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H. Bremes — Sept 19, 1967

Harvey Cohn — Sept 19, 1967

M. S. Cheema — Sept 19, 1967

S. Majumdar — Sept 19, 1967

W. E. Conway — Sept 19, 1967

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SIGNED: J. B. Smith

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ABSTRACT

In this paper two problems in approximation theory are considered. In the first part of this paper it is proved that if $\{f_k\} \epsilon L_p [0,1]$ converges to f in L_p norm and if Φ_k is the best L_p approximation to f_k (over polynomials of degree N) and Φ is the best L_p approximation to f (over polynomials of degree N) then Φ_k converges uniformly to Φ. This result is used to come arbitrarily close to the best L_2 approximant to a given $f \epsilon L_2 [0,1]$ in such a manner that numerical integration is not required.

In the second part of this paper Hermite-Fejér polynomials are used to approximate a given function continuous on $x \epsilon [-1,1]$. This approximation is integrated to provide a quadrature rule for estimation of $\int_{-1}^{1} f(x)\,dx$. This estimate is extended to functions of several variables and bounds are obtained for the error of the estimate.

Introduction

In the theory of approximation it is often desired to find the polynomial of given order which is closest, in some sense, to a given function. If the measure of closeness is the supremum norm the polynomial which is closest in this sense is called the best Tchebysheff approximant If the measure of closeness is the L_p norm the polynomial which is closest in this sense is called the best L_p approximant.

There are theorems [3] which characterize the best Tchebysheff approximant to $f(x)$ for $f(x) \in C[0, 1]$ and which characterize the best L_p approximant for $f(x) \in L_p [0,1]$. However, the problem of finding explicitly the best Tchebysheff or L_p approximant, for a given f, is an extremely difficult problem. In this paper it is proved that if $\{f_k\}$ is a sequence of functions converging in L_p norm to f, and if Φ_k is the best approximant (in the L_p norm) of f_k, then the Φ_k converge uniformly to Φ, the best approximant to f with $1 < p < \infty$. Uniform convergence (or convergence in the supremum norm) of f_k to f is also included. This allows one to get close to the best L_p approximant of a given function f by finding the best L_p approximant to a function which is sufficiently close (in L_p norm) to f. This may be a simpler problem since we may be able to find a polynomial which is sufficiently close to the function f, and it may be easier to find the best L_p approximant to the polynomial. This turns out to be the case for L_2 and theorems are proved and examples presented for this case.

In the second problem treated in this paper Hermite-Fejér polynomials are used to approximate a given continuous function. The uniform convergence of the Hermite-Fejér polynomials is used to find a quadrature rule, or approximation to the integral (over an interval), for the continuous function and this approximation converges to the integral of the function as N (the order of the Hermite-Fejér polynomial) becomes large.

The quadrature rule is denoted by

$$\int_{x=-1}^{x=1} f(x)\, dx \sim \sum_{\alpha=1}^{N} w_\alpha f(x_\alpha)\,.$$

and explicit expressions for the weights w_α are obtained. For functions with continuous first derivatives an error bound is obtained which depends on the maximum value of the first derivative of the function. The quadrature rule is generalized to functions of several variables. For functions with continuous first partial derivatives an error bound is obtained. In this paper the following symbolic notation will be used. For a set S $x \in S$ will mean x is contained in S and $x \notin S$ will indicate x not a element of S.

Definitions and Results From Functional Analysis and Approximation Theory

A norm on a linear vector space X is a real valued function, whose value at $y \in X$ is denoted by $||y||$, satisfying the following properties:

(1) $||x_1 + x_2|| \le ||x_1|| + ||x_2||$

(2) $||\alpha x|| = |\alpha| \; ||x||$ (for α a scalar)

(3) $||x|| \ge 0$

(4) $||x|| = 0$ if $x = 0$

A linear vector space on which a norm is defined becomes a metric space with metric defined by $d(x_1,x_2) = ||x_1 - x_2||$. A linear vector space which is a metric space (with metric induced by a norm) is called a normed linear space.

In this paper two normed linear spaces will be of particular interest.

The first is the space of continuous functions on $0 \le x \le 1$ with the norm defined by

$$||f(x)||_\infty = \sup_{0 \le x \le 1} |f(x)| = \max_{0 \le x \le 1} |f(x)|$$

for f(x) continuous on $[0,1]$. This normed linear space will be denoted by $C[0,1]$.

The second normed linear space of interest in this paper is the space of functions f(x) such that $\int_a^b |f(x)|^p \, dx < \infty$

where integration is in the Legesgue sense, a or b or both may be infinite, and $1 < p < \infty$. We denote this space by L_p and the norm in the space is

$$||f||_p = \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}$$

In this paper unless explicitly stated otherwise a = 0 and b = 1.

A vector space X is <u>finite-dimensional</u> if there is some positive integer N such that X contains N vectors which are linearly independent, while every set of N + 1 vectors in X is linearly dependent. For a finite-dimensional space we have the following results

(a) A finite-dimensional normed linear space is complete

(b) If X is a normed linear space, any finite-dimensional subspace of X is closed

(c) If X is a finite-dimensional normed linear space, each closed and bounded subset of X is compact.

If X is a vector space and $|| \quad ||'$ and $|| \quad ||''$ are norms on X, then $|| \quad ||'$ and $|| \quad ||''$ are said to be <u>equivalent</u> if there exists positive numbers a and b such that $a\, ||\, x\, ||' \leq ||\, x\, ||'' \leq b\, ||\, x\, ||'$ for all $x \in X$. We have the following result [1] : on a finite-dimensional space all norms are equivalent.

<u>Definition 2.1.</u> In a metric space (X,d) a subset A is relatively compact if $\bar{A}$ (the closure of A) is compact. We then have the following result [1].

A set A is relatively compact if and only if a convergent subsequence can be selected from every sequence of points in A.

Let X be a metric space and let $S \subset X$ be a subset of X. Let $s \in \bar{S}$ (the closure of S) that is s is a limit point of X.

Then if it can be shown [7] that there exists a sequence $\{s_i\} \epsilon S$ such that $\lim_{i \to \infty} s_i = s$.

Now consider for $f(x) \epsilon C[0,1]$ the problem of finding the Nth order polynomial which is closest to $f(x)$. That is the polynomial of the Nth order such that $p \epsilon P_N \; ||f(Z) - p(Z)||_\infty$ is a minimum where P_N denotes the space of at most Nth order real polynomials i.e. $P_N = \{a_o + a_1 x - - + a_N x^N\}$ a_i real. Then we have the following theorem due to Tonelli [2].

<u>Theorem 2.1</u>. Let S be a closed bounded set in the complex plane that contains more than N + 1 points. Let f(Z) be continuous on S and let

(1) $M = \min_{p \epsilon P_N} \max_{Z \epsilon S} |f(Z) - p(Z)|$. Let $p_N(Z)$ be any polynomial that realizes this extreme value and let

(2) $r(t) = f(Z) - p_N(Z)$

Then

(a) The number of distinct points of S at which $|r(Z)|$ takes on its maximum value is greater than N + 1

(b) There is a unique solution to (1).

This clearly implies the uniqueness of the best Nth order polynomial approximation (in supremum norm) to $f(x) \epsilon C[0,1]$. This is called the best Tchebysheff approximation to f.

We also have [3] that for $f(x) \epsilon L_p \; 1 < p < \infty$ there exists a unique $g \epsilon P_N$ such that $\inf_{x \epsilon P_N} ||f - x||_p = ||f - g||_p$.

That is in L_p space by Nth order polynomials there is a unique best Nth order polynomial approximation (in L_p norm).

A Convergence Theorem for A Sequence of Best L_p Approximations

Let $\lim_{k\to\infty} f_k = f$ in L_p norm $(1 < p < \infty)$ on the interval $[0,1]$. Let Φ_k be the best L_p approximant to f_k on $0 \le x \le 1$. Note that Φ_k is the best L_p approximant to f_k over the polynomials of the form $p_N(x) = a_N x^N + a_{N-1} x^{N-1} \ldots + a_o$ where N is an arbitrary but fixed positive integer. From the previous discussion we know that for $f_k \in L_p [0,1]$ Φ_k exists and is unique. Let Φ be the best L_p approximant to f (again this is over polynomials of degree N). Since the L_p spaces are complete we have $f \in L_p$ so we know Φ exists and is unique.

In the following discussion for $g \in L_p$ $||g||_p$ will denote the L_p norm of g. The previous statement that $\lim_{k\to\infty} f_k = f$ in L_p norm on $[0,1]$ is equivalent to $\lim_{k\to\infty} ||f_k - f||_p = 0$.

<u>Lemma 3.1.</u> $\lim_{k\to\infty} ||f_k - \Phi_k||_p = ||f - \Phi||_p$

<u>Proof:</u> Since Φ_k is the best L_p approximation to f_k we have

(1) $||f_k - \Phi_k||_p \le ||f_k - \Phi||_p = ||f_k - f + f - \Phi||_p$

$$\le ||f_k - f||_p + ||f - \Phi||_p .$$

Now, as Φ is the best L_p approximant to f we have

$$||f - \Phi||_p \le ||f - \Phi_k||_p \le ||f - f_k||_p + ||f_k - \Phi_k||_p$$

or

(2) $||f - \Phi||_p - ||f - f_k||_p \le ||f_k - \Phi_k||_p$

inequalities (1) and (2) together imply

(3) $||f - \Phi||_p - ||f - f_k||_p \le ||f_k - \Phi_k||_p \le ||f - \Phi||_p + ||f_k - f||_p$

but $\lim_{k\to\infty} ||f - f_k||_p = 0$

which together with (3) implies

$$\lim_{k\to\infty} ||f_k - \Phi_k||_p = ||f - \Phi||_p.$$

<u>Lemma 3.2.</u> $\lim_{k\to\infty} ||f - \Phi_k||_p = ||f - \Phi||_p$

<u>Proof:</u> Since Φ is the best L_p approximant to f we have

$$||f - \Phi||_p \le ||f - \Phi_k||_p = ||f - f_k + f_k - \Phi_k||_p$$
$$\le ||f - f_k||_p + ||f_k - \Phi_k||_p$$

so we have the inequality

(4) $||f - \Phi||_p \le ||f - \Phi_k||_p \le ||f - f_k||_p + ||f_k - \Phi_k||_p$ also

$\lim_{k\to\infty} ||f - f_k||_p = 0$ and by Lemma 3.1 $\lim_{k\to\infty} ||f_k - \Phi_k||_p = ||f - \Phi||_p$

which together with (4) implies that

$$\lim_{k\to\infty} ||f - \Phi_k||_p = ||f - \Phi||_p.$$

It is now convenient to define the subset

$$S = \{b\ f(x) + a_o + a_1 x + a_2 x^2 \ldots + a_N x^N\}$$

where b and the a_i are real numbers.

<u>Remark 3.1.</u> The subset S is a finite dimensional subspace of $L_p\,[0,1]$.

It is clear that for any choice of b and the a_i that the element

$b\ f(x) + a_o + a_1 x \ldots + a_N x^N$ will be an element of $L_p\,[0,1]$.

Now let

$$r(x) = b_1\ f(x) + a_o + a_1 x \ldots + a_N x^N$$

and

$$s(x) = c_1\ f(x) + d_o + d_1 x \ldots + d_N x^N$$

be two element of S then

$$\alpha r(x) + \beta\, s(x) = (\alpha\, b_1 + \beta\, c_1)\ f(x) + (\alpha\, a_o + \beta\, d_o)\ \ddot{x} \ldots$$
$$+ (\alpha\, a_N + \beta\, d_N)\ x^N$$

ϵ S so S is a subspace, clearly the finite collection of vectors $(f(x), 1, x, x^2, \ldots, x^N)$ span S and hence S is finite dimensional.

Remark 3.2. The subspace S is closed. This follows from the fact that S is a finite dimensional subspace of the normed linear space $L_p[0,1]$.

Lemma 3.3. The subset $T = \{f - \Phi_k\}$ is a bounded subset of S.

Proof: Clearly T is a subset of S. Now from Lemma 3.2

$$\lim_{k\to\infty} ||f - \Phi_k||_p = ||f - \Phi||_p$$

which implies that given $\epsilon > 0$ there exists a k_o such that $k > k_o$ implies

$$\big|\, ||f - \Phi_k||_p - ||f - \Phi||_p \,\big| < \epsilon$$

$$||f - \Phi||_p - \epsilon < ||f - \Phi_k||_p < ||f - \Phi||_p + \epsilon$$

if $k > k_o$ so the set of real numbers $\{||f - \Phi_k||_p\}$ is bounded by the larger of $||f - \Phi_k||_p$ $k = 1, 2, \ldots, k_o$ or $||f - \Phi||_p + \epsilon$.

Remark 3.3. The closure of T, denoted by $\overline{T}$, is also a bounded subset. Let $r \in \overline{T}$ be any element of $\overline{T}$. If r also contained in T then $||r||_p \le M$ as T is bounded. Now suppose $r \notin T$ then there is a sequence $\{t_j\}$ contained in T such that $\lim_{j\to\infty} t_j = r$ in L_p norm. This follows from the fact that $L_p[0,1]$ is a normed linear space, hence a metric space. Now we have

$$||r||_p = ||r - t_j + t_j||_p \le ||r - t_j||_p + ||t_j||_p$$

but as T is bounded $||t_j||_p \le M$ for all j and as $\lim_{j\to\infty} t_j = r$ we have $\lim_{j\to\infty} ||r - t_j||_p = 0$ so

$$||r||_p \le \lim_{j\to\infty} ||r - t_j||_p + M = 0 + M$$

or $||r||_p \le M$ for all $r \in \overline{T}$

hence $\overline{T}$ is bounded.

Lemma 3.4. The subset $T = \{f - \Phi_k\}$ is relatively compact.

Proof: By Remark 3.3 $\overline{T}$ is a closed bounded subset of S. But S is a finite dimensional normed linear space, and as $\overline{T}$ is a closed bounded subset of a finite dimensional normed linear space it is compact.

Remark 3.4. $T = \{f - \Phi_k\}$ contains a convergent subsequence. This follows from the fact that T is relatively compact.

Lemma 3.5. In the subset $R = \{f(x) + a_o + a_1 x \ldots + a_N x^N\} \subset S$ (where the a_i range over all real numbers) there exists a unique element $\beta = f(x) + b_o + b_1 x \ldots + b_N x^N$ such that

$$||\beta||_p = \inf_{r \in R} ||r||_p .$$

Proof: Since approximation in L_p space $(1 < p < \infty)$ over polynomials has a unique solution we know there is a unique polynomial such that $||b_o + b_1 x \ldots + b_N x^N + f(x)||_p$ = min over all polynomials of degree N (with real coefficients). This $\beta = f(x) + b_o + b_1 x \ldots + b_N x^N$ is clearly an element of R such that $||\beta||_p = \inf_{r \in R} ||r||_p$.

Now suppose there is in R another element

$$\delta = f(x) + d_o + d_1 x \ldots + d_N x^N \neq \beta$$

$$||\delta||_p = ||\beta||_p$$

then $d_o + d_1 + d_2 x^2 \ldots + d_N x^N$ would be another best L_p approximation to f(x). But this contradicts the uniqueness of the best L_p approximation over the space of polynomials of degree N.

Theorem 3.1. Let $f_k \rightarrow f$ in L_p norm where the f_k are in $L_p [0,1]$. Let Φ_k be the best L_p approximation to f_k (over polynomials of fixed degree N) and let Φ be the best L_p approximation to f (again over polynomials of fixed degree N) then $\lim_{k \to \infty} \Phi_k = \Phi$ where the limit is in the L_p norm.

<u>Proof:</u> Suppose that $\lim_{k\to\infty} ||\Phi_k - \Phi||_p \neq 0$ that is suppose the Φ_k do not converge to Φ in L_p norm. Then this implies that

$\lim_{k\to\infty} ||(f - \Phi_k) - (f - \Phi)||_p \neq 0$ which implies there exists for $\varepsilon > 0$ an infinite subsequence of $\{f - \Phi_k\}$, call this subsequence $\{f - \Phi_{k_j}\}$, such that $||(f - \Phi_{k_j}) - (f - \Phi)|| > \varepsilon$ for all j, but $\{f - \Phi_{k_j}\}$ is a bounded subset of S hence $\{f - \Phi_{k_j}\}$ has a convergent subsequence $\{f - \Phi_{k_{j_m}}\}$ that is $\lim_{m\to\infty} f - \Phi_{k_{j_m}} = e$ where $e \in S$ as S is compact (hence closed). Now as $\lim_{m\to\infty} ||(f - \Phi_{k_{j_m}}) - e||_p = 0$ we have that $\{f - \Phi_{k_{j_m}}\}$ is a Cauchy sequence hence

$||(f - \Phi_{k_{j_{m_1}}}) - (f - \Phi_{k_{j_{m_2}}})||_p < \varepsilon_1$ for all

$m_1, m_2 > \bar{m}$ (where $\bar{m}$ depends on ε_1).

But this implies that

$||\Phi_{k_{j_{m_2}}} - \Phi_{k_{j_{m_1}}}||_p < \varepsilon_1$ for all

$m_1, m_2 > \bar{m}$ or equivalently the sequence $\{\Phi_{k_{j_m}}\}$ is a Cauchy sequence.

This implies

$\lim_{m\to\infty} \Phi_{k_{j_m}} = d \in p_N$ (in L_p norm)

as each $\Phi_{k_{j_m}} \in p_N$ and p_N is a finite-dimensional subspace of a normed linear space, hence closed. Now clearly

$\lim_{m\to\infty} ||(f - \Phi_{k_{j_m}}) - (f - d)||_p = 0$

but as $\{f - \Phi_{k_{j_m}}\}$ is a subsequence of $\{f - \Phi_k\}$

$$\lim_{m\to\infty} ||f - \Phi_{k_{j_m}}||_p = ||f - \Phi||_p$$

since

$$\lim_{k\to\infty} ||f - \Phi_k||_p = ||f - \Phi||_p$$

We also have

$$\lim_{m\to\infty} ||f - \Phi_{k_{j_m}}||_p = ||f - d||_p$$

since

$$\lim_{m\to\infty} \Phi_{k_{j_m}} = d \text{ (in } L_p \text{ norm)}$$

But $d \neq \Phi$ as $\{f - \Phi_{k_{j_m}}\}$ is a subsequence of $\{f - \Phi_{k_j}\}$ so

$||(f - \Phi_{k_{j_m}}) - (f - \Phi)||_p > \epsilon$ for all m, which implies

$$\lim_{m\to\infty} ||\Phi - \Phi_{k_{j_m}}|| = ||\Phi - d|| > \epsilon,$$

but this implies there are two distinct points in R, namely f - d and f - Φ, such that

$$\inf_{r\epsilon R} ||f - r||_p = ||f - d||_p = ||f - \Phi||_p$$

with $d \neq \Phi$. But this contradicts Lemma 3.5 hence the assumption that $\lim_{k\to\infty} ||\Phi_k - \Phi||_p \neq 0$ was incorrect and the proof is complete.

Remark 3.5. There is a result due to Polya in the literature [5] which should be contrasted with Theorem 3.1. This result states that if f(x) is continuous on [0,1] and $\Phi_p(x)$ is the best L_p approximation to f(x) where

$\Phi p(x) = \sum_{i-1}^{N} a_i^{\,p}\ \psi_i(x)$. (Note all approximations are taken over the space spanned by $\{\psi_i(x)\}$ i = 1, 2,...N). Then $\Phi_p(x)$ (p = 1,2,...) contains a convergent subsequence as $p\to\infty$. Further the limit of any convergent subsequence of $\{\Phi_p(x)\}$ is a best Tchebysheff approximation to f(x).

In Theorem 3.1, p, the order of the approximation, is fixed. The convergence is on k where k is the index of the set of functions $\{f_k(x)\}$ which converge in L_p norm to f(x).

Remark 3.6. Note that the proof of Theorem 3.1 fails to go through for approximation by rational functions. This is because the rational functions of given order numerator and denominator do not for a subspace. To see this consider $R_1^1 = \{ \frac{ax + b}{c_x + d} \}$ a, b, c, d real. Then

$\frac{1}{x-1} \epsilon R_1^1$ $\frac{1}{x+1} \epsilon R_1^1$ but $(\frac{1}{x-1})-(\frac{1}{x+1}) = \frac{2}{x^2-1} \not\epsilon R_1^1$.

Now consider the space C $[0,1]$ of functions continuous on $[0,1]$ with norm $||f||_\infty = \sup_{0 \le x \le 1} |f(x)| = \max_{0 \le x \le 1} |f(x)|$ for f ϵ C$[0,1]$ then C $[0,1]$ is a normed linear space and we know by Tonelli's theorem [2] that approximation (in supremum norm) to an arbitrary element f ϵ C $[0,1]$ over the space of polynomials of degree N has a unique solution. There exists a unique $a_o + a_1x + a_2x^2 \dots a_Nx^N \epsilon P_N$ (where, as before, N is arbitrary but fixed) such that

$$\inf_{p\epsilon P_N} ||p - f||_\infty = ||a_o + a_1x \dots + a_Nx^N - f||_\infty .$$

Remark 3.7. We see that since the space C $[0,1]$ is a normed linear space in which approximation by polynomials in the supremum norm, is unique we may prove Theorem 3.1 without modification using C $[0,1]$ and we have the following: Let $\{f_k\}$ ϵ C $[0,1]$ and $\lim_{k\to\infty} ||f_k - f||_\infty = 0$ let Φ_k be the best Tschebysheff approximant to f_k, again over polynomials of degree N. (Note since C $[0,1]$ is complete we know f ϵ C $[0,1]$ so Φ exists and is unique). Then $\lim_{k\to\infty} ||\Phi_k - \Phi||_\infty = 0$, that is, the Φ_k

converge uniformly to Φ (the best Tchebysheff approximant to f).

Remark 3.8. The result of Remark 3.7 follows also from the following [5]: Let f(x) and g(x) be continuous on [0,1] and with best Tchebysheff approximations $\Phi_1(x)$ and $\Phi(x)$ respectively then there is a constant M which depends only on g(x) and $\{\psi_i(x)\}$ such that

$$\max_{x\epsilon[0,1]} |\Phi_1(x) - \Phi(x)| \le M \max_{x\epsilon[0,1]} |f(x) - g(x)|.$$

Note in the above theorem the set $\psi_i(x)$ is the set of functions over which the best approximation is being sought. Now consider the special case in which $\psi_i(x) = x^i$ $i = 0,1 \ldots N$ and suppose $\lim_{k\to\infty} ||g_k - g|| = 0$. Let Φ be the best Tchebysheff approximation to g(x) and let Φ_k be the best Tchebysheff approximation to $g_k(x)$ then from Theorem 2 above we have

$$\max_{x\epsilon[0,1]} |\Phi_k(x) - \Phi(x)| \le M \max_{x\epsilon[0,1]} |g_k(x) - g(x)|$$

or in norm notation

$$||\Phi_k - \Phi||_\infty \le M\ ||g_k - g||_\infty .$$

Now $\lim_{k\to\infty} ||g_k - g||_\infty = 0$

or $\lim_{k\to\infty} ||\Phi_k - \Phi||_\infty = 0$ that is the $\Phi_k(x)$ converge uniformly to $\Phi(x)$.

Note that the result of [5] at the first of this remark, states essentially that f close to g in supremum norm (f and g continuous) implies the best Tchebysheff approximant to f is close (in supremum norm) to the best Tchebysheff approximant to g. There is no result known to the author which generalizes this to L_p spaces.

Remark 3.9. Suppose we know how rapidly f_N converges to f. Specifically suppose $||f_k - f||_p \le g(k)$ where g(k) is ultimately monotone decreasing and $\lim_{k\to\infty} g(k) = 0$. Then from Lemma 2 we have (using L_p norms in place of the supremum norm)

$$||f - \Phi||_p \leq ||f - \Phi_k||_p \leq ||f - f_k||_p + ||f_k - \Phi_k||_p$$

But by Lemma 3.1

$$||f_k - \Phi_k||_p \leq ||f - \Phi||_p + ||f_k - f||_p$$

so we have

$$||f - \Phi_k||_p \leq 2||f - f_k||_p + ||f - \Phi||_p$$

or

$$||f - \Phi_k||_p - ||f - \Phi||_p \leq 2||f - f_k||_p = 2g(k).$$

Example 3.1. Consider the sequence of functions $f_N(x) = x^M + \frac{x}{N}$ and consider the best Tchebysheff approximation to $f_N(x)$ over the space of polynomials $p_{M-1}(x)$ (i.e. the space of polynomials of degree at most M-1). Then $\Phi_N(x)$, the best Tchebysheff approximation to $f_N(x)$ on $-1 \leq x \leq 1$, can be shown [3] to be $\Phi_N(x) = x^M - 2^{1-M}T_M(x) + \frac{x}{N}$ where $T_M(x)$ is the Tchebysheff polynomial of the first kind of order N. $f_N(x)$ converges uniformly to $f(x) = x^M$ and the best Tchebysheff approximation Φ to f is $\Phi(x) = x^M - 2^{1-M}\, T_M(x)$, since $|\Phi_N(x) - \Phi(x)| = |\frac{x}{N}| \leq \frac{1}{N}$ clearly we have $\Phi_N(x)$ converging uniformly to $\Phi(x)$. Note that the interval $[-1,1]$ as opposed to $[0,1]$ was chosen for convenience. The preceding theorems clearly apply to any finite interval, as well as for the interval $[0, 1]$.

<u>Counterexample 3.1.</u> To see that Theorem 3.1 does not hold for L_1 space consider the following example. Let f(x) be defined by

$$f(x) = 0 \qquad -1 \le x \le 0$$

$$f(x) = 1 \qquad 0 < x \le 1.$$

Then $||f||_1 = \int_{-1}^{1} |f(x)|\, dx = \int_{0}^{1} 1\, dx = 1$

so f(x) is contained in L_1. Now consider the subspace spanned by the polynomial p(x) = 1. That is functions of the form ap(x) = a where a is any real number. Denote by P_o the subspace $P_o = \{a \cdot 1\}$ where a is any real number. Then we have

$$||f(x) - a||_1 = \int_{-1}^{0} |0 - a|\, dx + \int_{0}^{1} |1 - a|\, dx.$$

For $1 \ge a \ge 0$ we have

$$||f(x) - a||_1 = a + (1-a) = 1.$$

For $a > 1$ we have

$$||f(x) - a||_1 = a + (a-1) = 2a - 1 > 1.$$

For $a < 0$ we have

$$||f(x) - a||_1 = -a + (1-a) = -2a + 1 > 1$$

so we have

$$\inf_{p(x) \in P_o} ||f(x) - p(x)||_1 = 1$$

and the minimum is attained for any $p(x) = a$, $0 \le a \le 1$. That is, approximation in L_1 space over polynomials of degree one is not unique.

Now consider the sequence of functions $f_k(x)$ defined by

$$f_k(x) = 0 \qquad -1 \le x \le 0$$

$$f_k(x) = 1 + \frac{1}{k} \qquad 0 < x \le 1.$$

Now we have that

$$||f(x) - f_k(x)||_1 = \int_0^1 |1 - (1 + \frac{1}{k})| \, dx$$

$$= \int_0^0 |- \frac{1}{k}| \, dx = \frac{1}{k}$$

so clearly f_k converges to f in L_1 norm. For $1 + \frac{1}{k} \ge a \ge 0$ we have

$$||f_k(x) - a||_1 = \int_{-1}^0 |0 - a| \, dx + \int_0^1 |1 + \frac{1}{k} - a| \, dx$$

$$= a + (1 + \frac{1}{k} - a) = 1 + \frac{1}{k} .$$

For $a > 1 + \frac{1}{k}$

$$||f_k(x) - a||_1 = a + (a - 1 + \frac{1}{k})$$

$$= 2a - (1 + \frac{1}{k}) > 1 + \frac{1}{k} .$$

For $a < 0$ we have

$$||f_k(x) - a|| = -a + (1 + \frac{1}{k} - a)$$

$$= (1 + \frac{1}{k}) - 2a > 1 + \frac{1}{k},$$

so the best L_1 approximation to $f_k(x)$ is given by $p(x) = b$ where $0 \le b \le 1 + \frac{1}{k}$. Now consider the sequence of functions

$$\Phi_k(x) = \frac{3}{4}$$

since we have $0 \le \frac{3}{4} \le 1 + \frac{1}{k}$ for all k, Φ_k is a best L_1 approximation to $f_k(x)$ for all k. Let

$$\Phi(x) = \frac{1}{2},$$

then $\Phi(x)$ is a best L_1 approximation to $f(x)$. Now

$$||\Phi_k - \Phi||_1 = \int_{-1}^{1} |\tfrac{3}{4} - \tfrac{1}{2}|\,dx = \tfrac{1}{2}$$

for all k. So $\{\Phi_k\}$ does not converge to Φ in L_1 norm. Note, however, that

$$||f_k - \Phi_k||_1 = 1 + \frac{1}{k} \text{ and}$$

$$||f - \Phi||_1 = 1$$

so we have

$$\lim_{k\to\infty} ||f_k - \Phi_k||_1 = ||f - \Phi||_1.$$

4

Application of the Convergence Theorem to a Problem in L_2 Space

In this chapter Theorem 3.1 will be extended to a form which will allow the approximation of the best L_2 approximant for a given continuous function f(x) in the event that a sequence of polynomials $p_N(x)$ is known which converge uniformly to f(x). The method of approximating the best L_2 approximant of f(x) will be such that no numerical integrations are required. To accomplish this we prove the following :

<u>Theorem 4.1.</u> Let $\{f_k\} \in L_p$ and $\lim_{k\to\infty} ||f - f_k||_p = 0$. Let Φ_k be the best approximation to f_k in L_p norm (over the space of polynomials of arbitrary but fixed degree N). Let Φ be the best approximation to f in L_p norm (again over polynomials of degree N). Then $\lim_{k\to\infty} ||\Phi - \Phi_k||_\infty = 0$. That is the Φ_k converge uniformly to Φ.

<u>Proof:</u> By Remark 5 we have that $\lim_{k\to\infty} ||\Phi_k - \Phi||_p = 0$. But as Φ and each Φ_k are polynomials of at most degree N (that is elements of a finite-dimensional subspace of a normed linear space) we have that $||\Phi_k - \Phi||_\infty \leq M\, ||\Phi_k - \Phi||_p$ since the two norms are equivalent on the finite-dimensional subspace. Therefore, since $\lim_{k\to\infty} ||\Phi_k - \Phi||_p = 0$ we have that $\lim_{k\to\infty} ||\Phi_k - \Phi||_\infty = 0$ and the theorem is proved.

For spaces of finite measure (specifically $0 \leq x \leq 1$) convergence in supremum norm implies convergence in L_p norm $1 < p < \infty$. To see this suppose $\lim_{k\to\infty} ||f - f_k||_\infty = 0$.

Then we have

$$||f - f_k||_p = \left(\int_o^1 |f - f_k|^p\right)^{\frac{1}{p}} \le \left(\int_o^1 ||f - f_k||_\infty^p\right)^{\frac{1}{p}}$$

$$\le (1)(||f - f_k||_\infty)$$

so $\lim_{k\to\infty} ||f - f_k||_\infty = 0$ which implies

$\lim_{k\to\infty} ||f - f_k||_p = 0.$

We therefore have the following:

Corollary 4.1. Let $\{f_k\} \varepsilon L_p$ and $\lim_{k\to\infty} ||f - f_k||_\infty = 0$. Let Φ_k be the best Tchebysheff approximation to f_k in L_p norm (over the space of polynomials of degree N). Let Φ be the best Tchebysheff approximation to f in L_p norm (over same space). Then $\lim_{k\to\infty} ||\Phi - \Phi_k||_\infty = 0$

Now consider the application of Theorem 4.1 and Corollary 4.1 to the space L_2. Suppose $f(x) \varepsilon L_2$ on $[0,1]$ and we seek the best L_2 approximant over polynomials of the form $a_o + a_1x + a_2x^2 \dots + a_Nx^N$. Call this subspace p_N. Then we seek the $p \varepsilon p_N$ such that $||f - p||_2$ is a minimum. Let $(\int_o^1 f\ g\ dx)$ be denoted by (f, g). Then it can be shown [2] that the a_i which minimize the integral are solutions of the normal equations:

$$a_o(1,1) + a_1(x,1) \dots + a_N(x^N,1) = (f,1)$$
$$a_o(1,x) + a_1(x,x) \dots + a_N(x^N,x) = (f,x)$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$a_o(1,x^N) + a_1(x,x^N) \dots + a_N(x^N,x^N) = (f, x^N).$$

It is seen that for a complicated f the N + 1 integrals (f,x^j) $j = 0, \dots, N$ must be evaluated numerically.

Now suppose we have a sequence $\{p_k\}$ of polynomials which converge uniformly to f. Note that p_k is the kth polynomial of the sequence $\{p_k\}$. It is not necessarily of order k. Using the p_k, we may get an approximation to the best least squares approximation f, in a fashion which does not require numerical integration.

Let $p_k = b_o + b_1 x + b_2 x^2 \ldots + b_m x^m$. Then we can solve, without numerical integration, for the best least squares approximant to p_k. We have

$$(1)\quad (p_k, x^j) = \left[\int_o^1 (b_o x^j + b_1 x^{j+2} \ldots . + b_m x^{m+j})\right] dx$$

$$\left[\frac{b_o}{j+1} + \frac{b_1}{j+2} \quad \ldots \quad + \frac{b_m}{j+(m+1)}\right]$$

for $j = 0, 1, 2, 3, \ldots N$.

So the coefficients $\{a_i\}$ $i = 0, N$ of the best least square approximant to $p_k(x)$ are the solutions of the set of equations (noting also that

$$(x^i, x^j) = \left[\int_o^1 x^i x^j \, d_x\right] = \left[\frac{1}{i+j+1}\right]$$

$$a_o (1) + a_1 \left(\frac{1}{2}\right) \ldots . + a_N \left(\frac{1}{N+1}\right) = (p_k, 1)$$

$$a_o \left(\frac{1}{2}\right) + a_1 \left(\frac{1}{3}\right) \ldots . + a_N \left(\frac{1}{N+2}\right) = (p_k, x)$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$a_o \left(\frac{1}{N+1}\right) + a_2 \left(\frac{1}{N+2}\right) \ldots . + a_N \left(\frac{1}{2N+1}\right) = (p_k, x)$$

The (p_k, x^j) are the expressions given by (1) and no numerical integration is necessary to evaluate them.

If we let

$p_1(x) = a_o + a_1 x + a_2 x^2 + \ldots . + a_N x^N$ be the best L_2 approximant to $p_k(x)$ and the best approximant for f (in L_2 norm over polynomials of degree N) by $p^*(x)$ we have from Remark 6 chapter 3 that

$$||f - p_1||_2 - ||f - p^*||_2 \le 2\, ||f - p_k||_2 \,,$$

but suppose $||f - p_k||_\infty \le \epsilon$ then $2\, ||f - p_k||_2 \le 2\epsilon$ so we have

(2) $||f - p_1||_2 - ||f - p^*||_2 \le 2\varepsilon$ (Note this is true only for a set of measure one).

Example 4.1. Consider the problem of finding the best approximation to e^x $0 \le x \le 1$ in L_2 norm over the space of polynomials of the form $p(x) = a_o + a_1 x$. Let $p^*(x) = c_o + c_1 x$ denote this unique best least squares approximant to e^x. Then the c_i are the solution to the normal equations

$$c_o(1,1) + c_1(1,x) = (1, e^x)$$

$$c_o(x,1) + c_1(x,x) = (x, e^x)$$

or

$$c_o + .5\ c_o = 1.71828$$

$$.5\ c_o + .33333\ c_1 = 1$$

upon solving these we find

$$c_o = .87312 \qquad c_1 = 1.69032$$

so $p^*(x) = .87312 + 1.69032x$

$$||e^x - p^*||_x = \left(\int_o^1 (e^x - .87312 + 1.69032x)^2\right)^{1/2}$$

$$= (.00396)^{1/2} = .06292.$$

Now consider the problem of approximating $p^*(x)$ by finding a polynomial which is sufficiently close to e^x and then finding the least squares approximant to the polynomial

Consider $p_k(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

on $(0 \le x \le 1)$ we have

$$|e^x - p_k(x)| \le .05162$$

so $||e^x - p_k(x)||_\infty = .05162$.

Now we have

$$(1,\ p_k(x)) = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 1.70833$$

and

$$(x,\ p_k(x)) = \frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{30} = .99167$$

These inner products may be written by inspection, no integration is necessary. If the best L_2 approximation to $p_k(x)$ is denoted by $p_1(x) = b_o + b_1x$ the b_i satisfy the normal equations

$$b_o(1,1) + b_1(1,x) = (1,p_k(x))$$

$$b_o(x,1) + b_1(x,x) = (x,p_k(x))$$

or

$$b_o + .5\ b_1 = 1.70833$$

$$.560b_o + .33333b_1 = .99167$$

so, $b_o = .883296$ and $b_1 = 1.65006$

so, $p_1(x) = .883296 + 1.65006x$

$$||e^x - p_1(x)||_2 = \left(\int_o^1 (e^x - .883296 - 1.65006x)^2\right)^{1/2}$$

$$= (.00417)^{1/2} = .06458.$$

Now we have

$$||e^x - p_1(x)||_2 - ||e^x - p^*(x)||_2$$

$$= .06458 - .06292 = .00166 \leq .01324$$

$$= 2||e^x - p_k(x)||_\infty$$

which is as predicted by equation 2.

<u>Remark 4.1.</u> It might be hoped that the above procedure could be used to approximate the best Tchebysheff approximant to a continuous function f(x) over the space of polynomials of degree N. That is suppose a polynomial $p_m(x)$ of degree $m > N$ is known such that $||p_m - f||_\infty < \varepsilon$. It might be hoped that the best Tchebysheff approximant to $p_m(x)$

might be found (as it was in L_2 case) and furnish an approximation to the best Tchebysheff approximant for f(x). However, for $m > N + 1$ the problem of finding the best Tchebysheff approximant to $p_m(x)$ has no known closed form solution,(as it does for the case of approximation in L_2 norm).

5

A Quadrature Rule Using Hermite-Fejér Approximation

Let $$H_{2N-1}(f,x) = \sum_{k=1}^{N} f(x_k^N)\, A_k^N(x)$$

where $$A_k^N(x) = (1 - x\,x_k^N)\left(\frac{T_N(x)}{N(x - x_k^N)}\right)^2 \quad (k = 1,2,\ldots N).$$

Then $H_{2N-1}(f, x_k^N) = f(x_k^N)$, $H'_{2N-1}(f, x_k^N) = 0$ where the x_k^N $(k = 1,2,\ldots N)$ are the N zeros of the Tscheybsheff polynomial $T_N(x)$. Then it can be shown [2] that

$\lim_{N\to\infty} H_{2N-1}(f,x) = f(x)$ uniformly on $[-1,1]$ if $f(x) \in C[-1,1]$

If we approximate f(x) by $H_{2N-1}(f,x)$ on $-1 \le x \le 1$ then by integrating $H_{2N-1}(f,x)$ we obtain an estimate of $\int_{-1}^{1} f(x)dx$.

In the following sections explicit formulas for the integration rule will be developed. It will be shown that

$$\lim_{N\to\infty} \int_{-1}^{1} H_{2N-1}(f,x)\,dx = \int_{-1}^{1} f(x)\,dx$$

uniformly for $f(x) \in C[-1,1]$ and a bound will be obtained for the error of the approximation.

(a) <u>Derivation of the Quadrature Rule</u>

If we approximate f(x) by $H_{2N-1}(f,x)$ we have

$$f(x) \sim H_{2N-1}(f,x) = \sum_{k=1}^{k=N} f(x_k^N)\, A_k^N(x)$$

and $$\int_{-1}^{1} f(x)\,dx \sim \int_{-1}^{1} \sum_{k=1}^{N} f(x_k^N)\, A_k^N(x)\,dx$$

$$= \sum_{k=1}^{N} \int_{-1}^{1} f(x_k^N)\, A_k^N(x)\,dx$$

$$= \sum_{k=1}^{N} f(x_k^N) \left(\int_{-1}^{1} A_k^N(x)\,dx \right)$$

or defining $w_k^N = \int_{-1}^{1} A_k^N(x)\,dx$

we have

$$(1) \quad \int_{x=-1}^{x=1} f(x)\,dx \sim \sum_{k=1}^{N} f(x_k^N)\, w_k^N .$$

So to determine the quadrature rule we must determine the w_k^N where

$$w_k^N = \int_{-1}^{1} A_k^N(x)\,dx = \int_{-1}^{1} (1-x\, x_k^N) \left(\frac{T_N(x)}{N(x - x_k^N)} \right)^2 dx$$

or

$$w_k^N = \frac{1}{N^2} \int_{-1}^{1} (1 - x\, x_k^N)\, \frac{(T_N(x))^2}{(x - x_k^N)^2}\, dx .$$

To evaluate the integral we need the following:

<u>Lemma 5.1.</u> $T_N(x) = \dfrac{T_{2N}(x) + 1}{2}$

<u>Proof:</u> $T_N(x) = \cos(N \cos^{-1} x)$ [1]

so, $(T_N(x))^2 = \cos^2 (N \cos^{-1} x)$

$$= \frac{\cos 2(N \cos^{-1} x) + 1}{2}$$

$$= \frac{\cos 2N \cos^{-1} + 1}{2}$$

$$= \frac{T_{2N} (x) + 1}{2}$$

and thus

$$w_k^N = \frac{1}{2N^2} \int_{-1}^{1} \frac{(1 - x\, x_k^N)}{(x - x_k^N)^2} \quad (T_{2N}(x) + 1)\, dx$$

Now using the fact [1] that

$$T_\alpha(x) = \frac{\alpha}{2} \sum_{m=0}^{[\frac{\alpha}{2}]} (-1)^m \frac{(\alpha-m-1)!}{m!(\alpha-2m)!} (2x)^{\alpha-2m}$$

we obtain (for $\alpha = 2N$)

$$T_{2N} (x) = N \sum_{m=0}^{m=N} (-1)^m \frac{(2N - m - 1)!}{m!(2N - 2m)!} (2x)^{2N - 2m} .$$

Now we have that

$$w_k^N = \frac{1}{2N^2} \int_{-1}^{1} \frac{(1 - x\, x_k^N)}{(x - x_k^N)^2} \quad (T_{2N}(x) + 1)\, dx$$

$$= \frac{1}{2N^2} \int_{-1}^{1} \frac{(1 - x\, x_k^N)}{(x - x_k^N)^2}\, dx + \int_{-1}^{1} \frac{T_{2x} (x)\, dx}{(x - x_k^N)^2}$$

$$- \int_{-1}^{1} \frac{x\, x_k^N \; T_{2N}(x)\, dx}{(x - x_n^N)^2} .$$

Now using the polynomial expression for $T_{2N}(x)$ this takes the form

$$w_k^N = \frac{1}{2N^2}\left(\int_{-1}^{1} \frac{(1 - x\,x_k^N)\,dx}{(x - x_k^N)^2}\right)$$

$$+ \frac{1}{2N^2}\left\{N \int_{-1}^{1}\left(\sum_{m=0}^{m=N} \frac{(2N - m - 1)\quad (2x)^{2N - 2m}}{m\quad (2N - 2m)} \frac{dx}{(x - x_k^N)^2}\right)\right\}$$

$$- \frac{1}{2N^2}\left\{\int_{-1}^{1} \frac{N\,x\,x_k^N}{(x - x_k^N)^2}\left(\sum_{m=0}^{m-N} (-1)^m \frac{(2N - m - 1)\,(2x)^{2N-2m}}{m\quad (2N - 2m)}\right)dx\right\}.$$

Defining

$$a_{jk}^N = \int_{-1}^{1} \frac{x^j\,dx}{(x - x_k^N)^2}$$

we obtain for $j = 0$

$$(2)\quad a_{0k}^N = \int_{-1}^{1} \frac{dx}{(x - x_k^N)^2} = -(x - x_k^N)^{-1}\Big|_{x=-1}^{x=1}$$

$$= \frac{-2}{(1 - (x_k^N)^2}\ .$$

For $j = 1$ we have

$$(3)\quad a_{1k}^N = \int_{x=-1}^{x=1} \frac{x\,dx}{(x - x_k^N)^2} = \frac{-x_k^N}{(x - x_k^N)}\Bigg|_{x=-1}^{x=1}$$

$$+ 1N\,|\,(x - x_k^N)\,|\,\Big|_{x=-1}^{x=1}$$

$$= \frac{-2\,x_k^N}{(1 - (x_k^N)^2)} + 1N\ \frac{1 - x_k^N}{1 + x_k^N}$$

For $j \geq 2$ we need the following integration formula [4] with $Z_1 = a + bx$

$$\int \frac{x^j\,dx}{Z_1^{\,2}} = \sum_{i=1}^{i=j-1} \frac{(-1)^{i-1}\, i\, a^{i-1}\, x^{j-1}}{(j-i)\, b^{i+1}}$$

$$+ \frac{(-1)^{j-1}\, a^j}{b^{j+1}\, Z_1} + \frac{(-1)^{j+1}\, j\, a^{j-1}}{b^{j+1}} \; \mathrm{lN}|Z_1|$$

using this formula we obtain for $j \geq 2$

$$(4) \quad a_{jk}^N = \sum_{i=1}^{i=j-1} \frac{(i)\,(-x_k^N)^{i-1}\,(-1)^{i-1}\left[(1)^{j-i} - (-1)^{j-i}\right]}{(j-i)}$$

$$+ (-1)^{j-1}\,(-x_k^N)^j\, \frac{2}{(1-(x_k^N)^2}$$

$$+ (-1)^{j+1}\, j(-x_k^N)^{j-1}\; \mathrm{lN}\, \frac{1-x_k^N}{1+x_k^N}$$

Using the previous notation we finally have

$$(5) \quad w_k^N = \frac{1}{2N^2} \left\{ a_{ok}^N - x_k^N\, a_{1k}^N + \sum_{m=0}^{m=N} N\, \frac{(2N-m-1)!}{m!\,(2N-2m)!}\, 2^{2N-2m}\, a_{2N-2m,k}^N \right.$$

$$\left. - \sum_{m=0}^{m=N} \frac{N\, x_k^N\,(2N-m-1)!}{m!\,(2N-2m)!}\, 2^{2N-2m}\, a_{2N-2m+1,k}^N \right\}$$

We now have the following

<u>Theorem 5.1.</u> $\int_{-1}^{1} H_{2N-1}(f,x)\,dx$ converges to $\int_{-1}^{1} f(x)\,dx$ as $N \longrightarrow \infty$

Proof: For $f(x) \in C\left[-1,1\right]$ since $H_{2N-1}\, f(x)$ converges to $f(x)\ \left(f(x) \in C\left[-1,1\right]\right)$ uniformly given $\frac{\varepsilon}{2} > 0$ there is a N_o such that

$N > N_o$ implies $\left|H_{2N-1}(f,x) - f(x)\right| \leq \frac{\varepsilon}{2}$
all $x \in \left[-1,1\right]$

$$\text{so } \left|\int_{-1}^{1} H_{2N-1}(f,x)\,dx - \int_{-1}^{1} f(x)\,dx\right|$$

$$= \left|\int_{-1}^{1} \left(H_{2N-1}(f,x) - f(x)\right) dx\right|$$

$$\leq \int_{-1}^{1} \left|H_{2N-1}(f,x) - f(x)\right| dx \leq \int_{-1}^{1} \frac{\varepsilon}{2}\, dx = \varepsilon \quad \text{all } N > N_o$$

$$\text{so } \int_{-1}^{1} H_{2N-1}(f,x)\,dx \text{ converges to } \int_{-1}^{1} f(x)\,dx$$

for all $f(x) \in C\left[-1,1\right]$.

(b) Derivation of an Error Bound for $f\ C^1\left[-1,1\right]$

To obtain a bound on the error we use the estimate [6]

$$\left|f(x) - H_N(f,x)\right| \leq \frac{4\lambda\pi}{N}\ (\alpha + \log N)$$

where $\alpha = \frac{1}{2} + C - \log 2$ (C is Euler's constant) and

$\lambda \geq 0 \quad \left|f(v) - f(u)\right| \leq \lambda\left|u-v\right|$ (λ the Lipschite constant for f)

$$\text{so } \left|\int_{-1}^{1} f(x)\,dx - \int_{-1}^{1} H_N(f,x)\,dx\right|$$

$$\leq \int_{-1}^{1} |f(x) - H_N(f,x)|\, dx \leq 2\left(\frac{4\lambda\pi}{N}(\alpha + \log N)\right)$$

$$= \frac{8\pi\lambda}{N}(\alpha + \log N)$$

Now if $f(x) \in C^1\left[-1,1\right]$

then $\lambda = \max_{-1 \leq x \leq 1} |f'(x)|$

so for $f(x) \in C^1\left[-1,1\right]$ we have

$$\left| \int_{-1}^{1} f(x)\, dx - \int_{-1}^{1} H_N(f,x)\, dx \right| \leq \max_{-1 \leq x \leq 1} |f'(x)| \quad 8\pi\, \frac{(\alpha + \log N)}{N}$$

so as the number of points N increases the error bound is always given in terms of the maximum value of the first derivative.

Note that if a given error bound is desired for a given numerical integration the order N of the quadrature rule may be determined if $f(x) \in C^1\left[-1,1\right]$ and a bound on $|f'(x)|$ $-1 \leq x \leq 1$ is known.

Generalization to Functions of More Than One Variable

Consider the function $f(x_1,x_2,\ldots x_p)$ defined for $-1 \le x_k \le 1$ $k = 1,2,\ldots,p$. Then the Hermite-Fejér approximation to $f(x_1, x_2,\ldots,x_p)$ is given by $H_{N_1,\ldots,N_p}(f, x_1, x_2,\ldots,x_p)$

$$= \sum_{h_1=1}^{N_1} \sum_{h_p=1}^{N_p} f(x_{h_1}^{N_1}, \ldots x_{h_p}^{N_p}) A_{h_1}^{N_1}(x_1) \ldots A_{h_p}^{N_p}(x_p)$$

where the $A_{h_i}^{N_i}$ $i = 1, 2, \ldots,p$

are defined previously. To obtain an approximation to

$$\int_{x_1=-1}^{x_1=1} \int_{x_2=-1}^{x_2=1} \cdots \int_{x_p=-1}^{x_p=1} f(x_1,x_2,\ldots x_p)\, dx_1\ldots dx_p$$

we may integrate $H_{N_1,\ldots,N_p}(f, x_1, x_2,\ldots x_p)$ over the corresponding volume. Performing this integration we obtain

$$\int_{x_1=-1}^{x_1=1} \cdots \int_{x_p=-1}^{x_p=1} H_{N_1,\ldots,N_p}(f, x_1, x_2,\ldots x_p)\, dx_1 \ldots dx_p$$

$$= \int_{x_1=-1}^{x_1=1} \cdots \int_{x_p=-1}^{x_p=1} \sum_{h_1=1}^{N_1} \cdots \sum_{h_p=1}^{N_p} f(x_{h_1}^{N_1} \ldots x_{h_p}^{N_p}) A_{h_1}^{N_1}(x_1) \ldots A_{h_p}^{N_p}(x_p)\, dx_1 \ldots dx_p$$

$$= \sum_{h_1=1}^{N_1} \cdots \sum_{h_p=1}^{N_p} f(x_{h_1}^{N_1}, \ldots x_{h_p}^{N_p}) \left(\int_{x_1=-1}^{x_1=1} A_{h_1}^{N_1}(x_1) dx_1 \right) \cdots \left(\int_{x_p=-1}^{x_p=1} A_{h_p}^{N_p}(x_p) dx_p \right)$$

$$= \sum_{h_1=1}^{N_1} \cdots \sum_{h_p=1}^{N_p} f(x_{h_1}^{N_1}, \ldots x_{h_p}^{N_p}) \, w_{h_1}^{N_1} \cdots w_{h_p}^{N_p}$$

where the $w_{h_i}^{N_i}$ $i = 1, 2, \ldots, p$ are as defined previously.

Now it has been shown [6] that if there exists λ_r such that

$$|f(x_1, \ldots, x_{r-1}, v, x_{r+1}, \ldots x_p) - f(x_1, \ldots, x_{r-1} \cdot u, x_{r+1}, \ldots x_p)| \le \lambda_r |v - u|$$

whenever x_q, u, v are in $[-1, 1]$ then for $N_1, \ldots, N_p$ positive integers

$$|f(x_1, \ldots x_p) - H_{N_1, N_2, \ldots N_p}(f, x_1, x_2, \ldots, x_p)|$$

$$\le \sum_{r=1}^{p} 4\lambda_r \pi N_r^{-1} (\alpha + \log N_r).$$ Then we have

$$(1) \quad \left| \int_V f(x_1, \ldots x_p) \, dv - \int_V H_{N_1 \cdots N_p}(f, x_i, x_2, \ldots x_p) \right|$$

$$\le 2^p \left(\sum_{r=1}^{p} 4\lambda_r \pi N_r^{-1} (\alpha + \log N_r) \right)$$

where $\int_V$ indicates integration over the hyper cube $-1 \le x_1 \le 1$, $-1 \le x_2 \le 1$, $x \le x_p \le 1$ if $f \in C^1$ and

$$M = \max_i \max_{-1 \le x_i \le 1} \left| \frac{\partial f}{\partial x_i} \right|.$$ Then we obtain the bound

$$(2)\quad \left| \int_V f(x_1,\dots x_p)\,dv - \int_V H_{N_1,\dots N_p}(f, x, \dots, x_p)\,dv \right|$$

$$\le 2^p M \left(\sum_{r=1}^{p} 4\,\pi\, N_r^{-1} \;(\alpha + \log N_r) \right).$$

(a) Some Numerical Examples

Example 6.1

Consider the problem of integrating $f(x) = \frac{x^4}{4} + \frac{e^x}{3}$ from -1 to 1. This example will be carried out for N = 3. For N = 3 we have [1].

$$x_k^N = \cos \frac{(2k-1)\pi}{2(N)}$$

so

$$x_1^3 = \cos \frac{(2-1)}{6}\pi = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} = .86603$$

$$x_2^3 = \cos \frac{(4-1)}{6}\pi = \cos \frac{\pi}{2} = 0$$

$$x_3^3 = \cos \frac{(6-1)}{6}\pi = \cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} = -\ .86603.$$

For N = 3 we obtain

$$w_1^3 = \frac{8}{15}$$

$$w_2^3 = \frac{14}{15}$$

$$w_3^3 = \frac{8}{15}.$$

Note $w_1^3 + w_2^3 + w_3^3 = 2$.

This is expected since the quadrature rule is exact for y = k (a constant). This follows as a result of $y'(x) \equiv 0$ so the error is bounded by

$$\max_{-1 \le x \le 1} |y'(x)| \; 8\,\pi \frac{(\alpha + \log N)}{N} = 0.$$

The exact value of the integral is

$$\int_{x=-1}^{x=1} f(x)\,dx = \int_{x=-1}^{x=1} \left(\frac{x^4}{4} + \frac{e^x}{3}\right) dx = 1.1287.$$

Using equation 1 chapter 5 we have as an approximate value of the integral

$$\int_{-1}^{1} f(x)\,dx \sim \left(\frac{(x_1^3)^4}{4} + \frac{e^{x_1^3}}{3}\right)\frac{8}{15} + \frac{14}{15}\left(\frac{(x_2^3)^4}{4} + \frac{e^{x_2^3}}{3}\right) +$$

$$\frac{8}{15}\left(\frac{(x_3^3)^4}{4} + \frac{e^{x_3^3}}{3}\right)$$

so $\int_{-1}^{1} f(x)\,dx \sim .9586$ so the error of the approximation is

1.1287 - .9586 = .1701. The bound on the error is given by

$\frac{4\lambda\pi}{N}(\alpha + \log N)$ in this case N = 3

$$\lambda = \max_{-1 \le x \le 1} |f'(x)| = \max_{-1 \le x \le 1} \left| x + \frac{e^x}{3} \right| \le 2$$

so $\frac{4\lambda\pi}{N}(\alpha + \log_N) = 5.5711$

In this example the actual error is considerably less than the error bound.

Example 6.2

Consider now the problem of integrating

$$f(x,y) = \frac{e^x y^2 + x^2 \sin y}{200} + e^{\frac{x+y}{10}}$$

over the volume $-1 \le x \le 1 \quad -1 \le y \le 1.$

We have

$$\int_{x=-1}^{x=1}\int_{y=-1}^{y=1}\left(\frac{e^x y^2 + x^2 \sin y}{200} + e^{\frac{x+y}{10}}\right) dxdy = 4.0265$$

for the exact value of the integral.

Choosing $N_1 = 3$ and $N_2 = 3$ we have, in the notation of the previous section

$$\int_{x=-1}^{x=1}\int_{y=-1}^{y=1} f(x,y) \sim \sum_{h_1=1}^{h_1=3}\sum_{h_2=1}^{h_2=3} f(x^3_{h_1}, y^3_{h_2})\, w^3_{h_1} w^3_{h_2}$$

where

$$w^3_1 = w^3_3 = \frac{8}{15}$$

$w^3_2 = \frac{14}{15}$. We have then

$$\sum_{h_1=1}^{h_1=3}\sum_{h_2=1}^{h_2=3} f(x^3_{h_1}, y^3_{h_2})\, w^3_{h_1} w^3_{h_2} = 4.0258.$$

We have also that

$$\frac{\partial f}{\partial x} \le \frac{4.9}{200} \qquad \frac{\partial f}{\partial y} \le \frac{6.7}{200}$$

so using equation 1 we obtain as a bound on the error of the approximation

$$\text{Error} \le 2^2 \left\{\frac{4\pi}{3}\left(\frac{4.9}{200}\right)(\alpha + \log 3) + \frac{4\pi}{3}\,\frac{(6.7)}{200}\,(\alpha + \log 3)\right\} \le .65.$$

The actual error is $|4.0265 - .40258| = .0007$ which is less than the bound on the error.

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