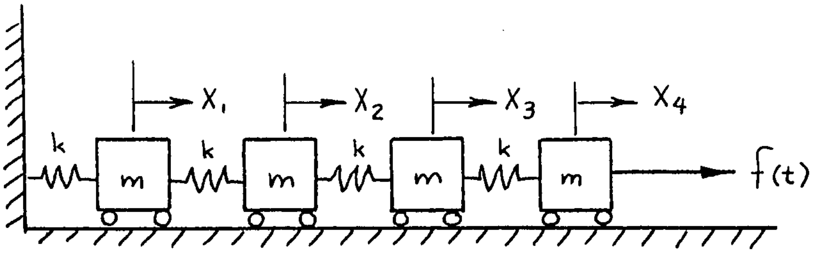

Fig. 2.3

Assume that x_3 is given and that f(t) is unknown. The equations of motion may be written as

$$m \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{Bmatrix} + k \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ f(t) \end{Bmatrix}$$

Dropping the forced equation and placing the terms involving the given coordinate on the right-hand side gives

$$m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_4 \end{Bmatrix} + k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_4 \end{Bmatrix} = - \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} m\,\ddot{x}_3 \\ k\,x_3 \end{Bmatrix}$$

To determine the eigenvectors of the homogeneous set of equations, assume

$$\{\ddot{x}\} = \{A\} \sin \omega t$$

Integration leads to the form of {x} as

$$\{x\} = -\frac{1}{\omega^2} \{A\} \sin \omega t$$

The resulting algebraic equations will possess a non-trivial solution only if the determinant of coefficients of the A's is zero.

Thus,

$$\begin{vmatrix} (\frac{2k}{\omega^2} - m) & -\frac{k}{\omega^2} & 0 \\ -\frac{k}{\omega^2} & (\frac{2k}{\omega^2} - m) & 0 \\ 0 & -\frac{k}{\omega^2} & -\frac{k}{\omega^2} \end{vmatrix} = 0$$

In this form, the characteristic equation is a cubic in $\frac{1}{\omega^2}$. The eigenvalues and eigenvectors are

$$\omega^2_{1,2,3} = \frac{k}{m};\ 3\ k/m;\ \infty$$

$$[\varphi] = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

The adjoint equations are

$$m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The resulting characteristic equation is

$$\begin{vmatrix} (\frac{2k}{\omega^2} - m) & -\frac{k}{\omega^2} & 0 \\ \frac{-k}{\omega^2} & (\frac{2k}{\omega^2} - m) & \frac{-k}{\omega^2} \\ 0 & 0 & \frac{-k}{\omega^2} \end{vmatrix} = 0$$

Therefore, the eigenvalues of the adjoint equations are the same as those of the reduced equations. The adjoint eigenvectors are

$$[\overline{\varphi}] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $\{x\} = [\varphi]\{q\}$ and premultiply the reduced equations by $[\overline{\varphi}]^T$.

The equations become

$$m \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + k \begin{bmatrix} -2 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = - \begin{bmatrix} 0 & -1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} m\ \ddot{x}_3 \\ k\ x_3 \end{Bmatrix}$$

which are uncoupled.

Example 3

The system for this example is the three degree of freedom lumped mass beam in bending shown in Fig. 2.4.

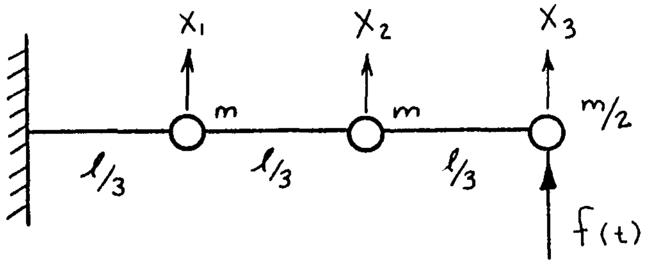

Fig. 2.4

Assume that x_2 is known and that f(t) is unknown. The equations of motion are

$$m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + k \begin{bmatrix} 80 & -46 & 12 \\ -46 & 44 & -16 \\ 12 & -16 & 7 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ f(t) \end{Bmatrix}$$

with

$$k = \frac{81}{13} \frac{EI}{\ell^3}$$

Removing the forced equation and placing the terms involving the given coordinate on the right-hand side yields the following reduced equations.

$$m \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_3 \end{Bmatrix} + k \begin{bmatrix} 80 & 12 \\ -46 & -16 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_3 \end{Bmatrix} = - \begin{bmatrix} 0 & -46 \\ 1 & 44 \end{bmatrix} \begin{Bmatrix} m\ \ddot{x}_2 \\ k\ x_2 \end{Bmatrix}$$

The adjoint equations are

$$m \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + k \begin{bmatrix} 80 & -46 \\ 12 & -16 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The two sets of eigenvectors are

$$[\varphi] = \begin{bmatrix} -.348 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and } [\bar{\varphi}] = \begin{bmatrix} 1.335 & 0 \\ 1 & 1 \end{bmatrix}$$

Substituting $\{x\} = [\varphi]\{q\}$ into the reduced equations and then premultiplying by $[\bar{\varphi}]^T$ gives the uncoupled equations

$$m \begin{bmatrix} -.464 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + k \begin{bmatrix} -21.1 & 0 \\ 0 & -16 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = - \begin{bmatrix} 1 & -17.4 \\ 1 & 44 \end{bmatrix} \begin{Bmatrix} m\ddot{x}_2 \\ kx_2 \end{Bmatrix}$$

Example 4

The complete details of this example are given in Appendix D. The system for this example is the quadruple pendulum shown in Fig. 2.5. This problem is of importance in that it shows that the method of uncoupling applies to systems that are inertially coupled as well as to those that are just elastically coupled.

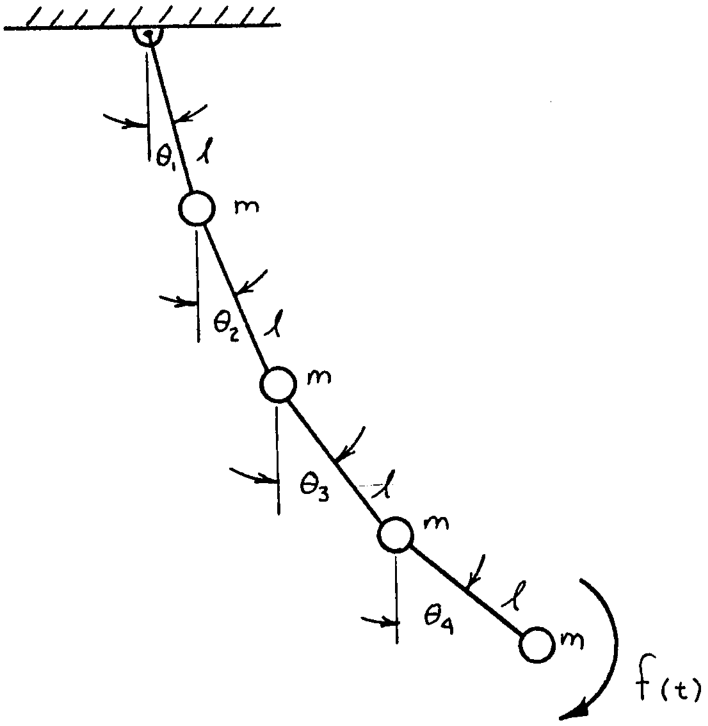

Fig. 2.5

In this case, f(t) represents an unknown couple applied to the last pendulum. Assume θ_3 is known. The equations of motion are

$$M \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \end{Bmatrix} + K \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ f(t) \end{Bmatrix}$$

where $M = m\ell^2$ $\quad K = mg\ell$

Removing the last equation and placing the terms containing θ_3 on the right-hand side gives the reduced equations

$$M \begin{bmatrix} 4 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 2 & 1 \end{bmatrix} \left\{ \begin{matrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_4 \end{matrix} \right\} + K \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left\{ \begin{matrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{matrix} \right\} = - \begin{bmatrix} 2 & 0 \\ 2 & 0 \\ 2 & 2 \end{bmatrix} \left\{ \begin{matrix} M\ \ddot{\theta}_3 \\ K\ \theta_3 \end{matrix} \right\}$$

The eigenvectors of these reduced equations are

$$[\varphi] = \begin{bmatrix} 0 & -.266 & .945 \\ 0 & -.234 & -1.445 \\ 1 & 1 & 1 \end{bmatrix}$$

and those of the adjoint equations are

$$[\bar{\varphi}] = \begin{bmatrix} 0 & -.533 & 1.87 \\ 0 & -.467 & -2.87 \\ 1 & 1 & 1 \end{bmatrix}$$

The uncoupled equations are

$$M \begin{bmatrix} 1 & 0 & 0 \\ 0 & .6425 & 0 \\ 0 & 0 & 2.258 \end{bmatrix} \left\{ \begin{matrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{matrix} \right\} + K \begin{bmatrix} 0 & 0 & 0 \\ 0 & .896 & 0 \\ 0 & 0 & 19.51 \end{bmatrix} \left\{ \begin{matrix} q_1 \\ q_2 \\ q_3 \end{matrix} \right\}$$

$$= - \begin{bmatrix} 2 & 2 \\ 0 & 2 \\ 0 & 2 \end{bmatrix} \left\{ \begin{matrix} M\ \ddot{\theta}_3 \\ K\ \theta_3 \end{matrix} \right\}$$

2.3 Solution of the Uncoupled Equations; Uniqueness

The equations of the inverse problem were reduced to a set of uncoupled equations in the preceding sections. These may now be solved one at a time. Consider the classical form of the solution, the sum of the particular solution and the homogeneous solution, written as

$$q_i = q_{hi} + q_{pi}$$

The constants of the homogeneous solution are evaluated from the initial conditions.

The homogeneous solution is generally harmonic or exponential motion involving the eigenvalues of the reduced system. These eigenvalues will not, in general, be true natural frequencies of the system, but merely artificial ones generated by the mathematics. As they are not genuine natural frequencies of the system, one would not expect them to appear in the final solution. In other words, from physical considerations

$$q_{hi} \equiv 0$$

Thus, the complete solution is expected to be just the particular solution, or

$$q_i = q_{pi}$$

The homogeneous solution may exist, however, if the forcing function includes components which involve the eigenvalues of the reduced system. In Sec. 2.2, the eigenvalues and the eigenvectors of the reduced equations were interpreted as the motion that results from a harmonic or exponential forcing function applied at the point of application of the unknown force such that a force-free node occurs at the point of known motion. Thus, it is physically possible for a force of this character to exist, without causing motion at the point of measurement. Hence, it is impossible to determine whether components of the forcing function that lead to homogeneous solutions exist or not. For this reason, the only part of the solution that can be evaluated is the particular solution and the solutions achieved are unique only if the unknown force contains no harmonic or exponential components involving the eigenvalues of the reduced system.

In a real situation, this uncertainty can be avoided since the eigenvalues and eigenvectors of the reduced system depend upon the location of the point of motion measurement. The complete solution can be established by taking measurements at two locations, computing the unknown force based on each set of data, and comparing the two solutions.

Example 5

This example is a continuation of the three degree of freedom beam in bending problem discussed in Example 3. Assume that the measured response of x_2 is

$$x_2=[.1729-.177 \cos \bar{\omega}_1 t+.00388 \cos \bar{\omega}_2 t+.000253 \cos \bar{\omega}_3 t] \frac{F_o L^3}{EI}$$

where

$$\bar{\omega}_1^2 = 3.73 \frac{EI}{ML^3}$$

$$\bar{\omega}_2^2 = 119 \frac{EI}{ML^3}$$

$$\bar{\omega}_3^2 = 738 \frac{EI}{ML^3}$$

and that the beam is initially at rest.

The uncoupled equations from Example 3 are

$$m \begin{bmatrix} -.464 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + K \begin{bmatrix} -21.1 & 0 \\ 0 & -16 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$= - \begin{bmatrix} 1 & -17.4 \\ 1 & 44 \end{bmatrix} \begin{Bmatrix} m \ddot{x}_2 \\ K x_2 \end{Bmatrix}$$

The initial conditions are

$$x_1(0) = \dot{x}_1(0) = 0$$

$$x_2(0) = \dot{x}_2(0) = 0$$

$$x_3(0) = \dot{x}_3(0) = 0$$

These, transformed, give the initial conditions in the new coordinates as

$$q_1(0) = \dot{q}_1(0) = 0$$
$$q_2(0) = \dot{q}_2(0) = 0$$

The first equation becomes

$$\ddot{q}_1+45.5\,\frac{K}{m}\,q_1=2.15\,\frac{F_o}{m}\,[.66 \cos \overline{\omega}_1 t-.462 \cos \overline{\omega}_2 t-.185 \cos \overline{\omega}_3 t]$$
$$-\,37.5\,\frac{F_oL^3}{EI}\,\frac{K}{m}\,[.1729-.177 \cos \overline{\omega}_1 t+.00388 \cos \overline{\omega}_2 t$$
$$+\,.000253 \cos \overline{\omega}_3 t]$$

The solution for q_1 is

$$q_1 = q_{h1} + q_{p1}$$

$$q_1 = q_{h1} + \left(\frac{F_oL^3}{EI}\right)\,(-.142+.153 \cos \overline{\omega}_1 t-.0116 \cos \overline{\omega}_2 t$$
$$+\,.001 \cos \overline{\omega}_3 t)$$

Since q_{p1} and $\dot{q}_{p1}$ are zero at $t = 0$, it is obvious that

$$q_{h1}(0) = 0$$
$$\dot{q}_{h1}(0) = 0$$

and hence

$$q_{h1} \equiv 0$$

The second equation yields q_2 directly.

$$Kq_2 = .062\,\frac{K\,F_oL^3}{EI}\,[.66 \cos \overline{\omega}_1 t-.462 \cos \overline{\omega}_2 t-.185 \cos \overline{\omega}_3 t]$$
$$+\,.275\,K\,\frac{F_oL^3}{EI}\,[.1729-.177 \cos \overline{\omega}_1 t+.00388 \cos \overline{\omega}_2 t$$
$$+\,.000253 \cos \overline{\omega}_3 t]$$

Thus,

$$q_2 = \frac{F_o L^3}{EI} [.475 - .48 \cos \bar{\omega}_1 t + .00616 \cos \bar{\omega}_2 t$$

$$- .001155 \cos \bar{\omega}_3 t]$$

In this case, the homogeneous equation is $q_2 = 0$. Thus, the homogeneous solution again vanishes.

The solution in the original coordinates is given by

$$\{x\} = [\varphi]\{q\}$$

The last of the original equations from Example 3 may be used to solve for f(t), written as

$$\frac{1}{2} m \ddot{x}_3 + K\ 12\ x_1 - 16\ K\ x_2 + 7\ x_3 = f(t)$$

The result is

$$f(t) = F_o \qquad \text{(a step input)}$$

The data (x_2 and initial conditions) for this problem and the final solution for f(t) correspond to the force-input problem number 3 in Appendix C.

Example 6

This example is a continuation of the quadruple pendulum problem of Example 4. It is given here to show the character of the homogeneous solution for an inertially coupled system.

Assume that the given motion is

$$\theta_3 = \frac{F_o}{K} [-.164 \cos \bar{\omega}_1 t - .0793 \cos \bar{\omega}_2 t + .223 \cos \bar{\omega}_3 t + .0187 \cos \bar{\omega}_4 t]$$

where

$$\bar{\omega}_1^2 = .322 \frac{K}{M} \qquad \bar{\omega}_3^2 = 4.53 \frac{K}{M}$$

$$\bar{\omega}_2^2 = 1.745 \frac{K}{M} \qquad \bar{\omega}_4^2 = 9.395 \frac{K}{M}$$

and that the system was initially at rest. These initial conditions transform to

$$q_1(0) = \dot{q}_1(0) = 0$$
$$q_2(0) = \dot{q}_2(0) = 0$$
$$q_3(0) = \dot{q}_3(0) = 0$$

The uncoupled equations from Example 4 are

$$M \begin{bmatrix} 1 & 0 & 0 \\ 0 & .6425 & 0 \\ 0 & 0 & 2.258 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + K \begin{bmatrix} 0 & 0 & 0 \\ 0 & .896 & 0 \\ 0 & 0 & 19.51 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

$$= - \begin{bmatrix} 2 & 2 \\ 0 & 2 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} M \ddot{\theta}_3 \\ K \theta_3 \end{Bmatrix}$$

The first equation is

$$M \ddot{q}_1 = - 2 M \ddot{\theta}_3 - 2 K \theta_3$$

Integration yields

$$q_1 = \frac{F_o}{K} [-.688 \cos \bar{\omega}_1 t + .068 \cos \bar{\omega}_2 t - .347 \cos \bar{\omega}_3 t$$

$$-.033 \cos \bar{\omega}_4 t + At + B]$$

From the initial conditions on q_1

$$A = 0 \quad \text{and} \quad B = 1.0$$

Thus,

$$q_1 = \frac{F_o}{K} [1-.690 \cos \bar{\omega}_1 t + .062 \cos \bar{\omega}_2 t - .347 \cos \bar{\omega}_3 t$$

$$- .0334 \cos \bar{\omega}_4 t]$$

The right-hand side of the equations for q_2 and q_3 contain only θ_3. As $\theta_3(0) = 0$, then

$$M \ddot{q}_p(0) + K q_p(0) = 0$$

Since the nature of the particular solution is cosines, the only way this equation can be satisfied is if

$$\ddot{q}_p(0) = q_p(0) = 0$$

From the initial conditions,

$$q(0) = 0$$

and thus

$$q_{h2} \equiv 0$$

and

$$q_{h3} \equiv 0$$

The rest of the solution is similar in nature to the solution in Example 5. The data for this example and the final result compare with that of the force-input problem number 5 in Appendix C.

Example 7

This example is a continuation of Example 2. Assume

$$x_3 = .333 \frac{F_o}{k} \sin 2 \sqrt{\frac{k}{m}}\, t$$

and that the initial conditions are

$$x_1(0) = 0 \qquad \dot{x}_1(0) = .222 \frac{F_o}{k}$$

$$x_2(0) = 0 \qquad \dot{x}_2(0) = -\ .444 \frac{F_o}{k}$$

$$x_3(0) = 0 \qquad \dot{x}_3(0) = .666 \frac{F_o}{k}$$

$$x_4(0) = 0 \qquad \dot{x}_4(0) = -\ .888 \frac{F_o}{k}$$

These initial conditions are the ones necessary for a pure steady-state solution for problem number 1 in Appendix C.

From Appendix D the particular solutions for this problem are

$$q_{p1} = .0555 \frac{F_o}{k} \sin 2\sqrt{\frac{k}{m}}\, t$$

$$q_{p2} = .1666 \frac{F_o}{k} \sin 2\sqrt{\frac{k}{m}}\, t$$

$$q_{p3} = -\ .666 \frac{F_o}{k} \sin 2\sqrt{\frac{k}{m}}\, t$$

Now,

$$\{x\} = [\varphi]\{q\}$$

so

$$x_{p1} = \frac{F_o}{k} (-.0555 \sin 2\sqrt{\frac{k}{m}}\, t + .1666 \sin 2\sqrt{\frac{k}{m}}\, t)$$

$$x_{p2} = \frac{F_o}{k} (-.0555 \sin 2\sqrt{\frac{k}{m}}\, t - .1666 \sin 2\sqrt{\frac{k}{m}}\, t)$$

$$x_{p4} = \frac{F_o}{k}(.0555 \sin 2\sqrt{\frac{k}{m}}\, t + .1666 \sin 2\sqrt{\frac{k}{m}}\, t - .666 \sin 2\sqrt{\frac{k}{m}}\, t)$$

Thus

$$x_{p1}(0) = 0$$

$$x_{p2}(0) = 0$$

$$x_{p4}(0) = 0$$

and

$$\dot{x}_{p1}(0) = .222 \frac{F_o}{k}$$

$$\dot{x}_{p2}(0) = - .444 \frac{F_o}{k}$$

$$\dot{x}_{p4}(0) = - .888 \frac{F_o}{k}$$

However,

$$x_i = x_{hi} + x_{pi}$$

but

$$x_i(0) = x_{pi}(0)$$

and

$$\dot{x}_i(0) = \dot{x}_{pi}(0)$$

Consequently,

$$x_{hi}(0) = \dot{x}_{hi}(0) = 0$$

and as a result

$$x_{hi} \equiv 0$$

This example further illustrates that the artificial eigenvalues do not appear in the homogeneous solution for non-homogeneous initial conditions.

CHAPTER 3

SYSTEMS FOR WHICH THE BASIC METHODS ARE ONLY PARTIALLY SUCCESSFUL

If the reduced equations of motion could always be uncoupled by the methods of Chap. 2, the inverse problem would be solved. Unfortunately, however, in some cases these methods are only partially successful. Chap. 3 will be concerned with these cases.

3.1 Close-Coupled Systems

For an n degree of freedom system, the number of reduced equations is n-1. If these equations are to be uncoupled by the methods outlined in Chap. 2, n-1 eigenvectors of the reduced equations and n-1 eigenvectors of the adjoint equations are required. For close-coupled systems, systems in which each mass is coupled only to its immediate neighbors, neither set of equations will in general possess the full set of n-1 eigenvectors.

According to Hildebrand (1), if the roots of the characteristic equation are not distinct there may be less than n-1 linearly independent eigenvectors. Examples indicate that if there are j distinct roots there will be only j eigenvectors. For close-coupled systems, the characteristic equation in general possesses multiple infinite roots. Only one eigenvector is associated with these multiple roots. Thus there will be an insufficient number of eigenvectors to completely uncouple the equations of motion.

Example 8

The system for this Example is the same as for Example 2. The only difference is that x_2 rather than x_3 is assumed to be the known coordinate, as shown in Fig. 3.1.

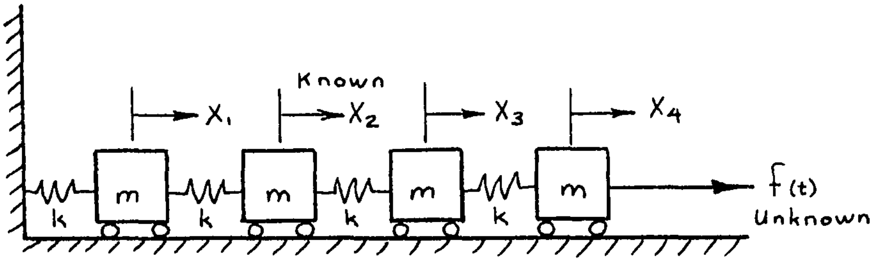

Fig. 3.1

Using the methods of Chap. 2, the reduced equations are

$$m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{Bmatrix} + k \begin{bmatrix} 2 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_3 \\ x_4 \end{Bmatrix} = - \begin{bmatrix} 0 & -1 \\ 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} m\ddot{x}_2 \\ kx_2 \end{Bmatrix} \tag{3-1}$$

The characteristic equation is

$$\begin{vmatrix} (\frac{-2k}{\omega^2} + m) & 0 & 0 \\ \frac{+k}{\omega^2} & \frac{+k}{\omega^2} & 0 \\ 0 & (\frac{-2k}{\omega^2} + m) & \frac{+k}{\omega^2} \end{vmatrix} = 0$$

or

$$\frac{1}{\omega^4} \left(\frac{2k}{m\omega^2} - 1\right) \left(\frac{k}{m}\right)^2 = 0$$

Then the eigenvalues are

$$\omega^2 = 2\,\frac{k}{m},\ \infty,\ \infty$$

The eigenvector associated with the finite eigenvalue is

$$\{\varphi\}_1 = \begin{Bmatrix} -1 \\ 1 \\ 0 \end{Bmatrix}$$

Further, the eigenvector associated with the pair of infinite eigenvalues is

$$\{\varphi\}_2 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

These are the only two independent eigenvectors of the system. Similarly, the adjoint system will have only two eigenvectors, given by

$$[\bar{\varphi}]_{1,2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Consequently, the three reduced equations cannot be completely uncoupled.

Δ[2]

Comparison of the results of Examples 2 and 8 shows that the number of distinct roots is determined by the relative locations of the known displacement and the unknown force. It appears from these examples that a close-coupled system will have all distinct roots only if the known coordinate is adjacent to the forced coordinate.

---

[2] The symbol Δ indicates the end of the example.

A closer examination of the behavior of close-coupled systems is needed. Consider a longitudinal system of n masses for which x_i is known, as shown in Fig. 3.2.

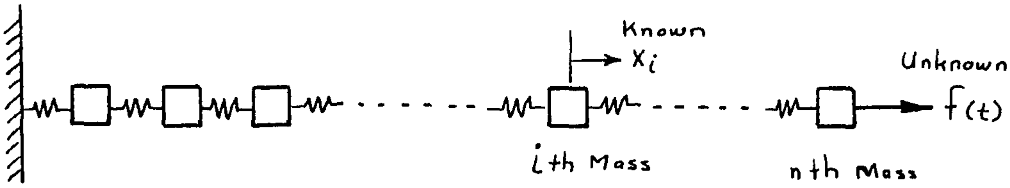

Fig. 3.2

The equations of motion are

$$\begin{bmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & & . \\ . & & & . \\ . & & & . \\ . & & & 0 \\ 0 & \dots & 0 & m_n \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ . \\ . \\ . \\ . \\ \ddot{x}_n \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} & 0 & \dots & 0 \\ k_{21} & k_{22} & & & . \\ 0 & & & & . \\ . & & & & 0 \\ . & & & & k_{n-1,n} \\ 0 & \dots & 0 & k_{n,n-1} & k_{n,n} \end{bmatrix} \begin{Bmatrix} x_1 \\ . \\ . \\ . \\ . \\ x_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ . \\ . \\ . \\ 0 \\ f(t) \end{Bmatrix}$$

Removing the last equation and placing the terms involving the given motion on the right side yields

$$
\left[\begin{array}{cccc|cccc}
m_1 & 0 & \cdots & 0 & & & & \\
0 & m_2 & & \cdot & & & & \\
\cdot & & \ddots & 0 & & 0 & & \\
0 & \cdots & 0 & m_{i-1} & & & & \\
\hline
 & & & & 0 & \cdots & 0 & 0 \\
 & & & & m_{i+1} & & \cdot & \cdot \\
 & 0 & & & \cdot & \ddots & \cdot & \cdot \\
 & & & & \cdot & \ddots & \cdot & \cdot \\
 & & & & \cdot & & \ddots & \cdot \\
 & & & & 0 \cdots 0 & & m_{n-1} & 0
\end{array}\right]
\left\{\begin{array}{c}
\ddot{x}_1 \\ \cdot \\ \cdot \\ \ddot{x}_{i-1} \\ \hline \ddot{x}_{i+1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \ddot{x}_n
\end{array}\right\}
$$

$$
+\left[\begin{array}{ccccc|ccccc}
k_{11} & k_{12} & 0 & \cdots & 0 & & & & & \\
k_{21} & k_{22} & & & 0 & & & & & \\
0 & & & & k_{i-2,i-1} & & & 0 & & \\
0 & \cdots & 0 & k_{i-1,i-2} & k_{i-1,i-1} & & & & & \\
\hline
0 & \cdots & \cdots & 0 & k_{i,i-1} & k_{i,i+1} & 0 & \cdots & \cdots & 0 \\
\cdot & & & & 0 & k_{i+1,i+1} & \ddots & & & \cdot \\
\cdot & & & & \cdot & k_{i+2,i+1} & \ddots & \ddots & & \cdot \\
\cdot & & & & \cdot & 0 & \ddots & \ddots & \ddots & \cdot \\
\cdot & & & & \cdot & \cdot & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 0 \cdots 0 & & k_{n-1,n-2} & k_{n-1,n-1} & k_{n-1,n}
\end{array}\right]
\left\{\begin{array}{c}
x_1 \\ \cdot \\ \cdot \\ x_{i-1} \\ \hline x_{i+1} \\ \cdot \\ \cdot \\ \cdot \\ x_n
\end{array}\right\}
$$

$$= -\left[\begin{array}{cc} 0 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & 0 \\ 0 & k_{i-1,i} \\ \hline m_i & k_{i,i} \\ 0 & k_{i+1,i} \\ \cdot & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & 0 \end{array}\right] \begin{Bmatrix} \ddot{x}_i \\ \\ x_i \end{Bmatrix} \qquad (3\text{-}2)$$

As indicated, the reduced equations have been partitioned into two sets of equations, the first i-1 equations and the remaining n-i equations.

The first set of i-1 equations is

$$\begin{bmatrix} m_1 & 0 & \cdots & \cdots & 0 \\ 0 & m_2 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & m_{i-1} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \vdots \\ \vdots \\ \vdots \\ \ddot{x}_{i-1} \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} & 0 & \cdots & 0 \\ k_{21} & k_{22} & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & k_{i-2,i-1} \\ 0 \cdots 0 & & & k_{i-1,i-2} & k_{i-1,i-1} \end{bmatrix} \begin{Bmatrix} x_1 \\ \vdots \\ \vdots \\ \vdots \\ x_{i-1} \end{Bmatrix}$$

$$= - \begin{bmatrix} 0 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & 0 \\ 0 & k_{i-1,i} \end{bmatrix} \begin{Bmatrix} \ddot{x}_i \\ x_i \end{Bmatrix} \tag{3-3}$$

This set of equations is not coupled to the other set and may be solved separately. More important, the coefficient matrices are symmetric. Consequently, these equations may be solved using classical methods (as in the case of Example 1). i-1 distinct eigenvalues and eigenvectors are associated with this set of equations. In Example 8, the first i-1 equations of the reduced equations, Eq. 3-1, is the first one equation. Note that it is not coupled to the remaining set of equations and that it may be solved by itself.

The last n-i equations can be written

$$\begin{bmatrix} 0 & \cdots & \cdots & 0 \\ m_{i+1} & & & \vdots \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 \cdots 0 & & m_{n-1} & 0 \end{bmatrix} \begin{Bmatrix} \ddot{x}_{i+1} \\ \vdots \\ \vdots \\ \ddot{x}_n \end{Bmatrix}$$

$$+ \begin{bmatrix} k_{i,i+1} & 0 & \cdots & \cdots & 0 \\ k_{i+1,i+1} & & & & \vdots \\ k_{i+2,i+1} & \ddots & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 \cdots 0 & k_{n-1,n-2} & & k_{n-1,n-1} & k_{n-1,n} \end{bmatrix} \begin{Bmatrix} x_{i+1} \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{Bmatrix}$$

$$= - \begin{bmatrix} k_{i,i-1} & m_i & k_{i,i} \\ 0 & 0 & k_{i+1,i} \\ \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_{i-1} \\ \ddot{x}_i \\ x_i \end{Bmatrix} \qquad (3\text{-}4)$$

These equations can be solved one at a time. Having solved the first i-1 equations, the coordinate x_{i-1} is known. Then the only unknown quantity in the i th equation of Eqs. 3-4 is the spring force $k_{i,i+1}\ x_{i+1}$. Thus, the coordinate x_{i+1} can be determined. Then the only unknown quantity in the (i+1) th equation is the spring force $k_{i+1,i+2}\ x_{i+2}$ and x_{i+2} can be determined. The solution for the remaining coordinates proceeds step-wise to the right. The equations to be solved are of algebraic character in the unknown coordinate, that is, they contain no derivatives of the unknown coordinate. If the methods of Chap. 2 are applied to this set of equations, the result will be n-i infinite eigenvalues. This fact can be observed by referring to Example 8. The two infinite eigenvalues are the result of the last n-i (two) reduced equations. Only one eigenvector is associated with the n-i infinite eigenvalues, which means that the system has a total of only i eigenvectors. Evidently, the n-1 reduced equations can only be simultaneously uncoupled if i = n-1. This is apparent from Examples 2 and 4 in Chap. 2.

On the other hand, beams in bending which have more remote coupling appear to have distinct eigenvalues regardless of the relative positions of the given coordinate and the forced coordinate. This fact will be illustrated in Example 9.

Example 9

The system for this example is the same three degree of freedom beam in bending as that in Example 3. Assume that x_1 is given rather than x_2 as in Example 3. Then the reduced equations are

$$m \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + K \begin{bmatrix} -46 & 12 \\ 44 & -16 \end{bmatrix} \begin{Bmatrix} x_2 \\ x_3 \end{Bmatrix} = - \begin{bmatrix} 1 & 80 \\ 0 & -46 \end{bmatrix} \begin{Bmatrix} m\ddot{x} \\ Kx_1 \end{Bmatrix}$$

The characteristic equation is

$$\begin{vmatrix} \frac{+46K}{\omega^2} & \frac{-12K}{\omega^2} \\ (\frac{-44K}{\omega^2} + m) & \frac{+16K}{\omega^2} \end{vmatrix} = 0$$

Thus

$$\omega^2 = - 17.32 \frac{K}{m}, \infty$$

The eigenvectors are

$$[\varphi] = \begin{bmatrix} .261 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad [\bar{\varphi}] = \begin{bmatrix} 1.335 & 1 \\ 1 & 0 \end{bmatrix}$$

Consequently, the reduced equations can be uncoupled as

$$m \begin{bmatrix} .261 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + K \begin{bmatrix} -4.5 & 0 \\ 0 & 12 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = - \begin{bmatrix} 1.335 & 61 \\ 1 & 80 \end{bmatrix} \begin{Bmatrix} m\ddot{x}_1 \\ Kx_1 \end{Bmatrix}$$

The appearance of the negative eigenvalue in this example should be noted. The significance of the homogeneous solutions of the uncoupled equations was discussed in Sec. 2.3. It was indicated in the discussion that unless the unknown force excitation contains terms which involve the eigenvalues of the reduced equations, the homogeneous solution must vanish. Thus, the appearance of negative eigenvalues in this example does not imply divergent motion.

This example suggests that for beams in bending, which are remote coupled, the methods of Chap. 2 are applicable regardless of the relative positions of the forced coordinate and the given coordinate.

3.2 Further Treatment of Close-Coupled Systems

It was shown in Sec. 3.1 that the reduced equations of motion for the longitudinal system of Fig. 3.2 can be divided into two sets. The first set, Eq. 3-3 can be solved conveniently using classical methods. As shown, only one eigenvector is associated with the second set of equations, Eqs. 3-4 and the methods of Chap. 2 are not fruitful. They can be solved step by step as indicated in Sec. 3.1, but that would be very time-consuming for a many-degree of freedom system. Therefore, a more efficient set of coordinates is desirable; that is, a set in which a number of coordinates can be ignored and still give acceptable approximate results. This section is concerned with the search for more efficient coordinates.

Consider, for example, the n degree of freedom longitudinal system of Fig. 3.2 with an unknown force at x_n and, for simplicity, the known motion at x_1. With i=1, the first set of equations, Eqs. 3-3, is not present. All n-1 of the reduced equations belong to the second set of equations, Eqs. 3-4. These equations can, of course, be solved one at a time. A complete solution for the motion requires the solution of all n-1 of the reduced equations. Substitution of the motion solution into the original nth equation yields the unknown applied force.

An alternate approach to the solution of this problem is to transform the original equations of motion to a more efficient set of coordinates.

The equations of motion are

$$\begin{bmatrix} m_1 & 0 & \cdots & \cdots & 0 \\ 0 & m_2 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & m_n \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \vdots \\ \vdots \\ \ddot{x}_n \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} & 0 & \cdots & \cdots & 0 \\ k_{21} & k_{22} & & & & \vdots \\ 0 & & & & & \vdots \\ \vdots & & & & & 0 \\ \vdots & & & & & k_{n-1,n} \\ 0 & \cdots & \cdots & 0 & k_{n,n-1} & k_{n,n} \end{bmatrix} \begin{Bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ f(t) \end{Bmatrix} \qquad (3\text{-}5)$$

Transform these equations to a new set of coordinates y with

$$\{x\} = [\Phi]\ \{y\}$$

Ideally, only one of the new coordinates, taken to be y_1, should contain the known coordinate, in this case x_1. Then y_1 will be the known coordinate in the new coordinates. Similarly, only one of the new coordinates, selected as y_n, should contain the coordinate x_n associated with the unknown force. This insures that after transformation only one equation will contain the unknown force. Thus, as in the method of Chap. 2, one unknown can be removed by dropping one equation. Hence the desired form of the transformation matrix is

$$[\Phi] = \begin{bmatrix} \Phi_{11} & 0 \ldots\ldots\ldots\ldots & 0 \\ \Phi_{21} & \ldots\ldots\ldots\ldots & \Phi_{2,n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \Phi_{n-1,1} & \ldots\ldots\ldots\ldots & \Phi_{n-1,n} \\ 0 & \ldots\ldots\ldots\ldots 0 & \Phi_{n,n} \end{bmatrix} \tag{3-6}$$

The choice of the shapes involved in the first and last columns of Eq. 3-6 is somewhat arbitrary. Examples indicate that an excellent choice is the static shape for a unit displacement at x_1 and no displacement at x_n for the first column and the static shape for a unit displacement at x_n and no displacement at x_1 for the last column. The remainder of the shapes are taken to be the normal mode shapes of the system with both x_1 and x_n fixed. It is well known that the normal mode shapes of a system serve as the basis for a very efficient set of coordinates. The actual transformation matrix may be written as

$$[\Phi] = \begin{bmatrix} 1 & 0 \cdots\cdots\cdots\cdots\cdots\cdots & 0 \\ \Phi_{21} & & \Phi_{2n} \\ \cdot & & \cdot \\ \cdot & \boxed{\quad \Phi_f \quad} & \cdot \\ \cdot & & \cdot \\ \Phi_{n-1,1} & & \Phi_{n-1,n} \\ 0 & \cdots\cdots\cdots\cdots\cdots\cdots 0 & 1 \end{bmatrix} \tag{3-7}$$

where the first and last columns are the two static shapes discussed and $[\Phi_f]$ is the matrix of normal mode shapes with $x_1 = x_n = 0$.

Premultiplication of the transformed equations by $[\Phi]^T$ yields

$$[\Phi]^T[m][\Phi]\{\ddot{y}\} + [\Phi]^T[k][\Phi]\{y\} = [\Phi]^T\{F\} \tag{3-8}$$

Let

$$[\Phi]^T[m][\Phi] = [M]$$

$[M]$ will have the form

$$\begin{bmatrix} M_{11} & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & M_{1n} \\ \cdot & & \cdot \\ \cdot & \boxed{\qquad M_f \qquad} & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ M_{n,1} & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & M_{nn} \end{bmatrix}$$

where $[M_f]$ is the diagonal generalized mass matrix of the system with x_1 and x_n fixed.

Also if

$$[K] = [\Phi]^T[K][\Phi]$$

then $[K]$ will have the form

$$\begin{bmatrix} K_{11} & \cdots\cdots\cdots & K_{1,n} \\ \vdots & & \vdots \\ \vdots & \boxed{\quad K_f \quad} & \vdots \\ \vdots & & \vdots \\ K_{n,1} & \cdots\cdots\cdots & K_{n,n} \end{bmatrix}$$

where $[K_f]$ is the diagonal generalized stiffness matrix of the system with x_1 and x_n fixed. Examples show that the only terms of $[K]$ that are non-zero are $[K_f]$, $K_{1\ 1}$, $K_{1\ n}$, $K_{n\ 1}$, and $K_{n\ n}$.

The form of $[K]$ becomes

$$\begin{bmatrix} K_{11} & 0\cdots\cdots\cdots 0 & K_{1,n} \\ 0 & & 0 \\ \vdots & \boxed{\quad K_f \quad} & \vdots \\ 0 & & 0 \\ K_{n,1} & 0\cdots\cdots\cdots 0 & K_{n,n} \end{bmatrix}$$

Since the only term in the last column of $[\Phi]^T$ is the 1, the new force matrix may be seen to be

$$[\Phi]^T\{F\} = \{F\}$$

The form of Eqs. 3-8 becomes

$$\begin{bmatrix} M_{11} & \cdots\cdots\cdots & M_{1,n} \\ \cdot & \begin{array}{|ccc|}\hline & & \\ & M_f & \\ & & \\ \hline\end{array} & \cdot \\ \cdot & & \cdot \\ M_{n,1} & \cdots\cdots\cdots & M_{n,n} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{y}_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \ddot{y}_n \end{Bmatrix}$$

$$+ \begin{bmatrix} K_{11} & 0 \cdots\cdots 0 & K_{1,n} \\ 0 & \begin{array}{|ccc|}\hline & & \\ & K_f & \\ & & \\ \hline\end{array} & 0 \\ \cdot & & \cdot \\ 0 & & 0 \\ K_{n,1} & 0 \cdots\cdots 0 & K_{n,n} \end{bmatrix} \begin{Bmatrix} x_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ f(t) \end{Bmatrix}$$

The unknown f(t) may now be removed by dropping the last equation and the terms containing the known motion, x_1, may be placed on the right side of the equation, leading to

$$\left[\begin{array}{ccc|c} M_{1,2} & \cdots & \cdots & M_{1,n} \\ \hline & & & \cdot \\ & M_f & & \cdot \\ & & & \cdot \\ & & & \cdot \\ & & & M_{n-1,n} \end{array}\right] \left\{\begin{array}{c} \ddot{y}_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \ddot{y}_n \end{array}\right\}$$

$$+ \left[\begin{array}{ccc|c} 0 & \cdots & 0 & K_{1,n} \\ \hline & & & 0 \\ & & & \cdot \\ & K_f & & \cdot \\ & & & \cdot \\ & & & 0 \end{array}\right] \left\{\begin{array}{c} y_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{array}\right\} = - \left[\begin{array}{cc} M_{11} & K_{11} \\ \cdot & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ M_{n-1,1} & 0 \end{array}\right] \left\{\begin{array}{c} \ddot{x}_1 \\ x_1 \end{array}\right\} \qquad (3\text{-}9)$$

Although these equations are still coupled, the coupling has been reduced to a minimum. Except for the first equation, each equation has only one coupling term. The left side of the first equation contains n terms: n-1 inertial terms and 1 elastic term. In many instances, this equation can be reduced to n-1 elastic terms and no inertial terms. This reduction is possible any time the reduced mass matrix is singular. If this condition exists, simple row operations will reduce the equations to the form

$$\left[\begin{array}{c|c} 0 \cdots\cdots\cdots\cdots & 0 \\ \hline & M_{2,n} \\ & \cdot \\ M_f & \cdot \\ & \cdot \\ & \cdot \\ & M_{n-1,n} \end{array}\right] \begin{Bmatrix} \ddot{y}_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \ddot{y}_n \end{Bmatrix} + \left[\begin{array}{c|c} R_1 \cdots\cdots R_{n-2} & K_{1,n} \\ \hline & 0 \\ & \cdot \\ K_f & \cdot \\ & \cdot \\ & \cdot \\ & 0 \end{array}\right] \begin{Bmatrix} y_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{Bmatrix}$$

$$= - \begin{bmatrix} S & K_{1,n} \\ M_{2,1} & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ M_{n-1,1} & 0 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ x_1 \end{Bmatrix} \qquad (3\text{-}10)$$

where

$$R_i = - K_{fii} \frac{M_{1,i+1}}{M_{fii}}$$

and

$$S = M_{11} - M_{2,1} \frac{M_{1,2}}{M_{f1,1}} \cdots\cdots M_{n-1,1} \frac{M_{1,n-1}}{M_{fn-1,n-1}}$$

Equations 3-10 are somewhat simpler than Eqs. 3-9 as the first equation of Eqs. 3-10 is of algebraic character, whereas, that of Eqs. 3-9 involves derivatives.

The primary advantage of having the reduced equations in the form of either Eqs. 3-9 or 3-10 is that the new coordinates are more efficient than the original ones. The coordinates $y_2, \ldots, y_{n-1}$ represent the motion in the normal modes of the system with $x_1 = x_n = 0$. Often, the response in the higher of these coordinates is negligible and these coordinates may be ignored. A corresponding number of the equations of motion may be dropped. Thus, the work involved in solving a many-degree of freedom problem can often be greatly reduced.

If some coordinate x_i other than x_1 had been given, Eqs. 3-10 in a slightly modified form could be used. The first i-1 equations can be solved by themselves as was explained earlier. The remaining equations are (from Eqs. 3-2)

$$\begin{bmatrix} m_i & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \cdot \\ \cdot & & \ddots & & \cdot \\ \cdot & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & m_n \end{bmatrix} \begin{Bmatrix} \ddot{x}_i \\ \cdot \\ \cdot \\ \cdot \\ \ddot{x}_n \end{Bmatrix} + \begin{bmatrix} k_{ii} & k_{i,i+1} & 0 & \cdots & 0 \\ k_{i+1,i} & k_{i+1,i+1} & & & \cdot \\ 0 & & & & \cdot \\ \cdot & & & & 0 \\ \cdot & & & & k_{n-1,n} \\ 0 & \cdots & 0 & k_{n,n-1} & k_{n,n} \end{bmatrix} \begin{Bmatrix} x_i \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{Bmatrix}$$

$$= \begin{Bmatrix} k_{i,i-1}\, x_{i-1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ f(t) \end{Bmatrix} \qquad (3\text{-}11)$$

These equations are of the same form as Eqs. 3-5. The only differences are:

1) the appearance of the term $k_{i,i-1}\, x_{i-1}$

2) i instead of 1

The $k_{i,i-1}\, x_{i-1}$ term is a known quantity (from the first i-1 equations) and thus it poses no problems. Eqs. 3-11 can, therefore, be transformed to a form similar to Eqs. 3-10.

Example 10

The system for this example is the same as for Example 2 and is shown in Fig. 3.3. Assume x_1 is known.

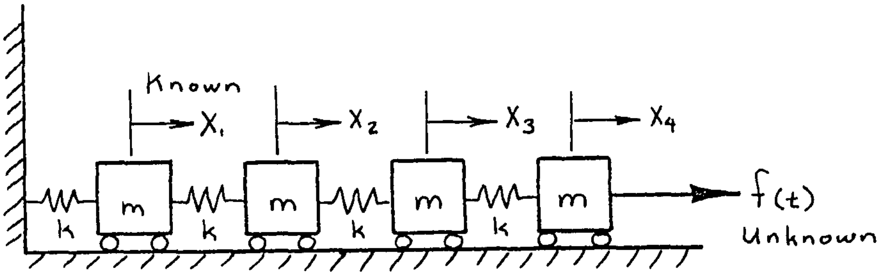

Fig. 3.3

The equations of motion are

$$m \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{Bmatrix} + k \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ f(t) \end{Bmatrix}$$

The transformation matrix is

$$[\Phi] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2/3 & 1 & -1 & 1/3 \\ 1/3 & 1 & 1 & 2/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

in accordance with Eq. 3-7.

Applying the transformation gives

$$[\Phi]^T[m][\Phi]\{\ddot{y}\} + [\Phi]^T[k][\Phi]\{y\} = [\Phi]^T\{F\}$$

or

$$m\begin{bmatrix} 1\;4/9 & 1 & -1/3 & 4/9 \\ 1 & 2 & 0 & 1 \\ -1/3 & 0 & 2 & 1/3 \\ 4/9 & 1 & 1/3 & 1\;4/9 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \end{Bmatrix}$$

$$+\;k\begin{bmatrix} 4/3 & 0 & 0 & -1/3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ -1/3 & 0 & 0 & 1/3 \end{bmatrix}\begin{Bmatrix} x_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ f(t) \end{Bmatrix}$$

f(t) can be removed by removing the last equation. The terms containing x_1 may be placed on the right-hand side of the equations, leading to

$$m\begin{bmatrix} 1 & -1/3 & 4/9 \\ 2 & 0 & 1 \\ 0 & 2 & 1/3 \end{bmatrix}\begin{Bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \end{Bmatrix} + k\begin{bmatrix} 0 & 0 & -1/3 \\ 2 & 0 & 0 \\ 0 & 6 & 0 \end{bmatrix}\begin{Bmatrix} y_2 \\ y_3 \\ y_4 \end{Bmatrix} = -\begin{bmatrix} 1\;4/9 & 4/3 \\ 1 & 0 \\ -1/3 & 0 \end{bmatrix}\begin{Bmatrix} m\ddot{x}_1 \\ kx_1 \end{Bmatrix}$$

Since the mass matrix is singular, the first equation can be simplified by row operations, giving

$$m \left[\begin{array}{cc|c} 0 & 0 & \multicolumn{1}{c}{0} \\ \cline{1-2} 2 & 0 & 1 \\ 0 & 2 & 1/3 \end{array}\right] \begin{matrix} \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \end{matrix} + k \left[\begin{array}{cc|c} -1 & 1 & \multicolumn{1}{c}{-1/3} \\ \cline{1-2} 2 & 0 & 0 \\ 0 & 6 & 0 \end{array}\right] \begin{matrix} y_2 \\ y_3 \\ y_4 \end{matrix} = - \begin{bmatrix} 1 & 4/3 \\ 1 & 0 \\ -1/3 & 0 \end{bmatrix} \begin{matrix} m\ddot{x}_1 \\ kx_1 \end{matrix}$$

These equations illustrate the efficiency of the new coordinates y. The stiffness associated with y_3 is three times the stiffness associated with y_2.

3.3 Solution to the Partially Uncoupled Equations

The partially uncoupled equations (Eqs. 3-9 or 3-10) may be solved by either of two methods. The simplest of the two, applicable to harmonically driven systems, involves substituting

$$\{y\} = \{A\} \sin \omega t \qquad \text{(or } \cos \omega t\text{)}$$

where ω corresponds to the measured frequency, into the equations and evaluating the constants A. This technique determines only the particular solution to the equations, but, as was discussed in Chap. 2, the homogeneous solution is expected to be identically zero.

Example 11

This example is a continuation of Example 10. The partially uncoupled equations are

$$m \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 1/3 \end{bmatrix} \begin{Bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \end{Bmatrix} + k \begin{bmatrix} -1 & 1 & -1/3 \\ 2 & 0 & 0 \\ 0 & 6 & 0 \end{bmatrix} \begin{Bmatrix} y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

$$= - \begin{bmatrix} 1 & 4/3 \\ 1 & 0 \\ -1/3 & 0 \end{bmatrix} \begin{Bmatrix} m\ddot{x}_1 \\ kx_1 \end{Bmatrix} \qquad (3\text{-}12)$$

Assume the measured value of x_1 to be

$$x_1 = .111 \frac{F_o}{k} \sin 2\sqrt{\frac{k}{m}}\, t$$

As the input to the equations is harmonic, assume

$$\begin{Bmatrix} y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} A_2 \\ A_3 \\ A_4 \end{Bmatrix} \sin 2\sqrt{\frac{k}{m}}\, t$$

Substituting these values into Eqs. 3-12 gives

$$-\frac{k}{m}\, m \begin{bmatrix} 0 & 0 & 0 \\ 8 & 0 & 4 \\ 0 & 8 & 4/3 \end{bmatrix} \begin{Bmatrix} A_2 \\ A_3 \\ A_4 \end{Bmatrix} \sin 2\sqrt{\frac{k}{m}}\, t + k \begin{bmatrix} -1 & 1 & -1/3 \\ 2 & 0 & 0 \\ 0 & 6 & 0 \end{bmatrix} \begin{Bmatrix} A_2 \\ A_3 \\ A_4 \end{Bmatrix} \sin 2\sqrt{\frac{k}{m}}\, t$$

$$= - \begin{bmatrix} -.444 & .148 \\ -.444 & 0 \\ .037 & 0 \end{bmatrix} \begin{Bmatrix} \frac{k}{m}\, m \\ k \end{Bmatrix} \frac{F_o}{k} \sin 2\sqrt{\frac{k}{m}}\, t$$

which reduces to

$$\begin{bmatrix} -1 & 1 & -1/3 \\ -6 & 0 & -4 \\ 0 & -2 & -4/3 \end{bmatrix} \begin{Bmatrix} A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} .296 \\ .444 \\ -.148 \end{Bmatrix} \frac{F_o}{k}$$

The solutions for A become

$$A_2 = .222 \frac{F_o}{k}$$

$$A_3 = .370 \frac{F_o}{k}$$

$$A_4 = .444 \frac{F_o}{k}$$

but

$$\{x\} = [\Phi]\{y\}$$

and thus

$$x_2 = -.222 \frac{F_o}{k} \sin 2\sqrt{\frac{k}{m}}\, t$$

$$x_3 = .333 \frac{F_o}{k} \sin 2\sqrt{\frac{k}{m}}\, t$$

$$x_4 = -.444 \frac{F_o}{k} \sin 2\sqrt{\frac{k}{m}}\, t$$

These values may be used in the last of the original equations of motion to yield the value of f(t).

$$f(t) = F_o \sin 2\sqrt{\frac{k}{m}}\, t$$

Δ

The second method of solution of the partially uncoupled equations is elimination. The first equation of Eqs. 3-10 is an algebraic equation. It may be used to solve for one of the unknowns in terms of the others. This result may then be used to reduce the number of the remaining equations by one. These equations possess a singular mass matrix also and may be reduced again. The process of elimination is continued until one equation in one unknown is obtained.

Example 12

This example treats the same system and the same given quantity as Example 11. With x_1 replaced by its given value, the equations become

$$m \begin{bmatrix} 0 & 0 & 0 \\ \hline 2 & 0 & 1 \\ 0 & 2 & 1/3 \end{bmatrix} \begin{Bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \end{Bmatrix} + k \begin{bmatrix} -1 & 1 & -1/3 \\ \hline 2 & 0 & 0 \\ 0 & 6 & 0 \end{bmatrix} \begin{Bmatrix} y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

$$= \begin{Bmatrix} .296 \\ .444 \\ -.148 \end{Bmatrix} F_o \sin 2\sqrt{\frac{k}{m}}\, t$$

The first equation, an algebraic one, is

$$-y_2 + y_3 - 1/3\ y_4 = .296\ \frac{F_o}{k} \sin 2\sqrt{\frac{k}{m}}\ t$$

or

$$y_4 = 3\ y_3 - 3\ y_2 - .888\ \frac{F_o}{k} \sin 2\sqrt{\frac{k}{m}}\ t$$

and

$$\ddot{y}_4 = 3\ \ddot{y}_3 - 3\ \ddot{y}_2 + 3.55\ \frac{F_o}{k}\ \frac{k}{m} \sin 2\sqrt{\frac{k}{m}}\ t$$

These values of y_4 and $\ddot{y}_4$ may be used to reduce the remaining equations to

$$m \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix} \begin{Bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \end{Bmatrix} + k \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \begin{Bmatrix} y_2 \\ y_3 \end{Bmatrix}$$

$$= \begin{Bmatrix} -3.111 \\ -1.333 \end{Bmatrix} F_o \sin 2\sqrt{\frac{k}{m}}\ t$$

The fact that the mass matrix is singular makes further elimination possible. Thus

$$m \begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \end{Bmatrix} + k \begin{bmatrix} 0 & 0 \\ -2 & 6 \end{bmatrix} \begin{Bmatrix} y_2 \\ y_3 \end{Bmatrix}$$

$$= \begin{Bmatrix} -3.111 \\ 1.778 \end{Bmatrix} F_o \sin 2\sqrt{\frac{k}{m}}\ t$$

The last equation gives

$$y_2 = 3\ y_3 - .888\ \frac{F_o}{k} \sin 2\sqrt{\frac{k}{m}}\ t$$

Eliminating again gives

$$y_3 = .370\ \frac{F_o}{k} \sin 2\sqrt{\frac{k}{m}}\ t$$

Thus

$$y_2 = .222\ \frac{F_o}{k} \sin 2\sqrt{\frac{k}{m}}\ t$$

and

$$y_4 = -.444\ \frac{F_o}{k} \sin 2\sqrt{\frac{k}{m}}\ t$$

These answers agree with the answers in Example 11.

CHAPTER 4

MANY DEGREE OF FREEDOM SYSTEMS

Chapter 4 will extend the ideas of Chaps. 2 and 3 to more complicated problems, problems for which hand calculation of the complete solution is too cumbersome. Specifically, this chapter will deal with computer techniques and approximate solutions.

4.1 Flexural Vibrations of a Beam; Remote-Coupled Systems

It was shown in Example 9 that the equations of motion of beams in bending which are remote-coupled may be uncoupled through application of the property of biorthogonality. This technique appears to be valid regardless of the number of degrees of freedom. For large systems, however, it is too involved for hand calculations. The most time-consuming part of the solution is the computation of the eigenvectors of the reduced equations and of the adjoint equations.

These eigenvectors can be determined through matrix iteration since the iteration process does not require that the matrices be symmetric. This technique assumes a solution of the form

$$\{\ddot{x}\} = \{A\} \sin \omega t$$

Substituting this into the reduced equations of motion, Eqs. 2-4, yields

$$[m_R]\{A\} \frac{-1}{\omega^2} [k_R]\{A\} = \{0\}$$

Premultiplying by $[k_R]^{-1}$ gives

$$\frac{1}{\omega^2} \{A\} = [k_R]^{-1}[m_R]\{A\} \tag{4-1}$$

The iteration process is started by assuming a trial vector for $\{A\}$ on the right-hand side of the equation. The operations on the right-hand side of Eqs. 4-1 yield an estimate for $\frac{1}{\omega^2}$ and a new trial vector. The iteration process is continued until it converges to an eigenvector. Examples indicate that the iteration of Eqs. 4-1 will converge to the eigenvector which corresponds to the eigenvalue of lowest absolute value.

In the usual iteration procedure, the eigenvector obtained is then eliminated, or swept out, from the equations by requiring the trial vectors for the next higher mode to be orthogonal to the eigenvector just determined with respect to the mass matrix. Since orthogonality does not exist for the reduced equations, the sweeping procedure must be based upon the property of biorthogonality.

The adjoint equations can be written in a form similar to that of Eqs. 4-1, leading to

$$\frac{1}{\omega^2}\{A_A\} = [k_R]^{T^{-1}}[m_R]^T\{A_A\} \qquad (4\text{-}2)$$

If these equations are now iterated to their fundamental eigenvector, the property of biorthogonality may be used to sweep the fundamental eigenvectors from Eqs. 4-1 and Eqs. 4-2. Once the fundamental eigenvectors have been swept from both sets of equations, both sets of equations may be iterated to their second eigenvectors. The process can be continued until all of the eigenvectors of the reduced equations and simultaneously those of the adjoint equations have been determined.

A discussion of the iteration process, the development of the sweeping procedure, and a listing of the Fortran computer program that was developed is located in Appendix B.

The iteration process as described does have one important limitation. Both the mass and the stiffness matrices must be nonsingular. This is a serious limitation as the majority of problems investigated have a singular reduced mass matrix. The equations of such a system can, however, be treated in modified form. Row operations may be performed on the reduced equations until one row of the mass matrix contains only zeros. That equation, now only elastically coupled, may be used to solve for one unknown

$$\frac{1}{\omega^2} \{A_A\} = [k_R]^{T^{-1}} [m_R]^T \{A_A\} \qquad (4\text{-}2)$$

If these equations are now iterated to their fundamental eigenvector, the property of biorthogonality may be used to sweep the fundamental eigenvectors from Eqs. 4-1 and Eqs. 4-2. Once the fundamental eigenvectors have been swept from both sets of equations, both sets of equations may be iterated to their second eigenvectors. The process can be continued until all of the eigenvectors of the reduced equations and simultaneously those of the adjoint equations have been determined.

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Example 13

The system is a four degree of freedom lumped mass beam in bending with an unknown force f(t) at x_4 and with x_1 known, as shown in Fig. 4.1.

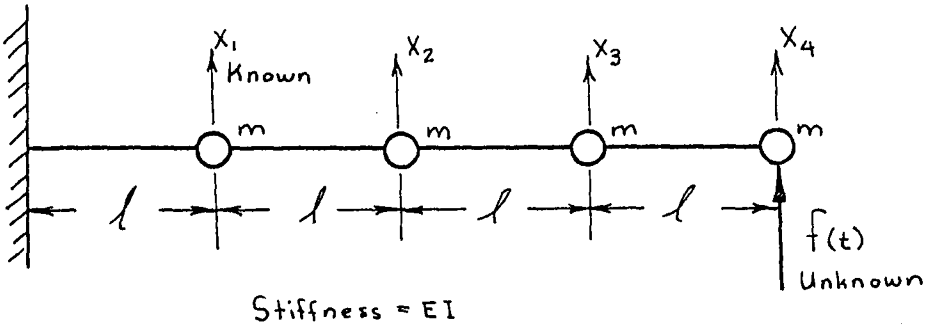

Fig. 4.1

The equations of motion are

$$m \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{Bmatrix} + K \begin{bmatrix} 18.8 & -11.8 & 4.45 & -.74 \\ -11.8 & 14.35 & -9.59 & 2.59 \\ 4.45 & -9.59 & 9.89 & -3.65 \\ -.74 & 2.59 & -3.65 & 1.61 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \\ f(t) \end{Bmatrix}$$

where $K = \frac{EI}{\ell^3}$

$f(t)$ can be removed by dropping the last equation. The terms involving the known coordinate x_1 can be placed on the right-hand side, as follows

$$m \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{Bmatrix} + K \begin{bmatrix} -11.8 & 4.45 & -.74 \\ 14.35 & -9.59 & 2.59 \\ -9.59 & 9.89 & -3.65 \end{bmatrix} \begin{Bmatrix} x_2 \\ x_3 \\ x_4 \end{Bmatrix}$$

$$= - \begin{bmatrix} 1 & 18.8 \\ 0 & -11.8 \\ 0 & 4.45 \end{bmatrix} \begin{Bmatrix} m\ddot{x}_1 \\ Kx_1 \end{Bmatrix}$$

The first equation yields

$$x_4 = - \frac{11.8}{.74} x_2 + \frac{4.45}{.74} x_3 + \frac{m}{.74} \frac{\ddot{x}_1}{K} + \frac{18.8}{.74} x_1$$

This equation can be used to eliminate x_4 from the remaining equations, as follows

$$m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + K \begin{bmatrix} -27.15 & 5.96 \\ 48.81 & -12.01 \end{bmatrix} \begin{Bmatrix} x_2 \\ x_3 \end{Bmatrix}$$

$$= \begin{bmatrix} -3.5 & -54.1 \\ 4.94 & 88.35 \end{bmatrix} \begin{Bmatrix} m\ddot{x}_1 \\ Kx_1 \end{Bmatrix}$$

These equations may be solved by the computer program as neither the mass nor the stiffness matrix is singular. Complete details of this example are given in Appendix D.

4.2 Longitudinal Vibrations of Bars; Close-Coupled Systems

A method was proposed in Chap. 3 for treating close-coupled systems. Consider for example the system shown in Fig. 3.2. The result was a set of uncoupled equations which describe that part of the system inboard of the point of known motion and a set of partially uncoupled equations which describe the system between the point of known motion and the point of application of the force. The method appears to be valid regardless of the number of degrees of freedom but becomes too involved for hand calculations as the system becomes large.

The first set of equations can be solved by classical methods, particularly by those which are amenable to computer techniques. Once the solutions to the first set of equations are known, the second set can be solved. The equations of the second set can either be solved one at a time as indicated in Sec. 3.1 or they can be transformed to a more efficient set of coordinates as discussed in Sec. 3.2. If the second set of equations involves many coordinates, the step-by-step solution becomes very tedious and the transformation outlined in Sec. 3.2 becomes desirable. As indicated in Sec. 3.2, the response in the higher modes of the solution to the transformed equations is expected to be small. Thus, in many instances the higher modes can be dropped, thereby reducing the number of calculations required. Generally, the resulting approximate solutions are acceptable.

Example 14

Consider the six degree of freedom system shown in Fig. 4.2.

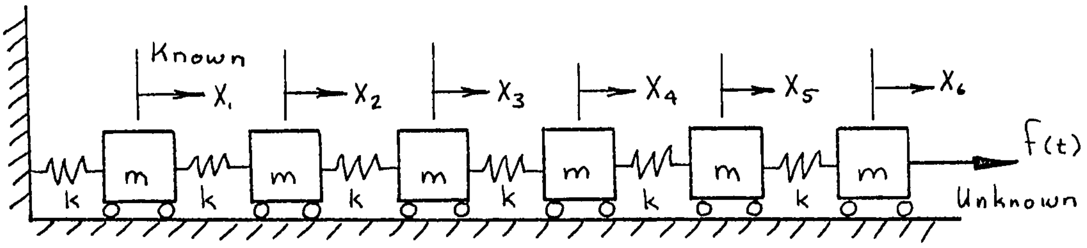

Fig. 4.2

Assume that x_1 is the measured coordinate. From Appendix D the transformed, partially uncoupled equations are written as

$$m \begin{bmatrix} 2.606 & -.724 & .382 & -.286 & .80 \\ 7.236 & 0 & 0 & 0 & 2.606 \\ 0 & 2.76 & 0 & 0 & .724 \\ 0 & 0 & 2.76 & 0 & .382 \\ 0 & 0 & 0 & 7.326 & .286 \end{bmatrix} \begin{Bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \\ \ddot{y}_5 \\ \ddot{y}_6 \end{Bmatrix}$$

$$+ k \begin{bmatrix} 0 & 0 & 0 & 0 & -.2 \\ 276 & 0 & 0 & 0 & 0 \\ 0 & 3.81 & 0 & 0 & 0 \\ 0 & 0 & 7.23 & 0 & 0 \\ 0 & 0 & 0 & 26.18 & 0 \end{bmatrix} \begin{Bmatrix} y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{Bmatrix} = - \begin{bmatrix} 2.2 & 1.2 \\ 2.606 & 0 \\ -.724 & 0 \\ .384 & 0 \\ -.286 & 0 \end{bmatrix} \begin{Bmatrix} m\ddot{x}_1 \\ kx_1 \end{Bmatrix}$$

The coordinates y are the result of the transformation

$$\{x\} = [\Phi]\{y\}$$

where the first column of $[\Phi]$ is the static shape with a unit displacement at x_1 and no displacement at x_n; the last column is the static shape with a unit displacement at x_n and no displacement at x_1. The intermediate columns of $[\Phi]$ consist of the eigenvectors of the system with x_1 and x_n fixed. It should be noted that the coordinates associated with the dynamic shapes are y_2, y_3, y_4, and y_5, and are therefore the only ones which can be dropped as higher modes.

Assume the measured response of x_1 is

$$x_1 = -.978 \frac{F_o}{k} \sin \tfrac{1}{2}\sqrt{\frac{k}{m}}\, t$$

For the substitution solution, assume

$$\{y\} = \{A\} \sin \tfrac{1}{2}\sqrt{\frac{k}{m}}\, t$$

The complete solution is

$$\begin{Bmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} -1.711 \\ -2.015 \\ -1.817 \\ -1.165 \\ -\ .222 \end{Bmatrix} \frac{F_o}{k} \sin \tfrac{1}{2}\sqrt{\frac{k}{m}}\, t$$

If the two highest frequency modes, y_4 and y_5 are dropped, the equations reduce to

$$m \begin{bmatrix} 2.606 & -.724 & .80 \\ 7.236 & 0 & 2.606 \\ 0 & 2.72 & .724 \end{bmatrix} \begin{Bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \end{Bmatrix} + k \begin{bmatrix} 0 & 0 & -.2 \\ 2.76 & 0 & 0 \\ 0 & 3.81 & 0 \end{bmatrix} \begin{Bmatrix} y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

$$= - \begin{bmatrix} 2.2 & 1.2 \\ 2.606 & 0 \\ -.724 & 0 \end{bmatrix} \begin{Bmatrix} m\ddot{x}_1 \\ kx_1 \end{Bmatrix}$$

The substitution solution to these equations is

$$\begin{Bmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} -1.694 \\ -2.028 \\ -1.827 \\ -1.152 \\ -\ .223 \end{Bmatrix} \frac{F_o}{k} \sin \tfrac{1}{2}\sqrt{\frac{k}{m}}\, t$$

which is an excellent approximation. Had one more truncation been performed (y_3, y_4, and y_5), the results would be

$$\begin{Bmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} -1.338 \\ -1.505 \\ -1.356 \\ -\ .892 \\ -\ .233 \end{Bmatrix} \frac{F_o}{k} \sin \tfrac{1}{2}\sqrt{\frac{k}{m}}\, t$$

Although this is not a particularly good approximation, it must be remembered that three of the four coordinates based on dynamic modes were dropped.

CHAPTER 5

CONTINUOUS SYSTEMS

The previous chapters have developed the basic ideas involved with the lumped mass inverse vibration problem. This chapter will extend these ideas to continuous systems. The major change is the shift from a set of ordinary differential equations in discrete coordinates to a partial differential equation involving continuous variables.

5.1 An Example of a Remote-Coupled System; Flexural Vibrations of a Beam

As in the lumped mass system, a beam in bending is an example of a remote-coupled system. It has been demonstrated in previous chapters that the reduced equations of a remote-coupled system can be uncoupled by using the eigenvectors of the adjoint system. A similar situation is true of the continuous case.

Consider a uniform, simply supported beam of length ℓ as shown in Fig. 5.1 with an unknown force applied at ξ_2 and with the motion W_g at ξ_1 given.

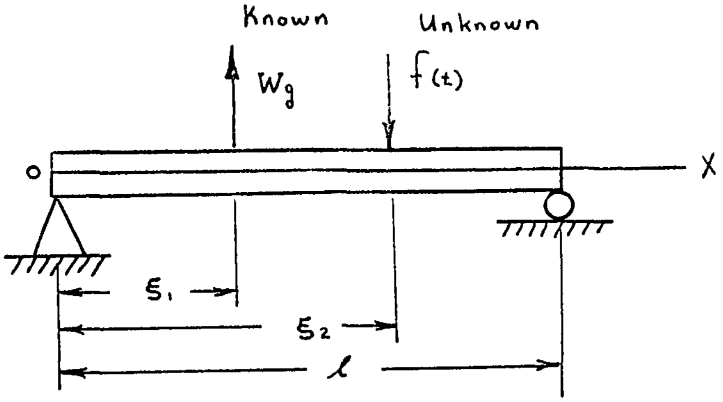

Fig. 5.1

The equation of motion

$$EI\, W^{\underline{\overline{IV}}} + \mu \ddot{W} = 0 \qquad (5\text{-}1)$$

is valid within each of the three segments, $x = 0$ to ξ_1, ξ_1 to ξ_2, and ξ_2 to ℓ. The boundary conditions at the ends of the segments are

a) $W(0,t) = 0$

b) $W(\ell,t) = 0$

c) $W''(0,t) = 0$

d) $W''(\ell,t) = 0$

e) $W(\xi_1^-,t) = W(\xi_1^+,t) = W_g$

f) $W'(\xi_1^-,t) = W'(\xi_1^+,t)$

g) $W''(\xi_1^-,t) = W''(\xi_1^+,t)$

h) $W'''(\xi_1^-,t) = W'''(\xi_1^+,t)$

i) $W(\xi_2^-,t) = W(\xi_2^+,t)$

j) $W'(\xi_2^-,t) = W'(\xi_2^+,t)$

k) $W''(\xi_2^-,t) = W''(\xi_2^+,t)$

l) $W'''(\xi_2^+,t)-W'''(\xi_2^-,t) = \dfrac{f(t)}{EI}$

(5-2)

Hence, there are thirteen boundary conditions (e) is actually two conditions), whereas, only twelve are required (four per segment). All of these conditions are homogeneous except e) and l). As condition l) contains the unknown exciting force f(t), the dropping of this boundary condition accomplishes the two desirable results:

1) there are now only twelve boundary conditions
2) one unknown is temporarily eliminated from the problem

Boundary condition e) can be made homogeneous by letting

$$W(x,t) = w(x,t) + \varphi_s(x)W_g(t) \qquad (5\text{-}3)$$

Substution of Eq. 5-3 into the equation of motion, Eq. 5-1, yields

$$EI\, w^{\overline{IV}} + \mu\ddot{w} = - EI\, \varphi_s^{\overline{IV}}\, W_g - \mu\varphi_s\, \ddot{W}_g$$

φ_s is selected as the static deflection of the system due to a load at ξ_2 such that unit displacement occurs at ξ_1. As φ_s is a static shape, it will satisfy

$$\varphi_s^{\overline{IV}} = 0$$

and the equation of motion becomes

$$EI\, w^{\overline{IV}} + \mu\ddot{w} = - \mu\varphi_s\, \ddot{W}_g \qquad (5\text{-}4)$$

The boundary conditions on φ_s are

a) $\varphi_s(0) = 0$

b) $\varphi_s(\ell) = 0$

c) $\varphi_s''(0) = 0$

d) $\varphi_s''(\ell) = 0$

e) $\varphi_s(\xi_1^-) = \varphi_s(\xi_1^+) = 1$

f) $\varphi_s'(\xi_1^-) = \varphi_s'(\xi_1^+)$

g) $\varphi_s''(\xi_1^- = \varphi_s''(\xi_1^+)$

h) $\varphi_s'''(\xi_1^-) = \varphi_s'''(\xi_1^+)$

i) $\varphi_s(\xi_2^-) = \varphi_s(\xi_2^+)$

j) $\varphi_s'(\xi_2^-) = \varphi_s'(\xi_2^+)$

k) $\varphi_s''(\xi_2^-) = \varphi_s''(\xi_2^+)$

l) $EI\ \varphi_s'''(\xi_2^+) - EI\ \varphi_s'''(\xi_2^-) = \text{constant}$

(5-5)

As a result, the boundary conditions on w are

a) $w(0,t) = 0$

b) $w(\ell,t) = 0$

c) $w''(0,t) = 0$

d) $w''(\ell,t) = 0$

e) $w(\xi_1^-,t) = w(\xi_1^+,t) = 0$

f) $w'(\xi_1^-,t) = w'(\xi_1^+,t)$

g) $w''(\xi_1^-,t) = w''(\xi_1^+,t)$

h) $w'''(\xi_1^-,t) = w'''(\xi_1^+,t)$

i) $w(\xi_2^-,t) = w(\xi_2^+,t)$

j) $w'(\xi_2^-,t) = w'(\xi_2^+,t)$

k) $w''(\xi_2^-,t) = w''(\xi_2^+,t)$

l) $w'''(\xi_2^+,t) - w'''(\xi_2^-,t)$

$$+ (\varphi_s'''(\xi_2^+) - \varphi_s'''(\xi_2^-))\ W_g = \frac{f(t)}{EI} \qquad (5\text{-}6)$$

Equation 5-4 and its boundary conditions in w are analogous to the reduced equations for lumped mass systems. The homogeneous part of the reduced equation, Eq. 5-4, can be solved for its eigenfunctions φ. In the lumped mass case, the eigenvectors of the reduced equations were not related to the natural mode shapes of the system; similarly, in the continuous case, the eigenfunctions of the reduced equation are not related to the actual mode shapes of the system. As in the lumped mass case, however, they may be physically interpreted as the shapes due to a harmonic force (exponential force for negative eigenvalues) applied at ξ_2 with force-free nodes at ξ_1.

The equation of motion for the adjoint system is

$$EI\,\overline{w}^{\underline{\overline{IV}}} + \mu\ddot{\overline{w}} = 0 \tag{5-7}$$

and the boundary conditions are

a) $\overline{w}(0,t) = 0$

b) $\overline{w}(\ell,t) = 0$

c) $\overline{w}''(0,t) = 0$

d) $\overline{w}''(\ell,t) = 0$

e) $\overline{w}(\xi_1^-,t) = \overline{w}(\xi_1^+,t)$

f) $\overline{w}'(\xi_1^-,t) = \overline{w}'(\xi_1^+,t)$

g) $\overline{w}''(\xi_1^-,t) = \overline{w}''(\xi_1^+,t)$

h) $EI\overline{w}'''(\xi_1^+,t) - EI\overline{w}'''(\xi_1^-,t) \neq 0$

i) $\overline{w}(\xi_2^-,t) = \overline{w}(\xi_2^+,t) = 0$

j) $\overline{w}'(\xi_2^-,t) = \overline{w}'(\xi_2^+,t)$

k) $\overline{w}''(\xi_2^-,t) = \overline{w}''(\xi_2^+,t)$

l) $\overline{w}'''(\xi_2^-,t) = \overline{w}'''(\xi_2^+,t)$ (5-8)

Dropping boundary condition h) will yield twelve homogeneous conditions for the adjoint equation. The adjoint eigenfunctions, $\overline{\varphi}$, can be interpreted as the shapes of the system with a harmonic force applied at ξ_1, and a force-free node at ξ_2.

Equation 5-4 is transformed to new coordinates q by the relation

$$w(x,t) = \sum_{j=1}^{\infty} \varphi_j(x) q_i(t) \tag{5-9}$$

The transformed equation is then multiplied by the i th eigenfunction, $\overline{\varphi}_i$, of the adjoint system. The result is integrated over the length ℓ with respect to x. The equation is

$$\int_0^{\ell} EI \sum_{j=1}^{\infty} \varphi_j^{\overline{\underline{IV}}} q_j \overline{\varphi}_i dx + \int_0^{\ell} \mu \sum_{j=1}^{\infty} \varphi_j \ddot{q}_j \overline{\varphi}_i dx = - \int_0^{\ell} \mu \ddot{\overline{W}}_g \varphi_s \overline{\varphi}_i dx \tag{5-10}$$

The first term of Eq. 5-10 can be simplified through integration by parts. The first term becomes

$$EI \sum_{j=1}^{\infty} \varphi_j''' \,\overline{\varphi}_i \, q_j \Bigg|_{o}^{\xi_1^-} + EI \sum_{j=1}^{\infty} \varphi_j''' \,\overline{\varphi}_i \, q_j \Bigg|_{\xi_1^+}^{\xi_2^-} + EI \sum_{j=1}^{\infty} \varphi_j''' \,\overline{\varphi}_i q_j \Bigg|_{\xi_2^+}^{\ell}$$

$$- EI \sum_{j=1}^{\infty} \varphi_j'' \,\overline{\varphi}_i' \, q_j \Bigg|_{o}^{\xi_1^-} - EI \sum_{j=1}^{\infty} \varphi_j'' \,\overline{\varphi}_i' q_j \Bigg|_{\xi_1^+}^{\xi_2^-}$$

$$- EI \sum_{j=1}^{\infty} \varphi_j'' \,\overline{\varphi}_i' q_j \Bigg|_{\xi_2^+}^{\ell} + \int_{o}^{\ell} EI \sum_{j=1}^{\infty} \varphi_j'' \,\overline{\varphi}_i'' q_j dx$$

The first six terms of this expression can be expanded as

$$\begin{aligned} EI \sum_{j=1}^{\infty} q_j \,[\varphi_j'''(\xi_1^-)\, \overline{\varphi}_i(\xi_1^-) &- \varphi_j'''(o)\, \overline{\varphi}_i(o) \\ + \varphi_j'''(\xi_2^-)\, \overline{\varphi}_i(\xi_2^-) &- \varphi_j'''(\xi_1^+)\, \overline{\varphi}_i(\xi_1^+) \\ + \varphi_j'''(\ell)\, \overline{\varphi}_i(\ell) &- \varphi_j'''(\xi_2^+)\, \overline{\varphi}_i(\xi_2^+) \\ + \varphi_j''(\xi_1^-)\, \overline{\varphi}_i'(\xi_1^-) &- \varphi_j''(0)\, \overline{\varphi}_i'(o) \\ + \varphi_j''(\xi_2^-)\, \overline{\varphi}_i'(\xi_2^-) &- \varphi_j''(\xi_1^+)\, \overline{\varphi}_i'(\xi_1^+) \\ + \varphi_j''(\ell)\, \overline{\varphi}_i'(\ell) &- \varphi_j''(\xi_2^+)\, \overline{\varphi}_i'(\xi_2^+)] \end{aligned}$$

Substitution of the boundary conditions, Eqs. 5-6 and 5-8, into these terms makes the expression equal to zero. Consequently, Eq. 5-10 becomes

$$\int_0^\ell EI \sum_{j=1}^{\infty} \varphi_j'' \, \bar{\varphi}_i'' \, q_j dx + \int_0^\ell \mu \sum_{j=1}^{\infty} \varphi_j \bar{\varphi}_i \ddot{q}_j dx = - \int_0^\ell \mu \, \ddot{W}_g \varphi_s \bar{\varphi}_i dx \tag{5-11}$$

The biorthogonality conditions, Eqs. A-17 and A-18 from Appendix A, indicate that all terms of the summation for i ≠ j are zero. Hence the first integral becomes

$$q_i \int_0^\ell EI \, \varphi_i'' \, \bar{\varphi}_i'' dx$$

and the second becomes

$$\ddot{q}_i \int_0^\ell \mu \, \varphi_i \bar{\varphi}_i dx$$

By defining

$$M_i = \int_0^\ell \mu \, \varphi_i \bar{\varphi}_i dx \tag{5-12}$$

and

$$K_i = \int_0^\ell EI \, \varphi_i'' \, \bar{\varphi}_i'' dx \tag{5-13}$$

Equation 5-11 becomes

$$M_i\ddot{q}_i + K_i q_i = - \ddot{W}_g \int_0^{\ell} \mu \, \varphi_s \bar{\varphi}_i dx \qquad (5\text{-}14)$$

which may be solved.

Once a solution in the generalized coordinates q is known, Eq. 5-9 can be used to obtain a solution in the original coordinates w. With w known, boundary condition 1) of Eqs. 5-6 can be used to evaluate f(t).

5.2 An Example of a Close-Coupled System--Longitudinal Vibrations of a Cantilever Bar

Just as in the lumped mass close-coupled system, the continuous close-coupled system can be only partially uncoupled. The ideas developed in Chap. 3, however, can be applied to effect a solution.

Consider the longitudinal vibrations of the uniform bar shown in Fig. 5.2.

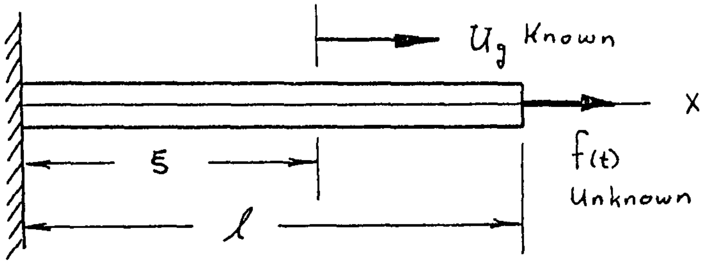

Fig. 5.2

Assume that f(t) is unknown and that the motion at the location ξ is U_g. The equation of motion is

$$EAU'' - \mu\ddot{U} = 0 \qquad (5\text{-}15)$$

and the boundary conditions are

a) $U(0,t) = 0$

b) $EAU'(\ell,t) = f(t)$

c) $U(\xi^{\pm},t) = U_g \qquad (5\text{-}16)$

As in the case of lumped mass systems, the solution is divided into two parts, the part of the bar to the left of ξ and the part to the right of ξ. The first part is shown in Fig. 5.3.

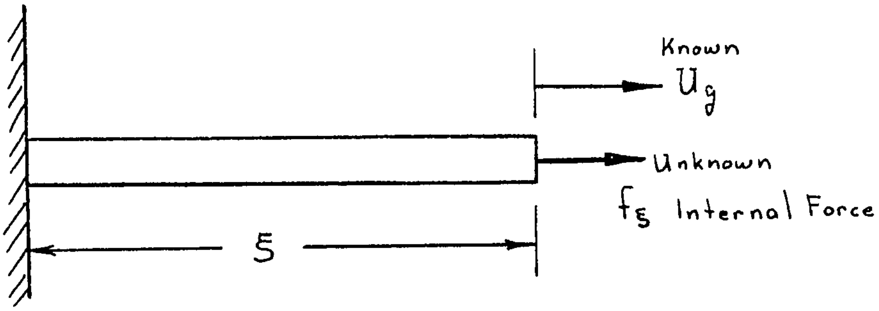

Fig. 5.3

The equation of motion for this part is

$$EAU'' - \mu\ddot{U} = 0 \qquad (5\text{-}15)$$

and the boundary conditions are

$$
\begin{aligned}
&\text{a)}\quad U(0,t) = 0\\
&\text{b)}\quad EAU'(\xi,t) = f_s\\
&\text{c)}\quad U(\xi,t) = U_g
\end{aligned}
\qquad (5\text{-}17)
$$

where f_s is the internal force in the bar at ξ.

f_s may be eliminated by dropping equation b) of Eqs. 5-17 since only two boundary conditions are required. Boundary condition c) of Eqs. 5-17 can be made homogeneous by letting

$$U(x,t) = u(x,t) + \varphi_s(x)\, U_g(t) \qquad (5\text{-}18)$$

in which φ_s is selected as the static deflection of the system due to a force at ξ such that the displacement at ξ is unity. φ_s satisfies

$$\varphi_s'' = 0$$

since it is a static shape. The boundary conditions on φ_s are

$$
\begin{aligned}
&\text{a)}\quad \varphi_s(0) = 0\\
&\text{b)}\quad \varphi_s(\xi) = 1
\end{aligned}
\qquad (5\text{-}19)
$$

As a result, the equation for $u(x,t)$ is

$$EAu'' - \mu\ddot{u} = \mu\varphi_s\, \ddot{U}_g \qquad (5\text{-}20)$$

with the boundary conditions

$$
\begin{aligned}
&\text{a)}\quad u(0,t) = 0\\
&\text{b)}\quad u(\xi,t) = 0
\end{aligned}
\qquad (5\text{-}21)
$$

The homogeneous part of Eqs. 5-20 with boundary conditions Eqs. 5-21 can be solved for its eigenfunctions $\varphi(x)$. The transformation

$$u(x,t) = \sum_{j=1}^{\infty} \varphi_j q_j$$

leads to a set of coupled equations. The equations can then be uncoupled by application of the usual orthogonality relationships. The term $\mu\varphi_s \ddot{U}_g$ can be treated as an externally applied force.

The uncoupled equations are

$$M_i \ddot{q}_i + K_i q_i = \int_0^s \mu \ddot{U}_g \varphi_s \varphi_i dx \tag{5-22}$$

where

$$M_i = \int_0^s \mu \varphi_i^2 \, dx$$

and

$$K_i = \int_0^s EA(\varphi_i')^2 \, dx$$

These equations can be solved individually and transformed back to u by

$$u(x,t) = \sum_{i=1}^{\infty} \varphi_i q_i$$

with $u(x,t)$ known, $U'(\xi,t)$ can be evaluated and then f_s can be determined from boundary condition Eqs. 5-17 b).

The solution for the right portion of the bar is not as simple as that for the left part, a situation encountered in the analysis of lumped mass systems. The right portion is shown in Fig. 5.4.

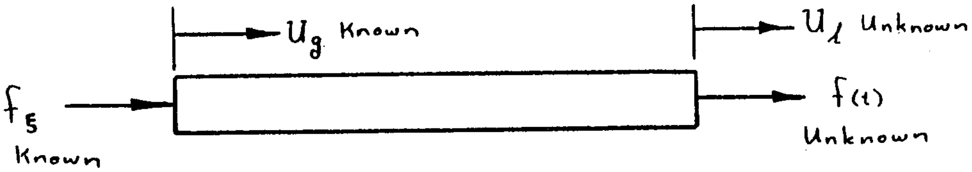

Fig. 5.4

The equation of motion is

$$EAU'' - \mu\ddot{U} = 0 \tag{5-15}$$

and the boundary conditions are

a) $U(\xi,t) = U_g$

b) $U(\ell,t) = U_\ell$

c) $EAU'(\xi,t) = f_\xi$

d) $EAU'(\ell,t) = f(t)$ (5-23)

Boundary conditions a) and b) can be made homogeneous by letting

$$U(x,t) = u(x,t) + \varphi_S(\xi)\, U_g + \varphi_\ell(x) U_\ell(t) \tag{5-24}$$

φ_S is the static shape due to a force at ξ with a unit displacement at ξ and zero displacement at ℓ. φ_ℓ is a static shape due to a force at ℓ with a unit displacement at ℓ and zero displacement at ξ. φ_s and φ_ℓ satisfy

$$\varphi_s'' = 0$$

$$\varphi_\ell'' = 0$$

and the boundary conditions

$$\begin{aligned} &\text{a)} \quad \varphi_s(\xi) = 1 \\ &\text{b)} \quad \varphi_s(\ell) = 0 \\ &\text{c)} \quad \varphi_\ell(\xi) = 0 \\ &\text{d)} \quad \varphi_\ell(\ell) = 1 \end{aligned} \tag{5-25}$$

Equation 5-15 becomes

$$EAu'' - \mu\ddot{u} = \mu\varphi_s \ddot{U}_g + \mu\varphi_\ell \ddot{U}_\ell \tag{5-26}$$

and the boundary conditions on u are

$$\begin{aligned} &\text{a)} \quad u(\xi,t) = 0 \\ &\text{b)} \quad u(\ell,t) = 0 \end{aligned} \tag{5-27}$$

The shapes chosen and the results are directly analogous to those for the lumped mass system analysis.

Equation 5-26 can be treated in the classical manner with the terms $\mu\varphi_s \ddot{U}_g$ and $\mu\varphi_\ell\ddot{U}_\ell$ representing externally applied forces. If the eigenfunctions of the homogeneous part of Eq. 5-26 with the boundary conditions Eqs. 5-27 are φ_{fj}, the transformation

$$u(x,t) = \sum_{j=1}^{\infty} \varphi_{fj}\, q_j$$

leads to coupled equations which can be uncoupled by applying the usual orthogonality relationships. The uncoupled equations can be written as

$$M_i\ddot{q}_i + K_i q_i - \ddot{U}_\ell \int_s^\ell \mu\varphi_s\varphi_{fi}dx = \ddot{U}_g \int_s^\ell \mu\varphi_s\varphi_{fi}dx \qquad (5\text{-}28)$$

where

$$M_i = \int_\xi^\ell \mu\ (\varphi_{fi})^2\ dx$$

and

$$K_i = \int_\xi^\ell EA\ (\varphi'_{fi})^2\ dx$$

Since U_ℓ is an unknown quantity, one more equation is required to yield a determinate system of equations. This one equation comes from the previously ignored boundary condition, Eq. 5-23 c).

$$EAU'(\xi,t) = f_\xi \qquad (5\text{-}23\text{-}c)$$

Applying Eq. 5-24 gives

$$EA[u'(\xi,t) + \varphi'_s(\xi)U_g(t) + \varphi'_\ell(\xi)U_\ell(t)] = f_\xi$$

or

$$EA[u'(\xi,t) + \varphi'_\ell(\xi)U_\ell(t)] = f_\xi - EA[\varphi'_s(\xi)U_g(t)] \qquad (5\text{-}29)$$

These equations are very similar to their counterparts in Chap. 3. In Eq. 5-28, each generalized coordinate q_i is inertially coupled to the unknown motion of the end of the bar U_ℓ. In Eq. 5-29, the motion U_ℓ is coupled to all of the coordinates q. Equation 5-28 represents a set which contains an infinite number of equations. This set would have to be truncated in order to obtain a solution. The method of solving the truncated set of equations would be similar to the method outlined in Chap. 4.

CHAPTER 6

SUMMARY AND RECOMMENDATIONS

6.1 Summary

Methods have been presented for analyzing the inverse problem. In Chap. 2, the equations of motion were written in the generalized coordinates associated with the eigenvectors of the reduced equations for the analysis of flexural vibrations of a lumped-mass beam (an example of a remote-coupled system). These equations of motion were uncoupled using the eigenvectors of the adjoint equations. It appeared from the examples that the largest part of the response of the beam tended to be in the coordinates associated with the smaller eigenvalues. In the usual vibration problem, this corresponds to the expected convergence of the response with the mode number. Thus it may often only be necessary to obtain the response in a few of the coordinates. Methods were presented in Chap. 4 for computerizing the solution for lumped-mass beams having many degrees of freedom.

In Chap. 3 the method which was successful with beams was found to be only partially successful with close-coupled systems, such as the longitudinal vibrations of a rod. It was found that the equations of motion can be separated into two sets. One of the sets can be transformed to a set of uncoupled equations using the methods of Sec. 2.1. It was shown that the second set of equations, although coupled, can be solved one at a time. For lumped-mass systems having many degrees of freedom, a method was given for transforming the second set of equations of motion to a nearly uncoupled set of equations in an efficient set of coordinates.

The methods were extended in Chap. 5 to consider examples of continuous systems.

6.2 Recommendations

A major area of the inverse problem that should be investigated further is the treatment of close-coupled systems. Attempts should be made to develop a procedure that would completely uncouple all of the reduced equations.

A basic problem, a category of the inverse problem, that should be examined is one in which an internal force (or constraint force) is known rather than a motion. Solution of this problem would permit unknown exciting forces to be calculated given strain gauge measurements of the internal forces.

APPENDIX A

THE BIORTHOGONALITY RELATIONS FOR LUMPED MASS SYSTEMS

The homogeneous part of the non-symmetric reduced equations, Eqs. 2-4, are of the form

$$[m_R]\{\ddot{x}\} + [k_R]\{x\} = \{0\} \tag{A-1}$$

The eigenvectors of these equations are the columns of $[\varphi]$. By definition, the adjoint equations are

$$[m_R]^T\{\ddot{x}\} + [k_R]^T\{x\} = \{0\} \tag{A-2}$$

and their eigenvectors are the columns of $[\bar{\varphi}]$.

For motion in the i th principle mode, Eq. A-1 becomes

$$-\ \omega_i^2\ [m_R]\ \{\varphi_i\} + [k_R]\{\varphi_i\} = \{0\} \tag{A-3}$$

where $\{\varphi_i\}$ is the i th eigenvector of Eq. A-1. For motion in the j th mode of the adjoint system, Eq. A-2 becomes

$$-\ \omega_j^2\ [m_R]^T\{\bar{\varphi}_j\} + [k_R]^T\{\bar{\varphi}_j\} = \{0\} \tag{A-4}$$

Premultiplication of Eq. A-3 by the transpose of the j th eigenvector of the adjoint equations yields

$$-\ \omega_i^2\ [\bar{\varphi}_j][m_R]\{\varphi_i\} + [\bar{\varphi}_j][k_R]\{\varphi_i\} = 0 \tag{A-5}$$

Similarly, premultiplication of Eq. A-4 by the transpose of the i th eigenvector of the reduced equations yields

$$-\omega_j^2 [\varphi_i][m_R]^T\{\bar{\varphi}_j\} + [\varphi_i][k_R]^T\{\bar{\varphi}_j\} = 0$$

Transposition of this equation gives

$$-\omega_j^2 [\bar{\varphi}_j][m_R]\{\varphi_i\} + [\bar{\varphi}_j][k_R]\{\varphi_i\} = 0 \qquad \text{(A-6)}$$

Subtraction of Eq. A-5 from Eq. A-6 yields

$$(\omega_i^2 - \omega_j^2)\ [\bar{\varphi}_j][m_R]\{\varphi_i\} = 0$$

Consequently,

$$[\bar{\varphi}_j][m_R]\{\varphi_i\} = 0 \text{ for } i \neq j \qquad \text{(A-7)}$$

and

$$[\bar{\varphi}_j][k_R]\{\varphi_i\} = 0 \text{ for } i \neq j \qquad \text{(A-8)}$$

These equations are the biorthogonality relationships for lumped mass systems.

THE BIORTHOGONALITY RELATIONS FOR THE FLEXURAL VIBRATIONS OF A CONTINUOUS BEAM

Equation 5-4 and its boundary conditions, Eqs. 5-6, form the reduced system of equations for the continuous system of Fig. A.1.

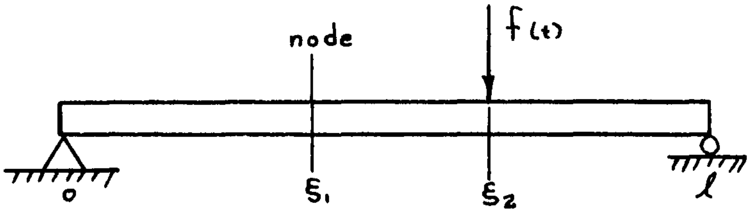

Fig. A.1

For motion in the j th mode, Eq. 5-4 becomes

$$EI\ \varphi_j^{\underline{IV}} - \mu\omega_j^2\ \varphi_j = 0 \qquad \text{(A-9)}$$

with boundary conditions

a) $\varphi_j(0) = 0$

b) $\varphi_j''(0) = 0$

c) $\varphi_j(\xi_1^+) = 0$

d) $\varphi_j(\xi_1^-) = 0$

e) $\varphi_j'(\xi_1^+) - \varphi_j'(\xi_1^-) = 0$

f) $\varphi_j''(\xi_1^+) - \varphi_j''(\xi_1^-) = 0$

g) $\varphi_j'''(\xi_1^+) - \varphi_j'''(\xi_1^-) = 0$

h) $\varphi_j(\xi_2^+) - \varphi_j(\xi_2^-) = 0$

i) $\varphi_j'(\xi_2^+) - \varphi_j'(\xi_2^-) = 0$

j) $\varphi_j''(\xi_2^+) - \varphi_j''(\xi_2^-) = 0$

k) $\varphi_j(\ell) = 0$

l) $\varphi_j''(\ell) = 0$ (A-10)

Equation 5-7 and its boundary conditions, Eqs. 5-8, form the adjoint system of equations for the continuous system illustrated in Fig. A.2.

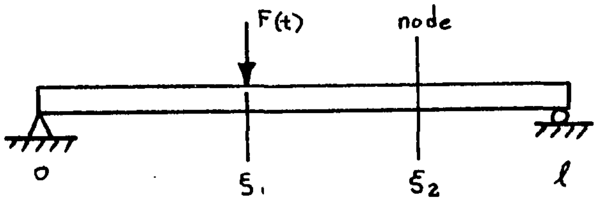

Fig. A.2

For motion in the i th mode, Eq. 5-7 becomes

$$EI\ \bar{\varphi}_i^{\overline{IV}} - \mu\omega_i^2\ \bar{\varphi}_i = 0 \qquad \text{(A-11)}$$

The boundary conditions are

a) $\bar{\varphi}_i(0) = 0$

b) $\bar{\varphi}_i''(0) = 0$

c) $\bar{\varphi}_i(\xi_1^+) - \bar{\varphi}_i(\xi_1^-) = 0$

d) $\varphi_i'(\xi_1^+) - \bar{\varphi}_i'(\xi_1^-) = 0$

e) $\varphi_i''(\xi_1^+) - \bar{\varphi}_i''(\xi_1^-) = 0$

f) $\bar{\varphi}_i(\xi_i^+) = 0$

g) $\bar{\varphi}_i(\xi_2^-) = 0$

h) $\bar{\varphi}_i'(\xi_2^+) - \bar{\varphi}_i'(\xi_2^-) = 0$

i) $\bar{\varphi}_i''(\xi_2^+) - \bar{\varphi}_i''(\xi_2^-) = 0$

j) $\bar{\varphi}_i'''(\xi_2^+) - \bar{\varphi}_i'''(\xi_2^-) = 0$

k) $\bar{\varphi}_i(\ell) = 0$

i) $\bar{\varphi}_i''(\ell) = 0$ (A-12)

Multiplying Eq. A-9 by $\bar{\varphi}_i$ and integrating over the three intervals 0 to ξ_1^-, ξ_1^+ to ξ_2^-, and ξ_2^+ to ℓ yields

$$\int_0^\ell EI\, \varphi_j^{\overline{IV}}\, \varphi_i dx = \omega_j^2 \int_0^\ell \mu \varphi_j \bar{\varphi}_i dx \qquad \text{(A-13)}$$

Integrating the left side by parts gives

$$EI\, \varphi_j'''\, \bar{\varphi}_i \Big|_0^{\xi_1^-} + EI\, \varphi_j'''\, \bar{\varphi}_i \Big|_{\xi_1^+}^{\xi_2^-} + EI\, \varphi_j'''\, \bar{\varphi}_i \Big|_{\xi_2^+}^{\ell}$$

$$- EI\, \varphi_j''\, \bar{\varphi}_i' \Big|_0^{\xi_1^-} - EI\, \varphi_j''\, \bar{\varphi}_i' \Big|_{\xi_1^+}^{\xi_2^-} - EI\, \varphi_j''\, \bar{\varphi}_i' \Big|_{\xi_2^+}^{\ell}$$

$$+ \int_0^\ell EI\, \varphi_i''\, \bar{\varphi}_j'' dx$$

Thus, Eq. A-13 can be rewritten as

$$\begin{aligned} EI\,[&\varphi_j'''(\xi_1^-)\,\bar{\varphi}_i(\xi_1^-) - \varphi_j'''(0)\,\bar{\varphi}_i(0) \\ &+ \varphi_j'''(\xi_2^-)\,\bar{\varphi}_i(\xi_2^-) - \varphi_j'''(\xi_1^+)\,\bar{\varphi}_i(\xi_1^+) \\ &+ \varphi_j'''(\ell)\,\bar{\varphi}_i(\ell) - \varphi_j'''(\xi_2^+)\,\bar{\varphi}_i(\xi_2^+) \\ &- \varphi_j''(\xi_1^-)\,\bar{\varphi}_i'(\xi_1^-) + \varphi_j''(0)\,\bar{\varphi}_i'(0) \\ &- \varphi_j''(\xi_2^-)\,\bar{\varphi}_i'(\xi_2^-) + \varphi_j''(\xi_1^+)\,\bar{\varphi}_i'(\xi_1^+) \\ &- \varphi_j''(\ell)\,\bar{\varphi}_i'(\ell) + \varphi_j''(\xi_2^+)\,\bar{\varphi}_i'(\xi_2^+)] \\ &+ \int_0^\ell EI\,\varphi_j''\,\bar{\varphi}_i''\,dx = \omega_j^2 \int_0^\ell \mu\varphi_j\bar{\varphi}_i\,dx \end{aligned} \qquad \text{(A-14)}$$

Due to the boundary conditions, Eqs. A-10 and A-12, all of the integrated terms are zero. As a result, Eq. A-13 becomes

$$\int_0^\ell EI\,\varphi_j''\,\bar{\varphi}_i''\,dx = \omega_j^2 \int_0^\ell \mu\varphi_j\bar{\varphi}_i\,dx \qquad \text{(A-15)}$$

A similar treatment of the adjoint equation, Eq. A-11, yields

$$\int_0^\ell EI\,\bar{\varphi}_i''\,\varphi_j''\,dx = \omega_i^2 \int_0^\ell \mu\varphi_j\bar{\varphi}_i\,dx \qquad \text{(A-16)}$$

Subtracting Eq. A-15 from Eq. A-16 gives

$$(\omega_i^2 - \omega_j^2) \int_0^\ell \mu\varphi_j\bar{\varphi}_i \, dx = 0$$

and hence

$$\int_0^\ell \mu\varphi_j\bar{\varphi}_i \, dx = 0 \qquad i \neq j \tag{A-17}$$

and

$$\int_0^\ell EI \, \varphi_j'' \, \bar{\varphi}_i'' \, dx = 0 \qquad i \neq j \tag{A-18}$$

which are the biorthogonality relationships for the flexural vibrations of a continuous beam.

Examination of Eq. A-14 shows that any homogeneous end conditions, conditions at $x = 0$ and $x = \ell$, will lead to the same reduction, that is, to Eq. A-15. Hence, the biorthogonality relationships are valid for any homogeneous end conditions.

USE OF THE BIORTHOGONALITY RELATIONSHIPS TO UNCOUPLE THE REDUCED EQUATIONS OF MOTION

The reduced equations of motion, Eq. 2-4, have the form

$$[m_R]\{\ddot{x}\} + [k_R]\{x\} = \{u\} \tag{A-19}$$

where $\{u\}$ is the matrix containing all of the terms involving the known coordinate. Equation A-19, written in generalized coordinates defined by

$$\{x\} = [\varphi]\{q\}$$

where $[\varphi]$ is the matrix of eigenvectors of the reduced equations, are of the form

$$[m_R][\varphi]\{\ddot{q}\} + [k_R][\varphi]\{q\} = \{u\}$$

Premultiplying these equations by the transpose of the adjoint eigenvector matrix yields

$$[\bar{\varphi}]^T[m_R][\varphi]\{\ddot{q}\} + [\bar{\varphi}]^T[k_R][\varphi]\{q\} = [\bar{\varphi}]^T\{u\} \qquad \text{(A-20)}$$

The only non-zero terms in the products

$$[\bar{\varphi}]^T[m_R][\varphi]$$

and

$$[\bar{\varphi}]^T[k_R][\varphi]$$

are those on the diagonal. All others are zero because of the biorthogonality conditions Eqs. A-7 and A-8. Define the generalized mass matrix and the generalized stiffness matrix as

$$\lceil M \rfloor = [\bar{\varphi}]^T[m_R][\varphi]$$

$$\lceil K \rfloor = [\bar{\varphi}]^T[k_R][\varphi]$$

Equation A-20 becomes

$$\lceil M \rfloor\{\ddot{q}\} + \lceil K \rfloor\{q\} = [\bar{\varphi}]^T\{u\} \qquad \text{(A-21)}$$

which is an uncoupled set of equations.

APPENDIX B

ITERATION PROCEDURE

In general, for free vibrations of an undamped system, the equations of motion can be written

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

If a solution of the form

$$\{x\} = \{A\} \sin \omega t$$

is assumed, the equations of motion become

$$- \omega^2[M]\{A\} + [K]\{A\} = \{0\}$$

or, for the reduced equations of motion

$$- \omega^2[m_R]\{A\} + [k_R]\{A\} = \{0\}$$

Rearranging gives

$$\frac{1}{\omega^2} \{A\} = [k_R]^{-1}[m_R]\{A\}$$

The product $[k_R]^{-1}[m_R]$ is usually called the dynamical matrix $[U]$; thus,

$$\frac{1}{\omega^2} \{A\} = [U]\{A\} \tag{B-1}$$

The iteration process is initiated by assuming some trial vector $\{A^{(o)}\}$ for the right-hand side of Eq. B-1. The result is

$$\{B\} = [U]\{A^{(o)}\}$$

If the vector {B} is normalized on one of its coordinates (the program listed at the end of this Appendix normalized on B_n), the expression can be written

$$(\frac{1}{\omega^{(1)}})^2\{A^{(1)}\} = [U]\{A^{(o)}\}$$

in which $(\omega^{(1)})^2$ is the first approximation of the fundamental (lowest in absolute magnitude) eigenvalue and $\{A^{(1)}\}$ is an improved trial vector. The second iteration gives

$$(\frac{1}{\omega^{(2)}})^2 \{A^{(2)}\} = [U]\{A^{(1)}\}$$

in which $(\omega^{(2)})^2$ and $\{A^{(2)}\}$ are improved approximations for the eigenvalue and eigenvector, respectively. The iteration process is continued until there is no significant difference between the trial vector and the subsequent improved vector. The final values of $(\frac{1}{\omega})^2$ and $\{A\}$ are the fundamental eigenvalue, $\frac{1}{\omega_1^2}$, and eigenvector, $\{\varphi_1\}$, of the reduced equations.

In order to obtain the second eigenvalue and eigenvector, the first mode must be swept from the equations of of motion. To accomplish this sweeping, the property of biorthogonality will be used. Consequently, the adjoint equations, Eqs. B-2, must be iterated to their fundamental eigenvector. The adjoint equations in a form to be iterated are

$$\frac{1}{\omega^2}\{\bar{A}\} = [k_R]^{T^{-1}}[m_R]^T\{\bar{A}\}$$

or

$$\frac{1}{\omega^2}\{\bar{A}\} = [\bar{U}]\{\bar{A}\} \tag{B-2}$$

Once $\{\varphi_1\}$ and $\{\bar{\varphi}_1\}$ are known, they may be used with the property of biorthogonality to sweep the first eigenvectors from Eqs. B-1 and B-2. The swept equations may then be iterated to the second eigenvectors, $\{\varphi_2\}$ and $\{\bar{\varphi}_2\}$. The alternate iterating-sweeping procedure can be continued until all of the eigenvectors have been obtained.

SWEEPING PROCEDURES FOR NONSYMMETRIC EQUATIONS

The first trial vector for the second eigenvector can be expressed as a linear combination of the eigenvectors of the reduced system, given by

$$\{A^{(o)}\} = \sum_{k=1}^{n} C_k \{\varphi_k\}$$

Removing the contribution of the first eigenvector gives

$$\{A_*^{(o)}\} = \sum_{k=1}^{n} C_k \{\varphi_k\} - C_1 \{\varphi_1\}$$

$\{A_*^{(o)}\}$ is a new trial vector which contains no trace of the first eigenvector. Thus,

$$\{A_*^{(o)}\} = \{A^{(o)}\} - C_1 \{\varphi_1\} \qquad \text{(B-3)}$$

Premultiplication of this expression by $\lfloor \bar{\varphi}_1 \rfloor [m]$ yields

$$\lfloor \bar{\varphi}_1 \rfloor [m]\{A_*^{(o)}\} = \lfloor \bar{\varphi}_1 \rfloor [m]\{A^{(o)}\} - C_1 \lfloor \bar{\varphi}_1 \rfloor [m]\{\varphi_1\}$$

but, since $\{A_*^{(o)}\}$ does not contain any contribution of $\{\varphi_1\}$, the property of biorthogonality requires that

$$\lfloor \bar{\varphi}_1 \rfloor [m]\{A_*^{(o)}\} = 0$$

Hence,

$$\lfloor \bar{\varphi}_1 \rfloor [m]\{A^{(o)}\} = C_1 \lfloor \bar{\varphi}_1 \rfloor [m]\{\varphi_1\}$$

The product on the right-hand side is the first generalized mass multiplied by C_1. Thus

$$\lfloor \bar{\varphi}_1 \rfloor [m]\{A^{(o)}\} = C_1 M_1$$

and

$$C_1 = \frac{\lfloor \bar{\varphi}_1 \rfloor [m]\{A^{(o)}\}}{M_1}$$

This expression can now be used in Eq. B-3, which yields

$$\{A_*^{(o)}\} = \{A^{(o)}\} - \{\varphi_1\}\frac{\lfloor\bar{\varphi}_1\rfloor[m]\{A^{(o)}\}}{M_1}$$

Rearranging and factoring gives

$$\{A_*^{(o)}\} = \left[[I] - \frac{\{\varphi_1\}\lfloor\bar{\varphi}_1\rfloor[m]}{M_1}\right]\{A^{(o)}\}$$

Applying this trial vector, which contains no part of the first eigenvector, to the right side of Eqs. B-1 yields

$$[U]\{A_*^{(o)}\} = \left[[U] - [U]\frac{\{\varphi_1\}\lfloor\bar{\varphi}_1\rfloor[m]}{M_1}\right]\{A^{(o)}\} \tag{B-4}$$

but

$$[U]\{\varphi_1\} = \frac{1}{\omega_1^2}\{\varphi_1\}$$

and Eq. B-4 becomes

$$[U]\{A_*^{(o)}\} = \left[[U] - \frac{\{\varphi_1\}\lfloor\bar{\varphi}_1\rfloor[m]}{\omega_1^2 M_1}\right]\{A^{(o)}\}$$

As a result, the swept dynamical matrix is

$$[U_s] = [U] - \frac{\{\varphi_1\}\lfloor\bar{\varphi}_1\rfloor[m]}{\omega_1^2 M_1}$$

and the equation to be iterated is

$$\frac{1}{\omega^2}\{A\} = [U_s]\{A\}$$

A similar treatment is used to sweep the first eigenvector from the adjoint equations. Also, similar procedures are used to sweep subsequent eigenvectors from the two sets of equations. For example, $[U_s]$ after the second eigenvector becomes

$$[U_s] = [U] - \frac{\{\varphi_1\}\lfloor\bar{\varphi}_1\rfloor[m]}{\omega_1^2 M_1} - \frac{\{\varphi_2\}\lfloor\bar{\varphi}_2\rfloor[m]}{\omega_1^2 M_2}$$

ACTUAL COMPUTING

The actual computing was performed by an IBM 7072-1401 digital computer at The University of Arizona. The details of the computational scheme are given in the following Fortran language program.

```
C    FORTRAN VIBRATIONS PROGRAM


      SUBROUTINE DIVCK (X,S)
C     THIS SUBROUTINE IS A DIVISION CHECK
C     THE NUMBER PRINTED FROM DIVCK SHOWS WHERE THE ERROR IS
C     DIGIT ONE TELLS WHICH PROGRAM OR SUBPROGRAM
C     DIGITS TWO AND THREE TELL WHICH STATEMENT IT FOLLOWS
C     DIGITS 4 AND 5 TELL HOW FAR IT FOLLOWS THE NUMBERED
C     STATEMENT
 3010FORMAT (///,5X,29HZERO DIVISOR AFTER STATEMENT
     1,F7.0,///)
      IF (X) 1,2,1
    2 PRINT 301,S
    1 RETURN
      END

      SUBROUTINE MAMUL (A,B,C,M,N,MM)
C     THIS SUBROUTINE IS FOR THE COMPUTATION OF MATRIX C
C     FROM C=A*B WHERE C IS AN M*MM MATRIX, A IS AN M*N,
C     AND B IS AN N*MM
      DIMENSION A(10,10),B(10,10),C(10,10)
      DO 3 I=1,M
      DO 3 J=1,MM
      C(I,J)=0.
      DO 3 K=1,N
    3 C(I,J)=C(I,J)+A(I,K)*B(K,J)
      RETURN
      END

      SUBROUTINE MTMUL(A,B,C,M,N,K)
      DIMENSION A(10,10),B(10,10),C(10,10),AT(1,10)
      DO 1 I=1,N
    1 AT(1,I)=A(I,1)
      DO 2 I=1,K
      C(1,I)=0.
      DO 2 J=1,N
    2 C(1,I)=C(1,I)+AT(1,J)*B(J,I)
      RETURN
      END

      SUBROUTINE MAINV (A,AINV,M)
C     THIS SUBROUTINE IS FOR COMPUTING MATRIX AINV WHICH IS
C     THE INVERSE OF MATRIX A
C     INVERSE OF MATRIX A
      DIMENSION A(10,10),B(10,20),AINV(10,10)
      N = 2*M
      DO 4 I=1,M
      DO 5 J=1,M
      B(I,J) = A(I,J)
    5 CONTINUE
    4 CONTINUE
```

```
      L=M+1
      DO 6 I=1,M
      DO 7 J=L,N
      IF (I-J+M) 8,9,8
    8 B(I,J) = 0.
      GO TO 7
    9 B(I,J) = 1.
    7 CONTINUE
    6 CONTINUE
      DO 10 J=1,M
      C = B(J,J)
      IF (C) 20,21,20
   20 DO 11 K=1,N
   11 B(J,K) = B(J,K) / C
      DO 14 L=1,M
      IF(L-J) 13,14,13
   13 D = B(L,J)
      DO 15 K=1,N
   15 B(L,K) = B(L,K) - B(J,K) * D
   14 CONTINUE
   10 CONTINUE
      L = M+1
      DO 16 I=1,M
      DO 17 J=1,M
      L=M+J
      AINV(I,J)=B(I,L)
   17 CONTINUE
   16 CONTINUE
      GO TO 23
   21 PRINT 22
   22 FORMAT(///,5X,21HZERO DIVISOR IN MAINV,///)
   23 RETURN
      END

      SUBROUTINE ITER (U,Q,OMEGA,M,NOTE)
      DIMENSION U(10,10),Q(10,1),A(10,1)
      X=0.
      DO 1 I=1,M
    1 Q(I)=1.
      GO TO (3,7),NOTE
    7 DO 8 I=1,M,2
    8 Q(I)=-1.
    3 CALL MAMUL(U,Q,A,M,M,1)
      RO=A(M)
      IF (RO) 20,21,20
   21 RO=A(M-1)
   20 CONTINUE
      IF (RO) 22,23,22
   23 RO=A(M-2)
   22 CONTINUE
      B=0.
      DO 6 I=1,M
```

```
    6 B=B+Q(I)
      C=0.
      CALL DIVCK (RO,20603.)
      DO 4 I=1,M
      Q(I)=A(I)/RO
    4 C=C+Q(I)
      X=X+1.
      IF(X-50.) 10,10,5
   10 IF (ABSF(B-C)-.0000001) 5,5,3
    5 OMEGA=1./RO
  300 FORMAT (10X,21HNUMBER OF ITERATIONS ,1PE15.7,//)
      PRINT 300,X
      RETURN
      END

C     THE PURPOSE OF THIS PROGRAM IS THE SOLN. OF M-DEGREE
C       OF FREEDOM, NON-SYMMETRIC, HARMONICALLY FORCED
C       VIBRATIONS PROBLEMS.  THE PROGRAM SOLVES FOR THE
C       EIGENVALUES AND EIGENVECTORS OF THE ORIGINAL SET OF
C       EQUATIONS AND OF THE ADJOINT SET OF EQUATIONS BY
C       ITERATIVE METHODS.  THE SWEEPING TECHNIQUE
C       UTILIZES THE PROPERTY OF BI-ORTHOGONALITY.
C     BASIC EQUATIONS  1./(OMEGA**2)*A=XKI*M*A
C       WHERE A ON THE RIGHT IS A TRIAL EIGENVECTOR
C       XKI IS THE INVERSE OF THE STIFFNESS MATRIX
C       M IS THE MASS MATRIX
C       A ON THE LEFT IS THE UNITIZED RESULT
C     SIZES    XK=M*M    XM=M*M   A=M*1
C    INPUT DATA
C      SIZE M AND INDEX---------        2I2
C      XK OR XKI -------                8F10.0
C      XM ------                        8F10.0
C      NUMBER OF FORCING TERMS N--------I2
C      FORCE MATRIX F--------           8F10.0
C      FORCING FREQUENCY OM----------   8F10.0
      DIMENSIONXK(10,10),XKI(10,10),U(10,10),UQ(10,10)
      DIMENSION XM(10,10),Q(10,1)
      DIMENSION PHI(10,10),PHIT(10,10)
      DIMENSION X(10,10),Y(10,10),Z(10,10)
      DIMENSION EXK(10,10)
      DIMENSIONFO(10,10),OM(10,1),OMEG(10,1),XX(10,10)
      DIMENSION COEFX(10,10)
      DIMENSION XKIT(10,10),XMT(10,10),UA(10,10),UQA(10,10)
      DIMENSION QA(10,1),PHITA(10,10)
      DIMENSION PHIA(10,10)
  100 FORMAT(1H1,4X,18HENTER NEW DATA SET,//)
  101 FORMAT(2I2)
  102 FORMAT(10X,2HM=,I2,5X,6HINDEX=,I2)
  103 FORMAT(8F10.0 )
  1040FORMAT(15X,16HSTIFFNESS MATRIX,10X,
     118HFLEXIBILITY MATRIX,/)
  105 FORMAT (10X,2I2,1PE15.7,8X,2I2,1PE15.7)
```

```
  106 FORMAT (26X,2I2,1PE15.7)
  1070FORMAT (15X,15HPHI (GIVEN SET),8X,
     121HPHI BAR (ADJOINT SET),/)
  109 FORMAT(5X,14HEND OF PROBLEM)
  110 FORMAT(//)
  1110FORMAT(15X,16HGENERALIZED MASS,8X,
     121HGENERALIZED STIFFNESS,/)
  112 FORMAT (30X,11HMASS MATRIX,/)
  113 FORMAT (I2)
  114 FORMAT (10X,6HOMEGA ,I2,9H SQUARED=,1PE15.7)
  115 FORMAT(14X,12HFORCE MATRIX,15X,17HDRIVING FREQUENCY)
  116 FORMAT(10X,23H(SINE OR COSINE COEFF.),/)
  117 FORMAT(11X,2I2,1PE15.7,12X,1PE15.7)
  118 FORMAT(13X,14HORIGINAL COORD,15X,17HGENERALIZED COORD)
  119 FORMAT(17X,6HOUTPUT,24X,6HOUTPUT,/)
  120 FORMAT (11X,2I2,1PE15.7,10X,2I2,1PE15.7)
C    READ INPUT DATA AND COMPUTE XKI
    1 READ 101,M,INDEX
      PRINT 100
      L=0
      NOTE=1
C     INDEX=1 MEANS THAT THE XK MATRIX IS TO BE READ
C     INDEX=2 MEANS THAT THE C=XKI MATRIX IS TO BE READ
      PRINT 102,M,INDEX
      PRINT 110
      GO TO (2,3),INDEX
    2 READ 103,((XK(I,J),I=1,M),J=1,M)
      CALL MAINV(XK,XKI,M)
      GO TO 4
    3 READ 103,((XKI(I,J),I=1,M),J=1,M)
      CALL MAINV(XKI,XK,M)
    4 CONTINUE
      READ 103,((XM(I,J),I=1,M),J=1,M)
      PRINT 104
      PRINT 105,((I,J,XK(I,J),I,J,XKI(I,J),I=1,M),J=1,M)
      PRINT 110
      PRINT 112
      PRINT 106,((I,J,XM(I,J),I=1,M),J=1,M)
      PRINT 110
C     FORM THE ADJOINT EQUATIONS
      DO 15 I=1,M
      DO 15 J=1,M
      XKIT(I,J)=XKI(J,I)
      XMT(I,J)=XM(J,I)
   15 EXK(I,J)=XK(I,J)
C    COMPUTE U=XKI*XM
      CALL MAMUL(XKI,XM,U,M,M,M)
      CALL MAMUL(XKIT,XMT,UA,M,M,M)
C    ITERATE
    5 CALL ITER(U,Q,OMEGA,M,NOTE)
      CALL ITER(UA,QA,FREQA,M,NOTE)
      L=L+1
```

```
      DO 10 I=1,M
      PHI(I,L)=Q(I)
      PHIA(I,L)=QA(I)
      PHITA(L,I)=QA(I)
   10 PHIT(L,I)=Q(I)
      OMEG(L,1) = OMEGA
      IF(L-M) 6,8,8
C     FORM NEW U=U-UQ BY USING BI-ORTH. RELATIONSHIPS
    6 CALL MAMUL(XM,Q,UQ,M,M,1)
      CALL MTMUL(QA,UQ,XK,1,M,1)
      DEN=XK(1,1)*(OMEGA)
      CALL MTMUL(QA,XM,XK,1,M,M)
      CALL MAMUL(Q,XK,UQ,M,1,M)
      CALL MAMUL (XMT,QA,UQA,M,M,1)
      CALL MTMUL(Q,UQA,XK,1,M,1)
      DENA=XK(1,1)*(OMEGA)
      CALL MTMUL (Q,XMT,XK,1,M,M)
      CALL MAMUL (QA,XK,UQA,M,1,M)
      CALL DIVCK (DEN,10614.)
      CALL DIVCK(DENA,10612.)
      DO 7 I=1,M
      DO 7 J=1,M
      UQA(I,J)=UQA(I,J)/DENA
      UA(I,J)=UA(I,J)-UQA(I,J)
      UQ(I,J)=UQ(I,J)/DEN
    7 U(I,J)=U(I,J)-UQ(I,J)
      AA=0.
      BB=0.
      DO 11 I=1,M
      AA=AA+1.
   11 BB=BB+Q(I)
      IF (AA-BB) 5,12,5
   12 NOTE=2
      GO TO 5
C     FIND GENERALIZED MASS AND STIFFNESS MATRICES TO TEST
C        FOR ACCURACY
    8 CALL MAMUL (XM,PHI,X,M,M,M)
      CALL MAMUL (PHITA,X,Y,M,M,M)
      CALL MAMUL (EXK,PHI,X,M,M,M)
      CALL MAMUL (PHITA,X,Z,M,M,M)
      PRINT 107
      PRINT 105,((I,J,PHI(I,J),I,J,PHIA(I,J),I=1,M),J=1,M)
      PRINT 110
      PRINT 114,(I,OMEG(I,1),I=1,M)
      PRINT 110
      PRINT 111
      PRINT 105, ((I,J,Y(I,J),I,J,Z(I,J),I=1,M),J=1,M)
      PRINT 110
C     READ FORCE DATA
      READ 113,N
      IF(N) 16,25,16
   16 READ 103, ((FO(I,J),I=1,M),J=1,N)
```

```
      CALL MAMUL (PHITA,FO,X,M,M,N)
      DO 19 J=1,N
      DO 19 I=1,M
      DIV=Y(I,I)
      CALL DIVCK (DIV,11900.)
   19 X(I,J)=X(I,J)/Y(I,I)
      READ 103, (OM(I,1),I=1,N)
C     COMPUTE GENERALIZED CO-ORD.
      DO 17 J=1,N
      DO 17 I=1,M
      DIV= OMEG(I,1)-OM(J,1)*OM(J,1)
      CALL DIVCK (DIV,11700.)
   17 COEFX(I,J)=1./DIV
      DO 18 J=1,N
      DO 18 I=1,M
   18 X(I,J)=X(I,J)*COEFX(I,J)
C     TRANSFORM TO ORIGINAL CO-ORD.
      CALL MAMUL (PHI,X,XX,M,M,N)
      PRINT 115
      PRINT 116
      PRINT 117,((I,J,FO(I,J),OM(J,1),I=1,M),J=1,N)
      PRINT 110
      PRINT 118
      PRINT 119
      PRINT120,((I,J,XX(I,J),I,J,X(I,J),I=1,M),J=1,N)
   25 CONTINUE
      PRINT 109
      GO TO 1
      END
```

APPENDIX C

FORCE-INPUT TYPE PROBLEMS

The problems that make up this Appendix are all of the usual force-input type. They were solved in order to obtain motion inputs for the inverse problem examples. They are included here to show the validity of the answers in the examples.

The problems included are:

1) four degree of freedom; longitudinal system; steady-state response due to a harmonic force applied to the end mass
2) six degree of freedom; longitudinal system; steady-state response due to a harmonic force applied to the end mass
3) three degree of freedom; beam in bending; response due to a step force applied to the end mass
4) four degree of freedom; beam in bending; steady-state response due to a harmonic force applied to the end mass
5) quadruple pendulum; response due to a step torque applied to the last pendulum.

Problem 1)

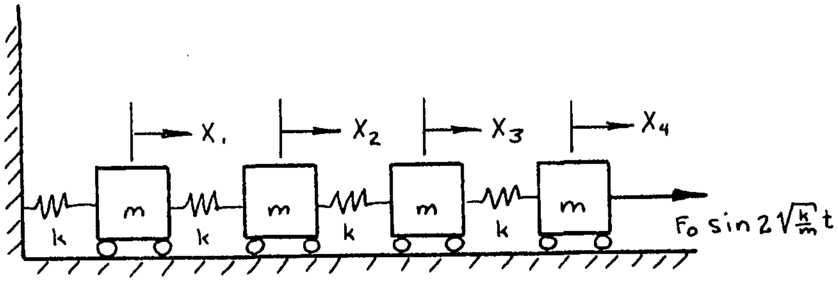

Fig. C.1

The equations of motion are

$$m \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{Bmatrix} + k \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ F_o \sin \Omega t \end{Bmatrix}$$

where $\Omega = 2\sqrt{\frac{k}{m}}$

The natural frequencies and normal mode shapes are

$$\omega^2 = (.1206;\ 1.0;\ 2.347;\ 3.532)\ \frac{k}{m}$$

$$[\varphi] = \begin{bmatrix} .347 & -1.0 & 1.532 & -1.879 \\ .652 & -1.0 & -\ .532 & 2.879 \\ .879 & 0 & -1.347 & -2.532 \\ 1. & 1. & 1. & 1. \end{bmatrix}$$

Transformed to the normal coordinates, the equations of motion are

$$m \begin{bmatrix} 2.319 & 0 & 0 & 0 \\ 0 & 3.0 & 0 & 0 \\ 0 & 0 & 5.445 & 0 \\ 0 & 0 & 0 & 19.23 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \ddot{q}_4 \end{Bmatrix}$$

$$+ k \begin{bmatrix} .279 & 0 & 0 & 0 \\ 0 & 3.0 & 0 & 0 \\ 0 & 0 & 12.78 & 0 \\ 0 & 0 & 0 & 67.93 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} F_o \sin \Omega t$$

Solving for the steady-state motion and transforming to the original coordinates yields

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} .111 \\ -.222 \\ .333 \\ -.444 \end{Bmatrix} \frac{F_o}{k} \sin 2\sqrt{\frac{k}{m}}\, t$$

Problem 2)

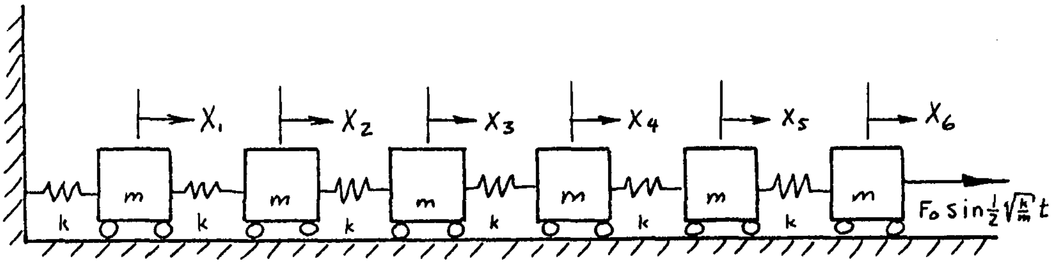

Fig. C.2

The equations of motion are

$$m \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \\ \ddot{x}_5 \\ \ddot{x}_6 \end{Bmatrix}$$

$$+ k \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ F_o \sin \Omega t \end{Bmatrix}$$

where $\Omega = \frac{1}{2}\sqrt{\frac{k}{m}}\, t$

The natural frequencies and normal mode shapes are

$$\omega^2 = (.058;\ .5029;\ 1.290;\ 2.241;\ 3.136;\ 3.77)\ \frac{k}{m}$$

$$[\varphi] = \begin{bmatrix} .241 & -.709 & 1.136 & -1.49 & 1.77 & -1.94 \\ .468 & -1.06 & .806 & .360 & -2.01 & 3.44 \\ .667 & -.880 & -.564 & 1.41 & .515 & -4.14 \\ .829 & -.256 & -1.20 & -.70 & 1.42 & 3.90 \\ .941 & .497 & -.290 & -1.24 & -2.13 & -2.77 \\ 1. & 1. & 1. & 1. & 1. & 1. \end{bmatrix}$$

Transformed to the normal coordinates, the equations of motion are

$$m \begin{bmatrix} 3.29 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.71 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4.79 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7.39 & 0 & 0 \\ 0 & 0 & 0 & 0 & 15.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 56.7 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \ddot{q}_4 \\ \ddot{q}_5 \\ \ddot{q}_6 \end{Bmatrix}$$

$$+ \ k \begin{bmatrix} .191 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.86 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6.19 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16.56 & 0 & 0 \\ 0 & 0 & 0 & 0 & 47.19 & 0 \\ 0 & 0 & 0 & 0 & 0 & 213.9 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} F_o \sin \Omega t$$

Solving for the steady-state motion and transforming to the original coordinates yields

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} -\ .978 \\ -1.711 \\ -2.017 \\ -1.818 \\ -1.165 \\ -\ .221 \end{Bmatrix} \frac{F_o}{k} \sin \frac{1}{2} \sqrt{\frac{k}{m}}\, t$$

Problem 3)

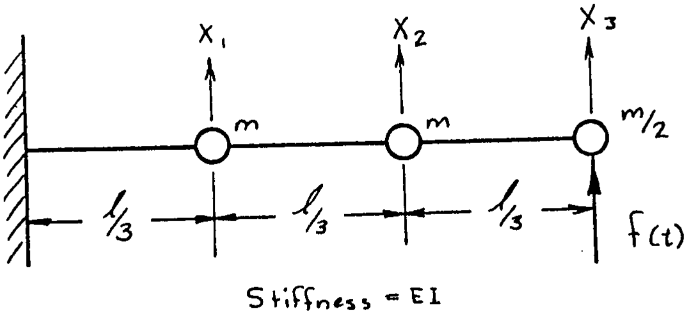

Fig. C.3

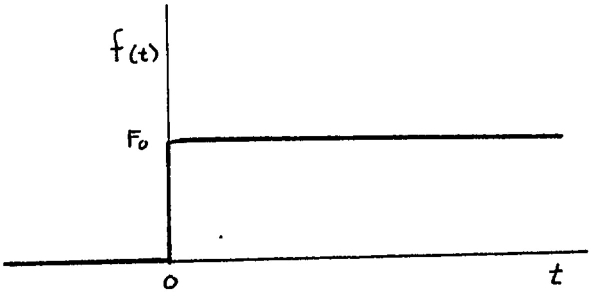

Fig. C.4

The equations of motion are

$$m\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\end{Bmatrix} + K\begin{bmatrix}80 & -46 & 12\\ -46 & 44 & -16\\ 12 & -16 & 7\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ f(t)\end{Bmatrix}$$

where $K = \frac{81}{13}\ \frac{EI}{\ell^3}$

The natural frequencies and normal mode shapes are

$$\omega^2 = (3.73;\ 119;\ 738)\ \frac{EI}{m\ell^3}$$

$$[\varphi] = \begin{bmatrix}.1617 & -.731 & 2.22\\ .540 & .707 & -1.59\\ 1. & 1. & 1.\end{bmatrix}$$

Transformed to the normal coordinates, the equations of motion are

$$m\begin{bmatrix}.818 & 0 & 0\\ 0 & 1.53 & 0\\ 0 & 0 & 7.98\end{bmatrix}\begin{Bmatrix}\ddot{q}_1\\ \ddot{q}_2\\ \ddot{q}_3\end{Bmatrix} + K\begin{bmatrix}.489 & 0 & 0\\ 0 & 29.2 & 0\\ 0 & 0 & 944.2\end{bmatrix}\begin{Bmatrix}q_1\\ q_2\\ q_3\end{Bmatrix} = \begin{Bmatrix}F_o\\ F_o\\ F_o\end{Bmatrix}$$

If homogeneous initial conditions are assumed, the solution in the original coordinates is

$$x_1 = \frac{F_o \ell^3}{EI} (.04938 - .053 \cos \omega_1 t + .0040 \cos \omega_2 t$$

$$- .00038 \cos \omega_3 t)$$

$$x_2 = \frac{F_o \ell^3}{EI} (.1729 - .177 \cos \omega_1 t + .00388 \cos \omega_2 t$$

$$+ .000253 \cos \omega_3 t)$$

$$x_3 = \frac{F_o \ell^3}{EI} (.333 - .328 \cos \omega_1 t - .00549 \cos \omega_2 t$$

$$- .00017 \cos \omega_3 t)$$

Problem 4)

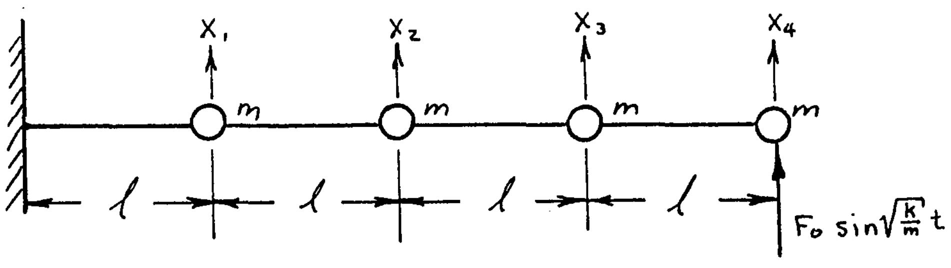

Fig. C.5

The equations of motion are

$$m \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{Bmatrix} + K \begin{bmatrix} 18.8 & -11.8 & 4.45 & -.74 \\ -11.8 & 14.35 & -9.59 & 2.59 \\ 4.45 & -9.59 & 9.89 & -3.65 \\ -.74 & 2.59 & -3.65 & 1.61 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \\ f(t) \end{Bmatrix}$$

where $K = \frac{EI}{\ell^3}$ and $f(t) = F_o \sin\sqrt{\frac{k}{m}}\, t$.

The natural frequencies and normal mode shapes are

$$\omega^2 = (.0308;\ 1.28;\ 10.21;\ 33.14)\ \frac{k}{m}$$

$$[\varphi] = \begin{bmatrix} .0925 & -.695 & 2.34 & -5.71 \\ .328 & -1.37 & .781 & 5.53 \\ .646 & -.748 & -2.27 & -3.53 \\ 1. & 1. & 1. & 1. \end{bmatrix}$$

Transformed to the normal coordinates, the equations of motion are

$$m \begin{bmatrix} 1.53 & 0 & 0 & 0 \\ 0 & 3.93 & 0 & 0 \\ 0 & 0 & 12.3 & 0 \\ 0 & 0 & 0 & 76.8 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \ddot{q}_4 \end{Bmatrix}$$

$$+ K \begin{bmatrix} .0473 & 0 & 0 & 0 \\ 0 & 5.04 & 0 & 0 \\ 0 & 0 & 125. & 0 \\ 0 & 0 & 0 & 2544. \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} F_o \sin\sqrt{\frac{k}{m}}\, t$$

Solving for the steady-state motion and transforming to the original coordinates yields

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} -\ .674 \\ -1.458 \\ -1.13 \\ .243 \end{Bmatrix} \frac{F_o}{K} \sin\sqrt{\frac{k}{m}}\, t$$

Problem 5)

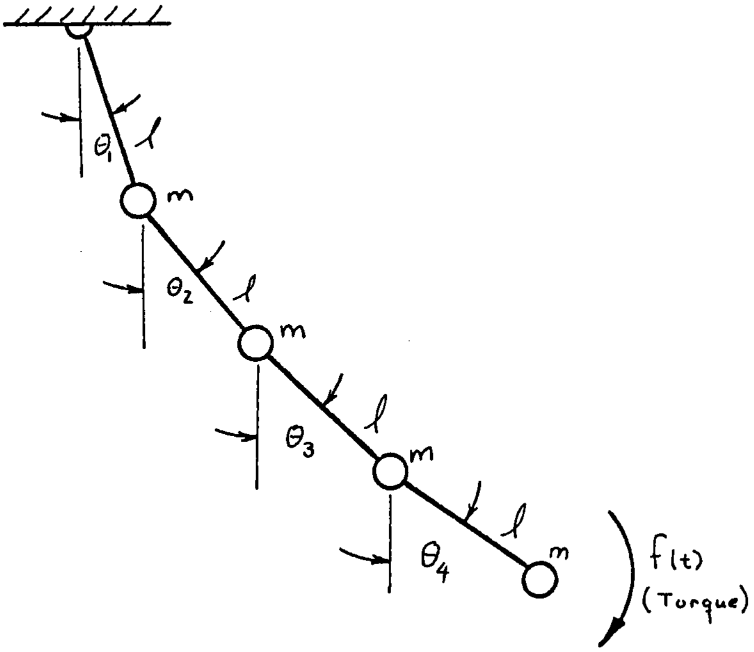

Fig. C.6

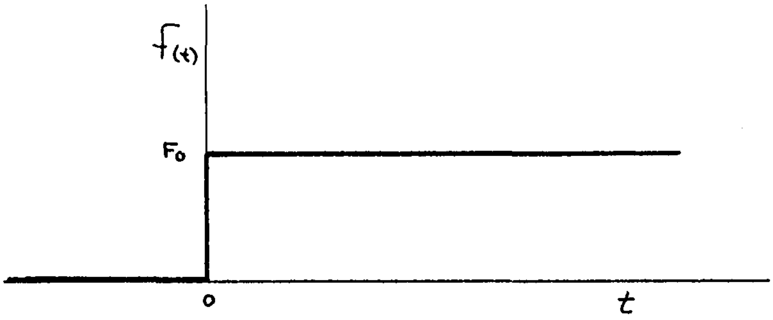

Fig. C.7

The equations of motion are

$$M \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \end{Bmatrix} + K \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ f(t) \end{Bmatrix}$$

where $M = m\ell^2$; $K = mg\ell$; $f(t) = F_o$

The natural frequencies and normal mode shapes are

$$\omega^2 = (.322;\ 1.745;\ 4.53;\ 9.395)\ \frac{K}{M}$$

$$[\phi] = \begin{bmatrix} .5668 & -.316 & .595 & -3.5 \\ .6948 & -.238 & -.106 & 6.3 \\ .839 & .127 & -1.26 & -3.7 \\ 1. & 1. & 1. & 1. \end{bmatrix}$$

Transformed to the normal coordinates, the equations of motion are

$$M \begin{bmatrix} 15.9 & 0 & 0 & 0 \\ 0 & .918 & 0 & 0 \\ 0 & 0 & 1.24 & 0 \\ 0 & 0 & 0 & 21. \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \ddot{q}_4 \end{Bmatrix} + K \begin{bmatrix} 5.14 & 0 & 0 & 0 \\ 0 & 1.6 & 0 & 0 \\ 0 & 0 & 5.67 & 0 \\ 0 & 0 & 0 & 197.3 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} F_o \\ F_o \\ F_o \\ F_o \end{Bmatrix}$$

If the pendulum starts from rest, the solution in the original coordinates is

$$\theta_1 = \frac{F_o}{K}(-.111 \cos \omega_1 t + .1975 \cos \omega_2 t - .105 \cos \omega_3 t + .01775 \cos \omega_4 t)$$

$$\theta_2 = \frac{F_o}{K}(-.1355 \cos \omega_1 t + .149 \cos \omega_2 t + .0187 \cos \omega_3 t - .0319 \cos \omega_4 t)$$

$$\theta_3 = \frac{F_o}{K}(-.164 \cos \omega_1 t - .0793 \cos \omega_2 t + .223 \cos \omega_3 t + .0187 \cos \omega_4 t)$$

$$\theta_4 = \frac{F_o}{K}(1. - .1955 \cos \omega_1 t - .625 \cos \omega_2 t - .1765 \cos \omega_3 t - .00507 \cos \omega_4 t)$$

APPENDIX D

INVERSE PROBLEMS

In this appendix, solutions for several inverse problems are worked out in detail. Excerpts from these problems appear as examples in the text.

The problems included are:

Problem 1) three degree of freedom beam in bending; unknown force at x_3; motion known at x_1; homogeneous initial conditions

Problem 2) quadruple pendulum; unknown torqe at θ_4; motion of θ_3 known; homogeneous initial conditions

Problem 3) four degree of freedom longitudinal system; unknown force at x_4; motion of x_3 known; steady-state solution only

Problem 4) four degree of freedom beam in bending; unknown force at x_4; motion of x_1 known; steady-state solution; computer solution

Problem 5) six degree of freedom longitudinal system; unknown force at x_6; motion of x_1 known; steady-state solution.

Problem 1)

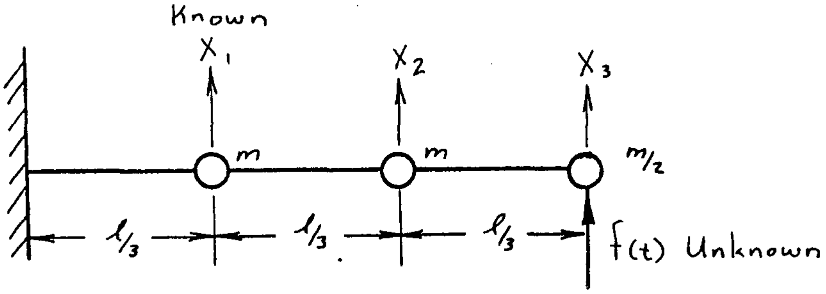

Fig. D.1

Assume
$$x_1 = (.04938 - .053 \cos \bar{\omega}_1 t + .0040 \cos \bar{\omega}_2 t - .00038 \cos \bar{\omega}_3 t) \frac{F_o \ell^3}{EI}$$

where

$$\bar{\omega}_1^{\,2} = 3.73 \frac{EI}{m\ell^3}$$

$$\bar{\omega}_2^{\,2} = 119. \frac{EI}{m\ell^3}$$

$$\bar{\omega}_3^{\,2} = 738. \frac{EI}{m\ell^3}$$

and that the beam is initially at rest.

The equations of motion are

$$m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + K \begin{bmatrix} 80 & -46 & 12 \\ -46 & 44 & -16 \\ 12 & -16 & 7 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ f(t) \end{Bmatrix} \tag{D-1}$$

where $K = \frac{81}{13} \frac{EI}{\ell^3}$

The methods of Sec. 2.2 can be used to obtain the reduced equations in the form

$$m \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + K \begin{bmatrix} -46 & 12 \\ 44 & -16 \end{bmatrix} \begin{Bmatrix} x_2 \\ x_3 \end{Bmatrix} = - \begin{bmatrix} 1 & 80 \\ 0 & -46 \end{bmatrix} \begin{Bmatrix} m\ddot{x}_1 \\ Kx_1 \end{Bmatrix}$$

The eigenvalues and eigenvectors are

$$\omega^2 = 107 \frac{EI}{m\ell^3};\ \infty$$

$$[\varphi] = \begin{bmatrix} .261 & 0 \\ 1 & 1 \end{bmatrix}$$

The adjoint equations can be written as

$$m \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + K \begin{bmatrix} -46 & 44 \\ 12 & -16 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The adjoint eigenvectors are $[\overline{\varphi}] = \begin{bmatrix} 1.335 & 1 \\ 1 & 0 \end{bmatrix}$

Following the procedure outlined in Sec. 2.2, the uncoupled equations of motion are

$$m \begin{bmatrix} .261 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + K \begin{bmatrix} -4.5 & 0 \\ 0 & 12 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$= - \begin{bmatrix} 1.335 & 61 \\ 1 & 80 \end{bmatrix} \begin{Bmatrix} m\ddot{x}_1 \\ Kx_1 \end{Bmatrix}$$

Substituting the given value of x_1 into the first of these equations yields

$$\ddot{q}_1 - 107 \frac{EI}{m\ell^3} q_1 = [-72. + 76.5 \cos \bar{\omega}_1 t - 3.37 \cos \bar{\omega}_2 t$$

$$- .882 \cos \bar{\omega}_3 t] \frac{EI}{m\ell^3}$$

Thus,

$$q_1 = (.672 - .691 \cos \bar{\omega}_1 t + .0149 \cos \bar{\omega}_2 t$$

$$+ .00104 \cos \bar{\omega}_3 t) \frac{F_o}{K}$$

The second equation yields directly

$$q_2 = (- .329 + .352 \cos \bar{\omega}_1 t - .0202 \cos \bar{\omega}_2 t$$

$$- .0012 \cos \bar{\omega}_3 t) \frac{F_o}{K}$$

Transforming this solution to the x coordinates yields

$$x_2 = (.1729 - .177 \cos \bar{\omega}_1 t + .00388 \cos \bar{\omega}_2 t$$

$$+ .000253 \cos \bar{\omega}_3 t) \frac{F_o}{K}$$

$$x_3 = (.333 - .328 \cos \bar{\omega}_1 t - .00549 \cos \bar{\omega}_2 t$$

$$- .00017 \cos \bar{\omega}_3 t) \frac{F_o}{K}$$

These values can be used in the last of Eqs. D-1 to yield the unknown force. Thus,

$$f(t) = \frac{1}{2} m\ddot{x}_3 + 12 Kx_1 - 16 Kx_2 + 7 Kx_3$$

or

$$f(t) = F_o \qquad \text{(a step function)}$$

Problem 2)

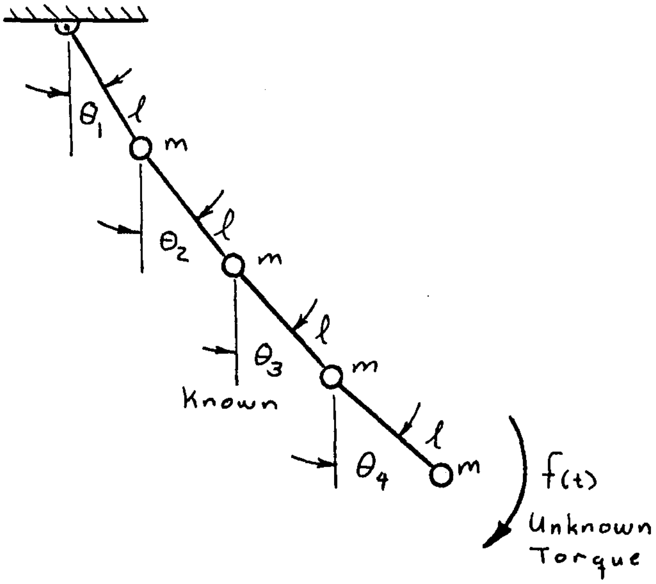

Fig. D.2

Assume

$$\theta_3 = \frac{F_o}{K}(-.164 \cos \bar{\omega}_1 t - .0793 \cos \bar{\omega}_2 t + .223 \cos \bar{\omega}_3 t + .0187 \cos \bar{\omega}_4 t)$$

where

$$\bar{\omega}_1{}^2 = .322 \frac{K}{M} \qquad \bar{\omega}_3{}^2 = 4.53 \frac{K}{M}$$

$$\bar{\omega}_2{}^2 = 1.745 \frac{K}{M} \qquad \bar{\omega}_4{}^2 = 9.395 \frac{K}{M}$$

and that the system is initially at rest. The equations of motion are

$$M \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \end{Bmatrix} + K \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ f(t) \end{Bmatrix}$$

where $M = m\ell^2$ and $K = mg\ell$. (D-2)

The reduced equations and their eigenvectors are

$$M \begin{bmatrix} 4 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_4 \end{Bmatrix} + K \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_4 \end{Bmatrix} = - \begin{bmatrix} 2 & 0 \\ 2 & 0 \\ 2 & 2 \end{bmatrix} \begin{Bmatrix} M\ddot{\theta}_3 \\ K\theta_3 \end{Bmatrix}$$

$$[\varphi] = \begin{bmatrix} 0 & -.266 & .945 \\ 0 & -.234 & -1.445 \\ 1 & 1 & 1 \end{bmatrix}$$

The adjoint equations and their eigenvectors are

$$M \begin{bmatrix} 4 & 3 & 2 \\ 3 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + K \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$[\overline{\varphi}] = \begin{bmatrix} 0 & -.533 & 1.87 \\ 0 & -.467 & -2.87 \\ 1 & 1 & 1 \end{bmatrix}$$

The uncoupled equations can be written

$$M \begin{bmatrix} 1 & 0 & 0 \\ 0 & .6425 & 0 \\ 0 & 0 & 2.258 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + K \begin{bmatrix} 0 & 0 & 0 \\ 0 & .896 & 0 \\ 0 & 0 & 19.51 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 0 & 2 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} M\ddot{\theta}_3 \\ K\theta_3 \end{Bmatrix} \qquad \text{(D-3)}$$

The first of these equations is

$$\ddot{q}_1 = -2\,\ddot{\theta}_3 - 2\,\frac{k}{m}\,\theta_3$$

the solution to which appears in detail in Example 6.

The result for q_1 is

$$q_1 = \frac{F_o}{K} (1.0 - .688 \cos \overline{\omega}_1 t + .068 \cos \overline{\omega}_2 t -.347 \cos \overline{\omega}_3 t - .033 \cos \overline{\omega}_4 t)$$

The solution to the second equation of Eqs. D-3 yields

$$q_2 = \frac{F_o}{K} (.477 \cos \overline{\omega}_1 t - .706 \cos \overline{\omega}_2 t + .222 \cos \overline{\omega}_3 t + .00728 \cos \overline{\omega}_4 t)$$

and the solution for the third equation is

$$q_3 = \frac{F_o}{K} (.0175 \cos \overline{\omega}_1 t + .0102 \cos \overline{\omega}_2 t - .0480 \cos \overline{\omega}_3 t + .0226 \cos \overline{\omega}_4 t)$$

This solution transformed to the θ coordinates is

$$\theta_1 = \frac{F_o}{K} (.111 \cos \overline{\omega}_1 t + .1975 \cos \overline{\omega}_2 t - .105 \cos \overline{\omega}_3 t + .01775 \cos \overline{\omega}_4 t)$$

$$\theta_2 = \frac{F_o}{K} (.1355 \cos \overline{\omega}_1 t + .149 \cos \overline{\omega}_2 t + .0187 \cos \overline{\omega}_3 t - .0319 \cos \overline{\omega}_4 t)$$

$$\theta_4 = \frac{F_o}{K} (1.0 - .1955 \cos \overline{\omega}_1 t - .625 \cos \overline{\omega}_2 t - .1765 \cos \overline{\omega}_3 t - .00507 \cos \overline{\omega}_4 t)$$

These values can be used in the last equation of Eq. D-2 to evaluate f(t) as

$$f(t) = M\,(\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{\theta}_3 + \ddot{\theta}_4) + K\,\theta_4$$

or as

$$f(t) = F_o \qquad \text{(a step function)}$$

<u>Problem 3)</u>

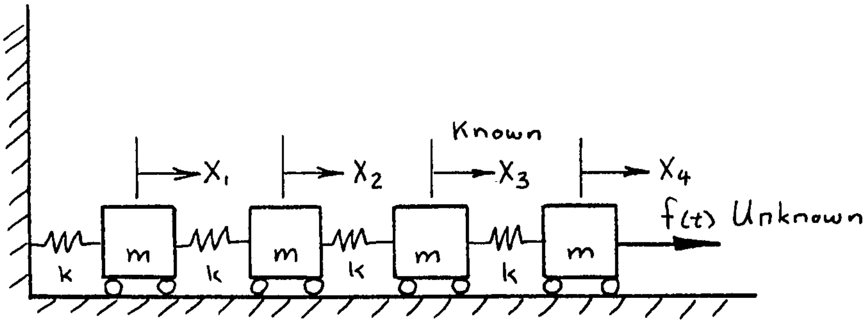

Fig. D.3

Assume $x_3 = .333\,\frac{F_o}{k}\sin 2\sqrt{\frac{k}{m}}\,t$

with initial conditions such that no transients exist. The equations of motion are

$$m\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\\ddot{x}_2\\\ddot{x}_3\\\ddot{x}_4\end{Bmatrix} + k\begin{bmatrix}2&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&1\end{bmatrix}\begin{Bmatrix}x_1\\x_2\\x_3\\x_4\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\\f(t)\end{Bmatrix} \tag{D-4}$$

The reduced equations and their eigenvectors are

$$m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_4 \end{Bmatrix} + k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_4 \end{Bmatrix} = - \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} m\ddot{x}_3 \\ kx_3 \end{Bmatrix}$$

$$[\varphi] = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

The adjoint equations and their eigenvectors are

$$m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$[\overline{\varphi}] = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The uncoupled equations are

$$m \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + k \begin{bmatrix} -2 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = - \begin{bmatrix} 0 & -1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} m\ddot{x}_3 \\ kx_3 \end{Bmatrix}$$

(D-5)

The solutions to these equations are

$$q_1 = .0555 \frac{F_o}{k} \sin 2\sqrt{\frac{k}{m}}\, t$$

$$q_2 = .1666 \frac{F_o}{k} \sin 2\sqrt{\frac{k}{m}}\, t$$

$$q_3 = -.666 \frac{F_o}{k} \sin 2\sqrt{\frac{k}{m}}\, t$$

These solutions transformed to the x coordinates are

$$x_1 = .111 \frac{F_o}{k} \sin 2\sqrt{\frac{k}{m}}\, t$$

$$x_2 = -\ .222 \frac{F_o}{k} \sin 2\sqrt{\frac{k}{m}}\, t$$

$$x_3 = -.444 \frac{F_o}{k} \sin 2\sqrt{\frac{k}{m}}\, t$$

The last equation of Eqs. D-4 is used to evaluate f(t) as

$$f(t) = F_o \sin 2\sqrt{\frac{k}{m}}\, t$$

Problem 4)

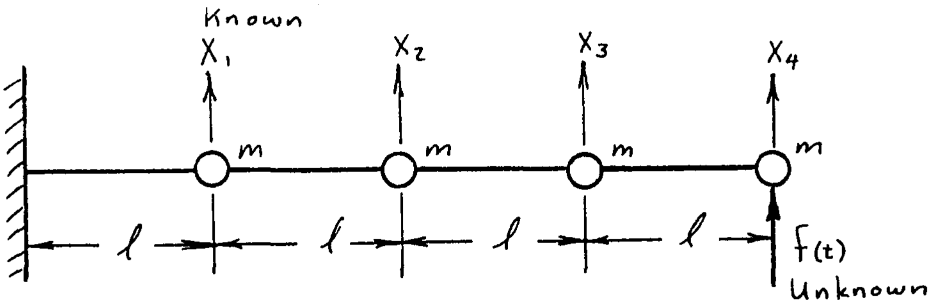

Fig. D.4

Assume

$$x_1 = -\ .674\ \frac{F_o}{K} \sin \sqrt{\frac{K}{m}}\ t$$

and initial conditions such that no transients exist.

The equations of motion are

$$m \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{Bmatrix} + K \begin{bmatrix} 18.8 & -11.8 & 4.45 & -\ .74 \\ -11.8 & 14.35 & -9.59 & 2.59 \\ 4.45 & -9.59 & 9.89 & -3.65 \\ -\ .74 & 2.59 & -3.65 & 1.61 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \\ f(t) \end{Bmatrix} \qquad \text{(D-6)}$$

where $K = \frac{EI}{\ell^3}$

The reduced equations are

$$m \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{Bmatrix} + K \begin{bmatrix} -11.8 & 4.45 & -\ .74 \\ 14.35 & -9.59 & 2.59 \\ -\ 9.59 & 9.89 & -3.65 \end{bmatrix} \begin{Bmatrix} x_2 \\ x_3 \\ x_4 \end{Bmatrix}$$

$$= -\begin{bmatrix} 1 & 18.8 \\ 0 & -11.8 \\ 0 & 4.45 \end{bmatrix} \begin{Bmatrix} m\ddot{x}_1 \\ Kx_1 \end{Bmatrix} \qquad \text{(D-7)}$$

These equations cannot be solved using the computer program in Appendix B due to the singular mass matrix. They can be modified, however, as discussed in Sec. 4.1 and Example 14 so that they can be solved. The result (from Example 14) is

$$m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + K \begin{bmatrix} -27.15 & 5.96 \\ 48.81 & -12.01 \end{bmatrix} \begin{Bmatrix} x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} -3.5 & -54.1 \\ 4.94 & 88.35 \end{bmatrix} \begin{Bmatrix} m\ddot{x}_1 \\ Kx_1 \end{Bmatrix}$$

These equations can be solved by using the computer program listed in Appendix B. The result is

$$x_2 = -\ 1.458\ \frac{F_o}{K} \sin\sqrt{\frac{K}{m}}\ t$$

$$x_3 = -\ 1.134\ \frac{F_o}{K} \sin\sqrt{\frac{K}{m}}\ t$$

These values can be used in the first equation of Eqs. D-7 to evaluate x_4. Thus,

$$x_4 = .2432\ \frac{F_o}{K} \sin\sqrt{\frac{K}{m}}\ t$$

These values for x_2, x_3, and x_4 can be used in the first equation of Eqs. D-6 to determine f(t). Hence,

$$f(t) = F_o \sin\sqrt{\frac{K}{m}}\ t$$

Problem 5)

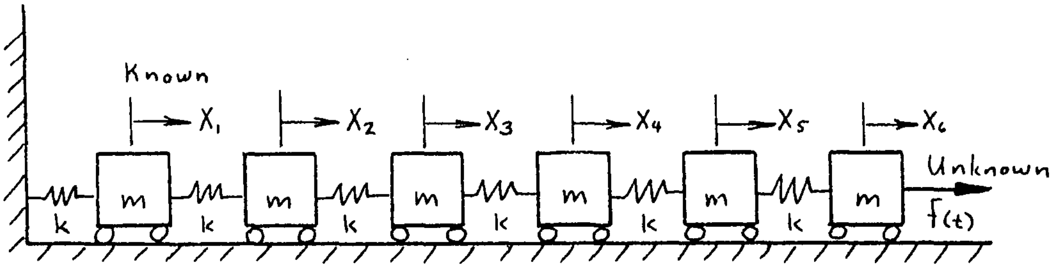

Fig. D.5

Assume

$$x_1 = - .978 \frac{F_o}{k} \sin \frac{1}{2} \sqrt{\frac{k}{m}}\, t$$

and that the initial conditions are such that no transients exist. The equations of motion are

$$m \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \\ \ddot{x}_5 \\ \ddot{x}_6 \end{Bmatrix}$$

$$+ \quad k \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ f(t) \end{Bmatrix}$$

In accordance with Sec. 3.2, the transformation is

$$\{x\} = [\Phi]\{y\}$$

where

$$[\Phi] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 4/5 & 1 & -1 & 1 & -1 & 1/5 \\ 3/5 & 1.618 & -.618 & -.618 & 1.618 & 2/5 \\ 2/5 & 1.618 & .618 & -.618 & -1.618 & 3/5 \\ 1/5 & 1 & 1 & 1 & 1 & 4/5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The equations, after being premultiplied by $[\Phi]^T$ can be written as

$$m \begin{bmatrix} 2.2 & 2.606 & -.724 & .382 & -.286 & .80 \\ 2.606 & 7.236 & 0 & 0 & 0 & 2.606 \\ -.724 & 0 & 2.76 & 0 & 0 & .724 \\ .382 & 0 & 0 & 2.76 & 0 & .382 \\ -.286 & 0 & 0 & 0 & 7.236 & .286 \\ .80 & 2.606 & .724 & .382 & .286 & 2.2 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \\ \ddot{y}_5 \\ \ddot{y}_6 \end{Bmatrix}$$

$$+ \; k \begin{bmatrix} 1.2 & 0 & 0 & 0 & 0 & -.2 \\ 0 & 2.76 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.81 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7.23 & 0 & 0 \\ 0 & 0 & 0 & 0 & 26.18 & 0 \\ -.2 & 0 & 0 & 0 & 0 & .2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ f(t) \end{Bmatrix}$$

After elimination of the last equation and the placing of terms containing x_1, (y_1) on the right side, the equations can be written

$$
m \left[\begin{array}{cccc|c}
2.606 & -.724 & .382 & -.286 & .80 \\
\hline
7.236 & 0 & 0 & 0 & 2.606 \\
0 & 2.76 & 0 & 0 & .724 \\
0 & 0 & 2.76 & 0 & .382 \\
0 & 0 & 0 & 7.236 & .286
\end{array}\right]
\begin{Bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \\ \ddot{y}_5 \\ \ddot{y}_6 \end{Bmatrix}
$$

$$
+ \; k \left[\begin{array}{cccc|c}
0 & 0 & 0 & 0 & -.2 \\
\hline
2.76 & 0 & 0 & 0 & 0 \\
0 & 3.81 & 0 & 0 & 0 \\
0 & 0 & 7.23 & 0 & 0 \\
0 & 0 & 0 & 26.18 & 0
\end{array}\right]
\begin{Bmatrix} y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{Bmatrix}
$$

$$
= - \begin{bmatrix}
2.2 & 1.2 \\
2.606 & 0 \\
-.724 & 0 \\
.382 & 0 \\
-2.86 & 0
\end{bmatrix}
\begin{Bmatrix} m\ddot{x}_1 \\ kx_1 \end{Bmatrix}
$$

The remaining details of this problem appear in Example 15.

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1) Hildebrand, F. B., Methods of Applied Mathematics, Prentice-Hall, pp. 59-83, 1952.

2) Halfman, R. L., Dynamics Vol. II, Addison-Wesley Pub. Co., pp. 421-424, 1962.